Sumsets Contained in Sets of Upper Banach Density 1

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Abstract Every set A of positive integers with upper Banach density 1 contains an infinite sequence of pairwise disjoint subsets $(B_i)_{i=1}^{\infty}$ such that B_i has upper Banach density 1 for all $i \in \mathbf{N}$ and $\sum_{i \in I} B_i \subseteq A$ for every nonempty finite set I of positive integers.

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1 Upper Banach Density

Let $\mathbf{N}$, $\mathbf{N}_0$, and $\mathbf{Z}$ denote, respectively, the sets of positive integers, nonnegative integers, and integers. Let $|S|$ denote the cardinality of the set S. We define the *interval of integers*

$$[x, y] = \{n \in \mathbf{N} : x \leq n \leq y\}.$$

Let A be a set of positive integers. Let $n \in \mathbf{N}$. For all $u \in \mathbf{N}_0$, we have

$$|A \cap [u, u + n - 1]| \in [0, n]$$

and so

$$f_A(n) = \max_{u \in \mathbf{N}_0} |A \cap [u, u + n - 1]|$$

exists. The *upper Banach density* of A is

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$$\delta(A) = \limsup_{n\to\infty} \frac{f_A(n)}{n}.$$

Let $n_1, n_2 \in \mathbf{N}$. There exists $u_1^* \in \mathbf{N}_0$ such that, with $u_2^* = u_1^* + n_1$,

$$\begin{aligned} f_A(n_1+n_2) &= \left|A \cap [u_1^*, u_1^* + n_1 + n_2 - 1]\right| \\ &= \left|A \cap [u_1^*, u_1^* + n_1 - 1]\right| + \left|A \cap [u_1^* + n_1, u_1^* + n_1 + n_2 - 1]\right| \\ &= \left|A \cap [u_1^*, u_1^* + n_1 - 1]\right| + \left|A \cap [u_2^*, u_2^* + n_2 - 1]\right| \\ &\leq f_A(n_1) + f_A(n_2). \end{aligned}$$

It is well known, and proved in the Appendix, that this inequality implies that

$$\delta(A) = \lim_{n\to\infty} \frac{f_A(n)}{n} = \inf_{n\in\mathbf{N}} \frac{f_A(n)}{n}.$$

2 An Erdős Sumset Conjecture

About 40 years ago, Erdős conjectured that if A is a set of positive integers of positive upper Banach density, then there exist infinite sets B and C of positive integers such that $B + C \subseteq A$. This conjecture has not yet been verified or disproved.

The *translation* of the set X by t is the set

$$X + t = \{x + t : x \in X\}.$$

Let B and C be sets of integers. For every integer t, if $B' = B + t$ and $C' = C - t$, then

$$B' + C' = (B + t) + (C - t) = B + C.$$

In particular, if C is bounded below and $t = \min(C)$, then $0 = \min(C')$ and $B' \subseteq B' + C'$. It follows that if B and C are infinite sets such that $B + C \subseteq A$, then, by translation, there exist infinite sets B' and C' such that $B' \subseteq A$ and $B' + C' \subseteq A$.

However, a set A with positive upper Banach density does not necessarily contain infinite subsets B and C with $B + C \subseteq A$. For example, let A be any set of odd numbers. For all sets B and C of odd numbers, the sumset $B + C$ is a set of even numbers, and so $A \cap (B + C) = \emptyset$. Of course, in this example, we have $B + C \subseteq A + 1$.

In this note, we prove that if A is a set of positive integers with upper Banach density $\delta(A) = 1$, then for every $h \geq 2$ there exist pairwise disjoint subsets $B_1, \ldots, B_h$ of A such that $\delta(B_i) = 1$ for all $i = 1, \ldots, h$ and

$$B_1 + \cdots + B_h \subseteq A.$$

Indeed, Theorem 2 states an even stronger result.

There are sets A of upper Banach density 1 for which no infinite subset B of A satisfies $2B \subseteq A + t$ for any integer t. A simple example is

$$A = \bigcup_{i=1}^{\infty} \left[4^i, 4^i + i - 1 \right].$$

The set A is the union of the infinite sequence of pairwise disjoint intervals

$$A_i = \left[4^i, 4^i + i - 1 \right].$$

Let $t \in \mathbf{N}_0$. There exists $i_0(t)$ such that $4^i - i > t$ for all $i \geq i_0(t)$. If $b_i \in A_i$ for some $i \geq i_0(t)$, then

$$4^i + i + t < 2 \cdot 4^i \leq 2b_i < 2 \cdot 4^i + 2i < 4^{i+1} - 2t \leq 4^{i+1} - t$$

and so $2b_i \notin 2A \pm t$. If B is an infinite subset of A, then for infinitely many i, there exist integers $b_i \in B \cap A_i$, and so $2B \not\subseteq A + t$ for all $t \in \mathbf{Z}$.

There are very few results about the Erdős conjecture. In 1980, Nathanson [9] proved that if $\delta(A) > 0$, then for every n there is a finite set C with $|C| = n$ and a subset B of A with $\delta(B) > 0$ such that $B + C \subseteq A$. In 2015, Di Nasso et al. [3] used nonstandard analysis to prove that the Erdős conjecture is true for sets A with upper Banach density $\delta(A) > 1/2$. They also proved that if $\delta(A) > 0$, then there exist infinite sets B and C and an integer t such that

$$B + C \subseteq A \cup (A + t).$$

It would be of interest to have purely combinatorial proofs of the results of Di Nasso et al.

For related work, see Di Nasso [1, 2], Gromov [4], Hegyvári [5, 6], Hindman [7], and Jin [8].

3 Results

The following result is well known.

Lemma 1 *A set of positive integers has upper Banach density 1 if and only if, for every d, it contains infinitely many pairwise disjoint intervals of d consecutive integers.*

Proof Let A be a set of positive integers. If, for every positive integer d, the set A contains an interval of d consecutive integers, then

$$\max_{u \in \mathbf{N}_0} \left(\frac{|A \cap [u, u+d-1]|}{d} \right) = 1$$

and so

$$\delta(A) = \lim_{d \to \infty} \max_{u \in \mathbf{N}_0} \left(\frac{|A \cap [u, u+d-1]|}{d} \right) = 1.$$

Suppose that, for some integer $d \geq 2$, the set A contains no interval of d consecutive integers. For every $u \in \mathbf{N}_0$, we consider the interval $I_{u,n} = [u, u+n-1]$. By the division algorithm, there are integers q and r with $0 \leq r < d$ such that

$$|I_{u,n}| = n = qd + r$$

and

$$q = \frac{n-r}{d} > \frac{n}{d} - 1.$$

For $j = 1, \ldots, q$, the intervals of integers

$$I_{u,n}^{(j)} = [u + (j-1)d, u + jd - 1]$$

and

$$I_{u,n}^{(q+1)} = [u + qd, u + n - 1]$$

are pairwise disjoint subsets of $I_{u,n}$ such that

$$I_{u,n} = \bigcup_{j=1}^{q+1} I_{u,n}^{(j)}.$$

We have

$$A \cap I_{u,n} = \bigcup_{j=1}^{q+1} (A \cap I_{u,n}^{(j)})$$

If A contains no interval of d consecutive integers, then, for all $j \in [1, q]$, at least one element of the interval $I_{u,n}^{(j)}$ is not an element of A, and so

$$|A \cap I_{u,n}^{(j)}| \leq |I_{u,n}^{(j)}| - 1.$$

It follows that

$$|A \cap I_{u,n}| = \sum_{j=1}^{q+1} \left|A \cap I_{u,n}^{(j)}\right| \leq \sum_{j=1}^{q} \left(\left|I_{u,n}^{(j)}\right| - 1\right) + \left|I_{u,n}^{(q+1)}\right|$$
$$= \sum_{j=1}^{q+1} \left|I_{u,n}^{(j)}\right| - q = |I_{u,n}| - q = n - q$$
$$< n - \frac{n}{d} + 1 = \left(1 - \frac{1}{d}\right) n + 1.$$

Dividing by $n = |I_{u,n}|$, we obtain

$$\max_{u \in \mathbf{N}_0} \frac{|A \cap I_{u,n}|}{n} \leq 1 - \frac{1}{d} + \frac{1}{n}.$$

and so

$$\delta(A) = \lim_{n \to \infty} \max_{u \in \mathbf{N}_0} \frac{|A \cap I_{u,n}|}{n} \leq 1 - \frac{1}{d} < 1$$

which is absurd. Therefore, A contains an interval of d consecutive integers for every $d \in \mathbf{N}$.

To prove that A contains infinitely many intervals of size d, it suffices to prove that if $[u, u+d-1] \subseteq A$, then $[v, v+d-1] \subseteq A$ for some $v \geq u+d$. Let $d' = u + 2d$. There exists $u' \in \mathbf{N}$ such that

$$[u', u' + d' - 1] = [u', u' + u + 2d - 1] \subseteq A.$$

Choosing $v = u' + u + d$, we have $v \geq u + d$ and

$$[v, v + d - 1] \subseteq [u', u' + u + 2d - 1] \subseteq A.$$

This completes the proof.

Let $\mathcal{F}(S)$ denote the set of all finite subsets of the set S, and let $\mathcal{F}^*(S)$ denote the set of all nonempty finite subsets of S. We have the fundamental binomial identity

$$\mathcal{F}^*([1, n+1]) = \mathcal{F}^*([1, n]) \cup \{\{n+1\} \cup J : J \in \mathcal{F}([1, n])\}. \tag{1}$$

Theorem 1 *Let A be a set of positive integers that has upper Banach density 1. For every sequence $(\ell_j)_{j=1}^{\infty}$ of positive integers, there exists a sequence $(b_j)_{j=1}^{\infty}$ of positive integers such that*

$$b_{j+1} \geq b_j + \ell_j$$

for all $j \in \mathbf{N}$, *and*

$$\sum_{j \in J} [b_j, b_j + \ell_j - 1] \subseteq A$$

for all $J \in \mathcal{F}^*(\mathbf{N})$.

Proof We shall construct the sequence $(b_j)_{j=1}^{\infty}$ by induction. For $n = 1$, choose $b_1 \in A$ such that $[b_1, b_1 + \ell_1 - 1] \subseteq A$.

Suppose that $(b_j)_{j=1}^{n}$ is a finite sequence of positive integers such that $b_{j+1} \geq b_j + \ell_j$ for $j \in [1, n-1]$ and

$$\sum_{j \in J} [b_j, b_j + \ell_j - 1] \subseteq A \tag{2}$$

for all $J \in \mathcal{F}^*([1, n])$. By Lemma 1, there exists $b_{n+1} \in A$ such that

$$b_{n+1} \geq b_n + \ell_n$$

and

$$\left[b_{n+1}, \sum_{j=1}^{n+1} (b_j + \ell_j) - 1 \right] \subseteq A.$$

It follows that

$$\left[b_{n+1}, b_{n+1} + \ell_{n+1} - 1 \right] \subseteq A.$$

Let $J \in \mathcal{F}([1, n])$. If

$$\begin{aligned} a &\in \sum_{j \in \{n+1\} \cup J} [b_j, b_j + \ell_j - 1] \\ &= \left[b_{n+1}, b_{n+1} + \ell_{n+1} - 1 \right] + \sum_{j \in J} [b_j, b_j + \ell_j - 1] \end{aligned}$$

then

$$\begin{aligned} b_{n+1} \leq a &\leq (b_{n+1} + \ell_{n+1} - 1) + \sum_{j \in J} \left(b_j + \ell_j - 1 \right) \\ &\leq \sum_{j=1}^{n+1} (b_j + \ell_j) - 1 \end{aligned}$$

and so $a \in A$ and

$$\sum_{j \in \{n+1\} \cup J} [b_j, b_j + \ell_j - 1] \subseteq \left[b_{n+1}, \sum_{j=1}^{n+1} (b_j + \ell_j) - 1 \right] \subseteq A. \tag{3}$$

Relations (1), (2), and (3) imply that

$$\sum_{j \in J} [b_j, b_j + \ell_j - 1] \subseteq A$$

for all $J \in \mathcal{F}^*([1, n+1])$. This completes the induction.

Theorem 2 *Every set A of positive integers that has upper Banach density 1 contains an infinite sequence of pairwise disjoint subsets* $(B_i)_{i=1}^{\infty}$ *such that* B_i *has upper Banach density 1 for all* $i \in \mathbf{N}$ *and*

$$\sum_{i \in I} B_i \subseteq A$$

for all $I \in \mathcal{F}^*(\mathbf{N})$.

Proof Let $(\ell_j)_{j=1}^{\infty}$ be a sequence of positive integers such that $\lim_{j \to \infty} \ell_j = \infty$, and let $(b_j)_{j=1}^{\infty}$ be a sequence of positive integers that satisfies Theorem 1. (For simplicity, we can let $\ell_j = j$ for all j.) Let $(X_i)_{i=1}^{\infty}$ be a sequence of infinite sets of positive integers that are pairwise disjoint. For $i \in \mathbf{N}$, let

$$B_i = \bigcup_{j \in X_i} [b_j, b_j + \ell_j - 1].$$

The set B_i contains intervals of ℓ_j consecutive integers for infinitely many ℓ_j, and so B_i has upper Banach density 1.

Let $I \in \mathcal{F}^*(\mathbf{N})$. If

$$a \in \sum_{i \in I} B_i \subseteq A$$

then for each $i \in I$ there exists $a_i \in B_i$ such that $a = \sum_{i \in I} a_i$. If $a_i \in B_i$, then there exists $j_i \in X_i$ such that

$$x_i \in \left[b_{j_i}, b_{j_i} + \ell_{j_i} - 1 \right].$$

We have $J = \{j_i : i \in I\} \in \mathcal{F}^*(\mathbf{N})$ and

$$a \in \sum_{j_i \in J} \left[b_{j_i}, b_{j_i} + \ell_{j_i} - 1 \right] \subseteq A.$$

This completes the proof.

Theorem 3 *Let A be a set of integers that contains arbitrarily long finite arithmetic progressions with bounded differences. There exist positive integers m and r, and an infinite sequence of pairwise disjoint sets $(B_i)_{i=1}^{\infty}$ such that B_i has upper Banach density 1 for all $i \in \mathbf{N}$ and*

$$m * \sum_{i \in I} B_i + r \subseteq A$$

for all $I \in \mathcal{F}^*(\mathbf{N})$.

Proof If the differences in the infinite set of finite arithmetic progressions contained in A are bounded by m_0, then there exists a difference $m \leq m_0$ that occurs infinitely often. It follows that there are arbitrarily long finite arithmetic progressions with difference m. Because there are only finitely many congruence classes modulo m, there exists a congruence class $r \pmod{m}$ such that A contains arbitrarily long sequences of consecutive integers in the congruence class $r \pmod{m}$. Thus, there exists an infinite set A' such that

$$m * A' + r \subseteq A$$

and A' contains arbitrarily long sequences of consecutive integers. Equivalently, A' has Banach density 1. By Theorem 2, the sequence A' contains an infinite sequence of pairwise disjoint subsets $(B_i)_{i=1}^{\infty}$ such that B_i has upper Banach density 1 for all $i \in \mathbf{N}$ and

$$\sum_{i \in I} B_i \subseteq A'$$

for all $I \in \mathcal{F}^*(\mathbf{N})$. It follows that

$$m * \sum_{i \in I} B_i + r \subseteq m * A' + r \subseteq A$$

for all $I \in \mathcal{F}^*(\mathbf{N})$. This completes the proof.

Appendix: Subadditivity and Limits

A real-valued arithmetic function f is *subadditive* if

$$f(n_1 + n_2) \leq f(n_1) + f(n_2) \tag{4}$$

for all $n_1, n_2 \in \mathbf{N}$.

The following result is sometimes called *Fekete's lemma*.

Lemma 2 *If f is a subadditive arithmetic function, then $\lim_{n\to\infty} f(n)/n$ exists, and*

$$\lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n\in\mathbf{N}} \frac{f(n)}{n}.$$

Proof It follows by induction from inequality (4) that

$$f(n_1 + \cdots + n_q) \leq f(n_1) + \cdots + f(n_q)$$

for all $n_1, \ldots, n_q \in \mathbf{N}$. Let $f(0) = 0$. Fix a positive integer d. For all $q, r \in \mathbf{N}_0$, we have

$$f(qd + r) \leq qf(d) + f(r).$$

By the division algorithm, every nonnegative integer n can be represented uniquely in the form $n = qd + r$, where $q \in \mathbf{N}_0$ and $r \in [0, d-1]$. Therefore,

$$\frac{f(n)}{n} = \frac{f(qd+r)}{n} \leq \frac{qf(d)}{qd} + \frac{f(r)}{n} = \frac{f(d)}{d} + \frac{f(r)}{n}.$$

Because the set $\{f(r) : r \in [0, d-1]\}$ is bounded, it follows that

$$\limsup_{n\to\infty} \frac{f(n)}{n} \leq \limsup_{n\to\infty} \left(\frac{f(d)}{d} + \frac{f(r)}{n} \right) = \frac{f(d)}{d}$$

for all $d \in \mathbf{N}$, and so

$$\limsup_{n\to\infty} \frac{f(n)}{n} \leq \inf_{d\in\mathbf{N}} \frac{f(d)}{d} \leq \liminf_{d\to\infty} \frac{f(d)}{d} = \liminf_{n\to\infty} \frac{f(n)}{n}.$$

This completes the proof.

References

1. M. Di Nasso, An elementary proof of Jin's theorem with a bound. Electron. J. Combin. **21**(2), Paper 2.37, 7 (2014)
2. M. Di Nasso, Embeddability properties of difference sets, Integers **14**, Paper No. A **27**, 24 (2014)
3. M. Di Nasso, I. Goldbring, R. Jin, S. Leth, M. Lupini, K. Mahlburg, On a sumset conjecture of Erdős. Canad. J. Math. **67**(4), 795–809 (2015)
4. M.L. Gromov, Colorful categories. Uspekhi Mat. Nauk **70**(4), 424, 3–76 (2015)
5. N. Hegyvári, On the dimension of the Hilbert cubes. J. Num. Theor. **77**(2), 326–330 (1999)
6. N. Hegyvári, On additive and multiplicative Hilbert cubes. J. Combin. Theor. Ser. A **115**(2), 354–360 (2008)

7. N. Hindman, Ultrafilters and combinatorial number theory, Number theory, Carbondale, Proceedings of Southern Illinois Conference, Southern Illinois University, Carbondale, Ill., Lecture Notes in Mathematics, vol. 751. Springer, Berlin **1979**, 119–184 (1979)
8. R. Jin, Standardizing nonstandard methods for upper Banach density problems, Unusual Applications of Number Theory, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 109–124 (2004)
9. M.B. Nathanson, Sumsets contained in infinite sets of integers. J. Combin. Theor. Ser. A **28**(2), 150–155 (1980)