Limits and Decomposition of de Bruijn's Additive Systems

Melvyn B. Nathanson

Abstract An *additive system* for the nonnegative integers is a family $(A_i)_{i \in I}$ of sets of nonnegative integers with $0 \in A_i$ for all $i \in I$ such that every nonnegative integer can be written uniquely in the form $\sum_{i \in I} a_i$ with $a_i \in A_i$ for all i and $a_i \neq 0$ for only finitely many i. In 1956, de Bruijn proved that every additive system is constructed from an infinite sequence $(g_i)_{i \in \mathbf{N}}$ of integers with $g_i \geq 2$ for all i or is a contraction of such a system. This paper discusses limits and the stability of additive systems and also describes the "uncontractable" or "indecomposable" additive systems.

Keywords Additive system · Additive basis · British number system

2010 Mathematics Subject Classification: 11A05 · 11B75

1 Additive Systems and de Bruijn's Theorem

Let $\mathbf{N}_0$ and $\mathbf{N}$ denote the sets of nonnegative integers and positive integers, respectively. For real numbers a and b, we define the interval of integers $[a, b) = \{x \in \mathbf{Z} : a \leq x < b\}$ and $[a, b] = \{x \in \mathbf{Z} : a \leq x \leq b\}$.

Let I be a nonempty finite or infinite set, and let $\mathcal{A} = (A_i)_{i \in I}$ be a family of sets of integers with $0 \in A_i$ and $|A_i| \geq 2$ for all $i \in I$. Each set A_i can be finite or infinite. The *sumset* $S = \sum_{i \in I} A_i$ is the set of all integers n that can be represented in the form $n = \sum_{i \in I} a_i$, where $a_i \in A_i$ for all $i \in I$ and $a_i \neq 0$ for only finitely many $i \in I$. If every element of S has a *unique* representation in the form $n = \sum_{i \in I} a_i$, then we call $\mathcal{A}$ a *unique representation system for* S, and we write $S = \bigoplus_{i \in I} A_i$.

In a unique representation system $\mathcal{A}$ for S, we have $A_i \cap A_j = \{0\}$ for all $i \neq j$. The condition $|A_i| \geq 2$ for all $i \in I$ implies that $A_i = S$ for some $i \in I$ if and only if $|I| = 1$. Moreover, if $I^\flat \subseteq I$ and $S = \sum_{i \in I^\flat} A_i$, then $S = \bigoplus_{i \in I^\flat} A_i$ and $I = I^\flat$.

M. B. Nathanson (✉)
Department of Mathematics, Lehman College (CUNY),
Bronx, NY 10468, USA
e-mail: melvyn.nathanson@lehman.cuny.edu

M. B. Nathanson (ed.), *Combinatorial and Additive Number Theory II*,
Springer Proceedings in Mathematics & Statistics 220,
https://doi.org/10.1007/978-3-319-68032-3_18

The family $\mathcal{A} = (A_i)_{i \in I}$ is an *additive system* if $\mathcal{A}$ is a unique representation system for the set of nonnegative integers, that is, if $\mathbf{N}_0 = \bigoplus_{i \in I} A_i$. The following lemma follows immediately from the definition of an additive system.

Lemma 1 *Let $\mathcal{B} = (B_j)_{j \in J}$ be an additive system. If $\{J_i\}_{i \in I}$ is a partition of J into pairwise disjoint nonempty sets, and if*

$$A_i = \sum_{j \in J_i} B_j$$

then $\mathcal{A} = (A_i)_{i \in I}$ is an additive system.

The additive system $\mathcal{A}$ obtained from the additive system $\mathcal{B}$ by the partition procedure described in Lemma 1 is called a *contraction* of $\mathcal{B}$. (In [1], de Bruijn called $\mathcal{A}$ a *degeneration* of $\mathcal{B}$.) If $I = J$ and if σ is a permutation of J such that $J_i = \{\sigma(i)\}$ for all $i \in J$, then $\mathcal{A}$ and $\mathcal{B}$ contain exactly the same sets. Thus, every additive system is a contraction of itself. An additive system $\mathcal{A}$ is a *proper contraction* of $\mathcal{B}$ if at least one set $A_i \in \mathcal{A}$ is the sum of at least two sets in $\mathcal{B}$.

Let X be a set of integers, and let g be an integer. The *dilation* of X by g is the set $g * X = \{gx : x \in X\}$.

Lemma 2 *Let $\mathcal{B} = (B_j)_{j \in J}$ be an additive system and let $I = \{i_0\} \cup J$, where $i_0 \notin J$. If*

$$A_{i_0} = [0, g)$$

and

$$A_j = g * B_j \qquad \textit{for all } j \in J$$

then $\mathcal{A} = (A_i)_{i \in I}$ is an additive system.

The additive system $\mathcal{A}$ obtained from the additive system $\mathcal{B}$ by the procedure described in Lemma 2 is called the *dilation* of $\mathcal{B}$ by g.

There are certain additive systems that de Bruijn called *British number systems*. A British number system is an additive system constructed from an infinite sequence of integers according to the algorithm in Theorem 1 below. de Bruijn [1] proved that British number systems are essentially the only additive systems.

Theorem 1 *Let $(g_i)_{i \in \mathbf{N}}$ be an infinite sequence of integers such that $g_i \geq 2$ for all $i \geq 1$. Let $G_0 = 1$ and, for $i \in \mathbf{N}$, let $G_i = \prod_{j=1}^{i} g_j$ and*

$$A_i = \{0, G_{i-1}, 2G_{i-1}, \ldots, (g_i - 1)G_{i-1}\} = G_{i-1} * [0, g_i).$$

Then $\mathcal{A} = (A_i)_{i \in \mathbf{N}}$ is an additive system.

Theorem 2 *Every additive system is a British number system or a proper contraction of a British number system.*

The proof of Theorem 2 depends on the following fundamental lemma.

Lemma 3 *Let* $\mathcal{A} = (A_i)_{i \in I}$ *be an additive system with* $|I| \geq 2$*, and let* i_1 *be the unique element of* I *such that* $1 \in I_{i_1}$*. There exist an integer* $g \geq 2$ *and a family of sets* $\mathcal{B} = (B_i)_{i \in I}$ *such that*

$$A_{i_1} = [0, g) \oplus g * B_{i_1}$$

and, for all $i \in I \setminus \{i_1\}$*,*

$$A_i = g * B_i.$$

If $B_{i_1} = \{0\}$*, then* $\mathcal{B} = (B_i)_{i \in I \setminus \{i_1\}}$ *is an additive system, and* $\mathcal{A}$ *is the dilation of the additive system* $\mathcal{B}$ *by the integer* g*. If* $B_{i_1} \neq \{0\}$*, then* $\mathcal{B} = (B_i)_{i \in I}$ *is an additive system and* $\mathcal{A}$ *is a contraction of the additive system* $\mathcal{B}$ *dilated by* g*.*

For proofs of Lemmas 1, 2, and 3 and Theorems 1 and 2, see Nathanson [4].

This paper gives a refinement of de Bruijn's theorem. Every additive system is a contraction of a British number system, but even a British number system can be a proper contraction of another British number system. An additive system that is not a proper contraction of another number system will be called *indecomposable*. In Sect. 3, we describe all indecomposable British number systems. Unsurprisingly, there is a one-to-one correspondence between indecomposable British number systems and infinite sequences of prime numbers.

In Sect. 4, we define the limit of a sequence of additive systems and discuss the stability of British number systems.

Maltenfort [2] and Munagi [3] have also studied de Bruijn's additive systems.

2 Decomposable and Indecomposable Sets

The set A of integers is a *proper sumset* if there exist sets B and C of integers such that $|B| \geq 2$, $|C| \geq 2$, and $A = B + C$. For example, if u and v are integers and $v - u \geq 3$, then the interval $[u, v)$ is a proper sumset:

$$[u, v) = [0, i) + [u, v + 1 - i)$$

for every $i \in [2, v - u)$.

The set A of integers is *decomposable* if there exist sets B and C such that (B, C) is a unique representation system for A, that is, if $|B| \geq 2$, $|C| \geq 2$, and $A = B \oplus C$. A decomposition $A = B \oplus C$ is also called a *tiling* of A by B. For example,

$$[0, 12) = \{0, 3\} \oplus \{0, 1, 2, 6, 7, 8\}.$$

If $A = B \oplus C$ is a decomposition, then $|A| = |B|\ |C|$ and so the integer $|A|$ is composite.

Let $n \geq 2$ and consider the interval of integers $A = [0, n)$. A *proper divisor* of n is a divisor d of n such that $1 < d < n$. Associated to every proper divisor d of n is the decomposition

$$[0, n) = [0, d) \oplus d * [0, n/d). \tag{1}$$

This is simply the division algorithm for integers. The number of decompositions of type (1) is the number of proper divisors d of n. There is exactly one such decomposition if and only if the integer n has a unique proper divisor if and only if n is the square of a prime number.

Lemma 4 *Let $n \geq 2$. The interval $[0, n)$ is indecomposable if and only if n is prime.*

Proof If n is prime then $[0, n)$ is indecomposable, and if n is composite, then $[0, n)$ is decomposable.

If $A = B \oplus C$ and g is a nonzero integer, then $g * A = g * B \oplus g * C$, and so every dilation of a decomposable set is decomposable.

The *translate* of the set A by an integer t is the set

$$A + t = \{a + t : a \in A\}.$$

Let $t_1, t_2 \in \mathbf{Z}$ with $t = t_1 + t_2$. If $A = B + C$, then

$$A + t = (B + t_1) + (C + t_2).$$

In particular, $A + t = (B + t) + C$. If $A = B \oplus C$, then $A + t = (B + t) \oplus C$, and so every translate of a decomposable set is decomposable. Similarly, if $A = B \oplus C$, then $A = (B - t) \oplus (C + t)$ for every integer t.

Let A be a set of nonnegative integers with $0 \in A$, and let B and C be sets of integers with $A = B \oplus C$. Let $t = \min(B)$. Defining $B' = B - t$ and $C' = C + t$, we obtain $A = B' \oplus C'$. Because $\min(B') = 0$, we obtain

$$0 = \min(A) = \min(B') + \min(C') = 0 + \min(C') = \min(C')$$

and so B' and C' are sets of nonnegative integers with $0 \in B' \cap C'$.

Not every set with a composite number of elements is decomposable. For example, the n-element set $\{0, 1, 2, 2^2, \ldots, 2^{n-2}\}$ is indecomposable for every $n \geq 2$. This is a special case of the following result.

Lemma 5 *Let $m \geq 2$. Let A be a set of integers that contains integers a_0 and a_1 such that $a_0 \not\equiv a_1 \pmod{m}$, and $a \equiv a_0 \pmod{m}$ for all $a \in A \setminus \{a_1\}$. The set A is indecomposable.*

Proof The distinct congruence classes $a_0 \pmod{m}$ and $a_1 \pmod{m}$ contain elements of A. Let B and C be sets of integers such that $A = B + C$ with $|B|, |C| \geq 2$. If B is contained in the congruence class $r \pmod{m}$ and C is contained in the congruence class $s \pmod{m}$, then $B + C$ is contained in the congruence class $r + s$

(mod m), and so $A \neq B + C$ (because A intersects two congruence classes). Therefore, at least one of the sets B and C must contain elements from distinct congruence classes modulo m. Let $b_1, b_2 \in B$ with $b_1 \not\equiv b_2 \pmod{m}$, and let $c_1, c_2 \in C$ with $c_1 \neq c_2$. We have $b_i + c_1 \in B + C$ for $i = 1, 2$ and $b_1 + c_1 \not\equiv b_2 + c_1 \pmod{m}$. Because A intersects only two congruence classes modulo m, and because the intersection with the congruence class $a_1 \pmod{m}$ contains only the integer a_1, we must have $b_i + c_1 = a_1$ for some $i \in \{1, 2\}$.

Similarly, $b_j + c_2 \in B + C$ for $j = 1, 2$ with $b_1 + c_2 \not\equiv b_2 + c_2 \pmod{m}$, and so $b_j + c_2 = a_1$ for some $j \in \{1, 2\}$. The equation $b_i + c_1 = b_j + c_2$ implies that $A \neq B \oplus C$. This completes the proof.

The following examples show that, in Lemma 5, the condition that the set A contains exactly one element of the congruence class $a_1 \pmod{m}$ is necessary.

Let $m \geq 2$, and let $R \subseteq [0, m)$ with $|R| \geq 2$. For every set J of integers with $|J| \geq 2$, we have

$$A = \{jm + r : j \in J \text{ and } r \in R\} = B \oplus C$$

where

$$B = \{jm : j \in J\} \qquad \text{and} \qquad C = R.$$

Let k, ℓ, and m be integers with $k \geq 2$, $\ell \geq 2$, and $m \geq 2$, and let u and v be integers such that $u \not\equiv v \pmod{m}$. Consider the set

$$A = \{im + u : i \in [0, \ell)\} \cup \{jm + v : j \in [0, k\ell)\}.$$

The sets

$$B = \{u\} \cup \{q\ell m + v : q \in [0, k)\}$$

and

$$C = \{im : i \in [0, \ell)\}$$

satisfy $|B| = 1 + k\ell \geq 2$, $|C| = \ell \geq 2$ and

$$A = B \oplus C.$$

3 Decomposition of Additive Systems

Contraction and dilation are two methods to construct new additive systems from old ones. Decomposition is a third method to produce new additive systems.

An additive system $\mathcal{A} = (A_i)_{i \in I}$ is called *decomposable* if the set A_{i_0} is decomposable for some $i_0 \in I$ and *indecomposable* if A_i is indecomposable for all $i \in I$.

Equivalently, an indecomposable additive system is an additive system that is not a proper contraction of another additive system.

Theorem 3 *Let $\mathcal{A} = (A_i)_{i \in I}$ be a decomposable additive system, and let A_{i_0} be a decomposable set in $\mathcal{A}$. Choose sets B and C of nonnegative integers such that $0 \in B \cap C$, $|B| \geq 2$, $|C| \geq 2$, and $A_{i_0} = B \oplus C$. Let*

$$I' = \{j_1, j_2\} \cup I \setminus \{i_0\}.$$

The family of sets $\mathcal{A}' = (A'_i)_{i \in I'}$ defined by

$$A'_i = \begin{cases} A_i & \text{if } i \in I \setminus \{i_0\} \\ B & \text{if } i = j_1 \\ C & \text{if } i = j_2 \end{cases}$$

is an additive system.

Proof This follows immediately from the definitions of additive system and indecomposable set.

We call $\mathcal{A}'$ a *decomposition* of the additive system $\mathcal{A}$.

Lemma 6 *Let a and b be positive integers, and let X be a set of integers. Then*

$$[0, ab) = [0, a) \oplus X \tag{2}$$

if and only if

$$X = a * [0, b).$$

Proof The division algorithm implies that $[0, ab) = [0, a) \oplus a * [0, b)$, and so $X = a * [0, b)$ is a solution of the additive set equation (2).

Conversely, let X be any solution of (2). Let $I = \{1, 2, 3\}$ and let $A_1 = [0, a)$, $A_2 = X$, and $A_3 = ab * N_0$. By the division algorithm, $\mathcal{A} = (A_i)_{i \in I}$ is an additive system. Applying Lemma 3 to $\mathcal{A}$, we obtain an integer $g \geq 2$ and sets B_1, B_2, and B_3 such that

$$\begin{aligned} [0, a) &= [0, g) \oplus g * B_1 \\ X &= g * B_2 \\ ab * \mathbf{N}_0 &= g * B_3. \end{aligned}$$

It follows that $g = a$, $B_1 = \{0\}$, $B_3 = b * \mathbf{N}_0$, and

$$\mathbf{N}_0 = B_2 \oplus B_3 = B_2 \oplus b * \mathbf{N}_0.$$

This implies that $B_2 = [0, b)$ and $X = a * [0, b)$.

There is also a nice polynomial proof of Lemma 6. Let

$$
\begin{aligned}
f(t) &= \sum_{i \in [0,ab)} t^i \\
g(t) &= \sum_{j \in [0,a)} t^j \\
h(t) &= \sum_{k \in [0,b)} t^{ak} \\
h_X(t) &= \sum_{x \in X} t^x.
\end{aligned}
$$

The set equation $[0, ab) = [0, a) \oplus a * [0, b)$ implies that

$$f(t) = g(t)h(t).$$

If $[0, ab) = [0, a) \oplus X$, then

$$f(t) = g(t)h_X(t)$$

and so

$$g(t)(h(t) - h_X(t)) = 0.$$

Because $g(t) \neq 0$, it follows that $h(t) = h_X(t)$ or, equivalently, $a * [0, b) = X$.

By Theorem 2, every additive system is a British number system or a proper contraction of a British number system. However, a British number system can also be a proper contraction of another British number system. Consider, for example, the British number systems $\mathcal{A}_2$ and $\mathcal{A}_4$ generated by the sequences $(2)_{i \in \mathbf{N}}$ and $(4)_{i \in \mathbf{N}}$, respectively:

$$
\begin{aligned}
\mathcal{A}_2 &= (\{0, 2^{i-1}\})_{i \in \mathbf{N}} = (2^{i-1} * [0, 2))_{i \in \mathbf{N}} \\
&= (\{0, 1\}, \{0, 2\}, \{0, 4\}, \{0, 8\}, \ldots)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_4 &= (\{0, 4^{i-1}, 2 \cdot 4^{i-1}, 3 \cdot 4^{i-1}\})_{i \in \mathbf{N}} = (4^{i-1} * [0, 4))_{i \in \mathbf{N}} \\
&= (\{0, 1, 2, 3\}, \{0, 4, 8, 12\}, \{0, 16, 32, 48\}, \{0, 64, 128, 192, 256\}, \ldots).
\end{aligned}
$$

Because

$$4^{i-1} * [0, 4) = \{0, 2^{2i-2}\} + \{0, 2^{2i-1}\} = 2^{2i-2} * [0, 2) + 2^{2i-1} * [0, 2)$$

we see that $\mathcal{A}_4$ is a contraction of $\mathcal{A}_2$.

de Bruijn [1] asserted the following necessary and sufficient condition for one British number system to be a contraction of another British number system.

Theorem 4 *Let $\mathcal{B} = (B_j)_{j\in\mathbf{N}}$ be the British number system constructed from the integer sequence $(h_j)_{j\in\mathbf{N}}$, and let $\mathcal{A} = (A_i)_{i\in\mathbf{N}}$ be the contraction of $\mathcal{B}$ constructed from a partition $(J_i)_{i\in\mathbf{N}}$ of $\mathbf{N}$ into nonempty finite sets. Then, $\mathcal{A}$ is a British number system if and only if J_i is a finite interval of integers for all $i \in \mathbf{N}$.*

Proof Let $(J_i)_{i\in\mathbf{N}}$ be a partition of $\mathbf{N}$ into nonempty finite intervals of integers. After re-indexing, there is a strictly increasing sequence $(u_i)_{i\in\mathbf{N}_0}$ of integers with $u_0 = 0$ such that $J_i = [u_{i-1} + 1, u_i]$ for all $i \in \mathbf{N}$.

If $\mathcal{B} = (B_j)_{j\in\mathbf{N}}$ is the British number system constructed from the integer sequence $(h_j)_{j\in\mathbf{N}}$, then $B_j = H_{j-1} * [0, h_j)$, where $H_0 = 1$ and $H_j = \prod_{k=1}^{j} h_k$. Let $G_0 = 1$. For $i \in \mathbf{N}$ we define

$$g_i = \frac{H_{u_i}}{H_{u_{i-1}}}$$

and

$$G_i = \prod_{j=1}^{i} g_j = \prod_{j=1}^{i} \frac{H_{u_j}}{H_{u_{j-1}}} = H_{u_i}.$$

We have

$$\begin{aligned} A_i &= \bigoplus_{j\in J_i} B_j = \bigoplus_{j=u_{i-1}+1}^{u_i} H_{j-1} * [0, h_j) \\ &= H_{u_{i-1}} * \bigoplus_{j=u_{i-1}+1}^{u_i} \frac{H_{j-1}}{H_{u_{i-1}}} * [0, h_j) \\ &= H_{u_{i-1}} * \big([0, h_{u_{i-1}+1}) + h_{u_{i-1}+1} * [0, h_{u_{i-1}+2}) \\ &\qquad + h_{u_{i-1}+1} h_{u_{i-1}+2} * [0, h_{u_{i-1}+3}) + \cdots \\ &\qquad + h_{u_{i-1}+1} \cdots h_{u_i-1} * [0, h_{u_i})\big) \\ &= H_{u_{i-1}} * \left[0, \frac{H_{u_i}}{H_{u_{i-1}}}\right) \\ &= G_{i-1} * [0, g_i) \end{aligned}$$

and so $\mathcal{A} = (A_i)_{i\in\mathbf{N}}$ is the British number system constructed from the integer sequence $(g_i)_{i\in\mathbf{N}}$.

Conversely, let $\mathcal{A} = (A_i)_{i\in\mathbf{N}}$ be a contraction of $\mathcal{B}$ constructed from a partition $(J_i)_{i\in\mathbf{N}}$ of $\mathbf{N}$ in which some set J_{i_0} is a not a finite interval of integers. Let $u = \min\left(J_{i_0}\right)$ and $w = \max\left(J_{i_0}\right)$. Because J_{i_0} is not an interval, there is a smallest integer v such that

$$u < v < w$$

and $[u, v-1] \subseteq J_{i_0}$, but $v \notin I_{j_0}$. Because

$$[u, v-1] \cup \{w\} \subseteq J_{i_0} \subseteq [u, v-1] \cup [v+1, w]$$

and

$$A_{i_0} = \sum_{j \in J_{i_0}} H_{j-1} * [0, h_j)$$

we have

$$\begin{aligned} H_{u-1} * [0, h_u) \cup H_{w-1} * [0, h_w) &\subseteq A_{i_0} \\ &\subseteq \sum_{j \in [u, v-1]} H_{j-1} * [0, h_j) + \sum_{j \in [v+1, w]} H_{j-1} * [0, h_j) \\ &\subseteq H_{u-1} * \left[0, \frac{H_{v-1}}{H_{u-1}}\right) + H_v * \left[0, \frac{H_w}{H_v}\right) \end{aligned}$$

Because $h_u \geq 2$ and $h_v \geq 2$, it follows that

$$H_{u-1} \in A_{i_0}$$

and

$$H_{w-1} = H_{u-1} \left(\frac{H_{w-1}}{H_{u-1}}\right) \in A_{i_0}.$$

The largest multiple of H_{u-1} in $H_{u-1} * \left[0, H_{v-1}/H_{u-1}\right)$ is $H_{u-1}(H_{v-1}/H_{u-1} - 1)$. The smallest positive multiple of H_{u-1} in $H_v * [0, H_w/H_v)$ is $H_v = H_{u-1}(H_v/H_{u-1})$. The inequality

$$1 \leq \frac{H_{v-1}}{H_{u-1}} - 1 < \frac{H_{v-1}}{H_{u-1}} < \frac{H_v}{H_{u-1}} \leq \frac{H_{w-1}}{H_{u-1}}$$

implies that the set A_{i_0} does not contain the integer $H_{u-1}(H_{v-1}/H_{u-1})$. In a British number system, every set consists of consecutive multiples of its smallest positive element. Because the set A_{i_0} lacks this property, it follows that $\mathcal{A}$ is not a British number system. This completes the proof.

Theorem 5 *There is a one-to-one correspondence between sequences $(p_i)_{i \in \mathbf{N}}$ of prime numbers and indecomposable British number systems. Moreover, every additive system is either indecomposable or a contraction of an indecomposable system.*

Proof Let $\mathcal{A}$ be a British number system generated by the sequence $(g_i)_{i \in \mathbf{N}}$, so that

$$\mathcal{A} = (G_{i-1} * [0, g_i))_{i \in \mathbf{N}}.$$

Suppose that g_k is composite for some $k \in \mathbf{N}$. Then $g_k = rs$, where $r \geq 2$ and $s \geq 2$ are integers. Construct the sequence $(g_i')_{i \in \mathbf{N}}$ as follows:

$$g'_i = \begin{cases} g_i & \text{if } i \leq k-1 \\ r & \text{if } i = k \\ s & \text{if } i = k+1 \\ g_{i-1} & \text{if } i \geq k+2. \end{cases}$$

Then,

$$G'_i = \prod_{j=1}^{i} g'_j = \begin{cases} G_i & \text{if } i \leq k-1 \\ rG_{k-1} & \text{if } i = k \\ G_k & \text{if } i = k+1 \\ G_{i-1} & \text{if } i \geq k+2 \end{cases}$$

and

$$\mathcal{A}' = \left(G'_{i-1} * [0, g'_i)\right)_{i \in \mathbf{N}}$$

is the British number system generated by the sequence $(g'_i)_{i \in \mathbf{N}}$. We have

$$G_{i-1} * [0, g_i) = \begin{cases} G'_{i-1} * [0, g'_i) & \text{if } i \leq k-1 \\ G'_i * [0, g'_{i+1}) & \text{if } i \geq k+1. \end{cases}$$

The identity

$$[0, g_k) = [0, rs) = [0, r) \oplus r * [0, s) = [0, g'_k) + \frac{G'_k}{G_{k-1}} * [0, g'_{k+1})$$

implies that

$$G_{k-1} * [0, g_k) = G'_{k-1} * [0, g'_k) + G'_k * [0, g'_{k+1}) = \sum_{i \in \{k, k+1\}} G'_{i-1} * [0, g'_i)$$

and so the British number system $\mathcal{A}$ is a contraction of the British number system $\mathcal{A}'$.

Conversely, if $\mathcal{A}$ is a contraction of a British number system $\mathcal{A}' = \left(G'_{i-1} * [0, g'_i)\right)_{i \in \mathbf{N}}$, then there are a positive integer k and a set I_k of positive integers with $|I_k| \geq 2$ such that

$$G_{k-1} * [0, g_k) = \sum_{i \in I_k} G'_{i-1} * [0, g'_i).$$

Therefore,

$$g_k = |G_{k-1} * [0, g_k)| = \left| \sum_{i \in I_k} G'_{i-1} * [0, g'_i) \right| = \prod_{i \in I_k} g'_i.$$

Because $|I_k| \geq 2$ and $|g_i'| \geq 2$ for all $i \in \mathbf{N}$, it follows that the integer g_k is composite. Thus, the British number system generated by $(g_i)_{i\in\mathbf{N}}$ is decomposable if and only if g_i is composite for at least one $i \in \mathbf{N}$. Equivalently, the British number system generated by $(g_i)_{i\in\mathbf{N}}$ is indecomposable if and only if $(g_i)_{i\in\mathbf{N}}$ is a sequence of prime numbers. This completes the proof.

Theorem 5 has also been observed by Munagi [3].

4 Limits of Additive Systems

Let $\mathcal{A} = (A_i)_{i\in\mathbf{N}_0}$ be an additive system, and let $(g_i)_{i\in[1,n]}$ be a finite sequence of integers with $g_i \geq 2$ for all $i \in [1, n]$. The *dilation* of $\mathcal{A}$ by the sequence $(g_i)_{i\in[1,n]}$ is the additive system defined inductively by

$$(g_i)_{i\in[1,n]} * \mathcal{A} = g_1 * \left((g_i)_{i\in[2,n]} * \mathcal{A}\right).$$

For $n = 1$, we have

$$\begin{aligned}\mathcal{A}^{(1)} &= (g_i)_{i\in[1,1]} * \mathcal{A} = g_1 * \mathcal{A}\\ &= [0, g_1) \cup (g_1 * A_i)_{i\in\mathbf{N}_0}\\ &= \left(A_i^{(1)}\right)_{i\in\mathbf{N}_0}\end{aligned}$$

where

$$A_1^{(1)} = [0, g_1)$$

and

$$A_i^{(1)} = g_1 * A_{i-1} \text{ for} i \geq 2.$$

For $n = 2$, we have

$$\begin{aligned}\mathcal{A}^{(2)} &= (g_i)_{i\in[1,2]} * \mathcal{A} = g_1 * (g_2 * \mathcal{A})\\ &= g_1 * \left([0, g_2) \cup (g_2 * A_i)_{i\in\mathbf{N}_0}\right)\\ &= [0, g_1) \cup (g_1 * [0, g_2)) \cup (g_1 g_2 * A_i)_{i\in\mathbf{N}_0}\\ &= \left(A_i^{(2)}\right)_{i\in\mathbf{N}_0}\end{aligned}$$

where

$$\begin{aligned}A_1^{(2)} &= [0, g_1)\\ A_2^{(2)} &= g_1 * [0, g_2)\end{aligned}$$

and

$$A_i^{(2)} = g_1 g_2 * A_{i-2} \text{ for } i \geq 3.$$

For $n = 3$, we have

$$g_3 * \mathcal{A} = [0, g_3) \cup (g_3 * A_i)_{i \in \mathbf{N}_0}$$
$$g_2 * (g_3 * \mathcal{A}) = [0, g_2) \cup g_2 * [0, g_3) \cup (g_2 g_3 * A_i)_{i \in \mathbf{N}_0}$$

and

$$\begin{aligned}\mathcal{A}^{(3)} &= (g_i)_{i \in [1,3]} * \mathcal{A} = g_1 * (g_2 * (g_3 * \mathcal{A})) \\ &= [0, g_1) \cup (g_1 * [0, g_2)) \cup (g_1 g_2 * [0, g_3)) \cup (g_1 g_2 g_3 * A_i)_{i \in \mathbf{N}_0} \\ &= \left(A_i^{(3)}\right)_{i \in \mathbf{N}_0}\end{aligned}$$

where

$$\begin{aligned}A_1^{(3)} &= [0, g_1) \\ A_2^{(3)} &= g_1 * [0, g_2) \\ A_3^{(3)} &= g_1 g_2 * [0, g_3) \\ A_i^{(3)} &= g_1 g_2 g_3 A_{i-3} \qquad \text{for } i \geq 4.\end{aligned}$$

Lemma 7 *Let $(g_i)_{i=1}^n$ be a sequence of integers such that $g_i \geq 2$ for all i. For every additive system $\mathcal{A} = (A_i)_{i \in \mathbf{N}}$,*

$$\mathcal{A}^{(n)} = (g_i)_{i=1}^n * \mathcal{A} = \left(A_i^{(n)}\right)_{i \in \mathbf{N}}$$

where

$$A_i^{(n)} = g_1 g_2 \cdots g_{i-1} * [0, g_i) \qquad \text{for } i = 1, \ldots, n$$

and

$$A_i^{(n)} = g_1 g_2 \cdots g_n * A_{i-n-1} \qquad \text{for } i \geq n+1.$$

Proof Induction on n.

Let $(\mathcal{A}^{(n)})_{n \in \mathbf{N}}$ be a sequence of additive systems. The additive system $\mathcal{A}$ is the *limit* of the sequence $(\mathcal{A}^{(n)})_{n \in \mathbf{N}}$ if it satisfies the following condition: The set S belongs to $\mathcal{A}$ if and only if S belongs to $\mathcal{A}^{(n)}$ for all sufficiently large n. We write

$$\lim_{n \to \infty} \mathcal{A}^{(n)} = \mathcal{A}$$

if $\mathcal{A}$ is the limit of the sequence $(\mathcal{A}^{(n)})_{n \in \mathbf{N}}$. The following result indicates the remarkable stability of a British number system.

Theorem 6 *Let $(g_i)_{i \in \mathbf{N}}$ be a sequence of integers such that $g_i \geq 2$ for all $i \in \mathbf{N}$, and let $\mathcal{G}$ be the British number system generated by $(g_i)_{i \in \mathbf{N}}$. Let $\mathcal{A}$ be an additive system and let $\mathcal{A}^{(n)} = (g_i)_{i \in [1,n]} * \mathcal{A}$. Then,*

$$\lim_{n \to \infty} \mathcal{A}^{(n)} = \mathcal{G}.$$

Proof If S is a set in $\mathcal{G}$, then $S = g_1 g_2 \cdots g_{i-1} * [0, g_i)$ for some $i \in \mathbf{N}$. By Lemma 7, S is a set in $\mathcal{A}^{(n)}$ for all $n \geq i$, and so $S \in \lim_{n \to \infty} \mathcal{A}^{(n)}$.

Conversely, let S be a set that is in $\mathcal{A}^{(n)}$ for all sufficiently large n. If S is finite, then $\max(S) < g_1 g_2 \cdots g_k$ for some integer k. If $n \geq k$ and $i \geq n+1$, then

$$\max\left(A_i^{(n)}\right) \geq g_1 \ldots g_n \geq g_1 \ldots g_k$$

and so $S \neq A_i^{(n)}$. Therefore, $S = A_i^{(n)}$ for some $i \leq n$, and so $S = g_1 g_2 \cdots g_{i-1} * [0, g_i)$ for some $i \leq n$.

If T is an infinite set in $\mathcal{A}^{(n)}$, then $T = g_1 g_2 \cdots g_n * A_{i-n-1}$ for some $i \geq n+1$, and so $\min(T \setminus \{0\}) \geq g_1 g_2 \cdots g_n \geq 2^n$. If $T \in \mathcal{A}^{(n)}$ for all $n \geq N$, then $\min(T \setminus \{0\}) \geq 2^n$ for all $n \geq N$, which is absurd. It follows that the set S is in $\mathcal{A}^{(n)}$ for all sufficiently large n if and only if S is finite and S is a set in the British number system generated by $(g_i)_{i \in \mathbf{N}}$. This completes the proof.

Corollary 8 *Let $(g_i)_{i \in \mathbf{N}}$ be a sequence of integers such that $g_i \geq 2$ for all $i \in \mathbf{N}$, and let $\mathcal{G}$ be the British number system generated by $(g_i)_{i \in \mathbf{N}}$. If $\mathcal{G}_n = (g_i)_{i \in [1,n]} * \mathbf{N}_0$, then*

$$\lim_{n \to \infty} \mathcal{G}_n = \mathcal{G}.$$

Acknowledgements Supported in part by a grant from the PSC-CUNY Research Award Program.

References

1. N.G. de Bruijn, On number systems. Nieuw Arch. Wisk. **4**(3), 15–17 (1956)
2. M. Maltenfort, Characterizing additive systems. Am. Math. Monthly **124**, 132–148 (2017)
3. A.O. Munagi, *k*-complementing subsets of nonnegative integers. Int. J. Math. Math. Sci. 215–224 (2005)
4. M.B. Nathanson, Additive systems and a theorem of de Bruijn. arXiv:1301.6208 (2013)