# Symmetrization Methods for Aggregation Functions

Radko Mesiar and Andrea Stupňanová(✉)

Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Bratislava, Slovak Republic
{radko.mesiar,andrea.stupnanova}@stuba.sk

**Abstract.** We introduce and discuss the concept of symmetrization methods for aggregation functions. Several symmetrization methods are exemplified. A particular stress is put on extremal symmetrization methods.

**Keywords:** Aggregation function · Anonymity · Permutation · Symmetric aggregation function · Symetrization method

## 1 Introduction

Symmetry of aggregation functions can be seen as a generalization of the commutativity of binary operations $x * y = y * x$, and it is known also as neutrality or anonymity. This important property indicates the equal treating of any considered input to be aggregated. It is crucial in any situation when the order of considered inputs is not known, for example when the inputs to be aggregated are evaluations of jury members stored after an anonymous procedure in a voting box (this example was motivating to call this property as anonymity in the field of multicriteria decision support). Two distinguished symmetrization methods, i.e., methods relating to a considered aggregation function some symmetric aggregation functions, can be found in [4], see also [5]. Namely, considering an $n$-ary real function $F$ and an $n$-ary input vector $\mathbf{x} = (x_1, \dots x_n)$, the function $F^+$ and $F^-$ given by

$$F^+(\mathbf{x}) = F(x_{(1)}, \dots, x_{(n)}) \quad \text{and} \quad F^-(\mathbf{x}) = F(x_{(n)}, \dots, x_{(1)}),$$

where $(.)$ is a permutation such that $x_{(1)} \le \dots \le x_{(n)}$, are symmetric. A generalization of these symmetrization methods based on a fixed permutation $\sigma \in \mathcal{P}_n$ (the set of all permutations on $\{1, \dots, n\}$) was introduced in [5], proposing a function $F_{(\sigma)}$ given by

$$F_{(\sigma)} = F(x_{(\sigma(1))}, \dots, x_{(\sigma(n))}).$$

Obviously $F_{(id)} = F^+$ and $F_{(rev)} = F^-$, where $id, rev \in \mathcal{P}_n$ are the identity permutation $(1, \dots, n)$ and the reversed permutation $(n, \dots, 1)$, respectively.


V. Torra et al. (Eds.): MDAI 2017, LNAI 10571, pp. 26–32, 2017.
DOI: 10.1007/978-3-319-67422-3_3

Observe that the permutation (.) depends on the input vector $\mathbf{x}$ and it need not be unique (this happens if there are some ties between the arguments $x_1, \ldots, x_n$). However, this possible non-uniqueness does not influence the fact that $F_\sigma$ is well defined and symmetric for any permutation $\sigma \in \mathcal{P}_n$ ($\sigma$ is independent of any considered input vector $\mathbf{x}$).

The aim of this contribution is to introduce and discuss an axiomatic approach to symmetrization of aggregation functions. The paper is organized as follows. In the next section, the necessary preliminaries are given. Section 3 brings our axiomatic characterization of symmetrization methods and offers several examples of symmetrization methods. In particular, two extremal symmetrization methods are described. In Sect. 4 we apply some of the introduced symmetrization methods to some non-symmetric aggregation functions, and especially to weighted arithmetic means. Some interesting observations are added in the concluding remarks.

## 2 Preliminaries

For a fixed $n \geq 1$, a mapping $A : [0,1]^n \to [0,1]$ is called an aggregation function whenever it is increasing in each coordinate and it satisfies the boundary conditions $A(\mathbf{0}) = A(0, \ldots, 0) = 0$ and $A(\mathbf{1}) = A(1, \ldots, 1) = 1$. Note that viewing $[0,1]^n$ and $[0,1]$ as bounded lattices, aggregation functions are just the order-homomorphisms.

The class of all $n$-ary aggregation functions is denoted as $\mathcal{A}_n$. Equipped with the partial order of $n$-ary real functions, $\mathcal{A}_n$ is a bounded lattice with the top element $A_\top$ and the bottom element $A_\perp$, given respectively by

$$A_\top(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0} \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad A_\perp(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{1} \\ 0 & \text{otherwise} \end{cases} .$$

For more details concerning aggregation functions we recommend monographs [1,2,5].

An aggregation function $A \in \mathcal{A}_n$ is called symmetric whenever

$$A(\mathbf{x}) = A(\mathbf{x}_\sigma) \quad \text{for any } \mathbf{x} \in [0,1]^n$$

and any permutation $\sigma \in \mathcal{P}_n$, where $\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.
Observe that the class $\mathcal{P}_n$ can be generated by two permutations, say $\sigma_1$ and $\sigma_2$ (i.e., any $\sigma \in \mathcal{P}_n$ can be obtained from $\sigma_1$ and $\sigma_2$, applying the composition operator consecutively), and then the symmetry of an aggregation function $A$ is characterized by the equality

$$A(\mathbf{x}) = A(\mathbf{x}_{\sigma_1}) = A(\mathbf{x}_{\sigma_2}) \quad \text{valid for each } \mathbf{x} \in [0,1]^n.$$

As one example recall $\sigma_1 = (2, 1, 3, \ldots, n)$ and $\sigma_2 = (2, 3, \ldots, n, 1)$.

The class of all n-ary symmetric aggregation functions is denoted as $\mathcal{A}_{ns}$. It is evident that $\mathcal{A}_{ns}$ is a sublattice of $\mathcal{A}_n$ with the top element $A_\top$ and the bottom element $A_\perp$. Both classes $\mathcal{A}_n$ and $\mathcal{A}_{ns}$ are closed under composition by means of $B \in \mathcal{A}_k$, i.e.,

∘ for any $k \geq 2$, $B \in \mathcal{A}_k$ and $A_1, \ldots, A_k \in \mathcal{A}_n$ $(A_1, \ldots, A_k \in \mathcal{A}_{ns})$ also the composite $B(A_1, \ldots, A_k) \in \mathcal{A}_n$ $(B(A_1, \ldots, A_k) \in \mathcal{A}_{ns})$.

Among several other properties of aggregation functions discussed in [1,2,5], we recall the idempotency. An aggregation function $A \in \mathcal{A}_n$ is called idempotent (averaging, compensative) whenever

$$A(\mathbf{c}) = A(c, \ldots, c) \quad \text{for any constant } c \in [0, 1].$$

Equivalently, the idempotency of aggregation functions can be characterized by the averaging property

$$\min(x_1, \ldots, x_n) \leq A(\mathbf{x}) \leq \max(x_1, \ldots, x_n).$$

## 3 Symmetrization Methods for Aggregation Functions

Any symmetrization method for ($n$-ary) aggregation functions should assign to an aggregation function $A \in \mathcal{A}_n$ some idempotent aggregation function $A^s \in \mathcal{A}_{ns}$. We expect that any such reasonable symmetrization method

- does not change the symmetric aggregation functions, i.e.,

$$A = A^s \quad \text{whenever} \quad A \in \mathcal{A}_{ns}, \text{ and}$$

- preserves the ordering of aggregation functions, i.e.,

$$\text{if } A, B \in \mathcal{A}_n,\ A \leq B, \quad \text{then} \quad A^s \leq B^s.$$

Formally, we propose the next axiomatic approach to symmetrization of aggregation functions.

**Definition 1.** *A mapping $\varphi : \mathcal{A}_n \to \mathcal{A}_{ns}$ is called a symmetrization method (for $n$-ary aggregation functions) whenever it is simultaneously*

*(i) an order homomorphism;*
*(ii) a projection.*

Hence, $\varphi : \mathcal{A}_n \to \mathcal{A}_{ns}$ is a symmetrization method whenever

$$A, B \in \mathcal{A}_n, A \leq B, \text{ implies } \varphi(A) \leq \varphi(B), \text{ and } \varphi(A) = A \text{ for any } A \in \mathcal{A}_{ns}.$$

Clearly, then $\varphi(\varphi(A)) = \varphi(A)$. All till now mentioned symmetrization methods (recall $F^+, F^-$ and $F_{(\sigma)}$) satisfy Definition 1. They are based on the permutation $(.) \in \mathcal{P}_n$ which depends on $\mathbf{x}$. This observation allows to split the class of all symmetrization methods into two subclasses:

- input dependent symmetrization methods;
- input independent symmetrization methods.

Note that though the input vector $\mathbf{x}$ is necessarily considered when processing $\varphi(A)(\mathbf{x})$ for any symmetrization method $\varphi$, the above partition indicates whether the introduction of $\varphi$ requires the determination of $\mathbf{x}$-dependent permutation (.) or not. The next results allow to introduce a rich variety of symmetrization methods of both kinds.

**Theorem 1.** *For a fixed $k \geq 1$, for any idempotent aggregation function $B \in \mathcal{A}_k$ and $k$-tuple $K = (\sigma_1, \ldots, \sigma_k) \in (\mathcal{P}_n)^k$ of permutations, the mapping $\varphi : \mathcal{A}_n \to \mathcal{A}_{ns}$ given by*

$$\varphi(A) = A_{B,K} = B(A_{(\sigma_1)}, \ldots, A_{(\sigma_k)}), \text{ i.e.,}$$
$$A_{B,K}(\mathbf{x}) = B(A(\mathbf{x}_{(\sigma_1)}), \ldots, A(\mathbf{x}_{(\sigma_k)}))$$

*is an input dependent symmetrization method.*

*Proof.*

(i) Note that if $k = 1$ then $B(x) = x, x \in [0,1]$, and $K = (\sigma) \in \mathcal{P}_n$, and thus $A_{B,K} = A_{(\sigma)}$.

(ii) For $k \geq 2$, the symmetry of $\varphi(A)$ was discussed in Sect. 2. Moreover, if $A \in \mathcal{A}_{ns}$ then $A_{(\sigma)} = A$ for any $\sigma \in \mathcal{P}_n$. Then the idempotency of $B$ ensures $\varphi(A) = B(A_{(\sigma_1)}, \ldots, A_{(\sigma_k)}) = B(A, \ldots, A) = A$, thus proving that $\varphi$ is a symmetrization method. Clearly, it is input dependent. □

To illustrate Theorem 1, consider $n = 3, k = 2, K = (id, rev)$ and $B \in \mathcal{A}_2$ given by $B(x_1, x_2) = \frac{x_1 + 2x_2}{3}$. Then, for any $A \in \mathcal{A}_3$,

$$A_{B,K}(x_1, x_2, x_3) = \frac{A(x_{(1)}, x_{(2)}, x_{(3)}) + 2A(x_{(3)}, x_{(2)}, x_{(1)})}{3}.$$

Suppose $A(x_1, x_2, x_3) = \sqrt[6]{x_1 {x_2}^2 {x_3}^3}$ (i.e., $A$ is a weighted geometric mean). Then

$$A_{B,K}(x_1, x_2, x_3) = \sqrt[6]{x_1 x_2 x_3}\, \frac{2\ \sqrt[6]{x_{(1)}} + \sqrt[3]{x_{(3)}}}{3}\, \sqrt[6]{x_{(2)}}$$
$$= \sqrt[6]{x_1 x_2 x_3}\, \frac{2\ \sqrt[3]{\min(x_1, x_2, x_3)} + \sqrt[3]{\max(x_1, x_2, x_3)}}{3}\, \sqrt[6]{\mathrm{med}(x_1, x_2, x_3)}.$$

The symmetry of $A_{B,K}$ is obvious.

**Theorem 2.** *Let $B \in \mathcal{A}_{n!}$ be an idempotent symmetric aggregation function of dimension $n!$. Then the mapping $\varphi : \mathcal{A}_n \to \mathcal{A}_{ns}$ given by $\varphi(A) = A_B$,*

$$A_B(\mathbf{x}) = B(A(\mathbf{x}_\sigma) | \sigma \in \mathcal{P}_n),$$

*where $\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, and $A_B$ is the $B$-aggregation of all $n!$ values $A(\mathbf{x}_\sigma), \sigma \in \mathcal{P}_n$, is an input independent symmetrization method.*

*Proof.* Evidently, $A_B$ is an $n$-ary aggregation function not dependent on $\mathbf{x}$-dependent permutation (.). Moreover, due to the facts that for any permutation $\tau \in \mathcal{P}_n$, it holds

$$\{\sigma \circ \tau | \sigma \in \mathcal{P}_n\} = \mathcal{P}_n, \quad \text{and} \quad (\mathbf{x}_\tau)_\sigma = \mathbf{x}_{\sigma \circ \tau},$$

we have

$$A_B(\mathbf{x}_\tau) = B(A(\mathbf{x}_{\sigma \circ \tau}) | \sigma \in \mathcal{P}_n) = B(A(\mathbf{x}_\sigma) | \sigma \in \mathcal{P}_n) = A_B(\mathbf{x}).$$

Hence $A_B$ is symmetric. Finally, if $A \in \mathcal{A}_{ns}$, the idempotency of $B$ ensures $A_B = A$. Summarizing, we have shown that $\varphi$ is an input independent symmetrization method. □

It is evident that that if $B_1, B_2 \in \mathcal{A}_{n!}, B_1 \leq B_2$, then also $A_{B_1} \leq A_{B_2}$ for any $A \in \mathcal{A}_n$.

Recall that the greatest idempotent aggregation function is the max operator, while the smallest one is the min operator. Moreover, both these functions are symmetric. These facts indicate the next interesting result.

**Theorem 3.** *Denote $\varphi^*(A) = A^* = A_{\max}$ and $\varphi_*(A) = A_* = A_{\min}$. Then for any symmetrization method $\varphi$ and any $A \in \mathcal{A}_n$ it holds*

$$\varphi_*(A) \leq \varphi(A) \leq \varphi^*(A),$$

*i.e., $\varphi^*$ is the greatest symmetrization method and $\varphi_*$ is the smallest symmetrization method.*

*Proof.* Recall that $A(\mathbf{x}_{id}) = A(\mathbf{x})$ and thus $A_* \leq A \leq A^*$. Due to the preservation of order of any symmetrization method $\varphi$ it holds

$$\varphi_*(A) = A_* = \varphi(A_*) \leq \varphi(A) \leq \varphi(A^*) = A^* = \varphi^*(A).$$ □

Note that extremal symmetrizations $A_*$ and $A^*$ were introduced and discussed in our recent paper [6].

As an interesting input independent symmetrization method we recall that the arithmetic mean AM satisfies all constraints of Theorem 3, and then

$$A_{\text{AM}}(\mathbf{x}) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} A(\mathbf{x}_\sigma).$$

## 4 Examples

As a prototypical aggregation function which is not symmetric we recall the weighted arithmetic mean $W_{\mathbf{w}} : [0,1]^n \to [0,1]$ given by

$$W_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i,$$

where the weighting vector $\mathbf{w} \in [0,1]^n$ satisfies $\sum_{i=1}^{n} w_i = 1$ and $\mathbf{w} \neq \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ (obviously, if $\mathbf{w} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ then $W_{\mathbf{w}} = \mathrm{AM}$ is the arithmetic mean which is a symmetric aggregation function). Then, applying different symmetrization methods, it holds:

- $(W_{\mathbf{w}})_{\mathrm{AM}} = \mathrm{AM}$;
- $(W_{\mathbf{w}})^* = \mathrm{OWA}_{\mathbf{w}^*}$ is the OWA operator [7], where $\mathbf{w}^* = (w_{[1]}, \ldots, w_{[n]})$, $[.] \in \mathcal{P}_n$ being a permutation such that $w_{[1]} \leq w_{[2]} \leq \cdots \leq w_{[n]}$, and then
$$(W_{\mathbf{w}})^*(\mathbf{x}) = \sum_{i=1}^{n} w_{[i]} x_{(i)};$$
note that vectors $\mathbf{w}^*$ and $(x_{(1)}, \ldots, x_{(n)})$ are increasing, i.e., we multiply the smallest weight $w_{[1]}$ and the smallest input $x_{(1)}$, and so one, till the product of the greatest weight $w_{[n]}$ and the greatest input $x_{(n)}$;
- $(W_{\mathbf{w}})_* = \mathrm{OWA}_{\mathbf{w}_*}$, where $\mathbf{w}_* = (w_{[n]}, \ldots, w_{[1]})$, and hence
$$(W_{\mathbf{w}})_*(\mathbf{x}) = \sum_{i=1}^{n} w_{[n-i+1]} x_{(i)};$$
here the greatest weight $w_{[n]}$ multiplies the smallest input $x_{(1)}$, etc.
- considering the input dependent symmetrization method $A_{B,K}$ introduced in the previous section for $B(x_1, x_2) = \frac{x_1 + 2x_2}{3}$ and $K = (id, rev)$, we have
$$(W_{\mathbf{w}})_{B,K}(\mathbf{x}) = \frac{1}{3}\left(\sum_{i=1}^{n} w_i x_{(i)} + 2\sum_{i=1}^{n} w_i x_{(n-i+1)}\right) = \sum_{i=1}^{n} v_i x_{(i)} = \mathrm{OWA}_{\mathbf{v}},$$
where, for $i = 1, \ldots, n$, the weight $v_i$ is given by $v_i = \frac{1}{3}(w_i + 2w_{n-i+1})$.

For $n = 3$, let $\mathbf{w} = (0.5, 0.3, 0.2)$ and $\mathbf{x} = (0.4, 0.8, 0.6)$. Then

$$\begin{aligned}
W_{\mathbf{w}}(\mathbf{x}) &= 0.56,\\
(W_{\mathbf{w}})_{\mathrm{AM}}(\mathbf{x}) &= 0.6,\\
(W_{\mathbf{w}})^*(\mathbf{x}) &= \mathrm{OWA}_{(0.2,0.3,0.5)}(0.4, 0.8, 0.6) = 0.66,\\
(W_{\mathbf{w}})_*(\mathbf{x}) &= \mathrm{OWA}_{(0.5,0.3,0.2)}(0.4, 0.8, 0.6) = 0.54,\\
(W_{\mathbf{w}})_{B,K}(\mathbf{x}) &= \frac{1}{3}\left(W_{\mathbf{w}}(0.4, 0.6, 0.8) + 2W_{\mathbf{w}}(0.8, 0.6, 0.4)\right)\\
&= \mathrm{OWA}_{(0.3,0.3,0.4)}(0.4, 0.8, 0.6) = 0.62.
\end{aligned}$$

As another example, consider the weighted geometric mean
$$G_{(w_1, 1-w_1)}(x_1, x_2) = {x_1}^{w_1} {x_2}^{1-w_1},$$
which is not symmetric whenever $w_1 \neq \frac{1}{2}$. Note that now $n = n! = 2$. Let $B_p \in \mathcal{A}_2$ be a power-root operator given by
$$B_p(x_1, x_2) = \left(\frac{{x_1}^p + {x_2}^p}{2}\right)^{\frac{1}{p}},$$

where $p \in \mathbb{R}\setminus\{0\}$, and $B_0 = G$ is the geometric mean. Then, following Theorem 3, we have

$$\left(G_{(w_1,1-w_1)}\right)_{B_p}(x_1,x_2) = \left(\frac{({x_1}^{w_1}{x_2}^{1-w_1})^p + ({x_1}^{1-w_1}{x_2}^{w_1})^p}{2}\right)^{\frac{1}{p}}$$
$$= (x_1x_2)^{\alpha} B_p({x_1}^{1-2\alpha}, {x_2}^{1-2\alpha}),$$

where $\alpha = \min(w_1, 1-w_1)$.

## 5 Concluding Remarks

We have introduced an axiomatic approach to symmetrization methods for aggregation functions. These methods belong either to input dependent methods (where the $\mathbf{x}$-dependent permutation $(.) \in \mathcal{P}_n$ is considered) or to input independent methods. We have also shown two extremal symmetrization methods. Proposed approaches were illustrated by some examples, with a particular stress on the symmetrization of the weighted arithmetic means. Note that the extremal symmetrized weighted arithmetic means $W^*$ and $W_*$ can be seen as solutions of optimization methods and they can be related to the Hungarian algorithm [3] known from the area of linear optimization. For more details see [6]. Note also that one can further extend the problem of symmetrization of aggregation functions related to weighting vectors, where the symmetrization is related to both input vector $\mathbf{x}$ and the weighting vector $\mathbf{w}$. This approach was initiated in [6] and we expect its deeper study in the near future.

**Acknowledgments.** The support of the grants APVV-14-0013 and VEGA 1/0682/16 is kindly announced.

## References

1. Beliakov, G., Bustince Sola, H., Calvo Sánchez, T.: A Practical Guide to Averaging Functions. Springer, New York (2016)
2. Beliakov, G., Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners. Springer, New York (2007)
3. Burkard, R.E., Dell'Amico, M., Martello, S.: Assignment Problems (Revised reprint). SIAM, Philadelphia (2012)
4. Calvo, T., Kolesárová, A., Komorníková, M., Mesiar, R.: Aggregation operators: properties, classes and construction methods. In: Aggregation Operators, Studies in Fuzziness and Soft Computing, vol. 97, pp. 3–104. Physica, Heidelberg (2002)
5. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge University Press, Cambridge (2009)
6. Mesiar, R., Stupňanová, A., Yager, R.R.: Extremal symmetrization of aggregation functions. Ann. Oper. Res. (2017). doi:10.1007/s10479-017-2471-x
7. Yager, R.R.: On ordered weighted averaging aggregation operators in multicriteria decisionmaking. IEEE Trans. Syst. Man Cybern. **18**(1), 183–190 (1988)