Comparison of Risk Averse Utility Functions on Two-Dimensional Regions

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Abstract. Weighted quasi-arithmetic means on two-dimensional regions are demonstrated, and risk averse conditions are discussed by the corresponding utility functions. For two utility functions on two-dimensional regions, we introduce a concept that decision making with one utility is more risk averse than decision making with the other utility. A necessary condition and a sufficient condition for the concept are demonstrated by their utility functions. Several examples are given to explain them.

1 Introduction

Weighted quasi-arithmetic means are important concept for mathematical theory such as the mean value theorems, and it is a fundamental tool for subjective estimation regarding information in management science, artificial intelligence and so on. Weighted quasi-arithmetic means of an interval are given mathematically by aggregation operations (Kolmogorov [4], Nagumo [6] and Aczél [1]). Bustince et al. [2] discussed aggregation operations on two-dimensional OWA operators, and Labreuche and Grabisch [5] demonstrated Choquet integral for aggregation in multicriteria decision making, and Torra and Godo [7] studied continuous WOWA operators for defuzzification. In micro-economics, subjective estimations with preference relations are formulated as utility functions (Fishburn [3]). From the view point of utility functions, Yoshida [8,9] have studied the relations between weighted quasi-arithmetic means on an interval and decision maker's behavior regarding risks. In one-dimensional cases, for twice continuously differentiable strictly increasing functions $\varphi, \psi : [a, b] \mapsto \mathbb{R}$ as decision makers' *utility functions* and a continuous function $\omega : [a, b] \mapsto (0, \infty)$ as a *weighting function*, *weighted quasi-arithmetic means* μ and ν on a closed interval $[a, b]$ are real numbers satisfying

$$\varphi(\mu) \int_a^b \omega(x)\, dx = \int_a^b \varphi(x)\, \omega(x)\, dx, \tag{1.1}$$

$$\psi(\nu) \int_a^b \omega(x)\, dx = \int_a^b \psi(x)\, \omega(x)\, dx \tag{1.2}$$

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in the *mean value theorem for integration*. Then it is said that decision making with utility function φ is *more risk averse* than decision making with utility function ψ if $\mu \le \nu$ for all closed intervals $[a, b]$. Its equivalent condition is

$$\frac{\varphi''}{\varphi'} \le \frac{\psi''}{\psi'} \tag{1.3}$$

on $\mathbb{R}$ (Yoshida [10,11]).

Yoshida [12] introduced weighted quasi-arithmetic means on two-dimensional regions, which are related to multi-object decision making. In this paper, using decision makers' utility functions we discuss relations between risk averse/risk neutral/risk loving conditions and the corresponding weighted quasi-arithmetic means on two-dimensional regions. In this paper we compare two decision makers' behaviors regarding risks by the weighted quasi-arithmetic means on two-dimensional regions and we give a characterization by their utility functions.

In Sect. 2 we introduce weighted quasi-arithmetic means on two-dimensional regions and we discuss their risk averse conditions. For two utility functions f and g on two-dimensional regions, we introduce a concept that decision making with utility f is more risk averse than decision making with utility g. Further we derive a necessary condition where decision making with utility f is more risk averse than decision making with utility g on two-dimensional regions, and we investigate the condition by several examples. In Sect. 3 we give sufficient conditions for the results in Sect. 2 when utility functions are quadratic.

2 Weighted Quasi-arithmetic Means on Two-Dimensional Regions

Let $\mathbb{R} = (-\infty, \infty)$ and let a domain D be a non-empty open convex subset of $\mathbb{R}^2$, and let $\mathcal{R}(D)$ be a family of closed convex subsets of D. Denote by $\mathcal{L}$ a family of twice continuously differentiable functions $f : D \mapsto \mathbb{R}$ which is strictly increasing, i.e. $f_x > 0$ and $f_y > 0$ on D, and denote by $\mathcal{W}$ a family of continuous functions $w : D \mapsto (0, \infty)$. For a closed convex set $R \in \mathcal{R}(D)$, *weighted quasi-arithmetic means* on region R with utility $f \in \mathcal{L}$ and weighting $w \in \mathcal{W}$ are given by a subset $M_w^f(R)$ of region R as follows.

$$M_w^f(R) = \left\{ (\tilde{x}, \tilde{y}) \in R \mid f(\tilde{x}, \tilde{y}) \iint_R w(x, y)\, dx\, dy = \iint_R f(x, y) w(x, y)\, dx\, dy \right\}. \tag{2.1}$$

Then we have $M_w^f(R) \neq \emptyset$ since f is continuous on R and

$$\min_{(\tilde{x},\tilde{y}) \in R} f(\tilde{x}, \tilde{y}) \le \iint_R f(x, y) w(x, y)\, dx\, dy \Big/ \iint_R w(x, y)\, dx\, dy \le \max_{(\tilde{x},\tilde{y}) \in R} f(\tilde{x}, \tilde{y}).$$

We introduce the following natural ordering on $\mathbb{R}^2$.

Definition 2.1 (A partial order $\preceq$ on $\mathbb{R}^2$).

(i) For two points $(\underline{x}, \underline{y}), (\overline{x}, \overline{y}) (\in \mathbb{R}^2)$, an order $(\underline{x}, \underline{y}) \preceq (\overline{x}, \overline{y})$ implies $\underline{x} \leq \overline{x}$ and $\underline{y} \leq \overline{y}$.
(ii) For two points $(\underline{x}, \underline{y}), (\overline{x}, \overline{y}) (\in \mathbb{R}^2)$, an order $(\underline{x}, \underline{y}) \prec (\overline{x}, \overline{y})$ implies $(\underline{x}, \underline{y}) \preceq (\overline{x}, \overline{y})$ and $(\underline{x}, \underline{y}) \neq (\overline{x}, \overline{y})$.
(iii) For two sets $A, B (\subset \mathbb{R}^2)$, an order $A \preceq B$ implies the following (a) and (b):
  (a) For any $(\underline{x}, \underline{y}) \in A$ there exists $(\overline{x}, \overline{y}) \in B$ satisfying $(\underline{x}, \underline{y}) \preceq (\overline{x}, \overline{y})$.
  (b) For any $(\overline{x}, \overline{y}) \in B$ there exists $(\underline{x}, \underline{y}) \in A$ satisfying $(\underline{x}, \underline{y}) \preceq (\overline{x}, \overline{y})$.

Let a closed convex region $R \in \mathcal{R}(D)$ and let a weighting function $w \in \mathcal{W}$. We define a point $(\overline{x}_R, \overline{y}_R)$ on region R by the following weighted quasi-arithmetic means:

$$\overline{x}_R = \iint_R x\, w(x, y)\, dx\, dy \Big/ \iint_R w(x, y)\, dx\, dy, \tag{2.2}$$

$$\overline{y}_R = \iint_R y\, w(x, y)\, dx\, dy \Big/ \iint_R w(x, y)\, dx\, dy. \tag{2.3}$$

Hence, $(\overline{x}_R, \overline{y}_R)$ is called an *invariant risk neutral point on R with weighting w* (Yoshida [12]). We separate the space $\mathbb{R}^2$ as follows. Let $R_{w,-}^{(\overline{x}_R, \overline{y}_R)} = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \prec (\overline{x}_R, \overline{y}_R)\} = \{(x, y) \in \mathbb{R}^2 \mid x \leq \overline{x}_R,\ y \leq \overline{y}_R, (x, y) \neq (\overline{x}_R, \overline{y}_R)\}$ and $R_{w,+}^{(\overline{x}_R, \overline{y}_R)} = \{(x, y) \in \mathbb{R}^2 \mid (\overline{x}_R, \overline{y}_R) \prec (x, y)\} = \{(x, y) \in \mathbb{R}^2 \mid x \geq \overline{x}_R,\ y \geq \overline{y}_R, (x, y) \neq (\overline{x}_R, \overline{y}_R)\}$. Then $R_{w,-}^{(\overline{x}_R, \overline{y}_R)}$ denotes a subregion of *risk averse points* and $R_{w,+}^{(\overline{x}_R, \overline{y}_R)}$ denotes a subregion of *risk loving points*. Let $R_{w}^{(\overline{x}_R, \overline{y}_R)} = R_{w,-}^{(\overline{x}_R, \overline{y}_R)} \cup R_{w,+}^{(\overline{x}_R, \overline{y}_R)} \cup \{(\overline{x}_R, \overline{y}_R)\}$. Now we introduce the following relations between decision maker's behavior and his utility.

Definition 2.2. Let a utility function $f \in \mathcal{L}$ and let a rectangle region $R \in \mathcal{R}(D)$.

(i) Decision making with utility f is called *risk neutral on R* if

$$f(\overline{x}_R, \overline{y}_R) \iint_R w(x, y)\, dx\, dy = \iint_R f(x, y) w(x, y)\, dx\, dy \tag{2.4}$$

for all density functions w.

(ii) Decision making with utility f is called *risk averse on R* if

$$f(\overline{x}_R, \overline{y}_R) \iint_R w(x, y)\, dx\, dy \geq \iint_R f(x, y) w(x, y)\, dx\, dy \tag{2.5}$$

for all density functions w.

(iii) Decision making with utility f is called *risk loving on R* if

$$f(\overline{x}_R, \overline{y}_R) \iint_R w(x, y)\, dx\, dy \leq \iint_R f(x, y) w(x, y)\, dx\, dy \tag{2.6}$$

for all density functions w.

Example 2.1. Let a domain $D = (-0.5, 1.25)^2$ and a region $R = [0, 1]^2$, and let a weighting function $w(x, y) = 1$ for $(x, y) \in D$. Then an invariant neutral point is $(\overline{x}_R, \overline{y}_R) = (0.5, 0.5)$ and $R_{w,-}^{(\overline{x}_R,\overline{y}_R)} = [0, 0.5]^2 \setminus \{(0.5, 0.5)\}$ and $R_{w,+}^{(\overline{x}_R,\overline{y}_R)} = [0.5, 1]^2 \setminus \{(0.5, 0.5)\}$. Let us consider two utility functions $f(x, y) = -x^2 - y^2 + 3x + 3y$ and $g(x, y) = 2x^2 + 2y^2 - 5x - 5y$ for $(x, y) \in D$. Then by Yoshida [12, Example 3.1(i), Lemma 2.2] decision making with utility function f is called risk averse on R with weighting w, and decision making with utility function g is also called risk loving on R with weighting w. Hence the corresponding weighted quasi-arithmetic means $M_w^f(R)$ and $M_w^g(R)$ are ordered by the order $\preceq$ in a restricted subregion $R_w^{(\overline{x}_R,\overline{y}_R)} = R_{w,-}^{(\overline{x}_R,\overline{y}_R)} \cup R_{w,+}^{(\overline{x}_R,\overline{y}_R)} \cup \{(\overline{x}_R, \overline{y}_R)\}$. However they can not be ordered on a subregion $R \setminus R_w^{(\overline{x}_R,\overline{y}_R)}$ (Fig. 1).

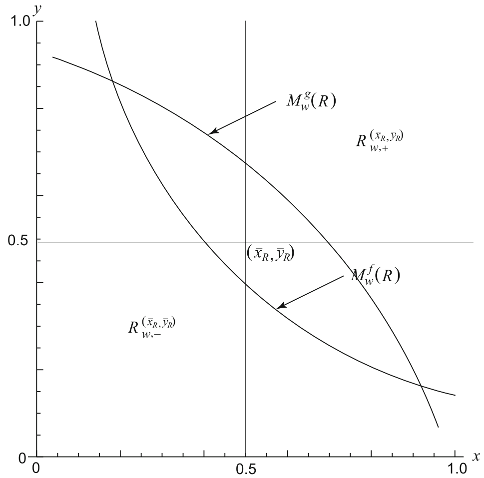

Fig. 1. $M_w^f(R) \cap R_w^{(\overline{x}_R,\overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R,\overline{y}_R)}$ $(f(x, y) = -x^2 - y^2 + 3x + 3y, g(x, y) = 2x^2 + 2y^2 - 5x - 5y,\ R = [0, 1]^2)$

It is natural that the order $\preceq$ should be given between weighted quasi-arithmetic means $M_w^f(R)$ of risk averse utility f and weighted quasi-arithmetic means $M_w^g(R)$ of risk loving utility g in Example 3.1. Therefore when we compare weighted quasi-arithmetic means $M_w^f(R)$ and $M_v^g(R)$, we discuss it on the meaningful restricted subregion $R_w^{(\overline{x}_R,\overline{y}_R)}$. Hence we introduce the following definition regarding the comparison of utility functions.

Definition 2.3. Let $f, g \in \mathcal{L}$ be utility functions on D. Decision making with utility f is *more risk averse than* decision making with utility g if it holds that

$$M_w^f(R) \cap R_w^{(\overline{x}_R,\overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R,\overline{y}_R)} \tag{2.7}$$

for all weighting functions $w \in \mathcal{W}$ on D and all closed convex regions $R \in \mathcal{R}(D)$.

Example 2.2. Let a domain $D = (-0.5, 1.25)^2$ and a region $R = [0, 1]^2$, and let a weighting function $w(x, y) = 1$ for $(x, y) \in D$. Then an invariant neutral point is $(\overline{x}_R, \overline{y}_R) = (0.5, 0.5)$ and $R_{w,-}^{(\overline{x}_R, \overline{y}_R)} = [0, 0.5]^2 \setminus \{(0.5, 0.5)\}$ and $R_{w,+}^{(\overline{x}_R, \overline{y}_R)} = [0.5, 1]^2 \setminus \{(0.5, 0.5)\}$. Let us consider two utility functions $f(x, y) = -x^2 - y^2 + 3x + 3y$ and $g(x, y) = -2x^2 - 2y^2 + 5x + 5y$ for $(x, y) \in D$. Then decision making with utility f is *more risk averse than* decision making with utility g as we see the relation (2.7) in Fig. 2.

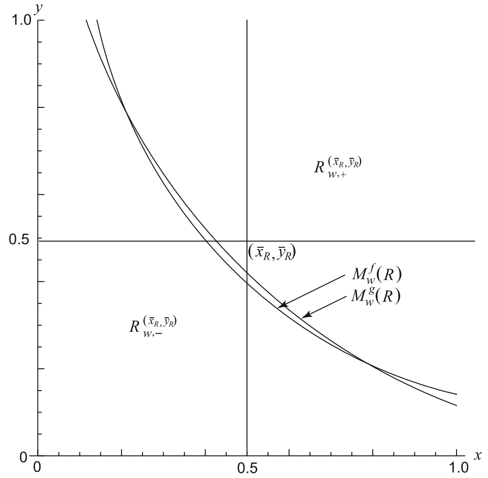

Fig. 2. $M_w^f(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)}$ $(f(x, y) = -x^2 - y^2 + 3x + 3y, g(x, y) = -2x^2 - 2y^2 + 5x + 5y,\ R = [0, 1]^2)$

Now we give a necessary condition for (2.7), i.e. decision making with utility f is more risk averse than decision making with utility g.

Theorem 2.1. *Let $f, g \in \mathcal{L}$ be utility functions on D. If decision making with utility f is more risk averse than decision making with utility g, then it holds that*

$$\frac{h^2 f_{xx} + 2rhk f_{xy} + k^2 f_{yy}}{h f_x + k f_y} \leq \frac{h^2 g_{xx} + 2rhk g_{xy} + k^2 g_{yy}}{h g_x + k g_y} \tag{2.8}$$

on D for all positive numbers h and k and all real numbers r satisfying $-1 \leq r \leq 1$.

From Theorem 2.1 we can easily obtain the following result, which is corresponding to [12, Theorem 3.1(i)].

Corollary 2.1. *Let $f, g \in \mathcal{L}$ be utility functions on D. If decision making with utility f is more risk averse than decision making with utility g, then it holds that*

$$\frac{f_{xx}}{f_x} \leq \frac{g_{xx}}{g_x} \quad \text{and} \quad \frac{f_{yy}}{f_y} \leq \frac{g_{yy}}{g_y} \quad \text{on } D. \tag{2.9}$$

Equation (2.8) in Theorem 2.1 gives a detailed relation between f and g rather than (2.9). A parameter r in necessary condition (2.8) depends on the shapes of closed convex regions $R \in \mathcal{R}(D)$. Now we investigate several examples with different shapes of regions R.

Example 2.3 (Rectangle regions)**.** Let h and k be positive numbers.Let rectangle regions

$$R_{h,k}^{\mathrm{Rect}}(a,b,t) = [a, a+ht] \times [b, b+kt] \tag{2.10}$$

for $(a,b) \in D$ and $t > 0$. Denote a family of rectangle regions by $\mathcal{R}_{h,k}^{\mathrm{Rect}}(D) = \{R_{h,k}^{\mathrm{Rect}}(a,b,t) \mid R_{h,k}^{\mathrm{Rect}}(a,b,t) \subset D,\ (a,b) \in D,\ t > 0\}(\subset \mathcal{R}(D))$, (Fig. 3).

Corollary 2.2. *If utility functions $f, g \in \mathcal{L}$ satisfy $M_w^f(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)}$ for all weighting functions $w \in \mathcal{W}$ on D and all rectangle regions $R \in \mathcal{R}_{h,k}^{\mathrm{Rect}}(D)$, then it holds that*

$$\frac{h^2 f_{xx} + k^2 f_{yy}}{h f_x + k f_y} \le \frac{h^2 g_{xx} + k^2 g_{yy}}{h g_x + k g_y} \tag{2.11}$$

on D.

Example 2.4 (Oval regions)**.** Let h and k be positive numbers. Let oval regions

$$R_{h,k}^{\mathrm{Oval}}(a,b,t) = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \frac{(x-a)^2}{h^2} + \frac{(y-b)^2}{k^2} \le t^2 \right\} \tag{2.12}$$

for $(a,b) \in D$ and $t > 0$. Denote a family of oval regions by $\mathcal{R}_{h,k}^{\mathrm{Oval}}(D) = \{R_{h,k}^{\mathrm{Oval}}(a,b,t) \mid R_{h,k}^{\mathrm{Oval}}(a,b,t) \subset D,\ (a,b) \in D,\ t > 0\}(\subset \mathcal{R}(D))$, (Fig. 3).

Corollary 2.3. *If utility functions $f, g \in \mathcal{L}$ satisfy $M_w^f(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)}$ for all weighting functions $w \in \mathcal{W}$ on D and all oval regions $R \in \mathcal{R}_{h,k}^{\mathrm{Oval}}(D)$, then it holds that*

$$\frac{h^2 f_{xx} + k^2 f_{yy}}{h f_x + k f_y} \le \frac{h^2 g_{xx} + k^2 g_{yy}}{h g_x + k g_y} \tag{2.13}$$

on D.

Example 2.5 (Triangle regions)**.** Let h and k be positive numbers. Let triangle regions

$$R_{h,k}^{\mathrm{Tri}}(a,b,t) = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x \ge a,\ y \ge b,\ \frac{x-a}{h} + \frac{y-b}{k} \le t \right\} \tag{2.14}$$

for $(a,b) \in D$ and $t > 0$. Denote a family of triangle regions by $\mathcal{R}_{h,k}^{\mathrm{Tri}}(D) = \{R_{h,k}^{\mathrm{Tri}}(a,b,t) \mid R_{h,k}^{\mathrm{Tri}}(a,b,t) \subset D,\ (a,b) \in D,\ t > 0\}(\subset \mathcal{R}(D))$, (Fig. 4).

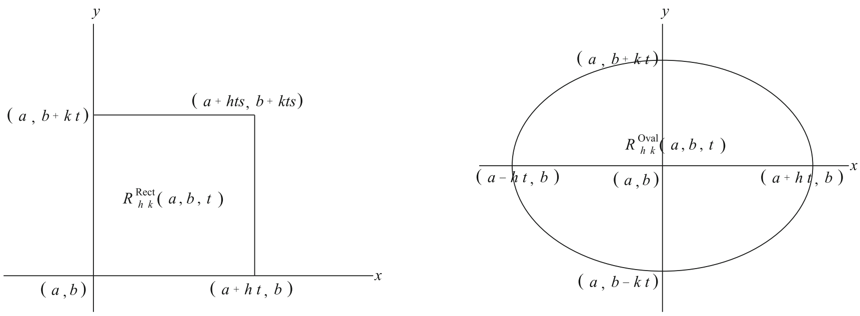

Fig. 3. Rectangle region $R^{\mathrm{Rect}}_{h,k}(a,b,t)$ and oval region $R^{\mathrm{Oval}}_{h,k}(a,b,t)$

Corollary 2.4. *If utility functions $f, g \in \mathcal{L}$ satisfy $M^f_w(R) \cap R^{(\overline{x}_R,\overline{y}_R)}_w \preceq M^g_v(R) \cap R^{(\overline{x}_R,\overline{y}_R)}_w$ for all weighting functions $w \in \mathcal{W}$ on D and all triangle regions $R \in \mathcal{R}^{\mathrm{Tri}}_{h,k}(D)$, then it holds that*

$$\frac{h^2 f_{xx} - hk f_{xy} + k^2 f_{yy}}{h f_x + k f_y} \leq \frac{h^2 g_{xx} - hk g_{xy} + k^2 g_{yy}}{h g_x + k g_y} \tag{2.15}$$

on D.

Example 2.6 (Parallelogram regions)**.** Let h and k be positive numbers. Let parallelogram regions

$$R^{\mathrm{Para}}_{h,k}(a,b,t) = \{(x,y) \mid |k(x-a) - 3h(y-b)| \leq 4hkt,\ |3k(x-a) - h(y-b)| \leq 4hkt\} \tag{2.16}$$

for $(a,b) \in D$ and $t > 0$. Denote a family of parallelogram regions by $\mathcal{R}^{\mathrm{Para}}_{h,k}(D) = \{R^{\mathrm{Para}}_{h,k}(a,b,t) \mid R^{\mathrm{Para}}_{h,k}(a,b,t) \subset D,\ (a,b) \in D,\ t > 0\} (\subset \mathcal{R}(D))$, (Fig. 4).

Corollary 2.5. *If utility functions $f, g \in \mathcal{L}$ satisfy $M^f_w(R) \cap R^{(\overline{x}_R,\overline{y}_R)}_w \preceq M^g_v(R) \cap R^{(\overline{x}_R,\overline{y}_R)}_w$ for all weighting functions $w \in \mathcal{W}$ on D and all parallelogram regions $R \in \mathcal{R}^{\mathrm{Para}}_{h,k}(D)$, then it holds that*

$$\frac{h^2 f_{xx} + \frac{3}{5} hk f_{xy} + k^2 f_{yy}}{h f_x + k f_y} \leq \frac{h^2 g_{xx} + \frac{3}{5} hk g_{xy} + k^2 g_{yy}}{h g_x + k g_y} \tag{2.17}$$

on D.

Example 2.3 (Rectangle regions) and Example 2.4 (Oval regions) are cases where $r = 0$ in (2.8), and Example 2.5 (Triangle regions) and Example 2.6 (Parallelogram regions) are cases where $r = -\frac{1}{2}$ and $r = \frac{3}{10}$ respectively in (2.8).

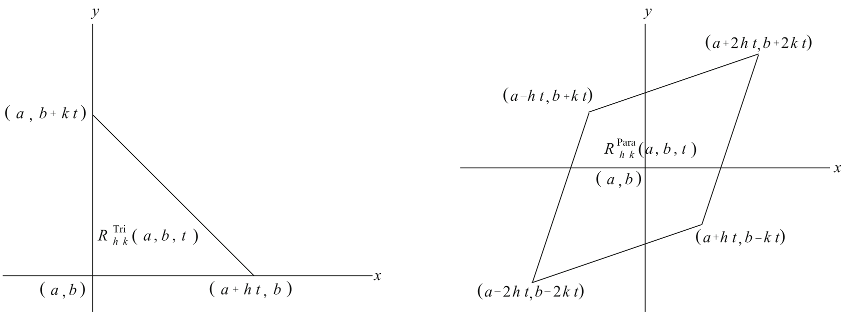

Fig. 4. Triangle region $R^{\rm Tri}_{h,k}(a,b,t)$ and parallelogram region $R^{\rm Para}_{h,k}(a,b,t)$

3 A Sufficient Condition

Let $f, g \in \mathcal{L}$ be utility functions on an open convex domain D. Theorem 2.1 gives a necessary condition that decision making with utility f is more risk averse than decision making with utility g. In this section, we discuss its sufficient condition. For a utility function $f \in \mathcal{L}$, its Hessian matrix is written by

$$H^f(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} \tag{3.1}$$

for $(x,y) \in D$. The the following proposition gives a sufficient condition for (2.8) in Theorem 2.1.

Proposition 3.1. *Let $f, g \in \mathcal{L}$ be utility functions on D. Then the following (i) and (ii) hold.*

(i) Matrices

$$\frac{1}{f_x(x,y)}H^f(x,y) - \frac{1}{g_x(x,y)}H^g(x,y) \ \textit{and}\ \frac{1}{f_y(x,y)}H^f(x,y) - \frac{1}{g_y(x,y)}H^g(x,y) \tag{3.2}$$

are negative semi-definite for all $(x,y) \in D$ if and only if a matrix

$$\frac{1}{hf_x(x,y)+kf_y(x,y)}H^f(x,y) - \frac{1}{hg_x(x,y)+kg_y(x,y)}H^g(x,y) \tag{3.3}$$

is negative semi-definite for all $(x,y) \in D$ and all positive numbers h and k.

(ii) If (3.2) are negative semi-definite at all $(x,y) \in D$, then (2.8) holds on D for all positive numbers h and k and all real numbers r satisfying $-1 \le r \le 1$.

From Proposition 3.1 implies that the condition (3.2) is stronger than the condition (2.8), however (3.2) is easier than (2.8) to check in actual cases. In this paper, utility functions $f (\in \mathcal{L})$ are called *quadratic* if the second derivatives

f_{xx}, f_{xy} and f_{yy} are constant functions. When utility functions are quadratic, the following theorem gives a sufficient condition for what decision making with utility f is more risk averse than decision making with utility g.

Theorem 3.1. *Let utility functions $f, g \in \mathcal{L}$ be quadratic on D. If*

$$\frac{1}{f_x(x,y)}H^f(x,y) - \frac{1}{g_x(x,y)}H^g(x,y) \quad \textit{and} \quad \frac{1}{f_y(x,y)}H^f(x,y) - \frac{1}{g_y(x,y)}H^g(x,y) \tag{3.4}$$

are negative semi-definite at all $(x,y) \in D$, then decision making with utility f is more risk averse than decision making with utility g, i.e.

$$M_w^f(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)}$$

for all weighting functions $w \in \mathcal{W}$ and all closed convex regions $R \in \mathcal{R}(D)$.

Now we give an example for Theorem 3.1.

Example 3.1 (Quadratic utility functions)**.** Let a domain $D = (-0.5, 1.5)^2$ and a region $R = [0,1]^2$, and let a weighting function $w(x,y) = 1$ for $(x,y) \in D$. Then an invariant neutral point is $(\overline{x}_R, \overline{y}_R) = (0.5, 0.5)$ and $R_{w,-}^{(\overline{x}_R, \overline{y}_R)} = [0, 0.5]^2 \setminus \{(0.5, 0.5)\}$ and $R_{w,+}^{(\overline{x}_R, \overline{y}_R)} = [0.5, 1]^2 \setminus \{(0.5, 0.5)\}$. Let us consider two quadratic utility functions $f(x,y) = -2x^2 - 2y^2 + 2xy + 8x + 8y$ and $g(x,y) = -x^2 - y^2 + xy + 5x + 5y$ for $(x,y) \in D$. Then f and g are increasing on D, i.e. $f_x(x,y) = -4x+2y+8 > 0$, $f_y(x,y) = 2x-4y+8 > 0$, $g_x(x,y) = -2x+y+5 > 0$ and $g_y(x,y) = x - 2y + 5 > 0$ on D. Their Hessian matrices are

$$H^f(x,y) = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \quad \text{and} \quad H^g(x,y) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3.5}$$

Let $A(x,y)$ and $B(x,y)$ by $A(x,y) = \frac{1}{f_x(x,y)}H^f(x,y) - \frac{1}{g_x(x,y)}H^g(x,y)$ and $B(x,y) = \frac{1}{f_y(x,y)}H^f(x,y) - \frac{1}{g_y(x,y)}H^g(x,y)$ for $(x,y) \in D$, and then we have

$$A(x,y) = \frac{1}{-4x+2y+8}\begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} - \frac{1}{-2x+y+5}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \tag{3.6}$$

$$B(x,y) = \frac{1}{2x-4y+8}\begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} - \frac{1}{x-2y+5}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3.7}$$

We can easily check $A(x,y)$ and $B(x,y)$ are negative definite for all $(x,y) \in D$. From Theorem 3.1, decision making with utility f is more risk averse than decision making with utility g on R and it holds that $M_w^f(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R, \overline{y}_R)}$ for all weighting functions $w \in \mathcal{W}$ (Fig. 5).

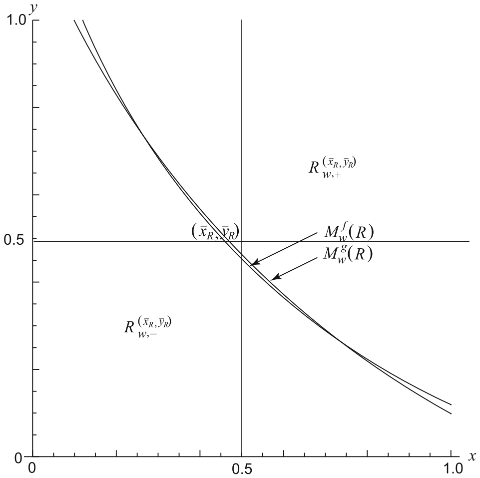

Fig. 5. $M_w^f(R) \cap R_w^{(\overline{x}_R,\overline{y}_R)} \preceq M_v^g(R) \cap R_w^{(\overline{x}_R,\overline{y}_R)}$ $(f(x,y) = -2x^2 - 2y^2 + 2xy + 8x + 8y, g(x,y) = -x^2 - y^2 + xy + 5x + 5y,\ R = [0,1]^2)$

Concluding Remark. When utility functions are quadratic, Theorem 3.1 gives a sufficient condition where decision making with utility f is more risk averse than decision making with utility g. It is an open problem whether (3.2) is a sufficient condition when utility functions are not quadratic but more general.

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