What a Fuzzy Set Is and What It Is not?

Enric Trillas and Rudolf Seising

Abstract Although in the literature there appear ‘type-one’ fuzzy sets, ‘type-two’ fuzzy sets, ‘intuitionistic’ fuzzy sets, etc., this theoretically driven paper tries to argue that only one type of fuzzy sets actually exists. This is due to the difference between the concepts of a fuzzy set” and a “membership function”.

1 Introduction

Although in the literature there appear ‘type-one’ fuzzy sets, ‘type-two’ fuzzy sets, ‘intuitionistic’ fuzzy sets, etc., this theoretically driven paper tries to argue that only one type of fuzzy sets actually exists. This is due to the difference between the concepts of a fuzzy set” and a “membership function”. Both concepts deserve to be clarified.

Fuzzy sets can, for instance, be contextually specified by a membership function with values in the real unit interval but, nevertheless, membership functions with values out of this interval can be, in some situations, significant, suitable and useful. Situations in which either the range of their values cannot be presumed to be totally ordered, or it is impossible to precisely determine the membership numerical values, or the linearly ordered real unit interval produces a drastic simplification of the meaning of the fuzzy set’s linguistic label by enlarging it through its ‘working’ meaning.

Indeed, this paper negates the existence of ‘other fuzzy sets’ than fuzzy sets, but it shows the possible suitability of designing their membership functions for

(*) To Professor Rudolf Kruse with the greatest esteem.

E. Trillas
University of Oviedo, Oviedo, Spain

R. Seising (✉)
Friedrich-Schiller University Jena, Jena, Germany
e-mail: R.seising@lrz.uni-muenchen.de

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sufficiently representing them, with as much as possible information available on the contextual behavior of the linguistic label. If, as Zadeh likes to say, ‘fuzzy logic is a matter of degree’, it is also a matter of design.

This paper starts with some historical notes on the development of the linguistic approach to fuzzy sets and fuzzy logic in the second half of the 1960s and in the first years of the 1970s. In the subsequent sections the paper presents a change of perspective by placing fuzzy sets in their natural domain, plain language; by going from ‘general definitions’ to ‘design’ in a given context, and depending on the meaning of the corresponding linguistic label inasmuch as a fuzzy set membership function should be carefully designed [19]. That is, they should be built up through a typical process of design, in which the simplicity of representation is a not to be forgotten practical value. Meaning is, indeed, only attributable to statements, and if simplicity is always considered as beautiful in science, ‘design’ is an art.

The main problem, to put it roughly, is representing words in a formal framework, similar to what Gottfried Wilhelm Leibniz had proposed more than 340 years ago. With his famous ‘Calculemus!’ he intended to resolve any differences of opinion: “The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate [calculemus], without further ado, to see who is right.” [6, p. 51].

We can find this idea of reducing reasoning to calculations already in the late 13th century in the work of the Catalonian, Ramon Llull. In his *Art Abreujada d’Atrobar Veritat* (“The Abbreviated Art of Finding Truth”), later published under the title *Ars generalis ultima* or *Ars magna* (“The Ultimate General Art” [Lull]. Leibniz had written his dissertation about Llull’s *Art magna* and he named it “ars combinatoria” [7, p. 30].

Llull and Leibniz’s arts have been steps on the plan for computing with concepts. All this deserves to be explained step-by-step; and in the first place, particularly the determination of words admitting of such a representation and where and by means of what is actually possible.

Remark What will not be taken into account in this paper are cases like that of the functions emerging from the aggregation of sets; that is, from aggregating their Characteristic Functions. For instance, if in the universe $X = \{1, 2, 3, 4\}$ the sets $A = \{1, 3\}$, and $B = \{1, 3, 4\}$ are aggregated by the mean $M(a, b) = (2a + 3b)/5$, what is obtained is not a set but the function $(2A + 3B)/2 = 1/1 + 0/2 + 1/3 + 0.6/4$, that is able to represent a fuzzy set provided a linguistic label for it (induced from those of A and B) can be known.

2 What Is It a Fuzzy Set?

2.1. Fuzzy Sets were launched in 1964 in three seminal papers by Lotfi A. Zadeh, a professor and chairman in the department of Electrical Engineering at the University of California, Berkeley. Zadeh construed his fundamental term of a "fuzzy set" without any non-mathematical meaning or application-oriented interpretation.

In the mid-summer of 1964 he was invited to a conference at Wright-Patterson Air Force Base in Dayton, Ohio. In his talk there, Zadeh considered problems of pattern classification, e.g. the process of representing the object patterns into a set of real variables which represent these patterns correctly and which would also be accepted by a computer.

Immediately after his travels he wrote on Fuzzy sets dealing with two problems:

- *Abstraction*—"the problem of identifying a decision function on the basis of a randomly sampled set", and
- *Generalization*—"referred to the use of the decision function identified during the abstraction process in order to classify the pattern correctly".

Zadeh first published this paper with his close friend Richard Bellman und Robert Kalaba as co-authors as a RAND-memo, and two years later in a scientific journal. Here he defined a "fuzzy set" as "a notion which extends the concept of membership in a set to situations in which there are many, possibly a continuum of, grades of membership." [2, 3, p. 1].

As a historically interested system theorist he had written the article "From Circuit Theory to System Theory" for the anniversary edition of the *Proceedings of the IRE* to mark the 50th year of the *Institute of Radio Engineers* in May 1962. In this article he stressed "the fundamental inadequacy of the conventional mathematics—the mathematics of precisely-defined points, functions, sets, probability measures, etc.—for coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions" [22].

Two years later he had found this new mathematics and he explained its concepts in his second seminal paper: "Essentially, these concepts relate to situations in which the source of imprecision is not a random variable or a stochastic process but rather a class or classes which do not possess sharply defined boundaries" [23, p. 29].

In his third seminal paper, "Fuzzy Sets", he motivated the need for his new theory as follows: "More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1. Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men, do not constitute classes or sets in the usual mathematical sense of these terms" [24].

2.2. After he had launched Fuzzy sets Zadeh proposed its use and applications to various disciplines. Computers and Computer Science (CS) have become part of Electrical Engineering (EE) Zadeh was very active to change his department's name from "EE" to "EECS". In 1969 in a talk at the conference "Man and Computer" in Bordeaux, France, he said: "As computers become more powerful and thus more influential in human affairs, the philosophical aspects of this question become increasingly overshadowed by the practical need to develop an operational understanding of the limitations of the machine judgment and decision making ability" [26, 27, p. 130]. He called it a paradox that the human brain is always solving problems by manipulating "fuzzy concepts" and "multidimensional fuzzy sensory inputs" whereas "the computing power of the most powerful, the most sophisticated digital computer in existence" is not able to do this. Therefore, he stated that "in many instances, the solution to a problem need not be exact", so that a considerable measure of fuzziness in its formulation and results may be tolerable. The human brain is designed to take advantage of this tolerance for imprecision whereas a digital computer, with its need for precise data and instructions, is not" [26, 27, p. 132]. He intended to push his theory of fuzzy sets to model the imprecise concepts and directives: "Although present-day computers are not designed to accept fuzzy data or execute fuzzy instructions, they can be programmed to do so indirectly by treating a fuzzy set as a data-type which can be encoded as an array […]" [26, 27, p. 132].

2.3. Already in 1968 Zadeh has presented "fuzzy algorithms" [25]. Usual algorithms depend upon precision. Each constant and variable is precisely defined; every function and procedure has a definition set and a value set. Each command builds upon them. Successfully running a series of commands requires that each result (output) of the execution of a command lies in the definition range of the following command, that it is, in other words, an element of the input set for the series. Not even the smallest inaccuracies may occur when defining these coordinated definition and value ranges. Zadeh now saw "that in real life situations people think […] like algorithms but not precisely defined algorithms". Inspired by this idea, he wrote: "The concept in question will be called fuzzy algorithm because it may be viewed as a generalization, through the process of fuzzification, of the conventional (nonfuzzy) conception of an algorithm. [25] To illustrate, fuzzy algorithms may contain fuzzy instructions such as:

(a) "Set y *approximately equal to* 10 if x is approximately equal to 5," or
(b) "If x is *large*, increase y by *several* units," or
(c) "If x is *large*, increase y by *several* units; if x is *small*, decrease y by *several* units; otherwise keep y unchanged."

The sources of fuzziness in these instructions are fuzzy sets which are identified by their names in italics.

To execute fuzzy algorithms by computers they have to receive an expression in fuzzy programming languages. Consequently, the next step for Zadeh was to define fuzzy languages. "All languages", he wrote in a paper on Architecture and Design of Digital Computers, "whether natural or artificial, tend to evolve and rise in level through the addition of new words to their vocabulary. These new words are, in effect, names for ordered subsets of names in the vocabulary to which they are added." [28, p. 16] He argued explicitly for programming languages that are—because of missing rigidness and preciseness and because of their fuzziness—more like natural languages. He mentioned the concept of stochastic languages that was published by the Finnish mathematician Paavo Turakainen in the foregoing year [21], being such an approximation to our human languages using randomizations in the productions, but however, he preferred fuzzy productions to achieve a formal fuzzy language. With his Ph. D student Edward T.-Z. Lee he co-authored a short sketch of a program to extend non-fuzzy formal languages to fuzzy languages in "Note on Fuzzy Languages" [5].

2.4. On the other hand, in the first years of Fuzzy sets Zadeh believed in successful applications of his new concepts in non-technical fields as he wrote in 1969: "What we still lack, and lack rather acutely, are methods for dealing with systems which are too complex or too ill-defined to admit of precise analysis. Such systems pervade life sciences, social sciences, philosophy, economics, psychology and many other "soft" fields" [25].

His search for application fields led to a period of interdisciplinary scientific exchange on the campus of his university between himself and the mathematician Hans-Joachim Bremermann, the psychologist Eleanor Rosch (Heider) and the linguist George Lakoff.

It was in these 1970s when psychologist Rosch developed her prototype theory on the basis of empirical studies. This theory assumes that people perceive objects in the real world by comparing them to prototypes and then subsequently ordering them. In this way, according to Rosch, the meanings of words are formed from prototypical details and scenes and then incorporated into lexical contexts depending on the context or situation. It could therefore be assumed that different societies process perceptions differently depending on how they go about solving problems [12]. When Lakoff heard about Rosch's experiments, he was working at the Center for Advanced Study in Behavioral Sciences at Stanford. During a discussion about prototype theory, someone there mentioned Zadeh's name and his idea of linking English words to membership functions and establishing fuzzy categories in this way. Lakoff and Zadeh met in 1971/72 at Stanford to discuss this idea after which Lakoff wrote his paper "Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts" [4]. In this work, Lakoff employed "hedges" (meaning barriers) to categorize linguistic expressions, he used the term "fuzzy logic" in his article and he therefore deserves credit for first introducing this expression in scientific literature. Based on his later research, however, Lakoff came to find that fuzzy logic was not an appropriate logic for linguistics: "It doesn't work for real natural languages, in traditional computer systems it works that way." [4]

However, "Inspired and influenced by many discussions with Professor G. Lakoff concerning the meaning of hedges and their interpretation in terms of fuzzy sets," Zadeh had also written an article in 1972 in which he contemplated "linguistic operators", which he called "hedges": "A Fuzzy Set-Theoretic Interpretation of

Hedges". Here he wrote: "A basic idea suggested in this paper in that a linguistic hedge such as very, more, more or less, much, essentially, slightly etc. may be viewed as an operator which acts on the fuzzy set representing the meaning of its operand [31].

2.5. Zadeh's occupation with natural and artificial languages gave rise to his studies in semantics. This intensive work let him to the question "Can the fuzziness of meaning be treated quantitatively, at least in principle?" [29, p. 160]. His 1971 article "Quantitative Fuzzy Semantics" [30] starts with a hint to these studies: "Few concepts are as basic to human thinking and yet as elusive of precise definition as the concept of »meaning«. Innumerable papers and books in the fields of philosophy, psychology, and linguistics have dealt at length with the question of what is the meaning of »meaning« without coming up with any definitive answers." [29, p. 159]

Zadeh started a new field of research "to point to the possibility of treating the fuzziness of meaning in a quantitative way and suggest a basis for what might be called quantitative fuzzy semantics" combining his results on fuzzy languages and fuzzy relations. In the section "Meaning" of this paper, he set up the basics: "Consider two spaces: (a) a universe of discourse, U, and (b) a set of terms, T, which play the roles of names of subsets of U. Let the generic elements of T and U be denoted by x and y, respectively. Then he started to define the meaning M(x) of a term x as a fuzzy subset of U characterized by a membership function μ(y|x) which is conditioned on x. One of his examples was: "Let U be the universe of objects which we can see. Let T be the set of terms white, grey, green, blue, yellow, red, black. Then each of these terms, e.g., red, may be regarded as a name for a fuzzy subset of elements of U which are red in color. Thus, the meaning of red, M (red), is a specified fuzzy subset of U."

In his "Outline of a new approach to the analysis of complex systems and decision processes" [32] and in the three-part article "The concept of a Lingustic Variable and its Application to Approximate Reasoning" [33], in "Fuzzy Logic and Approximate Reasoning" [34] and finally in "PRUF—a meaning representation language for natural languages" [35] Zadeh developed a linguistic approach to Fuzzy sets. He defined linguistic variables as those variables whose values are words or terms from natural or artificial languages. For instance, "not very large", "very large" or "fat", "not fat" or "fast", "very slow" are terms of the linguistic variables size, fatness and speed. Zadeh represented linguistic variables as fuzzy sets whose membership functions map the linguistic terms onto a numerical scale of values.

2.6. A fuzzy set is a concept associated by a linguistic label of which no mathematical axiomatic theory is currently known. A fuzzy set is nothing other than something just exhibited in plain languages, and through the usual forms of speaking; it is a concept actually well anchored in language, and is useful for roughly, economically, and quickly describing what once perceived by people is translated into words. At each context, a fuzzy set is seen by the speakers as a unique entity associated to its linguistic label's use that, usually, is context-dependent and purpose-driven.

The usual confusion between fuzzy set and membership function in a given universe of discourse is not sustainable, since there is not a one-to-one correspondence, but a one-to-many between fuzzy sets and its membership functions that can be given by expressions of different functions, even sharing some common properties among them. As it is well known, a linguistic label P in a universe of discourse X does not admit of a single membership function, but several such functions can be chosen, unless P is a precisely used label.

For instance, the use of the label 'big' in the closed interval [0, 10] and in plain language, can be represented by several membership functions like, for instance, $x/10$, $x^2/100$, etc., depending on the available additional information on its shape. It is commonly accepted that the use of 'big' in the universe [0, 10] can be described by the four rules:

(1) x is less big than y $\Longleftrightarrow$ $x \leq y$;
(2) 10 is totally big;
(3) 0 is not at all big;
(4) If x can be qualified as big, it exists $\varepsilon(x) > 0$, such that all the points in the interval $[x - \varepsilon(x), x]$ can also be qualified as big.

With them, the membership functions representing 'big' can be all those mappings $m_{big}: [0, 10] \to [0, 1]$, such that:

(1′) If $x \leq y$, then $m_{big}(x) \leq m_{big}(y)$;
(2′) $m_{big}(10) = 1$;
(3′) $m_{big}(0) = 0$;
(4′) m_{big} is continuous.

There are an enormous amount of all the strictly non-decreasing functions $[0, 10] \to [0, 1]$ joining the points of coordinates (0, 0) and (1, 10). Anyway, the four laws (1′)–(4′) cannot specify a single m_{big}; for specifying one of them, some additional information is required. For instance, provided it can be presumed that function m_{big} should be linear, then it only exists $m_{big}(x) = x/10$, but if what can be supposed is that m_{big} is quadratic several of them exist, with $m_{big}(x) = x^2/100$ among them. Notice that provided it were known that the curve defined by m_{big} passes through the point (5, 0.6), then neither $x/10$, nor $x^2/100$, are acceptable. Etc.

Clearly, fuzzy sets do not admit to being specified by a single membership function, with the only exception corresponding to the case in which the linguistic label is precise, that is, its use in X is describable by 'if and only if' rules. With this exception, fuzzy sets cannot be confused with the measures of their meaning, or membership functions. For instance, provided the use of 'big' in [0, 10] were described by just the precise rule,

$$'x \text{ is big} \Leftrightarrow 8 \leq x \leq 10',$$

then its only membership function will be,

$$m_{big}{*}(x) = 0, \text{ if } 0 \leq x < 8, \text{ and } m_{big}{*}(x) = 1, \text{ if } 8 \leq x \leq 10;$$

and, accordingly with the 'specification axiom' of set theory [4], 'big' is specified by the set $[8, 10] = (m_{big}{*})^{-1}(1)$.

It should be pointed out that function $m_{big}*$ verifies the former properties (1′), (2′), and (3′), but not property (4′) as a consequence of the failing of rule (4), since for instance, for any $\varepsilon > 0$, no point 8—ε can be qualified as big in the current use of this label. Function $m_{big}*$ is not a continuous one. It is rule (4) which distinguishes the imprecise from the precise uses of 'big' in [0, 10].

Notice that even defining the fuzzy set 'big' by each pair (big, m_{big}) instead of by only a function m_{big}, as it is also usually done [11], it not only follows that the linguistic label 'big' generates several fuzzy sets instead of a single one, but also a partial externalization of the concept of fuzzy set from language. The meaning of the imprecise word 'big', and what the functions m_{big} mean is hidden. For scientifically domesticating fuzzy sets, meaning and its measuring should be analyzed.

2.7. A linguistic label P names a property p the elements of the universe of discourse X enjoy, and which use is exhibited by the meaning attributed to the elemental statements 'x is P', for all x in X. Notice that the elements x can be physical or virtual, etc., but the new elements 'x is P' are just statements in the intellect, and belong to the set X [P] = {x is P; x in X} that is different of X; actually, meaning is attributed to statements, and it is usually context-dependent and purpose-driven. Meaning depends on the context on which the statements 'x is P' are used either in written form, or uttered, or gestured, etc., and depend on the purpose for such use [16, 17]. For instance, the same word P = odd, when used in the context of Arithmetic has a different meaning than when it is used in a social context, and, if in the first case it can only be used with purposes limited by the definition 'the rest of its division by two is one', in the second, it can be used with several and open purposes such as the descriptive, the insulting, etc. In the first case, its use is precise or rigid, but in the second it is imprecise or flexible. Analogously, in different contexts the word 'interesting' can be used with non coincidental purposes. Semantics is what really matters in language; without capturing the meaning of words, language is unintelligible.

The meaning of a word P is privative of a given universe of discourse; for instance, the same person can be tall in a population of pygmies, and short in one of giants. The concept of the meaning requires the joint consideration of both a universe of discourse X, and the word P, as it was formerly shown with the same word 'odd' in the respective universes of integer numbers, and of people.

2.8. To capture what it means P in X, it is necessary to know the relationship 'x is less P than y', expressing the recognition that x shows the property p named P less than y shows it [17, 18]; how the application of P varies along X. Of course, if P is precisely used in X, such a relationship just degenerates into 'x is equally P than y', and 'x is not equally P than y'. For instance, the numbers 3 and 21, 515 are equally odd, and the numbers 3 and 20 are not equally odd; in the universe of integer numbers there are only odd and not odd numbers.

Let's symbolically represent such relationship by '$x <_P y$'; it can be equivalently said that 'y is more P than x'. The symbol $<_P$ reflects a mathematical relation in X, of which it only can be quietly asserted that is reflexive, $x <_P x$, for all x in X; a property assuring that the relation $<_P \subseteq X \times X$ is not empty. The graph $(X, <_P)$ specifies the 'qualitative meaning' of P in X, and when the use of P is precise, the relations $<_P$ and $<_P^{-1}$ (whose intersection gives the relation $=_P$, 'equally P than'), just collapses in $=_P$. The qualitative meaning of a precisely used word P is the graph (X, $=_P$), between whose arcs are the loops at each element in X, and, of course, if P is imprecise in X, the graph (X, $<_P$) contains the graph (X, $=_P$).

These graphs agree with the common view that when telling something people introduce some organization, or rough order, in what is taken into account.

By another side, it should be observed that, in plain language, words do 'collectivize' in the universe of discourse. For instance, if X is the set of London's inhabitants, P = young facilitates in X the linguistic-collective of the 'Young Londoners'. Linguistic-collectives, such as 'ripe tomatoes', 'high mountains', 'comfortable chairs', 'tall people', etc., are well anchored in plain languages since its speakers not only easily use them, but immediately capture what they express. Such 'collectives' are linguistically generated in X by the corresponding meaning of P in X, and, by using an old philosophical expression, it allows us to say that the linguistic label has some 'extension' in the corresponding universe of discourse.

Of course, linguistic-collectives are not always sets; they are only sets if P is precise, as it is, for instance, in the collective of the 'thirty five year old Londoners', as it is guaranteed by the 'axiom of specification' of the theory of sets [4]. Even if they are a kind of cloudy entities, linguistic-collectives are empirically recognized, they exist in language; they are 'a reality' in language like clouds are in the atmosphere.

A fuzzy set in X with linguistic label P, is nothing else than the linguistic-collective generated by P in X; it can be specified by the graph (X, $<_P$) if P is imprecise, or the simpler graph (X, $=_P$) if P is precise. There is no difference between the concepts of linguistic-collective, fuzzy set, and qualitative-meaning; they just denote the same concept.

2.9. Once a qualitative meaning of P in X is recognized, it should be pointed out that its defining relation $<_P$ is not always a linear one; that is, there are often elements x and y in X such that it is neither x $<_P$ y, nor y $<_P$ x; elements that are 'not comparable' under $<_P$. Typical examples are obtained with P = interesting in a universe of possible businesses, with P = beautiful in a universe of paintings, with P = odd in a universe of people, with P = nice with houses, etc. In collections of paintings for instance, there are often pairs of which it is impossible to state that any one of them is less, or more beautiful than the other.

Once the qualitative meaning (X, $<_P$) is captured, it can be said that P is 'measurable' in X, since measures m_P of the extent up to which each x is P can be defined analogously to the former case of P = big in X = [0, 10]. If no relation $<_P$ can be even imagined, P is 'meaningless', or 'metaphysical' in X [17]. For instance, if $<_{big}$ is recognized as the linear order $\leq$ of the interval [0, 10] when such a word is used in it, there is no way of knowing $<_{big}$ seems to exist when 'big' is applied to dreams; hence 'big' is measurable in [0, 10], but it is in principle meaningless among dreams.

Although measurability is truly important for a scientific domestication of concepts, the metaphysical ones cannot be fully contemptible since at least they can have a true 'suggestive power'. What can be measured belongs to reasoning, and what cannot belongs to thinking; reasoning is but an organization of thinking for directing it towards a goal. A word that is currently metaphysical in X, can suggest to be applied in Y (different of X) in a measurable way. Anyway, this topic belongs to the kind of questions that are beyond what this paper is trying to consider.

3 How Fuzzy Sets Can Be Computationally Managed?

3.1. In the first place, it should be pointed out that a general definition of what is a measure should be liberated from the typical additive law always presumed in probability theory. It supposes that 'things grow' by superposition of non-overlapping pieces, and that the total measure is the sum of the measures of such pieces; something that cannot be always presumed, and less again with cloudy entities resembling linguistic-collectives.

From the background of game theory and decision-making, the Japanese physicist and engineer Michio Sugeno also had the idea that the property of additivity seemed to be too strong and therefore he reduced the integral form to monotonicity. In a later interview he recalled: "I put the adjective 'fuzzy' to this monotone measure simply because max-min-operations were used in its integral form as in fuzzy sets; this naming was later found to be not adequate. More precisely, I should have called it monotone measure or even 'non-additive measure'. I found that the monotonicity of the fuzzy measure well fits the calculations of max-min." Sugeno gave a mathematical foundation to this "fuzzy integral", it was later called the "Sugeno integral", at first in a Japanese journal in 1972 and in 1974 published in his doctoral thesis "Theory of Fuzzy Integrals and its Applications" [13–15].

Adapting Sugeno's concept of a 'fuzzy measure' [15] allows a general clear enough definition of a measure of meaning as follows [17, 18, 20].

Given the qualitative meaning (X, $<_P$), a measure of it is a mapping m_P: X [0, 1], such that:

(1) $x <_P y \Rightarrow m_P(x) \leq m_P(y)$,
(2) If z is maximal relatively to $<_P$, it is $m_P(z) = 1$,
(3) If z is minimal relatively to $<_P$, it is $m_P(z) = 0$.

This definition deserves some comments. Concerning (1), it just reflects that the growing variation of P along X, expressed by $<_P$, is translated into the growing of numbers in the unit interval given by its linear order $\leq$; what it does not reflect is how this growing is produced, something that each type of problem would require a particular form of decomposing elements in constitutive pieces.

Concerning (2), z is a maximal provided no other $x \in X$ exists such that $z <_P x$; a maximal is a 'prototype of P' in X, and its existence cannot be taken for granted, but if a single one exists it is called the maximum. Concerning (3), z is a minimal provided no other y exists such that $y <_P z$; a minimal is an 'anti-prototype of P' in

X, and its existence is also unsure, but if a single one exists it is called the minimum. When no maximal, or no minimal exist, then laws (2) or (3) cannot be applied; anyway, it does not imply that elements in X with measure one or zero are inexistent, such elements, if existing, can be respectively called 'working prototypes', and 'working anti-prototypes'. For instance, it is well known in probability theory there can be elements in the sigma-algebra of events that, not being an empty event, have a zero probability.

Nevertheless, the existence of prototypes and anti-prototypes seem to make consistent the recognition of the qualitative meaning; they allow the comparison of other elements in a form similar to the old Standard Meter Prototype for the (decimal system's) measuring of length. For instance concerning P = tall in a population, once recognized that Ruth is the tallest among inhabitants (Ruth is a prototype of tall), the tallness of the others is recognized by comparing them with Ruth; moreover, a stick with the same height as Ruth can serve for attaching relative and fractional measures of tallness. Of course, the same can be said once it is recognized that John is a less taller inhabitant, that he is an anti-prototype of tall in the population under consideration. For instance, in X = [0, 10], and respect to the former toy-example with P = big, there is initially recognized a unique maximal (the maximum 10), and a unique minimal (the minimum 0).

It should be pointed out that in the original Sugeno's definition of a 'fuzzy measure', X is a power-set 2^{Ω}, and the comparable relation (<) is the inclusion of sets $\subseteq$ with the maximum Ω, and the minimum Ø.

3.2. Let us recall that the three laws of a measure do not allow us to specify a single one unless P is precisely used in X; for instance, in the case of 'big' it was necessary to add some contextual information, or some reasonable hypothesis on the shape of the curve $y = m_{big}(x)$, as it could be, respectively, that it passes through (5, 0.6), or that it is linear, or quadratic, etc. If P is imprecise there is not a single measure for the meaning of P in X, but rather a set of them. It is similar to what happens, for example, with the probabilities of getting 'n points' ($1 \leq n$ 6) by throwing a single die, or with Sugeno's fuzzy measures containing many types of them as they are the big family of additive, sub-additive, and super-additive lambda-measures [15].

Only in the precise case is the measure unique. Since it is,

$$\text{If } x =_P y \Leftrightarrow x <_P y \ \& \ y <_P x => m_P(x) \leq m_P(y) \ \& \ m_P(y) \leq m_P(x) \Leftrightarrow$$
$$m_P(x) = m_P(y),$$

the measure preserves the relation $=_P$ that, in addition to reflexive is also symmetrical; provided $<_P$ were transitive, then $=_P$ is also transitive and, thus, an equivalence. Hence, because X is perfectly classified in the set specifying P (containing all the prototypes), and its complement (containing all the anti-prototypes), it only exists in the measure given by $m_P(x) = 1$ for all the first, and $m_P(x) = 0$ for all the second.

Notice that the only set with neither prototypes, nor anti-prototypes, is the empty set; it is a very 'odd set' since it is self-contradictory, $\emptyset \subseteq X = \emptyset^c$; indeed, it is the only self-contradictory set: $A \subseteq A^c \Longleftrightarrow A = \emptyset$. Its acceptance as a set derives at least, from two practical reasons; the first is for guaranteeing that the intersection of sets is always a set, and the second for denoting the 'extensionality' of statements that are non-applicable to a given universe, and as it is, for instance, 'getting nine' in the universe of the six elemental events {1, 2, 3, 4, 5, 6} obtainable when throwing a single die.

Hence, it can be said that the linguistic-collective given by the graph $(X, <_P)$, shows several contextual states $(X, <_P, m_P)$; each time a measure is specified, a 'contextual informational state' (in short, 'state') of the fuzzy set is manifested. Measures, or membership functions, represent known states of the fuzzy set; once designed, a membership function is practically employed to 'describe' the corresponding fuzzy set; they facilitate the management of fuzzy sets for computational purposes.

3.3. Once a measure m_P is specified, it defines the new relation in X,

$$x \leq_{mP} y \Leftrightarrow m_P(x) \leq m_P(y),$$

that is a linear ordering in X, and is obviously greater than $<_P$:

$$x < y = > m_P(x) \leq m_P(y) \Leftrightarrow x \leq_{mP} y; \text{ that is, } <_P \subseteq \leq_{mP}.$$

If there can be elements x and y for which it is neither $x <_P y$, nor $y <_P x$, nevertheless one of the two numbers m_P (x), m_P (y) will be greater than the other; there are no incomparable elements under $\leq_{mP}$. The difference set $\leq_{mP} - <_P$ is not always empty, even not necessarily if $<_P$ is linear.

Thus, the new and linear relation $\leq_{mP}$ enlarges the qualitative meaning, and gives the linear 'working' meaning $(X, \leq_{mP})$, only known after a measure m_P is specified, that is, a state of the fuzzy set is described; the working meaning is not unique and comes before specifying a measure. The process of designing a measure [19], can conduct (as it typically happens in the applications of fuzzy set theory) to only considering P through the 'membership function' $y = m_P$ (x), and, then, to the possibility of forgetting its qualitative meaning and just considering its working meaning.

It should be clear by now, that membership functions are (ideally) measures of words that are measurable in a universe of discourse; measures designed accordingly with what at each case is available on the relation $<_P$. It is said 'ideally', since in the design process some deviation from a measure can appear. It is something similar with stating that the probability of obtaining 'five points' in throwing a single die is 1/6, by presuming the die is perfectly constructed, and that the landing surface is perfectly smooth. Is an 'ideal die'.

Membership functions should be seen as 'designed approximations' to measures, a respect at which it lacks a definition that, for instance, can be the following: A designed membership function μ_P can be seen as a 'good enough' one provided for each $\varepsilon > 0$ it exists a measure m_P such that $Im_P (x - \mu_P (x)I < \varepsilon$, for all x in X. Obviously, when μ_P is itself a measure this definition is immediately satisfied.

Measures/membership functions/states of the fuzzy set, are always designed on the basis of the current knowledge of the relation 'less P than' available to the designer, and only if the linguistic label is precise can it be potentially considered as perfectly known. Nevertheless, it does not mean that the membership function's values of an either precise, or imprecise label can be easily computed; it suffices to think in the membership function of the precise linguistic label 'transcendental' for real numbers. It is very difficult to know if a somehow defined real number is, or is not, expressible by an integer followed by a denumerable number of digits without any pattern's periodicity. It is the case for proving that the numbers π, e, e^{π}, the Euler-Mascheroni constant γ, the Liouville constant $\Sigma 10^{-n!}$, the Dottie number x (such that cos $x = x$), the Chaitin halting constant Ω, etc., are actually non-algebraic, or transcendental. Proving that these numbers are transcendental requires sophisticated mathematical methods; the set of transcendental numbers has the power of continuum, is one-to-one and adjacent with all the real numbers of which they are a particular case. As Mathematics demonstrates, there are extremely complex precise concepts; precise is not a synonym of easy.

Notice that if relation $<_P$ is (artificially) identified with its sub-relation $=_P$, it means a forced 'precisification' of P in X, a change of its qualitative meaning that will imply just considering a {0, 1}—valued membership function for P, and avoiding the multiple measures that can exist for the graph (X, $<_P$). It is a risky change of the meaning of P in X, since it can imply an understanding of P in a different form than that in current use. It can happen, for instance, if 'x is big' is understood in [0, 10] and in plain language, as '$x > 8$'.

To summarize what has been said: Given a measurable linguistic label P in a universe of discourse X, its qualitative meaning generates in X a unique linguistic-collective, or fuzzy set, labeled P, that can be denoted by ***P***. To consider ***P*** in a form allowing for its scientific and practical management, a measure of its qualitative meaning/linguistic-collective/fuzzy set, should be carefully designed to represent its contextual informational state. Once anyone of such possible states/membership functions m_P is 'designed', thanks to the contextual information furnished by $<_P$ plus the additional information (or the reasonable hypotheses) the designer will be able to add (for example, that the measure is a linear function, a triangular one, a bell shaped one, etc.), the particular use of P in X, or the linguistic-collective/fuzzy set, is just 'seen' from such membership function m_P, that is, from the currently known state of the fuzzy set. There should be an awareness of the danger that can exist when forcing an imprecise P up to be precise.

Hence, there will be cases in which the qualitative meaning $<_P$ will not fully coincide with the corresponding state's working meaning $\leq_{mP}$; the second will actually enlarge the first, and, in addition, the working meaning is not only linear but can introduce into the graph (X, $<_P$) new arcs, that depending on the character of what is presumed for the design of the measure, can be spurious. Moreover, there can appear new working prototypes, or anti-prototypes than those that were initially recognized as such; that is, elements with a measure of one or zero that are not properly qualitative prototypes, or, respectively, qualitative anti-prototypes under $<_P$. Only seeing the meaning of P in X by a working meaning of it, implies some sort of risk.

It should still be pointed out that the usual identification of m_P (x) with a truth degree of the statement ‘x is P’, can result in identifying meaning with truth; that is, to simplify the concept of meaning to that of truth. Truth (= T) is a concept that, in its turn, has a meaning in the universe X [P] of the statements x is P, $x \in X$, and that should be formerly recognized for specifying a measure m_T for the meaning of T in X [P]. Actually, such identification means accepting m_P (x) = m_T (x is P), something very risky if no qualitative meaning of T is known, and that previously requires proving that m_T (x is P) is, for all x in X, a measure of the meaning of P in X. Surely, it will require establishing some criterion of ‘compatibility’ [17] between P in X, and T in X [P].

4 Towards Approaching Qualitative and Working Meanings

4.1. When the working meaning coincides with the qualitative meaning, it can be said that the graph (X, $\leq_{mP}$) perfectly reproduces the initially ‘observed’ graph (X, $<_P$); the measure is not adding a, may be spurious, information on the use of P in X. Because of the non-linear character of $<_P$, and the linear of $\leq_{mP}$, a perfect reproduction of the previously observed qualitative meaning is not always reached; the act of designing a measure valued in the unit interval can modify what was formerly observed.

Fuzzy sets can be practically and computationally managed thanks to their states, and seeing a fuzzy set through a current state can mean a true modification of the linguistic-collective produced by changing (X, $<_P$) by (X, $\leq_{mP}$). To ‘observe’ a fuzzy set is only possible under a ‘microscope’ showing its information’s states as best as possible.

Measuring can modify the initial qualitative meaning, and, obviously, the same can happen if [0, 1] were changed by whatsoever closed interval [a, b] in the real line, with prototypes taking the measure b, and anti-prototypes the measure a, and preserving property (1) of the measure.

Anyway, such a topic can be considered from two points of view. The first, is due to the practical fact that there are actual cases in which it is very difficult, if not impossible, to appreciate at each point that the measure is exactly some number in [0, 1]; for instance, sometimes it can be only recognized that such number belongs to some interval. The second, is that in plain language there are often proffered statements as, for example, ‘It is highly possible that he is rich’, or ‘It is barely possible that she is a gifted girl’, etc., whose (exact) numerical degree is surrounded by such an amount of uncertainty that it seems to better correspond to a blurred, approximate, number like ‘around 0.8’, ‘less than 0.5’, ‘between 0.4 and 0, 6’, ‘high’, etc.

Thus, there are actual situations in which it can be more suitable to adopt a range for the measure's values different from the real line, and not being linearly ordered. It is similar, for instance, to what happens when measuring the 'electrical impedance' by a complex number whose real part is the 'resistance', and its imaginary part is the 'reactance'. Recall that complex numbers are not linearly, but partially ordered, and that if its use was required by the two-fold physical phenomenon, often enough linguistic phenomena are very complex. Hence, reconsidering the range of values a measure can take cannot appear as something bizarre.

4.2. Instead of the unit interval for the values of the measure, it can be supposed a partially ordered set $(V, \leq)$, with maximum ω and minimum α. With it, the laws of a measure m ranging in V ($m: X \to V$) can be easily changed by preserving its first property, but placing ω instead of 1 in the first, and α instead of 0 in the second. With this change, the working meaning $\leq_{mP}$ is not more linear, but a partial order, whose coincidence with $<_P$ cannot be guaranteed, but it can be expected that more possibilities for it may appear. Preserved the inclusion $<_P \subseteq \leq_{mP}$, the difference set $\leq_{mP} - <_P$ have more chances to be either empty, or, at least, reduced to contain less arcs; in sum, to approach the first to the second relation. This depends on the particular problem, but it seems clear that the chances for reaching a perfect representation of the qualitative meaning by the working one will not decrease.

Two possibilities for V are the complex unit interval $\{a + ib; a, b \in [0, 1]\}$, and the set of the closed sub-intervals $[a, b] \subseteq [0, 1]$, that actually are not mathematically different sets since they can be seen as isomorphic [20, 33]. Anyway, the first can have the advantage of admitting the writing of complex (Cartesian) numbers $a + ib$ by its Euler's modulo-argument expression $\rho. e^{i\theta}$, with $\rho = \sqrt{(a^2 + b^2)}$, and $\theta = \tan^{-1}(b/a)$ that eventually can allow geometrical considerations to be added in a given problem.

Another candidate is the set of 'fuzzy numbers', those functions $[0, 1] \to [0, 1]$ specifying linguistic labels like 'high', 'around 0.6', 'bigger than 0.4', 'between 0.3 and 0.4', etc. It is clear that the subintervals [a, b] of [0, 1] can be seen as a particular type of fuzzy numbers; for instance, the interval [0.3, 0.4] is the same as the membership function equal to 1 in it, and to zero in $[0, 0.3) \cup (0.4, 1]$. The label 'high', for example, can be represented by the identity function, or by its square, etc.

Membership functions, and in particular those of fuzzy numbers in [0, 1], are almost always point-wise ordered by,

$$\mu \leq \sigma \Leftrightarrow \mu(x) \leq \sigma(x), \text{ for all x in } [0, 1],$$

a partial order that comprises the linear ordering of crisp numbers. Its minimum is the function $\mu_0(x) = 0$, and its maximum is $\mu_1(x) = 1$, both for all x in X. Of course, and even if it can go against simplicity, other orderings can be chosen for the set $[0, 1]^{[0, 1]}$ (containing, at least, all membership functions) and perhaps that can be better related with a given problem. In any case, when restricted to precise numbers, such orderings should coincide with the linear order of the unit interval, and before deciding to change the unit interval by the set of 'fuzzy numbers', the designer should try to refine and improve the design.

In this way a fuzzy set ***P***, labeled P, can be newly specified by the membership function of a fuzzy number,

$$m_P(x) = m_{\text{fuzzy number}};$$

this is, in essence, what was initially called a 'type-two fuzzy set' in the 3-part article from 1975 "The concept of a linguistic variable and its application to approximate reasoning" [33]. In the first part Zadeh introduced these type-2 fuzzy sets as follows: "… suppose that A is a fuzzy subset of a universe of discourse U, and the values of the membership function, μ_A, of A are allowed to be fuzzy subsets of the interval [0, 1]. To differentiate such fuzzy sets from those considered previously, we shall refer to them as fuzzy subsets of type 2 with the fuzzy sets whose membership functions are mappings from U to [0, 1] classified as type 1."

That is, the measure does not take numerical values at each point, but membership function ones; it is another and wider representation of the information's state of the fuzzy set, and requires to previously fix a partial ordering for the membership functions of fuzzy numbers for guaranteeing its measure's character. Since the fuzzy set is but a different name for the linguistic-collective, a 'type-two fuzzy set' refers to nothing else than a new representation of the states of the fuzzy set/linguistic-collective by means of functional values; that is, a particular type of non exclusively numerical values able to take into account the uncertainty associated to the difficulties for establishing a crisp number as a measure when it exists. It is a way of 'fuzzifying' the membership idea of 'being in' a fuzzy set, and for trying to approach the qualitative and the working meanings. More generally, in the same paper Zadeh defined then "A fuzzy set is of type n, n = 2, 3, …, if its membership function ranges over fuzzy sets of type n-1. The membership function of a fuzzy set of type 1 ranges over the interval [0, 1]" [33].

4.3. It is still possible to take V as a set of pure words, defining the partial ordering between them, in a form that can be associated with what they mean.

A particular and suggestive case is the following. If $<_P$ is a preorder (that is, a transitive relation in addition to its presumed reflexivity), with a maximum r, and a minimum t, then the relation $=_P$ is an equivalence generating the quotient set $X/=_P$, constituted by the equivalence classes $[x] = \{y \in X; y =_P x \Longleftrightarrow x <_P y \,\&\, y <_P x\}$. These classes inherit the order of X through the definition $[x] <^* [z] \Longleftrightarrow x <_P z$; $<^*$ is a partial order, and $(X/=_P, <^*)$ is a partially ordered set with maximum $[r] = \{r\}$, and minimum $[t] = \{t\}$.

Then, the mapping $m^*: X \to X/=_P$, given by $m^*(x) = [x]$, verifies the three laws of a measure valued in the partially ordered set $(X/=_P, <^*)$, and, obviously, $<^*_{m^*}$ is isomorphic with $<_P$, that is, the working meaning perfectly reflects the qualitative meaning. Measure m* can be named the 'natural measure'.

Provided each class [x] can be named by a word synthesizing what it represents, $(X/=_{P,} <^*)$ is isomorphic to the set of these words once it inherits the partial order $<^*$. Hence, and without going outside the problem's data, an example is obtained of a natural and perfect representation by words of the qualitative meaning.

4.4. There is again another view facilitated by the so-called ‘interval type-two fuzzy sets’ [11], employed in some applications and consisting in not considering the subintervals by their crisp membership functions, but by a kind of blurred triangular membership function for them, and more or less inspired in [33]. In this case, capturing the way of ordering these blurred functions, allowing to look at them as meaning’s measures, is still difficult. It is not even clear enough what happens with prototypes and anti-prototypes; that is, how their ‘measure values’ are defined, what are the ‘subintervals’ null and unity, and if they are respectively the minimum and the maximum among them; at the end, and when linguistically describing some system, precise words can also appear in between the imprecise ones, and the maximum and minimum values preserved for them.

Mendel’s ‘interval type-two fuzzy sets’ could be a different form of representing words, but its relation with their meaning and its measuring needs to be clarified. At least it should remain that the words considered for computing with words should be, in some clear sense, measurable.

5 Conclusion

5.1. For what has been said, expressions like ‘type-two fuzzy set’ are not properly appropriate since, after recognizing a qualitative meaning for the corresponding linguistic label, the fuzzy set is, in a given context, a unique, although nebulous, well anchored entity in language, or, if it is preferred, in thought. It is not to be forgotten that sets are also entities of thought; comparatively few sets that are of interest in science can be imagined like apples in a basket. As if somebody could ‘see’ the transcendental points between two of them in a straight line? Like sets, fuzzy sets are a creation of thoughts; like sets, fuzzy sets need some representational methodology for their practical management; for instance, that given for their membership functions/states.

Without previously knowing a qualitative meaning, it does not seem possible to follow a study allowing, in a form useful for computing with words, to practically manage the big amount of imprecise words permeating plain languages; these words should be measurable, not meaningless. Qualitative meaning, or fuzzy set, is what can be measured.

Nevertheless, to change the values for measuring the meaning’s extent from the real line into a different but partially ordered structure, is something that can eventually even conduct to approaching the working and the qualitative meanings. Anyway, the character of any kind of membership functions approaching ‘a measure of meaning’ should be preserved.

Such change, for instance, can be particularly made within the set of fuzzy numbers in [0, 1], instead of employing the unit interval, and once fuzzy numbers are represented by some functions in $[0, 1]^{[0.\ 1]}$, previously endowed with a partial ordering preserving the linear one formerly existing between the ‘crisp’ numbers included in it. Provided such ordering is not the usual point-wise, its definition can constitute an added problem. When there is great uncertainty about the precise values of the membership function, it can be a good option, provided the ordering is defined and it has a minimum and a maximum. The problem is even more difficult if intervals are considered through a blurred kind of a triangular membership functions whose ordered structure is not well established; it waits to be clarified.

5.2. In the light of what has been presented, instead of expressions like 'type-two fuzzy set', it can be better expressed as

'fuzzy set with type-two membership function',

or something similar. In any case, it can be suitable any name not introducing in the darkness the unique, and previously existing, fuzzy set/linguistic-collective which is to be managed by means of its states, this time given by the designed membership functions of those particular fuzzy sets in [0, 1] whose linguistic label denotes either an interval, or a blurred number, instead of a crisp number. There are neither 'type-one', nor 'type-two' fuzzy sets; there are only fuzzy sets that are a purely linguistic concept.

Once X and P are given, and the graph $(X, <_P)$ is known, the linguistic-collective ***P*** is commonly and empirically recognized to exist in X. Of course, for counting with an axiomatic theory of fuzzy sets, it lacks a definition of what it can mean that two linguistic-collectives ***P*** and ***Q*** coincide in X; something that, perhaps, could be achieved by defining that their respective qualitative meanings $(X, <_P)$ and $(X, <_Q)$ are isomorphic. A different topic is how the states should be represented for practically and computationally managing the fuzzy set in a form that can be suitable for a given problem, and as it is the case, for example, when it only can be asserted that m_P (x) belongs to an interval [a (x), b (x)], depending on x, or that the difference set $\leq_{mP} - <_P$ is too wide.

What has been said should only be understood as a theoretical prevention against using names like 'type-two fuzzy sets', or even 'intuitionistic fuzzy sets' [1]; names that can conduct to presume the existence of different types of linguistic-collectives. But, if such types actually existed, the differences among them arise from how its qualitative meaning is expressed, but not from how the informational states can be represented at each context in a given universe of discourse. In any case, and as simplicity is of utmost importance, advising the designer: 'Never change the unit interval before trying to improve the membership function values', does not seem to be a bad advice.

In one way or another, those names could be scientifically accepted as a shortening, but shortenings should be always explained!

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