Analysis of Perishable Queueing-Inventory System with Positive Service Time and $(S-1, S)$ Replenishment Policy

Agassi Melikov and Mammad Shahmaliyev[✉]

Deparment of Computer Sciences,
National Aviation Academy of Azerbaijan, Mardakan pr. 30, Baku, Azerbaijan
agassi.melikov@gmail.com, mammad.shahmaliyev@gmail.com

Abstract. In this paper, model of inventory system with positive service time and perishable inventory is studied. It is assumed that some demands do not acquire the item after service completion and order replenishment lead time is a positive random variable. $(S-1, S)$ order replenishment policy is applied. The exact and approximate methods are developed for calculation of joint distributions of the inventory level and number of customers in the system. The formulas for the system performance measures calculation are given as well. The high accuracy of formulas are confirmed by numerical experiments. The problem of choosing the optimal server rate to minimize the total cost is solved.

Keywords: Perishable inventory systems · Positive service time · $(S-1, S)$ order replenishment policy · Calculation methods

1 Introduction

Service time of demands in classical models of Inventory Systems (IS) is usually assumed to be equal to zero (or inconsiderable). However, in real systems this assumption does not always hold. Therefore, IS models where demand service time is a positive quantity were introduced. These models with positive demand service time are called Queueing-Inventory Systems (QIS) and were first studied in [1,2]. Detailed review of QIS models is given in [3].

In QIS model, usually, it is assumed that after service completion the inventory level decreases. However, in works [4,5] are given the real systems where this condition does not hold and the models of such QIS are studied.

In this paper, studied QIS models are different from the models in [4,5] in following moments. Firstly, unlike in [4,5], the QIS models with perishable inventory are studied (Perishable QIS, PQIS). Secondly, we assume that the arrived demands enter the queue even when the inventory level is zero, while they become impatient in the queue. Thirdly, the mean service time for demands that acquire the item is different than for the demands do not acquiring the item. Finally, it is assumed that the replenishment orders are placed according to $(S-1, S)$ policy.

PQIS models have been extensively studied and developed in a peer-reviewed scientific literature. Numerous literature references on this subject are given in the review works [6–9], as well as, in monography [10]. The results of analysis of PQIS models with positive service time performed in [11–17] could be found in [18]. It should be noted that, the order replenishment policies (ORP) used in the most works belong to a (s, S) policy class.

At the same time, studying PQIS models with positive service time using different ORP in order to find the most optimal policy is a popular research subject. In this paper, $(S - 1, S)$ policy is used. According to this policy, when inventory level decreases (after demand service completion or inventory perishing) an order of unit size is placed.

Some serious results of PQIS analysis with $(S-1, S)$ policy could be found in [19–21]. In these works, service time is assumed to be equal to zero, moreover in [19] the inventory level is right continuous. Analysis of available literature shows that the PQIS models with positive service time and $(S - 1, S)$ policy are not studied. Therefore, methods of exact and asymptotic analysis of PQIS model with finite queue length are given in this paper.

The paper is organized as follows. In Sect. 2, the description of the investigated PQIS model is presented and main performance measures are introduced. Exact and approximate methods to calculate the steady-state probabilities as well as performance measures are developed in Sect. 3. High accuracy of the developed approximate formulas by using numerical experiments are demonstrated in Sect. 4. The results of solution of the problem for choosing optimal server rate to minimize the total cost are shown as well. Conclusion remarks are given in Sect. 5.

2   Model Description and Problem Statement

The studied system has a finite storage warehouse of size S and continuous inventory level monitoring. Each inventory item independently perishes after a random time with exponential distribution function with parameter γ, $\gamma > 0$. At the same time, the item reserved for the demand service is not perishable. In other words, inventory level decreases not only after the demand service, but also because of the item perishing.

Demands are arriving into the system according to Poisson arrival process with the intensity λ for acquiring the inventory items. For the simplicity, we assume that the demands acquiring the item requires unit resource, that is, after service completion of such demands inventory level decreases for a single unit.

If the inventory level is positive upon arrival moment the demand is taken for the service with probability 1 if the server is idle by that time; otherwise, demand joins the queue. Demands are assumed to join the queue even if the inventory level is zero. If upon arrival moment of the demand the inventory level is zero, then according to Bernoulli trial with the parameter ϕ_1 it joins the queue and waits for inventory replenishment for a certain time, while with

the probability ϕ_2 demand leaves the system being unserved, $\phi_1 + \phi_2 = 1$. In that cases, demands in queue are impatient, that is, if inventory level is zero every demand independently waits in the queue for an exponentially distributed random time with mean τ^{-1}.

Queues with finite length is studied in this paper. In the model with finite queue, it is assumed that if at the moment of demand arrival there are N demands in the system (including the one that is being served) then it is lost with probability 1.

After service completion according to Bernoulli trial with parameter σ_1 demand refuses to acquire the item, while with probability σ_2 acquires, where, $\sigma_1 + \sigma_1 = 1$. If the demand refuses to acquire the item its service time has exponential distribution with mean μ_1^{-1}; otherwise its service time is exponentially distributed with mean μ_2^{-1}, $\mu_2 < \mu_1$.

Inventory replenishment is performed according to $(S - 1, S)$ policy with delay, that is, the order lead time is a positive random quantity that has an exponential distribution with mean ν^{-1}. So, if the number of pending orders at the moment is n, then the replenishment rate is $n\nu$.

Problem is to find the joint distributions of inventory level and number of demands in the system. Solution of this problem will allow to calculate the performance measures of PQIS model, as well as, to perform its cost analysis. The main performance measures are the average values of the following quantities: inventory level S_{av}, inventory perishing rate Γ_{av}, average reorder rate RR, loss rate of the customers due to balking RL_b, loss rate of the demands due to reneging RL_r, average queue length L_{av}.

3 Methods for Calculation of the System Performance Measures

System is modeled by 2-D MC with the states (m, n), where m - is inventory level, n - is number of demands in the system. State Space (SS) of the system is defined as follows:

$$E = \{(m, n) : m = 0, 1, \ldots, S, n = 0, 1, \ldots, N\} \tag{1}$$

Transition rate from the state $(m_1, n_1) \in E$ to $(m_2, n_2) \in E$ is denoted by $q((m_1, n_1), (m_2, n_2))$. All of these rates form generator matrix (Q-Matrix) of the given 2-D MC. Let's consider the problem of their calculation.

Transition between the states of the system are related to the following events: (i) demand arrival, (ii) service completion, (iii) product perishing, (iv) leaving the queue due to impatience and (v) inventory replenishment.

Taking into account assumed replenishment policy, following cases are considered while determining the initial state $(m_1, n_1) \in E$ of the system: (1) $m_1 > 0$; (2) $m_1 = 0$.

When $m_1 > 0$ transition from the state (m_1, n_1) because of the events (iv) is impossible, as in that case, demands in the queue are patient. Other transitions are defined as follows.

If the number of demands in system is less than N at the moment of demand arrival (event (i)) then the number of demands increases by one unit, that is, transition to the state $(m_1, n_1+1) \in E$ occurs; intensity of that transition is equal to λ. If after service completion the demand refuses to acquire the item (event of type (ii)), number of demands in the system is decreased by one, while inventory level remains unchanged, i.e. transition to the state $(m_1, n_1 - 1) \in E$ occurs and intensity of such transition is $\mu_1 \sigma_1$. If after service completion demand acquires the item (event of type (ii)), then both number of demands and inventory level decreases by one, that is, transition to the state $(m_1 - 1, n_1 - 1) \in E$ occurs; intensity of such transition is $\mu_2(1 - \sigma_1)$. After inventory item perishes (event of type (iii)) transition to the state $(m_1 - 1, n_1) \in E$ occurs, the intensity of such transition is equal to $m_1 \gamma$ for case $n_1 = 0$ and to $(m_1 - 1)\gamma$ for case $n_1 > 0$. At the moment of order replenishment (event of type (v)) transition to the state $(m_1 + 1, n_1) \in E$ occurs; intensity of such transition is equal to $(S - m_1)\nu$.

Consequently, for the case $m_1 > 0$, non-negative elements of Q-matrix are defined as follows:

$$q((m_1, n_1), (m_2, n_2)) = \begin{cases} \lambda, & \text{if } m_2 = m_1, \ n_2 = n_1 + 1 \\ \mu_1 \sigma_1, & \text{if } m_2 = m_1, \ n_2 = n_1 - 1 \\ \mu_2 \sigma_2, & \text{if } m_2 = m_1 - 1, \ n_2 = n_1 - 1 \\ m_1 \gamma, & \text{if } m_2 = m_1 - 1, \ n_2 = n_1 = 0 \\ (m_1 - 1)\gamma, & \text{if } m_2 = m_1 - 1, \ n_1 > 0, \ n_2 = n_1 \\ (S - m_1)\nu, & \text{if } m_2 = m_1 + 1, \ n_2 = n_1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Now, let at the initial state $(m_1, n_1) \in E$ holds the condition $m_1 = 0$. In this case transition from the current state because of the events (ii) and (iii) is impossible, as in these states demand service could not be performed because of zero inventory level. In these states transitions for the events (i) and (v) are defined analogously as in (2) and the arrived demand joins the queue with the probability ϕ_1. Transition intensities because of demand impatience (event of type (iv)) are defined as follows: after the demand leaves the system because of impatience demand count decreases by one unit, while, inventory level remains unchanged, that is, transition to the state $(0, n_1 - 1) \in E$ occurs; intensity of such transition is equal to $n_1 \tau$. At the moment of the order replenishment transition to the state $(1, n_1)$ occurs; intensity of such transition is equal to $S\nu$. So, for the case $m_1 = 0$ non-negative elements of Q-matrix are defined as follows:

$$q((0, n_1), (m_2, n_2)) = \begin{cases} \lambda \phi_1, & \text{if } m_2 = 0, \ n_2 = n_1 + 1 \\ n_1 \tau, & \text{if } m_2 = 0, \ n_2 = n_1 - 1 \\ S\nu, & \text{if } m_2 = 1, \ n_2 = n_1 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

It is clear from the formulas (2)−(3) that 2-D MC is irreducible and there exists stationary mode. Consequently, the steady-state probabilities $p(m, n)$,

$(m, n) \in E$ are the only solution of the system of balance equations (SBE), that are constructed based on the formulas (2) and (3). This SBE represents the set of linear equations of dimension $(S+1) \times (N+1)$. Due to its large dimension and obviousness the explicit form of SBE is not given in this work.

Required performance measures of the given PQIS are calculated through the steady state probabilities. So, the mean inventory level and the average number of demands in the system are calculated as the mathematical expectation of the corresponding random variables:

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{N} p(m, n); \tag{4}$$

$$L_{av} = \sum_{n=1}^{N} n \sum_{m=0}^{S} p(m, n); \tag{5}$$

As the inventory item reserved for the servicing demand could not perish the average perishing intensity is calculated as follows:

$$\Gamma_{av} = \gamma \left(\sum_{m=1}^{S} (mp(m, 0) + \sum_{n=1}^{N} (m - 1)p(m, n)) \right) \tag{6}$$

Replenishment order is placed either every time after servicing the demands that require the inventory item or after the item perishing. Consequently, the average reorder rate is calculated as follows:

$$RR = \sum_{m=1}^{S} (m\gamma p(m, 0) + ((m - 1)\gamma + \mu_2 \sigma_2)(1 - p(m, 0))) \tag{7}$$

As it is noted above, the balking occurs if at the moment of demand arrival the waiting hall (queue) is full. Therefore, the average loss rate of demands due to balking RL_b is given by:

$$RL_b = \lambda \left(\sum_{m=0}^{S} p(m, N) + \phi_2 \sum_{n=0}^{N-1} p(0, n) \right) \tag{8}$$

The reneging occurs only in the case of zero inventory level. Therefore, the average loss rate of the demands due to reneging RL_r is given by:

$$RL_r = \tau \sum_{n=1}^{N} p(0, n) \tag{9}$$

Analytic solution for the system could not be found due to the complexity of Q-matrix. The known numerical methods of linear algebra are only applicable for the Markov Chains of the moderate dimensions and become useless for the chains of larger dimension.

Therefore, approximate method [18] is used in this work that allows to perform asymptotic analysis of the performance measures of the given system for the large sizes of the inventory level and waiting hall for demands.

This method could be effectively applied for the models that work under large load; in other words, it is assumed that demand arrival intensity is far larger than the product perishing and replenishment rate, that is, $\lambda \gg max\{\gamma, \nu\}$. It should be noted that, this assumption is hold in many real PQIS. Moreover, as it was stated above, $\mu_1 \gg \mu_2$.

Taking into account the above conditions, let's consider the following split of the initial state space (1):

$$E = \bigcup_{m=0}^{S} E_m, \ E_{m_1} \bigcap E_{m_2} = 0, \ m_1 \neq m_2 \qquad (10)$$

where $E_m = \{(m, n) : n = 0, 1, \ldots, N\}, \ m = 0, 1, \ldots, S$.

Additionally, we conclude that the transition intensities inside a row are far larger than the transition intensities between the rows. Further, based on the split (10) of the initial state space (1), the following merge function is defined:

$$U((m, n)) = \langle m \rangle$$

where $\langle m \rangle$ is merged state that consists of all the states $E_m, \ m = 0, 1, \ldots, S$. Let's denote $\Omega = \{\langle m \rangle : m = 0, 1, \ldots, S\}$.

Approximate values of steady state probabilities $\tilde{p}(m, n), \ (m, n) \in E$ of the current model are defined as follows (see [18]):

$$\tilde{p}(m, n) = \rho_m(n)\pi(\langle m \rangle) \qquad (11)$$

where $\rho_m(n)$ - is the probability of state (m, n) inside the merged model with the state space E_m and $\pi(\langle m \rangle)$ - is the probability of a merged state $\langle m \rangle \in \Omega$.

Steady-state probabilities of the split and merged models are calculated as follows.

In the all states (m, n) within the split model with the state space E_m the first component is a constant, therefore, all the states of such class is determined only by the second component. Consequently, in the analysis of the split models with the state space E_m the state $(m, n) \in E_m$ could only be specified with the second component, so for the convenience, while studying the split models with the state space E_m its states (m, n) are simply denoted by $n, \ n = 0, 1, \ldots, N$.

It is concluded from the formulas (2) that the state probabilities within all the split models with the state space $E_m, \ m = 1, 2, \ldots, S$ are the same as in classical model $M/M/1/N$ with load $a = \lambda/\mu_1\sigma_1$, i.e.:

$$\rho_m(n) = a^n \frac{1 - a}{1 - a^{N+1}}, \ m = 1, 2, \ldots, S \qquad (12)$$

Remark 1. As the quantities $\rho_m(n)$ do not depend on the index m, $m = 1, 2, \ldots, S$ below these indexes are omitted.

It is concluded from the formula (3) that the state probabilities within the merged model with the state space E_0 are the same as in Erlang model $M/M/N/0$ with load $b = \lambda\phi_1/\tau$, that is:

$$\rho_0(n) = \frac{\theta(n)}{\sum\limits_{n=0}^{N} \theta(j)}, \quad n = 0, 1, \ldots, N \tag{13}$$

Further, the following notation is accepted: $\theta(j) = \dfrac{b_j}{j!}$

Let's denote the transition intensity from the merged state $\langle m_1 \rangle$ to another merged state $\langle m_2 \rangle$ with $q(\langle m_1 \rangle, \langle m_2 \rangle)$, $\langle m_1 \rangle, \langle m_2 \rangle \in \Omega$. According to [18] these parameters are defined as follows:

$$q(\langle m_1 \rangle, \langle m_2 \rangle) = \sum_{\substack{(m_1, n_1) \in E_{m_1}, \\ (m_2, n_2) \in E_{m_2}}} q((m_1, n_1), (m_2, n_2)) p(m_1, n_1) \tag{14}$$

Taking into account (2), (3) and (12), (13), (14) after some transformations we found that the given intensities are calculated as follows:

$$q(\langle m_1 \rangle, \langle m_2 \rangle) = \begin{cases} \Lambda(m_1), & \text{if } m_2 = m_1 - 1 \\ (S - m_1)\nu, & \text{if } m_2 = m_1 + 1 \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where $\Lambda(m_1) = m_1 \gamma \rho_0 + (1 - \rho(0))(\mu_2 \sigma_2 + (m_1 - 1)\gamma)$, $m_1 = 1, 2, \ldots, S$

Then from (15) we get:

$$\pi(\langle m \rangle) = \frac{S!\nu^m}{(S-m)!} \frac{\pi(0)}{\prod\limits_{i=1}^{m} \Lambda(i)}, \quad m = 1, 2, \ldots, S \tag{16}$$

where $\pi(0) = \left(\sum\limits_{m=0}^{S} \frac{S!\nu^m}{(S-m)!} \frac{1}{\prod\limits_{i=1}^{m} \Lambda(i)} \right)^{-1}$

Remark 2. We assume that $\prod\limits_{i=m}^{n} x_i = 1$, if $m > n$

Afterwards, taking into account (12), (13), (14), (15), (16) from (11) approximate joint distributions $\tilde{p}(m, n)$, $(m, n) \in E$ of inventory level and number of demands in the system are found. Using these probabilities after some transformations

from (4), (5), (6), (7), (8), (9) following formulas are obtained for the calculation of performance measures:

$$S_{av} \approx \sum_{m=1}^{S} m\pi(\langle m \rangle)$$

$$\Gamma_{av} \approx \gamma \sum_{m=1}^{S} \pi(\langle m \rangle)(m\rho(0) + (m-1)(1 - \rho(0)))$$

$$RR \approx \sum_{m=1}^{S} (m\gamma\rho(0)\pi(m) + (1 - \rho(0)\pi(\langle m \rangle))((m-1)\gamma + \mu_2\sigma_2))$$

$$RL_b \approx \lambda(\rho(N)(1 - \pi(\langle 0 \rangle)) + \pi(\langle 0 \rangle)(\rho_0(N) + \phi_2(1 - \rho_0(N))))$$

$$RL_r \approx \tau\pi(\langle 0 \rangle) \sum_{n=1}^{N} n\rho_0(n)$$

$$L_{av} \approx \pi(\langle 0 \rangle) \sum_{n=1}^{N} n\rho_0(n) + (1 - \pi(0)) \sum_{n=1}^{N} n\rho(n)$$

4 Numerical Results

Due to the limitations to the volume of the work, only accuracy of the steady-state probabilities of the initial 2-D MC and performance measures is considered. It should be noted that, the evaluation of the accuracy of given formulas analytically is impossible. Therefore, comparative analysis of the obtained numerical results is used. The accuracy of the approximate values are evaluated by using following norms:

Maximum absolute value of differences:

$$\|N\|_1 = \max_{n \in E} |p(n) - \tilde{p}(n)| \tag{17}$$

Cosine similarity:

$$\|N\|_2 = \frac{\sum\limits_{n \in E} p(n)\tilde{p}(n)}{\sqrt{\sum\limits_{n \in E} (p(n))^2} \sqrt{\sum\limits_{n \in E} (\tilde{p}(n))^2}} \tag{18}$$

Jaccard coefficient [22]:

$$\|N\|_3 = \frac{\sum\limits_{n \in E} \min\{p(n), \tilde{p}(n)\}}{\sum\limits_{n \in E} \max\{p(n), \tilde{p}(n)\}} \tag{19}$$

Results of the comparative analysis of the steady-state probabilities for the exact and approximate methods are given in Table 1. The initial parameters of

Table 1. Estimation of accuracy of the steady-state probabilities versus various norms

Values of parameters			Norms		
S	N	λ	$\|N\|_1$	$\|N\|_2$	$\|N\|_3$
10	10	60	0.009167	0.999201	0.930769
	30	60	0.009354	0.999193	0.931855
	50	60	0.00966	0.99913	0.928858
	70	60	0.010289	0.998972	0.914683
20	10	60	0.006392	0.999313	0.932304
	30	60	0.006392	0.999313	0.932337
	50	60	0.006393	0.999312	0.932285
	70	60	0.006393	0.999312	0.932011
	90	60	0.006393	0.999311	0.931672
	110	60	0.006394	0.999311	0.931521
30	10	60	0.005361	0.999334	0.932366
	30	60	0.005361	0.999334	0.932366
	50	60	0.005361	0.999334	0.932366
	70	60	0.005361	0.999334	0.932361
	90	60	0.005361	0.999334	0.932355
	110	60	0.005361	0.999334	0.932352
40	10	40	0.006973	0.998495	0.900238
		60	0.00465	0.999344	0.932367
	30	40	0.006973	0.998495	0.900238
		60	0.00465	0.999344	0.932367
	50	40	0.006973	0.998495	0.900237
		60	0.00465	0.999344	0.932367
	70	40	0.006973	0.998495	0.900237
		60	0.00465	0.999344	0.932367
	90	40	0.006973	0.998495	0.900237
		60	0.00465	0.999344	0.932367
	110	40	0.006973	0.998495	0.900237
		60	0.00465	0.999344	0.932367
	120	60	0.00465	0.999344	0.932367
50	10	40	0.006206	0.998509	0.900238
		60	0.00413	0.999351	0.932367
	30	40	0.006206	0.998509	0.900238
		60	0.00413	0.999351	0.932367
	50	40	0.006206	0.998509	0.900238
		60	0.00413	0.999351	0.932367
	70	40	0.006206	0.998509	0.900238
		60	0.00413	0.999351	0.932367
	90	40	0.006206	0.998509	0.900238
		60	0.00413	0.999351	0.932367
	110	40	0.006206	0.998509	0.900238
		60	0.00413	0.999351	0.932367

the system are assumed as follows: $\mu_1 = 15, \mu_2 = 3, \gamma = 2, \nu = 1, \tau = 0.5, \sigma_1 = 0.3, \phi_1 = 0.6$

The exact values of steady-state probabilities are calculated from corresponding balance equations using MATLAB package. Solving time of balance equations depends on its dimension and takes several hours for $S \times N > 5000$ (e.g.: for $S = 50$ and $N = 100$ with quad core CPU Core i7 2.40 Ghz and 8 GB RAM at least 3–4 h are required). It should be noted that, in the same PC approximately 3–4 s are needed for the calculation of performance measures for $S = 100$ and $N = 500$ while using the approximate method.

It is obvious from the Table 1 that the higher the arrival intensity is, the better accuracy of the calculated steady state probabilities of the model with respect to all norms is acquired, that is, with the increase of the arrival intensity the norm (17) is approaching 0, while the norms (18) and (19) are approaching 1. It is clear from split scheme (10) that with the increase of arrival intensity, the transition intensities between the state classes $E_m, m = 1, 2, \ldots, S$ decrease; the smaller intensities between the classes of states of split model we have, the more accurate state probabilities of initial model we get. For the above initial data the analysis of accuracy of the system performance measures was performed as well (see Tables 2 and 3). It should be noted that, the performance measures (4), (6), (7), (8), (9) are almost the same when using exact and approximate approaches (see Table 2). Only the small errors (less than 5%) are observed while calculating measure (5) and this is acceptable in engineering calculations (see Table 3).

Table 2. Estimation of accuracy of accuracy of the performance measures (4), (6) and (7) for $N \in [20, 120], \lambda \in [20, 60]$; EV - Exact Value, AV - Approximate Value

S	Sav		Γav		RR	
	EV	AV	EV	AV	EV	AV
10	3.300632	3.300632	4.639206	4.639206	111	111
20	6.633345	6.633345	11.26737	11.26737	422	422
30	9.966667	9.966667	17.933346	17.933346	933	933
40	13.3	13.3	24.6	24.6	1,644.00	1,644.00
50	16.633333	16.633333	31.266667	31.266667	2,555.00	2,555.00

Remark 3. Zero values in Table 3 are not exactly equal to zero and are obtained after rounding the numbers to the 6th order precision.

Now let's consider the problem of choosing the most optimal server. Let it is possible to choose the server from the predefined collection, where with the increase of the service rate the cost associated with the corresponding server increases as well. It is required to choose such server that will minimize the long-run expected total cost (TC).

$$TC = c_h S_{av} + c_r RR + c_b RL_b + c_g RL_r + c_p \Gamma_{av} + c_w L_{av} + c_s P_b \qquad (20)$$

Table 3. Estimation of accuracy of performance measures (5), (8) and (9)

S	N	λ	RL_b EV	AV	RL_r EV	AV	L_{av} EV	AV
10	10	60	53.434375	55.491991	0.090831	0.093377	9.873783	9.917505
	30	40	33.251864	35.313561	0.273342	0.271806	29.782387	29.850142
		60	53.248114	55.307103	0.277091	0.278264	29.864775	29.907871
	50	40	33.069064	35.168018	0.456141	0.417349	49.75709	49.761818
		60	53.063085	55.128522	0.462121	0.456846	49.851439	49.885624
	70	60	52.87945	54.974741	0.645755	0.610626	69.832213	69.813776
20	10	60	53.400653	55.499856	0.001592	0.001674	9.876329	9.918894
	30	40	33.397364	35.496658	0.004881	0.004873	29.802009	29.872825
		60	53.3973	55.496542	0.004944	0.004989	29.876169	29.918721
	50	40	33.394035	35.494048	0.00821	0.007482	49.801639	49.871242
		60	53.393956	55.49334	0.008288	0.00819	49.875961	49.918322
	70	40	33.390713	35.493373	0.011532	0.008158	69.80117	69.865791
		60	53.39062	55.490583	0.011625	0.010947	69.875698	69.917034
	90	40	33.387395	35.493368	0.01485	0.008162	89.800598	89.858999
		60	53.38729	55.489352	0.014954	0.012178	89.875373	89.912694
	110	40	33.38408	35.493368	0.018164	0.008162	109.79992	109.8522
		60	53.383966	55.489287	0.018278	0.012243	109.87498	109.90602
30	10	60	53.400012	55.499997	0.000028	0.00003	9.876403	9.918918
	30	60	53.399953	55.499939	0.000087	0.000088	29.8764	29.918915
	50	40	33.399895	35.499895	0.000145	0.000132	49.802382	49.873204
		60	53.399893	55.499882	0.000146	0.000145	49.876396	49.918908
	70	40	33.399835	35.499883	0.000204	0.000144	69.802374	69.873108
		60	53.399834	55.499833	0.000206	0.000194	69.876392	69.918886
	90	40	33.399776	35.499883	0.000264	0.000144	89.802366	89.872988
		60	53.399775	55.499812	0.000265	0.000215	89.876387	89.918809
	110	40	33.399717	35.499883	0.000323	0.000144	109.80236	109.87287
		60	53.399715	55.49981	0.000324	0.000217	109.87638	109.91869
40	10	60	53.4	55.5	0	0.000001	9.876404	9.918919
	30	60	53.399999	55.499999	0.000002	0.000002	29.876404	29.918919
	50	60	53.399998	55.499998	0.000003	0.000003	49.876404	49.918919
	70	60	53.399997	55.499997	0.000004	0.000003	69.876404	69.918918
	90	60	53.399996	55.499997	0.000005	0.000004	89.876404	89.918917
	110	60	53.399995	55.499997	0.000006	0.000004	109.8764	109.91892
	120	40	33.399994	35.499998	0.000006	0.000003	119.80239	119.87323
		60	53.399994	55.499997	0.000006	0.000004	119.8764	119.91891
50	10	60	53.4	55.5	0	0	9.876404	9.918919
	30	60	53.4	55.5	0	0	29.876404	29.918919
	50	60	53.4	55.5	0	0	49.876404	49.918919
	70	60	53.4	55.5	0	0	69.876404	69.918919
	90	60	53.4	55.5	0	0	89.876404	89.918919
	110	60	53.4	55.5	0	0	109.8764	109.91892

where P_b is probability that server is busy, i.e. $P_b = \sum\limits_{m=1}^{S} \sum\limits_{n=1}^{N} p(m, n)$. Here c_h is inventory carrying cost per unit item, c_r is setup cost per order, c_b is balking cost per customer, c_g is reneging cost per customer per unit time, c_p is the perishing cost per item per unit time, c_w is waiting time cost of a customer per unit time.

We assume that there are four possible options to choose the server: (1) $\mu_1 = 5, \mu_2 = 1$; (2) $\mu_1 = 8, \mu_2 = 2$; (3) $\mu_1 = 10, \mu_2 = 4$; (4) $\mu_1 = 15, \mu_2 = 5$. The values of the coefficients c_s when choosing the option $k, k = 1, 2, 3, 4$ are designated as $c_s^{(k)}$ and defined as: $c_s^{(1)} = 1, c_s^{(2)} = 2, c_s^{(3)} = 3, c_s^{(4)} = 4$. The values of other parameters in the (20) are constants: $c_h = 1, c_r = 0.1, c_b = 3, c_g = 2, c_p = 2, c_w = 1$.

Table 4. Results of solution of the problem (20) for $N = 150$

S	λ			
	5	10	15	30
10	4	4	4	4
50	4	2	2	2
80	1	1	1	1

Some results of the problem solution for the above data are given in Table 4., where the numbers 1, 2 and 4 indicate the index of the optimal server selection option. It is obvious from the Table 4. that if the inventory level is increasing the optimal option is to choose the server with the lesser service rate, and, vice versa, for the smaller values of the inventory level the optimal option will be the servers with the greater service rates.

5   Conclusion

PQIS model with perishable inventory and positive service time is studied in this paper. It is assumed that some demands do not acquire the item after service completion. When inventory level is zero, demands join or leave the system according to Bernoulli trial. Demands are impatient in the queue when the inventory level is zero. Order lead time, as well as, item perishing time are random variables with the exponential distributions and finite mean. Inventory replenishment policy belongs to $(S - 1, S)$ class. Exact and approximate formulas are given for calculation of steady-state probabilities of the given 2-D MC being the mathematical model of the studied system. Exact method is based on the solving of balance equations and is suitable for the moderate values of inventory level and length of the waiting hall for queuing the demands. Approximate approach is based on the state phase merging algorithms of 2-D Markov Chains and it is applicable for the systems of any dimension. High accuracy of the given formulas are shown using numerical experiments. Finally, the optimization problem of choosing optimal server for cost minimization is solved.

References

1. Sigman, K., Simchi-Levi, D.: Light traffic heuristic for an M/G/1 queue with limited inventory. Ann. Oper. Res. **40**, 371–380 (1992)
2. Melikov, A.Z., Molchanov, A.A.: Stock optimization in transport/storage systems. Cybernetics **27**(3), 484–487 (1992)
3. Krishnamoorthy, A., Lakshmy, B., Manikandan, R.: A survey on inventory models with positive service time. OPSEARCH **48**(2), 153–169 (2011)
4. Krishnamoorthy, A., Manikandan, R., Lakshmy, B.: Revisit to queueing-inventory system with positive service time. Ann. Oper. Res. **233**, 221–236 (2015)
5. Krishnamoorthy, A., Manikandan, R., Shajin, D.: Analysis of a multi-server queueing-inventory system. Adv. Oper. Res. (Hindawi Publ. Corp.) **2015**, 16 (2015). Article ID: 747328
6. Baron, O., Berman, O., Perry, D.: Continuous review inventory models for perishable items ordered in batches. Math. Methods Oper. Res. **72**, 217–247 (2010)
7. Lawrence, A.S., Sivakumar, B., Arivarignan, G.: A perishable inventory system with service facility and finite source. Appl. Math. Model. **37**, 4771–4786 (2013)
8. Goyal, S., Giri, B.: Recent trends in modeling of deteriorating inventory. Eur. J. Oper. Res. **134**, 1–16 (2011)
9. Karaesmen, I., Scheller-Wolf, A., Deniz, B.: Managing perishable and aging inventories: review and future research directions. In: Kempf, K., Keskinocak, P., Uzsoy, R. (eds.) Planning Production and Inventories in the Extended Enterprise. ISOR, vol. 151, pp. 393–436. Springer, Boston (2011). doi:10.1007/978-1-4419-6485-4_15
10. Nahmias, S.: Perishable Inventory Theory. Springer, Heidelberg (2011)
11. Sivakumar, B., Arivarignan, G.: A perishable inventory system with service facilities and negative customers. Adv. Model. Optim. **7**, 193–210 (2006)
12. Manuel, P., Sivakumar, B., Arivarignan, G.: A perishable inventory system with service facilities, MAP arrivals and PH-service times. J. Syst. Sci. Syst. Eng. **16**, 62–73 (2007)
13. Manuel, P., Sivakumar, B., Arivarignan, G.: A perishable inventory system with service facilities and retrial customers. Comput. Ind. Eng. **54**, 484–501 (2008)
14. Amirthakodi, M., Radhamami, V., Sivakumar, B.: A perishable inventory system with service facility and feedback customers. Ann. Oper. Res. **233**, 25–55 (2015)
15. Hamadi, H.M., Sangeetha, N., Sivakumar, B.: Optimal control of service parameter for a perishable inventory system maintained at service facility with impatient customers. Ann. Oper. Res. **233**, 3–23 (2015)
16. Berman, O., Sapna, K.P.: Optimal service rate of service facility with perishable inventory items. Nav. Res. Logist. **49**, 464–482 (2002)
17. Jajaraman, B., Sivakumar, B., Arivarignan, G.: A perishable inventory system with postponed demands and multiple server vacations. Model. Simul. Eng. (Hindawi Publ. Corp.) **2012**, 17 (2012). Article ID: 620960
18. Melikov, A.Z., Ponomarenko, L.A., Shahmaliyev, M.: Models of perishable queueing-inventory system with repeated customers. J. Autom. Inf. Sci. **48**(2), 22–38 (2016)
19. Kalpakam, S., Sapna, K.P.: (S-1, S) perishable systems with stochastic lead times. Math. Comput. Model. **21**(6), 95–104 (1995)
20. Kranenburg, A.A., van Houtum, G.J.: Cost optimization in the (S-1, S) lost sales inventory model with multiple demand classes. Oper. Res. Lett. **35**, 493–502 (2007)

21. Isotupa, K.P.S.: Cost analysis of an (S-1, S) inventory system with two demand classes and rationing. Ann. Oper. Res. **233**, 411–421 (2015)
22. Jaccard, P.: Etude de la distribution florale dans une portion des Alpes et du Jura. Bull. de la Soc. Vaud. des Sci. Nat. **37**, 547–579 (1901). (in French)