Recurrence Identities of b-ary Partitions

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Abstract Solving the b-ary partition problem, counting the number $p_b(n)$ of partitions of n into powers of b, is a pursuit which dates back to Euler. The function $p_b(n)$ satisfies a recurrence, and this note examines a family of identities which can be deduced by iterating the recurrence in a suitable way. These identities can then be used to calculate $p_b(n)$ for large values of n. Further, these identities correspond to generating function identities involving a sequence of polynomials which have suggestive connections to Eulerian polynomials.

Keywords Integer partitions · Partition functions · Recurrence · Congruences
Generating functions · Eulerian polynomials

1 History

A *partition* of a nonnegative integer n is an expression consisting of a sum of positive integers whose value is n. A *b-ary partition* of n is a partition of n where each term in the sum is a power of a base b. Denote the number of partitions of n as $p(n)$ and the number of b-ary partitions[1] as $p_b(n)$. For example, the partitions of 4 are $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, and therefore $p(4) = 5$ and $p_2(4) = 4$. The problem of calculating $p_b(n)$ dates to Euler [1] who first studied $p_2(n)$ in his celebrated 1748 paper *De partitione numerorum*. In 1918, Tanturri [2] examined the $p_2(n)$ problem, stating its recurrence and proving several identities. In that same year, Hardy and Ramanujan [3] published their asymptotic formula for the general partition function $p(n)$. To achieve this, they pioneered the circle method, noting that for the generating function for $p(n)$:

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[1]See Table 1 for values of $p_b(bn)$ for small values of b and n. The expression $p_b(bn)$ is chosen because by Theorem 3.1 the value of $p_b(n)$ is constant on runs of b.

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Table 1 Values of $p_b(n)$ for $2 \leq b \leq 9$ and $1 \leq n \leq 32$

n	$p_2(2n)$	$p_3(3n)$	$p_4(4n)$	$p_5(5n)$	$p_6(6n)$	$p_7(7n)$	$p_8(8n)$	$p_9(9n)$
1	2	2	2	2	2	2	2	2
2	4	3	3	3	3	3	3	3
3	6	5	4	4	4	4	4	4
4	10	7	6	5	5	5	5	5
5	14	9	8	7	6	6	6	6
6	20	12	10	9	8	7	7	7
7	26	15	12	11	10	9	8	8
8	36	18	15	13	12	11	10	9
9	46	23	18	15	14	13	12	11
10	60	28	21	18	16	15	14	13
11	74	33	24	21	18	17	16	15
12	94	40	28	24	21	19	18	17
13	114	47	32	27	24	21	20	19
14	140	54	36	30	27	24	22	21
15	166	63	40	34	30	27	24	23
16	202	72	46	38	33	30	27	25
17	238	81	52	42	36	33	30	27
18	284	93	58	46	40	36	33	30
19	330	105	64	50	44	39	36	33
20	390	117	72	55	48	42	39	36
21	450	132	80	60	52	46	42	39
22	524	147	88	65	56	50	45	42
23	598	162	96	70	60	54	48	45
24	692	180	106	75	65	58	52	48
25	786	198	116	82	70	62	56	51
26	900	216	126	89	75	66	60	54
27	1014	239	136	96	80	70	64	58
28	1154	262	148	103	85	75	68	62
29	1294	285	160	110	90	80	72	66
30	1460	313	172	119	96	85	76	70
31	1626	341	184	128	102	90	80	74
32	1828	369	199	137	108	95	85	78

Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; there is no obvious method of writing down a "dominant term."

In a 1940 paper, Mahler [4] established an oft-cited estimate which implies that $p_b(n)$ has intermediate growth, namely

$$\log p_b(n) \sim \frac{(\log n)^2}{2\log b}.$$

Later, in 1948, de Bruijn [5] made use of residue calculations to refine Mahler's work on the asymptotics of $p_b(n)$. Subsequently in 1966, Knuth [6] refined the asymptotic estimates of $p_2(n)$. Churchhouse [7] in 1969 proved theorems regarding congruences of $p_2(n)$ by iterating the recurrence. He also posited a conjecture related to these congruences. Then, Rødseth [8] in 1970 proved Churchhouse's conjecture as well as congruences in the cases where b is a prime. Many later authors adapted Rødseth's method, about which he says:

> The method we use below in proving the above results goes back to Ramanujan, and has been exploited since then by many writers, notably Watson. We use the technique of Atkin and O'Brien.

Building on Rødseth's work, Andrews [9] used generating function algebra to generalize Churchhouse's conjecture to all bases. This year also saw an independent proof of Churchhouse's conjecture by Gupta [10]. Then, in 1972, Gupta [11] proved Churchhouse's result in a simpler way by making use of Kemmer's identity. In 1975, Hirschhorn and Loxton [12] proved several congruences for $p_2(n)$ for n along certain arithmetic progressions. Dirdal [13, 14] also proved congruences for $p_b(n)$ realizing these as limits of congruences of $p_{b,d}(n)$, the number of partitions of n into powers of b repeating each power at most d times. Gupta and Pleasants [15] used Kemmer's identity and matrix methods in 1979 to prove properties of $p_b(n)$ based on the base b expansion of n. Then, in 1990, Reznick [16] proved properties of $p_{2,d}(n)$ while relating them to $p_2(n)$. His terrific bibliography in that paper is an excellent resource on the history of this subject. In a 2011 paper, Rødseth and Sellers [17] gave the problem a fresh look and proved congruences for $p_b(n)$ along the lines of Hirschhorn and Loxton.

2 Notation

Denote the set of integers by $\mathbb{Z}$ and the nonnegative integers by $\mathbb{N}$. Let $p_b(n)$ be the number of b-ary partitions of n, that is, the number of partitions of n whose parts are powers of b with no restriction on how often each power is used. Let $B_b(m,q)$ be the generating function of $p_b(b^m n)$, that is,

$$B_b(m,q) = \sum_{n\in\mathbb{Z}} p_b(b^m n)q^n.$$

Consider a sequence $(a_i)_{i\in I}$. Denote the length of the sequence as $|I|$, and note that a sequence can be interpreted as a $1 \times |I|$ matrix. Given a matrix M, denote its transposition as M^T.

The subsequent notations follow those of Graham et al. [18]. If S is any statement, then let $[[S]]$ denote the Iverson bracket:

$$[[S]] = \begin{cases} 1 \text{ if } S \text{ is true;} \\ 0 \text{ if } S \text{ is false.} \end{cases}$$

Denote the nth falling power of x as $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$. Let $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ be Stirling numbers of the first and second kind, respectively. In particular, define

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = [[n = 0]] \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix} = [[k = 0]]$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$$
$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}.$$

Further, let $\left\langle{n \atop k}\right\rangle$ denote the Eulerian numbers, that is,

$$\left\langle{n \atop k}\right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

3 The Recurrence

This section concerns itself with proving basic identities for $p_b(n)$.

Theorem 3.1 *The b-ary partition function satisfies the following recurrence:*

$$p_b(n) = 0 \textit{ for } n < 0$$
$$p_b(n) = 1 \textit{ for } 0 \le n < b$$
$$p_b(bn+i) = p_b(bn) \textit{ for } 0 \le i < b \qquad \text{(RI)}$$
$$p_b(bn) = p_b(bn-1) + p_b(n) \qquad \text{(RII)}$$

Proof Let $0 \le i < b$. Consider a b-ary partition of $bn+i$. Such a partition must contain at least i copies of 1. Let f be a map which removes i ones from a b-ary partition of $bn+i$, and similarly let g be a map which adds i ones to a b-ary partition of bn. These operations are inverses since removing i ones from a b-ary partition of $bn+i$, and then adding i ones to the result produces the initial b-ary partition, that is, fg is the identity map and therefore f is a bijection. Thus, $p_b(bn) = p_b(bn+i)$ which proves RI.

To see RII, partition the set of b-ary partitions of bn into two sets: those involving ones and those not. For those involving ones, removing a single one will result in a b-ary partition of $bn-1$, and vice versa. Note that there are $p_b(bn-1)$ such partitions. For those not, each part is a positive power of b, from which b may be factored out, and the resulting sum will be a b-ary partition of n. Similarly for any b-ary partition of n, multiplying each part by b will result in a b-ary partition of bn. As before, this defines a bijection from b-ary partitions of bn without ones to the b-ary partitions of n. Therefore, the number of b-ary partitions of bn without ones is $p_b(n)$. Consequently, $p_b(bn) = p_b(bn-1) + p_b(n)$.

The following corollary is the primary way the recurrence for $p_b(n)$ will be used in what is to follow.

Corollary 3.2 *The b-ary partition counting function $p_b(n)$ satisfies the following identity:*

$$p_b(bn) = p_b(bn-b) + p_b(n). \tag{RIII}$$

Proof Combining RI and RII reveals

$$\begin{aligned} p_b(bn) &= p_b(bn-1) + p_b(n) \text{ by } \mathbf{RII} \\ &= p_b(b(n-1)+b-1) + p_b(n) \text{ by } \mathbf{RI} \\ &= p_b(b(n-1)) + p_b(n) \end{aligned}$$

and hence the corollary.

4 Generalizations of Tanturri and Churchhouse

The following lemma is a generalization of an identity which goes back to Tanturri.

Lemma 4.1 *The b-ary partition counting function $p_b(n)$ satisfies the following identity:*

$$p_b(bn) = \sum_{k=0}^{n} p_b(n-k)$$

Proof By RIII, $p_b(n) = p_b(bn) - p_b(b(n-1))$, so

$$\sum_{k=0}^{n} p_b(b(n-k)) - p_b(b(n-1-k)) = \sum_{k=0}^{n} p_b(n-k)$$

where the left-hand side is a telescoping sum, leaving

$$p_b(bn) - p_b(-b) = \sum_{k=0}^{n} p_b(n-k)$$

hence

$$p_b(bn) = \sum_{k=0}^{n} p_b(n-k)$$

as desired.

Churchhouse extended this for $b = 2$ to calculate $p_2(2^m n)$. This may be further extended to all b.

Theorem 4.2 *There exist positive integers $C_{b,m}(k)$ such that*

$$p_b(b^m n) = \sum_{k=0}^{n} C_{b,m}(k) p_b(n-k). \qquad (\mathrm{IH}(m))$$

Proof The proof proceeds by induction on m, with the case $m = 1$ being provided by Lemma 4.1. The assertion IH$(m+1)$ can be shown by assuming IH(m), applying this to $p_b(b^m(bn))$, separating the first term, reindexing the remaining terms by setting $k = bj - i$, and using RI:

$$\begin{aligned} p_b(b^{m+1}n) &= p_b(b^m(bn)) \\ &= \sum_{k=0}^{bn} C_{b,m}(k) p_b(bn-k) \\ &= C_{b,m}(0) p_b(bn) + \sum_{k=1}^{bn} C_{b,m}(k) p_b(bn-k) \\ &= C_{b,m}(0) p_b(bn) + \sum_{j=1}^{n} \sum_{i=0}^{b-1} C_{b,m}(bj-i) p_b(bn-bj+i) \\ &= C_{b,m}(0) p_b(bn) + \sum_{j=1}^{n} \sum_{i=0}^{b-1} C_{b,m}(bj-i) p_b(bn-bj). \end{aligned}$$

Now, applying Lemma 4.1 and reindexing by setting $h = n - j - \ell$ reveals

$$\begin{aligned} p_b(bn - bj) &= p_b(b(n - j)) \\ &= \sum_{\ell=0}^{n-j} p_b(n - j - \ell) \\ &= \sum_{h=0}^{n-j} p_b(h) \end{aligned}$$

and therefore this yields

$$\begin{aligned} p_b(b^{m+1}n) &= C_{b,m}(0)p_b(bn) + \sum_{j=1}^{n}\sum_{i=0}^{b-1} C_{b,m}(bj - i)p_b(bn - bj) \\ &= C_{b,m}(0)p_b(bn) + \sum_{j=1}^{n}\sum_{i=0}^{b-1} C_{b,m}(bj - i)\sum_{h=0}^{n-j} p_b(h). \end{aligned}$$

This sum may be reordered by factoring out the sum indexed by h, extending the range of the sum indexed by h, making the substitution $s = n - h$, interchanging the sums indexed by j and s, limiting the range of the sum indexed by j, and recalling that $k = bj - i$, that is,

$$\begin{aligned} \sum_{j=1}^{n}\sum_{i=0}^{b-1} C_{b,m}(bj - i)\sum_{h=0}^{n-j} p_b(h) &= \sum_{j=1}^{n}\sum_{h=0}^{n-j} p_b(h)\sum_{i=0}^{b-1} C_{b,m}(bj - i) \\ &= \sum_{j=1}^{n}\sum_{h=0}^{n}[[h \le n - j]]p_b(h)\sum_{i=0}^{b-1} C_{b,m}(bj - i) \\ &= \sum_{j=1}^{n}\sum_{s=0}^{n}[[n-s \le n-j]]p_b(n - s)\sum_{i=0}^{b-1} C_{b,m}(bj - i) \\ &= \sum_{j=1}^{n}\sum_{s=0}^{n}[[j \le s]]p_b(n - s)\sum_{i=0}^{b-1} C_{b,m}(bj - i) \\ &= \sum_{s=0}^{n} p_b(n - s)\sum_{j=1}^{n}[[j \le s]]\sum_{i=0}^{b-1} C_{b,m}(bj - i) \\ &= \sum_{s=0}^{n} p_b(n - s)\sum_{j=1}^{s}\sum_{i=0}^{b-1} C_{b,m}(bj - i) \\ &= \sum_{s=0}^{n} p_b(n - s)\sum_{k=1}^{sb} C_{b,m}(k). \end{aligned}$$

Finally, the first term may be combined with this sum using Lemma 4.1:

$$\begin{aligned}
p_b(b^{m+1}n) &= C_{b,m}(0)p_b(bn) + \sum_{s=0}^{n} p_b(n-s)\sum_{k=1}^{sb} C_{b,m}(k) \\
&= C_{b,m}(0)\sum_{s=0}^{n} p_b(n-s) + \sum_{s=0}^{n} p_b(n-s)\sum_{k=1}^{sb} C_{b,m}(k) \\
&= \sum_{s=0}^{n} p_b(n-s)\sum_{k=0}^{sb} C_{b,m}(k) \\
&= \sum_{s=0}^{n}\sum_{k=0}^{bs} C_{b,m}(k)p_b(n-s)
\end{aligned}$$

Thus,

$$p_b(b^{m+1}n) = \sum_{s=0}^{n} C_{b,m+1}(s)p_b(n-s)$$

where

$$C_{b,m+1}(s) = \sum_{k=0}^{bs} C_{b,m}(k)$$

proving IH($m+1$) and hence the theorem.

The coefficients $C_{b,m}(k)$ are, in fact, more than simply coefficients, and they are indeed polynomials of degree $m-1$.

Theorem 4.3 *The values $C_{b,m}(k)$ are polynomials of degree at most $m-1$ evaluated at k.*

Proof Note that $C_{b,1} = 1$, a degree 0 polynomial in k. By the inductive hypothesis $C_{b,m}(k) = \sum_{i=0}^{m-1} \alpha_{m,i}k^i$, therefore[2]

$$\begin{aligned}
C_{b,m+1}(k) &= \sum_{j=0}^{bk} C_{b,m}(j) \\
&= \sum_{j=0}^{bk}\sum_{i=0}^{m-1} \alpha_{m-1,i}\, j^i \\
&= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{j=0}^{bk} j^i \\
&= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{j=0}^{bk}\sum_{l=0}^{i} \left\{ {i \atop l} \right\} j^{\underline{l}}
\end{aligned}$$

[2] Note that the Stirling numbers $\left[{n \atop k} \right]$ and $\left\{ {n \atop k} \right\}$ are defined on page 56.

by an identity in [18, p. 264] which gives powers as a sum of falling powers. Then, by interchanging the order of summation and using the power rule for falling powers (*Ibid.*, p. 50 (2.50)):

$$C_{b,m+1}(k) = \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \left\{ {i \atop l} \right\} \sum_{j=0}^{bk} j^{\underline{l}}$$
$$= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \left\{ {i \atop l} \right\} \frac{(bk)^{\underline{l+1}}}{l+1}.$$

Now, using an identity writing falling powers as a sum of powers (*Ibid.*, p. 264), noting that $l + 1 \leq m$ and interchanging the order of summation reveals:

$$C_{b,m+1}(k) = \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \sum_{j=0}^{l+1} \left[{l+1 \atop j} \right] (-1)^{l+1-j} (bk)^j$$
$$= \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \sum_{j=0}^{l+1} k^j b^j \left[{l+1 \atop j} \right] (-1)^{l+1-j}$$
$$= \sum_{j=0}^{m} k^j b^j \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \left[{l+1 \atop j} \right] (-1)^{l+1-j}$$

Thus, $C_{b,m+1}(k)$ is a polynomial in k of degree at most m with coefficients

$$\alpha_{m+1,j} = b^j \sum_{i=0}^{m-1} \alpha_{m,i} \sum_{l=0}^{i} \frac{1}{l+1} \left\{ {i \atop l} \right\} \left[{l+1 \atop j} \right] (-1)^{l+1-j}$$

concluding the proof.

Recall that the generating function for $p_b(b^m n)$ is $B_b(m, q) = \sum_{n \in \mathbb{Z}} p_b(b^m n) q^n$.

Lemma 4.4 *The generating function for $p_b(bn)$ satisfies the identity:*

$$(1 - q) B_b(1, q) = B_b(0, q)$$

Proof By RIII, $p_b(bn) = p_b(b(n-1)) + p_b(n)$ and therefore $p_b(n) = p_b(bn) - p_b(b(n-1))$, so multiplying by q^n on both sides and summing over all integers n

$$\begin{aligned}\sum_{n\in\mathbb{Z}} p_b(n)q^n &= \sum_{n\in\mathbb{Z}} p_b(bn)q^n - \sum_{n\in\mathbb{Z}} p_b(b(n-1))q^n \\ &= \sum_{n\in\mathbb{Z}} p_b(bn)q^n - q\sum_{n\in\mathbb{Z}} p_b(bn)q^n \\ &= (1-q)\sum_{n\in\mathbb{Z}} p_b(bn)q^n \\ B_b(0,q) &= (1-q)B_b(1,q)\end{aligned}$$

establishing the claim.

5 A Family of Generating Function Identities

This section contains a proof of the main theorem which reveals a family of generating function identities. These identities correspond to a sequence of polynomials which have suggestive connections to Eulerian polynomials.

First, this lemma shows the recurrence may be iterated to express any value of $p_b(n)$ as the sum of multiples of $p_b(b^m)$ for suitable m.

Lemma 5.1 *For all* $n, m \geq 1$, *and* $1 \leq k < b^m$,

$$p_b(b^m n + ub) = p_b(b^m n) + \sum_{k=1}^{u} p_b(b^{m-1}n + k).$$

Proof Let $k = ub + v$ with $0 \leq v < b$. It may be assumed that $v = 0$ because if $v > 0$, then by RI

$$p_b(b^m n + k) = p_b(b^m n + ub + v) = p_b(b^m n + ub).$$

Therefore, applying RIII once, twice, and finally a total of u times iteratively to the leading term, it may be seen that

$$\begin{aligned}p_b(b^m n + ub) &= p_b(b^m n + (u-1)b) + p_b(b^{m-1}n + u) \\ &= p_b(b^m n + (u-2)b) + p_b(b^{m-1}n + u-1) + p_b(b^{m-1}n + u) \\ &= p_b(b^m n + (u-2)b) + \sum_{j=0}^{1} p_b(b^{m-1}n + u - j) \\ p_b(b^m n + ub) &= p_b(b^m n) + \sum_{j=0}^{u-1} p_b(b^{m-1}n + u - j).\end{aligned}$$

Letting $\ell = u - j$ then reveals

$$p_b(b^m n + ub) = p_b(b^m n) + \sum_{\ell=1}^{u} p_b(b^{m-1}n + \ell)$$

concluding the proof.

Lemma 5.2 *For all n and $m \geq 2$,*

$$p_b(b^m n) = p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) + [[m > 2]]b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1) + ub)$$

Proof First apply RIII to $p_b(b^m n)$ to obtain

$$\begin{aligned} p_b(b^m n) &= p_b(b^m n - b) + p_b(b^{m-1}n) \\ &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1}n). \end{aligned}$$

Then, Lemma 5.1 may be applied to the first term resulting in

$$p_b(b^m(n-1) + (b^{m-1} - 1)b) = p_b(b^m(n-1)) + \sum_{k=1}^{b^{m-1}-1} p_b(b^{m-1}(n-1) + k)$$

which can then be substituted into the previous expression. Then, note that the first $(b-1)$ terms in the sum are identical by RI. When $m = 2$ these are the only terms, but if $m > 2$ there are more terms in the sum which is indicated by the factor $[[m > 2]]$ below.

$$\begin{aligned} p_b(b^m n) &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1}n) \\ &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + \sum_{k=1}^{b^{m-1}-1} p_b(b^{m-1}(n-1) + k) \\ &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}n) \\ &\quad + [[m > 2]] \sum_{k=b}^{b^{m-1}-1} p_b(b^{m-1}(n-1) + k) \end{aligned}$$

When $m > 2$, the summation stratifies by RI:

$$\sum_{k=b}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k) = \sum_{u=1}^{b^{m-2}-1} \sum_{v=0}^{b-1} p_b(b^{m-1}(n-1)+ub+v)$$
$$= b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub)$$

Therefore, in the general case, the expression becomes

$$p_b(b^m n) = p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) + [[m>2]]b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub)$$

as claimed.

Corollary 5.3 *The generating function for $p_b(b^2n)$ satisfies the identity:*

$$(1-q)^2 B_b(2,q) = (1+(b-1)q)B_b(0,q)$$

Proof When $m=2$, Lemma 5.2 becomes

$$p_b(b^2n) = p_b(b^2(n-1)) + p_b(bn) + (b-1)p_b(b(n-1))$$

that is,

$$p_b(b^2n) - p_b(b^2(n-1)) = p_b(bn) + (b-1)p_b(b(n-1)).$$

By passing to generating functions, the result is achieved.

$$\sum_{n\in\mathbb{Z}} p_b(b^2n)q^n - \sum_{n\in\mathbb{Z}} p_b(b^2(n-1))q^n = \sum_{n\in\mathbb{Z}} p_b(bn)q^n + \sum_{n\in\mathbb{Z}} (b-1)p_b(b(n-1))q^n$$
$$B_b(2,q) - qB_b(2,q) = B_b(1,q) + (b-1)qB_b(1,q)$$
$$(1-q)B_b(2,q) = (1+(b-1)q)B_b(1,q)$$
$$(1-q)B_b(2,q) = (1+(b-1)q)(1-q)^{-1}B_b(0,q)$$

Therefore,

$$(1+(b-1)q)B_b(0,q) = (1-q)^2 B_b(2,q)$$

as stated.

Lemma 5.4 *For all $n, m \geq 1$, and $1 \leq k < b^m$, there exist polynomials $g_{m,k,j}(x) = g_j(x)$ of degree j with integer coefficients for $0 \leq j \leq m-1$ such that*

$$p_b(b^m n + k) = p_b(b^m n) + \sum_{j=1}^{m-1} g_j(b)p_b(b^j n). \qquad \text{(IH}(m))$$

Proof The proof proceeds by induction on m. For $m = 1$, the induction hypothesis says $p_b(n+k) = p_b(n)$ for $1 \le k < b$ which is true by RI. Assume that IH(m') is true for all $m' < m$. From Lemma 5.1,

$$p_b(b^m n + ub) = p_b(b^m n) + \sum_{k=1}^{u} p_b(b^{m-1}n + k).$$

Then, by the induction hypothesis at $m - 1$,

$$p_b(b^{m-1} + k) = p_b(b^{m-1}n) + \sum_{l=1}^{m-2} g_{m,k,l}(b)p_b(b^l n).$$

Therefore,

$$\begin{aligned} p_b(b^m n + ub) &= p_b(b^m n) + \sum_{k=1}^{u}\left(p_b(b^{m-1}n) + \sum_{l=1}^{m-2} g_{m,k,l}(b)p_b(b^l n)\right) \\ &= p_b(b^m n) + up_b(b^{m-1}n) + \sum_{k=1}^{u}\sum_{l=1}^{m-2} g_{m,k,l}(b)p_b(b^l n). \end{aligned}$$

Finally, switching the order of summation reveals

$$p_b(b^m n + ub) = p_b(b^m n) + up_b(b^{m-1}n) + \sum_{l=1}^{m-2}\left(\sum_{k=1}^{u} g_{m,k,l}(b)\right)p_b(b^l n).$$

Let $w = b^{m-1} - u$, $g_{m-1}(x) = x^{m-1} - w$ and $g_j(x) = \left(\sum_{k=0}^{u-1} g_{k,j}(x)\right)$ for $1 \le j \le m-2$. Then, $u = b^{m-1} - w$ and $g_{m-1}(b) = b^{m-1} - w = u$, and therefore

$$p_b(b^m n + ub + v) = p_b(b^m n) + \sum_{j=1}^{m-1} g_j(b)p_b(b^j n)$$

as stated.

With this preparation, the main theorem may be proven. This allows the generating function $B_b(m, q)$ to be written in terms of $B_b(0, q)$.

Theorem 5.5 *For all m, there exists a polynomial $f_m(x, q)$ of degree $m - 1$ in q and degree $\binom{m}{2}$ in x such that*

$$f_m(b, q)B_b(0, q) = (1 - q)^m B_b(m, q).$$

Proof The proof proceeds by induction on m. The base case $m = 0$ is trivial, that is, $f_0(x, q) = 1$. Assume that the theorem holds for all $m' < m$. Applying RIII to

$p_b(b^m n)$ yields the following:

$$\begin{aligned} p_b(b^m n) &= p_b(b^m n - b) + p_b(b^{m-1} n) \\ &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1} n). \end{aligned}$$

By Lemma 5.4,

$$p_b(b^m(n-1) + b^m - b) = p_b(b^m(n-1)) + \sum_{j=1}^{m-1} g_j(b) p_b(b^j(n-1))$$

and therefore

$$p_b(b^m n) = p_b(b^m(n-1)) + \left(\sum_{j=1}^{m-1} g_j(b) p_b(b^j(n-1)) \right) + p_b(b^{m-1} n)$$

Then, multiplying by q^n on both sides and summing:

$$\begin{aligned} B_b(m,q) &= \sum_{n \in \mathbb{Z}} p_b(b^m n) q^n \\ &= \sum_{n \in \mathbb{Z}} p_b(b^m(n-1)) q^n + \sum_{n \in \mathbb{Z}} p_b(b^{m-1} n) q^n \\ &\quad + \sum_{n \in \mathbb{Z}} \sum_{j=1}^{m-1} g_j(b) p_b(b^j(n-1)) q^n \end{aligned}$$

and by reindexing in sums involving $n-1$ and combining the b^{m-1} terms,

$$\begin{aligned} B_b(m,q) &= q \sum_{n \in \mathbb{Z}} p_b(b^m n) q^n + \sum_{n \in \mathbb{Z}} p_b(b^{m-1} n) q^n + q \sum_{n \in \mathbb{Z}} \sum_{j=1}^{m-1} g_j(b) p_b(b^j n) q^n \\ &= q B_b(m,q) + (1 + g_{m-1}(b) q) \sum_{n \in \mathbb{Z}} p_b(b^{m-1} n) q^n \\ &\quad + q \sum_{j=1}^{m-2} g_j(b) \sum_{n \in \mathbb{Z}} p_b(b^j n) q^n \\ &= q B_b(m,q) + (1 + g_{m-1}(b) q) B_b(m-1,q) + q \sum_{j=1}^{m-2} g_j(b) B_b(j,q) \end{aligned}$$

Therefore,

$$(1-q)B_b(m,q) = (1+g_{m-1}(b)q)B_b(m-1,q) + q\sum_{j=1}^{m-2} g_j(b)B_b(j,q).$$

The induction hypothesis provides

$$(1-q)^j B_b(j,q) = B_b(0,q)f_j(q).$$

that is,

$$B_b(j,q) = \frac{(1-q)^{m-j-1}f_j(q)}{(1-q)^{m-1}}B_b(0,q).$$

Hence, substituting this into the previous sum and multiplying by $(1-q)^{m-1}$ reveals

$$\begin{aligned}(1-q)^m B_b(m,q) &= (1+g_{m-1}(b)q)f_{m-1}(q)B_b(0,q)\\ &\quad + q\sum_{j=1}^{m-2} g_j(b)(1-q)^{m-j-1}f_j(q)B_b(0,q)\\ &= \Bigg((1+g_{m-1}(b)q)f_{m-1}(q)\\ &\quad + q\sum_{j=1}^{m-2} g_j(b)(1-q)^{m-j-1}f_j(q)\Bigg)B_b(0,q).\end{aligned}$$

Consequently,

$$f_m(x,q) = \left((1+g_{m-1}(x)q)f_{m-1}(x,q) + q\sum_{j=1}^{m-2} g_j(x)(1-q)^{m-j-1}f_j(x,q)\right)$$

which is a polynomial of degree $m-1$ in q and degree $\binom{m}{2}$ in x, and therefore

$$f_m(b,q)B_b(0,q) = (1-q)^m B_b(m,q)$$

which proves the theorem.

6 The Polynomial Data

The polynomials in Theorem 5.5 provide a bridge between large values of $p_b(n)$ and its generating function identities. Lacking further theorems, evaluating these large values quickly exceeds the computational power of pencil and paper, but computers

Table 2 Polynomials $f_m(b, q)$ for $2 \le m \le 4$ and $2 \le b \le \binom{m}{2} + 2$

b	$f_2(b, q)$
2	$1+q$
3	$1+2q$

b	$f_3(b, q)$
2	$1+6q+q^2$
3	$1+19q+7q^2$
4	$1+42q+21q^2$
5	$1+78q+46q^2$

b	$f_4(b, q)$
2	$1+31q+31q^2+q^3$
3	$1+234q+447q^2+47q^3$
4	$1+1081q+2635q^2+379q^3$
5	$1+3702q+10218q^2+1704q^3$
6	$1+10335q+30735q^2+5585q^3$
7	$1+24896q+77801q^2+14951q^3$
8	$1+53669q+173747q^2+34727q^3$

Table 3 Polynomials $f_5(b, q)$ for $2 \le b \le 12$

b	$f_5(b, q)$
2	$1+196q+630q^2+196q^3+q^4$
3	$1+5822q+33504q^2+19040q^3+682q^4$
4	$1+79320q+561714q^2+387600q^3+19941q^4$
5	$1+642451q+5055891q^2+3835861q^3+231421q^4$
6	$1+3649340q+30621390q^2+24573740q^3+1621705q^4$
7	$1+16077981q+140871555q^2+117324441q^3+8201271q^4$
8	$1+58573732q+529473294q^2+452753140q^3+32941657q^4$
9	$1+184174970q+1704597594q^2+1486613030q^3+111398806q^4$
10	$1+515009556q+4855552326q^2+4299866676q^3+329571441q^4$
11	$1+1308822280q+12524820930q^2+11227696630q^3+876084760q^4$
12	$1+3072329216q+29763241530q^2+26948358536q^3+2133434941q^4$

are ideally suited to calculating these large values. Each $f_m(b, q)$ provides an identity which provides a new way to calculate values of the form $p_b(b^m n)$. Theorem 4.2 provides an alternate way of computing these numbers. The tools used for this work were primarily Python and Sage with double-checking provided by Mathematica. The City University of New York High Performance Computing Center at the College of Staten Island helpfully provided hardware for long-running computations, but with the optimizations provided by Theorems 4.2 and 5.5 retail consumer hardware is capable of calculating $f_m(b, q)$ for high values of m. Tables 2 and 3 contain identities of the form

$$f_m(q)B_b(0, q) = (1-q)^m \sum_{n \in \mathbb{Z}} p_b(b^m n)q^n$$

for various specific m and b.

Since $f_m(b, q)$ is a polynomial of degree $\binom{m}{2}$ in b, then so is each coefficient in q. Therefore, for a given m, by calculating $f_m(b, q)$ for $\binom{m}{2} + 1$ values of b, it is possible to determine a polynomial in b for each coefficient of q. This data

Table 4 Polynomials $f_m(b, q)$ for $1 \le m \le 4$.

m	$f_m(b,q)$
1	1
2	$bq - q + 1$
3	$\frac{1}{2}b^3q^2 + \frac{1}{2}b^3q - \frac{1}{2}b^2q^2 + \frac{1}{2}b^2q - bq^2 + bq + q^2 - 2q + 1$
4	$\frac{1}{6}b^6q^3 + \frac{2}{3}b^6q^2 - \frac{1}{4}b^5q^3 + \frac{1}{6}b^6q - \frac{1}{6}b^4q^3 + \frac{1}{4}b^5q - \frac{1}{6}b^4q^2 - \frac{1}{4}b^3q^3 + \frac{1}{3}b^4q - \frac{1}{2}b^3q^2$ $+\frac{1}{2}b^2q^3 + \frac{3}{4}b^3q - b^2q^2 + bq^3 + \frac{1}{2}b^2q - 2bq^2 - q^3 + bq + 3q^2 - 3q + 1$

Table 5 Coefficients of $f_5(b, q)$

	q^0	q^1	q^2	q^3	q^4
b^0	1	-4	6	-4	1
b^1	0	1	-3	3	-1
b^2	0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$
b^3	0	$\frac{3}{4}$	$-\frac{5}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{7}{8}$	$\frac{3}{8}$	$\frac{1}{24}$
b^5	0	$\frac{11}{24}$	$-\frac{5}{8}$	$-\frac{1}{8}$	$\frac{7}{24}$
b^6	0	$\frac{3}{8}$	$\frac{5}{24}$	$-\frac{13}{24}$	$-\frac{1}{24}$
b^7	0	$\frac{5}{24}$	$\frac{1}{8}$	$-\frac{3}{8}$	$\frac{1}{24}$
b^8	0	$\frac{1}{8}$	$\frac{5}{24}$	$-\frac{7}{24}$	$-\frac{1}{24}$
b^9	0	$\frac{1}{12}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{12}$
b^{10}	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$

determines $f_m(b, q)$ for a given m and all b. An alternate way of calculating this polynomial is to continue to iterate the recurrence. This method is demonstrated for $m = 3$ in Theorem 10.1. For the case $m = 4$, this approach works, but the argument is significantly longer than the $m = 3$ case.

Table 4 shows $f_m(b, q)$ for $1 \le m \le 4$, written out brutally as polynomials. This representation does not, at first glance, appear particularly illuminating, but it may be the case something may be learned from it. Along these lines, tables of coefficients for the monomials in $f_m(b, q)$ for $5 \le m \le 8$ are presented in Tables 5, 6, 7, and 8. These tables may also be thought of as matrices M_m so that

$$f_m(b, q) = Q_m M_m B_m$$

where $Q_m = ((q^i)_{i=0}^{m-1})^T$ and $B_m = (b^i)_{i=0}^{\binom{m}{2}}$. Perhaps, this representation will suggest a combinatorial interpretation of these coefficients.

The polynomials $f_m(b, q)$ may be seen from an alternate viewpoint as polynomials in b where each coefficient of b is a polynomial in q. This viewpoint (see Tables 9, 10, 11, and 12) proves its usefulness in revealing certain repeating structures. These poly-

Table 6 Coefficients of $f_6(b, q)$

	q^0	q^1	q^2	q^3	q^4	q^5
b^0	1	-5	10	-10	5	-1
b^1	0	1	-4	6	-4	1
b^2	0	$\frac{1}{2}$	-2	3	-2	$\frac{1}{2}$
b^3	0	$\frac{3}{4}$	-2	$\frac{3}{2}$	0	$-\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{4}{3}$	$\frac{5}{4}$	$-\frac{1}{3}$	$-\frac{1}{24}$
b^5	0	$\frac{25}{48}$	$-\frac{4}{3}$	$\frac{7}{8}$	$\frac{1}{6}$	$-\frac{11}{48}$
b^6	0	$\frac{1}{2}$	$-\frac{13}{24}$	$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{24}$
b^7	0	$\frac{13}{36}$	$-\frac{35}{72}$	$-\frac{5}{24}$	$\frac{31}{72}$	$-\frac{7}{72}$
b^8	0	$\frac{7}{24}$	$-\frac{1}{8}$	$-\frac{5}{8}$	$\frac{11}{24}$	0
b^9	0	$\frac{11}{48}$	$\frac{1}{12}$	$-\frac{19}{24}$	$\frac{5}{12}$	$\frac{1}{16}$
b^{10}	0	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{6}$	0
b^{11}	0	$\frac{23}{240}$	$\frac{17}{60}$	$-\frac{41}{120}$	$-\frac{1}{20}$	$\frac{1}{80}$
b^{12}	0	$\frac{1}{16}$	$\frac{7}{24}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{48}$
b^{13}	0	$\frac{5}{144}$	$\frac{17}{72}$	$-\frac{1}{12}$	$-\frac{13}{72}$	$-\frac{1}{144}$
b^{14}	0	$\frac{1}{48}$	$\frac{5}{24}$	0	$-\frac{5}{24}$	$-\frac{1}{48}$
b^{15}	0	$\frac{1}{120}$	$\frac{13}{60}$	$\frac{11}{20}$	$\frac{13}{60}$	$\frac{1}{120}$

nomials have been calculated for values of m up to 23, and unfortunately these pages are unable to contain them. Or, with apologies to Fermat, "Hanc *paginis* exiguitas non caperet." Fortunately, this data is available for download at the following URL: http://dakota.tensen.net/2015/rp/

The form of these polynomials suggests a conjecture containing an unexpected appearance of Eulerian numbers:

Conjecture 6.1 *The polynomial $f_m(b, q)$ has the form*

$$f_m(b, q) = \sum_{i=0}^{\binom{m}{2}} (1-q)^{m-y(i)} g_{m,i}(q) b^i$$

where $y(n) = \left\lfloor \frac{\sqrt{8n+1}}{2} \right\rfloor$ and $g_{m,i}(q)$ are polynomials. Further, with $\left\langle {n \atop k} \right\rangle$ denoting the Eulerian numbers[3]*:*

$$g_{m,\binom{m}{2}}(q) = \frac{q}{(m-1)!} \sum_{i=0}^{m-2} \left\langle {m-1 \atop i} \right\rangle q^i$$

[3] Note that the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ are defined on page 56.

Table 7 Coefficients of $f_7(b, q)$

	q^0	q^1	q^2	q^3	q^4	q^5	q^6
b^0	1	-6	15	-20	15	-6	1
b^1	0	1	-5	10	-10	5	-1
b^2	0	$\frac{1}{2}$	$-\frac{5}{2}$	5	-5	$\frac{5}{2}$	$-\frac{1}{2}$
b^3	0	$\frac{3}{4}$	$-\frac{11}{4}$	$\frac{7}{2}$	$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{43}{24}$	$\frac{31}{12}$	$-\frac{19}{12}$	$\frac{7}{24}$	$\frac{1}{24}$
b^5	0	$\frac{25}{48}$	$-\frac{89}{48}$	$\frac{53}{24}$	$-\frac{17}{24}$	$-\frac{19}{48}$	$\frac{11}{48}$
b^6	0	$\frac{17}{32}$	$-\frac{115}{96}$	$\frac{23}{48}$	$\frac{7}{16}$	$-\frac{17}{96}$	$-\frac{7}{96}$
b^7	0	$\frac{125}{288}$	$-\frac{331}{288}$	$\frac{109}{144}$	$\frac{41}{144}$	$-\frac{119}{288}$	$\frac{25}{288}$
b^8	0	$\frac{19}{48}$	$-\frac{13}{16}$	$\frac{1}{24}$	$\frac{19}{24}$	$-\frac{7}{16}$	$\frac{1}{48}$
b^9	0	$\frac{17}{48}$	$-\frac{71}{144}$	$-\frac{11}{18}$	$\frac{5}{4}$	$-\frac{67}{144}$	$-\frac{5}{144}$
b^{10}	0	$\frac{7}{24}$	$\frac{1}{16}$	$-\frac{23}{24}$	$\frac{7}{12}$	0	$\frac{1}{48}$
b^{11}	0	$\frac{313}{1440}$	$\frac{1}{32}$	$-\frac{127}{144}$	$\frac{113}{144}$	$-\frac{13}{96}$	$-\frac{23}{1440}$
b^{12}	0	$\frac{31}{180}$	$\frac{35}{144}$	$-\frac{71}{72}$	$\frac{19}{36}$	$\frac{5}{72}$	$-\frac{19}{720}$
b^{13}	0	$\frac{61}{480}$	$\frac{151}{480}$	$-\frac{69}{80}$	$\frac{21}{80}$	$\frac{27}{160}$	$-\frac{1}{96}$
b^{14}	0	$\frac{67}{720}$	$\frac{7}{18}$	$-\frac{53}{72}$	$-\frac{1}{18}$	$\frac{43}{144}$	$\frac{1}{90}$
b^{15}	0	$\frac{91}{1440}$	$\frac{43}{96}$	$-\frac{19}{144}$	$-\frac{49}{144}$	$-\frac{1}{32}$	$-\frac{11}{1440}$
b^{16}	0	$\frac{7}{180}$	$\frac{11}{36}$	$-\frac{11}{72}$	$-\frac{19}{72}$	$\frac{5}{72}$	$\frac{1}{360}$
b^{17}	0	$\frac{7}{288}$	$\frac{343}{1440}$	$-\frac{7}{240}$	$-\frac{181}{720}$	$\frac{23}{1440}$	$\frac{1}{480}$
b^{18}	0	$\frac{7}{480}$	$\frac{19}{96}$	$\frac{1}{16}$	$-\frac{13}{48}$	$-\frac{1}{96}$	$\frac{1}{160}$
b^{19}	0	$\frac{11}{1440}$	$\frac{13}{96}$	$\frac{19}{144}$	$-\frac{29}{144}$	$-\frac{7}{96}$	$-\frac{1}{1440}$
b^{20}	0	$\frac{1}{240}$	$\frac{5}{48}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{5}{48}$	$-\frac{1}{240}$
b^{21}	0	$\frac{1}{720}$	$\frac{19}{240}$	$\frac{151}{360}$	$\frac{151}{360}$	$\frac{19}{240}$	$\frac{1}{720}$

Another conjecture also suggests itself:

Conjecture 6.2 *Let $h_i(q)$ be defined by $qh_i(q) = g_{i+1,i}(q)$. Then,*

$$\begin{aligned} f_m(b, q) = (1-q)^{m-1} &+ q\sum_{i=0}^{m-1}(1-q)^{m-y(i)}h_i(q)b^i \\ &+ \sum_{i=m}^{\binom{m}{2}-1}(1-q)^{m-y(i)}g_{m,i}(q)b^i \\ &+ \frac{qb^{\binom{m}{2}}}{(m-1)!}\sum_{i=0}^{m-2}\left\langle {m-1 \atop i} \right\rangle q^i. \end{aligned}$$

Table 8 Coefficients of $f_8(b, q)$

	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7
b^0	1	-7	21	-35	35	-21	7	-1
b^1	0	1	-6	15	-20	15	-6	1
b^2	0	$\frac{1}{2}$	-3	$\frac{15}{2}$	-10	$\frac{15}{2}$	-3	$\frac{1}{2}$
b^3	0	$\frac{3}{4}$	$-\frac{7}{2}$	$\frac{25}{4}$	-5	$\frac{5}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$
b^4	0	$\frac{11}{24}$	$-\frac{9}{4}$	$\frac{35}{8}$	$-\frac{25}{6}$	$\frac{15}{8}$	$-\frac{1}{4}$	$-\frac{1}{24}$
b^5	0	$\frac{25}{48}$	$-\frac{19}{8}$	$\frac{65}{16}$	$-\frac{35}{12}$	$\frac{5}{16}$	$\frac{5}{8}$	$-\frac{11}{48}$
b^6	0	$\frac{17}{32}$	$-\frac{83}{48}$	$\frac{161}{96}$	$-\frac{1}{24}$	$-\frac{59}{96}$	$\frac{5}{48}$	$\frac{7}{96}$
b^7	0	$\frac{259}{576}$	$-\frac{161}{96}$	$\frac{137}{64}$	$-\frac{113}{144}$	$-\frac{89}{192}$	$\frac{13}{32}$	$-\frac{41}{576}$
b^8	0	$\frac{7}{16}$	$-\frac{137}{96}$	$\frac{127}{96}$	$\frac{11}{48}$	$-\frac{11}{12}$	$\frac{35}{96}$	$-\frac{1}{96}$
b^9	0	$\frac{27}{64}$	$-\frac{85}{72}$	$\frac{307}{576}$	$\frac{89}{72}$	$-\frac{823}{576}$	$\frac{7}{18}$	$\frac{17}{576}$
b^{10}	0	$\frac{329}{864}$	$-\frac{43}{72}$	$-\frac{131}{288}$	$\frac{127}{108}$	$-\frac{155}{288}$	$\frac{5}{72}$	$-\frac{31}{864}$
b^{11}	0	$\frac{911}{2880}$	$-\frac{389}{720}$	$-\frac{289}{576}$	$\frac{19}{12}$	$-\frac{623}{576}$	$\frac{161}{720}$	$\frac{1}{2880}$
b^{12}	0	$\frac{1189}{4320}$	$-\frac{323}{1440}$	$-\frac{13}{12}$	$\frac{787}{432}$	$-\frac{245}{288}$	$\frac{19}{480}$	$\frac{47}{2160}$
b^{13}	0	$\frac{329}{1440}$	$-\frac{1}{240}$	$-\frac{125}{96}$	$\frac{59}{36}$	$-\frac{47}{96}$	$-\frac{19}{240}$	$\frac{11}{1440}$
b^{14}	0	$\frac{325}{1728}$	$\frac{293}{1440}$	$-\frac{859}{576}$	$\frac{595}{432}$	$-\frac{23}{576}$	$-\frac{67}{288}$	$-\frac{43}{8640}$
b^{15}	0	$\frac{427}{2880}$	$\frac{281}{720}$	$-\frac{641}{576}$	$\frac{1}{2}$	$\frac{29}{576}$	$\frac{7}{720}$	$\frac{41}{2880}$
b^{16}	0	$\frac{61}{540}$	$\frac{521}{1440}$	$-\frac{1607}{1440}$	$\frac{1177}{2160}$	$\frac{179}{720}$	$-\frac{223}{1440}$	$\frac{11}{4320}$
b^{17}	0	$\frac{31}{360}$	$\frac{289}{720}$	$-\frac{127}{144}$	$\frac{1}{8}$	$\frac{13}{36}$	$-\frac{67}{720}$	$\frac{1}{720}$
b^{18}	0	$\frac{23}{360}$	$\frac{587}{1440}$	$-\frac{199}{288}$	$-\frac{25}{144}$	$\frac{67}{144}$	$-\frac{97}{1440}$	$-\frac{7}{1440}$
b^{19}	0	$\frac{131}{2880}$	$\frac{109}{288}$	$-\frac{247}{576}$	$-\frac{7}{16}$	$\frac{241}{576}$	$\frac{37}{1440}$	$-\frac{1}{576}$
b^{20}	0	$\frac{277}{8640}$	$\frac{11}{32}$	$-\frac{679}{2880}$	$-\frac{1283}{2160}$	$\frac{347}{960}$	$\frac{131}{1440}$	$\frac{13}{8640}$
b^{21}	0	$\frac{61}{2880}$	$\frac{5}{16}$	$\frac{13}{64}$	$-\frac{4}{9}$	$-\frac{5}{64}$	$-\frac{1}{80}$	$-\frac{1}{576}$
b^{22}	0	$\frac{269}{20160}$	$\frac{107}{504}$	$\frac{401}{4032}$	$-\frac{187}{504}$	$\frac{107}{4032}$	$\frac{13}{630}$	$-\frac{5}{4032}$
b^{23}	0	$\frac{1}{120}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{30}$	0
b^{24}	0	$\frac{43}{8640}$	$\frac{43}{360}$	$\frac{521}{2880}$	$-\frac{61}{270}$	$-\frac{269}{2880}$	$\frac{1}{72}$	$\frac{1}{8640}$
b^{25}	0	$\frac{1}{360}$	$\frac{4}{45}$	$\frac{29}{144}$	$-\frac{1}{6}$	$-\frac{5}{36}$	$\frac{1}{90}$	$\frac{1}{720}$
b^{26}	0	$\frac{1}{720}$	$\frac{1}{18}$	$\frac{13}{72}$	$-\frac{1}{18}$	$-\frac{23}{144}$	$-\frac{1}{45}$	0
b^{27}	0	$\frac{1}{1440}$	$\frac{7}{180}$	$\frac{49}{288}$	0	$-\frac{49}{288}$	$-\frac{7}{180}$	$-\frac{1}{1440}$
b^{28}	0	$\frac{1}{5040}$	$\frac{1}{42}$	$\frac{397}{1680}$	$\frac{151}{315}$	$\frac{397}{1680}$	$\frac{1}{42}$	$\frac{1}{5040}$

Table 9 Polynomial $f_4(b, q)$ as a polynomial in b

	$f_4(b,q)$	$=$	$(1-q)^3$		
$+$		q	$(1-q)^2$	b	
$+$	$(2)^{-1}$	q	$(1-q)^2$	b^2	
$+$	$(4)^{-1}$	q	$(1-q)$	$(3+q)\ b^3$	
$+$	$(6)^{-1}$	q	$(1-q)$	$(2+q)\ b^4$	
$+$	$(4)^{-1}$	q	$(1-q)$	$(1+q)\ b^5$	
$+$	$(6)^{-1}$	q	$\cdot$	$(1+4q+q^2)\ b^6$	

Table 10 Polynomial $f_5(b, q)$ as a polynomial in b

	$f_5(b,q)$	$=$	$(1-q)^4$	
$+$		q	$(1-q)^3$	b
$+$	$(2)^{-1}$	q	$(1-q)^3$	b^2
$+$	$(4)^{-1}$	q	$(1-q)^2$	$(3+q)\ b^3$
$+$	$(24)^{-1}$	q	$(1-q)^2$	$(11+q)\ b^4$
$+$	$(24)^{-1}$	q	$(1-q)^2$	$(11+7q)\ b^5$
$+$	$(24)^{-1}$	q	$(1-q)$	$(9+14q+q^2)\ b^6$
$+$	$(24)^{-1}$	q	$(1-q)$	$(5+8q-q^2)\ b^7$
$+$	$(24)^{-1}$	q	$(1-q)$	$(3+8q+q^2)\ b^8$
$+$	$(12)^{-1}$	q	$(1-q)$	$(1+4q+q^2)\ b^9$
$+$	$(24)^{-1}$	q	$\cdot$	$(1+11q+11q^2+q^3)\ b^{10}$

Assuming these conjectures indicates that the polynomials $f_b(m, q)$ are completely determined by the polynomials $(g_{m,i})_{i=m}^{\binom{m}{2}-1}$. The data so far obeys this conjecture, so these polynomials are given for $8 \le m \le 10$ (in Tables 13, 14, 15, 16, 17, 18) from which, by using the form above, one may construct $f_m(b, q)$.

7 A New Congruence

The conjectured form of $f_m(b, q)$ suggests several conjectures, including some regarding congruences of $p_b(n)$. For instance, $f_m(b, q) \equiv (1-q)^{m-1} \pmod b$ seems likely, and therefore Theorem 5.5 suggests

$$(1-q)^m B_b(m, q) \equiv (1-q)^{m-1} B_b(0, q) \pmod b$$

and hence

$$(1-q) B_b(m, q) \equiv B_b(0, q) \pmod b$$

Table 11 Polynomial $f_6(b, q)$ as a polynomial in b

$$\begin{aligned}
f_6(b,q) &= (1-q)^5 \\
&+ q\,(1-q)^4\, b \\
&+ (2)^{-1}\, q\,(1-q)^4\, b^2 \\
&+ (4)^{-1}\, q\,(1-q)^3\,(3+q)\, b^3 \\
&+ (24)^{-1}\, q\,(1-q)^3\,(11+q)\, b^4 \\
&+ (48)^{-1}\, q\,(1-q)^3\,(25+11q)\, b^5 \\
&+ (24)^{-1}\, q\,(1-q)^2\,(12+11q+q^2)\, b^6 \\
&+ (72)^{-1}\, q\,(1-q)^2\,(26+17q-7q^2)\, b^7 \\
&+ (24)^{-1}\, q\,(1-q)^2\,(7+11q)\, b^8 \\
&+ (48)^{-1}\, q\,(1-q)^2\,(11+26q+3q^2)\, b^9 \\
&+ (6)^{-1}\, q\,(1-q)\,(1+4q+q^2)\, b^{10} \\
&+ (240)^{-1}\, q\,(1-q)\,(23+91q+9q^2-3q^3)\, b^{11} \\
&+ (48)^{-1}\, q\,(1-q)\,(3+17q+5q^2-q^3)\, b^{12} \\
&+ (144)^{-1}\, q\,(1-q)\,(5+39q+27q^2+q^3)\, b^{13} \\
&+ (48)^{-1}\, q\,(1-q)\,(1+11q+11q^2+q^3)\, b^{14} \\
&+ (120)^{-1}\, q \cdot (1+26q+66q^2+26q^3+q^4)\, b^{15}
\end{aligned}$$

Fortunately, this can be proven independently of the conjectured form of $f_m(b, q)$ and this statement appears below as Theorem 7.2. Reducing Lemma 5.2 modulo b reveals the following corollary:

Corollary 7.1 *The partition counting function $p_b(n)$ satisfies the congruence:*

$$p_b(b^m n) \equiv p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)\,p_b(b^{m-1}(n-1)) \pmod{b}$$

This corollary can then be used to prove the following theorem.

Theorem 7.2 *The partition counting function $p_b(n)$ satisfies the congruence:*

$$p_b(b^m n) - p_b(b^m(n-1)) \equiv p_b(n) \pmod{b}$$

Proof Beginning with the statement Corollary 7.1,

$$p_b(b^m n) \equiv p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)\,p_b(b^{m-1}(n-1)) \pmod{b},$$

and applying Corollary 7.1 to the middle term $p_b(b^{m-1}n)$ reveals:

$$\begin{aligned}
p_b(b^m n) &\equiv p_b(b^m(n-1)) + p_b(b^{m-1}(n-1)) + p_b(b^{m-2}n) + (b-1)\,p_b(b^{m-2}(n-1)) \\
&\quad + (b-1)\,p_b(b^{m-1}(n-1)) \pmod{b} \\
&\equiv p_b(b^m(n-1)) + p_b(b^{m-2}n) + (b-1)\,p_b(b^{m-2}(n-1)) \pmod{b}
\end{aligned}$$

Table 12 Polynomial $f_7(b,q)$ as a polynomial in b

$$
\begin{aligned}
f_7(b,q) &= (1-q)^6 \\
&+ q\,(1-q)^5\, b \\
&+ (2)^{-1} q\,(1-q)^5\, b^2 \\
&+ (4)^{-1} q\,(1-q)^4\,(3+q)\, b^3 \\
&+ (24)^{-1} q\,(1-q)^4\,(11+q)\, b^4 \\
&+ (48)^{-1} q\,(1-q)^4\,(25+11q)\, b^5 \\
&+ (96)^{-1} q\,(1-q)^3\,(51+38q+7q^2)\, b^6 \\
&+ (288)^{-1} q\,(1-q)^3\,(125+44q-25q^2)\, b^7 \\
&+ (48)^{-1} q\,(1-q)^3\,(19+18q-q^2)\, b^8 \\
&+ (144)^{-1} q\,(1-q)^3\,(51+82q+5q^2)\, b^9 \\
&+ (48)^{-1} q\,(1-q)^2\,(14+31q+2q^2+q^3)\, b^{10} \\
&+ (1440)^{-1} q\,(1-q)^2\,(313+671q-241q^2-23q^3)\, b^{11} \\
&+ (720)^{-1} q\,(1-q)^2\,(124+423q+12q^2-19q^3)\, b^{12} \\
&+ (480)^{-1} q\,(1-q)^2\,(61+273q+71q^2-5q^3)\, b^{13} \\
&+ (720)^{-1} q\,(1-q)^2\,(67+414q+231q^2+8q^3)\, b^{14} \\
&+ (1440)^{-1} q\,(1-q)\,(91+736q+546q^2+56q^3+11q^4)\, b^{15} \\
&+ (360)^{-1} q\,(1-q)\,(14+124q+69q^2-26q^3-q^4)\, b^{16} \\
&+ (1440)^{-1} q\,(1-q)\,(35+378q+336q^2-26q^3-3q^4)\, b^{17} \\
&+ (480)^{-1} q\,(1-q)\,(7+102q+132q^2+2q^3-3q^4)\, b^{18} \\
&+ (1440)^{-1} q\,(1-q)\,(11+206q+396q^2+106q^3+q^4)\, b^{19} \\
&+ (240)^{-1} q\,(1-q)\,(1+26q+66q^2+26q^3+q^4)\, b^{20} \\
&+ (720)^{-1} q \cdot (1+57q+302q^2+302q^3+57q^4+q^5)\, b^{21}
\end{aligned}
$$

Subsequently applying the Corollary to the middle term $m-3$ more times produces

$$p_b(b^m n) \equiv p_b(b^m(n-1)) + p_b(bn) + (b-1)p_b(b(n-1)) \pmod b$$

and finally applying RIII to $p_b(bn)$ yields

$$
\begin{aligned}
p_b(b^m n) &\equiv p_b(b^m(n-1)) + p_b(b(n-1)) + p_b(n) + (b-1)p_b(b(n-1)) \pmod b \\
&\equiv p_b(b^m(n-1)) + p_b(n) \pmod b
\end{aligned}
$$

that is,

$$p_b(b^m n) - p_b(b^m(n-1)) \equiv p_b(n) \pmod b$$

as stated.

Table 13 Polynomials $g_{8,i}$ for $8 \le i \le 27$

m	i	$g_{m,i}$(q)
8	8	$(96)^{-1}\ (42 + 31q - q^2)$
8	9	$(576)^{-1}\ (243 + 292q + 17q^2)$
8	10	$(864)^{-1}\ (329 + 471q + 33q^2 + 31q^3)$
8	11	$(2880)^{-1}\ (911 + 1177q - 647q^2 - q^3)$
8	12	$(4320)^{-1}\ (1189 + 2598q - 453q^2 - 94q^3)$
8	13	$(1440)^{-1}\ (329 + 981q + 81q^2 - 11q^3)$
8	14	$(8640)^{-1}\ (1625 + 6633q + 2139q^2 + 43q^3)$
8	15	$(2880)^{-1}\ (427 + 1978q + 324q^2 + 110q^3 + 41q^4)$
8	16	$(4320)^{-1}\ (488 + 2539q - 231q^2 - 647q^3 + 11q^4)$
8	17	$(720)^{-1}\ (62 + 413q + 129q^2 - 65q^3 + q^4)$
8	18	$(1440)^{-1}\ (92 + 771q + 455q^2 - 111q^3 - 7q^4)$
8	19	$(2880)^{-1}\ (131 + 1352q + 1338q^2 + 64q^3 - 5q^4)$
8	20	$(8640)^{-1}\ (277 + 3524q + 4734q^2 + 812q^3 + 13q^4)$
8	21	$(2880)^{-1}\ (61 + 961q + 1546q^2 + 266q^3 + 41q^4 + 5q^5)$
8	22	$(20160)^{-1}\ (269 + 4549q + 6554q^2 - 926q^3 - 391q^4 + 25q^5)$
8	23	$(120)^{-1}\ (1 + 21q + 41q^2 + q^3 - 4q^4)$
8	24	$(8640)^{-1}\ (43 + 1075q + 2638q^2 + 686q^3 - 121q^4 - q^5)$
8	25	$(720)^{-1}\ (2 + 66q + 211q^2 + 91q^3 - 9q^4 - q^5)$
8	26	$(720)^{-1}\ (1 + 41q + 171q^2 + 131q^3 + 16q^4)$
8	27	$(1440)^{-1}\ (1 + 57q + 302q^2 + 302q^3 + 57q^4 + q^5)$

8 Sellers' Question

In a Spring 2014 talk at the New York Number Theory Seminar, Sellers presented the following identities:

$$\sum_{n\in\mathbb{Z}} p_3(81n+42)q^n = \frac{27(8q^2+17q+2)}{(1-q)^4} B_3(0,q)$$
$$\sum_{n\in\mathbb{Z}} p_3(81n+78)q^n = \frac{27(2q^2+17q+8)}{(1-q)^4} B_3(0,q)$$

and asked, "Why do the polynomial factors in the numerator come in such natural pairs as 'reciprocal polynomials'?" Given that $8q^2 + 17q + 2$ appears, why should its reciprocal polynomial, the polynomial with its coefficients reversed, that is, $2q^2 + 17q + 8$, appear?

Table 14 Polynomials $g_{9,i}$ for $9 \leq i \leq 24$

m	i	$g_{m,i}(q)$
9	9	$(1152)^{-1}\,(513 + 548q + 43q^2)$
9	10	$(3456)^{-1}\,(1463 + 1587q + 285q^2 + 121q^3)$
9	11	$(17280)^{-1}\,(6521 + 5607q - 3597q^2 + 109q^3)$
9	12	$(8640)^{-1}\,(3018 + 4631q - 1066q^2 - 103q^3)$
9	13	$(1920)^{-1}\,(597 + 1263q - 17q^2 - 3q^3)$
9	14	$(8640)^{-1}\,(2369 + 6981q + 1551q^2 + 79q^3)$
9	15	$(8640)^{-1}\,(2024 + 6103q - 513q^2 + 901q^3 + 125q^4)$
9	16	$(17280)^{-1}\,(3379 + 11660q - 4458q^2 - 2020q^3 + 79q^4)$
9	17	$(17280)^{-1}\,(2827 + 12542q - 732q^2 - 1750q^3 + 73q^4)$
9	18	$(17280)^{-1}\,(2325 + 13126q + 2988q^2 - 1854q^3 - 25q^4)$
9	19	$(17280)^{-1}\,(1867 + 12770q + 6684q^2 - 466q^3 + 25q^4)$
9	20	$(8640)^{-1}\,(743 + 6185q + 4971q^2 + 319q^3 + 22q^4)$
9	21	$(5760)^{-1}\,(383 + 3533q + 1974q^2 - 470q^3 + 331q^4 + 9q^5)$
9	22	$(60480)^{-1}\,(3054 + 31679q + 15070q^2 - 18864q^3 - 788q^4 + 89q^5)$
9	23	$(120960)^{-1}\,(4589 + 57611q + 54038q^2 - 21950q^3 - 3683q^4 + 115q^5)$
9	24	$(120960)^{-1}\,(3371 + 49719q + 63010q^2 - 11342q^3 - 4077q^4 + 119q^5)$

Table 15 Polynomials $g_{9,i}$ for $25 \leq i \leq 35$

m	i	$g_{m,i}(q)$
9	25	$(120960)^{-1}\,(2417 + 42599q + 74486q^2 + 5818q^3 - 4295q^4 - 65q^5)$
9	26	$(120960)^{-1}\,(1689 + 34745q + 75466q^2 + 20322q^3 - 1187q^4 + 5q^5)$
9	27	$(120960)^{-1}\,(1157 + 28445q + 79022q^2 + 39334q^3 + 3181q^4 + 61q^5)$
9	28	$(120960)^{-1}\,(761 + 21712q + 64153q^2 + 27480q^3 + 4667q^4 + 2168q^5 + 19q^6)$
9	29	$(120960)^{-1}\,(481 + 15384q + 47073q^2 + 8608q^3 - 10677q^4 - 408q^5 + 19q^6)$
9	30	$(120960)^{-1}\,(301 + 11232q + 40293q^2 + 15328q^3 - 6417q^4 - 288q^5 + 31q^6)$
9	31	$(10080)^{-1}\,(15 + 680q + 3011q^2 + 1856q^3 - 419q^4 - 104q^5 + q^6)$
9	32	$(120960)^{-1}\,(103 + 5528q + 28775q^2 + 25632q^3 + 1069q^4 - 632q^5 + 5q^6)$
9	33	$(20160)^{-1}\,(9 + 632q + 4097q^2 + 4832q^3 + 667q^4 - 152q^5 - 5q^6)$
9	34	$(60480)^{-1}\,(13 + 1112q + 8861q^2 + 14496q^3 + 5431q^4 + 328q^5 - q^6)$
9	35	$(10080)^{-1}\,(1 + 120q + 1191q^2 + 2416q^3 + 1191q^4 + 120q^5 + q^6)$

Table 16 Polynomials $g_{10,i}$ for $10 \le i \le 24$

m	i	$g_{m,i}(q)$
10	10	$(3456)^{-1}\ (1508 + 1479q + 366q^2 + 103q^3)$
10	11	$(17280)^{-1}\ (6971 + 4662q - 3057q^2 + 64q^3)$
10	12	$(17280)^{-1}\ (6736 + 8167q - 1772q^2 - 171q^3)$
10	13	$(25920)^{-1}\ (9437 + 15513q - 207q^2 + 97q^3)$
10	14	$(17280)^{-1}\ (5828 + 13307q + 2582q^2 + 243q^3)$
10	15	$(103680)^{-1}\ (31513 + 66086q - 10716q^2 + 15722q^3 + 1075q^4)$
10	16	$(34560)^{-1}\ (9287 + 22450q - 12300q^2 - 2258q^3 + 101q^4)$
10	17	$(17280)^{-1}\ (4103 + 12979q - 3099q^2 - 1091q^3 + 68q^4)$
10	18	$(34560)^{-1}\ (7159 + 28984q + 210q^2 - 3208q^3 - 25q^4)$
10	19	$(51840)^{-1}\ (9217 + 45515q + 12507q^2 - 1507q^3 + 148q^4)$
10	20	$(17280)^{-1}\ (2619 + 15673q + 7953q^2 + 315q^3 + 80q^4)$
10	21	$(32400)^{-1}\ (4111 + 25660q + 1585q^2 - 1870q^3 + 2915q^4 - q^5)$
10	22	$(120960)^{-1}\ (12709 + 90224q - 1458q^2 - 43888q^3 + 2869q^4 + 24q^5)$
10	23	$(362880)^{-1}\ (31337 + 267179q + 92282q^2 - 116054q^3 - 2579q^4 - 5q^5)$
10	24	$(120960)^{-1}\ (8470 + 84599q + 53646q^2 - 29488q^3 - 1388q^4 + 81q^5)$

Table 17 Polynomials $g_{10,i}$ for $25 \le i \le 34$

m	i	$g_{m,i}(q)$
10	25	$(241920)^{-1}\ (13535 + 158685q + 157158q^2 - 30134q^3 - 6837q^4 - 87q^5)$
10	26	$(120960)^{-1}\ (5334 + 71929q + 93434q^2 + 1956q^3 - 1360q^4 + 67q^5)$
10	27	$(362880)^{-1}\ (12460 + 195325q + 332566q^2 + 75308q^3 + 3926q^4 + 335q^5)$
10	28	$(241920)^{-1}\ (6355 + 108566q + 147413q^2 - 43284q^3 + 14869q^4 + 8062q^5 - 61q^6)$
10	29	$(362880)^{-1}\ (7178 + 138609q + 210603q^2 - 121954q^3 - 55980q^4 + 3057q^5 - 73q^6)$
10	30	$(120960)^{-1}\ (1781 + 39601q + 78646q^2 - 15078q^3 - 15247q^4 + 1013q^5 + 4q^6)$
10	31	$(725760)^{-1}\ (7807 + 200214q + 493533q^2 + 11980q^3 - 106623q^4 - 2178q^5 + 67q^6)$
10	32	$(80640)^{-1}\ (623 + 18268q + 54323q^2 + 15184q^3 - 7435q^4 - 332q^5 + 9q^6)$
10	33	$(362880)^{-1}\ (1970 + 66843q + 237309q^2 + 114242q^3 - 23184q^4 - 4029q^5 - 31q^6)$
10	34	$(60480)^{-1}\ (226 + 8760q + 36747q^2 + 27718q^3 + 2274q^4 - 126q^5 + q^6)$

Why should one expect that these sorts of identities exist in the first place? Some combinatorial insight is desired, but failing that Lemma 5.4 and Theorem 5.5 guarantee that *some* relationship exists, although they fall short of explaining why such reciprocal polynomials appear.

By Lemma 5.4, applying RI and RIII to an expression like $p_b(b^m n + k)$ will produce identities between its generating function and $B_b(0, q)$. Consider the results when Lemma 5.4 applied to Sellers' example:

Table 18 Polynomials $g_{10,i}$ for $35 \leq i \leq 44$

m	i	$g_{m,i}(q)$
10	35	$(725760)^{-1}\ (1831 + 82308q + 403011q^2 + 395512q^3 + 80853q^4 + 4068q^5 + 97q^6)$
10	36	$(241920)^{-1}$ $(399 + 20449q + 106983q^2 + 95433q^3 + 10453q^4 + 6939q^5 + 1269q^6 - 5q^7)$
10	37	$(725760)^{-1}$ $(763 + 43703q + 236763q^2 + 167071q^3 - 72007q^4 - 15363q^5 + 1969q^6 - 19q^7)$
10	38	$(120960)^{-1}\ (79 + 5281q + 33561q^2 + 33127q^3 - 6983q^4 - 4617q^5 + 31q^6 + q^7)$
10	39	$(362880)^{-1}\ (143 + 11137q + 80253q^2 + 99011q^3 + q^4 - 9297q^5 + 179q^6 + 13q^7)$
10	40	$(241920)^{-1}\ (55 + 5137q + 43875q^2 + 69397q^3 + 10597q^4 - 7533q^5$ $-575q^6 + 7q^7)$
10	41	$(1209600)^{-1}\ (151 + 16753q + 163431q^2 + 316465q^3 + 117805q^4 - 8181q^5$ $-1643q^6 + 19q^7)$
10	42	$(80640)^{-1}\ (5 + 723q + 8577q^2 + 20519q^3 + 10719q^4 + 9q^5 - 229q^6 - 3q^7)$
10	43	$(241920)^{-1}\ (7 + 1217q + 17163q^2 + 51757q^3 + 41957q^4 + 8595q^5 + 265q^6 - q^7)$
10	44	$(80640)^{-1}\ (1 + 247q + 4293q^2 + 15619q^3 + 15619q^4 + 4293q^5 + 247q^6 + q^7)$

$$p_3(81n + 42) = p_3(81n) + 14p_3(27n) + 30p_3(9n) + 9p_3(3n)$$
$$p_3(81n + 78) = p_3(81n) + 26p_3(27n) + 108p_3(9n) + 81p_3(3n)$$

Then, upon passing to generating functions

$$\begin{aligned}\sum_{n\in\mathbb{Z}} p_3(81n+42)q^n &= B_3(4,q) + 14B_3(3,q) + 30B_3(2,q) + 9B_3(1,q)\\ &= \left(\frac{f_4(3,q)}{(1-q)^4} + 14\frac{f_3(3,q)}{(1-q)^3} + 30\frac{f_2(3,q)}{(1-q)^2} + 9\frac{f_1(3,q)}{1-q}\right)B_3(0,q)\\ &= \frac{27(8q^2+17q+2)}{(1-q)^4}B_3(0,q)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\in\mathbb{Z}} p_3(81n+78)q^n &= B_3(4,q) + 26B_3(3,q) + 108B_3(2,q) + 81B_3(1,q)\\ &= \left(\frac{f_4(3,q)}{(1-q)^4} + 26\frac{f_3(3,q)}{(1-q)^3} + 108\frac{f_2(3,q)}{(1-q)^2} + 81\frac{f_1(3,q)}{1-q}\right)B_3(0,q)\\ &= \frac{27(2q^2+17q+8)}{(1-q)^4}B_3(0,q).\end{aligned}$$

A full understanding of identities like these seems to require a thorough understanding of the polynomials $f_m(b,q)$ as well as the polynomials $g_{m,k,j}(b)$ from Lemma 5.4.

9 Some Computations

Doing this work without computing $p_b(n)$ for large values of n would be a waste, so here are the values for a few choice n and their prime factorization.

$$\begin{aligned} p_2(2^{10}) &= 2320518948 \\ &= 2^2 \cdot 3 \cdot 11 \cdot 197 \cdot 89237 \\ &\text{See also [7]} \end{aligned}$$

$$\begin{aligned} p_2(2^{30}) &= 15252235216626126524825730422708790622448637721533\backslash \\ &\quad 7375091793655998185220930656974338568054217947023338 0 \\ &= 2^2 \cdot 5 \cdot 19 \cdot 31 \cdot 79 \cdot 1217 \cdot 46553987 \cdot 719224073 \\ &\quad \cdot 88243965275199121 \cdot 1201364132790744647 \\ &\quad \cdot 3793933910711600253501418262383058570580931 \end{aligned}$$

$$\begin{aligned} p_3(3^{27}) &= 350364423551707258416807382080740574025054741900008\backslash \\ &\quad 668600126882878615683202075701898785282388145497481\backslash \\ &\quad 04181920303840123935669522277987798995852 \\ &= 2^2 \cdot 875911058879268146042018455201851435062636854750 02\backslash \\ &\quad 1671500317207196539208005189254746963205970363743 7\backslash \\ &\quad 02604548007596003098391738056949694974896 3 \end{aligned}$$

10 Proving the Case $m = 3$

This section gives an iterative construction of the polynomial $f_3(b, q)$. The methods used here can be used to prove the $m = 4$ case, but the argument becomes significantly longer. It is likely that this method can be used to construct $f_m(b, q)$ for any fixed m, but the length of the argument becomes unwieldy.

Theorem 10.1 *The generating function for $p_b(b^3 n)$ satisfies the identity:*

$$f_3(b, q)B_b(q) = (1 - q)^3 B_b(3, q)$$

where

$$f_3(b, q) = (1 - q)^2 + q(1 - q)b - \tfrac{1}{2}q(1 - q)b^2 + \tfrac{1}{2}q(q + 1)b^3$$

Proof Begin as before, by iterating the recurrence via Lemma 5.2:

$$p_b(b^3n) = p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2(n-1)) + b\sum_{u=1}^{b-1} p_b(b^2(n-1)+ub)$$

The sum in the final term can be simplified further via Lemma 5.1:

$$\sum_{u=1}^{b-1} p_b(b^2(n-1)+ub) = \sum_{u=1}^{b-1} p_b(b^2(n-1)) + \sum_{k=1}^{u} p_b(b(n-1)+k) = (b-1)p_b(b^2(n-1)) + \sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)+k)$$

Now, $1 \le k \le b-1$ so by RI

$$\sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)+k) = \sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)) = p_b(b(n-1))\sum_{u=1}^{b-1}\sum_{k=1}^{u} 1 = p_b(b(n-1))\sum_{u=1}^{b-1} u = \binom{b}{2} p_b(b(n-1))$$

Finally, the original expression becomes

$$p_b(b^3n) = p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2(n-1)) + b(b-1)p_b(b^2(n-1)) + b\binom{b}{2}p_b(b(n-1))$$

Passing to generating functions by multiplying this identity by q^n and summing over all n yields

$$\sum_{n\in\mathbb{Z}} p_b(b^3n)q^n = \sum_{n\in\mathbb{Z}} p_b(b^3(n-1)) + \sum_{n\in\mathbb{Z}} p_b(b^2n) + \sum_{n\in\mathbb{Z}} (b-1)p_b(b^2(n-1))$$
$$+ \sum_{n\in\mathbb{Z}} b(b-1)p_b(b^2(n-1)) + \sum_{n\in\mathbb{Z}} b\binom{b}{2} p_b(b(n-1))$$
$$B_b(3,q) = qB_b(3,q) + B_b(2,q) + (b-1)qB_b(2,q)$$
$$+ b(b-1)qB_b(2,q) + b\binom{b}{2} qB_b(1,q)$$

After moving terms of $B_b(3, q)$ to the right-hand side, the above equation becomes

$$(1-q)B_b(3,q) = (1+(b-1)q + b(b-1)q)\, B_b(2,q) + b\binom{b}{2} qB_b(1,q)$$
$$= (1+(b^2-1)q)B_b(2,q) + b\binom{b}{2} qB_b(1,q)$$

Substituting in the results for $B_b(2, q)$ and $B_b(1, q)$ in Lemma 4.4, Corollary 5.3, and multiplying by $(1-q)^2$ yields

$$(1-q)^3 B_b(3,q) = (1{+}(b^2{-}1)q)\,((1{+}(b{-}1)q)B_b(0,q)) + b\binom{b}{2} q(1{-}q)B_b(0,q)$$
$$= \left((1{+}(b^2{-}1)q)\left((1{+}(b{-}1)q) + b\binom{b}{2} q(1{-}q)\right)\right) B_b(0,q)$$
$$= \left(1 + \tfrac{1}{2}(b-1)\left((b^2+2b+4)q + (b^2-2)q^2\right)\right) B_b(0,q)$$

Therefore,

$$(1-q)^3 B_b(3,q) = \left((1-q)^2 + q(1-q)b - \tfrac{1}{2}q(1-q)b^2 + \tfrac{1}{2}q(q+1)b^3\right) B_b(0,q)$$

as desired.

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