

Similarity Degree for Multi-Attribute Decision Making with Incomplete Dual Hesitant Fuzzy Sets

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Abstract. Due to the uncertainty and fuzziness of the real world, the dual hesitant fuzzy set (DHFS) has been proposed to express uncertain information during the process of multi-attribute decision making (MADM). In order to process the information with incomplete dual hesitant fuzzy elements (IDHFEs) in MADM, a new similarity degree for MADM with incomplete dual hesitant fuzzy sets (IDHFSs) is proposed. The concept of similarity degree of IDHFEs and similarity aggregation matrix are introduced. Then a complete dual hesitant fuzzy matrix (CDHFM) is obtained by using the maximum similarity to complement the data. An investment selection example is provided to illustrate the validity and applicability of the proposed method.

Keywords: Incomplete dual hesitant fuzzy element · Maximum similarity · Multi-attribute decision making

1 Introduction

MADM is a common process to human beings. In classical MADM, the assessments of alternatives are precisely known [1]. However, because of the inherent vagueness of human preferences and the uncertainty of objects, the attributes involved in decision making problems are better denoted by the fuzzy concept. Fuzzy sets have been introduced by Zadeh [2]. Subsequently, many scholars have done a lot of achievements on fuzzy decision making [3, 4, 5]. Hesitant fuzzy set (HFS), which has been introduced by Torra and Narukawa as an extension of fuzzy set [6, 7], describes the situations that permit the membership of an element to a given set having a few different values, which is a useful means to describe and deal with uncertain information in the process of MADM. However, the HFS only depicts people's membership and ignores the people's non-membership. Therefore, Zhu et al. have proposed the DHFS [8]. Comparing with the classical fuzzy set theory, DHFS increase the non-membership degree to describe the fuzzy nature of the objective world comprehensively and show a strong flexibility and practicality.

In recent years, hesitant fuzzy MADM has received more and more attention. Herrera-Videma et al. have emphasized fuzzy preference matrix and studied different forms of expression in fuzzy preference matrix approach [9]. Xu et al. have defined the distance and similarity measure of the HFSs [10, 11]. Zhang et al. have proposed hesitant fuzzy power aggregation operator to aggregate MADM evaluation values [12]. Xia and Xu also have developed some aggregation operators for hesitant fuzzy information and applied them in MADM problems under the hesitant fuzzy environment [13]. In the real life, people often judge the truth-values of a fuzzy proposition with language. Lin R has defined the hesitant fuzzy linguistic set by combining the advantages of linguistic evaluation values and HFS [14]. On the basis of multi-hesitant fuzzy sets, literature [15] has proposed the multi-hesitant fuzzy linguistic term sets.

The above studies are based on the assumption that the evaluation information is complete. However, due to the complexity of the decision making environment, the hesitation of the decision makers and the improper operation in the decision making process, the information is often incomplete. In order to solve the problem of MADM with information loss, the information is usually complemented by some methods. Therefore, this paper proposes a multi-attribute decision making method based on the maximum similarity of IDHFSs. We define the similarity of the IDHFEs in the IDHFSs and compare the similarity among individuals to complement the maximum data. An example is then provided to illustrate the proposed approach is more flexible and effective.

2 Preliminaries

DHFS, as a generalization of fuzzy set, permit the membership degree and the non-membership degree of an element to a set presented as several possible values between 0 and 1, which can describe the situations where people have hesitancy in providing their preferences over objects in the process of decision making.

Definition 1 [17]. Let X be a fixed set, then DHFS on X is defined as:

$$D = \{ \langle x, h(x), g(x) \mid x \in X \rangle \}$$

where $h(x)$ and $g(x)$ are two sets of several values in $[0, 1]$, representing the possible membership degrees and the non-membership degrees for $x \in X$. Also, there is

$$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1$$

where $\gamma \in h(x), \eta \in g(x), \gamma^+ \in h^+(x) = \cup_{\gamma \in h(x)} \max \{ \gamma \}$ and $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} \max \{ \eta \}$. The DHFS is composed of dual hesitant fuzzy elements (DHFEs), which is denoted by $d(x) = (h(x), g(x))$ ($d = (h, g)$ for short).

To compare the DHFSs, Zhu et al. have gave the following comparison laws:

Definition 2 [8]. Let $A_i = \{h_{A_i}, g_{A_i}\} (i = 1, 2)$ be any two DHFSs, $s_{A_i} = \frac{1}{l_h} \sum_{\gamma \in h} \gamma - \frac{1}{l_g} \sum_{\eta \in g} \eta (i = 1, 2)$ the score function of $A_i (i = 1, 2)$ and $p_{A_i} = \frac{1}{l_h} \sum_{\gamma \in h} \gamma + \frac{1}{l_g} \sum_{\eta \in g} \eta (i = 1, 2)$.

The accuracy function of $A_i(i = 1, 2)$, where l_h and l_g be the number of elements in h and g , respectively, then

- (i) if $s_{A_1} > s_{A_2}$, then A_1 is superior to A_2 , denoted by $A_1 \succ A_2$;
- (ii) if $s_{A_1} = s_{A_2}$, then
 - (1) if $p_{A_1} = p_{A_2}$, then A_1 is equivalent to A_2 , denoted by $A_1 \sim A_2$;
 - (2) if $p_{A_1} > p_{A_2}$, then A_1 is superior to A_2 , denoted by $A_1 \succ A_2$.

3 Similarity of Incomplete Dual Hesitant Fuzzy Elements

In the real life, due to the preference of experts and the influence of external factors, some values of the information can't be given. So there is a case where the DHFEs are incomplete when the values of the information are represented by DHFEs. Therefore, we divide the DHFEs into complete double hesitation fuzzy elements and incomplete double hesitation fuzzy elements. The corresponding definitions as follows:

Definition 3. Let $U = \{u_1, u_2, \dots, u_n\}$ be a fixed set and $A_i = (h_i, g_i)$ be any DHFEs on $U, i = 1, 2, \dots, n$.

- (1) if $n_i^h = (n_i)^* = n_i^g$, then the DHFEs are called complete dual hesitant fuzzy elements, denoted by CDHFEs; The matrix(s) composed of CDHFEs are called complete double hesitation fuzzy matrix(s), denoted by CDHFM(s); The set(s) composed of CDHFEs are called complete double hesitation fuzzy set(s), denoted by CDHFS(s).
- (2) if $n_i^h \neq (n_i)^* \neq n_i^g$, then the DHFEs are called incomplete dual hesitant fuzzy elements, denoted by IDHFEs. The matrix(s) composed of IDHFEs are called incomplete double hesitation fuzzy matrix(s), denoted by IDHFM(s); The set(s) composed of IDHFEs are called incomplete double hesitation fuzzy set(s), denoted by IDHFS(s).

Where n_i^h and n_i^g are the number of values in h_i and g_i , respectively. $(n_i)^*$ is the largest value of n_i^h and n_i^g .

Singh [16] has proposed the similarity measures of CDHFSs. However, it is no longer applicable for IDHFS(s). We improve the definition of similarity measures in [16] and give the definition of similarity degree of IDHFEs.

Definition 4. Let $A(u_\mu) = (h(u_\mu), g(u_\mu))$, and $B(u_\nu) = (h(u_\nu), g(u_\nu))$ be two IDHFEs, the similarity degree $s_{IDHFSs}(A, B)$ between A and B is defined as:

$$s_{IDHFSs}(A, B) = 2 \frac{\frac{1}{p} \sum_{k=1}^p h_{ik} \cdot \frac{1}{q} \sum_{s=1}^q h_{js} + \frac{1}{\alpha} \sum_{l=1}^{\alpha} g_{il} \cdot \frac{1}{\beta} \sum_{r=1}^{\beta} g_{jr}}{\left(\frac{1}{p} \sum_{k=1}^p h_{ik}\right)^2 + \left(\frac{1}{q} \sum_{s=1}^q h_{js}\right)^2 + \left(\frac{1}{\alpha} \sum_{l=1}^{\alpha} g_{il}\right)^2 + \left(\frac{1}{\beta} \sum_{r=1}^{\beta} g_{jr}\right)^2} \tag{1}$$

If we take weight of each element $x \in X$ for membership function and non-membership function into account, then

$$s_{IDHFEs}(A, B) = 2 \frac{\sum_{k=1}^p \omega_{\mu k} h_{\mu k} \cdot \sum_{s=1}^q \omega_{vs} h_{vs} + \sum_{l=1}^{\alpha} \omega_{\mu l} h_{\mu l} \cdot \sum_{r=1}^{\beta} \omega_{vr} h_{vr}}{(\sum_{k=1}^p \omega_{\mu k} h_{\mu k})^2 + (\sum_{s=1}^q \omega_{vs} h_{vs})^2 + (\sum_{l=1}^{\alpha} \omega_{\mu l} h_{\mu l})^2 + (\sum_{r=1}^{\beta} \omega_{vr} h_{vr})^2} \quad (2)$$

where $\sum_{k=1}^p \omega_{\mu k} h_{\mu k}$, $\sum_{s=1}^q \omega_{vs} h_{vs}$, $\sum_{l=1}^{\alpha} \omega_{\mu l} h_{\mu l}$, $\sum_{r=1}^{\beta} \omega_{vr} h_{vr}$ are called the weighted summation of each element $x \in X$ for membership function and non-membership function, respectively. Particularly, if each element has same importance, then (2) is reduced to (1).

According to the Definition 2, we can easily proof Definition 4 satisfies the following properties:

Proposition 1. Let A, B and C be any IDHFSs, then $s_{IDHFEs}(A, B)$ is the similarity degree, which satisfies the following properties:

- (i) $0 \leq s_{IDHFEs}(A, B) \leq 1$;
- (ii) $s_{IDHFEs}(A, B) = 1$ if and only if $A = B$;
- (iii) $s_{IDHFEs}(A, B) = s_{IDHFEs}(B, A)$;
- (iv) Let C be any IDHFS, if $A \subseteq B \subseteq C$, then $s_{IDHFEs}(A, B) \geq s_{IDHFEs}(A, C)$, $s_{IDHFEs}(B, C) \geq s_{IDHFEs}(A, C)$.

Definition 5. Let $A = (h_D(u_\mu), g_D(u_\mu))$ and $B = (h_D(u_\nu), g_D(u_\nu))$ be two IDHFEs, then the similarity matrix of A and B on attribute c_j is defined as:

$$R_{c_j} = (x_{\mu\nu})_{m \times m}, \text{ where } x_{\mu\nu} = s_{c_j}(A, B), 1 \leq j \leq n$$

Definition 6. Let $A = (h_D(u_\mu), g_D(u_\mu))$ and $B = (h_D(u_\nu), g_D(u_\nu))$ be two IDHFEs, then the similarity aggregation matrix R_{\aleph} is defined as:

$$R_{\aleph} = (r_{\mu\nu})_{m \times m} = \frac{\sum_{j=1}^n R_{c_j}(x_{\mu\nu})}{n}, \text{ where } r_{\mu\nu} = x_{c_1(\mu\nu)} \cdot x_{c_2(\mu\nu)} \cdot \dots \cdot x_{c_j(\mu\nu)}$$

4 Application of Proposed Similarity Degree in MADM

4.1 Method of Proposed Similarity Degree in MADM

When we complement the data to IDHFEs based on the maximum similarity, we need to operate and organize the system of the information many times. So we make the following signs:

- (1) Denote the similarity matrix as $R_{c_j}^t$ after the t th operations for the information system;

- (2) Denote the similarity aggregation matrix as $R'_{\aleph(c_j)}$ after the t th operations for the information system;
- (3) Denote the information system as D^t after the t th operations;
- (4) Denote the integral information system as D .

Note: Two IDHFEs, which similarity degree is the greatest, we complement the two IDHFEs by adding to the weighted summation each other until the IDHFEs get to CDHFEs, then ordering the membership degree and non-membership degree of the CDHFEs from the largest to the smallest.

Based on the above analysis, when the information of system is incomplete, we use the following method to find the best alternative in MADM problem.

Let $U = \{u_1, u_2, \dots, u_n\}$ be a set of alternatives and $C = \{c_1, c_2, \dots, c_m\}$ be a set of attributes, we construct the decision matrix $D^0 = (d_{\mu\nu})_{m \times n}$ where $d_{\mu\nu} = (h_{\mu\nu}, g_{\mu\nu})$ ($1 \leq \mu \leq m, 1 \leq \nu \leq n$). The method involves the following steps:

Step 1. Let $t = 1$ and calculate the similarity degrees $s_{c_j}(u_\mu, u_\nu)$ of any two alternatives u_μ and u_ν of attribute c_j in $U^{t-1} \in D^{t-1}$, we obtain the similarity matrix $R'_{c_j} = (x_{\mu\nu})_{n \times n}$ and the similarity aggregation matrix $R'_{\aleph(c_j)} = (r_{\mu\nu})_{n \times n}$;

Step 2. Scan similarity aggregation matrix $R'_{c_j} = (x_{\mu\nu})_{n \times n}$ and find the maximum of the similarity of any two alternatives with IDHFEs. According to the **Note**, we arrange the matrix D^{t-1} to get matrix D^t :

- (i) If $D^t \neq D, t = t + 1$, return **Step 1**,
- (ii) If $D^t = D$, then to **Step 3**;

Step 3. Calculate the similarity degrees among the alternatives $u^i_{c_j}$ and the ideal alternatives $u^*_{c_j}$ by using formula (2) ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$);

Step 4. Rank the alternatives according to the results of **Step 3**;

Step 5. Select the best alternative according to the **Step 4**.

4.2 Practical Example

Here, we take the example [16], to illustrate the utility of the proposed method. Also, we show that the results obtained using the proposed method are same as the results of Ref. [16].

There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: u_1 is a car company; u_2 is a food company; u_3 is a computer company; u_4 is an arms company. The investment company must make a decision according to the following three attributes: c_1 is the risk analysis; c_2 is the growth analysis; c_3 is the environmental impact analysis. The attribute weight vector for membership degree and nonmembership degree is given as $w = (0.35, 0.25, 0.40)^T$ and $z = (0.30, 0.40, 0.30)^T$, respectively. The four possible alternatives u_i ($i = 1, 2, 3, 4$) are to be evaluated using the dual hesitant fuzzy information by three decision makers under three attributes c_j ($j = 1, 2, 3$), as listed in the following dual hesitant fuzzy decision matrix D^0 (“ $\hat{\Delta}$ ” represent that the missing value):

$$D^0 = \begin{pmatrix} \{\{0.5, 0.4, 0.3\}, \{0.4, 0.3\}\} & \{\{0.6, 0.4, \widehat{\Delta}\}, \{0.4, 0.2\}\} & \{\{0.3, 0.2, 0.1\}, \{0.6, 0.5\}\} \\ \{\{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\} & \{\{0.7, 0.6, \widehat{\Delta}\}, \{0.3, 0.2\}\} & \{\{0.7, 0.6, 0.4\}, \{0.2, 0.1\}\} \\ \{\{0.6, 0.4, 0.3\}, \{0.3, \widehat{\Delta}\}\} & \{\{0.6, 0.5, \widehat{\Delta}\}, \{0.3, \widehat{\Delta}\}\} & \{\{0.6, 0.5, \widehat{\Delta}\}, \{0.3, 0.1\}\} \\ \{\{0.8, 0.7, 0.6\}, \{0.2, 0.1\}\} & \{\{0.7, 0.6, \widehat{\Delta}\}, \{0.2, \widehat{\Delta}\}\} & \{\{0.4, 0.3, \widehat{\Delta}\}, \{0.2, 0.1\}\} \end{pmatrix}$$

Step 1. Let $t = 1$ and calculate the similarity degrees $s_{c_j}(u_\mu, u_\nu)$ of any two alternatives u_μ and u_ν of attribute c_j to obtain the similarity matrix $R_{c_j}^1 = (x_{\mu\nu})_{4 \times 4}$, where $1 \leq \mu, \nu \leq 4, 1 \leq j \leq 3$. Then we get the similarity aggregation matrix $R_{N(c_j)}^1 = (r_{\mu\nu})_{4 \times 4}$ of c_j .

$$R_{c_1}^1 = \begin{pmatrix} 1 & 0.942 & 0.968 & 0.852 \\ 0.942 & 1 & 0.959 & 0.976 \\ 0.968 & 0.959 & 1 & 0.895 \\ 0.852 & 0.976 & 0.895 & 1 \end{pmatrix} \quad R_{c_2}^1 = \begin{pmatrix} 1 & 0.965 & 0.971 & 0.926 \\ 0.965 & 1 & 0.981 & 0.983 \\ 0.971 & 0.981 & 1 & 0.983 \\ 0.926 & 0.983 & 0.983 & 1 \end{pmatrix}$$

$$R_{c_3}^1 = \begin{pmatrix} 1 & 0.599 & 0.763 & 0.729 \\ 0.599 & 1 & 0.924 & 0.728 \\ 0.763 & 0.924 & 1 & 0.908 \\ 0.729 & 0.728 & 0.908 & 1 \end{pmatrix} \quad R_{N(c_j)}^1 = \begin{pmatrix} 1 & 0.835 & 0.900 & 0.835 \\ 0.835 & 1 & 0.954 & 0.895 \\ 0.900 & 0.954 & 1 & 0.928 \\ 0.835 & 0.895 & 0.928 & 1 \end{pmatrix}$$

Step 2. We can see $r_{23} = r_{32}$ is the largest from $R_{N(c_j)}^1$ easily, that is, the similarity of u_2 and u_3 is the largest. Arranging matrix D^0 , we can get matrix D^1 .

$$D^1 = \begin{pmatrix} \{\{0.5, 0.4, 0.3\}, \{0.4, 0.3\}\} & \{\{0.6, 0.4, \widehat{\Delta}\}, \{0.4, 0.2\}\} & \{\{0.3, 0.2, 0.1\}, \{0.6, 0.5\}\} \\ \left\{ \begin{array}{l} \{0.7, 0.6, 0.4\}, \\ \{0.3, 0.2, 0.105\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.7, 0.6, 0.38\}, \\ \{0.3, 0.2, 0.105\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.7, 0.6, 0.4\}, \\ \{0.2, 0.13, 0.1\} \end{array} \right\} \\ \left\{ \begin{array}{l} \{0.6, 0.4, 0.3\}, \\ \{0.3, 0.155, 0.155\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.6, 0.5, 0.45\}, \\ \{0.3, 0.155, 0.155\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.6, 0.57, 0.5\}, \\ \{0.3, 0.1, 0.095\} \end{array} \right\} \\ \{\{0.8, 0.7, 0.6\}, \{0.2, 0.1\}\} & \{\{0.7, 0.6, \widehat{\Delta}\}, \{0.2, \widehat{\Delta}\}\} & \{\{0.4, 0.3, \widehat{\Delta}\}, \{0.2, 0.1\}\} \end{pmatrix}$$

Thus, we can get $D^1 \neq D$, return **Step 1**.

Let $t = 2$ and calculate the similarity degrees $s_{c_j}(u_\mu, u_\nu)$ of any two alternatives u_μ and u_ν of attribute c_j to obtain the similarity matrix $R_{c_j}^2 = (x_{\mu\nu})_{4 \times 4}$, where $1 \leq \mu, \nu \leq 4, 1 \leq j \leq 3$. Then we get the similarity aggregation matrix $R_{N(c_j)}^2 = (r_{\mu\nu})_{4 \times 4}$ of c_j .

$$R_{c_1}^2 = \begin{pmatrix} 1 & 0.948 & 0.997 & 0.852 \\ 0.948 & 1 & 0.966 & 0.968 \\ 0.997 & 0.966 & 1 & 0.882 \\ 0.852 & 0.968 & 0.882 & 1 \end{pmatrix} R_{c_2}^2 = \begin{pmatrix} 1 & 0.901 & 0.932 & 0.926 \\ 0.901 & 1 & 0.996 & 0.948 \\ 0.932 & 0.996 & 1 & 0.956 \\ 0.926 & 0.948 & 0.956 & 1 \end{pmatrix}$$

$$R_{c_3}^2 = \begin{pmatrix} 1 & 0.650 & 0.682 & 0.729 \\ 0.650 & 1 & 0.998 & 0.730 \\ 0.682 & 0.998 & 1 & 0.737 \\ 0.729 & 0.730 & 0.737 & 1 \end{pmatrix} R_{N(c_j)}^2 = \begin{pmatrix} 1 & 0.833 & 0.871 & 0.835 \\ 0.833 & 1 & 0.987 & 0.882 \\ 0.871 & 0.987 & 1 & 0.858 \\ 0.835 & 0.882 & 0.858 & 1 \end{pmatrix}$$

Due to the similarity of u_2 and u_3 are complete, overlook $s(u_2, u_3)$. We can see $r_{24} = r_{42}$ is the largest from $R_{N(c_j)}^2$ easily, that is, the similarity of u_2 and u_4 is the largest. Arranging matrix D^1 , we can get matrix D^2 .

$$D^2 = \begin{pmatrix} \left\{ \{0.5, 0.4, 0.3\}, \{0.4, 0.3\} \right\} & \left\{ \{0.6, 0.4, \hat{\Delta}\}, \{0.4, 0.2\} \right\} & \left\{ \{0.3, 0.2, 0.1\}, \{0.6, 0.5\} \right\} \\ \left\{ \{0.7, 0.6, 0.4\}, \{0.3, 0.2, 0.105\} \right\} & \left\{ \{0.7, 0.6, 0.38\}, \{0.3, 0.2, 0.105\} \right\} & \left\{ \{0.7, 0.6, 0.4\}, \{0.2, 0.13, 0.1\} \right\} \\ \left\{ \{0.6, 0.4, 0.3\}, \{0.3, 0.155, 0.155\} \right\} & \left\{ \{0.6, 0.5, 0.45\}, \{0.3, 0.155, 0.155\} \right\} & \left\{ \{0.6, 0.57, 0.5\}, \{0.3, 0.1, 0.095\} \right\} \\ \left\{ \{0.8, 0.7, 0.6\}, \{0.2, 0.197, 0.1\} \right\} & \left\{ \{0.7, 0.6, 0.564\}, \{0.2, 0.197, 0.197\} \right\} & \left\{ \{0.4, 0.57, 0.3\}, \{0.2, 0.1, 0.1425\} \right\} \end{pmatrix}$$

Thus, we can get $D^2 \neq D$, return **Step 1**.

Let $t = 3$ and calculate the similarity degrees $s_{c_j}(u_\mu, u_\nu)$ of any two alternatives u_μ and u_ν of attribute c_j to obtain the similarity matrix $R_{c_j}^3 = (x_{\mu\nu})_{4 \times 4}$, where $1 \leq \mu, \nu \leq 4, 1 \leq j \leq 3$. Then we get the similarity aggregation matrix $R_{N(c_j)}^3 = (r_{\mu\nu})_{4 \times 4}$ of c_j .

$$R_{c_1}^3 = \begin{pmatrix} 1 & 0.948 & 0.997 & 0.871 \\ 0.948 & 1 & 0.966 & 0.979 \\ 0.997 & 0.966 & 1 & 0.898 \\ 0.871 & 0.979 & 0.898 & 1 \end{pmatrix} R_{c_2}^3 = \begin{pmatrix} 1 & 0.901 & 0.932 & 0.864 \\ 0.901 & 1 & 0.996 & 0.996 \\ 0.932 & 0.996 & 1 & 0.985 \\ 0.864 & 0.996 & 0.985 & 1 \end{pmatrix}$$

$$R_{c_3}^3 = \begin{pmatrix} 1 & 0.650 & 0.682 & 0.754 \\ 0.650 & 1 & 0.998 & 0.968 \\ 0.682 & 0.998 & 1 & 0.973 \\ 0.754 & 0.968 & 0.973 & 1 \end{pmatrix} R_{N(c_j)}^3 = \begin{pmatrix} 1 & 0.833 & 0.871 & 0.829 \\ 0.833 & 1 & 0.987 & 0.981 \\ 0.871 & 0.987 & 1 & 0.973 \\ 0.829 & 0.981 & 0.973 & 1 \end{pmatrix}$$

Due to the similarity of u_2, u_3, u_4 , are complete, overlook $s(u_2, u_3), s(u_2, u_4), s(u_3, u_4)$. We can see $r_{13} = r_{31}$ is the largest from $R_{N(c_j)}^3$ easily, that is, the similarity of u_1 and u_3 is the largest. Arranging matrix D^2 , we can get matrix D^3 .

$$D^3 = \begin{pmatrix} \left\{ \begin{array}{l} \{0.5, 0.4, 0.3\}, \\ \{0.4, 0.3, 0.2058\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.6, 0.515, 0.4\}, \\ \{0.4, 0.2058, 0.2\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.3, 0.2, 0.1\}, \\ \{0.6, 0.5, 0.168\} \end{array} \right\} \\ \left\{ \begin{array}{l} \{0.7, 0.6, 0.4\}, \\ \{0.3, 0.2, 0.105\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.7, 0.6, 0.38\}, \\ \{0.3, 0.2, 0.105\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.7, 0.6, 0.4\}, \\ \{0.2, 0.13, 0.1\} \end{array} \right\} \\ \left\{ \begin{array}{l} \{0.6, 0.4, 0.3\}, \\ \{0.3, 0.155, 0.155\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.6, 0.5, 0.45\}, \\ \{0.3, 0.155, 0.155\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.6, 0.57, 0.5\}, \\ \{0.3, 0.1, 0.095\} \end{array} \right\} \\ \left\{ \begin{array}{l} \{0.8, 0.7, 0.6\}, \\ \{0.2, 0.197, 0.1\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.7, 0.6, 0.564\}, \\ \{0.2, 0.197, 0.197\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.4, 0.57, 0.3\}, \\ \{0.2, 0.1, 0.1425\} \end{array} \right\} \end{pmatrix}$$

Thus, we can get $D^3 = D$, turn to **Step 3**.

Step 3. Calculate the similarity degrees among the alternatives $u_{c_j}^i$ and the ideal alternatives $u_{c_j}^*$ by using formula (2) ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$);

- (1) The ideal alternatives $u_{c_j}^*$: $u_{c_1}^* = (\{0.8, 0.7, 0.6\}, \{0.2, 0.155, 0.1\})$; $u_{c_2}^* = (\{0.7, 0.6, 0.564\}, \{0.2, 0.155, 0.105\})$; $u_{c_3}^* = (\{0.7, 0.6, 0.5\}, \{0.2, 0.1, 0.095\})$
- (2) The similarity degrees:

$$\begin{aligned} s_{DHFEs}(u_{c_1}^1, u_{c_1}^*) &= 0.852, s_{DHFEs}(u_{c_1}^2, u_{c_1}^*) = 0.978, s_{DHFEs}(u_{c_1}^3, u_{c_1}^*) = 0.897, \\ s_{DHFEs}(u_{c_1}^4, u_{c_1}^*) &= 0.999; \\ s_{DHFEs}(u_{c_2}^1, u_{c_2}^*) &= 0.962, s_{DHFEs}(u_{c_2}^2, u_{c_2}^*) = 0.993, s_{DHFEs}(u_{c_2}^3, u_{c_2}^*) = 0.980, \\ s_{DHFEs}(u_{c_2}^4, u_{c_2}^*) &= 0.997; \\ s_{DHFEs}(u_{c_3}^1, u_{c_3}^*) &= 0.598, s_{DHFEs}(u_{c_3}^2, u_{c_3}^*) = 0.998, s_{DHFEs}(u_{c_3}^3, u_{c_3}^*) = 0.955, \\ s_{DHFEs}(u_{c_3}^4, u_{c_3}^*) &= 0.955. \end{aligned}$$

Step 4. Rank the alternatives according to the results of **Step 3**;

$$\hat{s}_{u_2} > \hat{s}_{u_4} > \hat{s}_{u_3} > \hat{s}_{u_1}$$

Step 5. Select the best alternative according to the **Step 4**.

$$u_2 \succ u_4 \succ u_3 \succ u_1$$

Therefore, u_2 is the best alternative to choose.

4.3 Advantages of the Proposed Distance Measures

- (i) Singh has proposed the distance and similarity measures for MADM with DHFSs in [16]. The result of the proposed method is matched with existing method of Singh. But the method in [16] based on decision maker's preference parameter is known to make decision. The approach we proposed is to complement the missing data when the decision maker's risk preference is unknown, extracting rules from the existed information to solve the problem of missing information. Thus the proposed method is more practical and general.
- (ii) As mentioned above, we give the definition of the similarity degree for IDHFEs and give some related properties. Using the method of maximum similarity degree to complement the IDHFEs not only keeps the maximum similarity among the hesitant fuzzy information, but also solves the problem of information loss in IDHFSs.

5 Conclusions

In the real world, the HFS or DHFS is adequate for dealing with the vagueness of DM's judgment. However, due to the preference of experts and the influence of external factors, some information values can't be given, which results in the fuzzy information is not complete. In order to solve this problem, we propose a completion method of information values based on the maximum similarity degree in IDHFSs. An important advantage of the proposed method is that it not only solves the problem of information loss in IDHFSs, but also keeps the maximum similarity among the hesitant fuzzy information.

The next step of the research work is to extend the proposed method to the wider areas such as machine learning, clustering, reasoning and so on.

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References

1. Dyer, J.S., Fishburn, P.C.: Multiple criteria decision making, multiattribute utility theory: the next ten years. *Manag. Sci.* **38**(5), 645–654 (1992)
2. Zadeh, L.A.: Fuzzy sets. *Inf. Control* **8**(3), 338–353 (1965)
3. Cao, Q., Wu, J.: The extended COWG operators and their application to multiple attributive group decision making problems with interval numbers. *Appl. Math. Model.* **35**(5), 2075–2086 (2011)
4. Wan, S., Li, D.: Fuzzy mathematical programming approach to heterogeneous multiattribute decision-making with interval-valued intuitionistic fuzzy truth degrees. *Inf. Sci.* **325**, 484–503 (2015)

5. Joshi, D., Kumar, S.: Interval-valued intuitionistic hesitant fuzzy Choquet integral based TOPSIS method for multi-criteria group decision making. *Eur. J. Oper. Res.* **248**(1), 183–191 (2016)
6. Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. In: *IEEE International Conference on Fuzzy Systems*, pp. 1378–1382 (2009)
7. Torra, V.: Hesitant fuzzy sets. *Int. J. Intell. Syst.* **25**(6), 529–539 (2010)
8. Zhu, B., Xu, Z., Xia, M.: Dual hesitant fuzzy sets. *J. Appl. Math.* **2012**(11), 2607–2645 (2012)
9. Herreraviedma, E., Herrera, F., Chiclana, F., Luque, M.S.: Some issues on consistency of fuzzy preference relations. *Eur. J. Oper. Res.* **154**(1), 98–109 (2004)
10. Xu, Z., Xia, M.: On distance and correlation measures of hesitant fuzzy information. *Int. J. Intell. Syst.* **26**(5), 410–425 (2011)
11. Xu, Z., Xia, M.: Distance and similarity measures for hesitant fuzzy sets. *Inf. Sci.* **181**(11), 2128–2138 (2011)
12. Zhang, Z.: Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. *Inf. Sci.* **234**(10), 150–181 (2013)
13. Xia, M., Xu, Z.: Hesitant fuzzy information aggregation in decision making. *Int. J. Approx. Reason.* **52**(3), 395–407 (2011)
14. Lin, R., Zhao, X., Wei, G.: Models for selecting an ERP system with hesitant fuzzy linguistic information. *J. Intell. Fuzzy Syst.* **26**(5), 2155–2165 (2014)
15. Wang, J., Wang, J., Zhang, H., Chen, X.: Multi-criteria group decision-making approach based on 2-tuple linguistic aggregation operators with multi-hesitant fuzzy linguistic information. *Int. J. Fuzzy Syst.* **18**(1), 81–97 (2015)
16. Singh, P.: Distance and similarity measures for multiple-attribute decision making with dual hesitant fuzzy sets. *Comput. Appl. Math.* **36**, 1–16 (2015)
17. Zhao, N., Xu, Z., Liu, F.: Group decision making with dual hesitant fuzzy preference relations. *Cogn. Comput.* **8**, 1–25 (2016)