Chapter 5
Averaging with Exogenous Inputs and Electrical Networks

Abstract The dynamical models analyzed so far, with the exception of the noisy consensus models treated in Chap. 4, are autonomous systems with no input signals: Information enters the system only through the initial condition. Instead, there are a variety of different situations where it is natural to consider consensus models driven by exogenous input signals, including opinion dynamics in the presence of stubborn agents that do not modify their opinion, rendezvous problems with leader robots, and estimation algorithms based on pairwise measurements. A very useful tool to analyze these models is thinking of the graph as an electrical circuit with the exogenous signals interpreted as input currents or as nodes kept at a fixed voltage. In this chapter, we will first review the basic theory of electrical networks and their classical connection with reversible stochastic matrices: Sect. 5.1 concentrates on Green matrices and harmonic functions, while Sect. 5.2 is devoted to effective resistances. Afterward, we apply these tools to averaging dynamics with stubborn agents in Sect. 5.3 and to the problem of estimation from relative measurements in Sect. 5.4.

5.1 Electrical Networks and Harmonic Functions

There is a fundamental connection between reversible stochastic matrices, presented in Sect. 2.5, and electrical circuits: This connection sheds light on some of the concepts touched so far and, meanwhile, offers computational tools for new problems.

We start from a symmetric strongly connected graph $G = (V, E)$ with $|V| = N$ and a symmetric nonnegative matrix $C \in \mathbb{R}^{V\times V}$ called *conductance matrix* such that $\mathcal{G}_C = G$. We know that from C we can canonically construct a reversible stochastic matrix $P = D_{C\mathbf{1}}^{-1}C$. We now interpret G as an electrical circuit where edge (u, v) has electrical conductance $C_{uv} = C_{vu}$. We will refer to (G, C) as to an *electrical network*. Consider now a vector $\iota \in \mathbb{R}^V$ such that $\iota^*\mathbf{1} = 0$: We interpret ι_v as the *input current* injected at node v (if negative being an outgoing current). To the electrical network (G, C) and the input current ι, we can associate two functions $W \in \mathbb{R}^V$ (called the *voltage*) and $\phi \in \mathbb{R}^E$ (called the *current flow*) such that the following relations are satisfied

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$$\begin{cases} \sum_{v \in N_u} \phi_{uv} = \iota_u \ \forall u \in V \\ \phi_{uv} = C_{uv}(W_u - W_v) \ \forall (u, v) \in E. \end{cases} \tag{5.1}$$

The first relation is usually known as Kirchoff's law (sum of currents outgoing node u along the edges equals the incoming input current), while the second one is Ohm's law. The existence of a solution (W, ϕ) will follow by our considerations below, as well as uniqueness up to addition to W of multiples of $\mathbf{1}$. Notice that because of Ohm's law, it follows that $\phi_{uv} = -\phi_{vu}$ for all $(u, v) \in E$.

To the aim of rewriting in a more compact form relations (5.1), it is convenient to introduce some additional concepts. Denote by $\bar{E}$ the set of *undirected* edges of G: Namely $\bar{E}$ consists of those subsets $\{u, v\}$ of cardinality 2 such that $(u, v) \in E$ (possible self-loops present in G are disregarded in the construction of $\bar{E}$). An *incidence matrix* on G is any matrix $B \in \{0, +1, -1\}^{\bar{E} \times V}$ such that $B\mathbf{1} = 0$ and $B_{eu} \neq 0$ iff $u \in e$. It is immediate to see that given $e = \{u, v\}$, the e-th row of B has all zeroes except B_{eu} and B_{ev}: Necessarily one of them will be $+1$ and the other one -1 and this will be interpreted as choosing a direction in e from the node corresponding to $+1$ to the one corresponding to -1. Define $D_C \in \mathbb{R}^{\bar{E} \times \bar{E}}$ to be the diagonal matrix such that $(D_C)_{ee} = C_{uv} = C_{vu}$ if $e = \{u, v\}$. Observe that, for every $u \in V$,

$$(B^* D_C B)_{uu} = \sum_{e \in \bar{E}} (D_C)_{ee} B_{eu}^2 = (C\mathbf{1})_u - C_{uu}$$

while, if $u \neq v$,

$$(B^* D_C B)_{uv} = \sum_{e \in \bar{E}} B_{eu} (D_C)_{ee} B_{ev} = -C_{uv}$$

In other terms

$$B^* D_C B = D_{C\mathbf{1}} - C = L(C)\,.$$

In the special case when $C = A_G$ (the adjacency matrix of G), we thus obtain $B^* B = L_G$. Finally, define $\bar{\phi} \in \mathbb{R}^{\bar{E}}$ such that $\bar{\phi}_e = \phi_{uv} B_{eu}$ if $e = \{u, v\}$: According to this definition, $\bar{\phi}_e$ is the current flowing in the edge e, with the positive sign if flow is happening in the same direction of the conventional direction chosen on e by B. We can now rewrite relations (5.1) as

$$\begin{cases} B^* \bar{\phi} = \iota \\ D_C B W = \bar{\phi}. \end{cases} \tag{5.2}$$

These two equations together lead to the following equation for W:

$$L(C) W = \iota. \tag{5.3}$$

Recall that $L(C)$ is a symmetric matrix with rank $L(C) = N - 1$ and $L(C)\mathbf{1} = 0$ (see Chap. 1 for details). It thus admits the spectral representation

$$L(C) = \sum_{i \geq 2} \lambda_i x_i x_i^*,$$

where $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ are the nonzero eigenvalues with corresponding orthonormal eigenvectors $N^{-1/2}\mathbf{1}, x_1, \ldots, x_N$. The matrix

$$Z_C = \sum_{i \geq 2} \lambda_i^{-1} x_i x_i^*$$

is said to be the *Green matrix* associated with C. It has the properties

$$Z_C L(C) = L(C) Z_C = I - N^{-1}\mathbf{1}\mathbf{1}^*, \quad Z_C \mathbf{1} = 0. \tag{5.4}$$

If we consider $W = Z_C \iota$, using the property $\iota^*\mathbf{1} = 0$, we obtain $L(C)W = L(C)Z_C\iota = \iota$: Then, W solves (5.3). Notice that it also satisfies the relation $\mathbf{1}^* W = 0$ and that any other function $W + c\mathbf{1}$ also satisfies (5.3). Since the rank of $L(C)$ is $N - 1$, these are all the possible solutions. All pairs solving (5.3) are thus

$$W = Z_C \iota + c\mathbf{1}, \quad \bar{\phi} = CDBZ_C \iota.$$

We now give some insightful examples of computations of voltages.

Example 5.1 (*Line graph*) Consider the symmetric line graph $G = L_{N+1}$ (with vertex set $V = \{1, \ldots, N+1\}$) and with conductance matrix $C \in \mathbb{R}^{V \times V}$. Let $\iota \in \mathbb{R}^V$ be such that $-\iota_1 = 1 = \iota_{N+1}$ while $\iota_k = 0$ for all $k = 2, \ldots, N$. Using Kirchoff's law and a simple inductive argument, it follows that the current flow $\phi \in \mathbb{R}^{E \times E}$ is given by $\phi_{k,k+1} = -1$ for all $k = 1, \ldots, N$. Ohm's law then yields $W_{k+1} - W_k = C_{k,k+1}$ for all k. This yields $W_k - W_0 = \sum_{j=0}^{k-1} C_{j,j+1}$ In the special case when $C = A_G$ (all edges have conductance equal to 1), we obtain $W_k - W_0 = k$.

Example 5.2 (*Leaves and branches*) Let $G = (V, E)$ be a symmetric graph, C a conductance matrix and $\iota \in \mathbb{R}^V$ an input current (with $\iota^*\mathbf{1} = 0$). Let $v \in V$ be such that $\iota_v = 0$ and $d_v = 1$. Consider the longest path in G, $v_1 = v, v_2, \ldots, v_n$ with the property that $\iota_{v_k} = 0$ and $d_{v_k} = 2$ for all $k = 2, \ldots, n-1$. Since $\iota_{v_1} = 0$, Kirchoff law implies that no current can flow in the edge (v_1, v_2) and a simple inductive argument yields that the same happens in all edges (v_{k-1}, v_k) for $k = 3, \ldots n$. Ohm's law then implies that $W_{v_1} = \cdots = W_{v_n}$.

Example 5.3 (*Toroidal grid*) Consider the toroidal 2-grid $G = C_n \times C_n$ with unitary conductances ($C = A_G$). We know from Example 1.7 that its Laplace matrix $L(G) = L(C)$ has eigenvalues

$$\lambda_{(h,k)} = 4 - 2\cos\left(\frac{2\pi}{n}h\right) - 2\cos\left(\frac{2\pi}{n}k\right) \qquad (h, k) \in \{0, \ldots, n-1\}^2$$

with corresponding eigenvectors $x^{(h,k)}_{(v,w)} = \exp\left[i\frac{2\pi}{n}(vh+wk)\right]$. Therefore, the Green matrix can be represented as

$$(Z_C)_{(v_1,w_1)(v_2,w_2)} = \sum_{(h,k)\neq(0,0)} \frac{\exp\left[i\frac{2\pi}{n}((v_1-v_2)h+(w_1-w_2)k)\right]}{4-2\cos\left(\frac{2\pi}{n}h\right)-2\cos\left(\frac{2\pi}{n}k\right)}$$

If we consider an input current $\iota = e_{(0,0)} - e_{(\alpha,0)}$ (thus, supported on the two nodes (0, 0) and $(\alpha, 0)$), we obtain that the corresponding voltage is given, up to constants, by

$$\begin{aligned} W_{(v,w)} &= (Z_C)_{(v,w)(0,0)} - (Z_C)_{(v,w)(\alpha,0)} \\ &= \sum_{(h,k)\neq(0,0)} \frac{\left[1-\exp\left(-i\frac{2\pi}{n}\alpha h\right)\right]\exp\left[i\frac{2\pi}{n}(vh+wk)\right]}{4-2\cos\left(\frac{2\pi}{n}h\right)-2\cos\left(\frac{2\pi}{n}k\right)}. \end{aligned}$$

A similar explicit (but more complex) formula can be obtained for general Abelian Cayley graphs by applying Proposition 1.18. Another example is reported below.

Example 5.4 (*Hypercube*) Consider the hypercube graph H_n having node set $V = \{0,1\}^n$, defined in Example 1.3. Eigenvalues of the coincide with the numbers $2k$ for $k \in \{0,\dots,n\}$: Eigenvalue $2k$ has multiplicity $\binom{n}{k}$ and corresponding eigenvectors

$$\phi^{(x)}_v = (-1)^{\sum_i x_i v_i}, \quad x, v \in \{0,1\}^n, \sum_i x_i = k$$

Therefore,

$$(Z_C)_{vw} = \sum_{x\in\{0,1\}^n\setminus\{(0,\dots,0)\}} \frac{(-1)^{\sum_i x_i(v_i-w_i)}}{2\sum_i x_i}$$

If we consider an input current $\iota = e_{(0,0,\dots0)} - e_{(1,1,\dots,1)}$, we obtain that the corresponding voltage is given, up to constants, by

$$W_v = \sum_{x\in\{0,1\}^n\setminus\{(0,\dots,0)\}} \frac{(-1)^{\sum_i x_i v_i} - (-1)^{\sum_i x_i(1-v_i)}}{2\sum_i x_i} \tag{5.5}$$

Even though the Green matrix is a useful tool in constructing the theory, its explicit computation can be inconvenient. However, one key advantage of the electrical network approach is that there exist simple and powerful techniques to compute voltages without the need for an explicit computation of the Green matrix. For instance, the following result collects several useful tools that permit to simplify the computation of voltages and current flows by replacing a network by an equivalent simpler one. Preliminarily, notice that also graphs with multiple edges would be appropriate in this context: Kirchoff's and Ohm's law would remain valid and the theory developed so far would directly extend to this case.

Proposition 5.1 *Let (G, C) be an electrical network. Let $\iota \in \mathbb{R}^V$ be such that $\iota^* \mathbf{1} = 0$ and let (W, ϕ) be the corresponding voltage and current flow.*

- **Parallel law**. *Suppose e_1 and e_2 are two edges insisting on the same two vertices u and v. Consider the new electrical network $(\tilde{G}, \tilde{C})$ where $\tilde{G}$ only differs from G as the two edges e_1 and e_2 are replaced by a single edge e with conductance $\tilde{C}_{ee} = C_{e_1e_1} + C_{e_2e_2}$. Then, the voltage and the current flow in $(\tilde{G}, \tilde{C})$, corresponding to the same exogenous input currents ι, coincide with W, ϕ.*
- **Series law**. *Suppose that $v \in V$ is such that $\iota_v = 0$ and $d_v = 2$ with neighbors u_1 and u_2. Consider $(\tilde{G}, \tilde{C})$ where $\tilde{G}$ is a graph on $V \setminus \{v\}$ with same undirected edges as G but $\{u_1, v\}$ and $\{v, u_2\}$ replaced by $\{u_1, u_2\}$, and $\tilde{C}_{u_1u_2} = \tilde{C}_{u_2u_1} = (C^{-1}_{u_1v} + C^{-1}_{vu_2})^{-1}$. The voltage and current flow $\tilde{W}, \tilde{\phi}$ in $(\tilde{G}, \tilde{C})$ satisfy: $\tilde{W}_w = W_w$ for every $w \in V \setminus \{v\}$, $\tilde{\phi}_{w_1w_2} = \phi_{w_1w_2}$ for nodes in $V \setminus \{v\}$ such that $\{w_1, w_2\} \neq \{u_1, u_2\}$, while $\tilde{\phi}_{u_1,u_2} = \phi_{u_1,v} = \phi_{v,u_2}$.*
- **Glueing**. *Suppose that $W_u = W_v$. Consider the new electrical network $(\tilde{G}, \tilde{C})$ where $\tilde{G}$ is obtained from G by glueing together the two nodes u and v, while $\tilde{C} = C$ maintains the same conductances an all edges, and consider the input current $\tilde{\iota}$ defined by*

  $$\tilde{\iota}_w = \iota_w \; \forall w \in V \setminus \{u, v\}, \quad \tilde{\iota}_{u+v} = \iota_u + \iota_v$$

  where $u + v$ denotes the glued node in $\tilde{G}$. Then, the corresponding voltage $\tilde{W}$ and current flow $\tilde{\phi}$ on $(\tilde{G}, C)$ coincide with (W, ϕ), with the only change that $\tilde{W}_{u+v} = W_u = W_v$.

Proof Straightforward check that Kirchoff's and Ohm's laws are satisfied in the new networks. □

We will see in Sects. 5.3 and 5.4 that certain applications require to assign voltages in certain nodes. Below we show how to do it. Notice first of all that (5.3) can be rewritten as

$$L(P)W = D^{-1}_{C\mathbf{1}}\iota,$$

where $P = D^{-1}_{C\mathbf{1}}C$ is the canonical reversible stochastic matrix associated with C. Componentwise, this reads as

$$W_u - \sum_{v \in V} P_{uv} W_v = \frac{\iota_u}{\sum_w C_{uw}}$$

In particular, for each $u \in V$ such that $\iota_u = 0$, it holds that

$$W_u = \sum_{v \in V} P_{uv} W_v. \tag{5.6}$$

A function $W \in \mathbb{R}^V$ satisfying (5.6) (or equivalently (5.3)) for every u belonging to a subset $\tilde{V} \subseteq V$ is said to be *harmonic* on $\tilde{V}$. We have the following result.

Proposition 5.2 (Harmonic extension) *Let $\tilde{V} \subseteq V$ and let $\tilde{W} \in \mathbb{R}^{\tilde{V}}$. Then,*

(i) There exists exactly one $W \in \mathbb{R}^V$ harmonic on $V \setminus \tilde{V}$ and such that $W_{|\tilde{V}} = \tilde{W}$.

(ii) There exists a unique $\iota \in \mathbb{R}^V$ such that $\iota^*\mathbf{1} = 0$ and $\iota_v = 0$ for every $v \notin \tilde{V}$ such that W is the voltage generated by the input current ι.

Proof (i) Order vertices of V in such a way that those in $V \setminus \tilde{V}$ appear first and consider the corresponding block decomposition

$$P = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}.$$

Consider a vector of type $W = (x, \tilde{W})^*$ and impose it satisfies (5.6) for every $u \in V \setminus \tilde{V}$. This is equivalent to require

$$Qx + R\tilde{W} = x. \tag{5.7}$$

Since the graph is strongly connected, it is immediate to check, thanks to Proposition 2.4, that Q is an asymptotically stable sub-stochastic matrix. This implies that $I - Q$ is invertible, and therefore, (5.7) is solved by

$$x = (I - Q)^{-1} R\tilde{W}. \tag{5.8}$$

Consequently, $W = ((I - Q)^{-1} R\tilde{W}, \tilde{W})^*$ is the wanted harmonic extension of $\tilde{W}$.

(ii) It is sufficient to consider $\iota = L(C)W$. □

Remark 5.1 (*Harmonic extension is convex combination of assigned voltages*) The matrix $(I - Q)^{-1}R$ that appears in (5.8) has some important properties which we now discuss. First, notice that the inverse of $I - Q$ can be represented as a series $(I - Q)^{-1} = \sum_{n=0}^{\infty} Q^k$, and this implies that $(I - Q)^{-1}_{uv} \geq 0$ for all $u, v \in V \setminus \tilde{V}$. Since also R is nonnegative it follows that $(I - Q)^{-1}R$ is also nonnegative. Moreover, notice that since P is stochastic it holds $(I - Q)\mathbf{1} + R\mathbf{1} = 0$. But this implies that $(I - Q)^{-1}R\mathbf{1} = \mathbf{1}$. In other words, each row of $(I - Q)^{-1}R$ sums to 1. In general, however, we cannot say that $(I - Q)^{-1}R$ is a stochastic matrix since it is not a square matrix.

Example 5.5 (*Binary trees*) Consider a binary tree of depth t (see Fig. 5.1) with unitary conductances and assume that the root node v_0 is at voltage 0, while the 2^t leaves are at voltage 1. For symmetry reasons, all nodes at distance s from the root node have the same voltage, and thus, we can replace, by the glueing and parallel laws in Proposition 5.1, the network with an equivalent line graph L_{t+1} with set of nodes $\{v_0, v_1, \dots v_t\}$ and conductance matrix $C_{v_s v_{s+1}} = 2^{s+1}$ for $s = 0, \dots, t-1$. This implies that the total resistance between v_0 and v_t is given by

$$\sum_{s=0}^{t-1} \frac{1}{2^{s+1}} = 1 - 2^{-t}$$

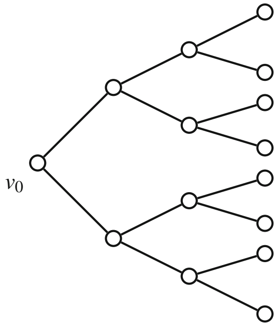

Fig. 5.1 A binary tree of depth 3: its root is labeled as v_0

The current along the line graph is thus by Ohm's law $(1 - 2^{-t})^{-1}$. Voltages at the various nodes can now be simply obtained by applying again Ohm's law:

$$W_{v_{s+1}} - W_{v_s} = (1 - 2^{-t})^{-1} 2^{-(s+1)}$$

and thus,

$$W_s = (1 - 2^{-t})^{-1} \sum_{k=0}^{s-1} 2^{-(k+1)} = \frac{1 - 2^{-s}}{1 - 2^{-t}}$$

Example 5.6 (*Voltages in barbell graphs*) Consider two complete graphs $K_i = (V_i, E_i)$ $(i = 1, 2)$ on the set of nodes V_i and unitary conductances. Fix two nodes $v_i \in V_i$ and consider the barbell graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{(v_1, v_2), (v_2, v_1)\}$. Consider two nodes $s_i \in (V_i \setminus v_i)$ and assign voltages $W_{s_1} = 0$ and $W_{s_2} = 1$. We want to compute the harmonic extension of W. For symmetry reasons, W will be constant at all nodes in $V_i \setminus \{s_i, v_i\}$ for $i = 1, 2$. By the glueing property and the parallel law, the electrical network can thus be replaced by a line with six nodes $s_1, w_1, v_1, v_2, w_2, s_2$ such that

$$C_{s_i,w_i} = C_{w_i,v_i} = N_i - 2\,,\ C_{v_1,v_2} = 1$$

where $N_i = |V_i|$. In order to compute the current, we can use the series law further reducing the electrical network to a single edge between s_1 and s_2 of conductance $[2(N_1 - 2)^{-1} + 2(N_2 - 2)^{-1} + 1]^{-1}$: The current coincides with the conductance. Using now Ohm's law we obtain

$$W_{w_1} = \frac{(N_1-2)^{-1}}{2(N_1-2)^{-1}+2(N_2-2)^{-1}+1}\,,\ W_{v_1} = \frac{(2(N_1-2))^{-1}}{2(N_1-2)^{-1}+2(N_2-2)^{-1}+1}$$
$$W_{v_2} = \frac{2(N_1-2)^{-1}+1}{2(N_1-2)^{-1}+2(N_2-2)^{-1}+1}\,,\ W_{w_2} = \frac{2(N_1-2)^{-1}+(N_2-2)^{-1}+1}{2(N_1-2)^{-1}+2(N_2-2)^{-1}+1}$$

5.2 Effective Resistance in Electrical Networks

A very useful concept in dealing with electrical networks is that of effective resistance between nodes. Formally, given an electrical network (G, C) and two nodes $u, v \in V$, we consider the input current $\iota = e_u - e_v$. The corresponding voltage up to translation is denoted by W, and the *effective resistance* between u and v is defined by

$$R_{\text{eff}}(u, v) := W_u - W_v.$$

The *average effective resistance* in the network is then defined as

$$R_{\text{ave}}(G, C) := \frac{1}{2N^2} \sum_{u,v \in V} R_{\text{eff}}(u, v).$$

The average effective resistance can be used as a measure of graph connectivity, in the sense that "well-connected" graphs will have small R_{ave}. We will return to this interpretation in Sect. 5.4 and in the Exercises.

Effective resistances can be characterized in terms of the Green matrix. Indeed, recalling that $W = Z_C \iota$, it holds

$$R_{\text{eff}}(u, v) = (e_u - e_v)^* Z_C (e_u - e_v) = (Z_C)_{uu} - 2(Z_C)_{uv} + (Z_C)_{vv} \tag{5.9}$$

Therefore, recalling that $Z_C \mathbf{1} = 0$, we also have

$$R_{\text{ave}}(G, C) = \frac{1}{2N^2} \sum_{u,v \in V} R_{\text{eff}}(u, v) = \frac{1}{N} \operatorname{tr}(Z_C) = \frac{1}{N} \sum_{i \geq 2} \frac{1}{\lambda_i}, \tag{5.10}$$

where $0 = \lambda_1, \dots, \lambda_N$ are the eigenvalues of $L(C)$.

Example 5.7 (*Effective resistance on line graphs*) Consider the symmetric line graph $G = L_{N+1}$ (with vertex set $V = \{1, \dots, N+1\}$) with conductance matrix $C \in \mathbb{R}^{V \times V}$. It immediately follows from Example 5.1 that

$$R_{\text{eff}}(1, N+1) = W_{N+1} - W_1 = \sum_{k=1}^{N} C_{kk+1}.$$

In the special case when $C = A_G$ (all edges have conductance equal to 1), we obtain $R_{\text{eff}}(1, N+1) = N$.

Example 5.8 (*Effective resistance on trees*) Let $G = (V, E)$ be a tree, C a conductance matrix and $\iota \in \mathbb{R}^V$ an input current such that $\iota_v = 1 = -\iota_w$ while $\iota_u = 0$ for every $u \neq v, w$. Consider the only path $v = v_1, \dots, v_{N+1} = w$ connecting v to w in G. From Example 5.2 and a repetition of glueing operations, it is immediate to check that all edges not contained in this path will have a current flow equal to 0. Consequently, a current flow equal to 1 will be flowing from v to w along the connecting path as if it was a line graph. From Example 5.7, it thus follows that $R_{\text{eff}}(v, w) = \sum_{k=1}^{N} C_{v_k v_{k+1}}$ In the special case when $C = A_G$, it follows that the effective resistance between two nodes coincides with their distance on the tree. Notice that it is also easy to compute the corresponding voltage at any vertex of the tree. Given a vertex $u \in V$, let v_k be the closest vertex of the path $v = v_1, \ldots, v_{N+1} = w$ to u. Then, $W_u = W_{v_k}$.

Example 5.9 (*Effective resistance on cycles*) Consider the graph C_n with node set $\mathbb{Z}_n$. By applying the parallel law and Example 5.7, we observe

$$R_{\text{eff}}(u, v) = \left(|v-u|^{-1} + (n - |v-u|)^{-1}\right)^{-1} = \frac{|v-u|\,(n-|v-u|)}{n}.$$

For general graphs, the computation of the effective resistance can be a complex problem and closed formulas can hardly be found. However, there are tools to efficiently estimate it. Before we can illustrate them, we need to introduce a further concept. Given an electrical circuit $(G = (V, E), C)$, a *flow* on it is any function $\phi \in \mathbb{R}^E$ such that $\phi_{uv} = -\phi_{vu}$. As before $\bar{E}$ will denote the set of undirected edges of G. Given an incidence matrix B of G, we can consider the flow defined on $\bar{E}$ and denote it by $\bar{\phi}$ (as we did for the current flow above). The *energy* of a flow ϕ is defined as

$$|\phi| = (1/2) \sum_{(uv)\in E} \frac{\phi_{uv}^2}{C_{uv}} = \sum_{\{uv\}\in \bar{E}} \frac{\bar{\phi}_{\{uv\}}^2}{C_{uv}}$$

Given $\iota \in \mathbb{R}^V$ such that $\iota^* \mathbf{1} = 0$, we say that a flow ϕ is compatible with ι if it satisfies Kirchoff's law $B^* \bar{\phi} = \iota$. The following variational principle holds true (a proof can be found in [20, Theorem 9.10]):

Lemma 5.1 (Thomson's principle) *Let (G, C) be an electrical network. Then,*

$$R_{\text{eff}}(u, v) = \inf\{|\phi| \;:\; \phi \textit{ is a flow compatible with } \iota = e_u - e_v\}.$$

Moreover the unique minimizer is the current flow induced by the input current $\iota = e_u - e_v$.

An immediate important consequence is the following result.

Corollary 5.1 (Raileigh's monotonicity law) *Let G be a symmetric graph and C' and C'' two conductance matrices on G such that $C'_{uv} \le C''_{uv}$ for all $u, v \in V$. Then, for any pair of vertices the corresponding effective resistances in the two networks satisfy*

$$R'_{\text{eff}}(u, v) \ge R''_{\text{eff}}(u, v).$$

Glueing nodes in an electrical network is equivalent to put conductance equal to ∞ between certain pairs of nodes. By virtue of Raileigh monotonicity law, this implies that the effective resistance, in the glueing operation, can never increase. The following is an example of application of this useful remark.

Example 5.10 (*Effective resistance on grids*) Consider a bidimensional grid $L_n \times L_n$ with set of nodes $\{1, \ldots, n\}^2$ and unitary conductances. Suppose we want to estimate the effective resistance between $(1, 1)$ and (n, n). Replace the network by a line network obtained by glueing together all nodes at distance d from $(1, 1)$ (and denote such super node by v_d). The nodes $(1, 1)$ and (n, n) become v_0 and v_{2n-2} in the new network. Let n_d be the number of nodes at distance d from $(0, 0)$. Since (x, y) is at distance d from $(0, 0)$ if and only if $x + y = d + 2$, we have that $n_d = d + 1$ if $d \le n - 1$. It follows that $C_{v_d v_{d+1}} = 2(d + 1)$ for all $d = 0, \ldots, n - 2$. Considering that the new network is specularly symmetric with respect to the node v_{n-1}, we have that

$$R_{\text{eff}}((1, 1), (n, n)) \ge R_{\text{eff}}(v_0, v_d) = \sum_{d=0}^{n-2} \frac{1}{d+1} \ge \int_1^n \frac{1}{x} \mathrm{d}x = \log n.$$

By constructing suitable flows and applying Thompson's principle, it can be shown that that it also holds $R_{\text{eff}}((1, 1), (n, n)) \le 2 \log n$. In case of grids of higher dimension L_n^d, instead, there exists $c_d > 0$ such that $R_{\text{eff}}(v, w) \le c_d$ for all $v, w \in \{1, \ldots, n\}^d$. More details can be found in [20].

When voltages are imposed at some nodes, all other voltages can be computed in terms of effective resistances.

Proposition 5.3 (Voltages and effective resistance) *Let (G, C) be an electrical network and v_0 and v_1 two distinct nodes in G. Let W be the voltage satisfying $W_{v_0} = 0$ and $W_{v_1} = 1$. Then,*

$$W_v = \frac{1}{2} + \frac{R_{\text{eff}}(v, v_0) - R_{\text{eff}}(v, v_1)}{2R_{\text{eff}}(v_0, v_1)} \quad \forall v \in V. \tag{5.11}$$

Proof Clearly, we can represent $W = Z_C \iota + c\mathbf{1}$ where ι is the input current given by $\iota = R_{\text{eff}}(v_0, v_1)^{-1}[e_{v_1} - e_{v_0}]$ and c a constant. Imposing $W_{v_0} = 0$, we obtain that $c = R_{\text{eff}}(v_0, v_1)^{-1}[(Z_C)_{v_0 v_0} - (Z_C)_{v_0 v_1}]$. For a generic $v \in V$, the voltage can thus be computed as follows (denoting Z_C as Z for conciseness):

$$\begin{aligned} W_v =& \frac{Z_{vv_1} - Z_{vv_0} + Z_{v_0 v_0} - Z_{v_0 v_1}}{R_{\text{eff}}(v_0, v_1)} \\ =& \frac{(-Z_{vv} + 2Z_{vv_1} - Z_{v_1 v_1}) + (Z_{vv} - 2Z_{vv_0} + Z_{v_0 v_0}) + (Z_{v_0 v_0} - 2Z_{v_0 v_1} + Z_{v_1 v_1})}{2R_{\text{eff}}(v_0, v_1)}. \end{aligned}$$

The result now follows from relation (5.9). □

We now present a significant example.

Example 5.11 (*Harmonic functions on line graphs*) Consider the symmetric line graph $G = L_{n+1}$ with vertex set $V = \{1, \ldots, n + 1\}$ and with conductance matrix

$C = A_G$. Let $\tilde{V} = \{1, n+1\}$ and put $\tilde{W}_1 = 0$ and $\tilde{W}_{n+1} = 1$. Let W be the harmonic extension of $\tilde{W}$. Using formula (5.11), we immediately obtain

$$W_k = \frac{1}{2}\frac{(k-1) - (n+1-k) + n}{n} = \frac{k-1}{n}, \ \forall k.$$

Notice that if the voltage assigned in nodes 1 and $n+1$ were different, namely $\tilde{W}_1 = \tilde{w}_1$ and $\tilde{W}_{n+1} = \tilde{w}_{n+1}$, then using the fact that the harmonic extension is a linear function of the boundary conditions, we would obtain the new voltage

$$\tilde{W}_k = \tilde{w}_1 + \frac{k-1}{n}(\tilde{w}_{n+1} - \tilde{w}_1).$$

Furthermore, it is important to be aware that Proposition 5.3 can be applied when the two nodes v_0 and v_1 are the outcome of glueing operations. Hence, its scope of application covers all cases where some nodes are connected to any *two* voltage levels.

5.3 Averaging Dynamics with Stubborn Agents

In the examples of consensus models studied so far, we have essentially assumed that all the agents are implementing the same dynamic law, all of them cooperating to reach a consensus. Very interesting models can however be obtained considering instead heterogeneous model where agents have different behaviors. As a special case, here we investigate the case when some of the agents maintain fixed initial state. These agents will be called *stubborn*. Several interpretations are possible. In robotic networks, these agents can be interpreted as leaders who are trying to keep the rest of the units within a certain region: In this context, we talk about the *containment* problem. In the context of opinion dynamics, stubborn agents play the role of opinion leaders or influencers.

Let $G = (V, E)$ be a strongly connected aperiodic graph endowed with a stochastic matrix $P \in \mathbb{R}^{V\times V}$ such that $G_P = G$. Consider the split $V = V^\ell \cup V^f$ with the understanding that agents in V^ℓ are the *leaders* while those in V^f are the *followers*. The dynamics we want to consider is given by the modified stochastic matrix $\tilde{P} \in \mathbb{R}^{V\times V}$ defined by

$$\tilde{P}_{uv} = \begin{cases} P_{uv} \text{ if } u \in V^f, \ v \in V \\ \delta_{uv} \ \text{ if } u \in V^\ell \end{cases}$$

If $V^\ell = \{v^\star\}$, the node $v^\star$ will be globally reachable and aperiodic for the graph $G_{\tilde{P}}$. Therefore, thanks to Theorem 2.2, there will be convergence to a consensus. Because of Proposition 2.5, it follows that the corresponding invariant probability for P will be $\pi = \delta_{v^\star}$ and therefore

$$\tilde{P}^t x(0) \to \mathbf{1}x(0)_{v^\star}$$

In other terms, consensus coincides with the initial (unchanged) state of the unique leader $v^\star$. If $|V^\ell| > 1$, the graph $G_{\tilde{P}}$ will not possess a globally reachable vertex, and therefore, consensus will not in general be achievable. Nevertheless, we would like to understand the behavior of $\tilde{P}^t x(0)$ for $t \to +\infty$.

If we order elements in V in such a way that followers come first, the matrix $\tilde{P}$ will have the block structure:

$$\tilde{P} = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

where $Q \in \mathbb{R}^{V^f \times V^f}$, $R \in \mathbb{R}^{V^f \times V^\ell}$, and where $I \in \mathbb{R}^{V^\ell \times V^\ell}$ is the identity matrix. If we split accordingly the state vector $x(t) = (x^f(t), x^\ell(t) \in \mathbb{R}^V$, we thus have dynamics

$$\begin{aligned} x^f(t+1) &= Qx^f(t) + Rx^\ell(t) \\ x^\ell(t+1) &= x^\ell(t) \end{aligned} \tag{5.12}$$

By the assumption made, it follows that Q is sub-stochastic satisfying the assumptions of Proposition 2.4. Hence, Q is asymptotically stable. These dynamics easily imply that $x^f(t)$ converges to $x^f(\infty) \in \mathbb{R}^{V^f}$ determined by the fixed point relation:

$$x^f(\infty) = Qx^f(\infty) + Rx^\ell(0)$$

which is equivalent to

$$(I - Q)x^f(\infty) = Rx^\ell$$

or since $I - Q$ is invertible, to

$$x^f(\infty) = (I - Q)^{-1}Rx^\ell \tag{5.13}$$

where $x^\ell = x^\ell(0)$. Notice in particular that the initial condition of the state of the followers, $x^f(0)$ does not play any role in the final state. If all the leaders share the same state, $x^\ell_v = c$ for every $v \in V^\ell$, then it is immediate to check that $x^f(\infty)_v = c$ for every $v \in V^f$, namely, they reach consensus. In general, however, $x_f(\infty)$ is not a consensus state. If we confront the formula for $x^f(\infty)$ above with (5.8), we see that, indeed, $x^f(\infty)$ can be interpreted as the harmonic extension of the leader assignment x^ℓ. If the original matrix P is a reversible stochastic matrix, we can then apply all the machinery from electrical networks for computing the vector $x^f(\infty)$.

Example 5.12 Consider the graph L_{N+1} with vertex set $\{1, \dots, N+1\}$ and leader nodes $\ell_1 = 1$ and $\ell_2 = N+1$. Let P be the SRW on the follower nodes. It follows from Example 5.11 that the followers' limit state is given by

$$x_k^f(\infty) = \frac{x_{\ell_2}^\ell - x_{\ell_1}^\ell}{N}(k-1) + x_{\ell_1}^\ell.$$

This formula also shows that there is consensus if and only if $x_{\ell_1}^\ell = x_{\ell_2}^\ell$.

Remark 5.2 (*Connectivity and influence*) Notice that $\left((I-Q)^{-1}R\right)_{hk} = \sum_n \left(Q^n R\right)_{hk}$ is not equal to 0 if and only if there exists a path in the graph connecting the follower h to the leader k. This implies that if a follower h can reach leader k only, then $x_h^f(\infty) = x_k^\ell(0)$.

Remark 5.3 (*Multi-dimensional case*) If the evolving state of each unit $x_v(t)$ is a vector (e.g., in $\mathbb{R}^2$ or $\mathbb{R}^3$) possibly indicating positions, the considerations above remain valid with the usual interpretation of the matrix multiplications done in previous chapter. Relation (5.13), in this case, has an even more vivid geometric representation. It says that the asymptotic state of each follower is a convex combination of the state of the leaders, in other words, each follower will eventually stay in the convex polyhedron whose vertices are the states of the leaders.

5.4 Estimation from Relative Measurements

Consider a set of agents V and a symmetric connected graph $G = (V, E)$. Each agent v possesses an attribute $\bar{x}_v \in \mathbb{R}^q$ (to be interpreted as position or quality, for instance) which is unknown to the agent itself. Any pair of agents $v, w \in V$, connected by an edge in G, make a cooperative measurements of their relative position

$$b_{\{v,w\}} = \bar{x}_v - \bar{x}_w + n_{\{v,w\}}, \tag{5.14}$$

where $n_{\{v,w\}}$ is a random variable modeling the measurement noise. We will assume that random variables are independent and identically distributed with mean 0 and variance σ^2. Also, we will assume that $q = 1$: This choice does not entail any loss of generality as the case of $q > 1$ can be captured working componentwise.

Notice that the measurements and the noises are naturally defined on the set of undirected edges $\bar{E}$ of the graph $b, n \in \mathbb{R}^{\bar{E}}$. However, also notice that the measurement model (5.14) assumes that a direction has been decided at the level of each pair v, w. If we consider the incidence matrix B of G corresponding to such chosen directions (e.g., $B_{\{v,w\}v} = 1$ in reference to (5.14)), then we can rewrite relations (5.14) in a more compact form as

$$b = B\bar{x} + n. \tag{5.15}$$

On the basis of the available measurements b, the goal is to obtain an estimate $\hat{x}$ of $\bar{x}$. A classical solution is the so called *least squares estimator*, defined as

$$\hat{x} := \operatorname{argmin}_{x \in \mathbb{R}^V} ||Bx - b||_2^2 \tag{5.16}$$

Notice that for sure the above minimum is not unique as $B\mathbf{1} = 0$. Indeed, notice that any translation of all real positions $\bar{x}$ of a vector $c\mathbf{1}$ would not change the vector b: In other terms, estimation can only be achieved modulo a translation addend. We will see below that this is the only "freedom" in the system so that (5.16) is uniquely defined up to this translation.

Remark 5.4 (*Maximum likelihood interpretation*) In the case when the variables $n_{\{v,w\}}$ are Gaussian, the least squares estimator coincide with the classical Maximum Likelihood (ML) estimator. Indeed, the density function of $b = Bx + n$ assuming x to be a parameter is given by

$$f(b|x) = \frac{1}{(2\pi\sigma^2)^{|\bar{E}|/2}} e^{-\frac{||b-Bx||_2^2}{(2\sigma^2)^{|\bar{E}|}}}$$

and therefore, the ML estimator is given by

$$\hat{x}_{\mathrm{ML}} := \mathrm{argmax}_{x\in\mathbb{R}^V} f(b|x) = \mathrm{argmin}_{x\in\mathbb{R}^V} ||Bx - b||_2^2$$

Consider the functional

$$J(x) = ||Bx - b||_2^2, \tag{5.17}$$

which is what we want to minimize. Notice that $J(x) = x^* B^* Bx - 2b^* Bx + ||b||_2^2$ is indeed a convex quadratic function and its minima coincide with its stationary points. Since its gradient is given by $\nabla J(x) = 2B^* Bx - 2B^* x$, its minima are the solutions of the equation $L_G x = B^* b$ (recall that $B^* B = L_G$). This equation is the voltage equation in the electrical network (G, A_G) and with input current $\iota = B^* b$ (notice that $\iota^* \mathbf{1} = b^* B\mathbf{1} = 0$ as required). Solutions are then given by

$$\hat{x} = Z_G B^* b + c\mathbf{1}, \tag{5.18}$$

provided we denote $Z_G = Z_{A_G}$. Notice that, since $\mathbf{1}^* Z_G = 0$, it follows that the solution $\hat{x} = Z_G B^* b$ is the (only) one satisfying $\mathbf{1}^* \hat{x} = 0$ (barycenter in the origin).

Remark 5.5 (*Trees*) In the special case when G is a tree, notice that B is an $(N-1) \times N$-matrix having rank equal to $N - 1$. Therefore, B is onto and there must exist $\hat{x}$ satisfying $B\hat{x} = b$; this is for sure a minimizer of $J(x)$, hence it must coincide with the solution (5.18). This implies that must hold $BZ_G B^* = I$.

We now want to study the performance of the least squares estimator. Particularly, we are interested in evaluating the effects of the noise and the topology of the graph. A natural performance measure is the minimum mean quadratic error, which considering the nonuniqueness of the solution, takes the form

$$J_{\mathrm{rel}} = \frac{1}{N}\mathbb{E}\|\hat{x} - \bar{x}\|_2^2 := \frac{1}{N}\min_{c\in\mathbb{R}} \mathbb{E}||(Z_G B^* b + c\mathbf{1}) - \bar{x}\|_2^2, \tag{5.19}$$

where the expectation is taken over the noise n. Notice that the optimal c is given by $c = N^{-1}\mathbf{1}^*\bar{x}$, which corresponds to the estimate $\hat{x}$ having the same barycenter of $\bar{x}$.

The following simple result shows that the mean-square error of the least squares solution depends only on the variance of the noise and on the topology of the graph through the Green matrix of the graph:

Proposition 5.4 (MSE formula) *Provided the undirected graph G is connected, cost* (5.19) *can be computed as*

$$J_{\mathrm{rel}} = \frac{\sigma^2}{N}\,\mathrm{tr}(Z_G). \tag{5.20}$$

Proof We compute as follows

$$\begin{aligned}
J_{\mathrm{rel}} &= \tfrac{1}{N}\min_{c\in\mathbb{R}} \mathbb{E}||(Z_G B^* b + c\mathbf{1}) - \bar{x}\|_2^2 \\
&= \tfrac{1}{N}\min_{c\in\mathbb{R}} \mathbb{E}||Z_G L_G \bar{x} - \bar{x} + c\mathbf{1} + Z_G B^* n||_2^2 \\
&= \tfrac{1}{N}\min_{c\in\mathbb{R}} \mathbb{E}||\mathbf{1}(-N^{-1}\mathbf{1}^*\bar{x} + c) + Z_G B^* n||_2^2 \\
&= \tfrac{1}{N}\min_{c\in\mathbb{R}} \mathbb{E}\left[N(-N^{-1}\mathbf{1}^*\bar{x} + c) + n^* B Z_G^2 B^* n\right] \\
&= \tfrac{1}{N}\mathbb{E}[n^* B Z_G^2 B^* n] = \tfrac{1}{N}\mathbb{E}\,\mathrm{tr}[Z_G B^* n n^* B Z_G] = \tfrac{1}{N}\,\mathrm{tr}[Z_G B^* \mathbb{E}[nn^*] B Z_G] = \tfrac{\sigma^2}{N}\,\mathrm{tr}(Z_G),
\end{aligned}$$

by using (5.4) and recalling $N^{-1}\mathbf{1}^*\bar{x} = c$. □

It follows from (5.10) that

$$J_{\mathrm{rel}} = \sigma^2 R_{\mathrm{ave}}(G, A_G).$$

Whenever we are able to compute or estimate the effective resistance in a graph, we will be able to estimate the performance of the mean-square position estimator. In particular, it follows from Examples 5.7 and 5.10 that for d-dimensional grids

$$J_{\mathrm{rel}} = \begin{cases} \Theta(N) & \text{for } d = 1 \\ \Theta(\log N) & \text{for } d = 2 \\ \Theta(1) & \text{for } d > 2 \end{cases} \tag{5.21}$$

as $N \to \infty$. We thus have strikingly different behaviors of the algorithm for $d \leq 2$ and $d > 2$, as in the first case performance degrades as N increases. A similarly poor performance affects trees: Example 5.8 implies that J_{rel} is linear in the diameter on trees with bounded degrees. This fact means that, even though the tree structure is sufficient to estimate absolute distances (as explained in Remark 5.5), the availability of additional measurements is essential to obtain good performance.

As shown by (5.18), the solution to the optimization problem (5.16) is easily obtained analytically. However, it is not immediately clear whether in practice such a solution can be computed in a distributed fashion by the nodes. We show now that the answer is positive, by presenting a distributed algorithm that allows each node v to compute its own component of the estimate $\hat{x}_v$.

Let us recall that $J(x)$ as defined in (5.17) is a convex function: Then its minima can be found by an iterative gradient descent algorithm. Let $x(t) \in \mathbb{R}^V$ be the vector of node estimates at iteration t. Then, we consider the following algorithm

$$x(t+1) = x(t) - \tau \nabla J(x(t)),$$

with $\tau > 0$ to be determined in order to ensure convergence. The recursive law can be rewritten as:

$$\begin{aligned} x(t+1) &= x(t) - \tau(L_G x(t) - B^* b) \\ &= (I - \tau L_G)x(t) + \tau B^* b \end{aligned}$$

Defining $P := I - \tau L_G \in \mathbb{R}^{V \times V}$ and $y := \tau B^* b \in \mathbb{R}^I$, we obtain the compact form

$$x(t+1) = Px(t) + y. \tag{5.22}$$

It is of note that the matrix P is inherently adapted to the measurement graph G, in the sense that $P_{uv} > 0$ only if (u, v) is an edge in G. This observation is key as it implies that the algorithm is naturally distributed over the graph which describes the problem, that is, there is *no need for communication between agents which do not share a measurement*.

The convergence properties of the algorithm are summarized in the next result.

Proposition 5.5 (Convergence) *Let G be symmetric and strongly connected. Choose τ such that $0 < \tau < \frac{1}{d_{\max}}$, where $d_{\max}$ denotes the largest degree in G. Then, the algorithm (5.22) is such that*

$$\lim_{t \to +\infty} x(t) = \hat{x},$$

where $\hat{x}$ is the solution in (5.18) characterized by the condition $\frac{1}{N}\mathbf{1}^*\hat{x} = \frac{1}{N}\mathbf{1}^* x(0)$.

Proof From the assumption on τ it follows that P is a symmetric, irreducible, aperiodic stochastic matrix. Then, we know from Chap. 2 that 1 is a simple eigenvalue whose eigenspace is spanned by $\mathbf{1}$, while all other eigenvalues are, in modulus, strictly less than 1. Since $\mathbf{1}^* y = 0$, it easily follows that $x(t)$ converges to a solution of the equation $x = Px + y$. Substituting the expression of P, we immediately get the result. Invariance of the barycenter simply follows by applying $\mathbf{1}^*$ to both sides of (5.22). □

We observe that, given an initial condition $x(0)$, the algorithm converges to a corresponding solution $\hat{x}$, specifically that one with the same average as $x(0)$. Then, in order to converge to the best solution, it is necessary to impose the same average of $\bar{x}$ to the initial condition $x(0)$.

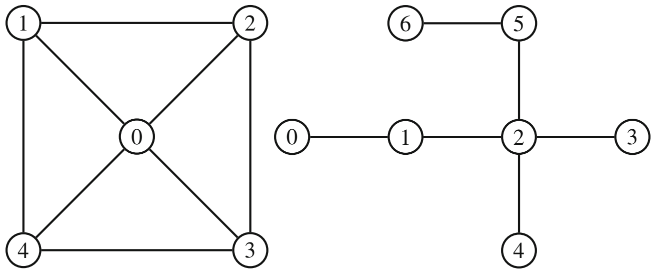

Fig. 5.2 Graphs G_1 and G_2 for Exercise 5.2

Exercises

Electrical Networks

Exercise 5.1 (*Notions of incidence matrix*) Compare the definition of incidence matrix given at the beginning of this chapter for undirected graphs and the notion defined in Exercise 1.12 for general weighted graphs.

Exercise 5.2 (*Potentials on small graphs*) Consider the graphs is Fig. 5.2. Compute the voltages

(i) W' defined on G_1 such that $W'_0 = 0$ and $W'_1 = 1$;
(ii) W'' defined on G_1 such that $W''_0 = 3$ and $W''_1 = 1$;
(iii) W''' defined on G_2 such that $W'''_0 = 0$ and $W'''_3 = 1$ and $W'''_6 = 2$.

Exercise 5.3 (*Voltages on a hypercube*) Consider the hypercube graph H_n having node set $V = \{0, 1\}^n$, defined in Example 1.3, with unitary conductances. Consider an input current $\iota = e_{(0,0,\ldots,0)} - e_{(1,1,\ldots,1)}$. Voltages at various nodes can be computed using the following method which is alternative to Example 5.4. First notice that for symmetry reasons, nodes at a certain distance d from $(0, 0, \ldots 0)$ will all have the same voltage. Transform consequently the electrical network into an equivalent line and compute voltages. Show that you obtain the same result as formula (5.5).

Exercise 5.4 (*Effective resistance is a distance*) Verify that the effective resistance satisfies the axioms (recalled in Exercise 1.11) to be a metric on the set of nodes of an electrical network.

Exercise 5.5 (*Average effective resistance*) Using either (5.20) or an "electrical" argument, compute $R_{\text{ave}}(G)$ of the following graphs (assume for simplicity to have unitary conductances):

(i) complete graph K_N;
(ii) cycle graph C_N;
(iii) complete bipartite graph K_{N_1,N_2};

(iv) hypercube graph H_k;
(v) barbell graph as defined in Example 5.6;
(vi) (toroidal) grid of dimension d, thereby proving (5.21).

Exercise 5.6 (*Effective resistance on binary tree*) Consider a binary tree of depth n with edges of unit conductance. Compute the effective resistance between the root and the leaves (glued together).

Exercise 5.7 (*Foster's equality*) Let the conductance matrix C have unit entries. Show that $\sum_{\{u,v\}\in \bar{E}} R_{\text{eff}}(u,v) = |V| - 1$.

Exercise 5.8 (*Green matrix of a stochastic matrix*) For any aperiodic irreducible stochastic matrix P having invariant probability π, the Green matrix can be defined as

$$Z_P := \sum_{t=0}^{+\infty}(P^t - \mathbf{1}\pi^*).$$

Show that $Z_P L(P) = L(P) Z_P = I - \mathbf{1}\pi^*$ and $(Z_P + \mathbf{1}\pi^*) = (L(P) + \mathbf{1}\pi^*)^{-1}$.

Exercise 5.9 (*J_x cost on symmetric matrices*) Let the N-dimensional stochastic irreducible aperiodic matrix P be symmetric and recall the cost

$$J_x(P) = \frac{1}{N}\operatorname{tr}\sum_{t\geq 0}(P^{2t} - \frac{1}{N}\mathbf{1}\mathbf{1}^*)$$

defined in (4.7). Verify that, following the notation from Exercise 5.8,

$$J_x(P) = \frac{1}{N}\operatorname{tr} Z_{P^2} = \frac{1}{N}\sum_{i=2}^{N}\left(1 - \mu_i(P^2)\right)^{-1},$$

and thus by virtue of (5.10)

$$J_x(P) = R_{\text{ave}}(\mathcal{G}_{P^2}, P^2).$$

Exercise 5.10 (*J_x cost on reversible matrices*) This exercise extends Exercise 5.9 to *reversible* irreducible aperiodic matrices. In that case, the cost takes the form $J_x(P) = \frac{1}{N}\sum_{t\geq 0}||P^t - \mathbf{1}\pi^*||_F^2$. For a reversible P, we can define the associated conductance matrix as

$$\Phi(P) = N \operatorname{diag}(\pi) P.$$

(i) Verify that $\Phi(P)$ is symmetric and $\mathbf{1}^*\Phi(P)\mathbf{1} = N$.
(ii) Verify that if C is a conductance matrix, then $\Phi(D_{C\mathbf{1}}^{-1}C) = (\mathbf{1}^*C\mathbf{1})^{-1} N C$.
(iii) [22, Theorems 3.1 and 3.2] If we assume that P is reversible and irreducible with positive diagonal, we let $D = \Phi(P^2)$ and we denote by A the adjacency matrix of $\mathcal{G}_P$, then

$$\frac{N^2\pi_{\min}^3}{\pi_{\max}} R_{\text{ave}}(\mathcal{G}_D, D) \le J_x(P) \le \frac{N^2\pi_{\max}^3}{\pi_{\min}} R_{\text{ave}}(\mathcal{G}_D, D),$$

and

$$\frac{N\pi_{\min}^3}{8p_{\max}^2 d_{\max}^2 \pi_{\max}^2} R_{\text{ave}}(\mathcal{G}_P, A) \le J_x(P) \le \frac{N\pi_{\max}^3}{8p_{\min}^2 \pi_{\min}^2} R_{\text{ave}}(\mathcal{G}_P, A),$$

where $\pi_{\min} \le \pi_v \le \pi_{\max}$ and $d_v \le d_{\max}$ for every node v and $p_{\min} \le P_{uv} \le p_{\max}$ for every $P_{uv} > 0$.

Exercise 5.11 (*J_x cost on example graphs*) Derive the scaling of J_x for the symmetric random walk matrix on the graphs of Exercise 5.5. To this goal, you can apply Exercises 5.9 or 5.10 depending on the graph.

Consensus with stubborn agents

Exercise 5.12 (*Asymptotic followers state*) Consider the two graphs (with leader and follower nodes) in Fig. 5.3, and the matrices corresponding to the Simple Random Walks on them.

(i) Referring to the graph in Fig. 5.3 (left), compute the limit states of the followers as a function of the states of the leaders l_0, l_1, l_2, l_3, l_4, as time goes to infinity.
(ii) Referring to the graph in Fig. 5.3 (right), compute the limit states of the followers as a function of the states of the leaders l_0 and $l_1 = l_2 = l_3 = l_4$. (Hint: take advantage of the symmetries to reduce the number of unknowns).

Exercise 5.13 (*Asymptotic followers state on trees*) Compute $x^f(\infty)$ as a function of $x^\ell(0)$ for the SRW on the trees with stubborn nodes in Fig. 5.4 Generalize the second graph when the lines departing from the common central node have any length n.

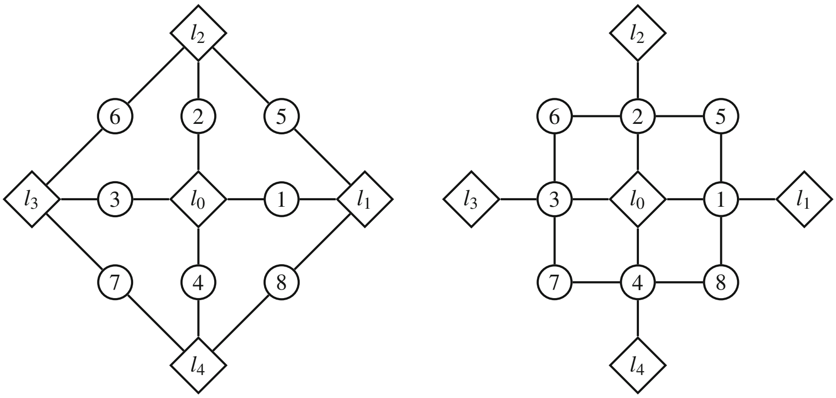

Fig. 5.3 Leader agents are represented by diamonds, follower agents by circles

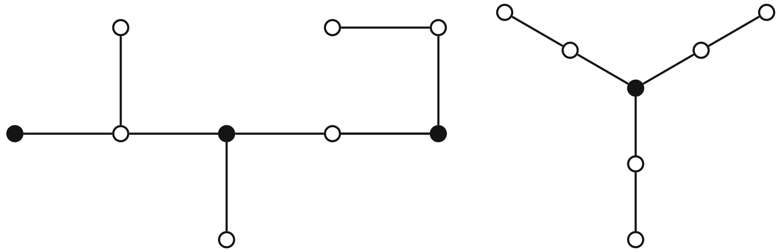

Fig. 5.4 Trees with stubborn agents (filled in black)

Exercise 5.14 (*Leaders in barbell graphs I*) Consider the graph $G = (V, E)$ where $V = V' \cup V''$ with $V' = \{v'_1, \dots, v'_N\}$ and $V'' = \{v''_1, \dots, v''_M\}$ and

$$E := \{(v'_i, v'_k) \mid i, k = 1, \dots, N\} \cup \{(v''_i, v''_k) \mid i, k = 1, \dots, M\} \cup \{(v'_1, v''_1), (v''_1, v'_1)\}$$

Assume that v'_a and v''_a are two leaders having opinion, respectively, equal to 0 and 1. Assume for the remaining nodes the consensus dynamics induced by the natural SRW and consider the asymptotic opinions as time goes to infinity.

(i) Compute the asymptotic opinions of the followers.
(ii) Compute the limits of such asymptotic opinions in the case when $N = M \to +\infty$.
(iii) Compute the limits of such asymptotic opinions in the case when $N = M^2 \to +\infty$.

Exercise 5.15 (*Leaders in barbell graphs II*) Consider the graph $G = (V, E)$ where $V = V' \cup V''$ with $V' = \{v'_1, \dots, v'_N\}$ and $V'' = \{v''_1, \dots, v''_N\}$ and

$$E := \{(v'_i, v'_k) \mid i, k = 1, \dots, N\} \cup \{(v''_i, v''_k) \mid i, k = 1, \dots, M\} \\ \cup \{(v'_h, v''_h), (v''_h, v'_h) \mid h = 1, \dots r\}$$

Assume that $\{v'_{r+1}, \dots v'_{r+s}\}$ and $\{v''_{r+1}, \dots v''_{r+s}\}$ are two set of leaders having opinion, respectively, equal to 0 and 1. Consider for the remaining nodes the consensus dynamics induced by a SRW on this graph.

(i) Compute the asymptotic opinions of the followers as functions of r and s.
(ii) Compute the limits of such asymptotic opinions in the case when $N \to +\infty$ and r, s are kept constant.
(iii) Compute the limits of such asymptotic opinions in the case when $\lceil r = \alpha N \rceil$, $\lceil s = \beta N \rceil$, and $N + \infty$.

Exercise 5.16 (*Optimal leader selection*) Let x denote the harmonic extension on a graph when $V^\ell = \{v_0, v_1\}$ and $x_{v_i} = i$. We think of the two leaders as competing to maximize their own influence. We define

$$H(v_0, v_1) = \frac{1}{N} \sum_{v \in V} x_v.$$

Note that $H(v_0, v_1) \in (0, 1)$.

(i) Show that

$$H(v_0, v_1) = \frac{1}{2} - \frac{\frac{1}{N} \sum_v R_{\text{eff}}(v, v_0) - \frac{1}{N} \sum_v R_{\text{eff}}(v, v_1)}{2R_{\text{eff}}(v_0, v_1)}$$

(ii) Consider the optimization problem

$$\min_{v_0 \in V} \max_{v_1 \in V \setminus \{v_0\}} H(v_0, v_1).$$

Show that the optimal value is not larger than $\frac{1}{2}$ and the optimal solution is the optimal solution of $\min_w \sum_v R_{\text{eff}}(v, w)$.

Bibliographical Notes

The most influential works for our treatment of electrical networks in connection with reversible stochastic matrices are the classical monograph [11] and the textbook [20], where the reader can find a more comprehensive treatment. A detailed analysis of the average resistance on d-dimensional graphs, which refines and extends Example 5.10 and Eq. (5.21), has been provided in various papers [2, 8, 31, 33]: For instance, it is known that the average resistance decreases with increasing d. The optimal choice of how to distribute conductances to minimize the average resistance is studied in [17].

Consensus with stubborn agents has attracted significant attention, in view of different applications. In robotic networks, it can be seen as a containment problem [19]. In social networks, stubborn agents that do not change their opinions are present, explicitly or implicitly, in a variety of models of opinion dynamics [1, 14, 16, 23, 25]. A recent survey of related literature has been given in [26]. Electrical networks have been used as a tool in this context by [10, 34]. Recently, the problem of the optimal placement of stubborn agents has recently attracted significant attention, also in relation with classical problems of actuator selection in control theory: Various objective functions have been considered, see [9, 12, 21]. The formulation used in Exercise 5.16 derives from [34, 35]. The paper [34] proposes a message-passing algorithm to effectively solve the placement problem for v_1 (after v_0 is in place). The algorithm was deduced within the electrical framework but has now been extended to nonreversible update matrices [29].

The problem of estimation from relative measurements studied in Sect. 5.4 has been brought to our attention by reading [3–5]. The electrical framework is a useful tool for its analysis [4, 32]. This estimation problem relates to various streams of applied research: It can be interpreted as a problem of relative localization between mobile robots [3], sensor calibration for wireless sensor networks [6], statistical ranking in machine learning [24], clock synchronization [18], or voltage estimation in power networks [13].

The simple distributed gradient algorithm is analyzed in [30], but several more sophisticated solutions have been proposed since at least [3]. We note that also randomized dynamics have been studied, which extend the ideas of Chap. 3 to consensus-like dynamics with inputs [7, 15, 27]. A general convergence analysis of such affine consensus-like randomized dynamics is given in [28].

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