

# Weak and Strong Solutions for Fuzzy Linear Programming Problems

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**Abstract.** This paper discusses some feasibility conditions for fuzzy linear programming problems. The selection of different membership functions in a fuzzy linear programming problem can lead to different solutions, including unbounded and infeasible solutions, so in this paper we generalize concepts of *weak* and *strong* solutions for this kind of problems. An application example is provided to illustrate our results.

## 1 Introduction and Motivation

Fuzzy Linear Programming (FLP) problems are among the most popular fuzzy optimization techniques, so its analysis provides valuable information to practitioners. The classical FLP model was proposed by Zimmermann [1], Zimmermann and Fullér [2], and Fiedler et al. [3]; Černý and Hladík [4], Hladík [5] have extended his results to two main families of fuzzy LPs: problems with fuzzy parameters/constraints, and problems with fuzzy parameters and crisp constraints. Hernández-Pérez and Figueroa-García [6] discussed some sensitivity issues for the Zimmermann's soft constraints model, and Figueroa-García et al. [7] have defined some feasibility conditions for fuzzy/crisp LPs.

Hladík [5] has defined basic concepts of weak/strong feasibility for interval-valued equations which we extend to FLPs. This way, we propose similar feasibility conditions for FLPs with fuzzy costs, parameters and constraints with nonlinear membership functions, which is a different problem of the issued by Zimmermann on his seminal work [1]. The FLP addressed here refers to an LP structure whose coefficients can be fuzzy sets with any linear/nonlinear shape (e.g. exponential, gaussian, quadratic, sigmoidal, etc.), including its constraints. To do so, we define feasibility over FLPs in two instances: feasibility regarding the support of all fuzzy parameters, and feasibility regarding  $\alpha$ -cuts. The case of infeasible FLPs is discussed as well.

This paper focuses on the analysis of weak/strong feasibility conditions for a general FLP model with nonlinear costs, technological coefficients, and constraints. Some examples are provided and its results are discussed. The paper is divided into six sections. Section 1 introduces the problem. In Sect. 2, some basics on fuzzy numbers are provided; in Sect. 3, description of the FLP model used

here and concepts of weak/strong feasibility, are introduced. Section 4 presents and explains some application examples (including feasible/infeasible examples); and Sect. 5 presents the concluding remarks of the study.

## 2 Basics on Fuzzy Numbers

FLP models are composed by three sets of fuzzy parameters: costs, technological coefficients, and constraints which are commonly defined as convex fuzzy sets (or fuzzy numbers) for which we provide some basic notations.  $\mathcal{P}(\mathbb{R})$  is the class of all crisp sets of  $X$  and  $\mathcal{F}(\mathbb{R})$  is the class of all fuzzy sets defined over the reals. A fuzzy set  $\tilde{A}$ ,  $\tilde{A} : X \rightarrow [0, 1]$  can be represented as a set of ordered pairs of an element  $x$  and its membership degree,  $\mu_{\tilde{A}}(x)$ , i.e.,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\} \tag{1}$$

The *support* of  $\tilde{A}$ ,  $supp(\tilde{A})$ , is composed by all the elements of  $X$  that have nonzero membership in  $\tilde{A}$ , this is:

$$supp(\tilde{A}) = \{x \mid \mu_{\tilde{A}}(x) > 0\} \forall x \in X \tag{2}$$

The  $\alpha$ -cut of  $\mu_{\tilde{A}}(x)$  namely  ${}^\alpha\tilde{A}$  represents the interval of all values of  $x$  which has a membership degree equal or greatest than  $\alpha$ , this means:

$${}^\alpha\tilde{A} = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\} \forall x \in X \tag{3}$$

$${}^\alpha\tilde{A} \in \left[ \inf_x {}^\alpha\mu_{\tilde{A}}(x), \sup_x {}^\alpha\mu_{\tilde{A}}(x) \right] = \left[ \check{A}_\alpha, \hat{A}_\alpha \right] \tag{4}$$

A *fuzzy number* is then a convex fuzzy set. Let  $\tilde{A} \in \mathcal{G}(\mathbb{R})$  where  $\mathcal{G}(\mathbb{R}) \in \mathcal{F}(\mathbb{R})$  is the class of all normal, upper semicontinuous, and fuzzy convex sets. Then,  $\tilde{A}$  is a Fuzzy Number (FN) iff there exists a closed interval  $[a, b] \neq \emptyset$  such that

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{for } x \in [a, b], \\ l(x) & \text{for } x \in (-\infty, a], \\ r(x) & \text{for } x \in [b, \infty) \end{cases} \tag{5}$$

where  $l : (-\infty, a) \rightarrow [0, 1]$  is monotonic non-decreasing, continuous from the right, and  $l(x) = 0$  for  $x < \omega_1$ , and  $r : (b, \infty) \rightarrow [0, 1]$  is monotonic non-increasing, continuous from the left, and  $r(x) = 0$  for  $x > \omega_2$ .

A graphical display of a nonlinear fuzzy set is given in Fig. 1. Its universe of discourse is the set of all values  $x \in \mathbb{R}$ , the *support* of  $\tilde{A}$ ,  $supp(\tilde{A})$  is the interval  $x \in [\check{A}, \hat{A}]$  and  $\mu_{\tilde{A}}$  is a triangular function with parameters  $\check{A}$ ,  $\hat{A}$  and  $\hat{A}$ .  $\alpha$  is the degree of membership that an specific value  $x$  has regarding  $\tilde{A}$  and the dashed region is an  $\alpha$ -cut done over  $\tilde{A}$ .

Note that any  $\alpha$ -cut done over a fuzzy number is monotonically increasing/decreasing, so for  $\alpha_1 < \alpha_2, \alpha \in [0, 1]$  then  ${}^{\alpha_2}\tilde{A} \subseteq {}^{\alpha_1}\tilde{A}$  and  ${}^\alpha\tilde{A} \subseteq supp(\tilde{A}), \forall \alpha \in [0, 1]$ .

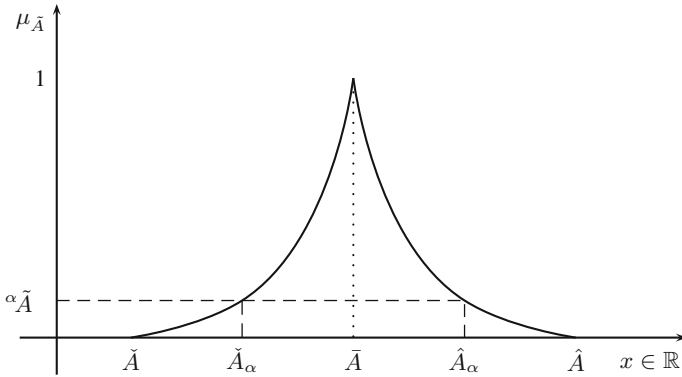


Fig. 1. Fuzzy set  $\tilde{A}$

### 3 Linear Programming with Fuzzy Parameters

The classical Linear Programming (LP) problem relates a set of  $Ax \leq b$  inequalities to a desired goal  $z = c'x$  for which we want to find a maxima of  $z$  through a set of decision variables  $x$ , this is  $\text{Max}\{z = c'x : Ax \leq b, x \geq 0\}$ , for short. In this model, all parameters  $c \in \mathbb{R}_n^+, A \in \mathbb{R}_{mn}^+, b \in \mathbb{R}_m^+$  are deterministic (e.g. constants).

Fuzzy LPs regard to a problem where its parameters cannot be defined as constants or singletons but as fuzzy sets which come from human like uncertainty. Most of available methods for FLPs are based on the fuzzy decision making principle (see Bellman and Zadeh [8]) and use linear membership functions and/or symmetrical shapes. A mathematical representation of an FLP is given as follows:

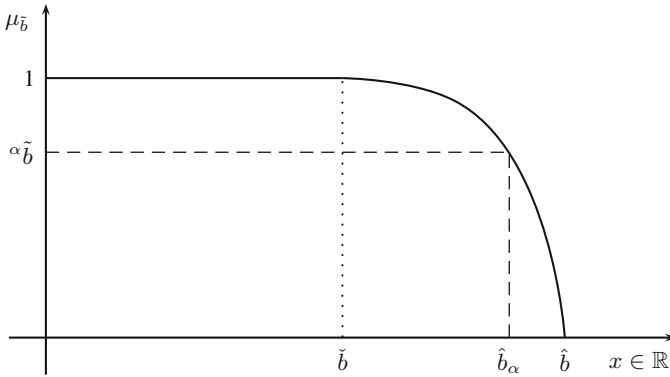
$$\begin{aligned}
 &\text{Max}_x \tilde{z} = \tilde{c}'x \\
 &\quad \text{s.t.} \\
 &\quad \tilde{A}x \lesssim \tilde{b} \\
 &\quad x \geq 0
 \end{aligned} \tag{6}$$

where  $\tilde{c} \in \mathcal{F}(\mathbb{R}), \tilde{A} \in \mathcal{F}(\mathbb{R}),$  and  $\tilde{b} \in \mathcal{F}(\mathbb{R}).$

The binary relation  $\lesssim$  for classical fuzzy sets has been proposed and investigated by Ramík and Řimánek [9], and the binary relation (fuzzy max order)  $\lesssim$  has been extended to Interval Type-2 fuzzy numbers by Figueroa-García et al. [10]. In this paper we analyze solutions for FLPs aside from the shapes of  $\tilde{c}, \tilde{A}$  and  $\tilde{b}.$

#### 3.1 Weak and Strong Solutions for FLPs

A fully solvable system  $\tilde{A}x \lesssim \tilde{b}$  implies that the fuzzy max order relation  $\lesssim$  holds for all  ${}^\alpha\tilde{A}x \leq {}^\alpha\tilde{b},$  which is a strong supposition since there is no any guarantee



**Fig. 2.** Fuzzy constraint  $\tilde{b}$

that the system is feasible for some  $\alpha \in [0, 1]$ , so we can say that the system  $\tilde{A}x \lesssim \tilde{b}$  is  $\alpha$ -feasible if there exists an  $\alpha \in [0, 1]$  for which the system  ${}^\alpha\tilde{A}x \leq {}^\alpha\tilde{b}$  is feasible. Finding a feasible  $\alpha$  could be a hard task, so we propose a way to find a feasible solution for an FLP.

Now, the system  $\tilde{A}x \lesssim \tilde{b}$  needs to be defined before solving (6). To do so, we use concepts of *weak/strong feasibility* for interval equations (see Fiedler et al. [3], Černý and Hladík [4], and Hladík [5]). In this paper, we only refer to weak/strong feasibility of fuzzy equations since the vector  $x$  in (6) is defined as non-negative, this is  $x \in \mathbb{R}_n^+$ , so hereinafter we refer to  $x$  as the set of non-negative solutions  $x \in \mathbb{R}_n^+$ .

**Definition 1.** Let  $\tilde{A}$  be a fuzzy matrix, and  $\tilde{b}$  a fuzzy vector,  $\{\tilde{A}, \tilde{b}\} \in \mathcal{F}(\mathbb{R})$ . Then the system  $\tilde{A}x \lesssim \tilde{b}$  is said to be **weak  $\alpha$ -feasible** if  $\exists x (\tilde{A}_\alpha x \leq \hat{b}_\alpha)$  for  $\alpha \in [0, 1]$ .

This means that given a value  $\alpha \in [0, 1]$ , a crisp coefficient matrix  $A_x \in [\check{A}_\alpha, \hat{A}_\alpha]$ , and  ${}^\alpha\tilde{b} \in [0, \hat{b}_\alpha]$  for which  $\check{A}_\alpha x \leq A_x x \leq \hat{A}_\alpha x$ , so if  $\exists x (A_x x \leq \hat{b}_\alpha)$  then

$$0 \leq \check{A}_\alpha x \leq A_x x \leq \hat{A}_\alpha x \leq \hat{b}_\alpha$$

and the binary order  $\check{A}_\alpha x \leq \hat{b}_\alpha$  holds. This also implies that all possible values of  $A \in [\check{A}_\alpha, \hat{A}_\alpha]$  satisfies  $Ax \leq \hat{b}_\alpha$  and  $x$  is said to be a weak solution of  $\check{A}_\alpha x \leq \hat{b}_\alpha$  since  $x$  only solves the system for  $\check{A}_\alpha, \hat{b}_\alpha$ .

**Definition 2.** Let  $A_x \in [\check{A}_\alpha, \hat{A}_\alpha]$  be a crisp coefficient matrix. An FLP is said to be  **$\alpha$ -infeasible** if  $\nexists x (\hat{A}_\alpha x \leq \hat{b}_\alpha)$ . This also implies that  $\nexists x (A_x x \leq \hat{b}_\alpha)$ .

**Definition 3.** Let  $\tilde{A}$  be a fuzzy matrix, and  $\tilde{b}$  a fuzzy vector,  $\{\tilde{A}, \tilde{b}\} \in \mathcal{F}(\mathbb{R})$ . Then the system  $\tilde{A}x \lesssim \tilde{b}$  is said to be **strong  $\alpha$ -feasible** if only if  $\exists x (\hat{A}_\alpha x \leq \check{b}_\alpha)$  for  $\alpha \in [0, 1]$ .

Definition 3 means that if  $\exists x(A_x x \leq \check{b}_\alpha)$  given  $\alpha \in [0, 1]$  and  $\check{A}_\alpha x \leq A_x x \leq \hat{A}_\alpha x$ , then we have that

$$\check{A}_\alpha x \leq A_x x \leq \hat{A}_\alpha x \leq \check{b}_\alpha$$

implies that the only solution  $x'$  for  $A_x x' \leq \check{b}_\alpha \forall A_x \in [\check{A}_\alpha, \hat{A}_\alpha]$  is  $\hat{A}_\alpha x'$ , and  $x'$  is then a strong solution for  $[\check{A}_\alpha, \hat{A}_\alpha]x' \leq \check{b}_\alpha$  since it solves  $A_x x' \leq \check{b}_\alpha \forall A_x \in [\check{A}_\alpha, \hat{A}_\alpha]$ .

### 3.2 Compact, Unbounded Solutions

Nonlinear gaussian, exponential, quadratic membership functions, etc. are among the most popular membership functions in decision making. This kind of membership functions have unbounded support, so they cannot provide a global solution of the problem. Thus, if  $\check{A}$  has unbounded support, this is  $\text{supp}(\check{A}) \in [-\infty, \infty]$ , then the system  $\check{A}x \leq \check{b}$  is untractable.

Feasibility of an FLP is constrained by the condition that  $\text{supp}(\check{A})$  and  $\text{supp}(\check{b})$  must be compact sets for which  $\inf\{\text{supp}(\check{A})\} = \check{A}$ ,  $\sup\{\text{supp}(\check{A})\} = \hat{A}$ ,  $\inf\{\text{supp}(\check{b})\} = 0$ ,  $\sup\{\text{supp}(\check{b})\} = \hat{b}$ , so  $\text{supp}(\check{A}) = [\check{A}, \hat{A}]$  and  $\text{supp}(\check{b}) = [0, \hat{b}]$  should be compact (see Fig. 2). This leads us to the following results.

**Definition 4.** Let  $\check{A}$  be a matrix of fuzzy sets and  $\check{b}$  a vector of fuzzy sets,  $\{\check{A}, \check{b}\} \in \mathcal{F}(\mathbb{R})$ . Then the linear system  $\check{A}x \lesssim \check{b}$  is said to be **compact** if and only if both  $\text{supp}(\check{A})$  and  $\text{supp}(\check{b})$  are compact sets.

**Definition 5.** Let  $\check{A}x \lesssim \check{b}$  be compact. It is said that  $\check{A}x \lesssim \check{b}$  is **weak feasible** if  $\exists x(\check{A}x \leq \hat{b})$ . Otherwise if  $\nexists x(\check{A}x \leq \hat{b})$ , it is said that  $\check{A}x \lesssim \check{b}$  is **infeasible**.

**Definition 6.** Let  $\check{A}x \lesssim \check{b}$  be compact. It is said that  $\check{A}x \lesssim \check{b}$  is **strong feasible** if  $\exists x(\check{A}x \leq \check{b})$ .

Note that Eq. (6) is fuzzy max ordered (see Ramík and Řimánek [9]) only for fully solvable FLPs, so  $\sum_j \check{A}_{ij} x_j \lesssim \check{b}_i, \forall i \in \mathbb{N}_m$  holds for every  $\alpha$ -cut. Then, Definitions 5 and 6 imply that the system  $\sum_j \alpha \check{A}_{ij} x_j \leq \alpha \check{b}_i, \forall i \in \mathbb{N}_m$  should be feasible at every boundary of  $\alpha \check{A}_{ij}$  and  $\alpha \check{b}_i$ , and this condition is ensured unless the problem is not feasible.

A more generalized concept about feasibility over FLPs comes from the idea that  $\text{Max}\{z = \check{c}x : \check{A}x \leq \check{b}, x \geq 0\}$  is feasible if there exists a combination of parameters into the supports of  $\check{A}, \check{b}$  that conforms a feasible solution, as shown as follows.

**Definition 7.** Let  $\check{A}$  be a fuzzy matrix,  $\check{b}$  be a fuzzy vector,  $\{\check{A}_{ij}, \check{b}_i\} \in \mathcal{F}(\mathbb{R})$ ,  $A_x$  be a crisp matrix and  $b_x$  be a crisp vector such that  $A_x \in \text{supp}(\check{A}), b_x \in \text{supp}(\check{b})$ . Then the system  $\check{A}x \lesssim \check{b}$  is said to be **feasible** if  $\exists x\{A_x x \leq b_x\}$ .

Feasibility in FLPs can be seen from a crisp point of view based on  $\alpha$ -cuts. Most of commercial optimizers such as CPLEX, AMLP, Gurobi, MATLAB, Xpress, etc. provide efficient routines to check feasible LPs (mostly based on

duality conditions) that can be applied to check feasibility of FLPs. What we recommend to readers is to check for weak feasibility before solving the entire problem.

Another important case to cover is unboundedness. It is clear that compact systems lead to bounded solutions as stated in Definition 4 unless the problem is infeasible, so unbounded FLP come from two sources: unbounded fuzzy parameters, or negative column vectors in the system  $A \leq b$ . Both conditions are considered in the following definitions.

**Definition 8.** Let  $\tilde{A}$  be a matrix of fuzzy sets and  $\tilde{b}$  a vector of fuzzy sets,  $\{\tilde{A}, \tilde{b}\} \in \mathcal{F}(\mathbb{R})$ . Then the linear system  $\tilde{A}x \lesssim \tilde{b}$  is said to be **unbounded** if one of the following conditions are satisfied:

- (i)  $\exists j(\text{supp}(\tilde{A}_{.j}) \in [-\infty, \infty])$ ,
- (ii)  $\exists j(\tilde{A}_{\alpha} \in \mathbb{R}^- \forall i \in \mathbb{N}_m)$ ,
- (iii)  $\text{supp}(\tilde{b}_i) \in [-\infty, \infty] \forall i \in \mathbb{N}_m$ .

In Definition 8, (i) means that an FLP is unbounded if  $j_{th}$  column vector is composed by unbounded sets, this is  $\text{supp}(\tilde{A}_{.j}) \in [-\infty, \infty]$ ; (ii) means that an FLP is unbounded if the  $j_{th}$  column vector is composed by bounded sets whose supports contain non-negative elements, this is  $\inf\{\text{supp}(\tilde{A}_{.j})\} = \tilde{A}_{\alpha} \in \mathbb{R}^-$ ; and (iii) means that if the constraints of an FLP are unbounded fuzzy sets, then the FLP is unbounded as well.

Other unboundedness conditions derived from (iii) can be defined for particular values  $A_x \in \text{supp}(\tilde{A})$  and  $b_x \in \text{supp}(\tilde{b})$ . The linear system  $A_x x \leq b_x$  is said to be **unbounded** if:

$$\exists y(A'_x y \geq 0, b'_x y < 0, y \geq 0).$$

This means that a convex combination of the dual variables  $y$  can obtain a negative column of  $A$  in its primal problem (see Farkas and Clark’s lemmas) leading to a unbounded primal LP. Also the linear system  $A_x x \leq b_x$  is unbounded if a convex combination of the columns of  $A_x$  leads to a negative value. This is:

$$\exists \lambda \left( \lambda_1 a_{i1} + \dots + \lambda_j a_{ij} + \dots + \lambda_n a_{in} \leq 0, \lambda_j \geq 0, \sum_{j \in \mathbb{N}_n} \lambda_j = 1 \right) \forall i \in \mathbb{N}_m,$$

where  $\lambda = \{\lambda_1, \dots, \lambda_j, \dots, \lambda_n\}$ ,  $\lambda_j \in \mathcal{P}(\mathbb{R})$ , and  $a_{ij}$  is the  $i, j$  element of  $A_x$ .

Fiedler et al. [3], Černý and Hladík [4], and Hladík [5] have proposed some algorithms to find solutions to unbounded problems. Although they proposed methods based on interval-valued LPs, their results can be applied without any restriction to FLPs since a fuzzy number can be decomposed into  $\alpha$ -cuts (a.k.a horizontal slices of  $\tilde{A}$ ).

### 4 Application Examples

**FLP with Weak and Strong Solutions.** Consider the following FLP:

$$\begin{aligned}
 \text{Max}_x \quad & z = \tilde{2}x_1 + \tilde{3}x_2 + \tilde{4}x_3 \\
 \text{s.t.} \quad & \\
 & \tilde{1}x_1 + \tilde{4}x_2 + \tilde{2}x_3 \lesssim \tilde{10} \\
 & \tilde{3}x_1 + \tilde{2}x_2 + \tilde{5}x_3 \lesssim \tilde{12} \\
 & \tilde{3}x_1 + \tilde{3}x_2 + \tilde{4}x_3 \lesssim \tilde{15} \\
 & x \geq 0
 \end{aligned}$$

The complete description of  $\tilde{A}$  is shown next:

$$\begin{aligned}
 \tilde{c}_1 &= T(1, 2, 5) & \tilde{c}_2 &= T(2, 3, 5) & \tilde{c}_3 &= T(2, 4, 7) \\
 \tilde{A}_{11} &= T(0, 1, 3) & \tilde{A}_{12} &= T(2, 4, 7) & \tilde{A}_{13} &= T(1, 2, 4) \\
 \tilde{A}_{21} &= T(1, 3, 5) & \tilde{A}_{22} &= T(0, 2, 5) & \tilde{A}_{23} &= T(2, 5, 7) \\
 \tilde{A}_{31} &= T(1, 3, 6) & \tilde{A}_{32} &= T(1, 3, 7) & \tilde{A}_{33} &= T(2, 4, 7) \\
 \tilde{b}_1 &= T_1(0, 10, 12) & \tilde{b}_2 &= T_1(0, 12, 15) & \tilde{b}_3 &= T_1(0, 15, 18)
 \end{aligned}$$

where  $T(a, b, c)$  denotes a triangular membership function, and  $T_1(a, b, c)$  denotes a linear semi-trapezoidal membership function.

To check strong feasibility (see Definition 6) we solve the following LP:

$$\begin{aligned}
 \text{Max}_x \quad & z = x_1 + 2x_2 + 2x_3 \\
 \text{s.t.} \quad & \\
 & 3x_1 + 7x_2 + 4x_3 \leq 10 \\
 & 5x_1 + 5x_2 + 7x_3 \leq 12 \\
 & 6x_1 + 7x_2 + 7x_3 \leq 15 \\
 & x \geq 0
 \end{aligned}$$

This problem has an optimal solution at  $x_1 = 0, x_2 = 0.7586, x_3 = 1.1724$  that reaches  $z = 3.8620$ . Note that this solution is called strong because it solves all possible combinations of  $\text{supp}(\tilde{A})$  and  $\text{supp}(\tilde{b})$  (please do the calculus).

To check for weak feasibility (see Definition 5) we solve the following LP:

$$\begin{aligned}
 \text{Max}_x \quad & z = 5x_1 + 5x_2 + 7x_3 \\
 \text{s.t.} \quad & \\
 & 2x_2 + x_3 \leq 10 \\
 & x_1 + 2x_3 \leq 12 \\
 & x_1 + x_2 + 2x_3 \leq 18 \\
 & x \geq 0
 \end{aligned}$$

This problem has an optimal solution at  $x_1 = 12, x_2 = 6, x_3 = 0$  that reaches  $z = 90$ . This solution is called weak because it does not solve other possible combinations of  $\text{supp}(\tilde{A})$  and  $\text{supp}(\tilde{b})$ , it only solves the system  $\tilde{A}x \leq \hat{b}$ .

**FLP with Infeasible Solutions.** Consider the following FLP:

$$\begin{aligned} \text{Max}_x z &= \tilde{2}x_1 + \tilde{3}x_2 + \tilde{4}x_3 \\ \text{s.t.} \\ \tilde{1}x_1 + \tilde{4}x_2 + \tilde{2}x_3 &\lesssim \tilde{10} \\ \tilde{3}x_1 + \tilde{2}x_2 + \tilde{5}x_3 &\lesssim \tilde{12} \\ \tilde{3}x_1 + \tilde{3}x_2 + \tilde{4}x_3 &\gtrsim \tilde{15} \\ x &\geq 0 \end{aligned}$$

The complete description of  $\tilde{A}, \tilde{b}, \tilde{c}$  are shown next:

$$\begin{aligned} \tilde{c}_1 &= T(1, 2, 5) & \tilde{c}_2 &= T(2, 3, 5) & \tilde{c}_3 &= T(2, 4, 7) \\ \tilde{A}_{11} &= T(0, 1, 3) & \tilde{A}_{12} &= T(2, 4, 7) & \tilde{A}_{13} &= T(1, 2, 4) \\ \tilde{A}_{21} &= T(1, 3, 5) & \tilde{A}_{22} &= T(0, 2, 5) & \tilde{A}_{23} &= T(2, 5, 7) \\ \tilde{A}_{31} &= T(1, 3, 6) & \tilde{A}_{32} &= T(0, 3, 7) & \tilde{A}_{33} &= T(1, 4, 7) \\ \tilde{b}_1 &= T_1(0, 10, 12) & \tilde{b}_2 &= T_1(0, 12, 15) & \tilde{b}_3 &= T_1(15, 18, \infty) \end{aligned}$$

To check strong feasibility (see Definition 6) we solve the following LP:

$$\begin{aligned} \text{Max}_x z &= x_1 + 2x_2 + 2x_3 \\ \text{s.t.} \\ 3x_1 + 7x_2 + 4x_3 &\leq 10 \\ 5x_1 + 5x_2 + 7x_3 &\leq 12 \\ x_1 + x_3 &\geq 18 \\ x &\geq 0 \end{aligned}$$

In this case, the problem is infeasible. This leads us to check for weak feasibility (see Definition 5). To do so, we have to solve the following LP:

$$\begin{aligned} \text{Max}_x z &= 5x_1 + 5x_2 + 7x_3 \\ \text{s.t.} \\ 2x_2 + x_3 &\leq 10 \\ x_1 + 2x_3 &\leq 12 \\ 6x_1 + 7x_2 + 7x_3 &\geq 15 \\ x &\geq 0 \end{aligned}$$

This problem has an optimal solution at  $x_1 = 15, x_2 = 6, x_3 = 0$  that reaches  $z = 105$ . Remember that this solution is called weak because it does not solve other possible combinations of  $\text{supp}(\tilde{A})$  and  $\text{supp}(\tilde{b})$ , it only solves the system  $\tilde{A}x \leq \hat{b}$ .



**FLP with Unbounded Solutions.** Consider the following FLP:

$$\begin{aligned}
 \text{Max}_x \quad & z = \tilde{5}x_1 + \tilde{4}x_2 + \tilde{5}x_3 \\
 \text{s.t.} \quad & \\
 & \tilde{3}x_1 + \tilde{4}x_2 + \tilde{2}x_3 \lesssim \tilde{15} \\
 & \tilde{4}x_1 + \tilde{5}x_2 + \tilde{3}x_3 \lesssim \tilde{18} \\
 & \tilde{3}x_1 + \tilde{4}x_2 + \tilde{3}x_3 \lesssim \tilde{16} \\
 & x \geq 0
 \end{aligned}$$

The complete description of  $\tilde{A}$  is shown next:

$$\begin{aligned}
 \tilde{c}_1 &= E(5, 2) & \tilde{c}_2 &= E(4, 1) & \tilde{c}_3 &= E(5, 1.5) \\
 \tilde{A}_{11} &= G(3, 1) & \tilde{A}_{12} &= G(4, 1) & \tilde{A}_{13} &= G(2, 0.5) \\
 \tilde{A}_{21} &= G(4, 2) & \tilde{A}_{22} &= G(5, 2) & \tilde{A}_{23} &= G(3, 1) \\
 \tilde{A}_{31} &= G(3, 0.5) & \tilde{A}_{32} &= G(4, 2) & \tilde{A}_{33} &= G(3, 1.5) \\
 \tilde{b}_1 &= QE(0, 15, 2) & \tilde{b}_2 &= QE(0, 18, 3) & \tilde{b}_3 &= QE(0, 16, 2)
 \end{aligned}$$

where  $E(a, b, c)$  denotes an exponential membership function,  $G(a, b)$  denotes a Gaussian membership function, and  $QE(a, b, c)$  denotes a quasi-exponential membership function, as shown as follows:

$$\begin{aligned}
 E(a, b) &= \begin{cases} \exp^{-\frac{(x-a)}{b}} & \text{for } x < a, \\ \exp^{-\frac{(a-x)}{b}} & \text{for } x > a, \end{cases} \\
 G(a, b) &= \exp^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} \quad \forall x \in (-\infty, \infty), \\
 QE(0, a, b) &= \begin{cases} 1 & \text{for } x < a, \\ \exp^{-\frac{(a-x)}{b}} & \text{for } x > a. \end{cases}
 \end{aligned}$$

Since all fuzzy sets are unbounded, it is clear that the problem is unbounded (see condition (i) in Definition 8). So we will check for  $\alpha$ -feasibility. To check strong  $\alpha$ -feasibility (see Definition 3) we select  $\alpha = 0.5$  to solve the following LP:

$$\begin{aligned}
 \text{Max}_x \quad & z = 6.38x_1 + 4.69x_2 + 6.03x_3 \\
 \text{s.t.} \quad & \\
 & 4.17x_1 + 5.17x_2 + 2.58x_3 \leq 16.38 \\
 & 6.35x_1 + 7.35x_2 + 4.17x_3 \leq 20.07 \\
 & 3.58x_1 + 6.35x_2 + 4.76x_3 \leq 17.38 \\
 & x \geq 0
 \end{aligned}$$

This problem has an optimal solution at  $x_1 = 1.5083, x_2 = 0, x_3 = 2.5122$  that reaches  $z = 24.80$ . Note that this solution is called  $\alpha$ -strong because it solves all possible combinations of  ${}^\alpha\tilde{A}$  and  ${}^\alpha\tilde{b}$  (please do the calculus). At this point, no need for checking weak feasibility of this problem since it has at least an  $\alpha$ -strong solution for  $\alpha = 0.5$ .

## 5 Concluding Remarks

We analyzed some necessary conditions to ensure feasibility of an FLP, using the supports of all fuzzy sets involved in the problem, or at least  $\alpha$ -feasibility. Note that our results apply to any kind of membership functions, and we have generalized some important results known for interval-valued optimization problems.

Some FLPs can use unbounded fuzzy sets, but it does not mean that the problem is always unbounded. In fact, it can be bounded for a given  $\alpha$  level, as shown in the examples. The proposed results also show that there is a chance of having elements  $A_x \in \text{supp}(\tilde{A})$  and  $b_x \in \text{supp}(\tilde{b})$  that could lead to infeasible solutions, but other elements can lead to feasible solutions.

We recommend to check weak feasibility of an FLP before finding any other kind of solutions of the problem. If a robust solution is needed, then a strong feasible solution will solve any combination of  $A_x$  and  $b_x$ .

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