# Research of Mathematical Model of Insurance Company in the Form of Queueing System in a Random Environment

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**Abstract.** The present paper is devoted to the research of the mathematical model of an insurance company in the form of the queueing system with an infinite number of servers. The arrival process of risks is regarded as a modulated Poisson arrival process. Applying the asymptotic analysis method under the condition of a high-rate arrivals, the characteristic function of the probability distribution for the two-dimensional process of the number of risks and the number of claims for insurance payments is obtained. It is shown that this probability distribution can be approximated by Gaussian distribution. These results can be applied to the estimation of functioning of the various economic systems.

**Keywords:** Mathematical model · Insurance company · Insurance payments · Queueing system · Characteristic function · Asymptotic analysis

## 1 Introduction

At present, the research and modeling of economic systems are paid a great deal of attention. These problems is usually related to research in the field of arrival processes. The results of these studies show that the classic models (for example, the Poisson ones) are not exactly modeling real arrival processes. Thus, the problem of the research models of economic systems with reference to this aspect becomes quite relevant. For example, the intensity of incoming risks into the insurance company is not a constant and it depends on the impact of external random factors such as season, state policy, probability of natural disasters, fashion, etc. On the whole, all papers focused on the research of mathematical models of insurance company include characteristics of the performance of a company with a stationary Poisson arrival process of risks. Thus, these models are reviewed in [1]. In the paper [2] the distribution of claims for insurance payments with a random value of contract duration is obtained. Applying an asymptotic analysis method, in [3] we obtain the two-dimensional probability distribution of the number of risks and the number of payments. The model with a possibility of reinsurance is investigated in [4]. Papers [5,6] cover the model with implicit advertisement and one-time insurance payment for limited


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and unlimited insurance coverage. In this paper, we consider the mathematical model of an insurance company in a random environment, when the rate of the arrival process, the rate of occurrences of the insured events and the contract duration are not constants and depend on the impact of external factors and change with time, which is undoubtedly present in real life.

## 2 Mathematical Model

Let us have a look at the model of an insurance company with infinite insurance coverage [7] (Fig. 1) in the form of a queueing system with an infinite number of servers. We can assume that risks (customers) coming into the company form high-rate modulated Poisson arrival process that is regulated by the random process $k(t)$ [8]. This process is a Markov chain with a continuous time that is defined by the matrix $N\mathbf{Q}$ of infinitesimal characteristics $Nq_{k\nu}$, where $k = 1, \ldots, K$, $\nu = 1, \ldots, K$ and $N$ has a large value (we suppose that $N \to \infty$ ).

Let us define the matrix $N\mathbf{\Lambda}$ with elements $N\lambda_k$ on the main diagonal. Here $N\lambda_k$ — the intensity of risks coming into the company, when Markov chain is in $k$ state, $\lambda_k$ — fixed value. Thus, the Markov chain $k(t)$ state defines the state of a random environment.

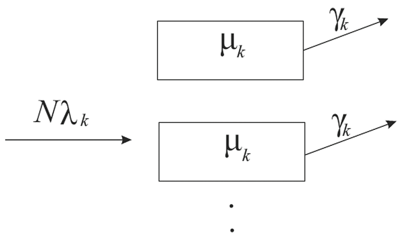


**Fig. 1.** Model of the insurance company in the form of queueing system with an infinite number of servers in a random environment.

After coming into the company the risk makes the insurance contract. The contract duration is the duration of serving a customer at a server. Each risk that is in the company during the contract duration generates claim for the insurance payment with intensity $\gamma_k$ independently from other risks. These intensities also depend on the environment state and form a diagonal matrix $\mathbf{\Gamma}$. Requirements for insurance payments also form a random process. It is natural to assume that the claim for payment is determined by the occurrence of the insured event. The contract duration for each risk in the company is considered to be random value, exponentially distributed with a parameter $\mu_k$, that is also dependent on the environment state. These values form the diagonal matrix $\mathbf{M}$.

Let us denote: $n(t)$ — number of claims for payments over the time interval $[0, t]$, $i(t)$ — number of insurance risks that are in the company at the moment $t$, $P_k(i, n, t) = \mathrm{P}\{i(t) = i, n(t) = n, k(t) = k\}$ — probability of a number of risks in the company at the moment $t$ equals to $i$, a number of claims for payments at the moment $t$ equals to $n$ and environment is in the $k$ state at the moment $t$. The problem is to obtain the expression for characteristic function of two-dimensional random process $(i(t),\, n(t))$ .

## 3 Kolmogorov Equations

Let us set up a system of Kolmogorov differential equations [9] for probability distribution $P_k(i,n,t)$. Using the notation $P_k(i,n,t) = \mathrm{P}\{i(t) = i, n(t) = n, k(t) = k\}$ and applying the formula of total probability, we can write the following equations

$$\begin{gathered} P_k(i,n,t+\Delta t) = P_k(i,n,t)(1 - N\lambda_k \Delta t)(1 - i\gamma_k \Delta t)(1 - i\mu_k \Delta t) \\ \times (1 + Nq_{kk}\Delta t) + N\lambda_k \Delta t P_k(i-1,n,t) + i\gamma_k \Delta t P_k(i,n-1,t) \\ + (i+1)\mu_k \Delta t P_k(i+1,n,t) + \sum_{\nu \neq k} P_\nu(i,n,t) N q_{\nu k} \Delta t + o(\Delta t)\ . \end{gathered} \tag{1}$$

for $k = 1, \ldots, K$. After performing some transformation, we derive the following system of the Kolmogorov differential equations for the probability distribution of the two-dimensional process $(i(t),\, n(t))$

$$\begin{gathered} \frac{\partial P_k(i,n,t)}{\partial t} = -(N\lambda_k + i\mu_k + i\gamma)P_k(i,n,t) + N\lambda P_k(i-1,n,t) \\ + (i+1)\mu_k P_k(i+1,n,t) + i\gamma_k P_k(i,n-1,t) + \sum_{\nu=1}^{K} P_\nu(i,n,t) N q_{\nu k}\ . \end{gathered} \tag{2}$$

To solve the system (2) let us consider partial characteristic functions:

$$H_k(u_1,u_2,t) = \sum_{i,n=0}^{\infty} e^{ju_1 i} e^{ju_2 n} P_k(i,n,t),$$

for $k = 1, \ldots, K$, $j$ — imaginary unit. Then, using system (2) and takint into account the properties of characteristic functions, we will obtain the first-order partial differential equation for $H_k(u_1,u_2,t)$ in the following form:

$$\begin{aligned} \frac{\partial H_k(u_1,u_2,t)}{\partial t} &= N\lambda_k(e^{ju_1} - 1)H_k(u_1,u_2,t) + \sum_{\nu=1}^{K} H_\nu(u_1,u_2,t) q_{\nu k} \\ &\quad + j\frac{\partial H_k(u_1,u_2,t)}{\partial u_1}(\mu_k - \mu_k e^{-ju_1} + \gamma_k - \gamma_k e^{ju_2})\ . \end{aligned} \tag{3}$$

Let us consider the vector characteristic function

$$\mathbf{H}(u_1,u_2,t) = \{H_1(u_1,u_2,t),\, H_2(u_1,u_2,t), \ldots, H_K(u_1,u_2,t)\}\ .$$

Thus, using (3) we can write the matrix differential equation for the function $\mathbf{H}(u_1, u_2, t)$

$$\frac{\partial \mathbf{H}(u_1,u_2,t)}{\partial t} = \mathbf{H}(u_1,u_2,t)[N\mathbf{\Lambda}(e^{ju_1}-1)+N\mathbf{Q}] \\ +j\frac{\partial \mathbf{H}(u_1,u_2,t)}{\partial u_1}[(1-e^{-ju_1})\mathbf{M}+(1-e^{ju_2})\mathbf{\Gamma}] , \tag{4}$$

matrixes $N\mathbf{\Lambda}$, $\mathbf{M}$, $\mathbf{\Gamma}$, $N\mathbf{Q}$ are defined above.

We will solve the Eq. (4) for vector characteristic function $\mathbf{H}(u_1, u_2, t)$ using the asymptotic analysis method [10] under conditions of high-rate arrival process and extremely often changes of a random environment states ($N \to \infty$).

## 4 The First-Order Asymptotic Analysis

Let us make the following changes to the variables in the Eq. (4):

$$\varepsilon = \frac{1}{N},\ u_1 = \varepsilon\omega_1,\ u_2 = \varepsilon\omega_2,\ \mathbf{H}(u_1,u_2,t) = \mathbf{F}(\omega_1,\omega_1,t,\varepsilon) . \tag{5}$$

Using these new variables we will write the equation for function $\mathbf{F}(\omega_1, \omega_1, t, \varepsilon)$:

$$\varepsilon\frac{\partial \mathbf{F}(\omega_1,\omega_2,t,\varepsilon)}{\partial t} = \mathbf{F}(\omega_1,\omega_2,t,\varepsilon)[\mathbf{\Lambda}(e^{j\omega_1\varepsilon}-1)+\mathbf{Q}] \\ +j\frac{\partial \mathbf{F}(\omega_1,\omega_2,t,\varepsilon)}{\partial \omega_1}[(1-e^{-j\omega_1\varepsilon})\mathbf{M}+(1-e^{j\omega_2\varepsilon})\mathbf{\Gamma}] . \tag{6}$$

Denote an asymptotic solution to this equation under the condition $\varepsilon \to 0$ by $\mathbf{F}(\omega_1, \omega_2, t)$:

$$\lim_{\varepsilon\to 0}\mathbf{F}(\omega_1,\omega_2,t,\varepsilon) = \mathbf{F}(\omega_1,\omega_2,t) .$$

Let us perform the asymptotic transition $\varepsilon \to 0$ in the Eq. (6). We will obtain

$$\mathbf{F}(\omega_1,\omega_2,t)\mathbf{Q} = \mathbf{0} . \tag{7}$$

Thus, the function $\mathbf{F}(\omega_1, \omega_2, t)$ is a solution for the homogeneous system of the linear algebraic Eq. (7). Solution for this system has the following form:

$$\mathbf{F}(\omega_1,\omega_2,t) = \mathbf{R}\Phi(\omega_1,\omega_2,t) , \tag{8}$$

where $\Phi(\omega_1, \omega_2, t)$ — some scalar function, $\mathbf{R}$ — a row vector of stationary probability distribution of Markov chain $k(t)$, that is defined by the equations system $\mathbf{RQ} = \mathbf{0}$ and a normalization condition $\mathbf{RE} = 1$, where $\mathbf{0}$ — a row vector with zeros and $\mathbf{E}$ — a column vector with enteries all equal to 1. To obtain function

$\Phi(\omega_1, \omega_2, t)$, we will sum up equations of the system (6). Taking into account condition $\mathbf{F}(\omega_1, \omega_2, t)\mathbf{Q} = \mathbf{0}$, we can write

$$\varepsilon \frac{\partial \mathbf{F}(\omega_1, \omega_2, t, \varepsilon)}{\partial t}\mathbf{E} = \mathbf{F}(\omega_1, \omega_2, t, \varepsilon)(e^{j\omega_1\varepsilon} - 1)\mathbf{\Lambda E}$$
$$+ j\frac{\partial \mathbf{F}(\omega_1, \omega_2, t, \varepsilon)}{\partial \omega_1}[(1 - e^{-j\omega_1\varepsilon})\mathbf{ME} + (1 - e^{j\omega_2\varepsilon})\mathbf{\Gamma E}] . \quad (9)$$

Let us divide left and right sides of the Eq. (9) by $\varepsilon$ and perform the asymptotic transition $\varepsilon \to 0$. We obtain the equation for $\mathbf{F}(\omega_1, \omega_2, t)$:

$$\frac{\partial \mathbf{F}(\omega_1, \omega_2, t)}{\partial t}\mathbf{E} = \mathbf{F}(\omega_1, \omega_2, t)j\omega_1\mathbf{\Lambda E}$$
$$- \omega_1\frac{\partial \mathbf{F}(\omega_1, \omega_2, t)}{\partial \omega_1}\mathbf{ME} + \omega_2\frac{\partial \mathbf{F}(\omega_1, \omega_2, t)}{\partial \omega_1}\mathbf{\Gamma E} . \quad (10)$$

Now we can write down the equation for the unknown scalar function $\Phi(\omega_1, \omega_2, t)$ considering $\mathbf{F}(\omega_1, \omega_2, t) = \mathbf{R}\Phi(\omega_1, \omega_2, t)$ and $\mathbf{RE} = 1$ in the following form:

$$\frac{\partial \Phi(\omega_1, \omega_2, t)}{\partial t} = \Phi(\omega_1, \omega_2, t)j\omega_1\mathbf{R\Lambda E}$$
$$-\omega_1\frac{\partial \Phi(\omega_1, \omega_2, t)}{\partial \omega_1}\mathbf{RME} + \omega_2\frac{\partial \Phi(\omega_1, \omega_2, t)}{\partial \omega_1}\mathbf{R\Gamma E} . \quad (11)$$

We have the first-order partial differential equation. Its solution is defined by solving a system of ordinary differential equations for characteristic curves [11]:

$$\frac{\partial t}{1} = \frac{\partial \Phi(\omega_1, \omega_2, t)}{\Phi(\omega_1, \omega_2, t)j\omega_1\kappa} = \frac{\partial \omega_1}{\omega_1\kappa_1 - \omega_2\kappa_2} , \quad (12)$$

where $\kappa = \mathbf{R\Lambda E}$, $\kappa_1 = \mathbf{RME}$, $\kappa_2 = \mathbf{R\Gamma E}$. Let us obtain first two integrals of this system. We can write following equation:

$$\frac{\partial t}{1} = \frac{\partial \Phi(\omega_1, \omega_2, t)}{\Phi(\omega_1, \omega_2, t)j\omega_1\kappa} . \quad (13)$$

The solution of Eq. (13) we will write down in the following form:

$$t = \frac{1}{\kappa_1}\ln(\omega_1\kappa_1 - \omega_2\kappa_2) - \ln C , \quad (14)$$

where $C$ is constant. Let us denote $C_1 = C^{\kappa_1}$, then $C_1 = (\omega_1\kappa_1 - \omega_2\kappa_2)e^{-t\kappa_1}$ . The other first integral we will obtain from the equation

$$\frac{\partial \Phi(\omega_1, \omega_2, t)}{\Phi(\omega_1, \omega_2, t)j\omega_1\kappa} = \frac{\partial \omega_1}{\omega_1\kappa_1 - \omega_2\kappa_2} . \quad (15)$$

The solution of the Eq. (15) has the following form:

$$\Phi(\omega_1, \omega_2, t) = e^{j\frac{\kappa\omega_1}{\kappa_1}} (\omega_1\kappa_1 - \omega_2\kappa_2)^{j\frac{\kappa\omega_2\kappa_2}{\kappa_1^2}} C_2 . \tag{16}$$

Let us introduce an arbitrary differentiated function $\phi(C_1) = C_2$. Then the general solution of the equation (15) will have the following form

$$\Phi(\omega_1, \omega_2, t) = e^{j\frac{\kappa\omega_1}{\kappa_1}} (\omega_1\kappa_1 - \omega_2\kappa_2)^{j\frac{\kappa\omega_2\kappa_2}{\kappa_1^2}} \phi((\omega_1\kappa_1 - \omega_2\kappa_2)e^{-t\kappa_1}) . \tag{17}$$

Let us define the partial solution with the help of initial conditions. We have to define $\Phi(\omega_1, \omega_2, 0)$ first. Let us write down value of functions $H_k(u_1, u_2, t)$, $k = 1 \ldots K$, at the moment $t = 0$:

$$H_k(u_1, u_2, 0) = \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} e^{ju_1 i} e^{ju_2 n} P_k(i, n, 0) = \sum_{i=0}^{\infty} e^{ju_1 i} P(i) ,$$

because at the initial moment of time (when the insurance company starts to work) there were no claims for insurance payments, thus $P(i, n, 0) = P(i)$ if $n = 0$ and $P(i, n, 0) = 0$ if $n > 0$. Let us denote the function $H_k(u_1, u_2, 0) = G_k(u_1)$ and the vector function $\mathbf{G}(u_1) = \{G_1(u_1), G_2(u_1), \ldots, G_K(u_1)\}$. Then we will write down the equation for $\mathbf{G}(u_1)$ in the following form:

$$\mathbf{G}(u_1)[N\mathbf{\Lambda}(e^{ju_1} - 1) + N\mathbf{Q}] + j\mathbf{G}^{'}(u_1)\mathbf{M}(1 - e^{-ju_1}) = \mathbf{0} . \tag{18}$$

We will solve Eq. (18) applying an asymptotic analysis method under similar asymptotic conditions and substitutions:

$$\varepsilon = \frac{1}{N},\ u_1 = \varepsilon\omega_1,\ \mathbf{G}(u_1) = \mathbf{F}(\omega_1, \varepsilon),\ \varepsilon \to 0 .$$

For the function $\mathbf{F}(\omega_1, \varepsilon)$ we can write

$$\mathbf{F}(\omega_1, \varepsilon)[\mathbf{\Lambda}(e^{ju_1} - 1) + \mathbf{Q}] + j\frac{\partial \mathbf{F}(\omega_1, \varepsilon)}{\partial \omega_1}\mathbf{M}(1 - e^{-ju_1}) = \mathbf{0} . \tag{19}$$

Let us denote

$$\lim_{\varepsilon \to 0} \mathbf{F}(\omega_1, \varepsilon) = \mathbf{F}(\omega_1)$$

and perform the asymptotic transition $\varepsilon \to 0$ in the Eq. (19). We will obtain $\mathbf{F}(\omega_1)\mathbf{Q} = 0$, therefore $\mathbf{F}(\omega_1) = \mathbf{R}\Psi(\omega_1)$, where scalar function $\Psi(\omega_1) = \Phi(\omega_1, \omega_2, 0)$, $\mathbf{R}$ — a row vector of stationary probability distribution of the Markov chain states. To obtain this function, we will sum up equations of the system (19), then divide by $\varepsilon$ and perform the asymptotic transition $\varepsilon \to 0$. We will obtain the equation for the unknown function $\Phi(\omega_1, \omega_2, 0) = \Psi(\omega_1)$ :

$$\Psi^{'}(\omega_1)\kappa_1 = j\Psi(\omega_1)\kappa , \tag{20}$$

where $\kappa = \mathbf{R\Lambda E}$, $\kappa_1 = \mathbf{RME}$.As a result, we obtain the following solution of this equation under initial condition $\Psi(0) = 1$ :

$$\Psi(\omega_1) = e^{j\frac{\kappa}{\kappa_1}\omega_1} . \tag{21}$$

Then we can write down the expression for function $\phi(C_1)$ :

$$\phi(C_1) = [e^{-t\kappa_1}(\omega_1\kappa_1 - \omega_2\kappa_2)]^{-j\omega_2\frac{\kappa\kappa_2}{\kappa_1^2}} . \tag{22}$$

Taking into account (22), the function $\Phi(\omega_1, \omega_2, t)$ will have the following form:

$$\Phi(\omega_1, \omega_2, t) = \exp\left\{j\omega_1\frac{\kappa}{\kappa_1} + j\omega_2\frac{\kappa\kappa_2}{\kappa_1}t\right\} . \tag{23}$$

Substituting this expression into (8), we obtain the expression for the function $\mathbf{F}(\omega_1, \omega_2, t)$ in the following form:

$$\mathbf{F}(\omega_1, \omega_1, t) = \mathbf{R}\exp\left\{j\omega_1\frac{\kappa}{\kappa_1} + j\omega_2\frac{\kappa\kappa_2}{\kappa_1}t\right\} . \tag{24}$$

For the function $\mathbf{H}(u_1, u_2, t)$ we can write down

$$\mathbf{H}(u_1, u_2, t) = \mathbf{F}(\omega_1, \omega_2, t, \varepsilon) \approx \mathbf{F}(\omega_1, \omega_2, t) = \mathbf{R}\exp\left\{j\omega_1\frac{\kappa}{\kappa_1} + j\omega_2\frac{\kappa\kappa_2}{\kappa_1}t\right\} .$$

Let us make in this formula substitutions that are inverse to changes (5). Using expression (8), we obtain the following expression for the vector characteristic function of the probability distribution of the two-dimensional process $(i(t),\ n(t))$

$$\begin{aligned}\mathbf{H}(u_1, u_2, t)\mathbf{E} &\approx \exp\left\{j\omega_1\frac{\kappa}{\kappa_1} + j\omega_2\frac{\kappa\kappa_2}{\kappa_1}t\right\} \\ &= \exp\left\{jNu_1\frac{\kappa}{\kappa_1} + jNu_2\frac{\kappa\kappa_2}{\kappa_1}t\right\} .\end{aligned} \tag{25}$$

## 5 The Second-Order Asymptotic Analysis

Let us denote the vector function $\mathbf{H}_2(u_1, u_2, t)$ satisfying the expression:

$$\mathbf{H}(u_1, u_2, t) = \mathbf{H_2}(u_1, u_2, t)\exp\left\{jNu_1\frac{\kappa}{\kappa_1} + jNu_2\frac{\kappa\kappa_2}{\kappa_1}t\right\} . \tag{26}$$

Substituting this expression in the Eq. (4), we obtain the equation for the function $\mathbf{H_2}(u_1, u_2, t)$ :

$$\begin{aligned}\frac{\partial\mathbf{H_2}(u_1, u_2, t)}{\partial t} &= \mathbf{H_2}(u_1, u_2, t)[N\mathbf{\Lambda}(e^{ju_1} - 1) + N\mathbf{Q}] \\ &- \mathbf{H_2}(u_1, u_2, t)\left[jNu_2\frac{\kappa\kappa_2}{\kappa_1}\mathbf{I} + N\frac{\kappa}{\kappa_1}\left((1 - e^{-ju_1})\mathbf{M} + (1 - e^{ju_2})\mathbf{\Gamma}\right)\right] \\ &+ j\frac{\partial\mathbf{H_2}(u_1, u_2, t)}{\partial u_1}[(1 - e^{-ju_1})\mathbf{M} + (1 - e^{ju_2})\mathbf{\Gamma}] ,\end{aligned} \tag{27}$$

where $\mathbf{I}$ — a diagonal unit matrix. Let us make the changes variables:

$$\varepsilon = \frac{1}{N^2},\ u_1 = \varepsilon\omega_1,\ u_2 = \varepsilon\omega_2,\ \mathbf{H_2}(u_1, u_2, t) = \mathbf{F_2}(\omega_1, \omega_1, t, \varepsilon)\ . \tag{28}$$

Using the new variables, we can rewrite the problem (27) in the form:

$$\varepsilon^2 \frac{\partial \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon)}{\partial t} = F_2(\omega_1, \omega_2, t, \varepsilon)[\mathbf{\Lambda}(e^{j\omega_1} - 1) + \mathbf{Q}]$$
$$- \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon), \left[ j\omega_2\varepsilon \frac{\kappa\kappa_2}{\kappa_1} I + \frac{\kappa}{\kappa_1} \left( (1 - e^{-j\omega_1\varepsilon})\mathbf{M} + (1 - e^{j\omega_2\varepsilon})\mathbf{\Gamma} \right) \right] \tag{29}$$
$$+ j\varepsilon \frac{\partial \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon)}{\partial \omega_1} [(1 - e^{-j\omega_1\varepsilon})\mathbf{M} + (1 - e^{j\omega_2\varepsilon})\mathbf{\Gamma}]\ .$$

Let us denote

$$\lim_{\varepsilon \to 0} F_2(\omega_1, \omega_2, t, \varepsilon) = F_2(\omega_1, \omega_2, t)\ .$$

Furthermore, we will perform the asymptotic transition $\varepsilon \to 0$ in (29), then we will obtain the equation $\mathbf{F_2}(\omega_1, \omega_2, t)\mathbf{Q} = \mathbf{0}$. Thus, the function $\mathbf{F_2}(\omega_1, \omega_2, t)$ can be written in the following form:

$$\mathbf{F_2}(\omega_1, \omega_2, t) = \mathbf{R}\Phi_2(\omega_1, \omega_2, t)\ , \tag{30}$$

where $\Phi_2(\omega_1, \omega_2, t)$ — some scalar function that will be defined below.

We will find the solution $\mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon)$ of the Eq. (29) in the following expansion form

$$\mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon) = \Phi_2(\omega_1, \omega_2, t) \left( \mathbf{R} + j\omega_1\varepsilon\mathbf{f_1} + j\omega_2\varepsilon\mathbf{f_2} + \mathbf{O}\left(\varepsilon^2\right) \right)\ , \tag{31}$$

where $\mathbf{f_1}$, $\mathbf{f_2}$ — some row vectors, $\mathbf{O}\left(\varepsilon^2\right)$ — a row vector that consist of the infinitesimals of the order $\varepsilon^2$.

Substituting (30) in the Eq. (29) and taking into account $\mathbf{RQ} = \mathbf{0}$, we obtain the matrix system of the equations for the row vectors $\mathbf{f_1}$, $\mathbf{f_2}$ :

$$\mathbf{f_1Q} = \frac{\kappa}{\kappa_1}\mathbf{RM} - \mathbf{R\Lambda}\ ,\ \mathbf{f_2Q} = \frac{\kappa}{\kappa_1}\mathbf{R}\kappa_2 - \frac{\kappa}{\kappa_1}\mathbf{R\Gamma}\ . \tag{32}$$

To obtain function $\Phi_2(\omega_1, \omega_2, t)$ let us sum up all equations of the system (29). We will obtain the following equation

$$\varepsilon^2 \frac{\partial \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon)}{\partial t}\mathbf{E} = \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon)[\mathbf{\Lambda}(e^{j\omega_1\varepsilon} - 1) + \mathbf{Q}]\mathbf{E}$$
$$- \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon) \left[ j\omega_2\varepsilon \frac{\kappa\kappa_2}{\kappa_1}\mathbf{I} + \frac{\kappa}{\kappa_1} \left( (1 - e^{-j\omega_1\varepsilon})\mathbf{M} + (1 - e^{j\omega_2\varepsilon})\mathbf{\Gamma} \right) \right] \mathbf{E} \tag{33}$$
$$+ j\varepsilon \frac{\partial \mathbf{F_2}(\omega_1, \omega_2, t, \varepsilon)}{\partial \omega_1} [(1 - e^{-j\omega_1\varepsilon})\mathbf{M} + (1 - e^{j\omega_2\varepsilon})\mathbf{\Gamma}]\mathbf{E}\ .$$

In the Eq. (33) let us substitute the expansion $e^{jw_1\varepsilon} = 1 + jw_1\varepsilon + \dfrac{(jw_1\varepsilon)^2}{2} + O\left(\varepsilon^3\right)$ and the expansion (31). We obtain the following equality

$$\varepsilon^2\frac{\partial\Phi_2(\omega_1,\omega_2,t)}{\partial t} = \Phi_2(\omega_1,\omega_2,t)\mathbf{R}\left[\boldsymbol{\Lambda}\left(j\omega_1\varepsilon - \frac{(\omega_1\varepsilon)^2}{2}\right) + \mathbf{Q} - j\omega_2\varepsilon\frac{\kappa\kappa_2}{\kappa_1}\mathbf{I}\right]\mathbf{E}$$
$$-\Phi_2(\omega_1,\omega_2,t)\mathbf{R}\frac{\kappa}{\kappa_1}\left[\left(j\omega_1\varepsilon + \frac{(\omega_1\varepsilon)^2}{2}\right)\mathbf{ME} + \left(-j\omega_1\varepsilon + \frac{(\omega_1\varepsilon)^2}{2}\right)\boldsymbol{\Gamma}\mathbf{E}\right]$$
$$-\Phi_2(\omega_1,\omega_2,t)\varepsilon(j\omega_1\mathbf{f_1} + j\omega_2\mathbf{f_2})$$
$$\times\left(j\frac{\kappa}{\kappa_1}\kappa_2\varepsilon\omega_2\mathbf{E} - \mathbf{QE} - j\omega_1\varepsilon\boldsymbol{\Lambda}\mathbf{E} + j\frac{\kappa}{\kappa_1}\varepsilon\omega_1\mathbf{ME} - j\frac{\kappa}{\kappa_1}\varepsilon\omega_2\boldsymbol{\Gamma}\mathbf{E}\right)$$
$$+\varepsilon^2\frac{\partial\Phi_2(\omega_1,\omega_2,t)}{\partial\omega_1}\mathbf{R}\left(\omega_1\mathbf{ME} - \omega_2\boldsymbol{\Gamma}\mathbf{E}\right) + O\left(\varepsilon^3\right)\ .$$

In the last expression let us divide left and right sides by $\varepsilon^2$, and after using the asymptotic transition $\varepsilon \to 0$, we obtain the equation for the function $\Phi_2(\omega_1,\omega_2,t)$:

$$\frac{\partial\Phi_2(\omega_1,\omega_2,t)}{\partial t} + \frac{\partial\Phi_2(\omega_1,\omega_2,t)}{\partial\omega_1}(\omega_1\kappa_1 - \omega_2\kappa_2) = \Phi_2(\omega_1,\omega_2,t)$$
$$\times\left[\omega_1^2(\mathbf{f_1}\mathbf{A_1} - \kappa) + \omega_1\omega_2(\mathbf{f_2}\mathbf{A_1} + \mathbf{f_1}\mathbf{A_2}) + \omega_2^2\left(\mathbf{f_2}\mathbf{A_2} - \frac{\kappa\kappa_2}{2\kappa_1}\right)\right]\ , \tag{34}$$

under the initial condition $\Phi_2(\omega_1,\omega_2,0) = \exp\left\{\dfrac{\mathbf{A_1}\mathbf{f_1} - \kappa}{2\kappa_1}\omega_1^2\right\}$ and where vectors $\mathbf{A_1}$, $\mathbf{A_2}$ are defined by expressions

$$\mathbf{A_1} = \left(\frac{\kappa}{\kappa_1}\mathbf{M} - \boldsymbol{\Lambda}\right)\mathbf{E}\ ,\ \mathbf{A_2} = \left(\frac{\kappa\kappa_2}{\kappa_1}\mathbf{I} - \frac{\kappa}{\kappa_1}\boldsymbol{\Gamma}\right)\mathbf{E}\ . \tag{35}$$

We will find a solution of the Eq. (34) in the following form:

$$\Phi_2(\omega_1,\omega_2,t) = \exp\left\{-\frac{1}{2}\left(K_{11}\omega_1^2 + 2K_{12}(t)\omega_1\omega_2 + K_{22}(t)\omega_2^2\right)\right\}\ . \tag{36}$$

Substituting this expression in the Eq. (34), we obtain the following system for $K_{11}$, $K_{12}(t)$, $K_{22}(t)$:

$$K_{11}\kappa_1 = \kappa - \mathbf{f_1}\mathbf{A_1}\ ,$$
$$K'_{12}(t) + \kappa_1 K_{12}(t) - \kappa_2 K_{11}(t) = -\mathbf{f_1}\mathbf{A_2} - \mathbf{f_2}\mathbf{A_1}\ , \tag{37}$$
$$\frac{1}{2}K'_{22}(t) - \kappa_2 K_{12}(t) = \frac{\kappa\kappa_2}{2\kappa_1} - \mathbf{f_2}\mathbf{A_2}\ ,$$

where vectors $\mathbf{f_1}$, $\mathbf{f_2}$ are defined by the system (32), $\mathbf{A_1}$, $\mathbf{A_2}$ are defined by expressions (35) and $\kappa = \mathbf{R}\boldsymbol{\Lambda}\mathbf{E}$, $\kappa_1 = \mathbf{RME}$, $\kappa_2 = \mathbf{R}\boldsymbol{\Gamma}\mathbf{E}$. Solving the system (37) under initial conditions $K_{12}(0) = 0$, $K_{22}(0) = 0$, we obtain the expressions for $K_{11}$, $K_{12}(t)$, $K_{22}(t)$:

$$K_{11} = \frac{\kappa - \mathbf{f_1}\mathbf{A_1}}{\kappa_1}\ , \tag{38}$$

$$K_{12}(t)=\left(1-e^{-\kappa_1 t}\right)\left[\frac{\kappa-\mathbf{A_1 f_1}}{\kappa_1^2}\kappa_2-\frac{\mathbf{A_1 f_2}+\mathbf{A_2 f_1}}{\kappa_1}\right], \tag{39}$$

$$\begin{aligned}K_{22}(t)=2t&\left[\frac{\kappa-\mathbf{A_1 f_1}}{\kappa_1^2}\kappa_2^2-\frac{\mathbf{A_1 f_2}+\mathbf{A_2 f_1}}{\kappa_1}\kappa_2-\left(\mathbf{A_2 f_2}-\frac{\kappa\kappa_2}{2\kappa_1}\right)\right]\\&+2\left(1-e^{-\kappa_1 t}\right)\left[\frac{\kappa-\mathbf{A_1 f_1}}{\kappa_1^3}\kappa_2^2-\frac{\mathbf{A_1 f_2}+\mathbf{A_2 f_1}}{\kappa_1^2}\kappa_2\right].\end{aligned} \tag{40}$$

Substituting these expressions into (30), we obtain the final form of the function $\mathbf{F_2}(\omega_1, \omega_2, t)$ as following expression

$$\mathbf{F_2}(\omega_1,\omega_2,t)=\mathbf{R}\exp\left\{-\frac{1}{2}\left(K_{11}\omega_1^2+2K_{12}(t)\omega_1\omega_2+K_{22}(t)\omega_2^2\right)\right\}. \tag{41}$$

Let us make in this formula substitutions that are inverse to changes (28). Using (26), we can write the expression for the vector characteristic function $\mathbf{H}(u_1, u_2, t)$ in the following form:

$$\begin{aligned}\mathbf{H}(u_1,u_2,t)=\mathbf{R}\exp\Big\{&-\frac{1}{2}\left(K_{11}(Nu_1)^2+2K_{12}(t)N^2u_1u_2\right.\\&\left.+K_{22}(t)(Nu_2)^2\right)+j\frac{N\kappa}{\kappa_1}u_1+j\frac{N\kappa\kappa_1}{\kappa_1}u_2t\Big\}.\end{aligned} \tag{42}$$

Thus, we have the following formula for second-order approximation $h_2(u_1, u_2, t)$ for the characteristic function $h(u_1, u_2, t) = \mathbf{H}(u_1, u_2, t)\mathbf{E}$ of the two-dimensional process $(i(t), n(t))$ under the condition that $N$ is large enough:

$$\begin{aligned}h(u_1,u_2,t)\approx h_2(u_1,u_2,t)=\exp\Big\{&-\frac{1}{2}\left(K_{11}(Nu_1)^2+2K_{12}(t)N^2u_1u_2\right.\\&\left.+K_{22}(t)(Nu_2)^2\right)+j\frac{N\kappa}{\kappa_1}u_1+j\frac{N\kappa\kappa_1}{\kappa_1}u_2t\Big\},\end{aligned} \tag{43}$$

where $K_{11}$, $K_{12}(t)$, $K_{22}(t)$ are defined by the expressions (38), (39) and (40).

## 6 Conclusions

In this paper we have researched the mathematical model of the insurance company in the form of queueing system with infinite number of servers with high-rate arrival process and in a random environment. We have shown that the probability distribution of the two-dimensional process of the insurance risks and the insurance payments under the above conditions can be approximated by the two-dimensional Gaussian distribution. These results can be used to analysis the activity of insurance companies and other economic systems.

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