On a $BMAP/G/1$ Retrial System with Two Types of Search of Customers from the Orbit

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Abstract. A single server retrial queueing model, in which customers arrive according to a batch Markovian arrival process (BMAP), is considered. An arriving batch, finding server busy, enters an orbit. Otherwise one customer from the arriving batch enters for service immediately while the rest join the orbit. The customers from the orbit try to reach the server subsequently and the inter-retrial times are exponentially distributed. Additionally, at each service completion epoch, two different search mechanisms are switched-on. Thus, when the server is idle, a competition takes place between primary customers, the customers coming by retrial and the two types of searches. It is assumed that if the type II search reaches the service facility ahead of the rest, all customers in the orbit are taken for service simultaneously, while in the other two cases, only a single customer is qualified to enter the service. We assume that the service times of the four types of customers namely, primary, repeated and those by the two types of searches are arbitrarily distributed with different distributions. Steady state analysis of the model is performed.

Keywords: Batch Markovian arrival process · Orbit · Retrials · Customers search · Group service

1 Intoduction

Retrial queues represent an important, challenging and complicated for mathematical analysis class of queueing systems. A retrial queueing system is characterised by the fact that a customer arriving when all servers accessible for him/her are busy, leaves the service area and joins a group of unsatisfied

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customers called orbit, but after a random amount of time he/she returns and repeats his/her demand for service. Retrial queueing systems arise frequently in the stochastic modelling of telecommunications, computer systems, contact centers, etc. Review of retrial queueing literature could be found in [1,2,24,25,27,32]. In the retrial set up, each service is preceded and followed by the server(s) idle time because of the ignorance of the status of the server(s) and orbital customers by each other.

We are interested in designing retrial queueing models that reduce the server(s) idle time. One way to achieve this is by the introduction of search of orbital customers immediately after a service completion. Search for orbital customers was introduced in [3] and the paper [20] generalizes the result in [3] by introducing a search time, two types of services to customers (primary/orbital) and by assuming the arrival process to be the batch Markovian process. The queueing model with customers search in the buffer (not in the orbit) was considered in [31] where after each service completion the server starts searching of a customer in the buffer and the rate of the exponentially distributed search time is proportional to the number of customers presenting in the system.

This paper generalizes the model discussed in [20] by introducing two types of search and different types of services to primary/orbital customers (retrial/type I/type II searches) retaining the assumption that the arrival process is batch Markovian process ($BMAP$). A particular case of the proposed model with batch Poisson arrival process has been considered in [11]. A retrial model with two types of search, in which the number of customers taken for service depends on the orbit size, and with the batch Poisson arrival process is considered in [12]. However, namely $BMAP$ suits well for modelling the correlated bursty traffic in the modern communication networks. Approximation of such flows in terms of the stationary Poisson process can cause huge errors in the evaluation of performance characteristics of the networks. Therefore, analysis of queueing models with the $BMAP$ is of a great importance. Chakravarthy S.R. in [7] provides a review of queueing models with the batch Markovian arrivals. Retrial models with $BMAP$ have been investigated, e.g., in the papers [4,8,9,16–18,26].

The present model is motivated, e.g. by the following practical situation: In Airport/Bus stations/Railway stations passengers individually get into transport vehicles to destinations. Also it is common that travel agencies arrange for bulk transport for all the customers. Broadcasting of information simultaneously to many customers is possible in various wireless communication networks. More motivations of group service can be found e.g. in [5,6,8,23].

2 The Mathematical Model

We consider a single server queueing system in which the arrivals occur according to a $BMAP$. The $BMAP$, a special class of tractable Markov renewal process, is a rich class of point processes that includes many well known processes such as Poisson, PH-renewal processes and Markov-modulated Poisson process. One of the most significant features of the $BMAP$ is the underlying Markovian structure and it fits ideally in the context of matrix-analytic solutions to stochastic models. As is well known, Poisson processes are the simplest and most tractable ones used extensively in stochastic modelling. The idea of the $BMAP$ is to significantly generalize the Poisson processes and still keep the tractability for modelling purposes.

The $BMAP$ is described as follows. Let the underlying Markov chain $\{\nu_t, t \geq 0\}$ be irreducible and let $Q^* = (q_{ij})$ be the generator of this Markov chain with state space $\{1, 2, \cdots, m\}$. At the end of a sojourn time in state i, that is exponentially distributed with parameter $\lambda_i \geq -q_{ii}$, one of the following two events could occur: with probability $P_{ij}(l), 1 \leq i, j \leq m$, the transition corresponds to an arrival of group size $l \geq 1$, and the underlying Markov chain $\{\nu_t, t \geq 0\}$ is in state j; with probability $P_{ij}(0)$, the transition corresponds to no arrival and the state of the process $\{\nu_t, t \geq 0\}$ is $j,\ j \neq i$. Note that the Markov chain $\{\nu_t, t \geq 0\}$ can go from state i to state i only through an arrival. For $l \geq 0$, define matrices $D_l = (d_{ij}(l))$ such that $d_{ii}(0) = -\lambda_i,\ 1 \leq i \leq m$; $d_{ij}(0) = \lambda_i P_{ij}(0),$ for $j \neq i,\ 1 \leq i, j \leq m,$ and $d_{ij}(l) = \lambda_i P_{ij}(l)$. Assuming D_0 to be a non-singular matrix, the interarrival times will be finite with probability one and the arrival process does not terminate. Hence, we see that D_0 is a stable matrix. The generator Q^* is then given by $Q^* = \sum_{l=0}^{\infty} D_l$. Let $D(z)$ be the matrix generating function of D_l. That is, $D(z) = \sum_{l=0}^{\infty} z^l D_l$.

Thus, the $BMAP$ is described by the matrices $\{D_l\}$ with D_0 governing the transitions corresponding to no arrival and D_l governing those corresponding to arrivals of group size $l, l \geq 1$. The point process described by the $BMAP$ is a special class of semi-Markov processes with transition probability matrix given by

$$\int_0^x e^{D_0 t} dt D_l = \left[I - e^{D_0 x}\right] (-D_0)^{-1} D_l, l \geq 1.$$

For use in the sequel, let **e**, **0** and I denote, respectively, the (column) vector of dimension m consisting of 1's, the (row) vector of dimension m consisting of 0's, and the identity matrix of order m.

Let $\boldsymbol{\theta}$ be the stationary probability vector of the associated Markov process with generator Q^*. That is, $\boldsymbol{\theta}$ is the unique (positive) probability vector satisfying $\boldsymbol{\theta} Q^* = 0,\ \boldsymbol{\theta}\mathbf{e} = 1$. The constant

$$\lambda = \boldsymbol{\theta} \sum_{k=1}^{\infty} k D_k \mathbf{e},$$

referred to as the *fundamental rate* gives the expected number of arrivals per unit of time in the stationary version of the $BMAP$. For further details on $BMAP$, its properties, particular cases and usefulness in stochastic modelling, we refer to [7,29].

The service mechanism of the present system is described in the following manner. The primary unit who meets the server idle is served with service times having the distribution function $B_0(t)$, while the rest join the orbit. Each unit in the arriving batch finding the server busy, enters the orbit and retry to access the server with the time between two successive retrials, exponentially distributed having intensity $\alpha_i, i \geq 0$, when the number of customers in the orbit is i. Additionally, at a service completion epoch, two different search mechanism are switched on. Thus, if the server is idle, a competition takes place among primary customer, retrial customers and those resulting in the two types of searches to access the server. If a retrial customer reaches the idle server first, the customer entering the service is served with service times having the distribution function $B_3(t)$ while if the type I search turned out to be successful, the selected customer is served according to the distribution function $B_1(t)$. If type II search succeeded, all units present in the orbit are taken for service simultaneously and the service time of the whole group follows distribution function $B_2(t)$. Denote by $\beta_i(s) = \int\limits_0^\infty e^{-st}dB_i(t),\ Re\ s > 0$, the Laplace-Stieltjes transform (LST) and $b_r^{(i)}$ (assumed to be finite), the r^{th} moment associated with the distribution function $B_i(t), i = 0, 1, 2, 3 :\ b_r^{(i)} = \int\limits_0^\infty t^r dB_i(t)$. The duration of the type I (type II) search is characterized by the distribution function $H_1(t)(H_2(t))$ with LST $h_1(s)(h_2(s))$ and finite expectations h_1 and h_2. Distribution functions $H_l(t),\ l = 1, 2$, may be arbitrary, however, we assume that duration of the searches cannot be both constant. Otherwise, the search with larger value of $h_l, l = 1, 2$, will never succeed to be finished earlier that the another search and the search with larger duration has to be excluded from consideration.

The presence of the additional search mechanism allows to minimize the idle time of the server. If holding cost (charge paid due to the customers stay in the system) and costs associated with the different types of the search and service of customers are introduced, optimal tuning of the parameters of search mechanism will be possible based on the analysis results of which are presented below.

3 The Stationary Distribution of the Embedded Markov Chain

Denote by t_n the n^{th} service completion epoch; i_n the number of customers in the orbit and ν_n the state of the $BMAP$ process ν_t at the moment $t_n + 0$. Then

$$\zeta_n = \{(i_n, \nu_n),\ n \geq 1\}$$

is a two-dimensional Markov chain with state space $\{(l, \nu)\ ;\ l \geq 0, \nu = 1, 2, \cdots m\}$. In the sequel, we need the following auxiliary matrices. Define

$$\Phi_i = \int\limits_0^\infty e^{(D_0 - \alpha_i I)t}(1 - H_1(t))(1 - H_2(t))dt,\ i > 0,\ \Phi_0 = (-D_0)^{-1},$$

$$F_i^{(1)} = \int_0^\infty e^{(D_0-\alpha_i I)t}(1-H_2(t))dH_1(t),\ F_i^{(2)} = \int_0^\infty e^{(D_0-\alpha_i I)t}(1-H_1(t))dH_2(t),$$

$$F_i^{(3)} = \alpha_i \Phi_i,\ \text{for } i>0 \text{ and } F_0^{(1)} = F_0^{(2)} = F_0^{(3)} = (-D_0)^{-1}.$$

Here $F_i^{(r)}, r = 1,2,3$, give the matrices of probabilities that the idle period of the server expires through type I search or type II search or retrial.

Let $\Omega_k^{(r)}$ be the matrix of probabilities that exactly k arrivals occur during a service time of the r^{th} type, $r = 0,1,2,3$. It is well-known, see, e.g., [29] that these matrices can be obtained as coefficient matrices in the following matrix generating function:

$$\Omega_r(z) = \beta_r(-D(z)) = \sum_{k=0}^{\infty} \Omega_k^{(r)} z^k = \int_0^\infty e^{D(z)t} dB_r(t), r = 0,1,2,3.$$

Let $P(i,l)$, for $i \geq 0, l \geq 0$, denote the matrix of the one-step transition probabilities of the Markov chain $\zeta_n, n \geq 1$, with the $(\nu,\nu')^{th}$ entry defined as

$$P\{i_{n+1} = l, \nu_{n+1} = \nu' | i_n = i, \nu_n = \nu\}, \nu, \nu' = 1, 2, \cdots, m.$$

The following lemma, whose proof follows immediately from the described customers access mechanism and the formula of total probability, gives expression for the matrices $P(i,l)$.

Lemma 1. *The matrices $P(i,l)$ are calculated as follows:*

$$P(0,l) = \Phi_0 \sum_{k=1}^{l+1} D_k \Omega_{l-k+1}^{(0)} \quad l \geq 0,$$

$$P(i,l) = \Phi_i \sum_{k=1}^{l-i+1} D_k \Omega_{l-i-k+1}^{(0)} + F_i^{(1)} \Omega_{l-i+1}^{(1)} + F_i^{(2)} \Omega_l^{(2)} + F_i^{(3)} \Omega_{l-i+1}^{(3)},\quad i \geq 1, l \geq i-1,$$

$$P(i,l) = F_i^{(2)} \Omega_l^{(2)},\quad i \geq 1, 0 \leq l < i-1.$$

From now on, we make the assumption that the retrial rate α_i does not depend on i. That is $\alpha_i = \alpha$ for $i > 0$. In this case, the matrices $\Phi_i,\ F_i^{(r)},\ r = 1,2,3$, do not depend on i and are denoted as $\Phi,\ F^{(r)},\ r = 1,2,3$, correspondingly.

It can be shown that, due to the possibility of simultaneous service of all customers presenting in the system, the system is stable under any set of the system parameters, therefore the stationary probabilities of the chain always exist.

Let us denote these probabilities by

$$\pi(i,\nu) = \lim_{n\to\infty} P\{i_n = i, \nu_n = \nu\},\ \nu = 1,\ldots,m,$$

and let us introduce the following row vectors:

$$\boldsymbol{\pi}_i = (\pi(i,1),\cdots\cdots,\pi(i,m)),\ i \geq 0.$$

Using the obtained transition probabilities, we get the system of linear algebraic equations (equilibrium equations) for the steady state probabilities as given below:

$$\boldsymbol{\pi}_l = \boldsymbol{\pi}_0 \Phi_0 \sum_{k=1}^{l+1} D_k \Omega^{(0)}_{l-k+1} + \sum_{i=1}^{\infty} \boldsymbol{\pi}_i F^{(2)} \Omega^{(2)}_l + \sum_{i=1}^{l+1} \boldsymbol{\pi}_i \left[\Phi \sum_{k=1}^{l-i+1} D_k \Omega^{(0)}_{l-i-k+1} + F^{(1)} \Omega^{(1)}_{l-i+1} F^{(3)} \Omega^{(3)}_{l-i+1} \right], \quad l \geq 0. \quad (1)$$

To solve this infinite system of equations, we introduce the vector probability generating function $\boldsymbol{\pi}(z) = \sum_{l=0}^{\infty} \boldsymbol{\pi}_l z^l, \quad |z| < 1.$

Multiplying each of the equations in (1) by the corresponding power of z, summing up and rearranging the terms, we get

$$z\boldsymbol{\pi}(z) = \boldsymbol{\pi}_0 \Phi_0 (D(z) - D_0) \Omega_0(z) + z(\boldsymbol{\pi}(1) - \boldsymbol{\pi}_0) F^{(2)} \Omega_2(z) + (\boldsymbol{\pi}(z) - \boldsymbol{\pi}_0) Y(z)$$

where

$$Y(z) = \Phi(D(z) - D_0)\Omega_0(z) + F^{(1)}\Omega_1(z) + F^{(3)}\Omega_3(z).$$

Thus, the vector generating function $\boldsymbol{\pi}(z)$ satisfies the following vector functional equation

$$\boldsymbol{\pi}(z)(zI - Y(z)) = \boldsymbol{\pi}(0)(\Phi_0(D(z) - D_0)\Omega_0(z) - zF^{(2)}\Omega_2(z) - Y(z)) + z\boldsymbol{\pi}(1)F^{(2)}\Omega_2(z). \quad (2)$$

This equation includes the unknown vector generating function $\boldsymbol{\pi}(z)$ at three points: z, 0 and 1. Next we make an attempt to eliminate the unknown vector $\boldsymbol{\pi}(1)$ from (2) in the trivial way, i.e., by substituting $z = 1$ in (2). Then we obtain the following relation between the vectors $\boldsymbol{\pi}(1)$ and $\boldsymbol{\pi}(0) = \boldsymbol{\pi}_0$:

$$\boldsymbol{\pi}(1)(I - Y(1) - F^{(2)}\Omega_2(1)) = \boldsymbol{\pi}_0(\Phi_0 \left[D(1) - D_0\right]. \quad (3)$$

However, we cannot eliminate the vector $\boldsymbol{\pi}(1)$ directly from Eq. (3) because it is possible to show that the matrix

$$A = Y(1) + F^{(2)}\Omega_2(1)$$

is irreducible stochastic and, consequently, the matrix $I - A$ in the right hand side of (3) is singular.

To overcome this difficulty, we apply the well-known trick by M. Neuts. Let $\boldsymbol{\rho}$ be the left probability eigenvector of the matrix A, i.e., it satisfies the equations

$$\boldsymbol{\rho} A = \boldsymbol{\rho}, \quad \boldsymbol{\rho}\mathbf{e} = 1.$$

Adding the vector $\boldsymbol{\pi}(1)\mathbf{e}\boldsymbol{\rho}$ to both sides of (3), observing that the matrix $I - A + \mathbf{e}\boldsymbol{\rho}$ is nonsingular and that

$$\boldsymbol{\rho}(I - A + \mathbf{e}\boldsymbol{\rho})^{-1} = \boldsymbol{\rho},$$

we obtain from (3) that

$$\boldsymbol{\pi}(1) = \boldsymbol{\rho} + \boldsymbol{\pi}_0(\Phi_0\,[D(1) - D_0]\,\Omega_0(1) - A)C,$$

where

$$C = (I - A + \mathbf{e}\boldsymbol{\rho})^{-1}.$$

Then, the vector functional Eq. (2) transforms into equation

$$\boldsymbol{\pi}(z)(zI - Y(z)) = \boldsymbol{\pi}_0\Big[\Phi_0(D(z) - D_0)\Omega_0(z) - zF^{(2)}\Omega_2(z) - Y(z)) + z(\Phi_0(D(1) - D_0)\Omega_0(1) - A)CF^{(2)}\Omega_2(z)\Big] + z\boldsymbol{\rho}F^{(2)}\Omega_2(z) \tag{4}$$

which includes the unknown vector generating function $\boldsymbol{\pi}(z)$ only at two points: z and 0.

The methodologies for solving equations of type (4) in the case when the matrix $Y(1)$ is stochastic are well-known. One of them is based on the use of M. Neuts' approach (see [30]) that exploits the matrix G which is the solution of the nonlinear matrix equation $G = Y(G)$. Another one uses reasonings of analyticity of the vector generating function $\boldsymbol{\pi}(z)$ in the unit disk of the complex plane, see, e.g. [15].

However, in (4) the matrix $Y(1)$ is the sub-stochastic, but not stochastic. Solution of Eq. (4) in this case can be derived using the results obtained during the analysis of $BMAP/SM/1$ queue with so called disasters, see [13,21] where the analyticity approach is properly adjusted or the papers [14,22] where the M. Neuts' approach is generalized to the corresponding class of multi-dimensional Markov chains. Disasters have the same effect (removal of all customers from the system) as simultaneous service of all customers from the orbit after type II search succeeds to win in competition with type I search and primary or orbital customers.

4 Stationary Distributions of the Number of Customers in the Orbit and in the System at Arbitrary Time

Denote by $\mathbf{p}(i,r), i \geq 0, r = 0, \ldots, 4$, the steady state probability vector that at an arbitrary time there are i customers in the system, and the current service is in the r^{th} mode. Note that $r = 4$ corresponds to the case when the server is idle. The following theorem gives expression for the steady state probability vectors.

Theorem 1. The stationary probability vector $\mathbf{p}(i,r)$ are calculated as follows:

$$\mathbf{p}(0,4) = \tau^{-1}\boldsymbol{\pi}_0(-D_0)^{-1},$$

$$\mathbf{p}(i,4) = \tau^{-1}\boldsymbol{\pi}_i\Phi, \quad i \geq 1,$$

$$\mathbf{p}(i,3) = \tau^{-1}\sum_{l=1}^{i}\boldsymbol{\pi}_l F^{(3)}\tilde{\Omega}_{i-l}^{(3)},$$

$$\mathbf{p}(i,2) = \tau^{-1}\sum_{l=1}^{i}\boldsymbol{\pi}_l F^{(2)}\tilde{\Omega}_{i-l}^{(2)},$$

$$\mathbf{p}(i,1) = \tau^{-1}\sum_{l=1}^{i}\boldsymbol{\pi}_l F^{(1)}\tilde{\Omega}_{i-l}^{(1)},$$

$$\mathbf{p}(i,0) = \tau^{-1}\boldsymbol{\pi}_0\sum_{k=1}^{i}(-D_0)^{-1}D_k\tilde{\Omega}_{i-k}^{(0)} + \sum_{l=1}^{i-1}\boldsymbol{\pi}_l\sum_{k=1}^{i-l}\Phi D_k\tilde{\Omega}_{i-l-k}^{(0)}, \quad i>0,$$

where the matrices $\tilde{\Omega}_m^{(r)}$ are the coefficients appearing in the matrix expansion

$$\tilde{\Omega}_r(z) = \sum_{k=0}^{\infty}\tilde{\Omega}_k^{(r)}z^k = \int_0^{\infty} e^{D(z)t}(1-B_r(t))dt,\ r=0,\dots,3,$$

and the average inter-departure time, τ, is given by formula

$$\tau = \boldsymbol{\pi}_0((-D_0)^{-1} + b_1^{(0)}I)\mathbf{e} + \sum_{i=1}^{\infty}\boldsymbol{\pi}_i\left(\sum_{j=1}^{3}F^{(j)}b_1^{(j)}\mathbf{e} + \Phi(I - D_0 b_1^{(0)})\mathbf{e}\right).$$

Proof follows from the theory of Markov renewal processes (see [10,31]).

In a similar manner, if we define $\mathbf{q}(i,r),\ i\geq 0,\ r=0,\dots,4$, as the steady state probability vectors at an arbitrary time that there are i customers in the orbit and the current service is in the r^{th} mode, we get the following result:

Theorem 2. Vectors $\mathbf{q}(i,r), i\geq 0, r=0,\dots,4$, are computed as follows:

$$\mathbf{q}(0,4) = \tau^{-1}\boldsymbol{\pi}_0(-D_0)^{-1},$$

$$\mathbf{q}(i,4) = \tau^{-1}\boldsymbol{\pi}_i\Phi, \quad i>0,$$

$$\mathbf{q}(i,3) = \tau^{-1}\sum_{l=1}^{i+1}\boldsymbol{\pi}_l F^{(3)}\tilde{\Omega}_{i-l+1}^{(3)},$$

$$\mathbf{q}(i,2) = \tau^{-1}\sum_{l=1}^{\infty}\boldsymbol{\pi}_l F^{(2)}\tilde{\Omega}_{i}^{(2)},$$

$$\mathbf{q}(i,1) = \tau^{-1}\sum_{l=1}^{i+1}\boldsymbol{\pi}_l F^{(1)}\tilde{\Omega}_{i-l+1}^{(1)},$$

$$\mathbf{q}(i,0) = \tau^{-1}\left(\boldsymbol{\pi}_0(-D_0)^{-1}\sum_{k=1}^{i+1}D_k\tilde{\Omega}_{i+1-k}^{(0)} + \sum_{l=1}^{i}\sum_{k=1}^{i+1-l}\Phi D_k\tilde{\Omega}_{i+1-l-k}^{(0)}\right), \quad i\geq 0.$$

5 Some Performance Measures

Let us introduce the following vector partial generating functions

$$\mathbf{P}(z,r)=\sum_{i=0}^{\infty} z^i\mathbf{p}(i,r),\ \mathbf{Q}(z,r)=\sum_{i=0}^{\infty} z^i\mathbf{q}(i,r),\ |z|<1,\ r=0,\dots,4.$$

Having computed the stationary distributions for both the system size and orbit size, we can calculate some important performance characteristics of the model as follows:

- Probability of the system being empty at an arbitrary moment is defined by $\mathbf{p}(0,4)\mathbf{e}$;
- Probability that the server is free at an arbitrary moment is defined by $\mathbf{P}(1,4)\mathbf{e}$;
- Probability that the server is working in the zero mode (primary customer service) at an arbitrary moment is defined by $\mathbf{P}(1,0)\mathbf{e}$;
- Probability that the server is working in mode 1 (orbital customer service after the type I search) at an arbitrary moment is defined by $\mathbf{P}(1,1)\mathbf{e}$;
- Probability that the server is working in mode 2 (orbital customer(s) service after the II search) at an arbitrary moment is defined by $\mathbf{P}(1,2)\mathbf{e}$;
- Probability that the server is working in mode 3 (retrial customer service) at an arbitrary moment is defined by $\mathbf{P}(1,3)\mathbf{e}$;
- Probability that the orbit is empty at an arbitrary moment is defined by $\sum_{j=0}^{4}\mathbf{q}(0,j)\mathbf{e}$;
- Probability of having i customers in the orbit at an arbitrary moment is defined by $\sum_{j=0}^{4}\mathbf{q}(i,j)\mathbf{e}$;
- Average number of customers in the system is defined by $\sum_{j=0}^{4}\mathbf{P}'(1,j)\mathbf{e}$;
- Average number of customers in the orbit is defined by $\sum_{j=0}^{4}\mathbf{Q}'(1,j)\mathbf{e}$.

Remark

Corresponding probabilities for an arbitrary batch arrival epochs are computed by the analogous formulas only the vector $\mathbf{e}$ has to be replaced with the vector $-D_0\mathbf{e}$. Probability of starting the service of an arbitrary customer immediately upon arrival (probability that the customer receives service in the system without visiting the orbit) is defined by

$$\lambda^{-1}\mathbf{P}(1,4)(-D_0)\mathbf{e}.$$

Some examples of computation of the key performance measures of the system for the considered model in case of the group Poisson arrival process are presented in [11]. Computations for the general case of the $BMAP$ are much more involved. But they can be successfully done based on the corresponding modules of software described in [19].

6 Conclusions

We considered retrial queueing model where the usual mechanism of customers access to the service via the competition of the primary and orbital customers is supplemented by the mechanisms of customers search in the orbit by the server. One option of the search leads to the individual service of a customer found in the orbit. Another one results in simultaneous service of all customers presenting in the orbit. Stationary distributions of the system states at the embedded service completion moments and arbitrary moments are computed along with some important performance measures of the system.

The results can be used for optimization of operation of the system if some cost criteria accounting the quality and cost of different kinds of customers service and access will be introduced. More types of customers search can be considered. The case when search times have a phase type distribution, in which the presented analytical results may be more easy implemented in the form of software, deserves more close consideration.

Extension of the analysis to the case when the total retrial intensity depends on the current number of customers in orbit is possible based on the results from [28] with modification that accounts possibility of emptying the system at the random moment, irrespectively to the current number of customers in the system.

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