White's Theorem

An Exposition of White's Characterization of Empty Lattice Tetrahedra

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Abstract We give an exposition of White's characterization of empty lattice tetra-hedra. In particular, we describe the second author's proof of White's theorem that appeared in her doctoral thesis (Rogers in Doctoral dissertation 1993) [7].

1 Introduction

The motivating example is the *lattice* tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, c)$ with c being an arbitrary positive integer. We denote this tetrahedron as $T_{1,1,c}$. Regardless of the size of c (and consequently the volume of $T_{1,1,c}$), $T_{1,1,c}$ does not contain any lattice points other than its vertices. This is in surprising contrast to the situation in $\mathbb{R}^2$ where a *lattice* triangle does not contain any lattice points, other than its vertices, if and only if it has area 1/2. (To see this we invoke Pick's theorem.)

Reeve [4] posed the problem of characterizing such tetrahedra. Some years later, White [10] solved this problem. Over the years, different authors have given proofs of White's theorem (see [1, 3, 5, 6, 8]). The second author gave a proof of White's theorem in her doctoral dissertation [7]. In this article, we give a detailed exposition of this proof.

Before stating the relevant theorems, we establish some notation and definitions. Let $a, b, c \in \mathbb{Z}$ with $0 \le a, b < c$. We will use d to denote the integer

$$d = (1 - a - b) \mod c, 0 \le d < c.$$

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Furthermore, $T_{a,b,c}$ will denote the lattice tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and (a, b, c).

Definition 1 Following Reznick [6], we call a lattice polyhedron that does not contain any lattice points other than its vertices an *empty lattice polyhedron*. Such a polyhedron belongs to a larger set of lattice polyhedra that do not contain any lattice points on their boundary other than the vertices. We call such polyhedra *clean lattice polyhedra*.

We insert a warning about the the terminology, particularly in the case of tetrahedra. Other names in the literature for empty tetrahedra are *fundamental*, *primitive*, *Reeve*.

Definition 2 An *affine unimodular map* is an affine map

$$L : \mathbb{R}^3 \to \mathbb{R}^3 \text{of the form} L(\mathbf{x}) = M\mathbf{x} + \mathbf{u},$$

where $M \in GL_3(\mathbb{Z})$, $\det(M) = \pm 1$ and $\mathbf{u} \in \mathbb{Z}^3$.

We now state the two theorems that we will prove.

Theorem 1 *Let T be an empty lattice tetrahedron. Then there is an affine unimodular map L such that $L(T) = T_{a,b,c}$, with $0 \le a, b < c$ and $\gcd(a, c) = \gcd(b, c) = \gcd(d, c) = 1$.*

Theorem 2 (**White**) *The lattice tetrahedron $T_{a,b,c}$ is empty if and only if $\gcd(a, c) = \gcd(b, c) = \gcd(d, c) = 1$ and at least one of the following hold:*

$$a = 1, b = 1, c = 1, d = 1.$$

We now state definitions and background results that will be used to prove the two theorems.

Definition 3 A set of lattice points $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{Z}^n$ is said to be *primitive* if it is a basis for the sublattice $\mathbb{Z}^n \cap (\mathbb{R}\mathbf{v}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{v}_k)$. Geometrically, this means that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is primitive if and only if the parallelepiped spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is empty.

The following is a list of standard results we will use. The proofs can be found in [9, Lectures V, VIII]. However, we have rephrased some of the statements. Consequently, the reader who consults [9] may need to read the relevant material carefully.

Theorem 3 *Every lattice has an integral basis.*

Theorem 4 *The property of being a lattice basis is preserved under the action of any unimodular transformation, that is, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis for $\mathbb{Z}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is an unimodular transformation, then $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ is also a basis of $\mathbb{Z}^n$. Furthermore, given two lattice bases, there is an unimodular transformation that maps one basis into the other.*

Theorem 5 *Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *be a linearly independent set of elements of* $\mathbb{Z}^n$, *and let* $H = \mathbb{Z}\mathbf{v}_1 \oplus \ldots \oplus \mathbb{Z}\mathbf{v}_n$. *Then the order of the quotient group* $\mathbb{Z}^n/H$ *equals*

$$\#(\mathbb{Z}^n/H) = |\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)|.$$

Theorem 6 *Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ *be a primitive set of* $\mathbb{Z}^n$. *Then* $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ *can be extended to a basis of* $\mathbb{Z}^n$.

We mention an interesting fact that emerges in the course of proving White's theorem. From Theorem 5, it follows that if $T_{a,b,c}$ is empty, then the parallelepiped spanned by $(1, 0, 0)$, $(0, 1, 0)$, (a, b, c) contains $(c - 1)$ lattice points in its interior. In the course of proving Theorem 2, we will find that *all of these points are coplanar*! More precisely, we have the following.

Corollary 1 *Let* $P_{a,b,c}$ *denote the parallelepiped spanned by* $(1, 0, 0)$, $(0, 1, 0)$, *and* (a, b, c). *If* $T_{a,b,c}$ *is empty, then* $P_{a,b,c}$ *contains* $(c - 1)$ *lattice points in its interior. If* $a = 1$, *then all of these lattice points lie on the plane* $x = 1$; *if* $b = 1$, *then all of these lattice points lie on the plane* $y = 1$; *if* $d = 1$, *then all of these lattice points lie on the plane* $x + y - z = 1$.

Warning: The co-planarity of these lattice points was mentioned in an article of the first author [2, Theorem 3.2]. Unfortunately, the description of the planes in [2] is completely incorrect! The author should have done his homework and not just relied on his faulty visualization skills!!

2 Proofs

We begin with some notation. Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{Z}^3$. We will denote the integer $\gcd(u_1, u_2, u_3)$ by $\gcd(\mathbf{u})$. Occasionally, we will use e_1, e_2, and e_3 to denote the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Proposition 1 *Let* $\mathbf{u}$, $\mathbf{v}$ *be two linearly independent elements in* $\mathbb{Z}^3$. *The following statements are equivalent.*

1. *P, the parallelogram spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *is an empty parallelogram.*
2. *T, the triangle spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *is an empty triangle.*
3. $\gcd(\mathbf{u} \times \mathbf{v}) = 1$.

Proof Clearly $(1) \Rightarrow (2)$. We prove the contrapositive to demonstrate that $(2) \Rightarrow (1)$. We assume that P contains a lattice point $\mathbf{x}$ that is not a vertex of P. Then either $\mathbf{x}$ or $(\mathbf{u} + \mathbf{v} - \mathbf{x})$ lies in T. Since neither lattice point can be a vertex of T, we conclude that T is not an empty triangle.

We now turn to proving that (1) and (3) are equivalent.

$(3) \Rightarrow (1)$: Since $\gcd(\mathbf{u} \times \mathbf{v}) = 1$, there exists, by the Extended Euclidean algorithm, $\mathbf{w} \in \mathbb{Z}^3$ such that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 1 = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$. By Theorem 5, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a

basis of $\mathbb{Z}^3$, and consequently they span an empty parallelepiped. We conclude that P is an empty parallelogram.

(1) $\Rightarrow$ (3): Since P is an empty parallelogram, $\{\mathbf{u}, \mathbf{v}\}$ is a primitive set of $\mathbb{Z}^3$, and consequently by Theorem 6 there is a lattice point $\mathbf{w}$ such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a basis of $\mathbb{Z}^3$. Consequently, $|\det(\mathbf{u}, \mathbf{v}, \mathbf{w})| = 1$. Since $\det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, we conclude that $\gcd(\mathbf{u} \times \mathbf{v}) = 1$.

Corollary 2 *The tetrahedron $T_{a,b,c}$ is clean if and only if $\gcd(a, c) = \gcd(b, c) = \gcd(d, c) = 1$.*

Proof Let $\triangle_1, \triangle_2, \triangle_3, \triangle_4$ denote the faces of $T_{a,b,c}$ where $\triangle_1$ is the triangle spanned by e_1 and e_2; $\triangle_2$ is the triangle spanned by e_1 and (a, b, c); $\triangle_3$ is the triangle spanned by e_2 and (a, b, c); and $\triangle_4$ is the triangle spanned by $(e_2 - e_1)$ and $((a, b, c) - e_1)$. $T_{a,b,c}$ is a clean tetrahedron if and only if $\triangle_1, \triangle_2, \triangle_3$, and $\triangle_4$ are all empty lattice triangles. Clearly $\triangle_1$ is an empty triangle. By Proposition 1, the triangles $\triangle_2, \triangle_3, \triangle_4$ are empty if and only if

$$\gcd(e_1 \times (a, b, c)) = \gcd(e_2 \times (a, b, c)) = \gcd((e_2 - e_1) \times ((a, b, c) - e_1)) = 1,$$

that is, $\gcd(b, c) = \gcd(a, c) = \gcd(d, c) = 1$.

Proof (Proof of Theorem 1) Let T be an empty lattice tetrahedron in $\mathbb{R}^3$. Without loss of generality we may assume that the origin is one of the vertices and the other 3 vertices are $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$. Since the triangle spanned by $\mathbf{u}$ and $\mathbf{v}$ is empty, by Proposition 1, the same holds for the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. Therefore, $\{\mathbf{u}, \mathbf{v}\}$ is a primitive set of $\mathbb{Z}^3$, and by Theorem 6 can be extended to a basis of $\mathbb{Z}^3$, $\mathbf{u}, \mathbf{v}, \mathbf{x}$. Now by Theorem 4, we have a unimodular transformation L_1 such that $L_1(\mathbf{u}) = e_1$, $L_2(\mathbf{v}) = e_2$, and $L_3(\mathbf{x}) = e_3$. Under this transformation, we see that the tetrahedron T is equivalent to the tetrahedron T_1 with vertices $0, e_1, e_2$ and (A, B, c) where $A, B, c \in \mathbb{Z}$ and $\text{vol}(T) = |c/6|$. If $c < 0$, we can compose L_1 with the unimodular transformation

$$L_2((x, y, z)) = (x, y, -z).$$

Consequently, we can assume that $c > 0$. We now use the division algorithm to express

$$A = q_1 c + a \text{ and } B = q_2 c + b, 0 \le a, b < c.$$

By acting on T_1 by the unimodular transformation

$$L_3((x, y, z)) = (x - q_1 z, y - q_2 z, z)$$

we get that T is equivalent to the tetrahedron T_2 with vertices $0, e_1, e_2$ and (a, b, c). Since T_2 is a clean tetrahedron, we invoke Corollary 2 to conclude that $\gcd(a, c) = \gcd(b, c) = \gcd(d, c) = 1$.

We now turn to the proof of White's theorem. Our proof is arranged in four parts. These are as follows:

Part 1: We prove that the tetrahedron $T_{a,b,c}$ is empty if and only if a system of equations involving a, b, d hold.

Part 2: This system of equations give an immediate proof of the ($\Leftarrow$) direction of White's theorem.

Part 3: The proof of the ($\Rightarrow$) direction of White's theorem is considerably more involved. We first develop a slight modification of the system of equations. This then leads us to define a finite set of arithmetic functions f_n. We then state and prove certain properties of these functions.

Part 4: We use the properties of f_n to complete the proof.

We will invoke the following identity in several places

Lemma 1 *Let $x \in \mathbb{R}$. If $x \notin \mathbb{Z}$, then*

$$\langle -x \rangle = 1 - \langle x \rangle. \tag{1}$$

We will typically invoke this identity in the following form:

$$\left\langle \frac{kl}{c} \right\rangle + \left\langle \frac{k(c-l)}{c} \right\rangle = 1 \tag{2}$$

for $0 < l < c, \gcd(l, c) = 1,$ and $k = 1, \ldots, c - 1$.

Proposition 2 *Let $c \in \mathbb{Z}$ with $c > 1$ and let $T_{a,b,c}$ be a clean lattice tetrahedron. Then, $T_{a,b,c}$ is empty if and only if*

$$\left\langle \frac{ka}{c} \right\rangle + \left\langle \frac{kb}{c} \right\rangle + \left\langle \frac{kd}{c} \right\rangle - \frac{k}{c} = 1 \tag{3}$$

holds for $k = 1, \ldots, c - 1$.

Proof (Proof of Part 1) Let P denote the parallelepiped spanned by e_1, e_2 and (a, b, c). Since volume$(P) = c$ and the faces of P are empty lattice parallelograms, we infer that P contains $(c - 1)$ lattice points in its interior. These lattice points are

$$\left\langle \frac{k(c-a)}{c} \right\rangle (1, 0, 0) + \left\langle \frac{k(c-b)}{c} \right\rangle (0, 1, 0) + \frac{k}{c}(a, b, c) \tag{4}$$

with $k = 1, \ldots, c - 1$.

$T_{a,b,c}$ is empty if and only if

$$1 < \left\langle \frac{k(c-a)}{c} \right\rangle + \left\langle \frac{k(c-b)}{c} \right\rangle + \frac{k}{c} < 2,$$

for $k = 1, \ldots, c - 1$. Some algebraic manipulation in conjunction with identity (1) gives the system of inequalities

$$0 < \left\langle \frac{ka}{c} \right\rangle + \left\langle \frac{kb}{c} \right\rangle - \frac{k}{c} < 1,$$

for $k = 1, \ldots, c - 1$. We now observe that

$$\left\langle \frac{ka}{c} \right\rangle + \left\langle \frac{kb}{c} \right\rangle - \frac{k}{c} \equiv \left\langle \frac{k(a + b - 1)}{c} \right\rangle \quad (\mathrm{mod}\ \mathbb{Z}), \tag{5}$$

for $k = 1, \ldots, c - 1$. Since both sides of the congruence are between 0 and 1, we conclude that we have a system of *equalities*

$$\left\langle \frac{ka}{c} \right\rangle + \left\langle \frac{kb}{c} \right\rangle - \frac{k}{c} = \left\langle \frac{k(a + b - 1)}{c} \right\rangle,$$

for $k = 1, \ldots, c - 1$. After a little more algebraic manipulation, we conclude that $T_{a,b,c}$ is empty if and only if

$$\left\langle \frac{ka}{c} \right\rangle + \left\langle \frac{kb}{c} \right\rangle + \left\langle \frac{kd}{c} \right\rangle - \frac{k}{c} = 1$$

for $k = 1, \ldots, c - 1$.

We can now easily prove ($\Leftarrow$) direction of White's theorem. The system of equations (3) in conjunction with the system of identities (2) allow us to conclude that the following tetrahedra are empty.

Corollary 3 *Let* $\gcd(a, c) = 1$. *Then the tetrahedra* $T_{1,a,c}$ *and* $T_{a,c-a,c}$ *are empty.*

To prove the ($\Rightarrow$) direction of White's theorem, we will work with a modification of (3). Define a set of arithmetic functions f_n for $n \in \mathbb{Z}^+$, $n < c$ and $\gcd(n, c) = 1$,

$$f_n : \{1, \ldots, c - 2\} \to \{0, 1\}$$

via

$$f_n(k) = \left\langle \frac{kn}{c} \right\rangle - \left\langle \frac{(k + 1)n}{c} \right\rangle + \frac{n}{c} = \left[\frac{(k + 1)n}{c} \right] - \left[\frac{kn}{c} \right]. \tag{6}$$

From (3), we obtain the system of equations

$$f_a(k) + f_b(k) + f_d(k) + \frac{1}{c} = \frac{a + b + d}{c}, \tag{7}$$

for $k = 1, \ldots, c - 2$. We now look at the case of $k = 1$ in (3) which shows that

$$\frac{a + b + d}{c} = 1 + \frac{1}{c}.$$

Thus, we can rewrite (7) as the system of equations

$$f_a(k) + f_b(k) + f_d(k) = 1, \tag{8}$$

for $k = 1, \ldots, c - 2$. We will work with this system (8) in conjunction with the properties of f_n to arrive at a proof of White's theorem.

Proposition 3 *The function f_n has the following properties.*

(i) $f_1^{-1}(\{1\}) = \emptyset$.
(ii) *For $n > 1$,*
$$f_n^{-1}(\{1\}) = \{[kc/n] : k = 1, \ldots, n - 1\}.$$

(iii) $f_{c-n} = 1 - f_n$.

Proof For $k = 1, \ldots, c - 2$,

$$f_1(k) = \left[\frac{k+1}{c}\right] - \left[\frac{k}{c}\right] = 0 - 0 = 0,$$

which proves (i).

We now prove statement (ii). If $l \in f_n^{-1}(\{1\})$ then there exists $k \in \mathbb{Z}^+$ such that

$$\frac{ln}{c} < k < \frac{(l+1)n}{c}.$$

It follows that $l = [kc/n]$. Conversely, if $l = [kc/n]$ for some integer k, with $1 \le k \le n - 1$, then we have that

$$l < \frac{kc}{n} < l + 1.$$

We now obtain that

$$\frac{ln}{c} < k < \frac{(l+1)n}{c}$$

and consequently $l \in f_n^{-1}(\{1\})$.

Statement (iii) is a consequence of identity (2).

$$\begin{aligned}
f_{c-n}(k) &= \left\langle \frac{k(c-n)}{c} \right\rangle - \left\langle \frac{(k+1)(c-n)}{c} \right\rangle + \frac{c-n}{c} \\
&= 1 - \left\langle \frac{kn}{c} \right\rangle - 1 + \left\langle \frac{(k+1)n}{c} \right\rangle + 1 - \frac{n}{c} \\
&= 1 - f_n(k).
\end{aligned}$$

We now complete the proof of White's theorem.

Proof (*Proof of Part 3*) Let $T_{a,b,c}$ be an empty tetrahedron with $c \ge 2$. We want to prove that either $a = 1$ or $b = 1$ or $d = 1$. We will argue by contradiction. So we

assume that $a, b, d \geq 2$. Consequently none of the sets $f_a^{-1}(\{1\})$, $f_b^{-1}(\{1\})$, $f_d^{-1}(\{1\})$ are empty. Since

$$f_a + f_b + f_d = 1,$$

can infer that a, b and d are distinct integers, and the sets

$$f_a^{-1}(\{1\}), \ f_b^{-1}(\{1\}), \ f_d^{-1}(\{1\})$$

are pairwise disjoint. (**Spoiler alert**: Our argument hinges crucially on the fact that $f_b^{-1}(\{1\}) \cap f_d^{-1}(\{1\}) = \emptyset$.) Without loss of generality, we can assume that $a > b > d$. It follows that $1 \in f_a^{-1}(\{1\})$, and consequently $1 \notin \left(f_b^{-1}(\{1\}) \cup f_d^{-1}(\{1\})\right)$. We now have that

$$f_b + f_d = f_{c-a}$$

and consequently

$$\left(f_b^{-1}(\{1\}) \cup f_d^{-1}(\{1\})\right) = f_{c-a}^{-1}(\{1\}),$$

that is,

$$\{ [kc/b] : k = 1, \ldots, b - 1 \} \cup \{ [kc/d] : k = 1, \ldots, d - 1 \}$$

$$= \{ [kc/(c - a)] : k = 1, \ldots, (c - a - 1) \} .$$

We now compare the smallest and largest elements in each of the 3 sets. Since $b > d \geq 2$ and $1 \notin f_{c-a}^{-1}(\{1\})$, we have that

$$2 \leq \left[\frac{c}{c-a}\right] = \left[\frac{c}{b}\right] < \left[\frac{c}{d}\right] \leq \left[\frac{(d-1)c}{d}\right] < \left[\frac{(b-1)c}{b}\right] = \left[\frac{(c-a-1)c}{c-a}\right].$$

We remark that the strict inequalities occur since

$$f_b^{-1}(\{1\}) \cap f_d^{-1}(\{1\}) = \emptyset.$$

Let s be the positive integer such that

$$\left[\frac{c}{d}\right] = \left[\frac{sc}{c-a}\right].$$

We now obtain that

$$\left[\frac{(s-1)c}{c-a}\right] = \left[\frac{(s-1)c}{b}\right] \text{ and } \left[\frac{(s+1)c}{c-a}\right] \leq \left[\frac{sc}{b}\right].$$

Combining these two observations, we get

$$\left[\frac{(s+1)c}{c-a}\right] - \left[\frac{(s-1)c}{c-a}\right] \le \left[\frac{sc}{b}\right] - \left[\frac{(s-1)c}{b}\right],$$

which implies the inequality

$$2\left[\frac{c}{c-a}\right] \le \left[\frac{c}{b}\right] + 1.$$

This leads to the contradiction that

$$\left[\frac{c}{c-a}\right] \le 1,$$

and consequently our assumption that $a, b, d \ge 2$ is false.

Proof (*Proof of Corollary* 1) Let $T_{a,b,c}$ be empty, with $c > 1$. By Theorem 2, we have that either $a = 1$ or $b = 1$ or $b = c - a$. If $a = 1$, then by replacing a by 1 in (4), we see that the x co-ordinate of each lattice point inside $P_{1,b,c}$ equals 1. The same argument works if $b = 1$. The only case that needs a little more work is, if $b = c - a$. In this case, (4) becomes

$$\left\langle\frac{k(c-a)}{c}\right\rangle (1, 0, 0) + \left\langle\frac{ka}{c}\right\rangle (0, 1, 0) + \frac{k}{c}(a, c-a, c). \tag{9}$$

If we now add the x and y co-ordinates and subtract the z co-ordinate, we get

$$\left\langle\frac{k(c-a)}{c}\right\rangle + \left\langle\frac{ka}{c}\right\rangle + \frac{ka}{c} + \frac{k(c-a)}{c} - k = \left\langle\frac{k(c-a)}{c}\right\rangle + \left\langle\frac{ka}{c}\right\rangle.$$

We now invoke the identities (2) to conclude that the RHS equals 1.

References

1. F. Breuer, F. von Heymann, Staircases in $\mathbb{Z}^2$. Integers **10**, 807–847 (2010)
2. M.R. Khan, A counting formula for primitive tetrahedra in $\mathbb{Z}^3$. Am. Math. Mon. **106**(6), 525–533 (1999)
3. D.R. Morrison, G. Stevens, Terminal Quotient Singularities in dimensions three and four. Proc. Amer. Math. Soc. **90**(1), 15–20 (1984)
4. J.E. Reeve, On the volume of lattice tetrahedra. Proc. London, Math. Soc. **7**(3) 378–395 (1957)
5. B. Reznick, Lattice point simplices. Discrete Math. **60**, 219–242 (1986)

6. B. Reznick, Clean lattice tetrahedra (preprint)
7. K.M. Rogers, Primitive simplices in $\mathbb{Z}^3$ and $\mathbb{Z}^4$, Doctoral dissertation, Columbia University, 1993
8. H.E. Scarf, Integral polyhedra in three space. Math. Oper. Res. **10**, 403–438 (1985)
9. C.L. Siegel, *Lectures on the Geometry of Numbers*, (Springer-Verlag, 1989)
10. G.K. White, Lattice Tetrahedra. Canad. J. Math **16**, 389–396 (1964)