Chapter 1
Graph Theory

Abstract This chapter is a self-contained and concise introduction to graph theory, which is essential to study the averaging dynamics over networks. After some basic notions in Sect. 1.1, the emphasis is on connectivity and periodicity properties, which are presented in Sects. 1.2 and 1.3, respectively. Section 1.4 introduces the adjacency and Laplacian matrices associated to a given graph and studies their spectra. Finally, Sect. 1.5 introduces some notable examples of graphs, such as circulant, Cayley, and De Bruijn graphs.

1.1 Basic Definitions and Examples

We begin with the definition of graph, which is central in our studies. A graph G is a pair (V, E) where V is a finite set, whose elements are said to be the *vertices* (or *nodes*) of G, and $E \subset V \times V$ is the set of *edges* (or *arcs*). The cardinality of V is said to be the *order* or the *size* of the graph. An edge of the form (u, u) is said to be a *self-loop*, or simply a *loop*. In a graph, every arc represents a connection or link between two nodes. It is customary to draw graphs by representing nodes as dots and arcs as arrows connecting the nodes in such away that for an edge $(u, v) \in V \times V$, we understand that u is the tail and v is the head of the arrow; see Fig. 1.1. When drawing a graph, we are thus implicitly assigning a location in the plane to each node. The trivial graph $E_V = (V, \emptyset)$ is said to be an *empty* graph. On the opposite extreme, the graph $K_V = (V, \{(u, v) : u \neq v\})$ is said to be a *complete* graph (note that self-loops have been excluded, see Fig. 1.1). Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if there exists a bijection $\psi : V \to V'$ such that

$$(v, w) \in E \;\Leftrightarrow\; (\psi(v), \psi(w)) \in E'.$$

For instance, two complete graphs are isomorphic when they have the same order. For this reason, we may also denote a complete graph of order n simply as K_n. Essentially, two isomorphic graphs simply differ by a different labeling of the vertices. Since in all the applications we will consider such differences will not play any role, we will consider two isomorphic graphs as identical in what follows. This equivalence also

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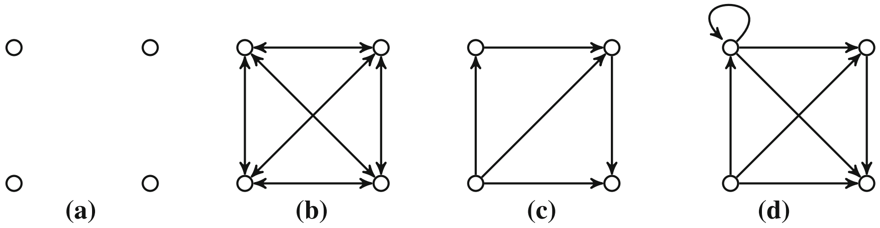


Fig. 1.1 Examples of graphs with four nodes: empty graph, complete graph, a graph without self-loops, and a graph with a self-loop

allows us to identify (when convenient) the vertex set with a set of numbers, writing for instance $V = \{1, \dots, |V|\}$.

Given a graph $G = (V, E)$, the *reverse* graph of G is the graph which is obtained by reversing all arcs. That is, $\text{rev}(G) = (V, \{(u, v) \in V \times V \ : \ (v, u) \in E\})$. The special case in which $\text{rev}(G) = G$ is very important, as it means that $(u, v) \in E$ if and only if $(v, u) \in E$. If a graph is such, it is said to be *symmetric*. When drawing a symmetric graph, there is no need to use pairs of arrows to connect nodes: In this case, we will rather use double-headed arrows or just segments; see Figs. 1.1 and 1.3.

If $(u, v) \in E$, then v is said to be a *out-neighbor* of u, and we write that $v \in \mathcal{N}_u^{\text{out}}$. Conversely, u is said to be a *in-neighbor* of v in the graph, and we write $u \in \mathcal{N}_v^{\text{in}}$. The number of out-neighbors of a node v is said to be its out-degree and is denoted by d_v^{out}. Correspondingly, the number of in-neighbors of a node v is said to be its in-degree and is denoted by d_v^{in}. Note that for every graph, the following identity holds true:

$$|E| = \sum_{v \in V} d_v^{\text{out}} = \sum_{w \in V} d_w^{\text{in}}. \tag{1.1}$$

A *source* is a node with no in-neighbors, and a *sink* is a node u with no out-neighbors. A graph is said to be d-(in/out-)regular if the (in/out-)degree of every node is d. A graph is *topologically balanced* if $d_v^{\text{out}} = d_v^{\text{in}}$ for all nodes v. Note that for a symmetric graph, there is no need to distinguish between in- and out-neighbors, so that any symmetric graph is topologically balanced and we will just talk about neighbors and degrees dropping the labels "in" and "out".

It is sometimes useful to identify certain relationships between graphs. The *intersection* and *union* of two graphs $G = (V, E)$ and $G' = (V', E')$ are denoted by, respectively, $G \cap G' = (V \cap V', E \cap E')$ and $G \cup G' = (V \cup V', E \cup E')$. On the other hand, we say that a graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$: this relation is denoted as $G' \subset G$. Furthermore, the subgraph G' is said to be *spanning* if $V' = V$ and is said to be the subgraph *induced* by V' if

$$E' = E \cap (V' \times V').$$

Fig. 1.2 Directed and undirected line graphs on four nodes

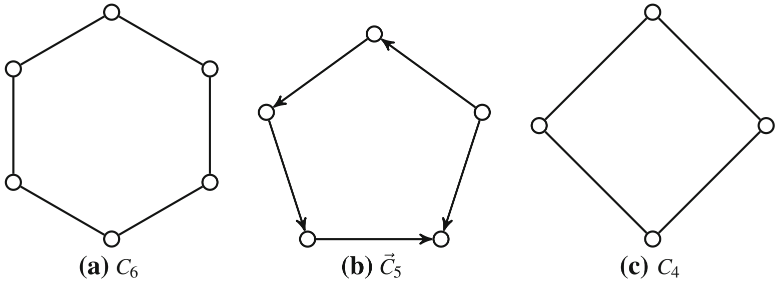


Fig. 1.3 Three examples of directed and undirected cycle graphs of small size

In this last case, we use the notation $G' = G_{|V'}$. Clearly, any graph on the vertex set V, without self-loops, is a subgraph of the complete graph K_V.

Next, we provide some more examples of graphs. Let $V = \{0, \ldots, n-1\}$.

(i) If $E = \{(u, v) \in V \times V \,:\, |v - u| = 1\}$, then (V, E) is said to be a *line* graph and is denoted as L_n.
(ii) If $E = \{(u, v) \in V \times V \,:\, v - u = 1\}$, then (V, E) is said to be a *directed line* graph and is denoted as $\vec{L}_n$.
(iii) If $E = \{(u, v) \in V \times V \,:\, v - u = 1 \mod n\}$, then (V, E) is said to be a *directed cycle* graph and is denoted as $\vec{C}_n$.
(iv) If $E = \{(u, v) \in V \times V \,:\, (v - u) \bmod n \in \{-1, +1\}\}$, then (V, E) is said to be a *cycle* graph and is denoted as C_n.

Some properties of line and cycle graphs can be immediately observed: For example, $\vec{L}_n \subset L_n$, and more precisely, $L_n = \vec{L}_n \cup \mathrm{rev}(\vec{L}_n)$. Correspondingly, $\vec{C}_n \subset C_n$ and $C_n = \vec{C}_n \cup \mathrm{rev}(\vec{C}_n)$. Moreover, the cycle graph C_n is 2-regular and the directed cycle graph $\vec{C}_n$ is topologically balanced. Examples of cycle and line graphs are drawn in Figs. 1.2 and 1.3.

1.2 Paths and Connectivity

In this section, we turn our attention to investigate the connectivity properties of graphs. The pictorial representation of graphs, which we have introduced above, makes the following definitions very natural.

Given a graph $G = (V, E)$ and a pair of nodes u, v, a *path* (of *length* l) from u to v on G is an ordered list of nodes $(w_0, \ldots, w_l)$ such that

(i) $w_0 = u$ and $w_l = v$;
(ii) $(w_i, w_{i+1}) \in E$ for every $i \in \{0, \ldots, l-1\}$.

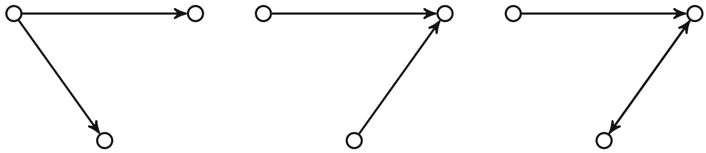


Fig. 1.4 Connectivity examples using graphs with three nodes: (weakly) connected without a globally reachable node, weakly connected with a globally reachable node, and strongly connected

The edges occurring in the definition of a path are said to *insist* on the path at hand. The path is said to be *simple* if the edges (w_i, w_{i+1}) are all distinct. If a path from u to v exists, we say that v is *reachable* from u. Given two nodes u and v, we say that they *communicate* if either $u = v$ or $u \neq v$ and there are both a path from u to v and one from v to u. It is easy to check—Exercise 1.2—that communication is an equivalence relation between nodes. These notions are instrumental to the following important definitions.

A graph $G = (V, E)$ is said to be

- *strongly connected* if every two nodes communicate;
- *connected* if for any pair of nodes (u, v), either u is reachable from v or v is reachable from u;
- *weakly connected* if $G \cup \mathrm{rev}(G)$ is strongly connected.

Note that these three definitions are equivalent for symmetric graphs. We also note that every graph can be seen as the disjoint union of weakly connected subgraphs, which we call *weakly connected components* or simply *connected components*.

A node v is said to be *globally reachable* if for every other node w there exists a path from w to v. Clearly, in a strongly connected graph, all nodes are globally reachable. A partial converse is given by the following result:

Proposition 1.1 (Connectivity and balance) *If G is topologically balanced and contains a globally reachable node, then G is strongly connected.*

Proof By contradiction, the graph $G = (V, E)$ is not strongly connected. Let R be the set of globally reachable nodes: By the assumptions, $\emptyset \subsetneq R \subsetneq V$. Consider the partition of nodes into R and $V \setminus R$, and note that there is no edge from R to $V \setminus R$ but there is at least one edge from $V \setminus R$ to R. Let v be the tail of such edge. By the balance property, there must be an edge (u, v) with $u \notin R$. In turns, the same remark implies that there exists an edge (t, u) with $t \notin R$. As the set $V \setminus R$ is finite, this iterative procedure must end after a finite number of steps, showing that there is at least one node in $V \setminus R$ that has different in-degree and out-degree, contradiction. □

Some examples of graph connectivity are given in Fig. 1.4.

The notion of path is also the ground to endow graphs with a natural distance between nodes. As we have defined above, the *length* of a path is the number of edges insisting on the path. Then, given two nodes u and v of a graph $G = (V, E)$, we can define the *distance from u to v* as the length of the shortest path which connects them. Precisely, we let[1]

$$\mathrm{dst}_G(u, v) = \min\{\ell : \text{ there exists in } G \text{ a path of length } \ell \text{ from } u \text{ to } v\}, \qquad (1.2)$$

provided $u \neq v$, and $\mathrm{dst}_G(u, u) = 0$. Note that the function $\mathrm{dst}_G(\cdot, \cdot)$ is symmetric in its arguments if G is a symmetric graph (cf. also Exercise 1.11). Furthermore, the *diameter* of the graph $G = (V, E)$ is defined as

$$\mathrm{diam}(G) = \max\{\mathrm{dst}_G(u, v) \; : \; u, v \in V\}.$$

Clearly, G is strongly connected if and only if $\mathrm{diam}(G)$ is finite. Moreover, for any strongly connected graph G of order n, it holds that $\mathrm{diam}(G) \leq n - 1$. It is easy to compute the diameter for the graph examples introduced above: For instance, for every $n \in \mathbb{N}$ we have $\mathrm{diam}(K_n) = 1$ and $\mathrm{diam}(C_n) = \lfloor n/2 \rfloor$.

A very important class of paths are "closed" paths: A path from a node to itself is said to be a *circuit*. For instance, loops are circuits of length one. A graph is said to be *circuit-free* if it contains no circuit. The following is a simple property of circuit-free graphs.

Proposition 1.2 (Source and sink) *Every circuit-free graph has at least one source and at least one sink.*

Proof By contradiction, we take a graph $G = (V, E)$ with no sink, that is such that $d_u^{\mathrm{out}} \geq 1$ for every $u \in V$. We pick any vertex and denote it as v_0. Then, we take one out-neighbor of v_0 and denote it by v_1. Then, recursively for $k \geq 1$, we take v_{k+1} among the out-neighbors of v_k. As the cardinality of V is finite, it must happen for a certain $\ell \in \mathbb{N}$ that $v_{\ell+1}$ belongs to $\{v_0, \ldots, v_\ell\}$, thus forming a circuit and providing the required contradiction. The existence of a source is proven similarly. □

When a graph $G = (V, E)$ is not strongly connected, we can consider its *strongly connected components*, which we define as follows. First, we have observed—see Exercise 1.2—that the relation of communication between nodes is an *equivalence relation*. Then, we can consider the partition of V into the corresponding equivalence classes $V = V_1 \cup V_2 \cup \cdots \cup V_s$ and the induced subgraphs $G_i = \big(V_i, E \cap (V_i \times V_i)\big)$, which are called the strongly connected components of G. If the graph G is symmetric, actually G is simply the union of these s subgraphs, in the sense that there is no further edge in the graph, the connected components being completely isolated from each other. For general graphs, the situation is more complicated: A useful way to describe what is left beyond the strongly connected components is the following concept of *condensation graph*, whose nodes represent the strongly connected components of G.

Definition 1.1 (*Condensation graph*) Given any graph $G = (V, E)$, consider its strongly connected components $G_k = (V_k, E_k)$, $k \in \{1, \ldots, s\}$. The condensation

[1] We understand that the minimum of an empty set is $+\infty$.

Fig. 1.5 An example of graph G (*left*) with its condensation graph $\mathscr{T}(G)$ (*right*)

graph of G is a graph $\mathscr{T}(G)$ with set of vertices $\{1, \ldots, s\}$ such that there is an arc in $\mathscr{T}(G)$ from h to k if $k \neq h$ and there is an arc in G from a vertex in V_k to a vertex in V_h.

The construction is illustrated in Fig. 1.5. We leave to the reader the task of proving the following properties of condensation graphs.

Proposition 1.3 (Condensation graphs) *Let G be any graph and $\mathscr{T}(G)$ its condensation graph. Then,*

(i) $\mathscr{T}(G)$ is circuit-free;
(ii) $\mathscr{T}(G)$ is (weakly) connected if and only if G is (weakly) connected;
(iii) G contains a globally reachable node if and only if $\mathscr{T}(G)$ has only one sink.

A *cycle* is a circuit of length at least 3, with no vertex repeated except the first and last one. A graph is said to be *cycle-free* if it contains no cycles, and *unicyclic* if it contains exactly one cycle. A *tree* is a symmetric cycle-free connected graph, and an cycle-free symmetric graph is also called a *forest*. The next result states some relevant properties of trees: other properties are presented in Exercise 1.5.

Proposition 1.4 (Trees) *Let $G = (V, E)$ be a symmetric graph. Then, the following four statements are equivalent.*

(i) G is a tree;
(ii) for any pair of distinct nodes u and v in V, there is exactly one path from u to v in G;
(iii) G is minimal connected*, that is, G is strongly connected and removing any edge makes the resulting graph not strongly connected;*
(iv) G is maximal cycle-free*, that is, G is cycle-free and adding one edge creates a cycle in G.*

Proof The key point of this proof is the equivalence between (i) and (ii). Indeed, assume G is a tree, that is, G is connected symmetric and cycle-free. Then, G is strongly connected, and thus, there is a path connecting u to v. Furthermore, if there was another path, the graph being symmetric would imply the existence of a cycle. Conversely, the existence of exactly one path implies connectedness and absence of cycles. Next, we can observe that removing any edge necessarily breaks at least one path, thus causing a graph satisfying (ii) to become not strongly connected. Conversely, property (iii) ensures that there are no multiple paths connecting the nodes, for otherwise strong connectivity would be robust to edge deletions. As we have noted that in symmetric graphs the absence of multiple paths is equivalent to the absence of cycle, we conclude that (iii) implies (ii). Proving the equivalence between (iii) and (iv) is left to the reader. □

1.3 Periodicity

Given a graph $G = (V, E)$ and $v \in V$, denote by L_v the set of lengths of the circuits in G to which v belongs. The period of v is the greatest common divisor (GCD) of the integers in L_v (if $L_v = \emptyset$, the period is undefined). The node is said to be *aperiodic* if its period is one. Notice that if a self-loop $(v, v) \in E$ is present, then $1 \in L_v$ and v is thus certainly aperiodic. The graph itself is said to be *aperiodic* if every node is aperiodic.

Example 1.1 In the directed cycle graph $\vec{C}_n$, each node has period equal to n. In the symmetric cycle graph C_n, instead, the period of each node is equal to $\mathrm{GCD}(2, n)$. In particular, symmetric cycle graphs C_n with n odd are all aperiodic.

Notice that, since circuits can be concatenated freely to obtain new circuits, it follows that the length sets L_v are closed under addition[2] ($\ell_1, \ell_2 \in L_v$ yield $\ell_1 + \ell_2 \in L_v$). For aperiodic nodes, something very strong can be stated about L_v. We start recalling the following well-known fact from algebra.

Lemma 1.1 (Bézout's identity) *Let $a_1, \dots, a_s \in \mathbb{N}$ and let $d \in \mathbb{N}$ be their GCD. Then, there exist s coefficients $\alpha_i \in \mathbb{Z}$ such that $\sum_i \alpha_i a_i = d$.*

By this lemma, we can prove the following key result.

Proposition 1.5 (Aperiodicity) *Let $G = (V, E)$ be a graph and let $v \in V$. The following conditions are equivalent.*

(i) v is aperiodic;
(ii) there exists $m \in \mathbb{N}$ such that $m, m+1 \in L_v$;
(iii) there exists $\ell \in \mathbb{N}$ such that for every $n \geq \ell$ it holds that $n \in L_v$.

Proof Clearly, (iii) ⇒ (ii) ⇒ (i).

(i) ⇒ (ii): Since v is aperiodic, we can find lengths $\ell_1, \ell_2, \dots, \ell_s \in L_v$ such that $1 = \mathrm{GCD}(\ell_1, \dots, \ell_s)$. Hence, by Lemma 1.1, we can find numbers $\alpha_i \in \mathbb{Z}$ such that $1 = \sum_{i=1}^{s} \alpha_i \ell_i$. Let $m = \sum_{i=1}^{s} |\alpha_i| \ell_i$ and notice that $m + 1 = \sum_{i=1}^{s} (|\alpha_i| + \alpha_i)\ell_i$. This shows that both $m, m+1 \in L_v$, yielding (ii).

(ii) ⇒ (iii): Notice first that if $m = 1$ in (ii), then (iii) is immediate. Suppose now that $m > 1$ and put $\ell = (m-1)m$. Let $n \geq \ell$. Dividing n by m, we obtain

[2] Note that this also implies closure under integer multiplication as $\ell \in L_v$ and $m \in \mathbb{N}$ yield $m\ell \in L_v$.

$n = mh + r = m(h - r) + (m + 1)r$. Since, by definition of rest, $r \leq m - 1$ and by our choice of ℓ the quotient satisfies $h \geq m - 1$, we have that $h - r \geq 0$. Since both m and $m + 1$ belong to L_v, the last inequality implies that $n \in L_v$. □

The result above shows that for every aperiodic node, there exists $\ell \in \mathbb{N}$ such that for every $n \geq \ell$ there exists a path of length n from the node to itself. In other words, for every aperiodic node, there exist paths of *any* length from the node to itself—possibly excluding lengths below a certain threshold. This fact clearly means a great "freedom of movement" in the graph. Furthermore, aperiodicity of a single node is easily inherited by the rest of the graph, as a consequence of the following result.

Proposition 1.6 (Aperiodic vertices) *Let $G = (V, E)$ be a graph and let $u, v \in V$ be two communicating nodes. Then, u is aperiodic if and only if v is aperiodic.*

Proof As the node u is aperiodic, there exists $\ell \in \mathbb{N}$ and two circuits from u to itself which have lengths ℓ and $\ell + 1$. By the communication assumption, there exist a path from u to v (of length m) and a path from v to u (of length n). Hence, there exist two circuits (possibly repeating vertices) from v to itself, which have lengths $m + \ell + n$ and $m + \ell + n + 1$, proving the thesis. □

As a corollary, if a graph is strongly connected and has an aperiodic vertex, the graph is aperiodic. The above discussion also allows us to conclude the following remarkable result.

Corollary 1.1 (Paths on strongly connected and aperiodic graphs) *If a graph $G = (V, E)$ is strongly connected and aperiodic, then there exists ℓ such that for any pair of nodes u, v and any length $m \geq \ell$ there is a path from u to v of length m.*

Next, we present an additional notion which relates to paths and connectivity. A graph (V, E) is said to be *bipartite* if the set V can be apportioned into two subsets V_1 and V_2 such that for all $(u, v) \in E$ either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$. We already know some examples of bipartite graphs. For instance, trees are bipartite and C_n is bipartite if and only if n is even. The following is another natural example.

Example 1.2 (*Complete bipartite*) Let A, B be two nonempty sets of cardinalities m and n, respectively. The *complete bipartite* graph $K_{m,n}$ is the graph with node set $A \cup B$ and an edge (u, v) if and only if $u \in A$ and $v \in B$ or $u \in B$ and $v \in A$.

An important characterization of bipartite graphs is given by the following result.

Proposition 1.7 (Bipartition condition) *A graph is bipartite if and only if every circuit has even length.*

Proof If the graph $G = (V, E)$ is bipartite with $V = V_1 \cup V_2$, every path u_0, $u_1, \ldots, u_p$ having $u_0 \in V_1$ is such that $u_i \in V_1$ if and only if i is even. Therefore, if $u_p = u_0$, then p must be even. In order to prove the converse statement, we construct the partition $\{V_1, V_2\}$. We take any circuit $u_0, u_1, \ldots, u_{p-1}, u_0$ and let u_i belong to

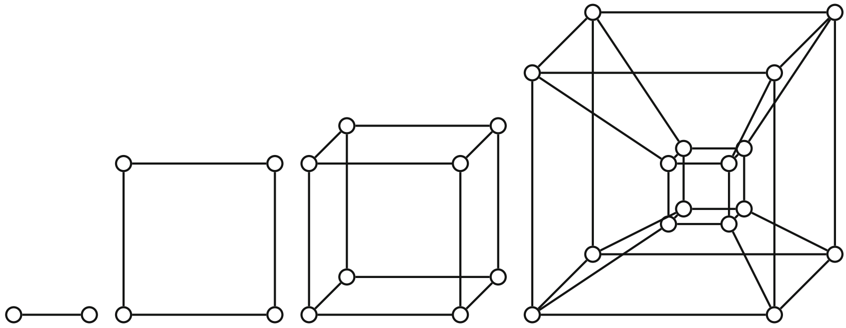


Fig. 1.6 Hypercubes of dimensions until four

V_1 if i is even, and to V_2 otherwise. Note that there is no edge connecting u_i and u_j when i and j are both even (or both odd), for otherwise the "shortcut" would create a circuit with odd length. Next, we move to another circuit $w_0, \dots, w_q$. If the circuit has no intersection with that examined before, we can just repeat the same reasoning. If instead, say, the new circuit contains a node s which has already been attributed to V_1, we denote $s = w_0$ and proceed as above. Note that the procedure can be performed without introducing contradictions, because of the absence of circuits of odd length. Iterating the procedure constructs the required partition. □

As a corollary, we note that any bipartite graph is not aperiodic.

Next, we introduce a remarkable family of graphs, which the reader may easily verify to be bipartite.

Example 1.3 (*Hypercube*) Let V be the set of the binary words of length n, that is, $V = \{0, 1\}^n$. Then, the hypercube H_n is the graph on V with an edge between two words whenever they differ in exactly one component, i.e., $E = \{(u, v) : \|u - v\|_1 = 1\}$. It is immediate to observe that $|V| = 2^n$, and that H_n is symmetric, n-regular, and bipartite. Hypercube graphs are so-called because they draw the vertices and edges of n-dimensional cubes: This can be observed from the examples in Fig. 1.6.

1.4 Matrices and Eigenvalues

This section introduces (i) relevant matrices which are used in the study of graphs, namely the adjacency and Laplacian matrices; (ii) the notion of graph associated with a matrix; and (iii) the definition of weighted graph that inherently involves a matrix. Relating graphs to matrices permits to take advantage of algebraic tools for the study of graphs, and conversely to express matrix properties in terms of graphs.

Furthermore, the spectrum of adjacency and Laplacian matrices conveys important information about the graphs: This study is the topic of spectral graph theory.

First, we provide the fundamental definition of *adjacency matrix*. Given a graph $G = (V, E)$, the adjacency matrix A (sometimes denoted as A_G) is a matrix in $\{0, 1\}^{V \times V}$ such that

$$\begin{cases} A_{uv} = 1 & \text{if } (u, v) \in E \\ A_{uv} = 0 & \text{if } (u, v) \notin E. \end{cases}$$

As an example, observe that the adjacency matrix of the third graph in Fig. 1.1 is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The adjacency matrix encodes all the information about the structure of the graph and is thus a very important notion. Furthermore, it permits to answer questions about paths and connectivity of the graph by purely algebraic computations. The next result is a chief example.

Proposition 1.8 (Adjacency matrix and paths) *Let $G = (V, E)$ be a graph with adjacency matrix A. Then, for all $u, v \in V$ and $k \in \mathbb{N}$, the u, v-entry of A^k equals the number of paths of length k (including paths with self-loops) from node u to node v.*

Proof The statement is proved by induction on the length k. By definition of adjacency matrix, the statement is true for $k = 1$. Next, we assume that the statement is true for k and we prove it for $k + 1$. Note that each path from u to v of length $k + 1$ consists of an edge (u, w) and a path from w to v of length k. Since we can write

$$(A^{k+1})_{uv} = \sum_{w \in V} A_{uw}(A^k)_{wv},$$

the statement follows by the inductive hypothesis. □

Proposition 1.8 and Corollary 1.1 imply the following fact.

Corollary 1.2 (Adjacency matrix of aperiodic graphs) *If a graph $G = (V, E)$ is strongly connected and aperiodic, then there exists $\ell \in \mathbb{N}$ such that, for every $m \geq \ell$, every entry of A^m is strictly positive.*

By writing the adjacency matrix, we associate a matrix to any graph: Conversely, we may associate a graph to any matrix. Let M be a square matrix with nonnegative entries, whose rows and columns are indexed in a set V. Then, the graph *associated* to M, denoted by $\mathcal{G}_M$, is the graph (V, E) such that

$$E = \{(u, v) \in V \times V \ : \ M_{uv} > 0\}.$$

$$M = \begin{bmatrix} 0.3 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

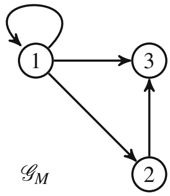


Fig. 1.7 A nonnegative matrix and its associated graph

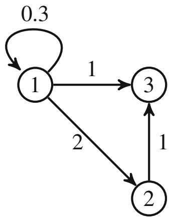


Fig. 1.8 The matrix M and the graph $\mathscr{G}_M = (V, E)$ in Fig. 1.7 drawn as a weighted graph (V, E, M)

An example is shown in Fig. 1.7. Based on this definition, we say that given a graph $G = (V, E)$, a matrix $M \in \mathbb{R}_{\geq 0}^{V \times V}$ is said to be *adapted to* G when $M_{uv} > 0$ and $u \neq v$ imply $(u, v) \in E$. Equivalently, we may say that $\mathscr{G}_M \subset G$ modulo self-loops. Of course, the adjacency matrix of a graph is an example of a matrix adapted to it.

Example 1.4 Consider the graph $\mathscr{G}_M$ in Fig. 1.7 and the matrices

$$M_1 \begin{bmatrix} 1 & 2.3 & 1 \\ 0 & 0 & 20 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0.1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 & 3.2 & 0 \\ 0.5 & 0 & 20 \\ 0 & 3 & 0 \end{bmatrix}.$$

Then, M_1 and M_2 are adapted to the graph, whereas M_3 is not.

Sometimes in the literature a graph $G = (V, E)$ together with a matrix $M \in \mathbb{R}_{\geq 0}^{V \times V}$, adapted to G, is called a *weighted graph* and is also denoted as (V, E, M). Weighted graphs are often depicted as graphs with each edge (v, w) labeled with M_{vw}, as in Fig. 1.8. Depending on the applications, M_{vw} may have a variety of different meanings: It can measure the capacity of a certain connection, a flow, a resistance, a distance, and so on.

We now propose the following fundamental construction. The Laplacian matrix associated with a matrix $M \in \mathbb{R}_{\geq 0}^{V \times V}$ is the matrix $L(M) \in \mathbb{R}^{V \times V}$ such that $L(M)_{uv} = -M_{uv}$ if $u \neq v$ and $L(M)_{uu} = \sum_{v: v \neq u} M_{uv}$. In matrix form, we may write that

$$L(M) = \mathrm{diag}(M\mathbf{1}) - M,$$

where the notation $\mathrm{diag}(x)$ denotes the square matrix whose main diagonal is the vector x. In the special case when M is the adjacency matrix of the graph G, the resulting Laplacian matrix $L(A_G)$ is simply called the *Laplacian* of G and denoted with the symbol L_G. Notice that in this case, L_G has the form

$$L_G = D_G - A_G$$

where D_G is a diagonal matrix such that $(D_G)_{uu} = d_u^{\text{out}}$ for every $u \in V$. As an immediate consequence, we observe that $L(M)$ does not depend on the diagonal values of M. In particular L_G is independent of the presence of self-loops in the graph. As well, it is immediate that the graph G is symmetric if and only if L_G is symmetric.

The spectrum of $L(M)$ plays an important role in graph theory and in many of the arguments which will be discussed later on. From the definition, it is immediate that for any matrix M, it holds that $L(M)\mathbf{1} = 0$, that is, 0 is an eigenvalue of $L(M)$ with eigenvector $\mathbf{1}$. We now propose a number of results in the case when M is symmetric. Possible extensions and refinements are outlined in Exercises 1.21 and 2.25. We start with the following basic fact.

Proposition 1.9 (Dirichlet form) *Let $M \in \mathbb{R}^{V\times V}$ be a symmetric matrix. For every $x \in \mathbb{R}^V$, it holds*

$$x^* L(M)x = \frac{1}{2}\sum_{u,v} M_{uv}(x_u - x_v)^2. \tag{1.3}$$

Proof By computing the quadratic form, we obtain

$$\begin{aligned} x^* L(M)x &= \sum_u \sum_{v:v\neq u} M_{uv}x_u^2 - \sum_{u,v:u\neq v} M_{uv}x_u x_v \\ &= \sum_{u,v:u\neq v} M_{uv}(x_u^2 - x_u x_v) \\ &= \tfrac{1}{2}\left[\sum_{u,v:u\neq v} M_{uv}(x_u^2 - x_u x_v) + \sum_{u,v:u\neq v} M_{vu}(x_v^2 - x_u x_v)\right] \\ &= \tfrac{1}{2}\sum_{u,v:u\neq v} M_{uv}(x_u^2 - 2x_u x_v + x_v^2). \end{aligned}$$

Notice the so-called "symmetrization" trick used in the third equality and the crucial role played by symmetry in the fourth equality. □

The previous result has a number of straightforward consequences.

Proposition 1.10 (Laplacian and connectivity) *Suppose M is symmetric. Then,*

(i) the Laplacian $L(M)$ is positive semidefinite;
(ii) the multiplicity of the eigenvalue 0 *equals the number of connected components of $\mathcal{G}_M$.*

Proof Exercise. Hint: to prove (ii), find suitable independent eigenvectors, each of them corresponding to a connected component. □

When M is symmetric of order n, eigenvalues of $L(M)$ are real and nonnegative: $0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$. Particularly relevant is the "second eigenvalue" λ_2, which admits the following variational characterization.

Proposition 1.11 (Variational characterization) *If M is symmetric, then it holds*

$$\lambda_2 = \min_{x \neq 0, x^* \mathbf{1} = 0} \frac{x^* L(M) x}{x^* x}. \tag{1.4}$$

Proof Since $L(M)$ is symmetric we can find an orthonormal basis of eigenvectors: $x_{(i)} \in \mathbb{R}^V$ for $i \in \{1, \ldots, n\}$ with $L(M)x_{(i)} = \lambda_i x_{(i)}$. We can assume that $x_{(1)} = n^{-1/2}\mathbf{1}$. $L(M)$ can be expressed as a combination of orthogonal projectors:

$$L(M) = \sum_{i \geq 2} \lambda_i x_{(i)} x_{(i)}^*$$

(this corresponds to diagonalizing $L(M)$ with respect to the basis of eigenvectors). Hence, if $y \in \mathbb{R}^V$ is such that $\mathbf{1}^* y = 0$

$$y^* L(M) y = \sum_{i \geq 2} \lambda_i (x_{(i)}^* y)^2 \geq \lambda_2 \sum_{i \geq 2} (x_{(i)}^* y)^2 = \lambda_2 ||y||_2^2$$

This yields $\leq$ in (1.4). On the other hand, it holds

$$\frac{x_{(2)}^* L(M) x_{(2)}}{x_{(2)}^* x_{(2)}} = \lambda_2$$

and thus also $\geq$ is proven. □

This result has an immediate consequence.

Proposition 1.12 (Monotonicity) *Let M and Q be two symmetric matrices such that $M_{uv} \geq Q_{uv}$ for every $u \neq v$. If we denote by $\lambda_2(M)$ and $\lambda_2(Q)$ the second eigenvalues of, respectively, $L(M)$ and $L(Q)$, then $\lambda_2(M) \geq \lambda_2(Q)$.*

Proof It easily follows combining (1.4) with the quadratic form given by (1.3). □

1.5 Examples of Graphs

This section regards notable families of graphs and their properties. We also introduce a notion of product between graphs, which is useful to construct further examples.

1.5.1 Circulant Graphs

An $n \times n$ matrix A is said to be *circulant* if it exists a vector $c \in \mathbb{R}^n$ such that

$$A = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & \dots \\ & & \vdots & & \\ c_1 & \cdots & & c_{n-1} & c_0 \end{bmatrix}.$$

This matrix can be denoted as $A = \text{circ}(c)$, where $c = [c_0, c_1, \dots, c_{n-1}]$. A graph is said to be *circulant* if its adjacency matrix is circulant. Examples include the directed cycle and cycle graphs introduced earlier.

Circulant matrices enjoy many interesting properties: Here we are interested in simple properties of their spectra, summarized in the following result.

Proposition 1.13 (Spectra of circulant matrices) *Let $c \in \mathbb{R}^n$ and consider the $n \times n$ circulant matrix $A = \text{circ}(c)$. Then, A has eigenvectors*

$$x^{(k)} = [1, \omega_k, \dots, \omega_k^{n-1}]^\top \qquad k \in \{0, \dots, n-1\},$$

and corresponding eigenvalues $\lambda_k = \sum_{\ell=0}^{n-1} c_\ell \omega_k^\ell$, where we have denoted $\omega_k = e^{\mathrm{i}\frac{2\pi}{n}k}$ with i the imaginary unit. Furthermore, the eigenvectors $x^{(k)}$ form an orthonormal basis of $\mathbb{C}^n$.

Proof Note that ω_k is such that $\omega_k^n = 1$: Indeed, ω_k is said to be a *n-th root of unity*. We leave to the reader to verify that the set of vectors $\{x^{(k)}\}_{k=0}^{n-1}$ is orthonormal, and we instead verify that the pair $(x^{(k)}, \lambda_k)$ satisfies the eigenvalue definition. To this goal, we compute

$$\begin{aligned} Ax^{(k)} &= c_0 \begin{bmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{n-1} \end{bmatrix} + c_1 \begin{bmatrix} \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{n-1} \\ 1 \end{bmatrix} + \dots + c_{n-1} \begin{bmatrix} \omega_k^{n-1} \\ 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \end{bmatrix} \\ &= \sum_{\ell=0}^{n-1} c_\ell \omega_k^\ell \begin{bmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{n-1} \end{bmatrix} \\ &= \sum_{\ell=0}^{n-1} c_\ell \omega_k^\ell x^{(k)} \end{aligned}$$

thus proving the thesis. □

As special cases, we can compute the spectra of cycle graphs.

Example 1.5 (*Spectra of cycles*) The graph $\vec{C}_n$ has Laplacian eigenvalues

$$\lambda_k = 1 - \exp\left(i\frac{2\pi}{n}k\right) \qquad k \in \{0, \dots, n-1\}.$$

The graph C_n has Laplacian eigenvalues

$$\lambda_k = 2 - 2\cos\left(\frac{2\pi}{n}k\right) \qquad k \in \{0, \dots, n-1\}.$$

Note that these eigenvalues are real, $\lambda_k = \lambda_{n-k}$, and indeed a basis of real eigenvectors can be found (exercise).

1.5.2 Product Graphs

In this paragraph, we introduce the binary operations of *Cartesian product* between matrices and between graphs. Consider two matrices $A \in \mathbb{R}^{V\times V}$ and $B \in \mathbb{R}^{H\times H}$. We start recalling the familiar Kronecker product of matrices $A \otimes B \in \mathbb{R}^{(V\times H)\times(V\times H)}$ defined by

$$(A \otimes B)_{uh,vk} = A_{uv}B_{hk}. \tag{1.5}$$

The Cartesian product is instead defined as the matrix $A \times B \in \mathbb{R}^{(V\times H)\times(V\times H)}$ such that

$$(A \times B)_{uh,vk} = A_{uv}\delta_{hk} + B_{hk}\delta_{uv}, \tag{1.6}$$

where δ_{hk} is the standard Kronecker delta ($\delta_{hh} = 1$ and $\delta_{hk} = 0$ if $h \neq k$). This definition can be conveniently rewritten, by using the Kronecker product, as

$$A \times B = A \otimes I_H + I_V \otimes B\,, \tag{1.7}$$

where I_V and I_H are the identity matrices. Interpreting $\mathbb{R}^{V\times H}$ as the space of real matrices with rows and columns labeled by, respectively, elements of V and H, the Kronecker and Cartesian products can be thought as linear applications acting on matrices. Given a matrix $M \in \mathbb{R}^{V\times H}$, they can be equivalently expressed as

$$(A \otimes B)M = AMB^\top \quad \text{and} \quad (A \times B)M = AM + MB^\top. \tag{1.8}$$

Both the Kronecker and the Cartesian product are associative and commutative up to relabeling vertices (see also Exercise 1.26).

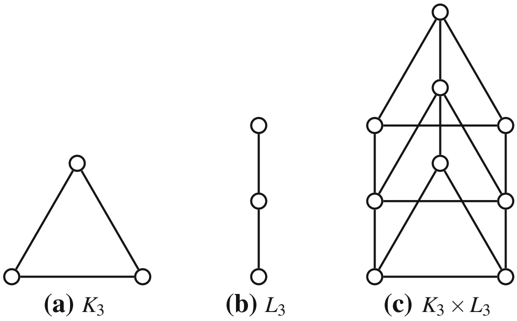


Fig. 1.9 Computing the Cartesian product between K_3 and L_3

The Cartesian product of matrices has a natural counterpart at the level of graphs. The (Cartesian) product of two graphs G and G' is the graph $G \times G'$, such that the vertex set is the Cartesian product of the vertex sets of G and G', and two vertices are adjacent when they agree in one coordinate and are adjacent in the other. A pictorial example is given in Fig. 1.9. We easily see that, as long as G and G' have no self-loops, the adjacency matrix of $G \times G'$ is $A_G \times A_{G'}$.

Furthermore, the degree of a node (u, h) in a product graph $G \times G'$ is the sum of the degrees of u in G and h in G'.

Proposition 1.14 (Product and Laplacian) *Let $A \in \mathbb{R}^{V \times V}$ and $B \in \mathbb{R}^{H \times H}$ be square matrices. Then,*

$$L(A \times B) = L(A) \times L(B).$$

Proof We start with a remark on products of diagonal operators. If $M \in \mathbb{R}^{V \times H}$, $\operatorname{diag}(M) \in \mathbb{R}^{(V \times H) \times (V \times H)}$ denotes the diagonal operator such that $\operatorname{diag}(M)_{uh,uh} = M_{uh}$. Given two vectors $x \in \mathbb{R}^V$ and $y \in \mathbb{R}^H$, it follows from definition (1.5) that $\operatorname{diag}(x) \otimes \operatorname{diag}(y) = \operatorname{diag}(xy^\top)$.

From

$$(A \times B)\mathbf{1}_V \mathbf{1}_H^\top = (A\mathbf{1}_V)\mathbf{1}_H^\top + \mathbf{1}_V (B\mathbf{1}_H)^\top$$

we then obtain

$$\operatorname{diag}\big((A \times B)\mathbf{1}_V \mathbf{1}_H^\top\big) = \operatorname{diag}(A\mathbf{1}_V) \otimes I_H + I_V \otimes \operatorname{diag}(B\mathbf{1}_H).$$

Thus,

$$\begin{aligned} L(A \times B) &= \operatorname{diag}(A\mathbf{1}_V) \otimes I_H + I_V \otimes \operatorname{diag}(B\mathbf{1}_H) - A \otimes I_H - I_V \otimes B \\ &= L(A) \otimes I_H + I_V \otimes L(B) = L(A) \times L(B), \end{aligned}$$

thereby proving the thesis. □

The spectrum of a product of matrices is determined by the spectra of the factors via a simple relation.

Proposition 1.15 (Spectrum of product matrices) *If A and B have eigenvalues λ and μ with corresponding eigenvectors $x \in \mathbb{R}^V$ and $y \in \mathbb{R}^H$ respectively, then $A \times B$ has eigenvalue $\lambda + \mu$ with eigenvector $xy^\top \in \mathbb{R}^{V \times H}$.*

Proof From the definition of Cartesian product, we observe

$$(A \times B)xy^\top = (Ax)y^\top + x(By)^\top = (\lambda + \mu)xy^\top,$$

which gives the thesis. □

The spectral properties of several families of graphs can be studied using the product operation defined above.

Example 1.6 (*Hypercube graph*) The hypercube H_n, defined in Example 1.3, is the Cartesian product of n factors K_2. The Laplace spectrum of K_2 is $\{0, 2\}$, and hence the Laplace spectrum of H_n consists of the numbers $2i$ with multiplicity $\binom{n}{i}$, for $i \in \{0, \ldots, n\}$.

Other examples are products of cycle graphs: The product of two cycles represents a square lattice on a two-dimensional torus.

Example 1.7 (*Bidimensional torus grid*) We know from Example 1.5 that the graph C_n has Laplace spectrum

$$\lambda_k = 2 - 2\cos\left(\frac{2\pi}{n}k\right) \qquad k \in \{0, \ldots, n-1\}.$$

Then, the product graph $C_n \times C_m$ has Laplace spectrum

$$4 - 2\cos\left(\frac{2\pi}{m}h\right) - 2\cos\left(\frac{2\pi}{n}k\right) \qquad h \in \{0, \ldots, m-1\},\ k \in \{0, \ldots, n-1\}.$$

The extension to k-dimensional grids is now natural.

Example 1.8 (*k-dimensional torus grid*) Since the graph C_m has Laplace spectrum $\lambda_k = 2 - 2\cos\left(\frac{2\pi}{m}k\right)$, $k \in \{0, \ldots, m-1\}$, the Laplace spectrum of the product graph $C_m^k = \underbrace{C_m \times \cdots \times C_m}_{k \text{ times}}$ is $2k - 2\sum_{i=1}^{k} \cos\left(\frac{2\pi}{m}h_i\right)$, $h_i \in \{0, \ldots, m-1\}$.

The reader may compute as an exercise the spectra of other multi-dimensional graphs, e.g., L_n^k (using Example 1.27) and $\vec{C}_n^k$ (using Example 1.5).

1.5.3 Cayley Graphs

A generalization of cycle graphs and toroidal grids is provided by the family of Abelian Cayley graphs, which are graphs whose set of nodes is an Abelian group and the set of edges is stable by translation operations. The formal definition is proposed below.

Definition 1.2 (*Abelian Cayley matrices and graphs*) Let Γ be an Abelian group (we use the additive notation) and let S be a subset of Γ. Then, the Γ-*Cayley graph* generated by S in Γ is the graph $\mathcal{G}(\Gamma, S)$ having Γ as node set and

$$E = \{(g, h) \in \Gamma \times \Gamma \ : \ h - g \in S\}$$

as edge set. In words, two nodes—i.e., two group elements—are neighbors if their difference is in S. When it causes no confusion, we shall simply refer to Cayley graphs without explicitly mentioning Γ. As well, a notion of Cayley matrix can be defined. Given a group Γ and a generating row vector $\pi \in \mathbb{R}^{\Gamma}$, we shall define the Γ-*Cayley matrix* generated by π as the matrix $\mathrm{cayl}(\pi) \in \mathbb{R}^{\Gamma \times \Gamma}$ defined by $\mathrm{cayl}(\pi)_{gh} = \pi_{h-g}$ for all h and g in Γ. Correspondingly, for a given Cayley matrix M, we shall denote by π^M the generating vector of the Cayley matrix M which is simply the row of M labeled by $g = 0$.

Clearly, the adjacency and Laplacian matrices of Γ-Cayley graphs are Γ-Cayley matrices. Conversely, if P is a Γ-Cayley matrix generated by π, then $\mathcal{G}_P$ is a Γ-Cayley graph with $S = \{h \in \Gamma \ : \ \pi_h \neq 0\}$.

Abelian Cayley graphs encompass several important examples.

Example 1.9 Let $\mathbb{Z}_n$ denote the cyclic group of integers modulo n.

(i) The *complete* graph on n nodes is $\mathcal{G}(\mathbb{Z}_n, \mathbb{Z}_n \setminus \{0\})$;
(ii) The *circulant* graphs (resp. matrices) are Abelian Cayley graphs (resp. matrices) on the group $\mathbb{Z}_n$: We have that $\mathrm{cayl}(\pi) = \mathrm{circ}(\pi)$. For instance, the *cycle* graph C_n is the circulant graph $\mathcal{G}(\mathbb{Z}_n, \{-1, 1\})$; its adjacency matrix is $A = \mathrm{circ}([0, 1, 0, \dots, 0, 1])$ and its Laplacian is $L = \mathrm{circ}([2, -1, 0, \dots, 0, -1])$.
(iii) The *grids* on a d-dimensional torus are $\mathcal{G}(\mathbb{Z}_n^d, \{e_i, -e_i\}_{i \in \{1,\dots,d\}})$, where e_i are the elements of the canonical basis of $\mathbb{R}^d$.
(iv) Keeping the same notation, the *d-dimensional hypercube* is $\mathcal{G}(\mathbb{Z}_2^d, \{e_i\}_{i \in \{1,\dots,d\}})$.

The algebraic structure of Cayley graphs and matrices implies strong properties. The next two results list some basic properties which can be proven as exercises.

Proposition 1.16 *Assume Γ is an Abelian group and $S \subset \Gamma$. Then, the following statements hold true.*

(i) $\mathcal{G}(\Gamma, S)$ is a symmetric graph if and only if S is inverse-closed, and is strongly connected if and only if S generates the group Γ.
(ii) $\mathcal{G}(\Gamma, S)$ is topologically balanced and $|S|$-regular.

Proposition 1.17 *If M and M' are Γ-Cayley matrices, then*

(i) their sum is $M + M'$ is Cayley and $\pi^{M+M'} = \pi^M + \pi^{M'}$;
(ii) M and M' commute and their product is Cayley. Namely, $\pi^{MM'} = \pi^M * \pi^{M'}$, where $*$ denotes convolution between vectors: $(v * v')_i = \sum_j v_j v'_{i-j}$. Moreover, $\pi^M * \pi^{M'} = M\pi^{M'}$.

The spectrum of a Cayley matrix can be computed by a discrete Fourier transform of its generating vector.

Proposition 1.18 (Spectrum of Cayley matrices) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$, so that $\sum_{\ell=1}^{d} n_\ell = N$. Let M be Γ-Cayley and $\pi \in \mathbb{R}^\Gamma$ be its generating vector. Then, the spectral structure of M can be described as follows:*

(i) the eigenvalues of M are

$$\lambda_h = \sum_{k\in\Gamma} \pi_k \exp\left(-i\, 2\pi \sum_{\ell=1}^{d} \frac{k_\ell h_\ell}{n_\ell}\right) \quad h \in \Gamma;$$

(ii) a corresponding orthogonal basis of eigenvectors $\chi^{(h)} \in \mathbb{R}^\Gamma$ is given by

$$\chi^{(h)}(k) = \exp\left(i\, 2\pi \sum_{\ell=1}^{d} \frac{k_\ell h_\ell}{n_\ell}\right) \tag{1.9}$$

(iii) the matrix M can be written as

$$M = \sum_{h\in\Gamma} \lambda_h N^{-1} \chi^{(h)} \chi^{(h)*}.$$

Proof To provide this proof, we first need to briefly review part of the theory of Fourier transforms on discrete groups: We refer to [25] for a comprehensive introduction. Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers. A *character* on Γ is a group homomorphism $\chi : \Gamma \to \mathbb{C}^*$, namely a function from Γ to $\mathbb{C}^*$ such that $\chi(g + h) = \chi(g)\chi(h)$ for all $g, h \in \Gamma$. We can interpret a character as a linear function $\chi : \Gamma \to \mathbb{C}^\Gamma$, i.e., as an N-dimensional vector of complex numbers. Since we have that $\chi(g)^N = \chi(Ng) = \chi(0) = 1$ for every $g \in \Gamma$, it follows that χ takes values on the Nth-roots of unity. The character $\chi_0(g) = 1$ for every $g \in \Gamma$ is called the trivial character (notice that χ_0 corresponds to $\mathbf{1}$). The set of all characters of the group Γ forms an Abelian group with respect to the entrywise multiplication. It is called the character group and denoted by $\hat{\Gamma}$. The trivial character is clearly the zero of $\hat{\Gamma}$. Moreover, $\hat{\Gamma}$ is isomorphic to Γ, and its cardinality is N. If we consider the vector space $\mathbb{C}^\Gamma$ of all functions from Γ to $\mathbb{C}$ with the canonical Hermitian form $\langle f_1, f_2 \rangle = \sum_{g\in\Gamma} f_1(g) f_2(g)$, it can be proved that the set $\{N^{-1/2}\chi \;:\; \chi \in \hat{\Gamma}\}$ is an orthonormal basis of $\mathbb{C}^\Gamma$. Then, it is possible to define the Fourier transform of a function $f : \Gamma \to \mathbb{C}$ as

$$\hat{f} : \hat{\Gamma} \to \mathbb{C} \qquad \hat{f}(\chi) = \sum_{g \in \Gamma} \chi(-g) f(g).$$

After this review, consider the Cayley matrix $M \in \mathbb{R}^{\Gamma \times \Gamma}$ generated by the vector $\pi \in \mathbb{R}^{\Gamma}$. The matrix M can be interpreted as a map from $\mathbb{C}^{\Gamma}$ to itself, and the spectral structure of M can be described as follows. Each character χ is an eigenfunction of M with eigenvalue $\hat{\pi}(\chi)$, because

$$\begin{aligned}(M\chi)(g) &= \sum_{h \in \Gamma} M_{gh} \chi(h) \\ &= \sum_{h \in \Gamma} \pi_{g-h} \chi(h) \\ &= \sum_{h \in \Gamma} \pi_{g-h} \chi(g) \chi(h-g) \\ &= \sum_{\ell \in \Gamma} \pi_{\ell} \chi(-\ell) \chi(g) \\ &= \hat{\pi}(\chi) \chi(g).\end{aligned}$$

Since the characters form an orthonormal basis, it follows that M is diagonalizable and its spectrum is given by $\{\hat{\pi}(\chi) \;:\; \chi \in \hat{\Gamma}\}$. Furthermore, the matrix can be rewritten as

$$M = \sum_{\chi \in \hat{\Gamma}} \hat{\pi}(\chi) N^{-1} \chi \chi^*,$$

where $\frac{1}{N} \chi \chi^*$ is a linear function from $\mathbb{C}^{\Gamma}$ to itself, projecting $\mathbb{C}^{\Gamma}$ onto the eigenspace generated by χ. Finally, a straightforward verification shows that characters of Γ are explicitly given by the $\chi^{(h)}$'s defined in (1.9). This completes the proof. □

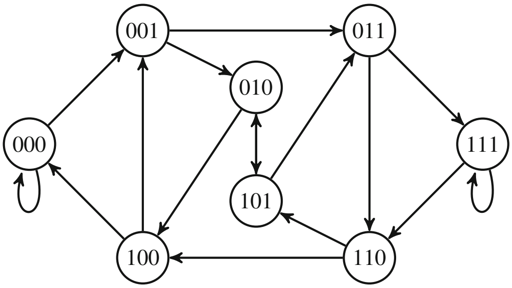
1.5.4 De Bruijn Graphs

We now present a remarkable class of graphs based on a combinatorial construction. A De Bruijn graph on k symbols of dimension h is defined as follows. The node set is the set all strings of length h of k given symbols, and there is an edge from u to v if v can be obtained from u by shifting all symbols by one place to the left and adding a new symbol at the rightmost place. More formally, $V = \{0, \ldots, k-1\}^h$ and

$$E = \{(u, v) \in V \times V \;:\; v_{i-1} = u_i \quad \text{for all } i \in \{1, \ldots, h\}\}.$$

See Fig. 1.10 for an example. An equivalent definition is given in Exercise 1.8.

De Bruijn graphs have notable properties: Here, we limit ourselves to observe the following facts, whose simple proofs are left to the reader (Exercise 1.32).



$$A = \begin{bmatrix} 1&1&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1 \\ 1&1&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1 \end{bmatrix}$$

Fig. 1.10 De Bruijn graph of dimension three on two symbols and its adjacency matrix

(i) If $h = 1$, then the De Bruijn graph is the complete graph.
(ii) Each vertex has exactly k incoming and k outgoing edges (counting self-loops), that is, $\mathbf{1}^* A = k\mathbf{1}^*$ and $A\mathbf{1} = k\mathbf{1}$.
(iii) The adjacency matrix A of a De Bruijn graph is such that $A^h = \mathbf{1}\mathbf{1}^*$.
(iv) The Laplacian eigenvalues of a De Bruijn graph are 0 and k, and k has multiplicity $n - 1$. In particular, the smallest nonzero Laplacian eigenvalue is equal to k.

Exercises

Exercises are divided into three groups, respectively devoted to basic graph theory (that is, the first three sections), algebraic graph theory (Sect. 1.4) and to significant examples of graphs (Sect. 1.5).

Basic Graph Theory

Exercise 1.1 (*Handshaking Lemma*) Let $G = (V, E)$ be a symmetric graph. Show that the sum of the degrees of all nodes is even.

Exercise 1.2 (*Strongly connected components*) Let there be a graph $G = (V, E)$. Verify that the relation between nodes "u and v communicate" is an *equivalence relation* in V, namely the relation is reflexive, symmetric, and transitive. Show that the strongly connected components of G are the equivalence classes of this relation.

Exercise 1.3 (*Periodicity and connectivity*) Let $G = (V, E)$ be a strongly connected graph.

(i) Prove that all nodes in G have the same period.
(ii) Prove that if at least one edge is symmetric (i.e., $\{(u, v), (v, u)\} \subset E$), then the period is either 1 or 2.
(iii) What can we argue about the period if $\vec{C}_3$ is a subgraph of G? And if $\vec{C}_4 \subset G$?

(iv) Prove Corollary 1.1.

Exercise 1.4 (*Trees have leaves*) In a symmetric graph, a vertex of degree one is said to be a *leaf*. Show that every tree has at least two leaves.

Exercise 1.5 (*Trees and number of edges*) Let $G = (V, E)$ be a symmetric graph and denote $v = |V|$ and $e = |E|/2$. Then, the following statements are true.

(i) If G is connected, then $e \geq v - 1$.
(ii) If G is connected and has at least one cycle, then $e \geq v$.
(iii) If G is a tree, then $e = v - 1$. Vice versa, if G is connected and $e = v - 1$, then G is a tree.
(iv) If G is a forest with k connected components, then $e = v - k$.
(v) If G is a connected unicycle, then $e = v$. Vice versa, if G is connected and $e = v$, then G is a unicycle.

Exercise 1.6 (*Globally reachable node*) Let $G = (V, E)$ be a graph of order at least two. Given a subset of nodes $U \subset V$, we say that v is an out-neighbor of U if there exists $(u, v) \in E$ with $u \in U$. Prove the following fact. The graph G has no globally reachable node if and only if there exist two disjoint nonempty subsets of nodes $U, W \subset V$ such that neither U nor W has an out-neighbor.

Exercise 1.7 (*Rooted and spanning trees*) A *rooted tree* is a circuit-free graph with the following property: There exists a vertex, called the *root*, such that any other vertex of the graph can be reached by one and only one directed path starting at the root. A *spanning tree* of a given graph is a spanning subgraph that is a rooted tree.

(i) A graph contains a spanning tree if and only if the reverse graph contains a globally reachable vertex.
(ii) If a graph is strongly connected, then it contains a globally reachable vertex and a spanning tree.

Exercise 1.8 (*Eulerian paths and circuits*) An Eulerian path is a path that visits all the graph edges exactly once. An Eulerian circuit is an Eulerian path which starts and ends at the same node. Show that the following properties hold for a graph G.

(i) If G has an Eulerian circuit, then G is topologically balanced.
(ii) If G is weakly connected and has an Eulerian circuit, then G is strongly connected.
(iii) G has an Eulerian circuit if and only if G is topologically balanced, and all of its vertices with nonzero degree belong to the same strongly connected component.
(iv) Let G be a weakly connected graph. Then, G is topologically balanced if and only if G has an Eulerian circuit.
(v) A weakly connected graph $G = (V, E)$ has an Eulerian path if and only if

    (a) at most one vertex $v \in V$ has $d_v^{\text{out}} - d_v^{\text{in}} = 1$;
    (b) at most one vertex $u \in V$ has $d_u^{\text{in}} - d_u^{\text{out}} = 1$; and
    (c) every other vertex has equal in-degree and out-degree.

Algebraic and Spectral Graph Theory

Exercise 1.9 (*Bipartite graph and adjacency matrices*) Observe that the adjacency matrix of bipartite graph $G = (V, E)$ can be written, up to a vertex permutation, in a block form

$$A_G = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

where $B \in \mathbb{R}^{V_1 \times V_2}$ and $C \in \mathbb{R}^{V_2 \times V_1}$, such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$.

(i) Compute $A_G^2, A_G^3, \ldots$
(ii) Prove that there is no path in G of odd length.

Exercise 1.10 (*Weight balance and connectivity*) Let G be a weighted graph. If G is weight-balanced and contains a globally reachable node, then G is strongly connected.

Exercise 1.11 (*Distance on a weighted graph*) Let $G = (V, E, A)$ be a weighted graph, and define the length of a path as the sum of the weights associated to the edges insisting on the path. Then, assuming that G is connected, define the distance function $\mathrm{dst} : V \times V \to \mathbb{R}_{\geq 0}$ as in (1.2). Observe that these definitions naturally extend the notions of path length and distance between nodes to weighted graphs. Prove that, if G is symmetric, then dst is a *metric* on V, that is

(i) $\mathrm{dst}(u, v) \geq 0$ and $\mathrm{dst}(u, v) = 0$ if and only if $u = v$ (positive definiteness);
(ii) $\mathrm{dst}(u, v) = \mathrm{dst}(v, u)$ for every $u, v \in V$ (symmetry);
(iii) $\mathrm{dst}(u, v) \leq \mathrm{dst}(u, w) + \mathrm{dst}(w, v)$ for every $u, v, w \in V$ (triangle inequality).

Exercise 1.12 (*Directed incidence matrix*) Let $G = (V, E, A)$ be a weighted graph and $L = L(A)$ the corresponding Laplacian matrix. Define the *directed incidence matrix* $\Phi \in \mathbb{R}^{E \times V}$ by

$$\Phi_{(u,v),w} = \begin{cases} 1 & \text{if } w = u \\ -1 & \text{if } w = v \\ 0 & \text{otherwise} \end{cases}$$

so that for each row, corresponding to an edge, there is a 1 corresponding to the tail and a -1 corresponding to the head of the edge. Then, show that

$$\Phi^* W \Phi = L + L^*,$$

where $W \in \mathbb{R}^{E \times E}$ is a diagonal matrix arranging all the weights of the edges, such that $W_{(u,v),(u,v)} = A_{uv}$.

Exercise 1.13 (*The 4-wheel*) Let $G = (V, E)$ be symmetric and $|V| = 5$.

(i) Prove or disprove the existence of a graph on V such that

   (a) all vertices have degree 3;
   (b) four vertices have degree 3 and one vertex has degree 4.

(ii) When such graph exists, may it be bipartite? May it contain a Eulerian circuit?
(iii) Write the adjacency matrix of such a graph, and compute its eigenvalues.

Exercise 1.14 (*Spectra of regular graphs*) Let G be a graph and assume G is out-regular with out-degree d. Then, λ is an adjacency eigenvalue if and only if $d - \lambda$ is a Laplacian eigenvalue.

Exercise 1.15 (*Complement graph*) Given a graph $G = (V, E)$, let the *complement* graph $\bar{G}$ be the graph having node set V of cardinality n and edge set $\bar{E} = \{(u, v) \in V \times V : u \neq v \text{ and } (u, v) \notin E\}$.

(i) Show that the Laplacian matrices of G and $\bar{G}$ are such that

$$L(G) + L(\bar{G}) = nI - \mathbf{1}\mathbf{1}^*.$$

(ii) For a given graph H, let $\lambda_j(H)$ denote the j-th smallest eigenvalue of $L(H)$. Conclude from (i) that for $2 \leq j \leq n$, it holds $\lambda_j(\bar{G}) = n - \lambda_{n+2-j}(G)$.
(iii) Prove that G and $\bar{G}$ cannot both be disconnected.

Exercise 1.16 (*Edge addition*) Let $G = (V, E)$ be a symmetric graph and $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ its Laplacian spectrum. Let $G \cup \{e\}$ denote the addition of a pair of edges $\{(u, v), (v, u)\}$ to the graph G. Then,

$$\lambda_2(G) \leq \lambda_2(G \cup \{e\}) \leq \lambda_2(G) + 2.$$

Exercise 1.17 (*Bounds on Laplacian eigenvalues*) Let $G = (V, E)$ be a symmetric graph and λ an eigenvalue of its Laplacian. Show that $\lambda \leq 2d_{\max}$ and that $\lambda \leq n$, where $d_{\max}$ is the largest degree and n the order of G. Find examples where the corresponding equalities hold.

Exercise 1.18 (*Algebraic connectivity and vertex connectivity*) For a connected symmetric graph G which is not complete, we define the *vertex connectivity* of G as

$$\kappa(G) = \min\{k \in \mathbb{N} : G \text{ can be disconnected by removing } k \text{ nodes}\}.$$

Then, $\lambda_2 \leq \kappa(G)$, where λ_2 denotes the smallest nonzero eigenvalue of L_G. For this reason, λ_2 also takes the name of *algebraic connectivity* of the graph.

Exercise 1.19 (*Information from spectrum*) Let $G = (V, E)$ be symmetric and regular, and assume that the spectrum of the adjacency matrix of G is

$$\{-3, -3, -1, -1, -1, -1, 1, 1, 0, 2, 2, 4\}.$$

(i) Compute the cardinality of V.
(ii) Compute the degree of G.
(iii) Is G connected?

(iv) Is G bipartite?
(v) Estimate the diameter of G.

Exercise 1.20 (*Bipartite graphs*) Let M be adapted to a bipartite graph. Show that if λ is an eigenvalue of M, then also $-\lambda$ is an eigenvalue of M.

Exercise 1.21 (*Properties of Laplacians*) Let $G = (V, E, A)$ be a weighted graph of order n and L be the (weighted) Laplacian of G. The following statements hold true.

(i) All eigenvalues of L have nonnegative real part.
(ii) The following properties are equivalent:

   (a) G is weight-balanced;
   (b) $\mathbf{1}^* L = 0$;
   (c) for all $x \in \mathbb{R}^V$, it holds $x^*(L + L^*)x = \sum_{u,v} A_{uv}(x_v - x_u)^2$;
   (d) $L + L^*$ is positive semidefinite.

(iii) G is weakly connected if and only if $\ker(L + L^*) = \mathrm{span}\{\mathbf{1}\}$.

Exercise 1.22 (*Diameter and Laplacian* [1, 20]) Given a connected symmetric graph G of order n, let λ_2 be its smallest nonzero Laplacian eigenvalue. Then, $\lambda_2 \geq \frac{4}{n(n-1)}$ and

$$\frac{4}{n\lambda_2} \leq \mathrm{diam}(G) \leq 2\left\lceil \sqrt{\frac{2d_{\max}}{\lambda_2}} \log_2 n \right\rceil.$$

Exercise 1.23 (*Diameter and normalized Laplacian*) Let $G = (V, E)$ be a symmetric graph of order n, A its adjacency matrix, and D its (diagonal) degree matrix. Let also $L = D - A$ be the Laplacian matrix and $M = I - D^{-1/2}AD^{-1/2}$ the *normalized Laplacian* (according to [10]). We denote by $0 = \mu_1 \leq \cdots \leq \mu_n \leq 2$ the eigenvalues of M. Then,

$$\frac{1}{\mu_2|E|} \leq \mathrm{diam}(G) \leq \frac{\log n}{\log \frac{\mu_n + \mu_2}{\mu_n - \mu_2}}.$$

Exercise 1.24 (*Paths of a certain length*) Let G be a graph of order n and assume there exists $h \in \mathbb{N}$ such that there is exactly one path of length h connecting any pair of nodes. Prove that there exists k such that

(i) $n = k^h$;
(ii) each vertex of G has exactly k in-neighbors and out-neighbors;
(iii) k is the only nonzero adjacency eigenvalue of G and has multiplicity one.

Example Families of Graphs

Exercise 1.25 (*Star graph*) The *star graph* S_n is a symmetric graph of order $n + 1$ such that one node, called the center, is the only neighbor to all the other n nodes; see

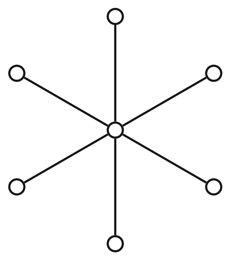


Fig. 1.11 The star graph S_6

Fig. 1.11. Observe that S_n is a tree and is isomorphic to $K_{1,n}$; its diameter is two for every n. Let $A_n = \begin{bmatrix} 0 & \mathbf{1}_n^* \\ \mathbf{1}_n & 0 \end{bmatrix}$ denote its adjacency matrix and L_n denote its Laplacian. Show that

(i) powers of A_n have the form

$$A_n^{2k-1} = n^{k-1} A_n, \qquad A_n^{2k} = n^{k-1} \begin{bmatrix} n & 0 \\ 0 & \mathbf{1}_n \mathbf{1}_n{}^* \end{bmatrix} \qquad \text{for all } k \in \mathbb{N};$$

(ii) the eigenvalues of L_n are the solutions of the polynomial

$$p(s) = s(s-1)^{n-1}(s-n).$$

Exercise 1.26 (*Kronecker product*) Verify the following useful properties of the Kronecker product of matrices as defined in (1.5). All matrices are assumed to be square.

(i) $AB \otimes CD = (A \otimes C)(B \otimes D)$;
(ii) $(A \otimes B)^* = (A^* \otimes B^*)$;
(iii) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, up to a permutation of indices;
(iv) $A \otimes B = B \otimes A$, up to a permutation of indices;
(v) $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$, provided A and B are invertible;
(vi) the eigenvalues of $A \otimes B$ are all possible products of an eigenvalue of A with an eigenvalue of B;
(vii) the eigenvectors of $A \otimes B$ are all possible Kronecker products of an eigenvector of A with an eigenvector of B;
(viii) $\operatorname{tr} A \otimes B = \operatorname{tr} A \operatorname{tr} B$;
(ix) $\det(A \otimes B) = (\det A)^n (\det B)^m$ where n and m are the dimensions, respectively, of A and B.

Exercise 1.27 (*Tridiagonal and augmented tridiagonal Toeplitz matrices* [6, Lemma 1.77 and (1.6.7)]) Let $V = \{1, \ldots, n\}$ and $a, b \in \mathbb{R}$. Consider the $n \times n$ *tridiagonal Toeplitz* matrix

$$\mathrm{Trid}_n(a,b) = \begin{bmatrix} b & a & 0 & \dots & & 0 \\ a & b & a & 0 & \cdots & 0 \\ 0 & a & b & a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ & \cdots & 0 & a & b & a \\ 0 & \cdots & & 0 & a & b \end{bmatrix}.$$

(i) The matrix $\mathrm{Trid}_n(a,b)$ has eigenvalues $\lambda_i = b + 2a\cos\left(\frac{i\pi}{n+1}\right)$, $i \in \{1,\dots,n\}$ and corresponding eigenvectors

$$x^{(i)} = \left[\sin\left(\frac{i\pi}{n+1}\right), \sin\left(\frac{2i\pi}{n+1}\right), \dots, \sin\left(\frac{ni\pi}{n+1}\right)\right]^*.$$

Furthermore, consider the $n \times n$ *augmented tridiagonal* Toeplitz matrix

$$\mathrm{ATrid}_n^{\pm}(a,b) := \mathrm{Trid}_n(a,b) \pm \begin{bmatrix} a & 0 & \dots & & 0 \\ 0 & 0 & \dots & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 0 & a \end{bmatrix}$$

(ii) The matrix $\mathrm{ATrid}_n^{\pm}(a,b)$ has eigenvalues

$$\begin{aligned} \lambda_i &= b + 2a\cos\left(\frac{i\pi}{n}\right) \qquad i \in \{1,\dots,n-1\}, \\ \lambda_n &= b \pm 2a. \end{aligned}$$

Exercise 1.28 (*Bidimensional grid*)

(i) Use Exercise 1.27 to verify that the graph L_m has adjacency spectrum

$$2\cos\left(\frac{k\pi}{m+1}\right) \qquad k \in \{1,\dots,m\}.$$

(ii) Show that the product graph $L_m \times L_n$ has adjacency spectrum

$$2\cos\left(\frac{2\pi}{m+1}h\right) + 2\cos\left(\frac{2\pi}{n+1}k\right) \qquad h \in \{1,\dots,m\},\ k \in \{1,\dots,n\}.$$

(iii) Find the Laplacian spectrum of L_m and $L_m \times L_n$.

Exercise 1.29 (*Cayley graph that is not a product*) Consider the Γ-Cayley graph G where $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$ and where

$$S = \{(\pm 1, 0),\ (0, \pm 1),\ \pm(1,1)\}.$$

Verify that the Laplacian eigenvalues are given by

$$6 - 2\cos\frac{2\pi h}{N} - 2\cos\frac{2\pi k}{N} - 2\cos\frac{2\pi(h+k)}{N}, \quad h, k = 0, \ldots, N-1.$$

Exercise 1.30 (*Algebraic connectivity of symmetric Cayley graphs*) Let λ_2 be the algebraic connectivity (cf. Exercise 1.18) of the Cayley graph $\mathscr{G}(\Gamma, S)$. Then, a nontrivial result in [7] implies that

$$\lambda_2 \leq \frac{C}{|\Gamma|^{2/|S|}},$$

where $C > 0$ is a constant independent of Γ and S. This inequality can easily be verified in examples of Cayley graphs presented in the text. For instance, you can check that

(i) the Laplacian eigenvalues of C_n are $\{2(1 - \cos\left(\frac{2\pi}{n}\ell\right))\}_{\ell \in \mathbb{Z}_n}$, and in particular

$$\lambda_2 = 2 - 2\cos\left(\frac{2\pi}{n}\right) \leq \frac{2\pi^2}{n^2};$$

(ii) the Laplacian eigenvalues of C_m^2 are $4 - 2\cos\left(\frac{2\pi}{m}h\right) + \cos\left(\frac{2\pi}{m}k\right)$, for $h, k \in \{0, \ldots, m-1\}$ and in particular

$$\lambda_2 = 4 - 4\cos\left(\frac{2\pi}{m}\right) \leq \frac{2\pi^2}{m^2};$$

(iii) the Laplacian eigenvalues of H_d are $\{2\ell\}_{\ell \in \{0,\ldots,d\}}$, and in particular $\lambda_2 = 2$.

Exercise 1.31 (*De Bruijn graphs*) Show that the De Bruijn graph on k symbols of dimension h denoted as B_k^h is the graph with order $n = k^h$ such that every node $u \in \{0, \ldots, n-1\}$ is connected to $ku, ku+1, ku+2, \ldots, ku+k-1$ (all mod k^h).

Exercise 1.32 (*Properties of De Bruijn graphs*) By using Exercise 1.24, prove the properties of De Bruijn graphs stated in Sect. 1.5.4.

Exercise 1.33 (*Geometric graphs*) Let V be a node set and $x \in (\mathbb{R}^d)^V$. The *r-disk graph* is a symmetric graph $\mathscr{G}_{r,\mathrm{disk}}(x) = (V, E(x))$ defined by $E(x) = \{(u, v) : \|x_v - x_u\| \leq r\}$. On the other hand, define the *distance graph* as the complete graph endowed with a weight matrix $W(x)$ such that $W_{uv}(x) = \|x_u - x_v\|$. Define the *Euclidean minimum spanning tree* $\mathscr{G}_{\mathrm{EMST}}(x)$ as the spanning tree of the distance graph of minimum weight (i.e., such that the sum of the weights of its edges is minimal). Show that $\mathscr{G}_{\mathrm{EMST}}(x) \subset \mathscr{G}_{r,\mathrm{disk}}(x)$ if and only if $\mathscr{G}_{r,\mathrm{disk}}(x)$ is connected.

Bibliographical Notes

Our account of graph theory is of course far from being a complete one. Instead, we have selected definitions and facts that will be needed in the following chapters. Hence, we expect that the reader may be interested in a broader introduction, for instance the books [3, 13]. Moreover, previous books on network coordination and robotic networks do contain introductions to graph theory, which partly overlap with ours [6, 19]. Our definition of graph is often referred to as *directed* graph in the literature, as opposed to *undirected* graphs, pairs $(V, \bar{E})$ in which the elements of $\bar{E}$ are unordered pairs of nodes $\{u, v\}$. Then, an undirected graph is equivalent (in our language) to a symmetric graph in which each pair of directed edges (u, v), (v, u) is counted as one. This notion of undirected graph will be used later in Chap. 5.

Matrices adapted to graphs are fundamental in this book: For this reason, we have devoted significant attention to algebraic graph theory, which studies graphs via certain matrices associated to them, especially the adjacency and the Laplacian matrices. The study of their spectra is the goal of spectral graph theory: Several books on this topic are available [5, 10, 11]. In our treatment, we have focused on the Laplacian spectrum: The properties of the adjacency spectrum are also notable [3, Sect. VIII], but less useful to our needs (for instance, insufficient to characterize connectivity [5, Sect. 1.3.7]).

Section 1.5 has been devoted to selected examples of graphs and associated matrices. We have also introduced the Cartesian product, a useful operation to construct graphs [9, 17]. We recall here the examples that we presented, together with some useful references. *Circulant* matrices are a standard topic in applied mathematics, covered for instance in the classical book [12]. A related class of matrices is that of *Toeplitz* matrices: General Toeplitz matrices are not important to us, but an example that is useful to compute the spectrum of line graphs is presented in Exercise 1.27. More generally, Cayley graphs have a long history in abstract mathematics and have been used in control theory to describe translation-invariant systems [24]. Our interest in *Abelian Cayley* topologies is motivated both by their algebraic properties, which allow for an elegant mathematical treatment [7, 25], and by their potential for the applications. Indeed, Abelian Cayley graphs are idealized representations of communication scenarios of practical interest. In particular, they describe communication patterns that are *local*, not only in the sense of a limited number of neighbors, but also with a bound on the geometric distance among connected nodes. In Abelian Cayley graphs, this constraint is abstracted into the definition of edge set [2, 4, 8, 15, 18]. For this reason, Abelian Cayley graphs are an alternative to other models of "local" communication, such as geometric graphs. *Geometric graphs* are graphs such that each node is endowed with a location and the edge set depends on these locations (see Exercise 1.33); various types of geometric graphs are presented in [6, Chap. 2]. If a geometric graph is constructed from a position vector which is a random variable, then its properties (e.g., connectivity) can be studied statistically. Such random geometric graphs have been studied extensively [22, 23] as a modeling paradigm to describe wireless communication networks [14, 16]. It is important to mention that other random families of graphs are used to describe different kinds of real-world networks, such as social networks, broadly referred to as to *complex networks*. Such graphs typically exhibit scale-free properties, small diameter and small spectral gap, thus being very different from geometric graphs. Their description is outside the scope of this book, but many sources are available to the interested reader [21, 26].

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