Chapter 4
DGFEMs for Pure Diffusion Problems

In this chapter we study the stability and *hp*-version a priori error analysis of the DGFEM discretization of a pure diffusion problem; referring back to (2.1), (2.3), this corresponds to the case when a is positive definite, $\mathbf{b} \equiv \mathbf{0}$, and $c \equiv 0$. In particular, we develop the underlying theory for two different sets of shape assumptions, which the polytopic elements forming the computational mesh $\mathscr{T}_h$ must satisfy. In the first instance, we assume that the number of faces each element κ, $\kappa \in \mathscr{T}_h$, possesses remains uniformly bounded under mesh refinement, but without the restriction of shape-regularity in the classical sense, cf. Assumption 25 below; see, also, [54]. We will then pursue the analysis in the case when this assumption is violated, i.e., when polytopic elements are permitted to have an arbitrary number of faces under mesh refinement; however, in this setting, a generalized shape-regularity assumption must be satisfied, cf. Assumption 30 below. This latter condition was first considered in [56]. The outline of this chapter is as follows. Upon recalling the diffusion model problem, we introduce the corresponding DGFEM in Sect. 4.1; the latter is based on the general scheme outlined in Sect. 2.4. For simplicity of presentation, here we focus on the symmetric version of the interior penalty DGFEM, though we stress that the analysis presented here naturally generalizes to both the IIP- and NIP-DGFEMs, as well as other DGFEMs proposed within the literature. Then, in Sects. 4.2 and 4.3 we pursue the error analysis under the two different mesh assumptions, respectively. Moreover, for the analysis of the second case, when elements with an arbitrary number of faces are permitted, we also prove the necessary trace inverse estimate, along with a polynomial approximation result. Finally, in Sect. 4.4 we discuss the relationship between these different mesh assumptions and conclude on the generality of polytopic meshes covered by our analysis.

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4.1 Model Problem and Discretization

Given an open bounded Lipschitz domain Ω in $\mathbb{R}^d$, $d \in \mathbb{N}$, with boundary $\partial\Omega$, we consider the following PDE boundary-value problem: find u such that

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \tag{4.1}$$

$$u = g_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \tag{4.2}$$

$$\mathbf{n} \cdot (a\nabla u) = g_{\mathrm{N}} \quad \text{on } \partial\Omega_{\mathrm{N}}. \tag{4.3}$$

Here, $f \in L^2(\Omega)$, $a = \{a_{ij}\}_{i,j=1}^{d}$, with $a_{ij} \in L^\infty(\Omega)$ and $a_{ij} = a_{ji}$, for $i, j = 1, \ldots, d$, and, at each $\mathbf{x}$ in $\bar{\Omega}$,

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\xi_i\xi_j \geq \theta|\boldsymbol{\xi}|^2 > 0, \tag{4.4}$$

where θ is a positive constant, for any vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ in $\mathbb{R}^d$. Here, the boundary of the computational domain Ω is subdivided into the two disjoint subsets $\partial\Omega_{\mathrm{D}}$ and $\partial\Omega_{\mathrm{N}}$ whose union is $\partial\Omega$, with $\partial\Omega_{\mathrm{D}}$ nonempty and relatively open in $\partial\Omega$. The well-posedness of the boundary value problem (4.1)–(4.3), under the uniform ellipticity condition (4.4) can be deduced, based on employing the Lax-Milgram Theorem; see, for example, [42, 64].

As in Chap. 3, we write $\mathscr{T}_h$ to denote a subdivision of the computational domain $\Omega \subset \mathbb{R}^d$, $d > 1$, into disjoint open polytopic elements κ constructed such that $\bar{\Omega} = \cup_{\kappa \in \mathscr{T}_h} \bar{\kappa}$. Recalling that $\mathscr{F}_h$ denotes the set of open $(d-1)$-dimensional simplicial element faces associated with the computational mesh $\mathscr{T}_h$, employing the notation introduced in Chap. 2, we write $\mathscr{F}_h = \mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^{\mathscr{B}}$, where $\mathscr{F}_h^{\mathscr{I}}$ denotes the set of interior element faces, and $\mathscr{F}_h^{\mathscr{B}}$ is the set of boundary element faces. For simplicity, we assume that $\mathscr{T}_h$ respects the decomposition of $\partial\Omega$ in the sense that each $F \in \mathscr{F}_h^{\mathscr{B}}$ belongs to the interior of exactly one of $\partial\Omega_{\mathrm{D}}$ or $\partial\Omega_{\mathrm{N}}$. Hence, we write $\mathscr{F}_h^{D}, \mathscr{F}_h^{N} \subset \mathscr{F}_h^{\mathscr{B}}$ to denote the subsets of boundary faces belonging to $\partial\Omega_{\mathrm{D}}, \partial\Omega_{\mathrm{N}}$, respectively.

To facilitate *hp*-adaptivity, to each element $\kappa \in \mathscr{T}_h$, we associate a local polynomial degree $p_\kappa \geq 1$, and collect the p_κ, $\kappa \in \mathscr{T}_h$, in the vector $\mathbf{p} := (p_\kappa : \kappa \in \mathscr{T}_h)$. With this notation, we define the finite element space $V^{\mathbf{p}}(\mathscr{T}_h)$ with respect to $\mathscr{T}_h$ and $\mathbf{p}$ by

$$V^{\mathbf{p}}(\mathscr{T}_h) := \{u \in L^2(\Omega) : u|_\kappa \in \mathscr{P}_{p_\kappa}(\kappa), \kappa \in \mathscr{T}_h\},$$

where, we recall that $\mathscr{P}_p(\kappa)$ denotes the space of polynomials of total degree p on κ. By construction, the local elemental polynomial spaces employed within the definition of $V^{\mathbf{p}}(\mathscr{T}_h)$ are defined in the physical space, without the need to map from a given reference or canonical frame, as is typically the case for classical FEMs; we refer to Chap. 6 concerning implementation aspects of the elementwise polynomial basis.

Following the derivation presented in Sect. 2.3, we recall the (SIP) DGFEM bilinear form

$$B_{\mathrm{d}}(w_h, v_h) := \sum_{\kappa\in\mathscr{T}_h}\int_\kappa a\nabla w_h \cdot \nabla v_h \,\mathrm{d}\mathbf{x} \\ - \int_{\mathscr{F}_h^{\mathscr{I}}\cup\mathscr{F}_h^{D}} (\{\!\{a\nabla w_h\}\!\} \cdot [\![v_h]\!] + \{\!\{a\nabla v_h\}\!\} \cdot [\![w_h]\!] - \sigma[\![w_h]\!] \cdot [\![v_h]\!]) \,\mathrm{d}s,$$

and linear functional

$$\ell(v_h) := \sum_{\kappa\in\mathscr{T}_h}\int_\kappa f v_h \,\mathrm{d}\mathbf{x} - \int_{\mathscr{F}_h^{D}} g_{\mathrm{D}}(a\nabla v_h \cdot \mathbf{n} - \sigma v_h)\,\mathrm{d}s + \int_{\mathscr{F}_h^{N}} g_{\mathrm{N}} v_h \,\mathrm{d}s,$$

for $w_h, v_h \in V^{\mathbf{p}}(\mathscr{T}_h)$. Thereby, the corresponding DGFEM is given by: find $u_h \in V^{\mathbf{p}}(\mathscr{T}_h)$ such that

$$B_{\mathrm{d}}(u_h, v_h) = \ell(v_h) \tag{4.5}$$

for all $v_h \in V^{\mathbf{p}}(\mathscr{T}_h)$.

The well-posedness and stability properties of the above method depend on the choice of the discontinuity-penalization function σ. These are analyzed in the next two sections based on employing different assumptions on the elements present in the computational mesh $\mathscr{T}_h$. Clearly, we expect that the choice of σ will be sensitive to the size of each face F, $F \in \mathscr{F}_h$, relative to the size of the element(s) which form F. In order to focus on the treatment of general polytopic subdivisions, throughout this chapter, we make the simplifying assumption that the entries of a are constant on each element κ, $\kappa \in \mathscr{T}_h$, i.e.,

$$a \in [V^{\mathbf{0}}(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}}. \tag{4.6}$$

We note that the proceeding results follow immediately, with only very minor changes, in the case when $a \in [V^{\mathbf{q}}(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}}$, where $\mathbf{q} := (q_\kappa : \kappa \in \mathscr{T}_h)$, such that $q_\kappa \in \mathbb{N}_0$ for all $\kappa \in \mathscr{T}_h$. The extension of the analysis to general positive (semi-)definite diffusion tensors will be treated later on in Chap. 5 when we consider the DGFEM discretization of general second-order PDEs with nonnegative characteristic form.

4.2 Error Analysis I: Bounded Number of Element Faces

We study the stability and a priori error analysis of the DGFEM (4.5) under the following assumption, which guarantees that the number of faces each element possesses remains bounded under mesh refinement.

Assumption 25 (Limited Number of Faces) *For each element $\kappa \in \mathscr{T}_h$, we define*

$$C_\kappa := card\Big\{F \in \mathscr{F}_h : F \subset \partial\kappa\Big\}.$$

We assume there exists a positive constant C_F, independent of the mesh parameters, such that

$$\max_{\kappa \in \mathscr{T}_h} C_\kappa \leq C_F.$$

We stress that *no* shape-regularity condition is required to be satisfied by $\mathscr{T}_h$ for the analysis in this section to hold.

4.2.1 Well-Posedness of the DGFEM

Firstly, we write $\sqrt{a} \in [V^{\mathbf{0}}(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}}$ to denote the unique (positive definite) square-root of the symmetric matrix a and $\bar{a}_\kappa := |\sqrt{a}|_2^2|_\kappa$, $\kappa \in \mathscr{T}_h$, where $|\cdot|_2$ denotes the matrix norm subordinate to the l_2-vector norm on $\mathbb{R}^d$, cf. [124]. With this notation, we define the discontinuity-penalization function $\sigma : \mathscr{F}_h \to \mathbb{R}$ in the following manner.

Definition 26 Assuming that (4.6) holds, the discontinuity-penalization function $\sigma : \mathscr{F}_h \to \mathbb{R}$ arising in (4.5) is given by

$$\sigma(\mathbf{x}) := \begin{cases} C_\sigma \max\limits_{\kappa \in \{\kappa^+,\kappa^-\}} \Big\{C_{\mathrm{INV}}(p_\kappa, \kappa, F)\, \dfrac{\bar{a}_\kappa p_\kappa^2 |F|}{|\kappa|}\Big\}, & \mathbf{x} \in F \in \mathscr{F}_h^{\mathscr{I}},\ F \subset \partial\kappa^+ \cap \partial\kappa^-, \\ C_\sigma C_{\mathrm{INV}}(p_\kappa, \kappa, F)\, \dfrac{\bar{a}_\kappa p_\kappa^2 |F|}{|\kappa|}, & \mathbf{x} \in F \in \mathscr{F}_h^{D},\ F \subset \partial\kappa. \end{cases} \tag{4.7}$$

Here, C_{INV} is the constant arising in the inverse inequality derived in Lemma 11, cf. (3.14), and C_σ is a positive constant independent of p_κ, $|F|$, and $|\kappa|$.

In accordance with the mesh terminology introduced in Sect. 3.1, we note that the value of the discontinuity-penalization function σ on a given elemental interface is independently determined on each constituent $(d-1)$-dimensional simplicial mesh face which forms the given interface. In this way, σ is independent of any local mesh size or polynomial degree quasi-uniformity assumption, as well as any local regularity condition on the location of hanging nodes on the boundary of each element κ, $\kappa \in \mathscr{T}_h$. In particular, for standard simplicial and tensor product meshes, which contain hanging nodes, the independent piecewise constant definition of the discontinuity-penalization function allows for the treatment of *irregular* hanging nodes, i.e., nodes which are arbitrarily positioned on the element interface, in a simple manner. This is in contrast to the usual error analysis of DGFEMs on meshes consisting of standard element shapes, whereby irregular hanging nodes are not permitted, since the discontinuity-penalization function definition typically relies on the face, and the corresponding interface which it belongs to, to be of comparable size to that of the element, cf. [124].

The first issue encountered when analyzing the DGFEM (4.5) is that this formulation is not well-defined for functions in $H^1(\Omega)$. Indeed, square-integrable functions, i.e., those that belong to $L^2(\Omega)$, do not have a well-defined trace on $\mathscr{F}_h$ and hence the terms $\{\!\{\nabla v\}\!\}$ are not well-defined for $v \in H^1(\Omega)$. Hence, unless we assume that the analytical solution of (4.1) possesses additional regularity, we cannot directly exploit Galerkin orthogonality. At first sight, this may not appear to be a pertinent issue in the context of a priori error bounds, whereby local solution regularity is routinely assumed to be sufficiently high. However, the presence of $\{\!\{\nabla v\}\!\}|_F$, for a face F, in the bilinear form, results in terms of the form $\{\!\{\nabla(u-\Pi u)\}\!\}|_F$, where $\Pi u \in V^{\mathbf{p}}(\mathscr{T}_h)$ is some approximation of u; this must then be estimated optimally to establish an (optimal) a priori error bound. Unfortunately, as the proof of Lemma 23 shows, to derive hp-approximation estimates for the $H^1(F)$-seminorm on polytopic meshes in the spirit of (3.30), we would additionally need an hp-approximation estimate on simplicial elements in the $W^{1,\infty}$-norm, (cf. (3.23)) which is neither available nor practical for it would increase further the solution regularity requirements artificially.

To overcome this issue, we introduce suitable extensions of the bilinear form $B_{\mathrm{d}}(\cdot,\cdot)$ and linear functional $\ell(\cdot)$. To this end, we write $\boldsymbol{\Pi}_{L^2} : [L^2(\Omega)]^d \to [V^{\mathbf{p}}(\mathscr{T}_h)]^d$ to denote the orthogonal L^2-projection onto the finite element space $[V^{\mathbf{p}}(\mathscr{T}_h)]^d$. Thereby, following [100, 147], we define the bilinear form

$$\tilde{B}_{\mathrm{d}}(w,v) := \sum_{\kappa\in\mathscr{T}_h}\int_\kappa a\nabla w\cdot\nabla v \,\mathrm{d}\mathbf{x} \tag{4.8}$$

$$-\int_{\mathscr{F}_h^{\mathscr{I}}\cup\mathscr{F}_h^D} \left(\{\!\{a\boldsymbol{\Pi}_{L^2}(\nabla w)\}\!\}\cdot[\![v]\!] + \{\!\{a\boldsymbol{\Pi}_{L^2}(\nabla v)\}\!\}\cdot[\![w]\!] - \sigma[\![w]\!]\cdot[\![v]\!]\,\mathrm{d}s\right), \tag{4.9}$$

and linear functional

$$\tilde{\ell}(v) := \sum_{\kappa\in\mathscr{T}_h}\int_\kappa f v\,\mathrm{d}\mathbf{x} - \int_{\mathscr{F}_h^D} g_{\mathrm{D}}(a\boldsymbol{\Pi}_{L^2}(\nabla v)\cdot\mathbf{n} - \sigma v)\,\mathrm{d}s + \int_{\mathscr{F}_h^N} g_{\mathrm{N}} v\,\mathrm{d}s$$

for all $v, w \in \mathscr{V} := H^1(\Omega) + V^{\mathbf{p}}(\mathscr{T}_h)$. Then the DGFEM formulation (4.5) may be rewritten in the following equivalent manner: find $u_h \in V^{\mathbf{p}}(\mathscr{T}_h)$ such that

$$\tilde{B}_{\mathrm{d}}(u_h, v_h) = \tilde{\ell}(v_h) \tag{4.10}$$

for all $v_h \in V^{\mathbf{p}}(\mathscr{T}_h)$.

Of course, for all $w, v \in V^{\mathbf{p}}(\mathscr{T}_h)$, we have $\tilde{B}_{\mathrm{d}}(w, v) = B_{\mathrm{d}}(w, v)$ and $\tilde{\ell}(v) = \ell(v)$, i.e., (4.10) and (4.5) give rise to the same DGFEM. However, the bilinear form $\tilde{B}_{\mathrm{d}}(\cdot, \cdot)$ is inconsistent due to the discrete nature of the projection operator $\boldsymbol{\Pi}_{L^2}$; thereby, Galerkin orthogonality no longer holds. Nevertheless, this formulation enables us to pursue the analysis without requiring $W^{1,\infty}$-norm approximation estimates, as we shall see below. Moreover, we are able to deduce both the coercivity and continuity of the (extended) bilinear form $\tilde{B}_{\mathrm{d}}(\cdot, \cdot)$ on $\mathscr{V} \times \mathscr{V}$.

We introduce the associated DGFEM energy norm given by

$$|||v|||_{\mathrm{DG}}^2 := \sum_{\kappa \in \mathscr{T}_h} \|\sqrt{a}\nabla v\|_{L^2(\kappa)}^2 + \int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^D} \sigma |[\![v]\!]|^2 \, \mathrm{d}s. \tag{4.11}$$

Here, and in the following, we shall often make use of the arithmetic-geometric mean inequality, written in the following form:

$$ab \leq a^2\epsilon + \frac{b^2}{4\epsilon}, \tag{4.12}$$

which holds for any $a, b \in \mathbb{R}$ and $\epsilon > 0$.

Lemma 27 *Given that Assumption 25 holds, with σ defined as in Definition 26, where C_σ is a sufficiently large positive constant, the bilinear form $\tilde{B}_{\mathrm{d}}(\cdot, \cdot)$ is coercive and continuous over $\mathscr{V} \times \mathscr{V}$, i.e.,*

$$\tilde{B}_{\mathrm{d}}(v, v) \geq C_{\mathrm{coer}} |||v|||_{\mathrm{DG}}^2 \quad \textit{for all} \quad v \in \mathscr{V}, \tag{4.13}$$

and

$$\tilde{B}_{\mathrm{d}}(w, v) \leq C_{\mathrm{cont}} |||w|||_{\mathrm{DG}} \, |||v|||_{\mathrm{DG}} \quad \textit{for all} \quad w, v \in \mathscr{V}, \tag{4.14}$$

respectively, where C_{coer} and C_{cont} are positive constants, independent of the local mesh sizes h_κ and local polynomial degree orders p_κ, $\kappa \in \mathscr{T}_h$.

Proof The proof is based on employing standard arguments, cf. [79], for example; in particular, the analysis exploits the inverse inequality stated in Lemma 11 for general polytopic elements. Firstly, to prove (4.13), we note that, for any $v \in \mathscr{V}$, we have the following identity

$$\tilde{B}_{\mathrm{d}}(v, v) = |||v|||_{\mathrm{DG}}^2 - 2 \int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^D} \{\!\{a \boldsymbol{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \, \mathrm{d}s. \tag{4.15}$$

We now proceed by bounding the second term on the right-hand side of (4.15). To this end, given $F \in \mathscr{F}_h^{\mathscr{I}}$, such that $F \subset \partial\kappa^+ \cap \partial\kappa^-$, $\kappa^\pm \in \mathscr{T}_h$, upon application of the Cauchy–Schwarz inequality and the arithmetic-geometric mean inequality, we deduce that

$$\begin{aligned}\int_F \{\!\{a\boldsymbol{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \,\mathrm{d}s &\le \frac{1}{2}\left(\|\frac{1}{\sqrt{\sigma}} a\boldsymbol{\Pi}_{L^2}(\nabla v^+)\|_{L^2(F)} + \|\frac{1}{\sqrt{\sigma}} a\boldsymbol{\Pi}_{L^2}(\nabla v^-)\|_{L^2(F)}\right) \\ &\quad \times \|\sqrt{\sigma}[\![v]\!]\|_{L^2(F)} \\ &\le \epsilon\left(\|\frac{1}{\sqrt{\sigma}} a\boldsymbol{\Pi}_{L^2}(\nabla v^+)\|^2_{L^2(F)} + \|\frac{1}{\sqrt{\sigma}} a\boldsymbol{\Pi}_{L^2}(\nabla v^-)\|^2_{L^2(F)}\right) \\ &\quad + \frac{1}{8\epsilon}\|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)}.\end{aligned}$$

Employing the inverse inequality stated in Lemma 11, the definition of the discontinuity-penalization function σ, the assumption on the diffusion tensor, cf. (4.6), and the stability of the L^2-projector $\boldsymbol{\Pi}_{L^2}$ in the L^2-norm, namely that $\|\boldsymbol{\Pi}_{L^2}\mathbf{v}\|_{L^2(\kappa)} \le \|\mathbf{v}\|_{L^2(\kappa)}$, for $\mathbf{v} \in [\mathscr{V}]^d$, $\kappa \in \mathscr{T}_h$, gives

$$\begin{aligned}&\int_F \{\!\{a\boldsymbol{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \,\mathrm{d}s \\ &\le \epsilon\Bigg(C_{\mathrm{INV}}(p_{\kappa^+}, \kappa^+, F)\frac{\bar{a}_{\kappa^+} p^2_{\kappa^+}|F|}{|\kappa^+|}\|\frac{1}{\sqrt{\sigma}}\sqrt{a}\boldsymbol{\Pi}_{L^2}(\nabla v)\|^2_{L^2(\kappa^+)} \\ &\quad + C_{\mathrm{INV}}(p_{\kappa^-}, \kappa^-, F)\frac{\bar{a}_{\kappa^-} p^2_{\kappa^-}|F|}{|\kappa^-|}\|\frac{1}{\sqrt{\sigma}}\sqrt{a}\boldsymbol{\Pi}_{L^2}(\nabla v)\|^2_{L^2(\kappa^-)}\Bigg) + \frac{1}{8\epsilon}\|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)} \\ &\le \frac{\epsilon}{C_\sigma}\left(\|\sqrt{a}\nabla v\|^2_{L^2(\kappa^+)} + \|\sqrt{a}\nabla v\|^2_{L^2(\kappa^-)}\right) + \frac{1}{8\epsilon}\|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)}. \end{aligned} \tag{4.16}$$

Similarly, for $F \in \mathscr{F}_h^D$, where $F \subset \partial\kappa$, $\kappa \in \mathscr{T}_h$, we get

$$\int_F \{\!\{a\boldsymbol{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \,\mathrm{d}s \le \frac{\epsilon}{C_\sigma}\|\sqrt{a}\nabla v\|^2_{L^2(\kappa)} + \frac{1}{4\epsilon}\|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)}. \tag{4.17}$$

Inserting (4.16) and (4.17) into (4.15), we deduce that

$$\tilde{B}_{\mathrm{d}}(v,v) \ge \left(1 - \frac{2C_F}{C_\sigma}\epsilon\right)\sum_{\kappa\in\mathscr{T}_h}\|\sqrt{a}\nabla v\|^2_{L^2(\kappa)} + \left(1 - \frac{1}{2\epsilon}\right)\sum_{F\in\mathscr{F}_h^{\mathscr{I}}\cup\mathscr{F}_h^D}\|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)},$$

since the number of elemental faces is uniformly bounded by Assumption 25. Hence the bilinear form $\tilde{B}_{\mathrm{d}}(\cdot,\cdot)$ is coercive over $\mathscr{V} \times \mathscr{V}$ if $C_\sigma > 2C_F\epsilon$ for some $\epsilon > 1/2$.

The proof of continuity of $\tilde{B}_{\mathrm{d}}(\cdot,\cdot)$ follows immediately, based on employing the Cauchy-Schwarz inequality, together with analogous arguments to establish upper bounds on the face terms, cf. (4.16) and (4.17) above.

The above analysis extends well-known results derived for meshes consisting of standard element shapes to the case when general polytopes are admitted. We stress that the proof is based on exploiting the new inverse inequality derived in Lemma 11, in order to provide a suitable upper bound on the face terms arising in the DGFEM (4.5), cf., also, (4.10), assuming that Assumption 25 holds. This approach has the crucial advantage of permitting very general polytopic meshes in the sense that shape-regularity of the underlying mesh $\mathscr{T}_h$ is not directly required.

4.2.2 A Priori Error Analysis

We now embark on the error analysis of the *hp*-version DGFEM (4.5), cf., also, (4.10). To this end, we quote the following abstract error bound, which is an instance of Strang's Second Lemma [64, 163], whereby the error is controlled by the sum of a quasi-optimal approximation term and a residual term.

Lemma 28 *Let $u \in H^1(\Omega)$ be the weak solution of (4.3) and $u_h \in V^{\mathbf{p}}(\mathscr{T}_h)$ the DGFEM solution defined by (4.5). Under the hypotheses of Lemma 27, the following abstract error bound holds*

$$|||u - u_h|||_{\mathrm{DG}} \leq \left(1 + \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}}\right) \inf_{v_h \in V^{\mathbf{p}}(\mathscr{T}_h)} |||u - v_h|||_{\mathrm{DG}} + \frac{1}{C_{\mathrm{coer}}} \sup_{w_h \in V^{\mathbf{p}}(\mathscr{T}_h)\setminus\{0\}} \frac{|\tilde{B}_{\mathrm{d}}(u, w_h) - \tilde{\ell}(w_h)|}{|||w_h|||_{\mathrm{DG}}}. \quad (4.18)$$

Proof Employing the triangle inequality gives

$$|||u - u_h|||_{\mathrm{DG}} \leq |||u - v_h|||_{\mathrm{DG}} + |||v_h - u_h|||_{\mathrm{DG}} \quad (4.19)$$

for all $v_h \in V^{\mathbf{p}}(\mathscr{T}_h)$. Thereby, we simply need to bound $|||v_h - u_h|||_{\mathrm{DG}}$; to this end, we exploit the coercivity and continuity of $\tilde{B}_{\mathrm{d}}(\cdot, \cdot)$ on $\mathscr{V} \times \mathscr{V}$, cf. Lemma 27, to obtain

$$\begin{aligned} |||v_h - u_h|||^2_{\mathrm{DG}} &\leq \frac{1}{C_{\mathrm{coer}}} \tilde{B}_{\mathrm{d}}(v_h - u_h, v_h - u_h) \\ &= \frac{1}{C_{\mathrm{coer}}} (\tilde{B}_{\mathrm{d}}(v_h - u, v_h - u_h) + \tilde{B}_{\mathrm{d}}(u - u_h, v_h - u_h)) \\ &\leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}} |||u - v_h|||_{\mathrm{DG}} |||v_h - u_h|||_{\mathrm{DG}} \\ &\quad + \frac{1}{C_{\mathrm{coer}}} (\tilde{B}_{\mathrm{d}}(u, v_h - u_h) - \tilde{\ell}(v_h - u_h)). \end{aligned}$$

Thereby, dividing both sides by $|||v_h - u_h|||_{\mathrm{DG}}$, and noting that $v_h \in V^{\mathbf{p}}(\mathscr{T}_h)$ is arbitrary, upon substitution into (4.19), we deduce the statement of the lemma.

The abstract error bound of Lemma 28 may now be employed to establish an *hp*-version a priori error bound for the DGFEM (4.5), based on exploiting the approximation results stated in Lemma 23. To this end, we assume that, given the polytopic mesh $\mathscr{T}_h$, there exists a shape-regular covering $\mathscr{T}_h^\sharp = \{\mathscr{K}\}$, cf. Definition 17, which satisfies Assumption 18. Furthermore, we assume that $u|_\kappa \in H^{l_\kappa}(\kappa)$, for some $l_\kappa > 1 + d/2$, for each $\kappa \in \mathscr{T}_h$, so that, by Theorem 21, $\mathfrak{E}u|_{\mathscr{K}} \in H^{l_\kappa}(\mathscr{K})$, where $\mathscr{K} \in \mathscr{T}_h^\sharp$, with $\kappa \subset \mathscr{K}$. Thereby, defining $\tilde{\Pi}_{\mathbf{p}}$ by $\tilde{\Pi}_{\mathbf{p}}|_\kappa := \tilde{\Pi}_{p_\kappa}$, for $\kappa \in \mathscr{T}_h$, upon application of Lemma 23, together with Assumption 25, we deduce that

$$
\begin{aligned}
\inf_{v_h \in V^{\mathbf{p}}(\mathscr{T}_h)} |||u - v_h|||_{\mathrm{DG}}^2 &\leq |||u - \tilde{\Pi}_{\mathbf{p}} u|||_{\mathrm{DG}}^2 \\
&\leq \sum_{\kappa \in \mathscr{T}_h} \left(\| \sqrt{a} \nabla (u - \tilde{\Pi}_{p_\kappa} u) \|_{L^2(\kappa)}^2 + 2 \sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} \sigma \| (u - \tilde{\Pi}_{p_\kappa} u)|_\kappa \|_{L^2(F)}^2 \right) \\
&\leq C \sum_{\kappa \in \mathscr{T}_h} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \left(\bar{a}_\kappa + \frac{h_\kappa^{-d+2}}{p_\kappa} \sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} C_m(p_\kappa, \kappa, F) \sigma |F| \right) \| \mathfrak{E}u \|_{H^{l_\kappa}(\mathscr{K})}^2,
\end{aligned}
\tag{4.20}
$$

with $s_\kappa = \min\{p_\kappa + 1, l_\kappa\}$ and C a positive constant, which depends on the shape-regularity of the covering $\mathscr{T}_h^\sharp$, but is independent of the discretization parameters; also, from Lemma 23, we recall that

$$
C_m(p, \kappa, F) = \min \left\{ \frac{h_\kappa^d}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, \frac{1}{p^{1-d}} \right\}.
$$

We now proceed to bound the residual term arising in (4.18). To this end, we first note that applying integration by parts elementwise, together with the identity (2.20), and noting that u is the analytical solution of (4.1), we get

$$
\begin{aligned}
\left| \tilde{B}_{\mathrm{d}}(u, w_h) - \tilde{\ell}(u, w_h) \right| &= \left| \int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^D} \{\!\{ a(\nabla u - \boldsymbol{\Pi}_{L^2}(\nabla u)) \}\!\} \cdot [\![w_h]\!] \, \mathrm{d}s \right| \\
&\leq \left(\int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^D} \sigma^{-1} | \{\!\{ a(\nabla u - \boldsymbol{\Pi}_{L^2}(\nabla u)) \}\!\} |^2 \, \mathrm{d}s \right)^{1/2} |||w_h|||_{\mathrm{DG}}.
\end{aligned}
$$

We write $\tilde{\boldsymbol{\Pi}}_{\boldsymbol{p}}$ to denote the vector-valued *hp*-projection operator obtained by applying the operator $\tilde{\Pi}_{\boldsymbol{p}}$ componentwise. Thereby, adding and subtracting $\tilde{\boldsymbol{\Pi}}_{\boldsymbol{p}}(\nabla u)$, gives

$$\begin{aligned}
&\int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^D} \sigma^{-1} |\{\!\{ a(\nabla u - \boldsymbol{\Pi}_{L^2}(\nabla u)) \}\!\}|^2 \, \mathrm{d}s \\
&\quad \le \int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^D} 2\sigma^{-1} (|\{\!\{ a(\nabla u - \tilde{\boldsymbol{\Pi}}_{\boldsymbol{p}}(\nabla u)) \}\!\}|^2 + |\{\!\{ a(\boldsymbol{\Pi}_{L^2}(\tilde{\boldsymbol{\Pi}}_{\boldsymbol{p}}(\nabla u) - \nabla u)) \}\!\}|^2) \, \mathrm{d}s \\
&\quad \equiv \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Using, as above, the approximation estimate (3.27) yields

$$\mathrm{I} \le C \sum_{\kappa \in \mathscr{T}_h} \bar{a}_\kappa^2 \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \frac{h_\kappa^{-d}}{p_\kappa^{-1}} \left(\sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} C_m(p_\kappa, \kappa, F) \sigma^{-1} |F| \right) \|\mathfrak{E}u\|_{H^{l_\kappa}(\mathscr{K})}^2.$$

Similarly, employing the inverse inequality (3.13), the L^2-stability of the projector $\boldsymbol{\Pi}_{L^2}$, and the approximation estimate (3.26), gives

$$\mathrm{II} \le C \sum_{\kappa \in \mathscr{T}_h} \bar{a}_\kappa^2 \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \frac{|\kappa|^{-1}}{p_\kappa^{-2}} \left(\sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} C_{\mathrm{INV}}(p_\kappa, \kappa, F) \sigma^{-1} |F| \right) \|\mathfrak{E}u\|_{H^{l_\kappa}(\mathscr{K})}^2.$$

Combining the above bounds, we deduce:

$$\begin{aligned}
&\sup_{w_h \in V^{\mathbf{p}}(\mathscr{T}_h) \setminus \{0\}} \frac{|\tilde{B}_{\mathrm{d}}(u, w_h) - \tilde{\ell}(u, w_h)|}{|||w_h|||_{\mathrm{DG}}} \\
&\quad \le \left(\mathrm{I} + \mathrm{II}\right)^{1/2} \\
&\quad \le C \Bigg(\sum_{\kappa \in \mathscr{T}_h} \bar{a}_\kappa^2 \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \\
&\qquad \times \left(\sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} \left(C_m(p_\kappa, \kappa, F) \frac{h_\kappa^{-d}}{p_\kappa^{-1}} + C_{\mathrm{INV}}(p_\kappa, \kappa, F) \frac{|\kappa|^{-1}}{p_\kappa^{-2}} \right) \sigma^{-1} |F| \right) \\
&\qquad \times \|\mathfrak{E}u\|_{H^{l_\kappa}(\mathscr{K})}^2 \Bigg)^{1/2}.
\end{aligned} \tag{4.21}$$

On the basis of the bounds (4.20) and (4.21), we state the following *hp*-version a priori error bound for the DGFEM (4.5).

Theorem 29 *Let $\mathscr{T}_h = \{\kappa\}$ be a subdivision of $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, consisting of general polytopic elements satisfying Assumptions 18 and 25, with $\mathscr{T}_h^\sharp = \{\mathscr{K}\}$ an* *associated covering of $\mathscr{T}_h$ consisting of shape-regular d-simplices, cf. Definition 17. Let $u_h \in V^{\mathbf{p}}(\mathscr{T}_h)$, with $p_\kappa \geq 1$ for all $\kappa \in \mathscr{T}_h$, be the corresponding DGFEM solution defined by (4.5), where the discontinuity-penalization function σ is given by (4.7). If the analytical solution $u \in H^1(\Omega)$ to (4.1)–(4.3) satisfies $u|_\kappa \in H^{l_\kappa}(\kappa)$, $l_\kappa > 1+d/2$, for each $\kappa \in \mathscr{T}_h$, such that $\mathfrak{E}u|_{\mathscr{K}} \in H^{l_\kappa}(\mathscr{K})$, where $\mathscr{K} \in \mathscr{T}_h^\sharp$ with $\kappa \subset \mathscr{K}$, then*

$$|||u - u_h|||_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathscr{T}_h} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \left(\bar{a}_\kappa + \mathscr{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|_{H^{l_\kappa}(\mathscr{K})}^2,$$

with $s_\kappa = \min\{p_\kappa + 1, l_\kappa\}$,

$$\begin{aligned}\mathscr{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa) := {} & \bar{a}_\kappa^2 p_\kappa h_\kappa^{-d} \sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} C_m(p_\kappa, \kappa, F)\sigma^{-1}|F| \\ & + \bar{a}_\kappa^2 p_\kappa^2 |\kappa|^{-1} \sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} C_{\mathrm{INV}}(p_\kappa, \kappa, F)\sigma^{-1}|F| \\ & + h_\kappa^{-d+2} p_\kappa^{-1} \sum_{F \subset \partial\kappa \setminus \partial\Omega_{\mathrm{N}}} C_m(p_\kappa, \kappa, F)\sigma|F|,\end{aligned}$$

where C is a positive constant, which depends on the shape-regularity of $\mathscr{T}_h^\sharp$, but is independent of the discretization parameters.

The above result generalizes well-known a priori bounds for DGFEMs defined on standard element shapes, cf. [124, 152], in two key ways. Firstly, meshes comprising of polytopic elements are admitted; secondly, elemental faces are allowed to degenerate. For $d = 3$, this also implies that positive measure interfaces may have degenerating (one-dimensional) edges. Thereby, this freedom is relevant to standard (simplicial/hexahedral) meshes with hanging nodes in the sense that no condition is required on the location of hanging nodes on the element boundaries. If, on the other hand, the diameter of the faces of each element $\kappa \in \mathscr{T}_h$ is of comparable size to the diameter of the corresponding element, for uniform orders $p_\kappa = p \geq 1$, $h = \max_{\kappa \in \mathscr{T}_h} h_\kappa$, $s_\kappa = s$, $s = \min\{p+1, l\}$, $l > 1 + d/2$, then the bound of Theorem 29 reduces to

$$|||u - u_h|||_{\mathrm{DG}} \leq C \frac{h^{s-1}}{p^{l-3/2}} \|u\|_{H^l(\Omega)},$$

cf. [54]. This coincides with the analogous result derived in [124] for standard meshes consisting of simplices or tensor-product elements. It is easy to check that the above a priori error bound is optimal in h and suboptimal in p by half an order, as expected, cf. [101].

4.3 Error Analysis II: Shape-Regular Polytopic Meshes

In this section, we pursue the error analysis on meshes which may potentially violate Assumption 25 in the sense that the number of faces that the elements possess may not be uniformly bounded under mesh refinement. We note that this may arise when sequences of coarser meshes are generated via element agglomeration of a given fine mesh $\mathscr{T}_h^{\text{fine}}$, cf. Sect. 6.1 and [15]. To this end, recalling Definition 7, we introduce the following assumption on the mesh $\mathscr{T}_h$.

Assumption 30 (Arbitrary Number of Faces) *For any $\kappa \in \mathscr{T}_h$, there exists a set of non-overlapping d-dimensional simplices $\{\kappa_\flat^F\}_{F\subset\partial\kappa} \subset \mathscr{F}_\flat^\kappa$ contained in κ, such that for all $F \subset \partial\kappa$, the following condition holds*

$$h_\kappa \le C_s \frac{d|\kappa_\flat^F|}{|F|}, \tag{4.22}$$

where C_s is a positive constant, which is independent of the discretization parameters, the number of faces that the element possesses, and the measure of F.

In Fig. 4.1 we present two potential polygons in $\mathbb{R}^2$ which satisfy the above mesh regularity assumption. We note that Assumption 30 does not place any restriction on either the number of faces that an element κ, $\kappa \in \mathscr{T}_h$, may possess, or the relative measure of its faces compared to the measure of the element itself. Indeed, shape-irregular simplices $\kappa_\flat^F$, with base $|F|$ of small size compared to the corresponding height, defined by $d|\kappa_\flat^F|/|F|$, are admitted. However, the height must be of comparable size to h_κ, cf. the polygon depicted in Fig. 4.1a. Furthermore, we note that the union of the simplices $\kappa_\flat^F$ does not need to cover the whole element κ, as in general it is sufficient to assume that

$$\bigcup_{F\subset\partial\kappa} \bar{\kappa}_\flat^F \subseteq \bar{\kappa}; \tag{4.23}$$

cf. the polygon given in Fig. 4.1b.

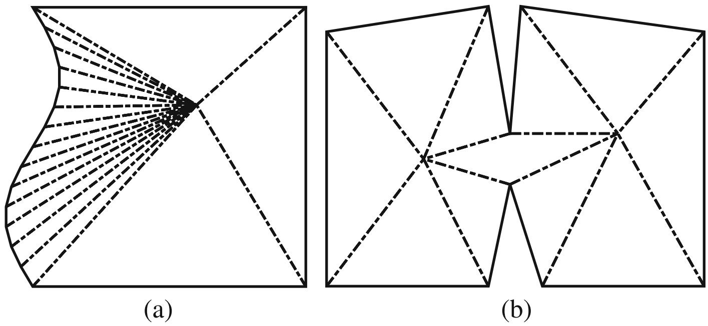

Fig. 4.1 (**a**) Polygon with many tiny faces; (**b**) Star-shaped polygon

Remark 31 We note that meshes consisting of elements which are (unions of) uniformly star-shaped polytopes satisfy Assumption 30. Moreover, it is clear that Assumption 30 is the natural generalization of the classical shape-regularity assumption, usually stated for simplicial and tensor product meshes, cf. [64] and Definition 2, to polytopic elements; in this setting, $\rho_\kappa := \min_{F\subset\partial\kappa} d|\kappa_\flat^F|/|F|$ denotes the radius of the largest inscribed ball.

Given Assumption 30 holds, we now develop an inverse estimate and polynomial approximation result on the boundary of each element κ in the computational mesh $\mathscr{T}_h$. To this end, we have the following two results, respectively.

Lemma 32 *Let $\kappa \in \mathscr{T}_h$; then assuming Assumption 30 is satisfied, for each $v \in \mathscr{P}_p(\kappa)$, the following inverse inequality holds*

$$\|v\|^2_{L^2(\partial\kappa)} \le C_s C_{\text{inv},1} d \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa)}. \tag{4.24}$$

Here, C_s is defined in (4.22), and is independent of v, $|\kappa|/\sup_{\kappa_\flat^F\subset\kappa}|\kappa_\flat^F|$, $|F|$, and p, cf. Assumption 30; moreover, $C_{\text{inv},1}$ is given in Lemma 6, and is independent of v, p, and h_κ.

Proof The proof is based on applying Lemma 6 over each simplex $\kappa_\flat^F$ contained within κ, together with (4.22) and (4.23); thereby, we get

$$\begin{aligned}
\|v\|^2_{L^2(\partial\kappa)} &\le \sum_{F\subset\partial\kappa} C_{\text{inv},1} p^2 \frac{|F|}{|\kappa_\flat^F|} \|v\|^2_{L^2(\kappa_\flat^F)} \\
&\le \sum_{F\subset\partial\kappa} C_s C_{\text{inv},1} d \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa_\flat^F)} \\
&\le C_s C_{\text{inv},1} d \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa)},
\end{aligned}$$

with $\kappa_\flat^F \in \mathscr{F}_\flat^\kappa$ as in Definition 7.

Lemma 33 *Let $\kappa \in \mathscr{T}_h$ and $\mathscr{K} \in \mathscr{T}_h^\sharp$ the corresponding simplex such that $\kappa \subset \mathscr{K}$, cf. Definition 17. Suppose that $v \in H^1(\Omega)$ is such that $\mathfrak{E}v|_{\mathscr{K}} \in H^{l_\kappa}(\mathscr{K})$, for some $l_\kappa > 1/2$. Then, given that Assumption 30 is satisfied, the following bound holds*

$$\|v - \tilde{\Pi}_p v\|_{L^2(\partial\kappa)} \le C_{\text{I},5} \frac{h_\kappa^{s_\kappa - 1/2}}{p^{l_\kappa - 1/2}} \|\mathscr{E} v\|_{H^{l_\kappa}(\mathscr{K})}, \tag{4.25}$$

where $s_\kappa = \min\{p+1, l_\kappa\}$ and $C_{\text{I},5}$ is a positive constant which depends on C_s from (4.22) and the shape-regularity of $\mathscr{K}$, but is independent of v, h_κ, p, and the number of faces per element.

Proof Exploiting Assumption 30, and in particular (4.22) and (4.23), together with the multiplicative trace inequality stated in Lemma 22, the arithmetic-geometric mean inequality, cf. (4.12) with $\epsilon = p/h_\kappa$, and the approximation result (3.26) given in Lemma 23, we deduce that

$$
\begin{aligned}
\|v - \tilde{\Pi}_p v\|^2_{L^2(\partial\kappa)} &\le \sum_{F\subset\partial\kappa} C_{\mathrm{t}} |F| \left(\frac{1}{|\kappa_\flat^F|} \|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} \right. \\
&\qquad \left. + \frac{h_{\kappa_\flat^F}}{|\kappa_\flat^F|} \|v - \tilde{\Pi}_p v\|_{L^2(\kappa_\flat^F)} \|\nabla (v - \tilde{\Pi}_p v)\|_{L^2(\kappa_\flat^F)} \right) \\
&\le C_{\mathrm{t}}\, C_s\, d \sum_{F\subset\partial\kappa} \left(\frac{1}{h_\kappa} \|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} \right. \\
&\qquad \left. + \|v - \tilde{\Pi}_p v\|_{L^2(\kappa_\flat^F)} \|\nabla (v - \tilde{\Pi}_p v)\|_{L^2(\kappa_\flat^F)} \right) \\
&\le C_{\mathrm{t}}\, C_s\, d \sum_{F\subset\partial\kappa} \left(\frac{p+1}{h_\kappa} \|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} + \frac{h_k}{4p} \|\nabla (v - \tilde{\Pi}_p v)\|^2_{L^2(\kappa_\flat^F)} \right) \\
&\le C_{\mathrm{t}}\, C_s\, d \left(\frac{p+1}{h_\kappa} \|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa)} + \frac{h_k}{4p} \|\nabla (v - \tilde{\Pi}_p v)\|^2_{L^2(\kappa)} \right) \\
&\le \frac{9}{4} C_{\mathrm{t}}\, C_s\, d\, C_{\mathrm{I},3}^2 \frac{h_\kappa^{2s_\kappa - 1}}{p^{2l_\kappa - 1}} \|\mathscr{E} v\|^2_{H^{l_\kappa}(\mathscr{K})}.
\end{aligned}
$$

The statement of the lemma now follows immediately with $C_{\mathrm{I},5} = 3C_{\mathrm{I},3}\sqrt{C_{\mathrm{t}}\, C_s\, d}/2$.

Remark 34 Crucially, the constants arising in the inverse inequality and approximation result derived in Lemmas 32 and 33, respectively, are independent of the number of faces that a given element possesses. Here, the proofs are based on applying inverse inequalities and approximation estimates, respectively, on each individual face F in such a manner to get a bound relative to each (anisotropic) simplex $\kappa_\flat^F$ related to the corresponding face F, $F \subset \partial\kappa$, rather than with respect to the element κ itself. Thereby, the resulting individual contributions may be summed to deduce the required inequality on κ. This process can be taken to the limit, in the sense of allowing $|F|$ to tend to zero; in this manner, elements with curved faces can be treated. We refer to [58, 153] for some recent results on the respective inverse and approximation estimates for general curved elements constructed in this spirit.

4.3.1 Stability and A Priori Error Analysis

In this section we develop the stability and a priori error analysis of the DGFEM (4.5) assuming now that Assumption 30 holds. To this end, we first deduce the following coercivity and continuity bounds for the bilinear form $\tilde{B}_{\mathrm{d}}(\cdot,\cdot)$ over $\mathscr{V} \times \mathscr{V}$.

Lemma 35 *Given that Assumption 30 holds, we define the discontinuity-penalization function* $\sigma : \mathscr{F}_h \to \mathbb{R}$ *by*

$$\sigma(\mathbf{x}) := \begin{cases} C_\sigma \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left\{ C_{\mathrm{inv},1} \dfrac{\bar{a}_\kappa p_\kappa^2}{h_\kappa} \right\}, & \mathbf{x} \in F \in \mathscr{F}_h^{\mathscr{I}},\ F = \partial\kappa^+ \cap \partial\kappa^-, \\ C_\sigma C_{\mathrm{inv},1} \dfrac{\bar{a}_\kappa p_\kappa^2}{h_\kappa}, & \mathbf{x} \in F \in \mathscr{F}_h^{D},\ F \subset \partial\kappa, \end{cases} \tag{4.26}$$

where C_σ *is a sufficiently large positive constant, which is independent of the number of faces per element. Then, the bilinear form* $\tilde{B}_{\mathrm{d}}(\cdot,\cdot)$ *is coercive and continuous over* $\mathscr{V} \times \mathscr{V}$*, i.e.,*

$$\tilde{B}_{\mathrm{d}}(v, v) \geq C_{\mathrm{coer}} |||v|||_{\mathrm{DG}}^2 \quad \textit{for all} \quad v \in \mathscr{V}, \tag{4.27}$$

and

$$\tilde{B}_{\mathrm{d}}(w, v) \leq C_{\mathrm{cont}} |||w|||_{\mathrm{DG}}\, |||v|||_{\mathrm{DG}} \quad \textit{for all} \quad w, v \in \mathscr{V}, \tag{4.28}$$

respectively, where C_{coer} *and* C_{cont} *are positive constants independent of the discretization parameters and the number of faces per element.*

Proof Recalling the second term on the right-hand side of (4.15) in the proof of Lemma 27, upon application of the arithmetic-geometric mean inequality (4.12), the proof of Lemma 32, and the L^2-stability of $\boldsymbol{\Pi}_{L^2}$, we deduce that

$$\begin{aligned} &\int_{\mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^{D}} \{\!\{ a \boldsymbol{\Pi}_{L^2}(\nabla v) \}\!\} \cdot [\![v]\!] \,\mathrm{d}s \\ &\leq \epsilon \sum_{\kappa \in \mathscr{T}_h} \sum_{F \subset \partial\kappa} \bar{a}_\kappa \left\| \frac{1}{\sqrt{\sigma}} \sqrt{a} \boldsymbol{\Pi}_{L^2}(\nabla v) \right\|_{L^2(F)}^2 + \frac{1}{4\epsilon} \sum_{F \in \mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^{D}} \|\sqrt{\sigma} [\![v]\!]\|_{L^2(F)}^2 \\ &\leq \epsilon \sum_{\kappa \in \mathscr{T}_h} \sum_{F \subset \partial\kappa} \sigma^{-1} \bar{a}_\kappa C_{\mathrm{inv},1} p_\kappa^2 \frac{|F|}{|\kappa_\flat^F|} \|\sqrt{a} \boldsymbol{\Pi}_{L^2}(\nabla v)\|_{L^2(\kappa_\flat^F)}^2 + \frac{1}{4\epsilon} \sum_{F \in \mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^{D}} \|\sqrt{\sigma} [\![v]\!]\|_{L^2(F)}^2 \\ &\leq \frac{\epsilon C_s d}{C_\sigma} \sum_{\kappa \in \mathscr{T}_h} \|\sqrt{a} \nabla v\|_{L^2(\kappa)}^2 + \frac{1}{4\epsilon} \sum_{F \in \mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^{D}} \|\sqrt{\sigma} [\![v]\!]\|_{L^2(F)}^2. \end{aligned} \tag{4.29}$$

Inserting (4.29) into (4.15) we get

$$\tilde{B}_{\mathrm{d}}(v, v) \geq \left(1 - \frac{2\epsilon C_s d}{C_\sigma}\right) \sum_{\kappa \in \mathscr{T}_h} \|\sqrt{a} \nabla v\|_{L^2(\kappa)}^2 + \left(1 - \frac{1}{2\epsilon}\right) \sum_{F \in \mathscr{F}_h^{\mathscr{I}} \cup \mathscr{F}_h^{D}} \|\sqrt{\sigma} [\![v]\!]\|_{L^2(F)}^2.$$

Hence, the bilinear form $\tilde{B}_{\rm d}(\cdot,\cdot)$ is coercive over $\mathscr{V}\times\mathscr{V}$ if $C_\sigma > 2C_s\epsilon d$ for some $\epsilon > 1/2$. Note that C_σ depends on C_s, but is independent of the number of faces per element. The continuity of $\tilde{B}_{\rm d}(\cdot,\cdot)$ follows analogously.

Finally, we derive an *hp*-version a priori error bound for the DGFEM (4.5), assuming that Assumption 30 holds. For brevity, we focus on the terms defined on the faces of the elements in the computational mesh, since they must be treated in a different manner to the analysis presented in Sect. 4.2. Thereby, employing (4.25) in Lemma 33, we deduce that

$$\begin{aligned}\int_{\mathscr{F}_h^{\mathscr{I}}\cup\mathscr{F}_h^{D}} \sigma[\![v-\tilde{\Pi}_p v]\!]^2\,{\rm d}s &\le 2\sum_{\kappa\in\mathscr{T}_h}\sum_{F\subset\partial\kappa\backslash\partial\Omega_{\rm N}} \sigma|_F\|v-\tilde{\Pi}_p v\|_{L^2(F)}^2\\ &\le 2\sum_{\kappa\in\mathscr{T}_h}\Big(\max_{F\subset\partial\kappa\backslash\partial\Omega_{\rm N}}\sigma|_F\Big)\|v-\tilde{\Pi}_p v\|_{L^2(\partial\kappa)}^2\\ &\le C\sum_{\kappa\in\mathscr{T}_h}\Big(\max_{F\subset\partial\kappa\backslash\partial\Omega_{\rm N}}\sigma|_F\Big)\frac{h_\kappa^{2s_\kappa-1}}{p^{2l_\kappa-1}}\|\mathscr{E}v\|_{H^{l_\kappa}(\mathscr{K})}^2, \qquad (4.30)\end{aligned}$$

where C is a positive constant, which is independent of the number of faces per element. Bounds on the remaining terms defined on $\mathscr{F}_h^{\mathscr{I}}\cup\mathscr{F}_h^{D}$ can be derived in a completely analogous fashion; for brevity the details are omitted. Hence, we arrive at the following a priori estimate.

Theorem 36 *Let $\mathscr{T}_h=\{\kappa\}$ be a subdivision of $\Omega\subset\mathbb{R}^d$, $d=2,3$, consisting of general polytopic elements satisfying Assumptions 18 and 30, with $\mathscr{T}_h^\sharp=\{\mathscr{K}\}$ an associated covering of $\mathscr{T}_h$ consisting of shape-regular d-simplices, cf. Definition 17. Let $u_h\in V^{\mathbf{p}}(\mathscr{T}_h)$, with $p_\kappa\ge 1$ for all $\kappa\in\mathscr{T}_h$, be the corresponding DGFEM solution defined by (4.5), where the discontinuity-penalization function is given by (4.26). If the analytical solution $u\in H^1(\Omega)$ to (4.1)–(4.3) satisfies $u|_\kappa\in H^{l_\kappa}(\kappa)$, $l_\kappa>3/2$, for each $\kappa\in\mathscr{T}_h$, such that $\mathfrak{E}u|_{\mathscr{K}}\in H^{l_\kappa}(\mathscr{K})$, where $\mathscr{K}\in\mathscr{T}_h^\sharp$ with $\kappa\subset\mathscr{K}$, then*

$$|\!|\!|u-u_h|\!|\!|_{\rm DG}^2\le C\sum_{\kappa\in\mathscr{T}_h}\frac{h_\kappa^{2(s_\kappa-1)}}{p_\kappa^{2(l_\kappa-1)}}\left(\bar{a}_\kappa+\mathscr{G}_\kappa(h_\kappa,p_\kappa)\right)\|\mathfrak{E}u\|_{H^{l_\kappa}(\mathscr{K})}^2,$$

where, $s_\kappa=\min\{p_\kappa+1,l_\kappa\}$,

$$\begin{aligned}\mathscr{G}_\kappa(h_\kappa,p_\kappa):=\ &\bar{a}_\kappa^2 p_\kappa h_\kappa^{-1}\max_{F\subset\partial\kappa\backslash\partial\Omega_{\rm N}}\sigma|_F^{-1}+\bar{a}_\kappa^2p_\kappa^2h_\kappa^{-1}\max_{F\subset\partial\kappa\backslash\partial\Omega_{\rm N}}\sigma|_F^{-1}\\ &+p_\kappa^{-1}h_\kappa\max_{F\subset\partial\kappa\backslash\partial\Omega_{\rm N}}\sigma|_F,\end{aligned}$$

and C is a positive constant, independent of the discretization parameters and the number of faces per element.

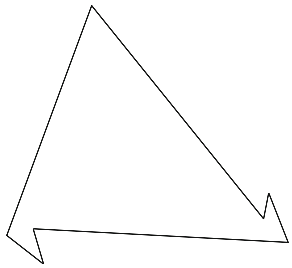

Fig. 4.2 As the small faces degenerate further, this element will not satisfy Assumption 30 but it is p-coverable and, of course, has a fixed number of faces

4.4 Mesh Assumptions for General Polytopic Elements

We conclude this chapter by discussing the relationship (or lack thereof) between the different mesh assumptions presented so far, each allowing for a proof of stability and convergence of the underlying DGFEM (4.5). On the one hand, Assumption 25 restricts the number of faces that each element in the mesh may possess; this assumption is necessary when applying the inverse estimate presented in Lemma 11 in a facewise fashion to establish stability of the DGFEM, cf. Lemma 27. On the other hand, Assumption 30, which removes this restriction on the number of element faces, is sufficient to deduce the inverse estimate stated in Lemma 32. Although one may be tempted at first sight to conclude that the latter setting includes the former, this is *not* necessarily the case. To substantiate this, consider the polygonal element depicted in Fig. 4.2; this cannot satisfy Assumption 30, with increasing degeneration of the small faces, yet it is p-coverable as these small faces degenerate for increasing p. Also the quadrilateral of Fig. 3.6 does not satisfy Assumption 30 as it fails the shape-regularity condition.

The above example shows that the two settings are applicable to different element shapes and, together, can allow for an extremely general class of admissible element shapes for which the above theory is valid. Furthermore, the two approaches may easily be combined within the same mesh allowing for mesh configurations and elements of unprecedented generality to be used to define provably convergent DGFEMs.