Some Reverse Hölder Type Inequalities Involving $(k, s)-$Riemann-Liouville Fractional Integrals

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Abstract. In this paper, we aim to present the improved version of the reverse Hölder type inequalities by taking $(k, s)-$Riemann-Liouville fractional integrals. Furthermore, we also discuss some applications of Theorem 1 using some types of fractional integrals.

Keywords: $(k, s)-$Riemann-Liouville fractional integrals · Holder inequality · Reverse Holder inequality

1 Introduction

Fractional integral inequalities involving $(k, s)-$ type integrals attract the attentions of many researchers due their diverse applications see, for examples, [1–4]. In [5], Farid *et al.* an integral inequality obtained by Mitrinovic and Pecaric was generalized to measure space as follows.

Theorem 1. *Let* $(\Omega_1, \Sigma_1, \mu_1)$,$(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* $\sigma-$*finite measures and let* $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, 2, 3, 4$ *be non-negative functions. Let* g *be the function having representation*

$$g(x) = \int_{\Omega_1} k(x,t) f(t) d\mu_1(t),$$

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where $k : \Omega_2 \times \Omega_1 \to \mathbb{R}$ is a general non-negative kernel and $f : \Omega_1 \to \mathbb{R}$ is real-valued function, and μ_2 is a non-decreasing function. If p, q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then

$$\int_{\Omega_2} f_1(x) f_2(x) g(x) d\mu_2(x) \tag{1}$$
$$\leq C \left(\int_{\Omega_2} f_3(x) g(x) d\mu_2(x) \right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_4(x) g(x) d\mu_2(x) \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_a^b k(x,t) f_1(x) f_2(x) d\mu_2(x) \right) \right. \tag{2}$$
$$\left. \left(\int_a^b k(x,t) f_3(x) d\mu_2(x) \right)^{\frac{-1}{p}} \left(\int_a^b k(x,t) f_4(x) d\mu_2(x) \right)^{\frac{-1}{q}} \right\}.$$

The following definitions and results are also required.

2 Preliminaries

Recently fractional integral inequalities are considered to be an important tool of applied mathematics and their many applications described by a number of researchers. As well as, the theory of fractional calculus is used in solving differential, integral and integro-differential equations and also in various other problems involving special functions [6–8].

We begin by recalling the well-known results.

1. The Pochhammer k-symbol $(x)_{n,k}$ and the k-gamma function Γ_k are defined as follows (see [9]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) \quad (n \in \mathbb{N};\ k > 0) \tag{3}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!\, k^n\, (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad \left(k > 0;\ x \in \mathbb{C} \backslash k\mathbb{Z}_0^-\right), \tag{4}$$

where $k\mathbb{Z}_0^- := \left\{kn : n \in \mathbb{Z}_0^-\right\}$. It is noted that the case $k = 1$ of equation ((3)) and equation ((4)) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt \quad (\Re(x) > 0). \tag{5}$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties: (i) $\Gamma_k(x+k) = x\,\Gamma_k(x)$; (ii) $\Gamma_k(k) = 1$; (iii) $\Gamma_k(x)$ is logarithmically convex. It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\frac{x}{k}\right) \quad (\Re(x) > 0;\ k > 0)\,. \tag{6}$$

2. Mubeen and Habibullah [10] introduced k-fractional integral of the Riemann-Liouville type of order α as follows:

$${}_kJ_a^{\alpha}\left[f\left(t\right)\right] = \frac{1}{\Gamma_k(\alpha)}\int_a^t \left(t-\tau\right)^{\frac{\alpha}{k}-1} f\left(\tau\right) d\tau, \left(\alpha > 0, x > 0, k > 0\right), \tag{7}$$

which, upon setting $k = 1$, is seen to yield the classical Riemann-Liouville fractional integral of order α:

$$J_a^{\alpha}\left\{f(t)\right\} := {}_1J_a^{\alpha}\left\{f(t)\right\} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau \quad (\alpha > 0;\ t > a)\,. \tag{8}$$

3. Sarikaya *et al.* [11] presented (k, s)-fractional integral of the Riemann-Liouville type of order α, which is a generalization of the k-fractional integral (7), defined as follows:

$${}_k^sJ_a^{\alpha}\left[f\left(t\right)\right] := \frac{\left(s+1\right)^{1-\frac{\alpha}{k}}}{k\Gamma_k\left(\alpha\right)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s f\left(\tau\right) d\tau,\ \tau \in \left[a, b\right], \tag{9}$$

where $k > 0, s \in \mathbb{R}\backslash\left\{-1\right\}$ and which, upon setting $s = 0$, immediately reduces to the k-integral (7).

4. In [11], the following results have been obtained. For f be continuous on $[a, b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then,

$${}_k^sJ_a^{\alpha}\left[{}_k^sJ_a^{\beta} f\left(t\right)\right] = {}_k^sJ_a^{\alpha+\beta} f\left(t\right) = {}_k^sJ_a^{\beta}\left[{}_k^sJ_a^{\alpha} f\left(t\right)\right], \tag{10}$$

and

$${}_k^sJ_a^{\alpha}\left[\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+\beta)}\left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1},$$

for all $\alpha, \beta > 0$, $x \in [a, b]$ and Γ_k denotes the $k-$gamma function.

5. Also, in [12], Akkurt *et al.* introduced $(k, H)-$fractional integral. Let (a, b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $h'(x)$ on (a, b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a, b]$ are defined by

$$\begin{aligned}&\left({}_kJ_{a^+,h}^{\alpha} f\right)(x) \\ &:= \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left[h(x)-h(t)\right]^{\frac{\alpha}{k}-1} h'(t) f(t) dt,\ k > 0\ , \Re(\alpha) > 0\end{aligned} \tag{11}$$

$$\left({}_kJ^{\alpha}_{b^-,h}f\right)(x) \tag{12}$$
$$:= \frac{1}{k\Gamma_k(\alpha)}\int_x^b [h(x)-h(t)]^{\frac{\alpha}{k}-1}h'(t)f(t)dt,\ k>0\ ,\Re(\alpha)>0.$$

Recently, Tomar and Agarwal [13] obtained following results for $(k,s)-$fractional integrals.

Theorem 2 (Hölder Inequality for (k,s)-fractional integrals). *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, for all $t>0,\ k>0,\ \alpha>0, s\in\mathbb{R}-\{-1\}$,*

$${}^s_kJ^{\alpha}_a \mid fg\,(t)| \le \left[{}^s_kJ^{\alpha}_a\,|f\,(t)|^p\right]^{\frac{1}{p}}\left[{}^s_kJ^{\alpha}_a\,|g\,(t)|^q\right]^{\frac{1}{q}}. \tag{13}$$

Lemma 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions and $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,k>0$ and $s\in\mathbb{R}-\{-1\}$, such that for $t\in[a,b]$, ${}^s_kJ^{\alpha}_a f^p(t)<\infty$, ${}^s_kJ^{\alpha}_a g^q(t)<\infty$. If*

$$0\le m\le\frac{f(\tau)}{g(\tau)}\le M<\infty, \tau\in[a,b], \tag{14}$$

then the inequality

$$\left[{}^s_kJ^{\alpha}_a f(t)\right]^{\frac{1}{p}}\left[{}^s_kJ^{\alpha}_a g(t)\right]^{\frac{1}{q}}\le\left(\frac{M}{m}\right)^{\frac{1}{pq}}\ {}^s_kJ^{\alpha}_a\left[f^{\frac{1}{p}}(t)g^{\frac{1}{q}}(t)\right] \tag{15}$$

holds.

Lemma 2. *Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions $\alpha,k>0$ and $s\in\mathbb{R}-\{-1\}$, such that for $t\in[a,b]$, ${}^s_kJ^{\alpha}_a f^p(t)<\infty$, ${}^s_kJ^{\alpha}_a g^q(t)<\infty$. If*

$$0\le m\le\frac{f^p(\tau)}{g^q(\tau)}\le M<\infty, \tau\in[a,b], \tag{16}$$

then we have

$$\left[{}^s_kJ^{\alpha}_a f^p(t)\right]^{\frac{1}{p}}\left[{}^s_kJ^{\alpha}_a g^q(t)\right]^{\frac{1}{q}}\le\left(\frac{M}{m}\right)^{\frac{1}{pq}}\ {}^s_kJ^{\alpha}_a\left(f(t)g(t)\right), \tag{17}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by this work, we establish in this paper some new extensions of the reverse Hölder type inequalities by taking $(k,s)-$Riemann-Liouville fractional integrals.

3 Reverse Hölder Type Inequalites

In this section we prove our main results (Theorems 3 and 4).

Theorem 3. *Let $f(x)$ and $g(x)$ be integrable functions and let $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequality holds*

$$ {}_k^sJ_a^\alpha \left| fg(t) \right| \geq {}_k^sJ_a^\alpha \left| f^p(t) \right|^{\frac{1}{p}} \ {}_k^sJ_a^\alpha \left| f^q(t) \right|^{\frac{1}{q}}. \tag{18} $$

Proof. Set $c = \frac{1}{p}$, $q = -pd$. Then we have $d = \frac{c}{c-1}$. By the Hölder inequality for $(k, s)-$fractional integrals, we have

$$ \begin{aligned} {}_k^sJ_a^\alpha \left| f^p(t) \right| &= {}_k^sJ_a^\alpha \left| fg(t) \right|^p \left| g^{-p}(t) \right| \\ &\leq \left[{}_k^sJ_a^\alpha \left| fg(t) \right|^{pc} \right]^{\frac{1}{c}} \left[{}_k^sJ_a^\alpha \left| g(t) \right|^{-pd} \right]^{\frac{1}{d}} \\ &= \left[{}_k^sJ_a^\alpha \left| fg(t) \right| \right]^{\frac{1}{c}} \left[{}_k^sJ_a^\alpha \left| g(t) \right|^q \right]^{1-p}. \end{aligned} \tag{19} $$

In equation (19), multiplying both sides by $\left({}_k^sJ_a^\alpha \left| g^q(t) \right| \right)^{p-1}$, we obtain

$$ \begin{aligned} & {}_k^sJ_a^\alpha \left| f^p(t) \right| \left({}_k^sJ_a^\alpha \left| g^q(t) \right| \right)^{p-1} \\ & \leq \left[{}_k^sJ_a^\alpha \left| fg(t) \right| \right]^p. \end{aligned} \tag{20} $$

Inequality (20) implies inequality

$$ {}_k^sJ_a^\alpha \left| fg(t) \right| \geq {}_k^sJ_a^\alpha \left| f^p(t) \right|^{\frac{1}{p}} \ {}_k^sJ_a^\alpha \left| f^q(t) \right|^{\frac{1}{q}} \tag{21} $$

which completes this theorem.

Theorem 4. *Suppose $p, q, l > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{l} = 1$. If f, g and h are positive functions such that*

i.) $0 < m \leq \frac{f^{\frac{p}{s}}}{g^{\frac{g}{s}}} \leq M < \infty$ for some $l > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$,
ii.) $0 < m \leq \frac{(fg)^s}{h^r} \leq M < \infty$,

then

$$ \begin{aligned} & \left({}_k^sJ_a^\alpha f^p(t) \right)^{\frac{1}{p}} \left({}_k^sJ_a^\alpha f^q(t) \right)^{\frac{1}{q}} \left({}_k^sJ_a^\alpha f^r(t) \right)^{\frac{1}{r}} \\ & \leq \left(\frac{M}{m} \right)^{\frac{1}{sr} + \frac{pq}{s^3}} \ {}_k^sJ_a^\alpha (fgh)(t). \end{aligned} \tag{22} $$

Proof. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ for some $s > 0$. Thus, $\frac{s}{p} + \frac{s}{q} = 1$ and $\frac{1}{s} + \frac{1}{r} = 1$. If we use *ii* and Lemma 2 for $H = fg$ and h, then we get

$$ \left({}_k^sJ_a^\alpha H^s(t) \right)^{\frac{1}{s}} \left({}_k^sJ_a^\alpha h^r(t) \right)^{\frac{1}{r}} \leq \left(\frac{M}{m} \right)^{\frac{1}{sr}} \left({}_k^sJ_a^\alpha (Hh)(t) \right) \tag{23} $$

which is equivalent to

$$\left({}_k^sJ_a^\alpha[f^s(t)g^s(t)]\right)^{\frac{1}{s}}\left({}_k^sJ_a^\alpha h^r(t)\right)^{\frac{1}{r}} \leq \left(\frac{M}{m}\right)^{\frac{1}{sr}}\left({}_k^sJ_a^\alpha(fgh)(t)\right). \tag{24}$$

Now, using i and the fact that $\frac{s}{p}+\frac{s}{q}=1$, and applying Lemma 2 to f^s and g^s, we also have

$$\left({}_k^sJ_a^\alpha f^p(t)\right)^{\frac{s}{p}}\left({}_k^sJ_a^\alpha g^q(t)\right)^{\frac{s}{q}} \leq \left(\frac{M}{m}\right)^{\frac{pq}{s^2}}\left({}_k^sJ_a^\alpha f^s(t)g^s(t)\right) \tag{25}$$

which is equivalent to

$$\left({}_k^sJ_a^\alpha f^p(t)\right)^{\frac{1}{p}}\left({}_k^sJ_a^\alpha g^q(t)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{pq}{s^3}}\left({}_k^sJ_a^\alpha f^s(t)g^s(t)\right)^{\frac{1}{s}}. \tag{26}$$

Combining equations (24) and (26), we obtain desired inequality equation (22), which is complete the proof.

4 Applications for Some Types Fractional Integrals

Here in this section, we discuss some applications of Theorem 1 in the terms of Theorems 5-7 and Corollary 1-5.

Theorem 5. *Let p,q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and let f be continuous on $[a,b]$, $k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then*

$$\int_a^b f_1(x)f_2(x){}_k^sJ_a^\alpha f(x)dx \tag{27}$$
$$\leq C\left(\int_a^b f_3(x){}_k^sJ_a^\alpha f(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_4(x){}_k^sJ_a^\alpha f(x)dx\right)^{\frac{1}{q}},$$

where

$$C=\sup_{t\in[a,b]}\left\{\left(\int_a^b\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_1(x)f_2(x)dx\right)\right. \tag{28}$$
$$\left.\left(\int_a^b\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_3(x)dx\right)^{\frac{-1}{p}}\left(\int_a^b\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_4(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Proof. In Theorem 1, if we take $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(t)=dt, d\mu_2(x)=dx$ and the kernel

$$k(x,t)=\begin{cases}\frac{(s+1)^{1-\frac{\alpha}{k}}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s}{k\Gamma_k(\alpha)} & \text{if } a\leq t\leq x\\ 0 & \text{if } x<t\leq b,\end{cases}$$

then $g(x)$ becomes ${}_k^sJ_a^\alpha f(t)$ and so we get desired inequality (27). This completes the proof of Theorem 5.

Corollary 1. *In Theorem 5, if we take $s = 0$, then we get*

$$\int_a^b f_1(x) f_2(x)\,{}_kJ_a^\alpha f(x)dx \tag{29}$$
$$\leq C\left(\int_a^b f_3(x)\,{}_kJ_a^\alpha f(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_4(x)\,{}_kJ_a^\alpha f(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1(x) f_2(x)dx\right)\right. \tag{30}$$
$$\left.\left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_3(x)dx\right)^{\frac{-1}{p}}\left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_4(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Remark 1. In Corollary 1, $\alpha = k = 1$, Theorem 1 reduces to Theorem 3.1 in [5].

Corollary 2. *In Theorem 5, if we take $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get*

$$\int_a^b f_1(x) f_2(x)\,{}_k^sJ_a^\alpha f(x)dx \tag{31}$$
$$\leq C\left(\int_a^b f_1^p(x)\,{}_k^sJ_a^\alpha f(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_2^q(x)\,{}_k^sJ_a^\alpha f(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^b \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} f_1(x) f_2(x)dx\right)\right. \tag{32}$$
$$\left.\left(\int_a^b \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} f_1^p(x)dx\right)^{\frac{-1}{p}}\left(\int_a^b \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Corollary 3. *In Corollary 2, if we take $s = 0$, then we get*

$$\int_a^b f_1(x) f_2(x)\,{}_kJ_a^\alpha f(x)dx \tag{33}$$
$$\leq C\left(\int_a^b f_1^p(x)\,{}_kJ_a^\alpha f(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_2^q(x)\,{}_kJ_a^\alpha f(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1(x) f_2(x)dx\right)\right. \tag{34}$$
$$\left.\left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1^p(x)dx\right)^{\frac{-1}{p}}\left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Remark 2. In Corollary 3, $\alpha = k = 1$, Corollary 3 reduces to Corollary 3.2 in [5].

Theorem 6. *Let (a, b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Let $h(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $h'(x)$ on (a, b). Also, let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and let f be continuous on $[a, b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then*

$$\int_a^b f_1(x) f_2(x) \left({}_kJ^{\alpha}_{a^+,h} f \right)(x) dx \tag{35}$$
$$\leq C \left(\int_a^b f_3(x) \left({}_kJ^{\alpha}_{a^+,h} f \right)(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_4(x) \left({}_kJ^{\alpha}_{a^+,h} f \right)(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k} - 1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k} - 1} h'(t) f_3(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k} - 1} h'(t) f_4(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{36}$$

Proof. Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(t) = dt, d\mu_2(x) = dx$ and the kernel

$$k(x, t) = \begin{cases} \frac{(h(x) - h(t))^{\frac{\alpha}{k} - 1} h'(t)}{k \Gamma_k(\alpha)} & \text{if } a \leq t \leq x \\ 0 & \text{if } x < t \leq b, \end{cases}$$

then $g(x)$ becomes $\left({}_kJ^{\alpha}_{a^+,h} f \right)(x)$ and so we get desired inequality (35). This completes the proof of Theorem 6.

Corollary 4. *In Theorem 6, setting $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, we get*

$$\int_a^b f_1(x) f_2(x) \left({}_kJ^{\alpha}_{a^+,h} f \right)(x) dx \tag{37}$$
$$\leq C \left(\int_a^b f_1^p(x) \left({}_kJ^{\alpha}_{a^+,h} f \right)(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x) \left({}_kJ^{\alpha}_{a^+,h} f \right)(x) dx \right)^{\frac{1}{q}},$$

where

$$C=\sup_{t\in[a,b]}\left\{\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_1(x)f_2(x)dx\right)\right.$$
$$\times\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_1^p(x)dx\right)^{\frac{-1}{p}}$$
$$\left.\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}. \tag{38}$$

Theorem 7. *Under the assumptions of Theorem 6, we have*

$$\int_a^b f_1(x)f_2(x)\left({}_kJ^{\alpha}_{b^-,h}f\right)(x)dx \tag{39}$$
$$\leq C\left(\int_a^b f_3(x)\left({}_kJ^{\alpha}_{b^-,h}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_4(x)\left({}_kJ^{\alpha}_{b^-,h}f\right)(x)dx\right)^{\frac{1}{q}},$$

where

$$C=\sup_{t\in[a,b]}\left\{\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_1(x)f_2(x)dx\right)\right.$$
$$\times\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_3(x)dx\right)^{\frac{-1}{p}}$$
$$\left.\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_4(x)dx\right)^{\frac{-1}{q}}\right\}. \tag{40}$$

Proof. In contrast to Theorem 6, if we take the kernel

$$k(x,t)=\begin{cases}\frac{(h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)}{k\Gamma_k(\alpha)} & \text{if } x\leq t\leq b\\ 0 & \text{if } a<t\leq x,\end{cases}$$

we obtain desired inequality.

Corollary 5. *In Theorem 7, setting $f_3(x)=f_1^p(x)$ and $f_4(x)=f_2^q(x)$, we get*

$$\int_a^b f_1(x)f_2(x)\left({}_kJ^{\alpha}_{b^-,h}f\right)(x)dx \tag{41}$$
$$\leq C\left(\int_a^b f_1^p(x)\left({}_kJ^{\alpha}_{b^-,h}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_2^q(x)\left({}_kJ^{\alpha}_{b^-,h}f\right)(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \times \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k}-1} h'(t) f_1^p(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k}-1} h'(t) f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{42}$$

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