Periodic Solution of Linear Autonomous Dynamic System

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Abstract. A method for the study of periodic solutions of autonomous dynamic systems described by ordinary differential equations with phase and integral constraints is supposed. General problem of periodic solution is formulated in the form of the boundary value problem with constraints. The boundary problem is reduced to the controllability problem of dynamic systems with phase and integral constraints by introducing a fictitious control. Solution of the controllability problem is reduced to a Fredholm integral equation of the first kind. The necessary and sufficient conditions for existence of the periodic solution are obtained and an algorithm for constructing periodic solution to the limit points of minimizing sequences is developed. Scientific novelty of the results consists in a completely new approach to the study of periodic solutions for linear systems focused on the use of modern information technologies is offered. The existence of periodic solution and its construction are solved together.

Keywords: Linear autonomous system · Dynamic system · Periodic solution · Ordinary differential equation · Boundary value problem · Controllability problem · Controllable system

1 Problem Statement

We consider a linear autonomous system

$$\dot{x} = Ax, t \in (-\infty, +\infty), \tag{1}$$

where A is a constant matrix of $n \times n$ order. The problems are set:

Problem 1. Find necessary and sufficiently conditions for existence of T_* periodic solution of system (1).

Problem 2. Find T_* periodic solution of system (1)
Solving these problems is of interest for system (1) of $(n > 4)$ higher order.

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We assume, that system (1) has a periodic solution $x_*(t) = x_*(t+T_)$, $t \in I = (-\infty, +\infty)$, where T_* is period. Let $x_*(0) = x_0$ be a value. Then $x_*(T_*) = x_0$.

Since the periodic solution is defined by values of the phase coordinates in the period limits, then for constructing of periodic solution it should be considered the value $t \in I_* = [0, T_*]$.

We represent the matrix $A = A_1 + B_1P$, where A_1, B, P are matrixes of $n \times n$, $n \times m$, $m \times n$ orders, accordingly. Then the boundary value problem (1) is written in the form

$$\dot{x} = A_1x + B_1Px, t \in I_* = [0, T_*], x(0) = x(T_*) = x_0. \tag{2}$$

Linear controllable system corresponding to system (2) has the form (2)

$$\dot{y} = A_1y + B_1u(t), t \in I_* = [0, T_*], \tag{3}$$

$$y(0) = y(T_*) = x(0) = x(T_*) = x_0, u(\cdot) \in L_2(I, R^m), \tag{4}$$

where T_* is period, a unknown value. We note, if $u(t) = Px(t)$, $t \in I_*$, then system (3), (4) coincides to the origin (2).

2 Solution of a Linear Controllable System

We assume that the matrixes A_1, B_1 such that the matrix

$$W_*(0, T_*) = \int_0^{T_*} e^{-A_1t}B_1B_1^*e^{-A_1^*t}dt \tag{5}$$

of $n \times n$ order is positively defined.

In the case, when the matrix $A_1 = 0$, $P = I_n$, the matrix $B_1 = A$, relation (5) is written as $W_*(0, T_*) = \int_0^{T_*} AA^*dt$. We note, that the matrix $W_*(0, T_*) > 0$ is equivalent to the fact, that the rank of the matrix $\|B_1, A_1B_1, \ldots, A_1^{n-1}B_1\|$ is equal to n.

Theorem 1. *Let $W_*(0, T_*) > 0$ be a matrix. Then control $u(\cdot) \in L_2(I, R^m)$ transfers the trajectory of system (3) from any initial point $y(0) = x_0 \in R^n$ to any finite state $y(T_*) = x_0$ if and only if, when*

$$\begin{gathered} u(t) \in U = \{u(\cdot) \in L_2(I, R^m)/u(t) = v(t) + \lambda_1(t, x_0, x_0) + \\ +N_1(t)z(T_*, v), \forall v, v(\cdot) \in L_2(I, R^m)\}, \end{gathered} \tag{6}$$

where

$$\lambda_1(t_1, x_0, x_0) = B_1^*e^{-A_1^*t}W_*^{-1}(0, T_*)a, a = e^{-A_1T_*}x_0 - x_0,$$

$$N_1(t) = -B_1^*e^{-A_1^*t}W_*^{-1}(0, T_*)e^{-A_1T_*}, t \in I_*,$$

the function $z(t, v_*)$, $t \in I_*$ is a solution of the differential equation

$$\dot{z} = A_1z + B_1v, z(0) = 0, v(\cdot) \in L_2(I, R^m). \tag{7}$$

The solution of the differential equation (3) corresponding to control $u(t) \in U$ is defined by formula

$$y(t) = z(t,v) + \lambda_2(t,x_0,x_0) + N_2(t)z(T_*,v), t \in I_*, \tag{8}$$

where

$$\lambda_2(t,x_0,x_1) = e^{A_1 t}W_*(t,T_*)W_*^{-1}(0,T_*)x_0 + e^{A_1 t}W_*(0,t)W_*^{-1}(0,T_*)e^{-A_1 T_*}x_0,$$

$$N_2(t) = -e^{A_1 t}W_*(0,t)W_*^{-1}(0,T_*)e^{-A_1 T_*}, W_*(0,T_*) = \int_0^t e^{-A_1 \tau}B_1 B_1^* e^{-A_1^* \tau},$$

$$W_*(t,T_*) = W_*(0,T_*) - W_*(0,t), t \in I_*.$$

Lemma 1. *Let $W_*(0,T_*) > 0$ be a matrix. The boundary value problem (2) is equivalent to the problem*

$$v(t) + T(t)x_0 + N_1(t)z(T_*,v) = Py(t), t \in I_*, x_0 \in R^n, \tag{9}$$

$$\dot{z} = A_1 z + B_1 v(t), z(0) = 0, t \in I_*, v(\cdot) \in L_2(I,R^m), \tag{10}$$

where

$$T(t) = B_1 e^{-A_1^* t}W_*^{-1}(0,T_*)[e^{-A_1 T_*} - I_n],$$

$$y(t) = z(t,v) + C(t)x_0 + N_2(t)z(T_*,v), t \in I, \tag{11}$$

$$C(t) = e^{A_1 t}[W_*(t,T_*)W_*^{-1}(0,T_*) + W_*(0,t)W_*^{-1}(0,T_*)e^{-A_1 T_*}].$$

Proof of the Lemma follows from relations (6)-(10), at $u(t) \in U$, $u(t) = Py(t)$, $t \in I_*$.

3 Necessary and Sufficient Condition for Existence of a Solution of the Boundary Value Problem

Theorem 2. *Let $W_*(0,T_*) > 0$ be a matrix. In order the boundary value problem (2) to have a solution, it is necessary and sufficient that the value $I(v_*,x_{0_*}) = 0$, where $(v_*,x_{0_*}) \in H = L_2(I,R^m) \times R^n$ is a solution of optimization problem*

$$I(v,x_0) = \int_0^{T_*} |v(t) + T(t)x_0 + N_1(t)z(T_*,v) - Py(t)| \to \inf \tag{12}$$

under conditions

$$\dot{z} = A_1 z + B_1 v(t), z(0) = 0, t \in I_*, \tag{13}$$

$$v(\cdot) \in L_2(I_*,R^m), x_0 \in R^n. \tag{14}$$

Proof of the Theorem follows from Theorem 1, Lemma 1 and relations (9)-(11).

Lemma 2. *Suppose $W_*(0,T_*) > 0$ is a matrix, the function*

$$F_*(q,t) = v + T(t)x_0 + N_1(t)z(T_*,v) - Py,$$

where $y = z + C(t)x_0 + N_2(t)z(T_*,v)$, $q = (v,x_0,z,z(T_*)) \in R^m \times R^n \times R^n \times R^n$.

Then the partial derivatives

$$\begin{gathered} F_{*v}(q,t) = 2[v + T(t)x_0 + N_1(t)z(T_*,v) - Py], \\ F_{*x_0}(q,t) = [2T^*(t) + 2C^*(t)P^*][v + T(t)x_0 + N_1(t)z(T_*) - Py], \\ F_{*z}(q,t) = -2P^*(t)[v + T(t)x_0 + N_1(t)z(T_*) - Py], \\ F_{*z(T_*)}(q,t) = [2N_1^*(t) - 2N_2^*(t)P^*][v + T(t)x_0 + N_1(t)z(T_*) - Py]. \end{gathered} \tag{15}$$

Lemma 3. *Let $W_*(0,T_*) > 0$ be a matrix. Then:*
1) functional (12) under conditions (13), (14) is convex
2) derivative $F_{*q}(q,t) = (F_{*v}, F_{*x_0}, F_{*z}, F_{*z(T_*)})$ satisfies to the Lipshitz condition

$$\|F_{*q}(q+\Delta q,t) - F_{*q}(q,t)\| \le M\|\Delta q\|, \forall q, q+\Delta q \in R^{m+4n}.$$

Theorem 3. *Let $W_*(0,T_*) > 0$ be a matrix. Then functional (12), under conditions (13), (14) continuously differentiable by Freshet, gradient of functional*

$$I'(v,x_0) = (I'_v(v,x_0), I'_{x_0}(v,x_0)) \in H = L_2(I_*,R^m) \times R^n$$

in any point $(v,x_0) \in H$ is computed by the formula

$$\begin{gathered} I'_v(v,x_0) = F_{*v}(q(t),t) - B_1^*\psi(t) \in L_2(I_*,R^m), \\ I'_{x_0}(v,x_0) = \int_0^{T_*} F_{*x_0}(q(t),t)dt \in R^n, \end{gathered} \tag{16}$$

where partial derivatives are defined by formula (15), $q(t) = (v(t),x_0,z(t,v), z(T_*,v))$, the function $z(t)$, $t \in I$ is a solution of the differential equation (12), for $v = v(t)$, $t \in I$, and function $\psi(t)$, $t \in I_*$ is a solution of the adjoint system

$$\dot{\psi} = F_{*z}(q(t),t) - A_1^*\psi, \psi(t_1) = -\int_0^{T_*} F_{*z(T_*)}(q(t),t)dt. \tag{17}$$

Moreover, the gradient $I'(v,x_0)$, $(v,x_0) \in H$ satisfies the Lipshitz condition

$$\left\|I'(v^1,x_0^1) - I'(v^2,x_0^2)\right\| \le K_*(\left\|v^1 - v^2\right\|^2 + \left|x_0^1 - x_0^2\right|^2)^{1/2}, \tag{18}$$

where $K_* = const > 0$ is a Lipshitz constant.

It should be noted, that for a linear system with constant coefficients (3) the following statements are valid:
1) rank of the matrix $\left\|B_1, A_1B_1, \ldots, A_1^{n-1}B_1\right\|$ is equal to n;
2) for any $T > 0$, the matrix

$$W_*(0,T) = \int_0^T e^{-A_1t}B_1B_1^*e^{-A_1^*t}dt$$

is positively defined.

Consequently, for any sequence $\{T_i\} \subset R^1$, $0 < T_1 < T_2 < \ldots < T_k < \ldots$, the matrix $W_*(0, T_k) > 0$.

Let $\{T_k\} \subset R^1$, $0 < T_1 < T_2 < \ldots < T_k <$ be a sequence. We construct the sequences

$$\begin{array}{c} v^k_{n+1}(t) = v^k_n(t) - \alpha_n I'_v(v^k_n, x^k_{0n}), \;\; x^k_{0_{n+1}}(t) = x^k_0(t) - \alpha_n I'_{x_0}(v^k_n, x^k_{0_n}), \\ n = 0, 1, 2, \ldots \;\; 0 < \varepsilon_0 \le \alpha_n \le \frac{2}{K_* + 2\varepsilon_1}, \;\; \varepsilon_1 > 0, \end{array} \tag{19}$$

on the base of formulas (6)-(8), where $t \in [0, T_k]$, $I'_v(v^k, x^k_0)$,$I'_{x_0}(v^k, x^k_0)$ are defined by formula (16) by substituting $W_*(0, T_*)$, T_* on $W_*(0, T_k)$, T_k, accordingly.

In other words, we fix a value $T_k > 0$ from sequence $\{T_i\} \subset R^1, 0 < T_1 < T_2 < \ldots < T_k < \ldots$ and compute the Freshet derivative for functional (12), under the conditions (12), (13) by formulas (16)-(18), by substituting T_*, $W_*(0, T_*)$ on T_k, $W_*(0, T_k)$, accordingly. The result is the sequences (19).

Theorem 4. *Let $W_*(0, T_k) > 0$ be a matrix, $\{T_k\} \subset R^1$, $0 < T_1 < T_2 < \ldots < T_k < \ldots$, the sequences $\{v^k_n\}$, $\{x^k_{0n}\}$ are defined by formula (19), the set*

$$\Lambda_k = \{(v, x_0) \in H / I_k(v, x_0) \le I_k(v_0, x_{00})\}$$

is bounded, where functional is defined by

$$I_k(v, x_0) = \int_0^{T_k} |v(t) + T(t)x_0 + N_1(t)z(T_k, v) - Py(t)| dt.$$

Then for any fixed $T_k > 0$ statements are valid:
3) The sequence $\{v^k_n, x^{kn}_0\}$ is minimizing, i.e.

$$\lim_{n\to\infty} I_k(v^k_n, x^{kn}_0) = I_k(v^k_*, x^{k*}_0) = \inf_{(v,x_0)\in \Lambda_k} I_k(v, x_0);$$

4) The sequences $\{v^k_n\}$, $\{x^{kn}_0\}$ are weakly converged to the points $v^k_n \xrightarrow{A;} v^k_*$, $x^k_{0n} \xrightarrow{A;} x^k_{0*}$ at $n \to \infty$, $(v^k_*, x^{k*}_0) \in X^*_k$;
5) The estimation of the convergence rate is valid

$$0 < I_k(v^k_n, x^k_{0n}) - I(v^k_*, x^k_{0*}) \le \tfrac{C_k}{n}, c_k = const > 0, n = 1, 2, \ldots;$$

6) For system (2) to have a periodic solution it is necessary and sufficient, that for some $T_k = T_*$ there exists the value $I_k(v^k_*, x^k_{0*}) = 0$.
7) Periodic solution of system (12) is defined by the formula

$$x_*(t) = y_*(t) = z(t, v^k_*) + C(t)x^k_{0*} + N_2(t)z(T_k, v^k_*), t \in [0, T_k = T_*],$$

where $T_k = T_*$ is a period, $I_k(v^k_*, x^k_{0*}) = 0$.

4 Algorithm for Constructing a Periodic Solution

We can formulate the following algorithm for constructing periodic solution of system (1) based on Theorems 1-4, Lemmas 1-3.

1. We present the matrix A as the sum $A = A_1 + B_1 P$ such that the matrix

$$W_*(0, T_k) = \int_0^{T_k} e^{-A_1 t} B_1 B_1^* e^{-A_1^* t} dt$$

   will be positively defined, where $T_k > 0$ is a number. We note, that in order to $W_*(0, T_k) > 0$ necessary and sufficiently, that the rank of the matrix $\|B_1, A_1 B_1, \ldots, A_1^{n-1} B_1\|$ is equal to n.
2. We choose the sequence $\{T_k\} \subset R^1, 0 < T_1 < T_2 < \ldots < T_k < \ldots$. We note, if the rank $\|B_1, A_1 B_1, \ldots, A_1^{n-1} B_1\| = n$, then for any $T_k > 0$ the matrix $W_*(0, T_k) > 0$.
3. We solve the optimization problem: minimize the functional

$$I_k(v, x_0) = \int_0^{T_k} |v(t) + T(t)x_0 + N_1(t)z(T_k, v) - Py(t)|^2 \, dt \to \inf \qquad (20)$$

   under conditions

$$\dot{z} = A_1 z + B_1 v(t), z(0) = 0, t \in [0, T_k] = I, \qquad (21)$$

$$v(\cdot) \in L_2(I, R^m), x_0 \in R^n. \qquad (22)$$

   We note, that: 1) the value $I_k(v, x_0) \geq 0$, consequently, functional is bounded from below; 2) functional (20) under conditions (21), (22) is convex; 3) to solve optimization problem (20) – (22) we construct the sequences (19). As a result, we find the solution of optimization problem (20)-(22): $(v_*^k, x_{0*}^k) \in \Lambda_k$, $I_k(v_*^k, x_{0*}^k)$ at fixed T_k.
4. We repeat items 1 - 3. Finally, the values $I_k(v_*^k, x_{0*}^k)$, $k = 1, 2, \ldots$ are known. If for value T_{k_*} the value $I_{k_*}(v_*^{k_*}, x_{0*}^{k_*}) = 0$, then $T_{k_*} = T_*$ is a period of the origin periodic solution, and periodic solution

$$x_*(t) = z(t, v_*^{k_*}) + C(t)x_{0*}^{k_*} + N_2(t)z(T_{k_*}, v_*^{k_*}), t \in [0, T_{k_*}] = [0, T_*].$$

5. If the value $I_k(v_*^k, x_{0*}^k) > 0$ for any sequences $\{T_k\} \subset R^1$,$0 < T_1 << T_2 < \ldots < T_k < \ldots$, then the origin system (2) has no any periodic solution.

The results obtained above can be applied for construction of periodic solutions in non-autonomous systems.

We consider a linear non-autonomous system

$$\dot{x} = A(t)x + \mu(t), t \in (-\infty, +\infty), \qquad (23)$$

where elements of the matrix $A(t)$ and vector function $\mu(t)$ are periodic functions with period T_* i.e. $A(t) = A(t + T_*)$, $\mu(t) = \mu(t + T_*)$, $\forall t$, $t \in (-\infty, +\infty)$, T_* is the known function.

The questions arise: Does the system (23) have periodic solution with a period equal to T_*? Find periodic solution of (23) with a period T_*.

Let $x_*(t)$ be a periodic solution of system (23) with a period T_* i.e. $x_*(t) = x(t+T_*)$, $\forall t$, $t \in (-\infty, +\infty)$. Then

$$A(t)x_*(t) = A(t+T_*)x_*(t+T_*), \mu(t) = \mu(t+T_*), t \in (-\infty, +\infty).$$

For constructing a periodic solution it is enough to consider a solution of system (23) for values $t \in [0, T_*]$ in view of the invariance of solution by any displacement on t. Let $x_*(0) = x_*(T_*) = x_0$.

By applying the results above, we get:

1) the matrix $A(t) = A_1(t) + B_1(t)P$, where $W_1(0,T_*) = \int_0^{T_*} \Phi(0,t)B_1(t)B_1^*(t)\Phi^*(0,t)dt > 0$;
2) linear controllable system has the form

$$\dot{y} = A_1(t)y + B_1(t)u(t) + \mu(t), t \in I_* = [0, T_*],$$

$$y(0) = y(T_*) = x_*(0) = x_*(T) = x_0, u(\cdot) \in L_2(I, R^m);$$

3) optimization problem is written: minimize the functional

$$I(v, x_0) = \int_0^{T_*} |v(t) + T(t)x_0 + \bar{\mu}(t) + N_1(t)z(T_*, v) - Py(t)|^2 \, dt \to \inf$$

under conditions

$$\dot{z} = A_1(t)z + B_1(t)v(t), z(0) = 0, t \in [0, T_*] = I_*,$$

$$v(\cdot) \in L_2(I_*, R^m), x_0 \in R^n.$$

4) Necessary and sufficient conditions for existence of a periodic solution of system (23) with period T_* is defined by equality $I(v_*, x_{0*}) = 0$, where (v_*, x_{0*}) is a solution of the optimization problem.
5) Optimal solution (v_*, x_{0*}) is defined by constructing the minimizing sequences.

5 Conclusion

A more general problem of periodic solution of the boundary value problem of ordinary differential equations with phase and integral constraints is formulated on the base of a review of scientific research on the periodic solutions of autonomous dynamical systems [1]-[4].

The boundary value problem is reduced to the problem of controllability of dynamic systems with phase and integral constraints by introducing a fictitious boundary control [5]. Solution of the controllability problem is reduced to a Fredholm integral equation of the first kind. The necessary and sufficient conditions for the solvability of the Fredholm integral equation of the first kind are obtained and the general solution of the integral equation is found.

The results of fundamental research on the controllability theory of dynamic systems, as well as new results on the solvability and construction the solution of the Fredholm integral equation of the first kind enable to reduce solutions of the general problem of periodic solution to the special initial problem of optimal control.

The necessary and sufficient condition for the existence of periodic solution of autonomous dynamic system in the form of requirements on a non-negative functional values is obtained.

The algorithm for constructing periodic solution to the limit points of minimizing sequences is developed. The estimation of the convergence rate is obtained.

Scientific novelty of the results consists in a completely new approach to the study of periodic solutions of autonomous dynamical systems, focused on the use of modern information technologies is offered. The existence of periodic solution and its construction are solved together.

A distinctive feature of the proposed method from the known methods of investigation of periodic solutions is that: firstly, the properties of analytic right-hand sides, the differential equations are not required; secondly, there is no need for small parameter system.

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