Research of Heterogeneous Queueing System $SM|M^{(n)}|\infty$

Ekaterina Pankratova[1(✉)], Mais Farkhadov[1], and Erol Gelenbe[2]

[1] V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia
pankate@sibmail.com, mais@ipu.ru
[2] Imperial College, London EEE Bldg Rm 1009, South Kensington Campus, London, England
e.gelenbe@imperial.ac.uk

Abstract. One of the modifications of the mathematical models used to describe processes in multi-service communication networks and telecommunication systems is the queueing system with heterogeneous servers. As a rule, for simulation of such processes the system with non-Poisson input flows is used. We consider the queuing system with infinite number of servers of n different types and exponential service time. Incoming flow is a Semi Markovian Process (SM-flow). Investigation of n-dimensional stochastic process characterizing the number of occupied servers of different types is performed using the initial moments method.

Keywords: Queueing system · Incoming sm-flow · Heterogeneous servers · Method of initial moments

1 Introduction

Systems with heterogeneous servers [4,5,10,11,13,14,18] and non-Poisson incoming flows [12,20,21] are suitable to simulate the functioning of real information systems. Such systems include queueing systems with non-ordinary Poisson incoming flows and exponential service time [2,8,9]; systems with parallel functioning blocks [3,6,7,15,19]. These papers deal with different configurations of parallel-service systems: single-line queueing systems with finite and infinite buffer, priority maintenance, impatient applications and a common ordinary incoming flow; queueing systems with two or more service blocs with a finite number of servers and a common final queue. Mathematical models of inhomogeneous infinite-linear systems with different types of servicing devices allow taking into account the heterogeneity of incoming applications requiring different maintenance time, which more adequately describes real information systems [16,17]. In this paper we study a heterogeneous queueing system with SM-incoming flow and exponential service time.

A. Dudin et al. (Eds.): ITMM 2017, CCIS 800, pp. 122–132, 2017.
DOI: 10.1007/978-3-319-68069-9_10

2 Statement of the Problem

Consider the queuing system with infinite number of servers of n different types and exponential service time. Incoming flow is a Semi Markovian Process (SM-flow) which given by matrix $\mathbf{A}(x)$ consisting of elements $A_{k_1k_2}(x)(k_1 = 0,\ldots,N, k_2 = 0,\ldots,N)$.

$$A_{k_1k_2}(x) = F(k_2, x; k_1) = P\left\{\xi(k+1) = k_2, \tau(k+1) < x|\xi(k) = k_1\right\}, \quad (1)$$

where $\xi(k)$ — the Markov chain with discrete time and the transition probability matrix $\mathbf{P}$, $\tau(k)$ — non-Markov process for which

$$F(x) = P\left\{\tau(k) < x\right\} = \sum_{i=0}^{N} A_i(x) r(i), \quad (2)$$

$r(i)$ — stationary probability distribution of the Markov chain $\xi(k)$.

At the time of occurrence of the event in this stream only one customer flows in the system. The type of incoming customer is defined as i-type with probability p_i $(i = 1,\ldots,n)$. It goes to the appropriate device type, where its' service is performed during a random time having an exponential distribution function with parameter μ_i $(i = 1,\ldots,n)$ corresponding to the type of the customer.

Set the problem of exploring of n-dimensional stochastic process $\{l_1(t),\ldots,l_n(t)\}$ describing the number of occupied units of i-type at time t. Incoming flow is not Poisson, hence the n-dimensional process $\{l_1(t),\ldots,l_n(t)\}$ is non-Markov. Consider a $(n+2)$-dimensional Markov process $\{s(t), z(t), l_1(t),\ldots,l_n(t)\}$, here $z(t)$ — the time from t until the occurrence of the following event of SM-flow, $s(t)$ — the process is defined as follows

$$s(t) = \xi(k+1) \text{ if } t_k < t < t_{k+1},\ t_k = \sum_{i=1}^{k} \tau(i).$$

For the joint probability distribution

$$P(s, z, l_1,\ldots,l_n, t) = P\{s(t) = s, z(t) < z, l_1(t) = l_1,\ldots,l_n(t) = l_n\}$$

we can write

$$P(s, z - \Delta t, l_1,\ldots,l_n, t + \Delta t)$$
$$= \left[P(s, z, l_1,\ldots,l_n, t) - P(s, \Delta t, l_1,\ldots,l_n, t)\right]\prod_{i=1}^{n}(1 - l_i\mu_i)$$
$$+ \sum_{\nu=1}^{K} P(\nu, \Delta t, l_1 - 1,\ldots,l_n, t) A_{\nu s}(z) p_1 + \ldots \quad (3)$$
$$+ \sum_{\nu=1}^{K} P(\nu, \Delta t, l_1,\ldots,l_n - 1, t) A_{\nu s}(z) p_n + P(s, z, l_1 + 1,\ldots,l_n, t)(l_1 + 1)\mu_1 \Delta t + \ldots$$
$$+ P(s, z, l_1,\ldots,l_n + 1, t)(l_n + 1)\mu_n \Delta t + o(\Delta t),\ s = 1,\ldots,K.$$

System of Kolmogorov differential equations for the probability distribution $P\{s, z, l_1, \ldots, l_n, t\}$ is the following:

$$\frac{\partial P(s,z,l_1,\ldots,l_n,t)}{\partial t} = \frac{\partial P(s,z,l_1,\ldots,l_n,t)}{\partial z} - \frac{\partial P(s,0,l_1,\ldots,l_n,t)}{\partial z}$$

$$- \sum_{i=1}^{n} l_i \mu_i P(s,z,l_1,\ldots,l_n,t) \quad (4)$$

$$+ p_1 \sum_{\nu=1}^{K} \frac{\partial P(\nu,0,l_1-1,\ldots,l_n,t)}{\partial z} A_{\nu s} + \ldots + p_n \sum_{\nu=1}^{K} \frac{\partial P(\nu,0,l_1,\ldots,l_n-1,t)}{\partial z} A_{\nu s}$$

$$+ \mu_1(l_1+1)P(s,z,l_1+1,\ldots,l_n,t) + \ldots + \mu_n(l_n+1)P(s,z,l_1,\ldots,l_n+1,t),$$

$$s = 1,\ldots,K.$$

We will find the solution of the system (4) during stationary operation of the system. Denote $\lim_{t\to\infty} P(s,z,l_1,\ldots,l_n,t) = \Pi(s,z,l_1,\ldots,l_n),\ s = 1,\ldots,K.$

Then the equation (4) takes the form

$$\frac{\partial \Pi(s,z,l_1,\ldots,l_n)}{\partial z} - \frac{\partial \Pi(s,0,l_1,\ldots,l_n)}{\partial z} - \sum_{i=1}^{n} l_i \mu_i \Pi(s,z,l_1,\ldots,l_n)$$

$$+ p_1 \sum_{\nu=1}^{K} \frac{\partial \Pi(\nu,0,l_1-1,\ldots,l_n)}{\partial z} A_{\nu s} + \ldots$$

$$+ p_n \sum_{\nu=1}^{K} \frac{\partial \Pi(\nu,0,l_1,\ldots,l_n-1)}{\partial z} A_{\nu s} \quad (5)$$

$$+ \mu_1(l_1+1)\Pi(s,z,l_1+1,\ldots,l_n) + \ldots$$

$$+ \mu_n(l_n+1)\Pi(s,z,l_1,\ldots,l_n+1) = 0$$

$$s = 1,\ldots,K.$$

Introduce partial characteristic functions [1]:

$$H(s,z,u_1,\ldots,u_n) = \sum_{l_1=0}^{\infty} \ldots \sum_{l_n=0}^{\infty} e^{ju_1 l_1} \times \ldots \times e^{ju_n l_n} \Pi(s,z,l_1,\ldots,l_n),$$

where $s = 1,\ldots,K,\ j = \sqrt{-1}$ — imaginary unit.

In view of

$$\frac{\partial H(s,z,u_1,\ldots,u_n)}{\partial u_i} = j \sum_{l_1=1}^{\infty} \cdots \sum_{l_n=1}^{\infty} l_i e^{ju_1 l_1} \times \cdots \times e^{ju_n l_n} \Pi(s,z,l_1,\ldots,l_n),$$

$$i = 1,\ldots,n,\ s = 1,\ldots,K,$$

and using (4) write the system of differential equations for partial characteristic functions $H(s,z,u_1,\ldots,u_n)$

$$\frac{\partial H(s,z,u_1,\dots,u_n)}{\partial z} - \frac{\partial H(s,0,u_1,\dots,u_n)}{\partial z} \tag{6}$$

$$+j\sum_{i=1}^{n}\mu_i(1-e^{ju_i})\frac{\partial H(s,z,u_1,\dots,u_n)}{\partial u_i} + \sum_{i=1}^{n}p_i e^{ju_i}\sum_{\nu=1}^{K}\frac{\partial H(\nu,0,u_1,\dots,u_n)}{\partial z}A_{\nu s}(z)=0,$$
$$s=1,\dots,K,$$

which we rewrite in the form of the vector-matrix equation

$$\begin{gathered}\frac{\partial \mathbf{H}(z,u_1,\dots,u_n)}{\partial z} + j\sum_{i=1}^{n}\mu_i(1-e^{-ju_i})\frac{\partial \mathbf{H}(z,u_1,\dots,u_n)}{\partial u_i} \\ + \frac{\partial \mathbf{H}(0,u_1,\dots,u_n)}{\partial z}\left(\sum_{i=1}^{n}p_i e^{ju_i}\mathbf{A}(z)-\mathbf{I}\right)=0,\end{gathered} \tag{7}$$

$\mathbf{H}(z,u_1,\dots,u_n) = [H(1,z,u_1,\dots,u_n), H(2,z,u_1,\dots,u_n),\dots,H(K,u_1,\dots,u_n)]$ — row vector consisting of characteristic functions of the random process $\{s(t),z,(t)l_1(t),\dots,l_n(t)\}$ for each state of the process $s(t)$,

$$\frac{\partial \mathbf{H}(0,u_1,\dots,u_n)}{\partial z} = \left.\frac{\partial \mathbf{H}(z,u_1,\dots,u_n)}{\partial z}\right|_{z=0}. \tag{8}$$

The solution $\mathbf{H}(z,u_1,\dots,u_n)$ of system (7) satisfies condition

$$\mathbf{H}(z,0,\dots,0)=\mathbf{r}(z)$$

and determines the characteristic function of the number of occupied servers in the stationary mode for the system SM|M$^{(n)}$|∞ by the equality

$$Me^{j\sum\limits_{i=1}^{n}u_i l_i(t)} = \mathbf{H}(\infty,u_1,\dots,u_n)\mathbf{e}. \tag{9}$$

$\mathbf{r}(z)$ — stationary probability distribution of a two-dimensional stochastic process $\{s(t),z(t)\}$, which has the form

$$\mathbf{r}(z)=\kappa_1\mathbf{r}\int\limits_0^z(\mathbf{P}-\mathbf{A}(x))\,dx, \tag{10}$$

where $\mathbf{r}$ — stationary probability distribution of the Markov chain $\xi(k)$, $k=1,\dots,K$, $\kappa_1=\frac{1}{\mathbf{rAe}}$, $\mathbf{A}=\int\limits_0^\infty(\mathbf{P}-\mathbf{A}(x))\,dx$.

The equation (7) will be considered as the basis for further research.

3 The Main Probabilistic Characteristics for System SM|M$^{(n)}$|∞

Theorem 1. *For the initial moments of number of employed devices of each type for the steady-state functioning of the heterogeneous system $SM|M^{(n)}|\infty$ the following statements are true:*

Statement 1

The average value of number employed devices of the i-th type $fm_i (i = 1, \dots, n)$ in the heterogeneous system $SM|M^{(n)}|\infty$ has the form:

$$fm_i = \frac{p_i}{\mu_i}\lambda, \tag{11}$$

where $\lambda = \mathbf{r}'(0)\mathbf{e}$, $\mathbf{e} = [1, \dots, 1]^T$ — a unit column vector.

Statement 2

Initial moments of the second order of number of employed devices of the i-th type $sm_i\ (i = 1, \dots, n)$ in the heterogeneous system $SM|M^{(n)}|\infty$ has the form:

$$sm_i = \frac{p_i}{\mu_i}\left[\lambda + p_i\mathbf{r}'(0)\mathbf{A}^*(\mu_i)(\mathbf{I} - \mathbf{A}^*(\mu_i))^{-1}\mathbf{e}\right],$$

where $\mathbf{A}^*(\alpha) = \int\limits_0^\infty e^{-\alpha z} d\mathbf{A}(z)$.

Statement 3

Correlation moment of number of employed devices of the i-th and g-th types $cm_{ig}\ (i = 1, \dots, n,\ g = 1, \dots, n,\ i \neq g)$ in the heterogeneous system $SM|M^{(n)}|\infty$ has the form:

$$cm_{ig} = \frac{p_i p_g}{\mu_i + \mu_g}\mathbf{r}'(0)\left[\mathbf{A}^*(\mu_i)\left(\mathbf{I} - \mathbf{A}^*(\mu_i)\right)^{-1} + \mathbf{A}^*(\mu_g)\left(\mathbf{I} - \mathbf{A}^*(\mu_g)\right)^{-1}\right]\mathbf{e}. \tag{12}$$

Proof. Denote:

- $\mathbf{fm}_i(z) = [fm_i(1, z), fm_i(2, z), \dots, fm_i(K, z)]$ — row-vector of conditional mathematical expectations of number employed devices of i-th type ($i = 1, \dots, n$);
- $\mathbf{sm}_i(z) = [sm_i(1, z), sm_i(2, z), \dots, sm_i(K, z)]$ — row-vector of conditional moments of the second order of number employed devices of i-th type ($i = 1, \dots, n$);
- $\mathbf{cm}_{ig}(z) = [cm_{ig}(1, z), cm_{ig}(2, z), \dots, cm_{ig}(K, z)]$ — row-vector of correlation moments of number employed devices of i-th and g-th types ($i = 1, \dots, n$, $g = 1, \dots, n$, $i \neq g$).

We use the following properties of the characteristic function:

$$\left.\frac{\partial \mathbf{H}(z, u_1, \dots, u_n)}{\partial u_i}\right|_{u_1=0, \dots, u_n=0} = j\mathbf{fm}_i(z),$$

$$\left.\frac{\partial^2 \mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i^2}\right|_{u_1=0,\ldots,u_n=0} = j^2\mathbf{sm}_i(z), \tag{13}$$

$$\left.\frac{\partial^2 \mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i u_g}\right|_{u_1=0,\ldots,u_n=0} = j^2\mathbf{cm}_{ig}(z),$$

$$i=1,\ldots,n,\ g=1,\ldots,n,\ i\neq g.$$

Initial moments of the first order.

The average number of occupied devices of each type in the system is determined as follows:

$$fm_i = \mathbf{fm}_i(\infty)\mathbf{e},\ i=1,\ldots,n,\ \mathbf{e}=[1,\ldots,1]^T. \tag{14}$$

Differentiate equation (7) with respect to u_i, $i=1,\ldots,n$.

$$\left.\frac{\partial^2 \mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i \partial z}\right|_{u_1=0,\ldots,u_n=0} + j^2\mu_i e^{-ju_i}\left.\frac{\partial \mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i}\right|_{u_1=0,\ldots,u_n=0}$$
$$+\,j\sum_{\nu=1}^{n}\mu_\nu(1-e^{-ju_\nu})\left.\frac{\partial^2 \mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i \partial u_\nu}\right|_{u_1=0,\ldots,u_n=0} \tag{15}$$
$$+\frac{\partial^2 \mathbf{H}(0,u_1,\ldots,u_n)}{\partial u_i \partial z}\left.\left(\sum_{i=1}^{n}p_i e^{ju_i}\mathbf{A}(z)-\mathbf{I}\right)\right|_{u_1=0,\ldots,u_n=0}$$
$$+\,j\left.\frac{\partial \mathbf{H}(0,u_1,\ldots,u_n)}{\partial z}p_i e^{ju_i}\mathbf{A}(z)\right|_{u_1=0,\ldots,u_n=0} = 0,\ i=1,\ldots,n,$$

taking into account (13) we obtain

$$\mathbf{fm}_i'(z)-\mu_i\mathbf{fm}_i(z)+\mathbf{fm}_i'(0)\left(\mathbf{A}(z)-\mathbf{I}\right)+p_i\mathbf{r}'(0)\mathbf{A}(z)=0,\ i=1,\ldots,n. \tag{16}$$

This equation will be solved by the conversation of Laplace-Stieltjes, denoting

$$\mathbf{\Phi}_i(\alpha)=\int_0^\infty e^{-\alpha z}d\mathbf{fm}_i(z),\ i=1,\ldots,n,\ \ \mathbf{A}^*(\alpha)=\int_0^\infty e^{-\alpha z}d\mathbf{A}(z). \tag{17}$$

Completing the conversation of Laplace-Stieltjes in (16), we obtain the equality

$$(\mu_i-\alpha)\mathbf{\Phi}_i(\alpha)=\mathbf{fm}_i'(0)\left(\mathbf{A}^*(\alpha)-\mathbf{I}\right)+\mathbf{r}'(0)p_i\mathbf{A}^*(\alpha),\ i=1,\ldots,n, \tag{18}$$

putting in which $\alpha=\mu_i$, $i=1,\ldots,n$, we find the form of the vector $\mathbf{fm}_i'(0)$

$$\mathbf{fm}_i'(0)=p_i\mathbf{r}'(0)\mathbf{A}^*(\mu_i)\left(\mathbf{I}-\mathbf{A}^*(\mu_i)\right)^{-1}. \tag{19}$$

Substituting the expression (19) in the (18) we obtain

$$\mathbf{\Phi}_i(\alpha)=\frac{1}{\mu_i-\alpha}\left\{\mathbf{fm}_i'(0)\left(\mathbf{A}^*(\alpha)-\mathbf{I}\right)+p_i\mathbf{r}'(0)\mathbf{A}^*(\alpha)\right\},\ i=1,\ldots,n. \tag{20}$$

Since $\mathbf{fm}_i(\infty) = \mathbf{\Phi}_i(0)$ and $\mathbf{A}^*(\infty) = \mathbf{P}$ then putting $\alpha = 0$ in (20) we obtain

$$\mathbf{\Phi}_i(0) = \mathbf{fm}_i(\infty) = \frac{1}{\mu_i}\left\{\mathbf{fm}_i'(0)\left(\mathbf{P} - \mathbf{I}\right) + p_i\mathbf{r}'(0)\mathbf{P}\right\},\ i = 1,\ldots,n. \tag{21}$$

Thus we have the following expression for the average value of number employed devices of the i-th type fm_i, $(i = 1,\ldots,n)$:

$$fm_i = \mathbf{fm}_i(\infty)\mathbf{e} = \frac{p_i}{\mu_i}\mathbf{r}'(0)\mathbf{e} = \frac{p_i}{\mu_i}\lambda,\ i = 1,\ldots,n,\ \mathbf{e} = [1,\ldots,1]^T.$$

Initial moments of the second order.

To find the second-order moment of the number of employed devices, we differentiate with respect to u_i, $i = 1,\ldots,n$ the equality (15).

$$\begin{aligned}
&\left.\frac{\partial^3\mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i^2\partial z}\right|_{u_1=0,\ldots,u_n=0} + j\mu_i e^{-ju_i}\left.\frac{\partial\mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i}\right|_{u_1=0,\ldots,u_n=0}\\
&+2j^2\mu_i e^{-ju_i}\left.\frac{\partial^2\mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i^2}\right|_{u_1=0,\ldots,u_n=0}\\
&+j\sum_{\nu=1}^{n}\mu_\nu(1-e^{-ju_\nu})\left.\frac{\partial^3\mathbf{H}(z,u_1,\ldots,u_n)}{\partial u_i^2\partial u_\nu}\right|_{u_1=0,\ldots,u_n=0}\\
&+\frac{\partial^3\mathbf{H}(0,u_1,\ldots,u_n)}{\partial u_i^2\partial z}\left.\left(\sum_{i=1}^{n}p_i e^{ju_i}\mathbf{A}(z)-\mathbf{I}\right)\right|_{u_1=0,\ldots,u_n=0}\\
&+2j\left.\frac{\partial^2\mathbf{H}(0,u_1,\ldots,u_n)}{\partial u_i\partial z}p_i e^{ju_i}\mathbf{A}(z)\right|_{u_1=0,\ldots,u_n=0}\\
&+j^2\left.\frac{\partial\mathbf{H}(0,u_1,\ldots,u_n)}{\partial z}p_i e^{ju_i}\mathbf{A}(z)\right|_{u_1=0,\ldots,u_n=0} = 0,\ i = 1,\ldots,n,
\end{aligned} \tag{22}$$

taking into account (13), we obtain the differential equation to find $\mathbf{sm}_i(z)$, $i = 1,\ldots,n$

$$\begin{aligned}
&\mathbf{sm}_i'(z) + \mu_i\mathbf{fm}_i(z) - 2\mu_i\mathbf{sm}_i(z) + \mathbf{sm}_i'(0)\left(\mathbf{A}(z) - \mathbf{I}\right)\\
&+p_i\left\{\mathbf{fm}_i'(0) + \mathbf{r}'(0)\right\}\mathbf{A}(z) = 0,\ i = 1,\ldots,n.
\end{aligned} \tag{23}$$

We will solve equation (23) using the conversation of Laplace-Stiltjes. Denote

$$\mathbf{\Psi}_i(\alpha) = \int_0^\infty e^{-\alpha z}d\mathbf{sm}_i(z),\ i = 1,\ldots,n, \tag{24}$$

then the equation (23) takes the form

$$\begin{aligned}
&(2\mu_i - \alpha)\mathbf{\Psi}_i(\alpha) = \mu_i\mathbf{\Phi}_i(\alpha) + \mathbf{sm}_i'(0)(\mathbf{A}^*(\alpha) - \mathbf{I})\\
&+p_i\left\{2\mathbf{fm}_i'(0) + \mathbf{r}'(0)\right\}\mathbf{A}^*(\alpha),\ i = 1,\ldots,n,
\end{aligned} \tag{25}$$

$\mathbf{A}^*(\alpha)$ is determined by the expression (17).

Let $\alpha = 2\mu_i$ in (25), we obtain the system of differential equations for $\mathbf{sm}_i'(0)$, $i = 1, \ldots, n$

$$\mathbf{sm}_i'(0) = [\mu_i \boldsymbol{\Phi}_i(2\mu_i) + p_i \left\{2\mathbf{fm}_i'(0) + \mathbf{r}'(0)\right\} \mathbf{A}^*(2\mu_i)\Big] (\mathbf{I} - \mathbf{A}^*(2\mu_i))^{-1}, \; i = 1, \ldots, n. \tag{26}$$

It follows from (25) that

$$\boldsymbol{\Psi}_i(\alpha) = \frac{1}{2\mu_i - \alpha} [\mu_i \boldsymbol{\Phi}_i(\alpha) + \mathbf{sm}_i'(0)(\mathbf{A}^*(\alpha) - \mathbf{I}) + p_i \left\{2\mathbf{fm}_i'(0) + \mathbf{r}'(0)\right\} \mathbf{A}^*(\alpha)\Big], \; i = 1, \ldots, n, \tag{27}$$

and taking into account that

$$\mathbf{sm}_i(\infty) = \boldsymbol{\Psi}_i(0) = \frac{1}{2\mu_i} [\mu_i \mathbf{fm}_i(\infty) + \mathbf{sm}_i'(0) (\mathbf{P} - \mathbf{I}) + p_i \left\{2\mathbf{fm}_i'(0) + \mathbf{r}'(0)\right\} \mathbf{P}\Big], i = 1, \ldots, n. \tag{28}$$

we can write

$$sm_i = \mathbf{sm}_i(\infty)\mathbf{e} = \frac{1}{2\mu_i} \mu_i \mathbf{fm}_i(\infty)\mathbf{e} + \frac{p_i}{2\mu_i} \left\{2\mathbf{fm}_i'(0) + \mathbf{r}'(0)\right\} \mathbf{Pe} = \frac{p_i}{\mu_i} \left(\mathbf{fm}_i'(0)\mathbf{e} + \lambda\right), \; i = 1, \ldots, n. \tag{29}$$

Thus, taking into account (19) we have expression for initial moment of the second order

$$sm_i = \frac{p_i}{\mu_i} \left[\lambda + p_i \mathbf{r}'(0)\mathbf{A}^*(\mu_i)(\mathbf{I} - \mathbf{A}^*(\mu_i))^{-1}\mathbf{e}\right], \; i = 1, \ldots, n.$$

Correlation moment.

Differentiate the equality (15) respect to u_g, $g = 1, \ldots, n, g \neq i$.

$$\begin{aligned}
&\frac{\partial^3 \mathbf{H}(z, u_1, \ldots, u_n)}{\partial u_i \partial u_g \partial z}\bigg|_{u_1=0,\ldots,u_n=0} + j\mu_i e^{-ju_i} \frac{\partial^2 \mathbf{H}(z, u_1, \ldots, u_n)}{\partial u_i \partial u_g}\bigg|_{u_1=0,\ldots,u_n=0} \\
&+ j^2 \mu_g e^{-ju_g} \frac{\partial^2 \mathbf{H}(z, u_1, \ldots, u_n)}{\partial u_i \partial u_g}\bigg|_{u_1=0,\ldots,u_n=0} \\
&+ j \sum_{\nu=1}^{n} \mu_\nu (1 - e^{-ju_\nu}) \frac{\partial^3 \mathbf{H}(z, u_1, \ldots, u_n)}{\partial u_i \partial u_\nu \partial u_g}\bigg|_{u_1=0,\ldots,u_n=0} \\
&+ \frac{\partial^3 \mathbf{H}(0, u_1, \ldots, u_n)}{\partial u_i \partial u_g \partial z} \left(\sum_{i=1}^{n} p_i e^{ju_i} \mathbf{A}(z) - \mathbf{I}\right)\bigg|_{u_1=0,\ldots,u_n=0} \\
&+ \frac{\partial^2 \mathbf{H}(0, u_1, \ldots, u_n)}{\partial u_i \partial z} j p_g e^{ju_g} \mathbf{A}(z)\bigg|_{u_1=0,\ldots,u_n=0}
\end{aligned} \tag{30}$$

$$+j\frac{\partial^2 \mathbf{H}(0,u_1,\ldots,u_n)}{\partial u_g \partial z} p_i e^{ju_i}\mathbf{A}(z)\Big|_{u_1=0,\ldots,u_n=0} = 0,$$
$$i=1,\ldots,n,\ g=1,\ldots,n, g\neq i,$$

taking into account (13):

$$\begin{aligned}&\mathbf{cm}'_{ig}(z)-(\mu_i+\mu_g)\mathbf{cm}_{ig}(z)+\mathbf{cm}'_{ig}(0)\left(\mathbf{A}(z)-\mathbf{I}\right)\\ &+\left\{p_g\mathbf{fm}'_i(0)+p_i\mathbf{fm}'_g(0)\right\}\mathbf{A}(z)=0,\ i=1,\ldots,n,\ g=1,\ldots,n, g\neq i.\end{aligned} \tag{31}$$

We will solve equation (31) using the conversation of Laplace-Stiltjes. Denote

$$\mathbf{\Theta}_{ig}(\alpha)=\int_0^\infty e^{-\alpha z} d\mathbf{cm}_{ig}(z),\ i=1,\ldots,n, g=1,\ldots,n, g\neq i, \tag{32}$$

then the equation (31) takes the form

$$\begin{aligned}(\mu_i+\mu_g-\alpha)\mathbf{\Theta}_{ig}(\alpha)&=\mathbf{cm}'_{ig}(0)\left(\mathbf{A}^*(\alpha)-\mathbf{I}\right)\\ &+\left\{p_g\mathbf{fm}'_i(0)+p_i\mathbf{fm}'_g(0)\right\}\mathbf{A}^*(\alpha)=0,\\ &i=1,\ldots,n,\ g=1,\ldots,n, g\neq i,\end{aligned} \tag{33}$$

$\mathbf{A}^*(\alpha)$ is determined by the expression (17).

Put $\alpha=\mu_i+\mu_g$ in (33), we obtain the system of differential equations for $\mathbf{cm}'_{ig}(0)$, $i=1,\ldots,n,\ g=1,\ldots,n, g\neq i$

$$\begin{aligned}\mathbf{cm}'_{ig}(0)=\left\{p_g\mathbf{fm}'_i(0)+p_i\mathbf{fm}'_g(0)\right\}\mathbf{A}^*(\mu_i+\mu_g)\left(\mathbf{I}-\mathbf{A}^*(\mu_i+\mu_g)\right)^{-1}=0,\\ i=1,\ldots,n,\ g=1,\ldots,n, g\neq i.\end{aligned} \tag{34}$$

Since $\mathbf{cm}_{ig}(\infty)=\mathbf{\Theta}_{ig}(0)$, it follows from (33) that the expression for the correlation moment cm_{ig} is as follows

$$\begin{aligned}cm_{ig}=\mathbf{cm}_{ig}(\infty)\mathbf{e}&=\frac{1}{\mu_i+\mu_g}\left[\mathbf{cm}'_{ig}(0)\left(\mathbf{P}-\mathbf{I}\right)\right.\\ &\left.+\left\{p_g\mathbf{fm}'_i(0)+p_i\mathbf{fm}'_g(0)\right\}\mathbf{P}\right]\mathbf{e}\\ =\frac{p_ip_g}{\mu_i+\mu_g}\mathbf{r}'(0)&\left[\mathbf{A}^*(\mu_i)\left(\mathbf{I}-\mathbf{A}^*(\mu_i)\right)^{-1}+\mathbf{A}^*(\mu_g)\left(\mathbf{I}-\mathbf{A}^*(\mu_g)\right)^{-1}\right]\mathbf{e},\\ &i=1,\ldots,n,\ g=1,\ldots,n, g\neq i.\end{aligned} \tag{35}$$

□

We can write the expression for finding the variance of the number of occupied servers of each types in the heterogeneous system SM|M$^{(n)}$|∞

$$\begin{aligned}Var_i&=sm_i-[fm_i]^2,\ i=1,\ldots,n,\\ Var_i&=\frac{p_i}{\mu_i}\lambda+\frac{p_i^2}{\mu_i}\mathbf{r}'(0)\mathbf{A}^*(\mu_i)\left(\mathbf{I}-\mathbf{A}^*(\mu_i)\right)^{-1}\mathbf{e},\ i=1,\ldots,n.\end{aligned} \tag{36}$$

Now, using the obtained expressions for the main probabilistic characteristics, we can write the equality for the correlation coefficient r_{ig} of the number of different types devices employed in system SM|M$^{(n)}$|∞

$$r_{ig}=\frac{cov_{ig}}{\sqrt{Var_iVar_g}}=\frac{cm_{ig}-fm_ifm_g}{\sqrt{Var_iVar_g}},\ i=1,\ldots,n,\ g=1,\ldots,n, g\neq i. \tag{37}$$

4 Conclusion

In this paper we construct and investigate a mathematical model as a queueing system with the Semi Markovian incoming flow and heterogeneous service. The main probabilistic characteristics are found for the system under investigation, namely, the initial moments of the first and the second order of the number of employed devices of different type. Furthermore, we found an expression for the correlation coefficient between the number of different types devices employed. The resulting correlation coefficient indicates that the processes of change in the number of employed devices of different type in the system are dependent. Therefore, we can conclude that this infinitely linear queuing system with n types of servers can not be considered as a set of n separate systems with only one type of servers.

In the future it is planned to apply the asymptotic methods of investigation for finding moments of a higher order and for studying the functioning of the system under different special conditions. This may include the development of methods for investigating heterogeneous systems, for example, in the asymptotic condition of: high intensity of the incoming flow or an equivalent increase in the service time on devices of different type or extremely rare changes in special flow states (MMPP, MAP, SM).

There is great interest in the studying of various modifications of heterogeneous queueing systems: heterogeneous queueing systems with returns, with different volumes of applications of special incoming flows, and many others.

References

1. Artalejo, J.R., Gómes-Coral, A.: Retrial Queueing systems: A Computational Approach. Springer, Heidelberg (2008). 318 p
2. Chechelnitsky, A.A., Kucherenko, O.V.: Stationary characteristics of parallel queueing systems with two-dimensional input flow. Collection of Scientific Articles, Minsk, vol. 2, pp. 262–268 (2009). (in Russian)
3. Down, D.G., Wu, R.: Multi-layered round robin routing for parallel servers. Queueing Syst. **53**(4), 177–188 (2006)
4. Efrosinin, D., Farhadov, M., Kudubaeva, S.: Performance analysis and monotone control of a tandem queueing system. In: Vishnevsky, V., Kozyrev, D., Larionov, A. (eds.) DCCN 2013. CCIS, vol. 279, pp. 241–255. Springer, Cham (2014). doi:10.1007/978-3-319-05209-0_21
5. Efrosinin, D., Sztrik, J.: Performance analysis of a two-server heterogeneous retrial queue with threshold policy. Qual. Technol. Quant. Manage. **8**(3), 211–236 (2011)
6. Iravani, S.M.R., Luangkesorn, K.L., Simchi-Levi, D.: A general decomposition algorithm for parallel queues with correlated arrivals. Queueing Syst. **47**(4), 313–344 (2004)
7. Kargahi, M., Movaghar, A.: Utility accrual dynamic routing in real-time parallel systems. IEEE Trans. Parallel Distrib Syst. (TDPS) **21**(12), 1822–1835 (2010)
8. Lebedev, E.: On Asymptotic enlargement problem for stochastic networks. In: Mathematicals Methods and Optimization of Telecommunication Networks: Queues Flows, Systems, Networks, Minsk, pp. 108–109 (2011)

9. Lebedev, E., Chechelnitsky, A.: Limit diffusions for multi-channel networks with interdependent inputs. In: Modern Probabilistic Methods for Analysis and Optimization of Information and Telecommunication Networks, Minsk, pp. 133–136 (2011)
10. Lisovskaya, E., Moiseeva, S.: Study of the queuing systems $M|GI|N|\infty$. In: Dudin, A., Nazarov, A., Yakupov, R. (eds.) ITMM 2015. CCIS, vol. 564, pp. 175–184. Springer, Cham (2015). doi:10.1007/978-3-319-25861-4_15
11. Lisovskaya, E., Moiseeva, S., Pagano, M.: The total capacity of customers in the infinite-server queue with MMPP arrivals. In: Vishnevskiy, V.M., Samouylov, K.E., Kozyrev, D.V. (eds.) DCCN 2016. CCIS, vol. 678, pp. 110–120. Springer, Cham (2016). doi:10.1007/978-3-319-51917-3_11
12. Lozhkhovsky, A.G., Kaptu, V.A., Verbanov, O.V., Kolchar, V.M.: A mathematical model of packet traffic. Bul. Tomsk Polytech. Univ. **9**, 113–119 (2011). TPU, Tomsk (in Russian)
13. Moiseev, A.: Asymptotic analysis of the queueing network $SM-(GI/\infty)^K$. In: Dudin, A., Nazarov, A., Yakupov, R. (eds.) ITMM 2015. CCIS, vol. 564, pp. 73–84. Springer, Cham (2015). doi:10.1007/978-3-319-25861-4_7
14. Moiseeva, S., Zadiranova, L.: Feedback in infinite-server queuing systems. In: Vishnevsky, V., Kozyrev, D. (eds.) DCCN 2015. CCIS, vol. 601, pp. 370–377. Springer, Cham (2016). doi:10.1007/978-3-319-30843-2_38
15. Movaghar, A.: Analysis of a dynamic assignment of impatient customers to parallel queues. Queueing Syst. **67**(3), 251–273 (2011)
16. Pankratova, E., Moiseeva, S.: Queueing system with renewal arrival process and two types of customers. In: IEEE International Congress on Ultra Modern Telecommunications and Control Systems (ICUMT), pp. 514–517. IEEE, St. Petersburg (2015)
17. Pankratova, E., Moiseeva, S.: Queueing system $GI|GI|\infty$ with n types of customers. In: Dudin, A., Nazarov, A., Yakupov, R. (eds.) ITMM 2015. CCIS, vol. 564, pp. 216–225. Springer, Cham (2015). doi:10.1007/978-3-319-25861-4_19
18. Rykov, V., Efrosinin, D.: On the slow server problem. Autom. Remote Control **70**(12), 2013–2023 (2009)
19. Sheu, R.S., Ziedins, I.: Asymptotically optimal control of parallel tandem queues with loss. Queueing Syst. **65**(3), 211–227 (2010)
20. Taqqu, M.S., Leland, W.E., Willinger, W., Wilson, D.V.: Self-similarity in high-speed packet traffic: analysis and modeling of ethernet traffic measurements. Stat. Sci. **10**, 67–85 (1995)
21. Willinger, W.: The discovery of self-similar traffic. In: Haring, G., Lindemann, C., Reiser, M. (eds.) Performance Evaluation: Origins and Directions. LNCS, vol. 1769, pp. 513–527. Springer, Heidelberg (2000). doi:10.1007/3-540-46506-5_24