Springer Proceedings in Mathematics & Statistics

Tynysbek Sh. Kalmenov
Erlan D. Nursultanov
Michael V. Ruzhansky
Makhmud A. Sadybekov *Editors*

Functional Analysis in Interdisciplinary Applications

Astana, Kazakhstan, October 2017

Springer

Springer Proceedings in Mathematics & Statistics

Volume 216

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Tynysbek Sh. Kalmenov
Erlan D. Nursultanov · Michael V. Ruzhansky
Makhmud A. Sadybekov
Editors

Functional Analysis in Interdisciplinary Applications

Astana, Kazakhstan, October 2017

Springer

Editors
Tynysbek Sh. Kalmenov
Institute of Mathematics
and Mathematical Modeling
Almaty
Kazakhstan

Michael V. Ruzhansky
Department of Mathematics
Imperial College London
London
UK

Erlan D. Nursultanov
Department of Mechanics
and Mathematics
Kazakhstan Branch of Lomonosov
Moscow State University
Astana
Kazakhstan

Makhmud A. Sadybekov
Institute of Mathematics
and Mathematical Modeling
Almaty
Kazakhstan

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-319-67052-2 ISBN 978-3-319-67053-9 (eBook)
https://doi.org/10.1007/978-3-319-67053-9

Library of Congress Control Number: 2017952024

Mathematics Subject Classification (2010): 46–XX, 26Axx, 26Bxx, 34Bxx, 34Kxx, 35–XX

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Nowadays, by functional analysis one usually understands the branch of analysis devoted to studying infinite dimensional topological spaces and mappings between them. Historically, the main examples of such spaces were spaces of functions, thus confirming the term of "functional" analysis.

Probably for the first time, the term "functional" was used by Hadamard in his 1910 book devoted to the calculus of variations. As such, the French mathematician Jacques Salomon Hadamard founded the modern school of linear functional analysis further developed by the Hungarian mathematician Frigyes Riesz and the Polish mathematician Stefan Banach.

By the beginning of the XXI century the functional analysis has grown so much penetrating various branches of mathematics that it is now difficult to define precisely the subject of this discipline. At the same time the functional analysis was enriched by numerous more specialised branches based on topics of the more classical analysis. For example, in Wikipedia, it is defined[1] as follows: "Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear functions defined on these spaces and respecting these structures in a suitable sense. The historical roots of functional analysis lie in the study of spaces of functions and the formulation of properties of transformations of functions such as the Fourier transform as transformations defining continuous, unitary etc. operators between function spaces. This point of view turned out to be particularly useful for the study of differential and integral equations".

The functional analysis nowadays finds its applications in many theoretical and applied branches of mathematics. Many theoretical constructions, important for the development of mathematics, are described in the language of the functional analysis. Its applications are encountered in numerous areas

[1]https://en.wikipedia.org/wiki/Functionalanalysis

such as the theories of differential and integral equations, mathematical and theoretical physics, control and optimisation theories, probability theory and mathematical statistics. The Fourier analysis combined with techniques of functional analysis led to the development of the theory of distributions and the theory of pseudo-differential operators.

Motivated by their large applicability for real life problems, applications of functional analysis have been the purpose of an intensive research effort in the last decades, yielding significant progress in the theory of functions and functional spaces, in differential equations and boundary value problems, in differential and integral operators and spectral theory, and in mathematical methods in physical sciences.

The present volume is devoted to these investigations. The publication of this collection of papers is based on the materials of the conference "Functional analysis in interdisciplinary applications" organised in the framework of the VI Congress of the Turkic World Mathematical Society (2–5 October, 2017, Astana, Kazakhstan). The aim of the conference is to unite mathematicians working in the areas of functional analysis and its interdisciplinary applications to share new trends of applications of the functional analysis. This conference is also dedicated to the 75th anniversary of the outstanding expert in differential operators and spectral problems Prof. Mukhtarbay Otelbaev, doctor of physical and mathematical sciences, professor of the Gumilyov Eurasian National University, the vice-director of the Kazakhstan branch of the Lomonosov Moscow State University, the chief researcher of the Institute of Mathematics and Mathematical Modelling, the laureate of the State Prize of the Republic of Kazakhstan, an academician of the National Academy of Sciences of the Republic of Kazakhstan. Professor Mukhtarbay Otelbaev is deservedly considered the founder of research on functional analysis in Kazakhstan.

The present volume contains a citation for Mukhtarbay Otelbaev devoted to his 75th anniversary. Moreover, the volume also contains a citation for Professor Erlan Nursultanov, doctor of physical and mathematical sciences, the Head of Department of Mathematics and Computer Science of the Kazakhstan Branch of the Lomonosov Moscow State University, for his 60th anniversary. He is a specialist in the areas of Fourier series, quadrature formulae, interpolation of function spaces, and stochastic processes. The 44 articles collected in the present volume are selected from the talks by conference participants. In fact, they are the outgrowth and further development of the talks presented at the conference by participants from different countries, including Germany, India, Iraq, Kazakhstan, Russia, Serbia, Tajikistan, Turkey, United Kingdom and Uzbekistan. All of them contain new results and went thorough a refereeing process. The volume reflects the latest developments in the area of functional analysis and its interdisciplinary applications.

This volume contains four different chapters. The first chapter contains the contributed papers focusing on various aspects of the theory of functions and functional spaces, including problems of approximation of function classes, conditions for boundedness and compactness for operators in different function spaces, results on multipliers of Fourier series for various spaces, results concerning the evolution of invariant Riemannian metrics on generalised Wallach spaces, results of almost null and almost convergent double sequence spaces. The second chapter is devoted to the research on differential equations and boundary value problems. Correct and ill-posed problems for linear and nonlinear partial differential equations, construction problems and properties of their solutions are considered. The third chapter contains the results of studies on differential and integral operators and on the spectral theory. Various questions for ordinary differential operators, for second-order and high-order partial differential operators, volume potential for elliptic differential equations, correct restrictions and extensions of linear operators, properties for root functions of various differential operators, spectral geometry issues are considered. The fourth chapter is focused on the simulation of problems arising in real-world applications of physical sciences, such as electromagnetic fields, continuum mechanics and complex flows. Direct and inverse problems are considered as well as the Stefan problem for parabolic and pseudo-parabolic equations, dynamic problems, problems of a program of basic control systems. The volume editors thank the authors for their high-quality contributions.

The editors want to sincerely thank Gulnara Igissinova and Sholpan Balgimbayeva from the Institute of Mathematics and Mathematical Modeling, who have contributed greatly to the editing and technical design of this volume.

Almaty, Kazakstan — Tynysbek Sh. Kalmenov
Astana, Kazakhstan — Erlan D. Nursultanov
London, UK — Michael V. Ruzhansky
Almaty, Kazakhstan — Makhmud A. Sadybekov
June 2017

Contents

Professor Mukhtarbay Otelbaev: Citation for His 75th Birthday

Mukhtarbay Otelbaev is the professor of the Department of Fundamental and Applied Mathematics of L.N. Gumilyov Eurasian National University, the director of the Eurasian Mathematical Institute at the Eurasian National University, the deputy director of the Kazakhstan branch of M.V. Lomonosov Moscow State University, the Chief Researcher of the Institute of Mathematics and Mathematical Modelling, the Laureate of the State Prize of the Republic of Kazakhstan, Doctor of Sciences in physics and mathematics, the academician of the National Academy of Sciences of the Republic of Kazakhstan. He was born on October 3, 1942 in the village Karakemer of the Kordai district of the Zhambyl region, Kazakhstan.

He started his working life as a tractor-driver in his native village. After graduating from the evening school in 1962 in the village Karakonyz (now Masanchi), he entered the Kyrgyz State University in Frunze (now Bishkek). In 1962–1965, he served in the Soviet Army. In 1965–1966 he worked as a teacher of mathematics at Chapaev evening school in the village Karakemer of the Kordai district of the Zhambyl region.

After that he continued his studies at the Faculty of Mechanics and Mathematics of the Moscow State University which he graduated in 1969.

In the same year he entered postgraduate studies at the same faculty under supervision of the famous scientists, Profs. B.M. Levitan and A.G. Kostyuchenko. In 1972 he defended the Ph.D. thesis titled "About the spectrum of some differential operators".

Since 1973 M. Otelbaev was in Alma-Ata, where he worked as a junior researcher, a senior researcher, and then as the head of a laboratory at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR.

In 1978, he brilliantly defended the Doctor of Sciences thesis titled "Estimates of the spectrum of elliptic operators and related embedding theorems" at the Dissertation Council number 1 of the Faculty of Mechanics and Mathematics of the Moscow State University headed by Prof. A.N. Kolmogorov, a prominent mathematician, the academician of the Academy of Sciences of the USSR.

In 1989, M. Otelbaev was elected a corresponding member of the Academy of Sciences of the Kazakh SSR, and in 2004 he became a real member of the National Academy of Sciences of the Republic of Kazakhstan.

Professor M. Otelbaev is an expert in the field of functional analysis and its applications, the author of 3 monographs and over 200 scientific papers and inventions widely recognized both in Kazakhstan and abroad. More than 70 of his works were published in rated international scientific journals (with the impact-factor Journal Citation Reports Web of Science or included in the SCOPUS database).

His main works are grouped around the following fields:

- Spectral theory of differential operators;
- Embedding theory and approximation theory;
- Separability and coercive estimates for differential operators;
- General theory of boundary value problems;
- Theory of generalized analytic functions;
- Computational mathematics;
- Nonlinear evolutional equations;
- Theoretical physics;
- Other fields of mathematics.

Let us briefly recapture the main results of Prof. M. Otelbaev.

Spectral Theory of Differential Operators

M. Otelbaev developed new methods for studying the spectral properties of differential operators, which are the result of consistent and skilled implementation of the general idea of the localization of the considered problems. In particular, he invented a construction of averaging coefficients well describing those features of their behaviour which influence the spectral properties of a differential operator. This construction, known under the notation q^*, made it possible to answer many of the hitherto open questions of the spectral theory of the Schrödinger operator and its generalisations.

The function q^* and its different variants have a number of remarkable properties, which allow applying this function to a wide range of problems. Here we note some problems solved by M. Otelbaev by using the function q^* on the basis of sophisticated analysis of the properties of differential operators.

(1) A criterion for the membership of the resolvent of the Schrödinger type operator with a non-negative potential in the class $\sigma_p, (1 \le p \le \infty)$ was found (previously only a criterion for the membership in σ_∞ was known), and two-sided estimates for the eigenvalues of this operator were obtained under the minimal assumptions of the smoothness of the coefficients.
(2) The general localization principle was proved for the problems of selfadjointness and of the maximal dissipativity (simultaneously with the American mathematician P. Chernov) which provided significant progress in this area.
(3) Examples were given showing that the classical Carleman-Titchmarsh formula for the distribution function $N(\lambda)$ of the eigenvalues of the Sturm-Liouville operator is not always correct even in the class of monotonic potentials and a new formula was found valid for all monotonic potentials.
(4) The following result of M. Otelbaev is principally important: for $N(\lambda)$ there is no universal asymptotic formula.
(5) From the time of Carleman, who found the asymptotics for $N(\lambda)$ and, by using it, the asymptotics of the eigenvalues themselves, many mathematicians worked on finding the asymptotics for $N(\lambda)$, but as a result they could not get rid of the so-called Tauberian conditions. M. Otelbaev was the first one, when looking for the asymptotics of the eigenvalues, who omitted the interim step of finding the asymptotics for $N(\lambda)$, which allowed getting rid of all unessential conditions for the problem including Tauberian conditions.
(6) The two-sided asymptotics for $N(\lambda)$ for the Dirac operator showed for the first time that $N_-(\lambda)$ and $N_+(\lambda)$ may be not equivalent. The results of M. Otelbaev on the spectral theory were included as separate chapters in the monographs of B.M. Levitan and I.S. Sargsyan "Sturm-Liouville and Dirac operators" (Moscow: Nauka, 1985), and of A.G. Kostyuchenko and I.S. Sargsyan "Distribution of eigenvalues" (Moscow: Nauka, 1979), which became classical books on the subject.

Recently, M. Otelbaev, jointly with Prof. V. I. Burenkov, described a wide class of non-selfadjoint elliptic operators of order $2l$ with general boundary conditions, whose singular numbers have the same asymptotics as the eigenvalues of the l^{th} power of the Laplace operators with the Dirichlet boundary conditions.

Embedding Theory and Approximation Theory

This field of mathematics has developed as a separate branch in the works of S.L. Sobolev in 1930s. Beginning with the works of L.D. Kudryavtsev (around 1960) a new era of weighted function spaces used in the theory of differential operators with variable coefficients arises.

M. Otelbaev began research in this field already being a mature mathematician and managed to create a new method of proving embedding theorem which is, in form and essence, a local approach to such problems. In the theory of weighted Sobolev spaces, the most used weighted function spaces, M. Otelbaev obtained the following fundamental results.

(1) A criterion for an embedding and for the compactness of an embedding.
(2) Two-sided estimates for the norm of an embedding operator.
(3) Two-sided estimates for Kolmogorov's width and for the approximation numbers of an embedding operator, and a criterion for the membership for an embedding operator in the classes $\sigma_p, (1 \leq p \leq \infty)$. It turned out that one of the variants of the function q^* is an adequate tool for description of the exact conditions ensuring an embedding. For applications it is particularly important that all the estimates were given in terms of weight functions thus allowing taking into account the characteristics of their local behaviour.

Separability and Coercive Estimates for Differential Operators

The term "separability" was suggested by the famous English mathematicians Everitt and Geertz around 1970s, who investigated the smoothness of solutions to the Sturm-Liouville equation.

Soon after that, M. Otelbaev was involved in the research on this topic. He developed a method for studying the separability of more general, multi-dimensional operators and variable type operators, as well for the smoothness of solutions to nonlinear equations. In particular, by using this method one can study the separability of general differential operators in weighted, not necessarily Hilbert, spaces. With his interest in solving problems in the most general setting, M. Otelbaev obtained

(1) weighted estimates not only of the derivatives of solutions of the highest order, but also of intermediate derivatives for a wide class of linear and nonlinear equations;
(2) estimates of the approximation numbers of separable operators exact in a certain class of coefficients.

General Theory of Boundary Problems

The classical formulation of the boundary value problem is as follows: given an equation and boundary conditions, to investigate the solvability of this problem and the properties of the solution, if it exists (in the sense of being in a certain space). Beginning with M.I. Vishik (1951), there is another, more general approach: given an equation and a space to which the right-hand side and the solution should belong, to describe all boundary conditions for which the problem is correctly solvable in this space.

In this problem as well, despite the numerous previous studies, M. Otelbaev has obtained new results remarkable in depth and transparency. The rich mathematical intuition, the depth of thinking and extensive knowledge, coupled with rejection of traditional constraints on the considered operators and spaces, allowed him to develop an abstract theory of extension and restriction of not necessarily linear operators in linear topological spaces.

Using this theory, M. Otelbaev and his students were the first to describe all correct boundary value problems for such "pathological" operators as the Bitsadze-Samarskii operator, the ultrahyperbolic operator, the pseudoparabolic operator, the Cauchy-Riemann operator and others (For some of them previously no correct boundary value problems were known!). Moreover, considerations were carried out in non-Hilbert spaces L_p and C. This theory also allowed describing the structural properties of the spectrums of correct restrictions of a given differential operator.

Theory of Generalized Analytic Functions

In the theory of generalized analytic functions, developed by the well-known scientist I.N. Vekua, a member of the Academy of Sciences of the USSR, the main facts are:

(a) a theorem on the representation of a solution;
(b) a theorem on the continuity of a solution;
(c) a theorem on the Fredholm property.

All other facts of the theory are deduced from (a), (b), and (c). Various authors have gradually widened the class of spaces in which the Vekua theory was valid.

M. Otelbaev found the widest space among the spaces close to the so-called ideal spaces, to which the coefficients and the right-hand side should belong, so that the facts (a), (b) and (c) remain valid.

Computational Mathematics

M. Otelbaev proposed a new numerical method for solving boundary value problems (as well as general operator equations). By using the embedding and extension theorems, he reduced the considered boundary value problem to the problem of minimising a functional. The boundary conditions and also nonlinearities are "hidden" in the integral expressions. Moreover, by this method the problem of "the choice of a basis" was solved, in which many prominent mathematicians have been interested for a long time.

The method of M. Otelbaev can be easily algorithmised and allows finding the solution with the required accuracy. Moreover, the procedure of finding a numerical solution is stable. Computer calculations conducted by his students and students of Prof. Sh. Smagulov showed the effectiveness of the method.

M. Otelbaev developed a method of approximate calculation of eigenvalues and eigenvectors of non-selfadjoint matrices, based on a variational principle. The method reduces the problem to the analogous problem for self-adjoint matrices, for which there is a well-developed theory. Unlike other methods, for example, the maximum gradient method, this method

(1) provides global convergence,
(2) is convenient for calculating the initial approximation,
(3) allows calculating the eigenvalues with the smallest real part,
(4) can be used in the general case of a compact non-self adjoint operator.

M. Otelbaev obtained a two-sided estimate for the smallest eigenvalue of a difference operator which is important for computational mathematics. Due to the need for cumbersome calculations, methods for the parallelization are actively developed in the world. M. Otelbaev offered an effective algorithm of parallelization for approximate solutions of boundary value problems and the Cauchy problem for various classes of differential equations.

In addition, Prof. M. Otelbaev gave a new approximate method for solving a linear algebraic system with a poorly conditioned matrix and parallelizing the solution process.

Nonlinear Evolution Equations

In hydrodynamics for describing a laminar flow of an incompressible fluid, as well as a turbulent flow the system of the Navier-Stokes equations is used. However, mathematically, the existence of a global solution has not been proved yet. Therefore, there are some uncertainties concerning using this system as an understood mathematical model.

M. Otelbaev managed to reduce the existence problem of a global solution to the Navier-Stokes equation to other equivalent problems, in particular, to the problem of the existence of the so-called “dividing function”. He obtained a criterion for strong solvability of nonlinear evolution equations, similar to the Navier-Stokes equation, and also constructed examples of equations not globally strongly solvable to which the system of Navier-Stokes type equations reduce.

The paper of Prof. M. Otelbaev in which he published a full proof of the Clay Navier-Stokes Millennium Problem obtained a high resonance: first, the paper was published in the Kazakhstan scientific journal “Mathematical Journal” (No. 4, 2013) in Russian language. However, in the process of analysing his proof a mistake in calculations was found, which was acknowledged by M. Otelbaev. Notwithstanding that the proof was incorrect, it is generally recognised that the work of M. Otelbaev has brought a new push in the progress of research on the Navier-Stokes equation. In particular, after the publication of this work, a change has been made to the statement of the problem by the Clay Institute: an additional condition of pressure periodicity has been added. Also based on the incorrect solution of the problem by M. Otelbaev, Terence Tao published a substantial work devoted to disproving the fact that the Navier-Stokes problem can be solved in an abstract form.

Theoretical Physics

M. Otelbaev obtained a number of interesting mathematical results in this area. He

(a) found explicit formulae for the n-particle motion in the space (in the framework of Einstein’s relativity theory);
(b) derived an integral formula of the matter motion;
(c) proposed a new transformation of the type of the well-known Lorentz transformation which works both for $v<c$ and for $v>c$. If $v<c$ the Otelbaev transformation coincides with the Lorentz transformation;
(d) proved mathematically that one can obtain the results of physics arising from the special Einstein’s relativity theory staying within the classical wave theory.

Other Fields of Mathematics

The research interests of M. Otelbaev are extremely diverse. The following topics complete their partial description.

(1) M. Otelbaev chose a certain nonlinear integral operator, for which he proved a criterion of continuity. This operator turned out to be an important model in the theory of nonlinear integral operators, based on which one can develop and test new methods. Consequently, M. Otelbaev together with Professor R. Oinarov obtained a necessary and sufficient condition ensuring the Lipschitz property (contractibility) of the Uryson operator in the spaces of summable and continuous functions.
(2) He investigated spectral characteristics and smoothness of solutions to equations of mixed type. A criterion of coinciding of the generalised Neumann and Dirichlet problems for degenerate elliptic equations was found.
(3) In the recent years, the problem of oscillatory and non-oscillatory solutions to differential equations has become a fashionable topic in mathematics. Already in the late 80s, M. Otelbaev obtained a sufficient condition ensuring the non-oscillation property of solutions to the Sturm-Liouville problem, close to a necessary one.
(4) M. Otelbaev studied the problem of controlling a laser heat source. He showed that under the usual formulation, it does not even have a generalized solution. Consequently, he proposed a new formulation of the problem in terms of "order" and "admittance precision" for surface treatment. He proved the solvability of this problem in such formulation, and solved some optimization problems without using the known methods of optimal control. In addition, jointly with Prof. A. Hasanoglu, he solved an inverse identification problem of an unknown time source, on the basis of the measured output data, when the boundary conditions are given in the Dirichlet or Neumann form, as well as in the form of the final overdetermination.

Summing up the review of the scientific creativity of M. Otelbaev, one should note as characteristics features of his work the diversity of his scientific interests, the fundamentality of research, the interest in solving problems in the most general formulation and obtaining solutions of the level of a criterion.

A large number of publications of M. Otelbaev characterise his high efficiency, diligence, and research productivity. He was a participant of numerous international scientific conferences, which took place in Kazakhstan, Russia, Ukraine, Poland, Czechoslovakia, Germany, Morocco, Turkey, Greece, and Japan.

M. Otelbaev has carried out great work in preparing highly qualified researchers and university teachers. Over 35 years he was giving lectures for students of various universities of the Republic of Kazakhstan, organised a series of seminars and study groups for graduate students, interns, Master and Ph.D. students. The courses “Extensions and restrictions of differential operators”, “The theory of divisibility”, “Embedding theorems”, “Modern numerical methods”, and many others developed by M. Otelbaev, are well known.

He has created a large mathematical school in Kazakhstan: 70 postgraduate students have defended Ph.D. thesa under his supervision, 9 of them later defended Doctor of Sciences theses.

M. Otelbaev made a significant contribution to the organisation and development of science and education in Kazakhstan. In 1985–1986, he was the rector of the Zhambul Pedagogical Institute, from 1991 to 1993 he organised and worked as the director of the new Institute of Applied Mathematics of the Academy of Sciences and the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda, in 1994–1995 he was the head of the Department at Aerospace Agency of the Republic of Kazakhstan.

Since 2001 he is the deputy director of the Kazakhstan branch of the Moscow State University, and simultaneously the director of the Eurasian Mathematical Institute at the L.N. Gumilyov Eurasian National University.

For a number of years, M. Otelbaev is a member of the editorial board of the Kazakhstan scientific journal “Mathematical Journal”, published by the Institute of Mathematics and Mathematical Modelling, of the “Proceedings of the Academy of Sciences of the Republic of Kazakhstan, series in Physics and Mathematics” and of the international scientific journal “Applied and Computational Mathematics” of the National Academy of Sciences of the Republic of Azerbaijan. Since 2010 he is an editor-in-chief, together with academician V.A. Sadovnichy and Prof. V.I. Burenkov, of the “Eurasian Mathematical Journal” (Included in the Scopus database), which is published in English language by the Gumilyov Eurasian National University, together with the Moscow State University, the Peoples’ Friendship University of Russia, and the University of Padua.

He was the chairman of the international scientific conference “Modern Problems of Mathematics”, held at Gumilyov Eurasian National University in 2002, and was a member of program committees of 10 international scientific conferences devoted to problems of mathematics and computer science held at the Kazakh National University, Karaganda State University, the Institute of Mathematics of the Ministry of Education and Sciences of the Republic of Kazakhstan, Pavlodar State University, and Semei University. In 2007, he was elected the Vice-President of the Turkic World Mathematical society.

In 2004, Prof. M. Otelbaev became a Laureate of the Economic Cooperation Organization in the category “Science and technology”. In 2006 and 2011, he was awarded the state grant “The best university teacher”.

In 2007, Prof. M. Otelbaev was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology.

Almaty, Kazakhstan — Tynysbek Sh. Kalmenov
Astana, Kazakhstan — Erlan D. Nursultanov
London, UK — Michael V. Ruzhansky
Almaty, Kazakhstan — Makhmud A. Sadybekov

References

1. Otelbaev, M.: 2-sided estimates of diameters and their applications. Dokl. Akad. Nauk SSSR. **231**:4, 810–813 (1976)
2. Otelbaev, M.: Bounds for eigenvalues of singular differential operators. Math. Notes **20**:5–6, 1038–1043 (1976)
3. Bliev, N.K., Otelbaev, M.: Dense sets and certain estimates for norms of embedding operator in Banach-spaces. Sib. Math. J. **17**:4, 559–564 (1976)
4. Otelbaev, M.: Divisibility of elliptic operators. Dokl. Akad. Nauk SSSR. **231**:3, 540–543 (1977)
5. Otelbaev, M.: Weight theorems of embedding. Dokl. Akad. Nauk SSSR. **231**:6, 1265–1268 (1977)
6. Levitan, B.M., Otelbaev, M.: Conditions of self-adjointness of schrodinger and dirac operators. Dokl. Akad. Nauk SSSR. **235**:4, 768–771 (1977)
7. Otelbaev, M.: Kolmogorov estimates of diameters for one class of weight spaces. Dokl. Akad. Nauk SSSR. **235**:6, 1270–1273 (1977)
8. Mazya, V.G., Otelbaev, M.: Imbedding theorems and spectrum of a pseudodifferential operator. Siberian Mathematical Journal. **18**:5, 758–770 (1977)
9. Apyshev, O.D., Otelbaev, M.: Spectrum of one class of binomial operators. Dokl. Akad. Nauk SSSR. **248**:2, 265–268 (1979)
10. Otelbaev, M.: Estimates of approximative numbers of operators connected with elliptic equations. Dokl. Akad. Nauk SSSR. **248**:4, 783–787 (1979)
11. Otelbaev, M.: Estimates of s-numbers and completeness conditions for systems of root vectors of non-self-adjoint Sturm-Liouville operators. Math. Notes **25**:3–4, 216–221 (1979)
12. Otelbaev, M.: Criterion for the resolvent of a Sturm-Liouville operator to be kernel. Math. Notes **25**:3–4, 296–297 (1979)
13. Otelbaev, M., Suvorchenkova, G.A.: Necessary and sufficient condition for boundedness and continuity of a certain class of Urysohn operators. Sib. Math. J. **20**:2, 307–310 (1979)
14. Oinarov, R., Otelbaev, M.: Compressibility criterion for the Uryson operator. Dokl. Akad. Nauk SSSR. **255**:6, 1316–1318 (1980)
15. Apyshev, O.D., Otelbaev, M.: On the spectrum of a class of differential-operators and some imbedding theorems. Math. USSR-Izvestiya. **15**:1, 1–24 (1980)
16. Lizorkin, P.I., Otelbaev, M.: Imbedding theorems and compactness for spaces of sobolev type with weights. Math. USSR-Sbornik. **36**:3, 331–349 (1980)

17. Kalmenov, T.S., Otelbaev, M.: Regular boundary-value-problems for the Lavrentev-Bitsadze equation. Differ. Equ. **17**:5, 578–588 (1981)
18. Otelbaev, M.: Asymptotic formulas of eigenvalues of the Sturm-Liouville operator. Dokl. Akad. Nauk SSSR. **259**:1, 42–44 (1981)
19. Otelbaev, M.: A criterion for the coincidence of extensions, corresponding to Dirichlet and Neumann problems, of an elliptic operator. Math. Notes **29**:5–6, 442–446 (1981)
20. Lizorkin, P.I., Otelbaev, M.: Imbedding theorems and compactness for spaces of Sobolev type with weights .2. Math. USSR-Sbornik. **40**:1, 51–77 (1981)
21. Otelbaev, M., Shynybekov A.N.: The correct problems of Bitsadze-Samarskii type. Dokl. Akad. Nauk SSSR. **265**:4, 815–819 (1982)
22. Kokebaev, B.K., Otelbaev, M., Shynybekov, A.N.: Some questions of expansion and restriction of operators. Dokl. Akad. Nauk SSSR. **271**:6, 1307–1310 (1983)
23. Otelbaev, M.: Asymptotic formulas for the eigenvalues of the Sturm-Liouville operator. Siberian Mathematical Journal. **24**:4, 586–598 (1983)
24. Oinarov, R., Otelbaev, M.: Lipschitz and contractiveness criteria for nonlinear integral-operators. Siberian Mathematical Journal. **25**:6, 925–935 (1984)
25. Ospanov, K.N., Otelbaev, M.: Boundary-problems for the generalized Cauchy-Riemann system with nonsmooth coefficients. Dokl. Akad. Nauk SSSR. **283**:1, 46–49 (1985)
26. Otelbaev, M.: The asymptotic of the transport-operator spectrum in the slab geometry. Dokl. Akad. Nauk SSSR. **284**:1, 51–53 (1985)
27. Oinarov, R., Otelbaev, M.: A criterion for a general Sturm-Liouville operator to have a discrete spectrum, and a related imbedding theorem. Differ. Equ. **24**:4, 402–408 (1988)
28. Muratbekov, M.B., Otelbaev, M.: Smoothness and approximate properties of solutions of Schrodinger-type one class nonlinear equations. Izvestiya Vysshikh Uchebnykh Zavedenii Matematika. 3, 44–47 (1989)
29. Ospanov, K.N., Otelbaev, M.: Cauchy-Riemann generalized system with rough coefficients. Izvestiya Vysshikh Uchebnykh Zavedenii Matematika. 3, 48–56 (1989)
30. Biyarov, B.N., Otelbaev, M.: A description of normal extensions. Math. Notes **53**:5–6, 474–478 (1993)
31. Otelbaev, M., Smagulov, S.: On a new method for approximate solution of boundary value problems in arbitrary domains. Dokl. Math. **63**:3, 364–367 (2001)
32. Baldybek, Z., Otelbaev, M., Smagulov, S.: A method for the approximate solution of an initial-boundary value problem for the Navier-Stokes equations. Dokl. Math. **66**:2, 206–209 (2002)
33. Otelbaev, M., Rysbaiuly, B., Suranchiev, A.Z.: Calculation of the eigenvalues and eigenvectors of non-self-conjugate matrices. Dokl. Math. **67**:3, 333–335 (2003)
34. Otelbaev, M., Durmagambetov, A.A., Seitkulov, E.N.: Existence conditions for a global strong solution to one class of nonlinear evolution equations in a Hilbert space. Dokl. Math. **73**:3, 391–393 (2006)
35. Otelbaev, M., Durmagambetov, A.A., Seitkulov, Ye.N.: Conditions for the existence of a global strong solution to a class of nonlinear evolution equations in a Hilbert space. Proc. Steklov Inst. Math. **260**:1, 194–203 (2008)

36. Otelbaev, M., Durmagambetov, A.A., Seitkulov, Ye.N.: Conditions for existence of a global strong solution to one class of nonlinear evolution equations in Hilbert space. Sib. Math. J. **49**:3, 498–511 (2008)
37. Otelbaev, M., Durmagambetov, A.A., Seitkulov, Ye.N.: Conditions for existence of a global strong solution to one class of nonlinear evolution equations in Hilbert space. II. Sib. Math. J. **49**:4, 684–691 (2008)
38. Otelbaev, M., Zhapsarbaeva L.K.: Continuous dependence of the solution of a parabolic equation in a Hilbert space on the parameters and initial data. Diff. Equ. **45**:6, 836–861 (2009)
39. Otelbaev, M., Hasanov, A., Akpayev, B.: Control problem with a point source of heat used as a controller. Dokl. Math. **82**:3, 971–973 (2010)
40. Hasanov, A., Otelbaev, M., Akpayev, B.: Inverse heat conduction problems with boundary and final time measured output data. Inverse Problems in Science and Engineering. **19**:7, 985–1006 (2011)
41. Otelbaev, M.: Examples of equations of Navier-Stokes type not strongly solvable in the large. Math. Notes **89**:5-6, 726–733 (2011)
42. Otelbaev, M., Hasanov, A., Akpayev, B.: A source identification problem related to mathematical model of laser surface heating. Appl. Math. Lett. **25**:10, 1480–1485 (2012)
43. Otelbaev, M.: Existence of a strong solution of the Navier-Stokes equation. Math. J. (Almaty). **13**:4, 5–104 (2013)
44. Kal'menov, T.Sh., Otelbaev, M.: Boundary Criterion for Integral Operators. Doklady Mathematics. **93**:1, 58–61 (2016)
45. Muratbekov, M., Otelbaev, M.: On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. Eurasian Math. J. **7**:1, 50–67 (2016)
46. Burenkov, V.I., Nursultanov, E.D., Kalmenov, T.Sh., Oinarov, R., Otelbaev, M., Tararykova, T.V., Temirkhanova, A.M. and EMJ.: EMJ: from Scopus Q4 to Scopus Q3 in two years?! Eurasian Math. J. **7**:3, 6 (2016)

Professor Erlan Nursultanov: Citation for His 60th Birthday

Professor Erlan Nursultanov is a well-known mathematician whose research achievements and contributions to the mathematical community are very valuable. Specialists in the field of function theory are well aware of works of E. Nursultanov, where he not only solved several important problems, but also created new research methods that are actively used nowadays in the literature. Among his most important achievements one can mention the theory of net spaces. On its basis, he developed a method for obtaining lower bounds for the norms of integral operators, which provides several applications. In particular, E. Nursultanov has significantly extended classical results such as the Hardy-Littlewood theorem on the Fourier coefficients, the Hörmander theorem on Fourier multipliers and O'Neill's theorem on the convolution operator in the multidimensional case.

E. Nursultanov introduced and developed a method of multiparametric interpolation, generalising the real interpolation method of Lyons and Peetre. This method allows solving the problem of reiteration of the real method. He also constructed an interpolation theory for anisotropic function spaces. In particular, the problem of obtaining a Marcinkiewicz-type interpolation theorem for Lebesgue spaces with a mixed metric was solved in this way.

Moreover, E. Nursultanov and A.G. Kostuchenko developed a new theory studying integral operators based on analysis of the singularities of kernels.

This theory provides a new approach for dealing with optimal control of resonance problems.

Professor Nursultanov studied the problem of multipliers of Fourier series, namely, he found a sufficient condition that depends essentially on the parameter p for the multipliers of trigonometric Fourier series in the Lebesgue spaces L^p. He also solved the problem of constructing the optimal recovery operator for classes of functions with dominating mixed derivatives.

Erlan Nursultanov has published more than 150 scientific papers. In 2016, he received the Top Author award from Springer Nature. Moreover, 2 doctoral dissertations, 4 Ph.D. thesa, and 9 Candidate's thesa have been defended under his supervision.

Erlan Nursultanov was born on May 25, 1957, in Karaganda. He graduated from the Faculty of Mathematics of Karaganda State University in 1979. In 1979–1982 he studied at the graduate school of the Moscow State University. In 1983, E. Nursultanov defended his thesis for the degree of the Candidate of Physical and Mathematical Sciences at the Moscow State University. In 1999, he defended his Doctor of Science' thesis at V.A. Steklov Mathematical Institute.

During the period of 1983–1989 Prof. Nursultanov worked at the Department of Mathematical Analysis at the Karaganda State University under E.A. Buketov first as an assistant, then as a senior lecturer, and finally as a docent. He continued working at the department as an adjunct professor until 1999. In the period 1992–1999 he worked as the head of the laboratory "Applied functional analysis" at the Institute of Applied Mathematics. In 1999–2001 he was the Chair of the Department of Mathematics and Methods of Modelling at the Karaganda State University. Since 2001, Prof. Nursultanov has been working at the Head of the Department of Mathematics and Informatics at the Kazakhstan branch of the Moscow State University.

In addition to his valuable contributions to the research, Prof. Nursultanov also contributes to the mathematical community in several other ways. These include organising and participating in the conferences, serving as a member in expert commissions and as an Editorial board member in several mathematical journals. He also participates actively in joint research projects and international collaborations, gives lectures at different international universities and supervises young researchers.

Almaty, Kazakhstan — Tynysbek Sh. Kalmenov
London, UK — Michael V. Ruzhansky
Almaty, Kazakhstan — Makhmud A. Sadybekov

References

1. Nursultanov, E.D.: Certain properties of the roots of a weakly hyperbolic operator pencil. Funct. Anal. Appl. **23**:1, 68–69 (1989)
2. Nursultanov, E.D.: A many-parameter interpolation functor and the Lorentz space $L_{p,q}, q = (q_1, , q_n)$. Funct. Anal. Appl. **31**:2, 136–138 (1997)
3. Nursultanov, E.D.: On a lower bound for a multiplier of the Fourier transform from $M(L_p \to L_q)$. Math. Notes **62**:5–6, 792–796 (1997)
4. Nursultanov, E.D.: Interpolation properties of some anisotropic spaces and Hardy-Littlewood type inequalities. Dedicated to P. L. Ul' yanov. East. J. Approx **4**:2, 243–275 (1998)
5. Nursultanov, E.D., Bekmagambetov, K.A.: The multiparametric interpolation method and embedding theorems for the Besov spaces $B^{\overleftarrow{\alpha}}_{\overleftarrow{p}}[0, 2\pi]$. Anal. Math. **24**:4, 241–263 (1998) (Russian)
6. Nursultanov, E.D.: Network spaces and inequalities of Hardy-Littlewood type. Sb.Math. **189**:3–4, 399–419 (1998)
7. Nursultanov, E.D.: On multipliers of Fourier series in a trigonometric system. Math. Notes **63**:1–2, 205–214 (1998)
8. Kostyuchenko, A.G., Nursultanov, E.D.: Theory of the control of "catastrophes". Rus. Math. Surv. **53**:3, 628–629 (1998)
9. Nursultanov, E.D.: Net spaces and the Fourier transform. (Russian) Dokl. Akad. Nauk. **361**:5, 597–599 (1998)
10. Nursultanov, E.D.: On multipliers of Fourier series in a trigonometric system. Math. Notes **65**:5–6, 801 (1999)
11. Nursultanov, E.D., Kostyuchenko, A.G.: On integral operators in L_p-spaces. Fundam. Prikl. Mat. **5**:2, 475–491 (1999) (Russian)
12. Nursultanov, E.D., Tleukhanova, N.T.: Multipliers of multiple Fourier series. Trudi MIRAN im. V.A. Steklova. **227**, 53–59 (1999) (Russian)
13. Nursultanov, E.D., Tleukhanova, N.T.: Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces. Funct. Anal. Appl. **34**:2, 151–153 (2000)
14. Nursultanov, E.D.: On the coefficients of multiple Fourier series from L_p-spaces. Izv. Math. **64**:1, 93–120 (2000)
15. Nursultanov, E.D., Tleukhanova, N.T.: On the approximate computation of integrals for functions in the spaces $W_p^\alpha[0, 1]^n$. Rus. Math. Surv. **55**:6, 1165–1167 (2000)
16. Nursultanov, E.D., Djumabaeva, D.G.: On the convergence of double Fourier series of functions in $L_p[0, 1]^2$, where $p = (p_1, p_2)$. Anal. Math. **28**:2, 89–101 (2000)
17. Nursultanov, E.D., Saidahmetov, K.S.: On lower bound of the norm of integral convolution operator. Fundam. Prikl. Mat. **8**:1, 141–150 (2002) (Russian)
18. Nursultanov, E.D., Tleukhanova, N.T.: Quadrature formulae for classes of functions of law smoothness. Sbornik Math. **194**:9–10, 1559–1584 (2003)
19. Nursultanov, E.D., Aubakirov, T.U.: Hardy-Littlewood theorem for Fourier series. Math. Notes **73**:3–4, 340–347 (2003)
20. Nursultanov, E.D.: Application of interpolational methods to the study of properties of functions of several variables. Math. Notes **75**:3–4, 341–351 (2004)

21. Nursultanov, E.D.: Interpolation theorems for anisotropic function spaces and their applications. Dokl. Math. **69**:1, 16–19 (2004)
22. Nursultanov, E.D.: The method of net spaces. Mathematical Institute of the Academy of Sciences of the Czech Republic, Praha. 201–212 (2005)
23. Nursultanov, E.D.: Nikol'skii's inequality for different metrics and properties of the sequence of norms of the Fourier sums of a function in the Lorentz Space. Trudi MIRAN im. V.A.Steklova. **255**, 197–215 (2006) (Russian)
24. Aubakirov, T.U., Nursultanov, E.D.: Spaces of stochastic processes, interpolation theorems. Rus. Math. Surv. **61**:6, 1167–1169 (2006)
25. Nursultanov, E.D., Bekmaganbetov, K.A.: On Interpolation and Embedding Theorems for the Spaces B-pT(sigma q)(Omega). Math. Notes **84**:5–6, 733–736 (2008)
26. Nursultanov, E.D., Bekmaganbetov, K.A.: Embedding theorems for anisotropic Besov spaces $B_{pr}^{aq}[0,2p)^n$. Izvestiya Math. **73**:4, 655–668 (2009)
27. Dyachenko, M.I., Nursultanov, E.D.: Hardy-Littlewood theorem for trigonometric series with alpha-monotone coefficients. Sb. Math. **200**:11–12, 1617–1631 (2009)
28. Bekmaganbetov, K.A., Nursultanov, E.D.: Interpolation Besov and Lizorkin-Triebel Spaces. Anal. Math. **35**:3, 169–188 (2009)
29. Nursultanov, E.D., Tikhonov, S., Tleukhanova, N.T.: Norm inequalities for convolution operators. C. R. Math. **347**:23–24, 1385–1388 (2009)
30. Nursultanov, E.D., Aubakirov, T.U.: Interpolation theorem for stochastic processes. Eurasian Math. J. **1**:1, 8–16 (2010)
31. Burenkov, V.I., Nursultanov, E.D.: Description of Interpolation Spaces for Local Morrey-Type Spaces. Proc. Steklov Inst. Math. **269**:1, 46–56 (2010)
32. Nursultanov, E.D., Kopezhanova, A.N., Persson, L.E.: Relations between summability of the Fourier coefficients in regular systems and functions from some Lorentz type spaces. Proc. A. Razmadze Math. Inst. **152**, 73–88 (2010)
33. Nursultanov, E.D., Tikhonov, S.: Net Spaces and Boundedness of Integral Operators. J. Geom. Anal. **21**:4, 950–981 (2011)
34. Nursultanov, E.D., Tikhonov, S.U.: Convolution inequalities in Lorentz spaces. J. Fourier Anal. Appl. **17**:3, 486–505 (2011)
35. Kopezhanova, A.N., Nursultanov, E.D., Persson, L.-E.: On inequalities for the Fourier transform of functions from Lorentz spaces. Math. Notes **90**:5–6, 767–770 (2011)
36. Nursultanov, E., Aubakirov, T.: Interpolation methods for stochastic processes spaces. Abstr. Appl. Anal. Art. ID 152043, (2013)
37. Nursultanov, E., Tikhonov, S.: A sharp Remez inequality for trigonometric polynomials. Constr. Approx. **38**:1, 101–132 (2013)
38. Burenkov, V.I., Darbayeva, D.K., Nursultanov, E.D.: Description of interpolation spaces for general local Morrey-type spaces. Eurasian Math. J. **4**:1, 46–53 (2013)
39. Dyachenko, M.I., Nursultanov, E.D., Zhantakbayeva, A.M.: Hardy-Littlewood type theorems. Eurasian Math. J. **4**:2, 140–143 (2013)
40. Dyachenko, M.I., Nursultanov, E.D., Tikhonov, S.Yu.: Global and local smoothness of Hilbert transforms. Proc. Steklov Inst. Math. **280**:1, 169–180 (2013)

41. Burenkov, V.I., Nursultanov, E.D., Chigambayeva, D.: Description of the Interpolation Spaces for a Pair of Local Morrey-Type Spaces and Their Generalizations. Proc. Steklov Inst. Math. **284**:1, 97–128 (2014)
42. Zhantakbaeva, A.M., Nursultanov, E.D.: Paley Inequality for Bellman Transform of Multiple Fourier Series. Moscow University Mathematics Bulletin. **69**:3, 94–99 (2014)
43. Dyachenko, M., Nursultanov, E., Kankenova, A.: On summability of Fourier coefficients of functions from Lebesgue space. J. Math. Anal. Appl. **419**:2, 959–971 (2014)
44. Nursultanov, E.D., Tleukhanova, N.T.: On the recovery of multiplicative transformations of functions from anisotropic spaces. Sibirsk. Mat. Zh. **55**:3, 482–497 (2014)
45. Nursultanov, E.D., Ruzhansky, M.V., Tikhonov, S.Yu.: Nikolskii inequality and functional classes on compact Lie groups. Func. Anal. Appl. **49**:3, 226–229 (2015)
46. Nursultanov, E., Tikhonov, S.: Weighted norm inequalities for convolution and Riesz potential. Potential Anal. **42**:2, 435–456 (2015)
47. Dyachenko, M.I., Nursultanov, E.D., Nursultanov, M.E.: The Hardy-Littlewood theorem for multiple Fourier series with monotone coefficients. Math. Notes **99**:3–4, 503–510 (2016)
48. Akylzhanov, R., Nursultanov, E., Ruzhansky, M.: Hardy-Littlewood-Paley inequalities and Fourier multipliers on SU(2). Studia Mathematica. **234**:1, 1–29 (2016)
49. Nursultanov, E.D., Ruzhansky, M.V., Tikhonov, S.Yu.: Nikolskii Inequality and Besov, Tribel-Lizorkin and Beurling Spaces son Compact Homogeneous manifolds. Ann. Sc. Norm. Super. Pisa CI. Sci (5). **16**:3, 981–1017 (2016)
50. Akylzhanov, R., Nursultanov, E., Ruzhansky, M.: Hardy-Littlewood-Paley-type inequalities on compact Lie groups. Math. Notes **100**:1–2, 309–312 (2016)
51. Burenkov, V.I., Nursultanov, E.D., Kalmenov, T.Sh., et al. EMJ: from Scopus Q4 to Scopus Q3 in two years?!. Eurasian Math. J. **7**:3, 6–6 (2016)
52. Nursultanov, E., Persson, L.E., Kopezhanova, A.: Some new two-sided inequalities concerning the Fourier transform. Math. Inequal. Appl. **20**:3, 855–864 (2017)
53. Dzhumabaeva, D.G., Dyachenko, M.I., Nursultanov, E.D.: On convergence of multiple trigonometric series with monotone coefficients. Sibirsk. Mat. Zh. **58**:2, 205–214 (2017)
54. Dyachenko, M., Mukanov, A., Nursultanov, E.: A Boas-type theorem for alpha-monotone functions. Math. Scand. **120**:1, 39–58 (2017)

Part I

Theory of Functions and Functional Spaces

On Evolution of Invariant Riemannian Metrics on Generalized Wallach Spaces Under the Normalized Ricci Flow

Nurlan Abiev(✉)

M. Kh. Dulaty Taraz State University, 60 Tole bi st., Taraz 080000, Kazakhstan
abievn@mail.ru

Abstract. The aim of this paper is to discuss some results of [2,3] relating to the study of the evolution of invariant Riemannian metrics on generalized Wallach spaces with $a_1 = a_2 = a_3 = a$, where $a \in (0, 1/2)$. We proved that for the Wallach spaces $SU(3)/T_{\max}$, $Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$, and $F_4/Spin(8)$, the normalized Ricci flow evolves all generic invariant Riemannian metrics with positive sectional curvature into metrics with mixed sectional curvature. Moreover, we obtained general results concerning the evolution of invariant Riemannian metrics on generalized Wallach spaces with $a \in (0, 1/2) \setminus \{1/4\}$ under the normalized Ricci flow. The very special case $a = 1/4$ is also considered.

Keywords: Wallach space · Generalized Wallach space · Riemannian metric · Normalized Ricci flow · Sectional curvature · Ricci curvature · Scalar curvature · Panar dynamical system · Singular point · Bifurcation value

Introduction and the Main Results

The study of evolution of a 1-parameter family of Riemannian metrics $\boldsymbol{g}(t)$ in a Riemannian manifold $\mathscr{M}^n$ under the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\boldsymbol{g}(t) = -2\,\mathrm{Ric}_{\boldsymbol{g}} + 2\boldsymbol{g}(t)\frac{S_{\boldsymbol{g}}}{n}\,, \tag{1}$$

where $\mathrm{Ric}_{\boldsymbol{g}}$ and $S_{\boldsymbol{g}}$ are the Ricci tensor and the scalar curvature of the Riemannian metric $\boldsymbol{g}$, respectively, was initiated by R. Hamilton in [17], and since then it has been continuing successfully in the case of homogeneous spaces, where one of the important problems is to investigate whether or not the positiveness of the sectional curvature or positiveness of the Ricci curvature of Riemannian metrics is preserved under the (normalized) Ricci flow [9,17]. A recent survey on the evolution of positively curved Riemannian metrics under the Ricci flow could be found in [23]. Interesting results on the evolution of invariant

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_1

Riemannian metrics could also be found in the papers [10–12, 18, 20, 21, 27, 29] and the references therein.

Similar questions were considered in [2, 3] with respect to the *Wallach spaces*

$$\begin{aligned} &W_6 := SU(3)/T_{\max}, \\ &W_{12} := Sp(3)/Sp(1) \times Sp(1) \times Sp(1), \\ &W_{24} := F_4/Spin(8) \end{aligned} \tag{2}$$

that admit invariant Riemannian metrics of positive sectional curvature [29] and *generalized Wallach spaces* (or *three-locally-symmetric spaces*) which are characterized as compact homogeneous spaces G/H whose isotropy representation decomposes into a direct sum $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ of three $\mathrm{Ad}(H)$-invariant irreducible modules satisfying $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$, $i \in \{1, 2, 3\}$ (see the original papers [22, 24, 26] for definitions and details). Every generalized Wallach space can be characterized by a triple of real numbers $a_i := A/d_i \in [0, 1/2]$, $i = 1, 2, 3$, where A is a non negative number, $d_i := \dim(\mathfrak{p}_i)$. Note that $a_1 = a_2 = a_3 =: a$ and $\dim(\mathfrak{p}_1) = \dim(\mathfrak{p}_2) = \dim(\mathfrak{p}_3) =: \mathbf{d}$ for the Wallach spaces W_6, W_{12}, and W_{24}. Moreover, for these spaces, a is equal to 1/6, 1/8, 1/9 and $\mathbf{d}$ is equal to 2, 4, 8, respectively. It should also be noted that the classification of generalized Wallach spaces is obtained recently (independently) in the papers [13, 25]. Moreover, the classification suggested by [25] is complete, whereas [13] assumes simpleness of the Lie group G.

For a fixed bi-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of the Lie group G, any G-invariant Riemannian metric $\boldsymbol{g}$ on G/H is determined by an $\mathrm{Ad}(H)$-invariant inner product

$$(\cdot, \cdot) = x_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + x_2 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_2} + x_3 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_3}, \tag{3}$$

where x_1, x_2, x_3 are positive real numbers. Therefore, the space of such metrics is 2-dimensional up to a scale factor. Any metric with $x_1 = x_2 = x_3$ is called *normal*, whereas the metric with $x_1 = x_2 = x_3 = 1$ is called *standard* or *Killing*. The subspace of invariant metrics satisfying $x_i = x_j$ for some $i \neq j$, is invariant under the normalized Ricci flow, because these special metrics have a larger connected isometry group. Indeed, such a metric (x_1, x_2, x_3) admits additional isometries generated by the right action of the group $K \subset G$ with the Lie algebra $\mathfrak{k} := \mathfrak{h} \oplus \mathfrak{p}_k$, $\{i, j, k\} = \{1, 2, 3\}$, see details in [25]. All such metrics are related to the above mentioned submersions of the form $K/H \to G/H \to G/K$, coming from inclusions $H \subset K \subset G$, see e. g. [9, Chapter 9]. In what follows we call these metrics *exceptional* or *submersion metrics*. These metrics constitute three one-parameter families up to a homothety. All other metrics we call *generic* or *non-exceptional*. Note that the Wallach spaces (2) are the total spaces of the following submersions: $S^2 \to W_6 \to \mathbb{CP}^2$, $S^4 \to W_{12} \to \mathbb{HP}^2$, $S^8 \to W_{24} \to \mathrm{Ca}\mathbb{P}^2$. The first main result obtained in [3] is the following

Theorem 1. *On the Wallach spaces W_6, W_{12}, and W_{24}, the normalized Ricci flow evolves all generic metrics with positive sectional curvature into metrics with mixed sectional curvature.*

Moreover, the normalized Ricci flow removes every generic metric from the set of metrics with positive sectional curvature in a finite time and does not return it back to this set. This finite time depends of the initial points and could be as long as we want, see details in Section 2.1.

Theorem 1 easily implies the following result obtained in [14]: on the Wallach spaces W_6, W_{12}, and W_{24}, the normalized Ricci flow evolves some metrics with positive sectional curvature into metrics with mixed sectional curvature.

The second main result of [3] is related to the evolution of metrics with positive Ricci curvature.

Theorem 2. *On the Wallach spaces W_{12} and W_{24}, the normalized Ricci flow evolves all generic metrics with positive Ricci curvature into metrics with mixed Ricci curvature.*

Moreover, the normalized Ricci flow removes every generic metric from the set of metrics with positive Ricci curvature in a finite time and does not return it back to this set. This finite time depends of the initial points and could be as long as we want. Note also that the normalized Ricci flow can evolve some metrics with mixed Ricci curvature to metrics with positive Ricci curvature. Moreover, there is a non-extendable integral curve of the normalized Ricci flow with exactly one metric of non-negative Ricci curvature, see details in Section 2.2.

In the paper [11], C. Böhm and B. Wilking studied (in particular) some properties of the (normalized) Ricci flow on the Wallach space W_{12}. They proved that the (normalized) Ricci flow on W_{12} evolves certain positively curved metrics into metrics with mixed Ricci curvature, see [11, Theorem 3.1]. The same assertion for the space W_{24} obtained by M. W. Cheung and N. R. Wallach in [14, Theorem 3]. On the other hand, it was proved in [14, Theorem 8] that every invariant metric with positive sectional curvature on the space W_6 retains positive Ricci curvature under the Ricci flow. Hence, Theorem 2 fails for W_6. Note also that for some invariant metrics with positive Ricci curvature on W_6, the Ricci flow can evolve them to metrics with mixed Ricci curvature, see [14, Theorem 3] or Remark 6 below. The principal difference between W_6 and two other Wallach spaces is explained in Lemma 5 and Remark 4. We emphasize that the special status of W_6 follows from Proposition 1 and the description of the boundary of R, the set of metrics with positive Ricci curvature (19). In [3] the following theorem was also proved demonstrating that Theorem 2 can be extended to some other generalized Wallach spaces.

Theorem 3. *Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 =: a$, where $a \in (0, 1/4) \cup (1/4, 1/2)$. If $a < 1/6$, then the normalized Ricci flow evolves all generic metrics with positive Ricci curvature into metrics with mixed Ricci curvature. If $a \in (1/6, 1/4) \cup (1/4, 1/2)$, then the normalized Ricci flow evolves all generic metrics into metrics with positive Ricci curvature.*

For instance, the spaces $Sp(3k)/Sp(k) \times Sp(k) \times Sp(k)$ correspond to the case $a = \frac{k}{6k+2} < 1/6$, whereas the spaces $SO(3k)/SO(k) \times SO(k) \times SO(k)$, $k > 2$, correspond to the case $1/6 < a = \frac{k}{6k-4} < 1/4$. Note also that $SO(3)$

correspond to $a = 1/2$, the maximal possible value for $a = a_1 = a_2 = a_3$, see details in [4,5]. It is interesting also that $1/9$ is the minimal possible value for $a = a_1 = a_2 = a_3$ among non-symmetric generalized Wallach spaces, see [25]. It should also be noted that there are many generalized Wallach spaces with $a = 1/6$, for example, the spaces $SU(3k)/S(U(k) \times U(k) \times U(k))$. All these spaces are Kähler C-spaces, see [25]. The following result was obtained in [3], that generalizes Theorem 8 in [14].

Theorem 4. *Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 = 1/6$. Suppose that it is supplied with the invariant Riemannian metric* (3) *such that $x_k < x_i + x_j$ for all indices with $\{i, j, k\} = \{1, 2, 3\}$, then the normalized Ricci flow on G/H with this metric as the initial point, preserves the positivity of the Ricci curvature.*

It should be noted that $x_k = x_i + x_j$ is just the unstable manifold of the Kähler – Einstein metric for all generalized Wallach spaces with $a = 1/6$.

And, finally, the very special case of generalized Wallach spaces was studied in [2] corresponding to $a = 1/4$. In [2] the following theorem was proved

Theorem 5. *Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 = 1/4$. Then the normalized Ricci flow evolves all generic metrics into metrics with positive Ricci curvature.*

According to [25, Theorem 1] infinitely many generalized Wallach spaces correspond to $a_1 = a_2 = a_3 := a = 1/4$, more precisely if $G = F \times F \times F \times F$ and $H = \operatorname{diag}(F) \subset G$ for some connected and simply connected compact simple Lie group F, then G/H is a generalized Wallach space corresponding to the value $a = 1/4$ and having the following description on the Lie algebra level $(\mathfrak{g}, \mathfrak{h}) = \big(\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f},\ \operatorname{diag}(\mathfrak{f}) = \{(X, X, X, X) \mid X \in \mathfrak{f}\}\big)$, where $\mathfrak{f}$ is the Lie algebra of F and (up to permutation) $\mathfrak{p}_1 = \big\{(X, X, -X, -X) \mid X \in \mathfrak{f}\big\}$, $\mathfrak{p}_2 = \big\{(X, -X, X, -X) \mid X \in \mathfrak{f}\big\}$, $\mathfrak{p}_3 = \big\{(X, -X, -X, X) \mid X \in \mathfrak{f}\big\}$. Other example of a generalized Wallach space with $a = 1/4$ is $SO(6)/SO(2) \times SO(2) \times SO(2)$.

The paper is organized as follows: In Sect. 1 we reduce the normalized Ricci flow equation (1) to the system of ODE's (ordinary differential equations) (10). In Sect. 2 we demonstrate the main idea of proving Theorems 1–4 based on the detailed description of metrics admitting positive sectional or positive Ricci curvature and the analysis of asymptotical behavior of solutions of the system (10). The visual illustrations of results will also be given. In Sect. 3 we discuss the special case $a = 1/4$ and expose proof of Theorem 5 briefly.

1 Preliminaries

1.1 Some Basic Facts from the Theory of ODE's

Consider a dynamical system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{4}$$

where P and Q are real polynomials of degree $\leq m$. A point $(x^0, y^0) \in \mathbb{R}^2$ is said to be a *singular point* of the system (4), if $P(x^0, y^0) = Q(x^0, y^0) = 0$. Such (x^0, y^0) is called *degenerate*, if the Jacobian matrix $J = J(x^0, y^0)$ of the system (4) evaluated at (x^0, y^0) has at least one zero eigenvalue. The singular point (x^0, y^0) is called *linearly zero* if $J = 0$, see [16].

A differentiable function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be an *invariant* of (4), if there exist a polynomial $k(x, y)$ of degree $\leq (m-1)$ such that

$$P(x,y)\,\frac{\partial \varphi(x,y)}{\partial x} + Q(x,y)\,\frac{\partial \varphi(x,y)}{\partial y} = k(x,y)\,\varphi(x,y).$$

The polynomial k is called a *co-factor* of the invariant φ. A curve defined by $\varphi(x, y) = 0$ is called an *invariant curve* of (4). If $k \equiv 0$, then φ is a *first integral* of the system (4). Curves, defined by $P(x, y) = 0$ and $Q(x, y) = 0$, are called the *main isoclines* of the system (4).

1.2 Reduction of the Normalized Ricci Flow Equation to a System of ODE's

For the Ricci operator Ric and the scalar curvature S of the metric (3) the following expressions are known in the case of generalized Wallach spaces

$$\begin{aligned} \mathrm{Ric} &= \boldsymbol{r_1}\ \mathrm{Id}|_{\mathfrak{p}_1} + \boldsymbol{r_2}\ \mathrm{Id}|_{\mathfrak{p}_2} + \boldsymbol{r_3}\ \mathrm{Id}|_{\mathfrak{p}_3}, \\ S &= d_1\boldsymbol{r_1} + d_2\boldsymbol{r_2} + d_3\boldsymbol{r_3}, \end{aligned}$$

where the principal Ricci curvatures $\boldsymbol{r_1}, \boldsymbol{r_2}$ and $\boldsymbol{r_3}$ can be evaluated by the formulas

$$\mathbf{r_i} = \frac{1}{2x_i} + \frac{a_i}{2}\left(\frac{x_i}{x_j x_k} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_i x_k}\right), \quad \{i,j,k\} = \{1,2,3\}, \tag{5}$$

obtained in Lemma 2 of [24].

By using the above equalities and taking into account the equality $n = d_1 + d_2 + d_3$, the (volume) normalized Ricci flow equation (1) can be reduced to a system of ODE's of the following form

$$\frac{dx_i}{dt} = F_i := -2x_i\left(\mathbf{r_i} - \frac{S}{n}\right), \quad i = 1,2,3. \tag{6}$$

Observe that singular points of the system (6) are identical to invariant Einstein metrics of the space under consideration. Indeed, if (x_1, x_2, x_3) is a singular point of (6), then $\boldsymbol{r_1} = \boldsymbol{r_2} = \boldsymbol{r_3}$. The converse is obvious.

It suffices to consider only invariant metrics with

$$\mathrm{Vol} := x_1^{1/a_1} x_2^{1/a_2} x_3^{1/a_3} \equiv 1,$$

because the metric (3) has the same volume as the standard metric if and only if $\mathrm{Vol} \equiv 1$. Indeed, the case of general volume is reduced to this one by a suitable homothety. This observation is the main argument to apply the (volume) normalized Ricci flow instead of the non-normalized Ricci flow in the case of the Wallach spaces, as far as in the case of generalized Wallach spaces (see details e. g. on pp. 259–260 of [17] and in [4,5]).

It is easy to show that the function $\mathrm{Vol} := x_1^{1/a_1} x_2^{1/a_2} x_3^{1/a_3}$ is a first integral of (6). Therefore, we can reduce (6) to the following system of two differential equations on the surface $\mathrm{Vol} \equiv 1$, see details in [4,5]:

$$\frac{dx_1}{dt} = F_1\big(x_1, x_2, \varphi(x_1, x_2)\big), \quad \frac{dx_2}{dt} = F_2\big(x_1, x_2, \varphi(x_1, x_2)\big), \tag{7}$$

where $\varphi(x_1, x_2) := x_1^{-\frac{a_3}{a_1}} x_2^{-\frac{a_3}{a_2}}$.

Assume further that $a_1 = a_2 = a_3 := a$. Then (7) takes the form

$$\begin{aligned} \tfrac{dx_1}{dt} &= x_1 x_2^{-1} + x_2 x_1^2 - 2 - 2a x_1\big(2x_1^2 - x_2^2 - x_1^{-2} x_2^{-2}\big), \\ \tfrac{dx_2}{dt} &= x_2 x_1^{-1} + x_1 x_2^2 - 2 - 2a x_2\big(2x_2^2 - x_1^2 - x_1^{-2} x_2^{-2}\big). \end{aligned} \tag{8}$$

For our goals we need also a system of ODE's obtaining in scale invariant variables

$$w_1 := \frac{x_3}{x_1}, \quad w_2 := \frac{x_3}{x_2}. \tag{9}$$

Observing that

$$\frac{1}{w_i}\frac{dw_i}{dt} = \frac{1}{x_3}\frac{dx_3}{dt} - \frac{1}{x_i}\frac{dx_i}{dt} = -2(\mathbf{r_3} - \mathbf{r_i}), \quad i = 1, 2,$$

the system (6) can be reduced to the following system for $w_1 > 0$ and $w_2 > 0$:

$$\begin{aligned} \tfrac{dw_1}{dt} &= x_3^{-1}(w_1 - 1)(w_1 - 2aw_1w_2 - 2aw_2), \\ \tfrac{dw_2}{dt} &= x_3^{-1}(w_2 - 1)(w_2 - 2aw_1w_2 - 2aw_1). \end{aligned}$$

Since the preceding system is autonomous we can introduce the new time-parameter $t := t/x_3$ not changing integral curves and their orientation ($x_3 > 0$). Then we get

$$\begin{aligned} \tfrac{dw_1}{dt} &= f(w_1, w_2) := (w_1 - 1)(w_1 - 2aw_1w_2 - 2aw_2), \\ \tfrac{dw_2}{dt} &= g(w_1, w_2) := (w_2 - 1)(w_2 - 2aw_1w_2 - 2aw_1). \end{aligned} \tag{10}$$

Since the result of reducing of the system (10) does not depend on the concrete value of x_3, we can put $x_3 = x_1^{-1} x_2^{-1}$ according to $\mathrm{Vol} \equiv 1$. This assumption and (9) imply that there exist the following homeomorphism between the coordinate systems (x_1, x_2) and (w_1, w_2)

$$(x_1, x_2) \mapsto (w_1, w_2) := \big(x_1^{-2} x_2^{-1},\, x_1^{-1} x_2^{-2}\big), \tag{11}$$

which provides the topological equivalence of phase portraits of the systems (8) and (10) at every fixed value of the parameter a.

1.2.1 Invariant Curves and Main Isoclines of the System (10)

The following straight lines c_1, c_2 and c_3 determined by the equations

$$w_1 = 1, \quad w_2 = 1, \quad w_2 = w_1,$$

respectively, are invariant for the system (10). Indeed, it suffices to note that the polynomial $-2aw_1w_2 - 2a + w_1 + w_2 - 1$ is a co-factor for $w_2 = w_1$. Co-factors of the other lines are obvious. Note that the curves c_1, c_2 and c_3 have the common point $E_0 = (1, 1)$ and separate the domain $(0, \infty)^2$ into 6 connected invariant components (see Fig. 1).

The study of normalized Ricci flow in each pair of these components are equivalent due to the following property of the Wallach spaces: there is a finite group of isometries fixing the isotropy and permuting the modules $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$. Therefore, it suffices to study solutions of (10) with initial points given only in the following set

$$\Omega := \left\{(w_1, w_2) \in \mathbb{R}^2 \mid w_2 > w_1 > 1\right\}. \tag{12}$$

It follows directly from (11) that on the plane (x_1, x_2) the curves

$$x_2 = x_1^{-2}, \quad x_1 = x_2^{-2}, \quad x_2 = x_1$$

correspond to c_1, c_2 and c_3 being the invariant sets of the system (8). Clearly, these curves have the common point $F := (1, 1)$ and separate the domain $\left\{(x_1, x_2) \mid x_i > 0\right\}$ into 6 connected invariant components.

A simple analysis of the right hand sides of the system (10) provides elementary tools for studying the behavior of its integral curves. For instance, we can

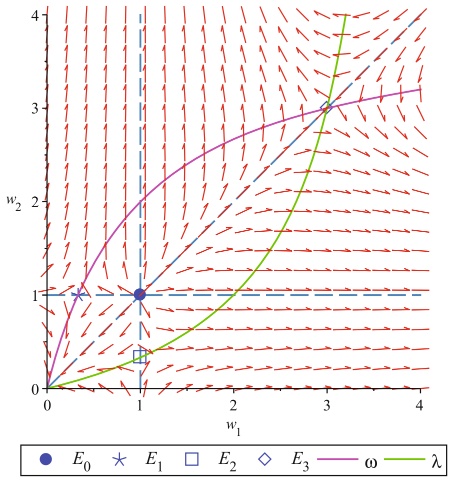


Fig. 1. The case $a = 1/8$: The singular points E_0, E_1, E_2, E_3, the isoclines ω, λ and the phase portrait of the system (10)

predict the slope of integral curves of (10) in Ω and interpret them geometrically. According to this observations, let us consider the main isoclines (10) (see Fig. 1)

$$\omega := \left\{(w_1, w_2) \in \mathbb{R}_+^2 \mid y_1 := w_1 - 2aw_1w_2 - 2aw_2 = 0\right\},$$
$$\lambda := \left\{(w_1, w_2) \in \mathbb{R}_+^2 \mid y_2 := w_2 - 2aw_1w_2 - 2aw_1 = 0\right\},$$

where $\mathbb{R}_+$ means the set of positive real numbers.

Let $(w_1(t), w_2(t))$ be any integral curve of (10) given in Ω. Then $w_1' < 0$ (respectively $w_1' > 0$) over (respectively under) ω and $w_2' > 0$ (respectively $w_2' < 0$) over (respectively under) λ. Clearly, $w_1' = 0$ on ω and $w_2' = 0$ on λ.

1.2.2 Singular Points of the System (10) at $a \neq 1/4$

The following lemma can be easily proved by direct calculations (see the left panel of Figure 1)

Lemma 1. *Let $a \neq 1/4$. Then the system (10) has exactly four singular points $E_0 = (1,1)$, $E_1 = (q,1)$, $E_2 = (1,q)$, $E_3 = \left(q^{-1}, q^{-1}\right)$, which are non degenerate, where $q := 2a(1-2a)^{-1}$. Moreover, E_1, E_2 and E_3 are saddles and E_0 is an unstable node.*

1.2.3 Asymptotic Behavior of Solutions of the System (10)

Now we should study integral curves of (10) in Ω estimating their "curvature" as $w_1 \to 1+0$ and $w_2 \to +\infty$ more precisely. Since $f \neq 0$ in Ω, any solution $(w_1(t), w_2(t))$ of the system (10) represents some differentiable function $w_2 = \phi(w_1)$ being the unique solution of the following initial value problem

$$\frac{dw_2}{dw_1} = \frac{g}{f} = \frac{(w_2-1)(w_2 - 2aw_1w_2 - 2aw_1)}{(w_1-1)(w_1 - 2aw_1w_2 - 2aw_2)}, \qquad w_2|_{w_1=w_1^0} = w_2^0, \tag{13}$$

where

$$\frac{g}{f} \to -\infty \quad \text{as} \quad w_1 \to 1+0 \quad \text{and} \quad w_2 \to +\infty.$$

We are going to reformulate revised versions of two important statements formulated and proved in [3].

Lemma 2. *Let $w_2 = \phi(w_1)$ be a solution of (13), where $(w_1, w_2) \in \Omega$. Then for any small $\varepsilon > 0$ there exist constants $C_1, C_2 > 0$ such that*

$$C_1(w_1-1)^{-\frac{(1-\varepsilon)(1-2a)}{4a}} \leq \phi(w_1) \leq C_2(w_1-1)^{-\frac{(1+\varepsilon)(1-2a)}{4a}}$$

for w_1 sufficiently close to 1 and $w_1 > 1$.

Proof. An easy analysis shows that

$$\lim_{\substack{w_1\to 1+0\\ w_2\to+\infty}} \frac{4a}{2a-1}\,\frac{w_1-1}{w_2}\,\frac{g}{f}=1.$$

Therefore, by definition of the limit for any sufficiently small $\varepsilon>0$ there exists $\delta>0$ such that

$$1-\varepsilon \le \frac{4a}{2a-1}\,\frac{w_1-1}{w_2}\,\frac{dw_2}{dw_1}\le 1+\varepsilon$$

for all $1<w_1<1+\delta$ and $w_2>\frac{1}{\delta}$. Taking w_1' and w_1'' close to 1 (assuming $1<w_1'<w_1''<1+\delta$) and integrating the preceding inequalities on the segment $[w_1', w_1'']$, we get

$$(1-\varepsilon)\int\limits_{w_1'}^{w_1''}\frac{dw_1}{w_1-1}\le \frac{4a}{2a-1}\int\limits_{\phi(w_1')}^{\phi(w_1'')}\frac{dw_2}{w_2}\le (1+\varepsilon)\int\limits_{w_1'}^{w_1''}\frac{dw_1}{w_1-1}$$

which is equivalent to

$$\left(\frac{w_1''-1}{w_1'-1}\right)^{1-\varepsilon}\le\left(\frac{w_2''}{w_2'}\right)^{\frac{4a}{2a-1}}\le\left(\frac{w_1''-1}{w_1'-1}\right)^{1+\varepsilon}.$$

This means that for any small $\varepsilon>0$ there exist constants $C_1, C_2>0$ such that

$$C_1(w_1-1)^{-(1-\varepsilon)}\le w_2^{\frac{4a}{1-2a}}\le C_2(w_1-1)^{-(1+\varepsilon)}$$

for w_1 sufficiently close to 1 (at fixed w_1'' and $w_1':=w_1$). □

Proposition 1. *Suppose that a curve γ determined in Ω by an equation $w_2:=\psi(w_1)$ satisfies the asymptotic equality*

$$\psi(w_1)\sim(w_1-1)^{-\alpha}\quad as\quad w_1\to 1+0,$$

where $\alpha>0$. Then the following assertion holds: if $\frac{1-2a}{4a}<\alpha$ (respectively, $\frac{1-2a}{4a}>\alpha$), then every integral curve $w_2=\phi(w_1)$ of (13) in Ω lies under (respectively, over) γ for sufficiently large t.

Proof. Recall that $w_1\to 1+0$ and $w_2\to+\infty$ as $t\to+\infty$ on every integral curve of (10) originated in Ω. Let us introduce the function

$$\Phi(t):=\frac{\phi(w_1(t))}{\psi(w_1(t))}.$$

Note that Φ is continuous for all t. In Lemma 2 we may take $\varepsilon>0$ such that $\varepsilon<\left|1-\frac{4a\alpha}{1-2a}\right|$.

If $\frac{1-2a}{4a} < \alpha$, then $\frac{(1+\varepsilon)(1-2a)}{4a} < \alpha$. This means that

$$\lim_{t\to+\infty} \Phi(t) = \lim_{w_1\to 1+0} \frac{\phi(w_1)}{\psi(w_1)} = \lim_{w_1\to 1+0} \frac{\phi(w_1)}{(w_1-1)^{-\alpha}} \le C_2 \lim_{w_1\to 1+0} \frac{(w_1-1)^{-\frac{(1+\varepsilon)(1-2a)}{4a}}}{(w_1-1)^{-\alpha}} = 0.$$

Since $\Phi(t) > 0$ for all $t \ge 0$, it follows that $\lim_{t\to+\infty} \Phi(t) = 0$. By definition of the limit there exists sufficiently large $\Delta > 0$ such that $\Phi(t) < 1$ whenever $t > \Delta$. The case when the initial point (w_1^0, w_2^0) of the trajectory lies under γ is obvious. Suppose that (w_1^0, w_2^0) lies over γ. Then $\Phi(0) > 1$, hence by the intermediate value theorem there exists $t = T \in (0, \Delta]$ such that $\Phi(T) = 1$ and $\Phi(t) < 1$ for all $t > T$. In other words, for any integral curve $w_2 = \phi(w_1)$ initiated from $\Omega \setminus R$ there exists a finite time $t = T$ such that $w_2 = \phi(w_1)$ intersects the curve γ from up to down and lies *under* γ for all $t > T$ (for all w_1 sufficiently close to 1 and $w_1 > 1$).

Assume now $\frac{1-2a}{4a} > \alpha$. Then $\frac{(1-\varepsilon)(1-2a)}{4a} > \alpha$. Similarly, this means that

$$+\infty = C_1 \lim_{w_1\to 1+0} \frac{(w_1-1)^{-\frac{(1-\varepsilon)(1-2a)}{4a}}}{(w_1-1)^{-\alpha}} \le \lim_{w_1\to 1+0} \frac{\phi(w_1)}{(w_1-1)^{-\alpha}}$$

and the integral curve lies *over* the curve γ for all w_1 sufficiently close to 1 and $w_1 > 1$. □

1.3 The set D of Invariant Metrics with Positive Sectional Curvature

A detailed description of invariant metrics of positive sectional curvature on the Wallach spaces (2) was given by F. M. Valiev in [28]. We reformulate his results in our notation. Let us fix a Wallach space G/H (i. e. consider $a = 1/6$, $a = 1/8$, or $a = 1/9$). Recall that we deal with only positive x_i. Let us consider the functions

$$\gamma_i = \gamma_i(x_1, x_2, x_3) := (x_j - x_k)^2 + 2x_i(x_j + x_k) - 3x_i^2,$$

where $\{i, j, k\} = \{1, 2, 3\}$. Note that under the restrictions $x_i > 0$, the equations $\gamma_i = 0$, $i = 1, 2, 3$, determine cones congruent each to other under the permutation $i \to j \to k \to i$. Note also that these cones have the empty intersections pairwise.

According to results of [28] and the symmetry in γ_1, γ_2, and γ_3 under permutations of x_1, x_2, and x_3, the set of metrics with non-negative sectional curvature is the following:

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3_+ \mid \gamma_1 \ge 0,\ \gamma_2 \ge 0,\ \gamma_3 \ge 0\right\}. \qquad (14)$$

By Theorem 3 in [28] and the above mentioned symmetry, the set of metrics with positive sectional curvature is the following:

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3_+ \mid \gamma_1 > 0,\ \gamma_2 > 0,\ \gamma_3 > 0\right\} \setminus \left\{(t, t, t) \in \mathbb{R}^3 \mid t > 0\right\}. \qquad (15)$$

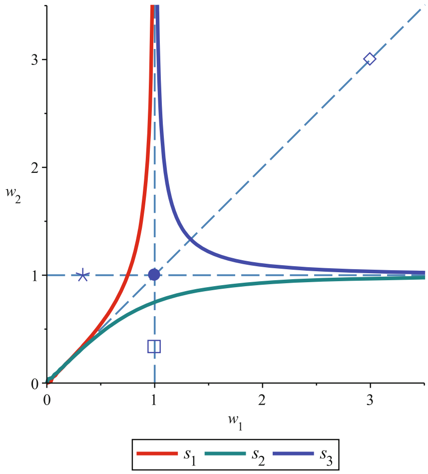


Fig. 2. The curves s_1, s_2, s_3 corresponding to the system (10)

Let us describe the domain D in the coordinates (w_1, w_2). Denote by s_i curves on the plane (w_1, w_2) determined by the equations $\gamma_i\bigl(\frac{1}{w_1}, \frac{1}{w_2}, 1\bigr) = 0$ (see Fig. 2).

For $w_1 > 0$ and $w_2 > 0$, these equations are respectively equivalent to

$$
\begin{aligned}
l_1 &:= \quad w_1^2w_2^2 - 2w_1^2w_2 + 2w_1w_2^2 + w_1^2 + 2w_1w_2 - 3w_2^2 = 0,\\
l_2 &:= \quad w_1^2w_2^2 + 2w_1^2w_2 - 2w_1w_2^2 - 3w_1^2 + 2w_1w_2 + w_2^2 = 0, \qquad (16)\\
l_3 &:= -3w_1^2w_2^2 + 2w_1^2w_2 + 2w_1w_2^2 + w_1^2 - 2w_1w_2 + w_2^2 = 0.
\end{aligned}
$$

It is easy to check that (14) is a connected set with a boundary consisting of the union of the cones $\gamma_1 = 0, \gamma_2 = 0$ and $\gamma_3 = 0$. Therefore, solving the system of inequalities $\gamma_i\bigl(\frac{1}{w_1}, \frac{1}{w_2}, 1\bigr) > 0$, $i = 1, 2, 3$, we get a connected domain on the plane (w_1, w_2) bounded by the curves s_1, s_2 and s_3. Let us denote it by D. We also observe that $s_i \cap s_j = \emptyset$ for $w_1 > 0$ and $w_2 > 0$, where $i \neq j$.

Remark 1. Taking into account homotheties, it suffices to prove Theorem 1 for invariant metrics $\bigl(\frac{1}{w_1}, \frac{1}{w_2}, 1\bigr)$ with $(w_1, w_2) \in D \setminus \{(1, 1)\}$ in the coordinates (w_1, w_2).

1.4 The Set R of Invariant Metrics with Positive Ricci Curvature

Let us describe the set R of invariant metrics with positive Ricci curvature on the given generalized Wallach space. Using the expressions (5) for the principal Ricci curvatures $\boldsymbol{r_i}$, we introduce the functions $k_i := x_jx_k + a(x_i^2 - x_j^2 - x_k^2)$, where $x_i > 0$, $i \neq j \neq k \neq i$, $i, j, k \in \{1, 2, 3\}$. Then clearly the sets of invariant metrics with non-negative and positive Ricci curvature are respectively the following:

$$\bigl\{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid k_1 \geq 0,\ k_2 \geq 0,\ k_3 \geq 0\bigr\}, \qquad (17)$$

$$\bigl\{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid k_1 > 0,\ k_2 > 0,\ k_3 > 0\bigr\}. \qquad (18)$$

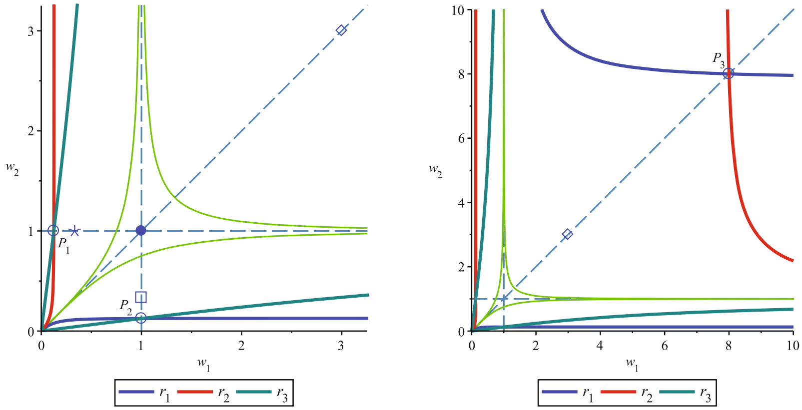


Fig. 3. The curves r_1, r_2, r_3 and the points P_1, P_2, P_3 corresponding to the system (10) at $a = 1/8$

Now, we will describe the domain R in the coordinates (w_1, w_2). Denote by r_i curves determined by the equations $k_i\big(\frac{1}{w_1}, \frac{1}{w_2}, 1\big) = 0$, respectively (see Fig. 3). For $w_1 > 0$ and $w_2 > 0$, these equations are respectively equivalent to

$$\begin{aligned} \rho_1 &:= -aw_1^2w_2^2 - aw_1^2 + aw_2^2 + w_1^2w_2 = 0, \\ \rho_2 &:= -aw_1^2w_2^2 + aw_1^2 - aw_2^2 + w_1w_2^2 = 0, \\ \rho_3 &:= \quad aw_1^2w_2^2 - aw_1^2 - aw_2^2 + w_1w_2 = 0. \end{aligned} \tag{19}$$

Since the set (17) is connected and its boundary is a part of the union of the cones $k_1 = 0$, $k_2 = 0$ and $k_3 = 0$, we easily get on the plane (w_1, w_2) a connected domain R bounded by the curves r_1, r_2 and r_3 solving the system of inequalities $k_i\big(\frac{1}{w_1}, \frac{1}{w_2}, 1\big) > 0$, $i = 1, 2, 3$.

Below we reveal some useful properties of the curves r_i. It is clear that each of the curves r_i, $i = 1, 2, 3$, consists of two disjoint connected components. In general we will use the description of r_i's given by $k_i\big(\frac{1}{w_1}, \frac{1}{w_2}, 1\big) = 0$, but we will concretize the component of r_i in cases when it is necessary.

Let us show that $r_i \cap s_j = \emptyset$ for $i, j \in \{1, 2, 3\}$ and $w_1 > 0$, $w_2 > 0$. By symmetry, we will confirm the equality $r_1 \cap s_3 = \emptyset$ only. In fact, eliminating w_2 from the system of the equations $\rho_1 = 0$ and $l_3 = 0$, we get the quadratic equation $(10a+3)(2a-1)w_1^2 - (2a-1)^2w_1 - 16a^2 = 0$ which has no real solution since its discriminant is negative at $a \in [\frac{1}{9}, \frac{1}{2})$: $(18a-1)(2a-1)(1+6a)^2 < 0$.

Next, easy calculations show that $c_1 \cap r_2 \cap r_3 = \{P_1\}$, $c_2 \cap r_1 \cap r_3 = \{P_2\}$ and $c_3 \cap r_1 \cap r_2 = \{P_3\}$ (see Fig. 3), where

$$P_1 := (a, 1), \quad P_2 := (1, a), \quad P_3 := \left(a^{-1}, a^{-1}\right). \tag{20}$$

It is easy to see that c_3 is tangent to the curves r_1 and r_2 at the point $(0, 0)$, whereas the pairs (r_1, r_3) and (r_2, r_3) have the asymptotes c_2 and c_1, respectively.

Remark 2. By analogy with the case of the sectional curvature, it suffices to prove Theorems 2, 3, and 4 only for invariant metrics $\left(\frac{1}{w_1}, \frac{1}{w_2}, 1\right)$ with $(w_1, w_2) \in R$ in the coordinates (w_1, w_2).

2 Proof of Theorems 1–4

2.1 Evolution of Invariant Metrics with Positive Sectional Curvature at $a \in \{1/9, 1/8, 1/6\}$

Lemma 3. *If $a \in (0, 1/4)$, then every trajectory of the system (10) originated in $D \setminus (c_1 \cup c_2 \cup c_3)$ reaches the boundary $s_1 \cup s_2 \cup s_3$ of D in finite time and leaves D. This finite time could be as long as we want.*

The corresponding picture is depicted in Fig. 4.

Proof. Without loss of generality consider only the part $D \cap \Omega$ of D, where Ω is given by (12). Consider any trajectory $\big(w_1(t), w_2(t)\big)$ of (10) initiated at $(w_1^0, w_2^0) \in D \cap \Omega$. The equation of s_3 has an unique positive solution, see (16)

$$w_2 \sim \frac{1}{2}(w_1 - 1)^{-1/2} \quad \text{as} \quad w_1 \to 1 + 0.$$

Therefore, we have $\alpha = 1/2$ in Proposition 1. Since $\frac{1-2a}{4a} > \alpha = 1/2$ whenever $0 < a < 1/4$ the trajectory $\big(w_1(t), w_2(t)\big)$ lies over the curve s_3 for $w_1 \to 1 + 0$ (corresponding to $t \to +\infty$). By continuity there exists a point on the curve $s_3 \cap \Omega$ at which $\big(w_1(t), w_2(t)\big)$ intersects $s_3 \cap \Omega$ and leaves the set D. □

Let us consider the vector field $V := (f, g)$, associated with the system (10), and the gradient $\nabla l_i \equiv \left(\frac{\partial l_i}{\partial w_1}, \frac{\partial l_i}{\partial w_2}\right)$, that is, the normal vector of the curve s_i (see (16)), $i = 1, 2, 3$.

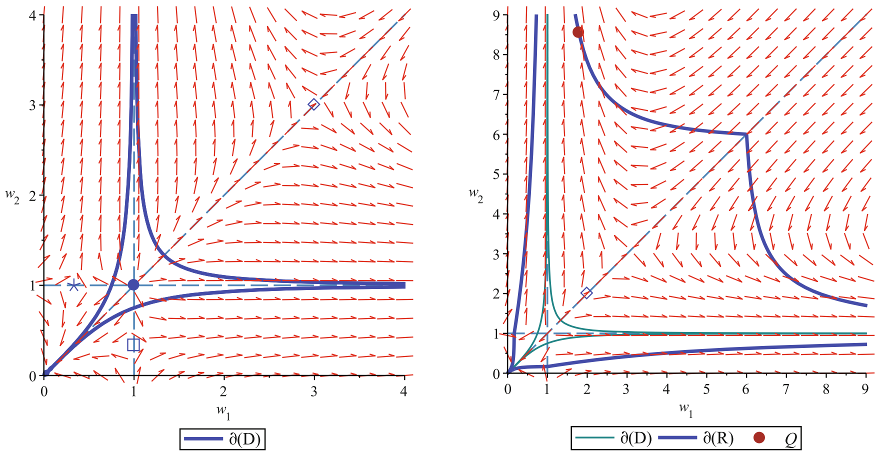


Fig. 4. The domains D, R and the phase portrait of the system (10) at $a = 1/8$

Lemma 4. *No trajectory of the system (10), $a \in \{1/9, 1/8, 1/6\}$ could return back to the domain D leaving D once.*

Proof. Consider points $(w_1, w_2) \in \partial(D) \cap \Omega$ without loss of generality. It is required to prove that the inequality $(V, \nabla l_3) < 0$ holds at every point of the part $s_3 \cap \Omega$ of the curve s_3 (in fact the mentioned inequality holds at every point of s_3 as we will see below). Here, $(V, \nabla l_3)$ means the usual inner product of the vectors V and ∇l_3 in the plane (w_1, w_2). By direct calculations we get

$$(V, \nabla l_3) = 2(w_1 - 1)(w_2 - 1)W,$$

where $W := 12aw_1^2w_2^2 - 3(w_1 + w_2)w_1w_2(1 - 2a) - (w_1 - w_2)^2(1 + 2a)$.

Substituting the expression $3w_1^2w_2^2 = 2(w_1 + w_2)w_1w_2 + (w_1 - w_2)^2$ which is equivalent to $l_3 = 0$ into W yields

$$W = (14a - 3)(w_1 + w_2)w_1w_2 - (w_1 - w_2)^2(1 - 2a) < 0.$$

To complete the proof of the lemma note that the normal vector ∇l_3 of the curve s_3 is inner for the set D since

$$\frac{\partial l_3}{\partial w_2} = -2(w_1 - 1)(3w_1w_2 + (w_2 - w_1)) < 0$$

on the curve s_3 (the curve s_3 has no singularities). □

Remark 3. Actually, we have proved a more strong assertion in the proof of Lemma 4: No one integral curve of the system (10) initiated outside D, could reach the set D (see Fig. 4). In particular, the normalized Ricci flow could not evolve metrics with mixed sectional curvature to metrics with positive sectional curvature.

Proof of Theorem 1 According to (9), we can consider the set $D \setminus \{(1, 1)\}$ in the plane (w_1, w_2) instead of the set (15) of invariant metrics with positive sectional curvature as it was noted in Remark 1. Now, it suffices to apply Lemmas 3 and 4 to complete the proof of the theorem and additional assertions just after Theorem 1.

2.2 Evolution of Invariant Metrics with Positive Ricci Curvature at $a \in (0, 1/2) \setminus \{1/4\}$

Lemma 5. *If $a \in (0, 1/6)$, then every integral curve of the system (10), initiated in $R \setminus (c_1 \cup c_2 \cup c_3)$, reaches the boundary $r_1 \cup r_2 \cup r_3$ of R in finite time and leaves R. This finite time could be as long as we want.*

The corresponding phase portraits are depicted in Fig. 4.

Proof. It is sufficient to consider only the set $R \cap \Omega$, where Ω is given by (12). Consider any trajectory $\big(w_1(t), w_2(t)\big)$ of the system (10) initiated at an arbitrary point $(w_1^0, w_2^0) \in R \cap \Omega$. The equation $\rho_1 = 0$ for the curve r_1 (see (19)) has the solution

$$w_2 \sim \frac{1}{2a}(w_1 - 1)^{-1} \quad \text{as} \quad w_1 \to 1 + 0\,,$$

corresponding to the "upper" part $\gamma := r_1 \cap \Omega$ of the curve r_1 (see the right-hand panel of Fig. 3). Note that $\frac{1-2a}{4a} > 1$ for all $0 < a < 1/6$. Then according to Proposition 1 the trajectory $\big(w_1(t), w_2(t)\big)$ lies over γ for $w_1 \to 1 + 0$ (corresponding to $t \to +\infty$). Hence by continuity there exists a point on γ at which $\big(w_1(t), w_2(t)\big)$ must intersect γ and leave R. Finally, we see that for initial points close to the point of the type $(w_1, w_2) \in c_i$, $i = 1, 2, 3$, the time for leaving the set of metrics with positive Ricci curvature could be as long as we want. □

Remark 4. Note that for $a = 1/6$ we get the equality $\frac{1-2a}{4a} = 1$. Hence, the arguments in the above proof do not work for the space W_6. Moreover, we know that Lemma 5 is failed for this space, see Theorem 8 of [14].

Remark 5. The equation of r_1 (see (19)) has also an another solution $w_2 = a + O(w_1 - 1)$ corresponding to the "lower" part of the curve r_1 (see the left-hand panel of Figure 3). Note that in this case we have exactly the point $P_2 = (1, a)$ (see (20)) as $w_1 \to 1$.

Lemma 6. *No trajectory of the system (10), $a \in \{1/6, 1/8, 1/9\}$ could return back to the domain R leaving R once.*

Proof of Lemma 6 is too long, so we will omit it here. Readers can find it in [3].

Remark 6. Actually, we have proved a more strong assertion in the proof of Lemma 6: Some integral curves of the system (10), initiated outside the domain R, could reach R (e. g. through the part of the curve r_1 between the points P_3 and Q, intersecting $r_1 \subset \partial(R)$ from up to down), see the right-hand panel of Fig. 4. But later these trajectories will leave R irrevocably, if will reach $\partial(R)$ (e. g. in $R \cap \Omega$, this could happen about the part of r_1 situated from the left of the point Q). Note that this effect follows also from Lemma 5 for $a = 1/8$ and $a = 1/9$. Hence, in particular, the normalized Ricci flow can evolve some metrics with mixed Ricci curvature to metrics with positive Ricci curvature.

Proof of Theorem 2 According to (9), we can consider the set R instead of the set (18) of invariant metrics with positive Ricci curvature as it was noted in Remark 2. Now, it suffices to apply Lemmas 5 and 6 to complete the proof of the theorem and additional assertions just after Theorem 2.

Proof of Theorem 3 It is sufficient to work with the set Ω given by (12). The equation $\rho_1 = 0$ for the curve r_1 (see (19)) has the solution

$$w_2 \sim \frac{1}{2a}(w_1 - 1)^{-1} \quad \text{as} \quad w_1 \to 1 + 0\,,$$

corresponding to the "upper" part γ of the curve r_1, which is the "upper" part of the boundary of $R \cap \Omega$, the set of metric with positive Ricci curvature in Ω (see the right-hand panel of Fig. 3).

Consider the case $a \in (0, 1/6)$ and any trajectory $\big(w_1(t), w_2(t)\big)$ of the system (10) initiated at a point of $R \cap \Omega$. Note that $\frac{1-2a}{4a} > 1$ for all $0 < a < 1/6$. Then according to Proposition 1 the trajectory $\big(w_1(t), w_2(t)\big)$ lies over γ for $w_1 \to 1 + 0$ (corresponding to $t \to +\infty$).

Now, consider the case $a \in (1/6, 1/4) \cup (1/4, 1/2)$. Clearly, $\frac{1-2a}{4a} < 1$ for all $a \in (1/6, 1/2)$. Proposition 1 implies that the normalized Ricci flow evolves every initial metric in Ω into metrics with positive Ricci curvature. This proves the theorem.

Proof of Theorem 4 First, note that the set of metrics with the property $x_i = x_j + x_k$ is an invariant set of the system (6) with right hand sides $F_i := -2x_i(t)\left(\mathbf{r_i} - \frac{S}{n}\right)$ for $a = 1/6$. Indeed, if we consider any metric with $x_3 = x_1 + x_2$, then direct calculations show that $F_1 + F_2 - F_3 \equiv 0$ for $a = 1/6$. Note also that every non-normal Einstein metric on the space under consideration is such that $x_i = x_j + x_k$ for suitable indices.

Hence, in the scale invariant coordinates (w_1, w_2) we have an invariant curve $w_1^{-1} + w_2^{-1} = 1$ of the system (10) passing through the point $E_3 = (2, 2)$. Since E_3 is a saddle of the system (10), the curve $w_1^{-1} + w_2^{-1} = 1$ is necessarily one of the separatrices (more exactly, the unstable manifold) of this point E_3 by uniqueness of a solution of the initial value problem (obviously the line $w_2 = w_1$ is the second separatrix).

For submersion metrics the proof is easy and follows from the discussion in Introduction. Let us consider the case of generic metrics. Without loss of generality we may suppose that the initial metric is in Ω. By the above discussion, the set $\left\{(w_1, w_2) \,|\, w_2 < \frac{w_1}{w_1-1}\right\} \cap \Omega$ is an invariant set of the system (10). Simple calculations show that the curve $\left\{(w_1, w_2) \,|\, w_2 = \frac{w_1}{w_1-1}\right\} \cap \Omega$ lies under the curve $r_1 \cap \Omega \subset \partial(R)$. Hence, every trajectory of (10) initiated in the set $\left\{(w_1, w_2) \,|\, w_2 < \frac{w_1}{w_1-1}\right\} \cap \Omega$ remains in the domain $R \cap \Omega$, that proves the theorem.

Remark 7. For W_6, the metrics (3) with $x_i = x_j + x_k$ constitute the set of Kähler invariant metrics, see e. g. [9, Chapter 8]. The general result that the set of Kähler metrics is invariant under the Ricci flow on every manifold is obtained in [7].

Remark 8. Note that conditions of Theorem 4 are valid for metrics from D, the set of metrics with positive sectional curvature on the space W_6. Hence, we get the generalization of Theorem 8 in [14].

Finally, we reproduce additional illustrations suggested us by Wolfgang Ziller. We reproduce in Fig. 5 the domains of positive sectional, positive Ricci, and positive scalar curvatures (denoted by D, R, and S, respectively) of the system (6) in the plane $x_1 + x_2 + x_3 = 1$ for $a = 1/6$, because the space W_6 admits Kähler invariant metrics, that constitute a small triangle in Fig. 5. Note also that three non-normal Einstein metrics in this case are Kähler – Einstein and

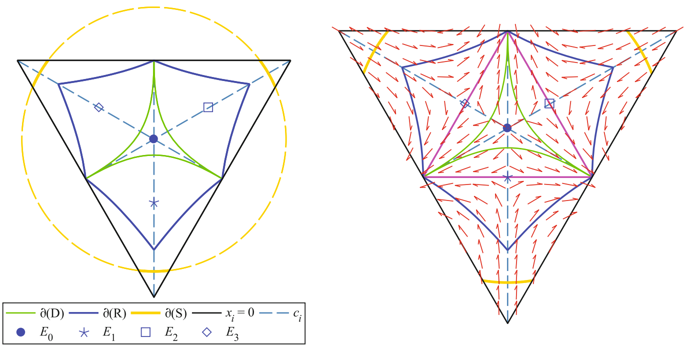


Fig. 5. The case $a = 1/6$: The domains of positive sectional, positive Ricci, and positive scalar curvatures, Kähler metrics, the phase portrait of the system (6) in the plane $x_1 + x_2 + x_3 = 1$

one can easily get main properties of the Kähler – Ricci flow on the space W_6 using this picture. Note that Riemannian metrics constitute a triangle and the set S is bounded by a circle.

3 Proof of Theorem 5

3.1 The System (10) at $a = 1/4$

3.1.1 The Unique Singular Point of the System (10)

Lemma 7. *Let $a = 1/4$. Then the system (10) has a unique singular point $E = (1, 1)$, which is a linearly zero saddle. Moreover, the lines $w_1 = 1$, $w_2 = 1$ and $w_2 = w_1$ are separatrices of this saddle.*

Proof. Using results of [19] it has been proved in [4, Theorem 2] that in the case $a = 1/4$ the system (8) has a unique singular point $F = (1, 1)$, which is a linearly zero saddle with six hyperbolic sectors around it. By homeomorphism (11), a unique singular point $E = (1, 1)$ (see Fig. 6) of the system (10), corresponding to $a = 1/4$, is also a saddle of the same type as F. Since the invariant lines (solutions) $w_1 = 1$, $w_2 = 1$ and $w_2 = w_1$ of the system (10) pass through the point E, they are separatrices of the saddle E at $a = 1/4$. □

Remark 9. The value $a = 1/4$ is a bifurcation value for the system (10) causing a qualitative reorganization of its phase portrait: four isolated non degenerate (simple) singular points of (10) shown in Lemma 1 merge as $a \to 1/4$ into the unique degenerate (complicated) singular point $(1, 1)$, described in Lemma 7.

The value $a = 1/4$ is also interesting from the point of view of algebraic geometry: the point $(1/4, 1/4, 1/4)$ is an *elliptic umbilic* (in the sense of Darboux [15]) or a *point of the type* D_4^- (in the terminology of [6]) of a specific surface

$$\Omega := \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid Q(a_1, a_2, a_3) = 0\} \subset \mathbb{R}^3$$

introduced in [4], where $Q(a_1, a_2, a_3)$ is a symmetric polynomial in the variables a_1, a_2, a_3 of degree 12:

$$\begin{aligned} Q(a_1,a_2,a_3) \;=\; & (2s_1+4s_3-1)(64s_1^5-64s_1^4+8s_1^3+12s_1^2-6s_1+1 \\ & +240s_3s_1^2-240s_3s_1-1536s_3^2s_1-4096s_3^3+60s_3+768s_3^2) \\ & -8s_1(2s_1+4s_3-1)(2s_1-32s_3-1)(10s_1+32s_3-5)s_2 \\ & -16s_1^2(13-52s_1+640s_3s_1+1024s_3^2-320s_3+52s_1^2)s_2^2 \\ & +64(2s_1-1)(2s_1-32s_3-1)s_2^3+2048s_1(2s_1-1)s_2^4, \end{aligned}$$

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = a_1a_2 + a_1a_3 + a_2a_3, \quad s_3 = a_1a_2a_3.$$

The importance of Ω is due to the need to develop a special apparatus for studying general properties of degenerate singular points of Ricci flows initiated in [4,5], so according to these works Ω includes the following set

$$\big\{(a_1,a_2,a_3) \in (0,1/2]^3 \mid \text{system (7) has at least one degenerate singular point}\big\}.$$

As shown in [1] *the set* $(0,1/2]^3 \cap \Omega$ *is connected* and *the set* $(0,1/2]^3 \setminus \Omega$ *consists of three connected components* with respect to the standard topology of $\mathbb{R}^3$. It should also be noted that a more detailed description of the surface Ω was obtained in [8].

3.1.2 Attracting and Repelling Manifolds of the Saddle (1, 1) of (10)

According to Lemma 7 integral curves of (10) are determined by influences of the saddle $(1,1)$ and its separatrices c_1, c_2, c_3 only. Our purpose is to detect orientations of these integral curves. To answer the question it is enough to establish what parts of c_1, c_2 and c_3 may be stable or unstable manifolds for E. Since we are restricted by integral curves of (10) belonging to Ω, it suffices to consider the sets $\{w_1 = 1, w_2 \geq 1\}$ and $\{w_2 = w_1 \geq 1\}$ which bound the set Ω.

Lemma 8. *Let* $a = 1/4$. *Then for every integral curve of (10), initiated in* Ω, *the sets* $\{w_2 = w_1 \geq 1\}$ *and* $\{w_1 = 1, w_2 \geq 1\}$ *are respectively an attracting and a repelling manifolds of the saddle* $(1,1)$.

Proof. Consider the main isoclines ω and λ of (10). Let $(w_1(t), w_2(t))$ be any trajectory of (10) with any initial point from Ω. It is clear that ω and λ separate $\mathbb{R}^2_+$ into four disjoint domains (see Fig. 6), in every of which each of the functions y_1 and y_2 preserve its sign by continuity. For example, at the point $(w_1, w_2) = (1,2)$ we observe that $y_1 < 0$ and $y_2 > 0$. Hence, $w_1' < 0$ in Ω (in Ω trajectories are oriented from right to left). Analyzing the second equation in (10) we get $w_2' > 0$ (< 0) over (under) the curve $\lambda \cap \Omega$. Therefore, trajectories are directed to up (down) at points in Ω, situated over (under) λ. □

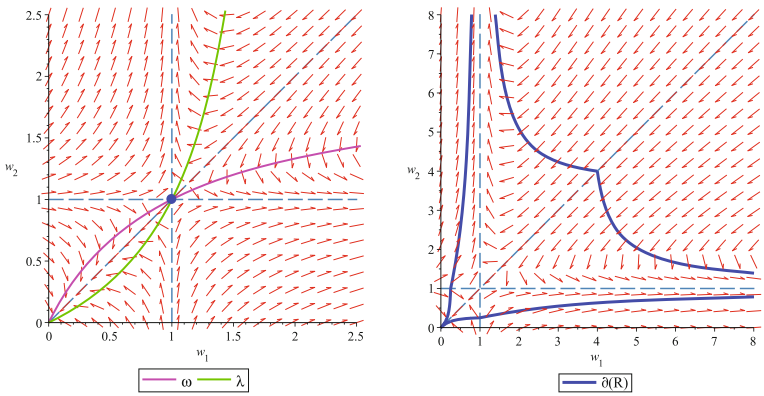


Fig. 6. The case $a = 1/4$: The main isoclines ω, λ, the domain R and the phase portrait of (10)

3.2 Evolution of Invariant Metrics with Positive Ricci Curvature at $a = 1/4$

Lemma 9. *At $a = 1/4$ every integral curve of the system (10), initiated in $\Omega \setminus R$, attains $R \cap \Omega$ in finite time and remains in $R \cap \Omega$. This finite time could be as long as we want.*

Lemma 9 can be proved by the same way as Lemma 5 using Proposition 1.

Since the intermediate value theorem mentioned in Proposition 1 can not guarantee the uniqueness of the intersection point of an integral curve with the boundary γ of the domain $R \cap \Omega$, Lemma 9 does not answer the question could integral curves intersect or touch γ several time. The following lemma refutes such a possibility.

Lemma 10. *At $a = 1/4$ every integral curve of the system (10), initiated in Ω, can admit at most one common point with the boundary γ of the domain $R \cap \Omega$.*

Proof of Lemma 10 can be found in [2].

Remark 10. Lemma 10 fails in general. For example, at $a = 1/8$ or $a = 1/9$ the system (10) admits integral curves which intersect the curve γ twice (see Remark 6 above).

Proof of Theorem 5 According to our agreements consider invariant Riemannian metrics $\left(w_1^{-1}, w_2^{-1}, 1\right)$ only, satisfying $(w_1, w_2) \in \Omega$. Now it suffices to apply Lemmas 9 and 10.

Acknowledgements. The author is grateful to Prof. Yuriĭ Nikonorov for fruitful collaboration. The project was supported by Grant 1452/GF4 of Ministry of Education and Sciences of the Republic of Kazakhstan. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Abiev, N.A.: On topological structure of some sets related to the normalized Ricci flow on generalized Wallach spaces. Vladikavkaz. Math. J. **17**(3), 5–13 (2015) (Russian)
2. Abiev, N.A.: On evolution of invariant Riemannian metrics on a class of generalized Wallach spaces under the normalized Ricci flow. Mat. Trudy. **20**(1), 1–18 (2017) (Russian, in press)
3. Abiev, N.A., Nikonorov, YuG: The evolution of positively curved invariant Riemannian metrics on the Wallach spaces under the Ricci flow. Ann. Glob. Anal. Geom. **50**(1), 65–84 (2016)
4. Abiev, N.A., Arvanitoyeorgos, A., Nikonorov, Yu.G., Siasos, P.: The dynamics of the Ricci flow on generalized Wallach spaces. Differ. Geom. Appl. **35**(Suppl.), 26–43 (2014)
5. Abiev, N.A., Arvanitoyeorgos, A., Nikonorov, Yu.G., Siasos, P.: The Ricci flow on some generalized Wallach spaces. In: Rovenski, V., Walczak, P. (eds.) Geometry and its Applications. Springer Proceedings in Mathematics and Statistics, vol. 72, pp. 3–37. Springer, Switzerland (2014)
6. Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differentiable maps. Monographs in Mathematics, 82. vol. 1. Birkhäuser, Boston-Basel-Stuttgart (1985)
7. Bando, S.: On the classification of three-dimensional compact Kaehler manifolds of nonnegative bisectional curvature. J. Differ. Geom. **19**(2), 283–297 (1984)
8. Batkhin, A.B., Bruno, A.D.: Investigation of a real algebraic surface. Prog. Comput. Softw. **41**(2), 74–83 (2015)
9. Besse, A.L.: Einstein Manifolds. Springer, Berlin (1987)
10. Böhm, C.: On the long time behavior of homogeneous Ricci flows. Comment. Math. Helv. **90**, 543–571 (2015)
11. Böhm, C., Wilking, B.: Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature. GAFA, Geom. Func. Anal. **17**, 665–681 (2007)
12. Buzano, M.: Ricci flow on homogeneous spaces with two isotropy summands. Ann. Glob. Anal. Geom. **45**(1), 25–45 (2014)
13. Chen, Zh., Kang Y., Liang, K.: Invariant Einstein metrics on three-locally-symmetric spaces. Commun. Anal. Geom. (to appear)
14. Cheung, M.W., Wallach, N.R.: Ricci flow and curvature on the variety of flags on the two dimensional projective space over the complexes, quaternions and octonions. Proc. Am. Math. Soc. **143**(1), 369–378 (2015)
15. Darboux, G.: Leçons sur la théorie générale des surfaces et les applications géometriques du calcul infinitésimal, vol. 4. Gauthier-Villars, Paris (1896)
16. Dumortier, F., Llibre, J., Artes, J.: Qualitative Theory of Planar Differential Systems. Universitext. Springer, Berlin (2006)
17. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**, 255–306 (1982)
18. Jablonski, M.: Homogeneous Ricci solitons. J. Reine Angew. Math. **699**, 159–182 (2015)
19. Jiang, Q., Llibre, J.: Qualitative classification of singular points. Qual. Theory Dyn. Syst. **6**(1), 87–167 (2005)
20. Lafuente, R.: Scalar curvature behavior of homogeneous Ricci flows. J. Geom. Anal. **25**(4), 2313–2322 (2014)

21. Lauret, J.: Ricci flow on homogeneous manifolds. Math. Z. **274**(12), 373–403 (2013)
22. Lomshakov, A.M., Nikonorov, YuG, Firsov, E.V.: Invariant Einstein metrics on three-locally-symmetric spaces. Sib. Adv. Math. **14**(3), 43–62 (2004)
23. Ni, L.: Ricci flow and manifolds with positive curvature. In: Howe, R., Hunziker, M., Willenbring, J.F. (eds.) Symmetry: representation theory and its applications. In honor of N.R. Wallach. Progress in Mathematics, vol. 257, pp. 491–504. Birkhäuser/Springer, New York (2014)
24. Nikonorov, YuG: On a class of homogeneous compact Einstein manifolds. Sib. Math. J. **41**(1), 168–172 (2000)
25. Nikonorov, YuG: Classification of generalized Wallach spaces. Geom. Dedicata. **181**(1), 193–212 (2016)
26. Nikonorov, YuG, Rodionov, E.D., Slavskii, V.V.: Geometry of homogeneous Riemannian manifolds. J. Math. Sci. (New York) **146**(7), 6313–6390 (2007)
27. Payne, T.L.: The Ricci flow for nilmanifolds. J. Mod. Dyn. **4**(1), 65–90 (2010)
28. Valiev, F.M.: Precise estimates for the sectional curvature of homogeneous Riemannian metrics on Wallach spaces. Sib. Math. J. **20**, 176–187 (1979)
29. Wallach, N.R.: Compact homogeneous Riemannian manifolds with strictly positive curvature. Ann. Math., Second Series. **96**(2), 277–295 (1972)

Hyperbolic Cross Approximation of Some Multivariate Function Classes with Respect to Wavelet System with Compact Supports

Sholpan A. Balgimbayeva(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
sholpan.balgyn@gmail.com

Abstract. In this paper we obtain estimates sharp in order hyperbolic cross approximation w.r.t. d−multiple wavelet system with compact supports $\psi^{(\mathtt{d})}$ of the Nikol'skii – Besov and Lizorkin – Triebel type classes associated with this system in the space $L_q([0,1]^d)$ for a number of relations between the parameters of the classes and the space.

Keywords: Hyperbolic cross · The Nikol'skii – Besov type space · The Lizorkin – Triebel type space · Wavelet system with compact supports

1 Introduction

Let $L_q = L_q([0,1]^d)\,(1 \leq q \leq \infty, 2 \leq d \in \mathbb{N})$ be the space of all (equivalence classes of) measurable functions $f : [0,1]^d \to \mathbb{C}$ that are q power integrable (essentially bounded for $q = \infty$) on $[0,1]^d$, endowed with the standard norm

$$\| f \mid L_q \| = \| f \mid L_q([0,1]^d) \| = \left(\int_{[0,1]^d} |\, f(x) \,|^q dx \right)^{1/q} (1 \leq q < \infty),$$

$$\| f \mid L_\infty \| = \| f \mid L_\infty([0,1]^d) \| = \text{ess sup} \{ |\, f(x) \,| : x \in [0,1]^d \}.$$

Let $\Phi = \{\phi_\imath \,|\, \imath \in \mathrm{J}\}$ be a countable collection of functions in L_∞ which are orthonormal in L_2 and let $\{J(u) \subset \mathrm{J} \,|\, u \in \mathbb{N}\}$ be such that $J(u) \subset J(u+1)$ and $\# J(u) < \infty$ for all $u \in \mathbb{N}$, $\cup_{u \in \mathbb{N}} J(u) = \mathrm{J}$.

For $f \in L_1$, we consider Fourier sums w. r. t. system Φ of the form

$$S_u^\Phi(f,x) = \sum_{\imath \in J(u)} \langle f, \phi_\imath \rangle \phi_\imath(x),$$

where $\langle f, g \rangle = \int_{[0,1]^d} f(x)\overline{g}(x)dx$ ($\overline{z}$ is the number complex conjugate to $z \in \mathbb{C}$).

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_2

For a set $\mathrm{F} \in L_q$, we denote

$$E_u(\mathrm{F}, \Phi, L_q) = \sup\{\|f - S_u^{\Phi}(f, x) \,|\, L_q\| \mid f \in \mathrm{F}\}. \tag{1}$$

In the present paper, we study quantities (1) in the special case of $\mathtt{d}-$multiple wavelet system $\psi^{(\mathtt{d})}$ with compact supports and so called hyperbolic crosses used as Φ and $J(u)$, respectively (for definitions see below). Main goal is to give estimates (sharp in order) for hyperbolic cross approximation w. r. t. system $\psi^{(\mathtt{d})}$ of the Nikol'skii – Besov and Lizorkin – Triebel type classes associated with that system (for definition see sect. 2) in the space $L_q([0,1]^d)$ for a number of relations between the parameters of the classes and the space.

First we introduce some notations. Let $d \in \mathbb{N}$, $\mathrm{z}_d = \{1, \ldots, d\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$.

Let $k \in \mathbb{N}\colon\ k \le d$. Fix a multi-index $\mathtt{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$ such that $d_1 + \cdots + d_k = d$ (thus, $\mathtt{d} = d$ if $k = 1$, while if $k = d$, then $\mathtt{d} = \overrightarrow{1} = (1, \ldots, 1) \in \mathbb{N}^d$) and representation of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ in the form $x = (x^1, \ldots, x^k)$, where $x^\kappa \in \mathbb{R}^{n_\kappa}$, $\kappa \in \mathrm{z}_k$.

Denote

$$\mathrm{E}^d = \mathrm{E}^d(0) = \{\mathit{0}, \mathit{1}\}^d, \quad \mathrm{E}^d(1) = \mathrm{E}^d \setminus \{(\mathit{0}, \ldots, \mathit{0})\};$$
$$\Lambda(d, j) = \mathbb{Z}^d \cap [0, 2^j - 1]^d,\ j \in \mathbb{N}_0.$$

Let univariate scaling function $\psi^{(0)}$ and the associated wavelet $\psi^{(1)}$ be compactly supported:

$$\operatorname{supp} \psi^{(0)} \cup \operatorname{supp} \psi^{(1)} \subset [0, 2N - 1] \ \text{ for some } \ N > 0;$$

$\psi^{(0)}$, $\psi^{(1)} \in C^r(\mathbb{R})$.

Further, $\mathtt{d}-$multiple wavelet system is defined as follows:

$$\psi^{(\mathtt{d})} = \{\psi^{\iota}_{\alpha\lambda}(x) \ \mid\ \iota \in \mathrm{E}^{\mathtt{d}}(\alpha),\ \lambda \in \Lambda(\mathtt{d}, \alpha),\ \alpha \in \mathbb{N}_0^k\},$$

where

$$\psi^{\iota}_{\alpha\lambda}(x) = \prod_{\kappa=1}^{k} \psi^{\iota^\kappa}_{\alpha_\kappa \lambda^\kappa}(x^\kappa), \quad \psi^{\iota^\kappa}_{j\lambda^\kappa}(x^\kappa) = 2^{\frac{j d_\kappa}{2}} \psi^{\iota^\kappa}(2^j x^\kappa - \lambda^\kappa);$$

$$\psi^{\iota^\kappa}(x^\kappa) = \prod_{\nu \in \mathrm{K}_\kappa} \psi^{(\iota_\nu)}(x_\nu);$$

here

$$\mathrm{E}^{\mathtt{d}}(\alpha) = \mathrm{E}^{d_1}(\operatorname{sign}(\alpha_1)) \otimes \cdots \otimes \mathrm{E}^{d_k}(\operatorname{sign}(\alpha_k)),$$
$$\Lambda(\mathtt{d}, \alpha) = \Lambda(d_1, \alpha_1) \times \ldots \times \Lambda(d_k, \alpha_k).$$

It is clear that the system $\psi^{(\mathtt{d})}$ is orthonormal in $L_2([0,1]^d)$. Furthermore, the system $\psi^{(\mathtt{d})}$ is an unconditional basis in $L_q([0,1]^d)$ with $1 < q < \infty$: in the case $k = 1$ this fact is proved in [8, ch. 8], general case $1 \le k \le d$ follows from here because the system $\psi^{(\mathtt{d})}$ (w.r.t. variable x) is a tensor product of the systems $\psi^{(d_\kappa)}$ (w.r.t. variables x^κ), $\kappa \in \mathrm{z}_k$.

Define operators Δ_α^ψ $(\alpha \in \mathbb{N}_0^k)$ as follows: for $f \in L_1$, let

$$\Delta_\alpha^\psi(f,x) = \sum_{\iota \in \mathrm{E}^{\mathtt{d}}(\alpha)} \sum_{\lambda \in \Lambda(\mathtt{d},\alpha)} f_{\alpha\lambda}^\iota \psi_{\alpha\lambda}^\iota(x)$$

be its dyadic packet, where

$$f_{\alpha\lambda}^\iota = \int_{[0,1]^d} f(x)\, \psi_{\alpha\lambda}^\iota(x)\, dx.$$

Furthermore, for $f \in L_1$, we define its so called hyperbolic cross Fourier sum w. r. t. system $\psi^{(\mathtt{d})}$ (with fixed $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}_+^k$) by the formula

$$S_u^{\psi,\gamma}(f,x) = \sum_{\alpha\gamma \le u} \Delta_\alpha^\psi(f,x) \quad (u \in \mathbb{N})$$

(here $\alpha\gamma := \alpha_1\gamma_1 + \cdots + \alpha_k\gamma_k$). In the sequel, we will denote quantity (1) with $S_u^{\psi,\gamma}$ instead of S_u^Φ simply as $E_u^\gamma(\mathrm{F}, L_q)$.

2 Function Classes

Let $1 \le p$, $\theta \le \infty$ and let $\ell_\theta \equiv \ell_\theta(\mathbb{N}_0^k)$ be a space of complex number sequences $(c_\alpha) = (c_\alpha \,|\, \alpha \in \mathbb{N}_0^k)$ with the finite norm

$$\|(c_\alpha)\,|\,\ell_\theta\,\| = \Big(\sum_{\alpha \in \mathbb{N}_0^k} |\,c_\alpha\,|^\theta\Big)^{1/\theta} \ (1 \le \theta < \infty), \quad \|(c_\alpha)\,|\,\ell_\infty\,\| = \sup_{\alpha \in \mathbb{N}_0^k} |\,c_\alpha\,|;$$

let $\ell_\theta(L_p) \equiv \ell_\theta(L_p([0,1]^d))$ (respectively $L_p(\ell_\theta) \equiv L_p([0,1]^d;\ell_\theta)$) be a space of function sequences $(g_\alpha(x)) = (g_\alpha(x)\,|\,\alpha \in \mathbb{N}_0^k)$ $(x \in [0,1]^d)$ with the finite norm

$$\|\,(g_\alpha(x))\,|\,\ell_\theta(L_p)\,\| = \|\,(\,\|\,g_\alpha\,|\,L_p\,\|)\,|\,\ell_\theta\,\|$$

(respectively

$$\|\,(g_\alpha(x))\,|\,L_p(\ell_\theta)\,\| = \|\,\|\,(g_\alpha(\cdot))\,|\,\ell_\theta\,\|\,|\,L_p\,\|).$$

Now we are in position to introduce function spaces (as well as classes) under consideration.

Definition 1. Let $s = (s_1, \dots, s_k) \in \mathbb{R}_+^k$, $1 \le p, \theta \le \infty$. Then

i) the Nikol'skii – Besov type space $\psi B_{p\,\theta}^{s\,\mathtt{d}} \equiv \psi B_{p\,\theta}^{s\,\mathtt{d}}([0,1]^d)$ associated with the system $\psi^{(\mathtt{d})}$ consists of all functions $f \in L_p$ for which the norm

$$\|\,f\,|\,\psi B_{p\,\theta}^{s\,\mathtt{d}}\,\| \;=\; \|(2^{\alpha s}\Delta_\alpha^\psi(f,x))\,|\,\ell_\theta(L_p)\| \tag{2}$$

is finite;

ii) the Lizorkin – Triebel type space $\psi L_{p\,\theta}^{s\,\mathtt{d}} \equiv \psi L_{p\,\theta}^{s\,\mathtt{d}}([0,1]^d)$ associated with the system $\psi^{(\mathtt{d})}$ consists of all functions $f \in L_p$ for which the norm

$$\|\,f\,|\,\psi L_{p\,\theta}^{s\,\mathtt{d}}\,\| \;=\; \|(2^{\alpha s}\Delta_\alpha^\psi(f,x))\,|\,L_p(\,\ell_\theta)\| \tag{3}$$

is finite.

Unit balls $\psi \mathrm{B}_{p\,\theta}^{s\,\mathtt{d}} \equiv \psi \mathrm{B}_{p\,\theta}^{s\,\mathtt{d}}([0,1]^d)$ and $\psi \mathrm{L}_{p\,\theta}^{s\,\mathtt{d}} \equiv \psi \mathrm{L}_{p\,\theta}^{s\,\mathtt{d}}([0,1]^d)$ of these spaces will be called the Nikol'skii – Besov and Lizorkin – Triebel classes associated with the system $\psi^{(\mathtt{d})}$, respectively.

3 Main Result

Put $p_* = \min\{p, 2\}$, $\varsigma_\kappa = \frac{s_\kappa}{d_\kappa}$; without loss of generality, we assume that $\varsigma \equiv \min\{\varsigma_\kappa \mid \kappa \in \mathrm{z}_k\} = \varsigma_1 = \ldots = \varsigma_\omega < \varsigma_\kappa$, $\kappa \in \mathrm{z}_k \setminus \mathrm{z}_\omega$ for certain $\omega \in \mathrm{z}_k$. Take a vector $\varsigma' = (\varsigma'_1, \ldots, \varsigma'_k)$ such that $\varsigma = \varsigma'_1 = \ldots = \varsigma'_\omega < \varsigma'_\kappa < \varsigma_\kappa$, $\kappa = \omega+1, \ldots, k$, and put $s' = (s'_1, \ldots, s'_k)$ with $s'_\kappa = \varsigma'_\kappa d_\kappa$ $(\kappa \in \mathrm{z}_k)$ and $\gamma = \frac{1}{\varsigma} s'$.

Below, we will use the symbols $\ll$ and $\asymp$ to show relations between the orders of quantities : for functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ we write $F(u) \ll H(u)$ as $u \to \infty$, if there exists a constant $C = C(F, H) > 0$ such that the inequality $F(u) \le CH(u)$ holds for $u \ge u_0 > 0$, and $F(u) \asymp H(u)$, if $F(u) \ll H(u)$ and $H(u) \ll F(u)$, simultaneously.

For a number $a \in \mathbb{R}$ set $a_+ = \max\{a, 0\}$; below, log is logarithm to the base 2.

Theorem 1. *Let $1 < q \le p < \infty$, $1 \le \theta \le \infty$ and $s \in \mathbb{R}^k_+$. Then*

$$E^\gamma_u(\psi \mathrm{B}^{s\,\mathtt{d}}_{p\,\theta}, L_q) \asymp 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{p_*} - \frac{1}{\theta})_+};$$

$$E^\gamma_u(\psi \mathrm{L}^{s\,\mathtt{d}}_{p\,\theta}, L_q) \asymp 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{2} - \frac{1}{\theta})_+}.$$

Remark 1. If $s < r$ then we have classes analogues to the classical Nikol'skii – Besov and Lizorkin – Triebel classes and this theorem is analog of the theorem from [6]. If $s_j > r_j$ for some $j \in \{1, \ldots, d\}$ then we have another classes and this theorem is analog of Theorem 4.1 from [2] and analog of Theorem 1 from [3] for the Nikol'skii – Besov and Lizorkin – Triebel type classes associated with system $\psi^{(\mathtt{d})}$.

4 Preliminaries

In this section, we collect known facts that are important for further consideration.

First we formulate the Littlewood – Paley – type theorem related to the system $\psi^{(\mathtt{d})}$ and its corollary.

Theorem 2. *Let $1 < p < \infty$. Then there exists a constant $C = C(\mathtt{d}, p) > 0$ such that*

$$C^{-1} \| f \mid L_p \| \le \| (\varDelta^\psi_\alpha(f, x)) \mid L_p([0,1]^d; \ell_2) \| \le C \| f \mid L_p \|,$$

$$C^{-1} \| f \mid L_p \| \le \| \Big(\sum_\alpha \sum_\iota \sum_\lambda |\langle f, \psi^\iota_{\alpha\lambda} \rangle \, \psi^\iota_{\alpha\lambda}|^2 \Big)^{1/2} \mid L_p([0,1]^d) \| \le C \| f \mid L_p \|.$$

for all functions $f \in L_p$.

Since the system $\psi^{(\mathtt{d})}$ is an unconditional basis in L_p, Theorem 4 can easily be proved by arguments involving the classical Khinchin inequality for Rademacher functions (see, for instance, [8, ch. 8], where the case $k = 1$ is analyzed).

Corollary 1. *Let* $1 < p < \infty$, $p_* = \min(p, 2)$. *Then, for any function* $f \in L_p$

$$\| f \,|\, L_p \| \le C(v, m, p) \, \| \, (\Delta_\alpha^\psi(f, x)) \,|\, \ell_{p_*}(L_p([0,1]^d)) \, \|.$$

Below, when proving upper bounds from Theorem 1 and estimating the dimensions of the corresponding subspaces, we will systematically use the following lemma from [2] (for proof see [6]).

Lemma 1. *Let* $\beta, \gamma \in \mathbb{R}_+^d$ *be such that* $\beta_\nu = \gamma_\nu$ *for* $\nu \in \mathrm{z}_\omega$ *and* $\beta_\nu > \gamma_\nu$ *for* $\nu \in \mathrm{z}_n \setminus \mathrm{z}_\omega$, *and let* $L > 0$. *Then the following relations are valid:*

$$\mathrm{I}_L^{\beta,\gamma}(u) \equiv \sum_{\alpha \in \mathbb{N}_0^d : \alpha\gamma > u} 2^{-L\alpha\beta} \asymp 2^{-Lu} u^{\omega - 1} \qquad as \qquad u \to +\infty; \tag{4}$$

$$\mathrm{J}_L^{\gamma,\beta}(u) \equiv \sum_{\alpha \in \mathbb{N}_0^d : \alpha\beta \le u} 2^{L\alpha\gamma} \asymp 2^{Lu} u^{\omega - 1} \qquad as \qquad u \to +\infty. \tag{5}$$

5 Proof of Upper Bounds in Theorem 1

B. We begin with upper estimates for the classes $\psi \mathrm{B}_{p\,\theta}^{s\,\mathtt{d}}$.

a) Let, first, $p = q = 1$. Then for $f \in \psi \mathrm{B}_{p\,\theta}^{s\,\mathtt{d}}$, we have

$$\| f - S_u^{\psi,\gamma}(f) \,|\, L_1 \| = \| \sum_{\alpha s' > u\varsigma} \Delta_\alpha^\psi(f) \,|\, L_1 \| \le$$

$$\le \| \, (\Delta_\alpha^\psi(f))_{\alpha s' > u\varsigma} \,|\, \ell_1(L_1) \, \| = \| \, (2^{-\alpha s} 2^{\alpha s} \Delta_\alpha^\psi(f))_{\alpha s' > u\varsigma} \,|\, \ell_1(L_1) \, \| \equiv \Im_1(u).$$

By the definition (2) of the norm in space $\psi B_{p\,\theta}^{s\,\mathtt{d}}$, for $\theta = 1$, we obviously have ($\alpha s \ge \alpha s' > u\varsigma$)

$$\Im_1(u) \le 2^{-\varsigma u} \| \, (2^{\alpha s} \Delta_\alpha^\psi(f))_{\alpha s' > u\varsigma} \,|\, \ell_1(L_1) \, \| \ll 2^{-\varsigma u} \, \| \, f \,|\, \psi B_{1\,1}^{s\,m} \, \| \le 2^{-\varsigma u}.$$

For $1 < \theta < \infty$, successively applying the Hölder inequality for series $\big(\| \, (c_j d_j) \,|\, \ell_1(J) \, \| \le \| \, (c_j) \,|\, \ell_a(J) \, \| \cdot \| \, (d_j) \,|\, \ell_b(J) \, \|$ with $1 \le a \le \infty$, $\frac{1}{a} + \frac{1}{b} = 1 \big)$ (c $a = \theta$) and relation (4) and taking into account the definition (2) of the norm in $\psi B_{p\,\theta}^{s\,\mathtt{d}}$, we obtain

$$\Im_1(u) \le \left(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s \theta'} \right)^{1/\theta'} \| \, (2^{\alpha s} \Delta_\alpha^\psi(f))_{\alpha s' > u\varsigma} \,|\, \ell_\theta(L_1) \, \| \ll$$

$$\ll (2^{-\varsigma u \theta'} u^{\omega - 1})^{1/\theta'} \; \| \, f \,|\, \psi B_{1\,\theta}^{s\,\mathtt{d}} \, \| \le 2^{-\varsigma u} u^{(\omega - 1)(1 - \frac{1}{\theta})}.$$

Finally, if $\theta=\infty$, then, using again (4) and (2), we find

$$\Im_1(u) \le \Bigg(\sum_{\alpha s'>u\varsigma} 2^{-\alpha s}\Bigg) \|\,(2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_\infty(L_1)\,\| \ll$$

$$\ll 2^{-\varsigma u}u^{(\omega-1)}\,\|\,f\,|\,\psi B^{s\,\mathtt{d}}_{1\,\theta}\,\| \le 2^{-\varsigma u}u^{(\omega-1)}.$$

The required upper estimate is proved.

b) Now, let $1<p<\infty$ and $1\le q\le p$. By Corollary 1, for $f(x)\in\psi\mathrm{B}^{s\,\mathtt{d}}_{p\,\theta}$ we have

$$\|\,f-S_u^{\psi,\gamma}(f)\,|\,L_q\,\| = \|\sum_{\alpha s'>u\varsigma}\varDelta_\alpha^\psi(f)\,|\,L_q\,\| \le$$

$$\le \|\sum_{\alpha s'>u\varsigma}\varDelta_\alpha^\psi(f)\,|\,L_p\,\| \le \|\,(\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_{p_*}(L_p)\,\| \equiv \Im_2(u).$$

For $1\le\theta\le p_*$, using Jensen's inequality for series $\big(\|\,(c_j)\,|\,\ell_a(J)\,\|\le\|\,(c_j)\,|\,\ell_b(J)\,\|$ for $0<b\le a\le\infty\big)$ and the definition (2) of the norm in $\psi B^{s\,\mathtt{d}}_{p\,\theta}$, we obtain

$$\Im_2(u) \le \|\,(\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_\theta(L_p)\,\| \le \|\,(2^{-\alpha s}2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_\theta(L_p)\,\|$$

$$\ll 2^{-\varsigma u}\|\,(2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_\theta(L_p)\,\| \ll 2^{-\varsigma u}\,\|\,f\,|\,\psi B^{s\,\mathtt{d}}_{p\,\theta}\,\| \le 2^{-\varsigma u}.$$

For $p_*<\theta<\infty$, applying the Hölder inequality for series with exponents $a=\frac{\theta}{p_*}$ and $b=\frac{\theta}{\theta-p_*}$ and relation (4) and taking into account (2), we find

$$\Im_2(u) = \|\,(2^{-\alpha s}2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_{p_*}(L_p)\,\| \le \Bigg(\sum_{\alpha s'>u\varsigma}2^{-\alpha s\frac{\theta p_*}{\theta-p_*}}\Bigg)^{\frac{\theta-p_*}{\theta p_*}}\times$$

$$\times\|\,(2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_\theta(L_p)\,\| \ll 2^{-\varsigma u}u^{(\omega-1)(\frac{1}{p_*}-\frac{1}{\theta})}\,\|\,f\,|\,\psi B^{s\,\mathtt{d}}_{p\,\theta}\,\| \le$$

$$\le 2^{-\varsigma u}u^{(\omega-1)(\frac{1}{p_*}-\frac{1}{\theta})}.$$

If $\theta=\infty$, then (4) and (2) obviously imply

$$\Im_2(u) = \|\,(2^{-\alpha s}2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_{p_*}(L_p)\,\| \le$$

$$\le \Bigg(\sum_{\alpha s'>u\varsigma}2^{-\alpha s p_*}\Bigg)^{1/p_*}\times\|\,(2^{\alpha s}\varDelta_\alpha^\psi(f))_{\alpha s'>u\varsigma}\,|\,\ell_\infty(L_p)\,\| \ll$$

$$\ll 2^{-\varsigma u}u^{(\omega-1)\frac{1}{p_*}}\,\|\,f\,|\,\psi B^{s\,\mathtt{d}}_{p\,\infty}\,\| \le 2^{-\varsigma u}u^{(\omega-1)\frac{1}{p_*}}.$$

Thus, the upper estimate is proved.

c) Let $1\le q<p=\infty$. In this case $p_*=2$. We also denote $q^*=\max(q,2)$. Since $\|\cdot|\,L_q\,\|\le\|\cdot|\,L_{q^*}\,\|\le\|\cdot|\,L_\infty\,\|$, we have the elementary embedding $\psi B^{s\,\mathtt{d}}_{\infty\,\theta}\subset\psi B^{s\,\mathtt{d}}_{q^*\,\theta}$, $1\le\theta\le\infty$, and the inequality

$$E_u^\gamma(\psi\mathrm{B}^{s\,\mathtt{d}}_{\infty\,\theta},L_q) \le E_u^\gamma(\psi\mathrm{B}^{s\,\mathtt{d}}_{q^*\,\theta},L_{q^*}).$$

Therefore, it follows from what has been proved in item b) that $(\min(q^*,2)=2)$

$$E_u^{\gamma}(\psi \mathrm{B}_{\infty\,\theta}^{s\,\mathtt{d}}, L_q) \ll 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{2}-\frac{1}{\theta})_+};$$

i.e., the required upper estimate is proved.

Thus, all the upper estimates of the theorem for the classes $\psi \mathrm{B}_{p\,\theta}^{s\,\mathtt{d}}$ are established.

L. Now, we obtain upper estimates for the classes $\psi \mathrm{L}_{p\,\theta}^{s\,\mathtt{d}}$.

a) Let, first, $p=q=1$. For $\theta=1$, the upper estimate is already obtained, because $\psi \mathrm{L}_{1\,1}^{s\,\mathtt{d}} = \psi \mathrm{B}_{1\,1}^{s\,\mathtt{d}}$, (see the estimates above for the classes $\psi \mathrm{B}_{1\,1}^{s\,\mathtt{d}}$).

Consider the case $1<\theta\le\infty$. Let $f\in \psi \mathrm{L}_{1\,\theta}^{s\,\mathtt{d}}$. Then

$$\| f - S_u^{\psi,\gamma}(f) \,|\, L_1 \| = \| \sum_{\alpha s' > u\varsigma} \Delta_\alpha^{\psi}(f) \,|\, L_1 \| \le$$

$$\le \| (2^{-\alpha s} 2^{\alpha s} \Delta_\alpha^{\psi}(f))_{\alpha s' > u\varsigma} \,|\, L_1(\ell_1) \| \equiv \Im_3(u).$$

For $1<\theta<\infty$, applying the Hölder inequality for series, relation (4) and the definition (3) of the norm in $\psi L_{p\,\theta}^{s\,\mathtt{d}}$, we obtain

$$\Im_3(u) \le \Bigg(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s \theta'} \Bigg)^{1/\theta'} \| (2^{\alpha s} \Delta_\alpha^{\psi}(f))_{\alpha s' > u\varsigma} \,|\, L_1(\ell_\theta) \| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{\theta})} \| f \,|\, \psi L_{1\,\theta}^{s\,\mathtt{d}} \| \le 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{\theta})}.$$

If $\theta=\infty$, then in a similar way we arrive at

$$\Im_3(u) \le \sum_{\alpha s' > u\varsigma} 2^{-\alpha s} \| (2^{\alpha s} \Delta_\alpha^{\psi}(f))_{\alpha s' > u\varsigma} \,|\, L_1(\ell_\infty) \| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)} \| f \,|\, \psi L_{1\,\infty}^{s\,\mathtt{d}} \| \le 2^{-\varsigma u} u^{(\omega-1)}.$$

Thus, the required upper estimate

$$E_u^{\gamma}(\psi \mathrm{L}_{1\theta}^{s,\mathtt{d}}, L_1) \ll 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{\theta})}$$

is established for $1\le\theta\le\infty$.

b) Now, let $1<p<\infty$ and $1\le q\le p$. For $f\in \psi \mathrm{L}_{p\,\theta}^{s\,\mathtt{d}}$, by Theorem 4 we find

$$\| f - S_u^{\psi,\gamma}(f) \,|\, L_q \| = \| \sum_{\alpha s' > u\varsigma} \Delta_\alpha^{\psi}(f) \,|\, L_q \| \le$$

$$\le \| \sum_{\alpha s' > u\varsigma} \Delta_\alpha^{\psi}(f) \,|\, L_p \| \asymp \| (2^{-\alpha s} 2^{\alpha s} \Delta_\alpha^{\psi}(f))_{\alpha s' > u\varsigma} \,|\, L_p(\ell_2) \| \equiv \Im_4(u).$$

For $1\le q\le 2$, applying Jensen's inequality for series and taking into account (3), we obtain $(\alpha s \ge \alpha s' > u\varsigma)$

$$\Im_4(u) \le \| (2^{-\alpha s} 2^{\alpha s} \Delta_\alpha^{\psi}(f))_{\alpha s' > u\varsigma} \,|\, L_p(\ell_\theta) \| \le$$

$$\leq 2^{-\varsigma u} \| \, (2^{\alpha s} \Delta^{\psi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_\theta) \, \| \ll 2^{-\varsigma u} \| \, f \, | \, \psi L^{s\,\mathtt{d}}_{p\,\theta} \, \| \, \leq 2^{-\varsigma u}.$$

If $2 < \theta < \infty$, then, successively applying the Hölder inequality for series with exponents $a = \frac{\theta}{\theta - 2}$ and $b = \frac{\theta}{2}$, relation (4) and (3), we obtain

$$\Im_4(u) \leq \left(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s \frac{2\theta}{\theta - 2}} \right)^{\frac{\theta - 2}{2\theta}} \| \, (2^{\alpha s} \Delta^{\psi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_\theta) \, \|$$

$$\ll 2^{-\varsigma u} u^{(\omega - 1)(\frac{1}{2} - \frac{1}{\theta})} \| \, f \, | \, \psi L^{s\,\mathtt{d}}_{p\,\theta} \, \| \, \ll 2^{-\varsigma u} u^{(\omega - 1)(\frac{1}{2} - \frac{1}{\theta})}.$$

If $q = \infty$, then, as above, relation (4) combined with (3) yields

$$\Im_4(u) \leq \left(\sum_{\alpha s' > u\varsigma} 2^{-2\alpha s} \right)^{1/2} \| \, (2^{\alpha s} \Delta^{\psi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_\infty) \, \| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega - 1)\frac{1}{2}} \| \, f \, | \, \psi L^{s\,\mathtt{d}}_{p\,\infty} \, \| \, \leq 2^{-\varsigma u} u^{(\omega - 1)\frac{1}{2}},$$

i.e., the required upper estimate is established.

Thus, all the upper estimates in the theorem for the classes $\psi \mathrm{L}^{s\,m}_{p\,\theta}$ are also proved.

6 Proof of Lower Bounds in Theorem 1

To obtain lower bounds in Theorem 1 we need the notion of Fourier width. Recall that $N-$th Fourier width (or, which is the same, orthowidth) of a set $\mathrm{F} \subset L_q$ is defined as

$$\varphi_N(\mathrm{F}, L_q) = \inf_{\{g_j\}_{j=1}^N} \sup_{f \in \mathrm{F}} \| f - \sum_{j=1}^{N} \langle f, g_j \rangle g_j \ | \ L_q \|,$$

where the infimum is taken over all orthonormal (in L_2) systems $\{g_j\}_{j=1}^N \subset L_\infty$.

Notion of Fourier width was invented by V.N. Temlyakov [6] in 1982. Remark that extensive literature has been devoted to the estimates sharp in order for Fourier widths of various classes of smooth functions in one and several variables; here we only refer to monographs [7,8] and articles [1,3,4,6], where one can also find detailed history of the problem and comprehensive references.

By virtue of (5) (with $L = 1$), the dimension $\psi(\gamma, u)$ of the linear span of the set $\{\psi^{\iota}_{\alpha\lambda} \, | \, \iota \in \mathrm{E}^{\mathtt{d}}(\alpha), \ \lambda \in \Lambda(\mathtt{d}, \alpha), \, \alpha \in \mathbb{N}_0^k : \alpha\gamma \leq u\}$ is of order $2^u u^{\omega - 1}$. Hence, choosing $N \geq \psi(\gamma, u)$, we get the inequality

$$E^{\gamma}_{u}(\mathrm{F}, L_q) \geq \varphi_N(\mathrm{F}, L_q). \tag{6}$$

Under the hypotheses of Theorem 1 we have:

$$\varphi_N(\psi \mathrm{B}^{s\,\mathtt{d}}_{p\,\theta}, L_q) \asymp \left(\frac{\log^{\omega - 1} N}{N} \right)^{\sigma} (\log^{\omega - 1} N)^{(\frac{1}{p_*} - \frac{1}{\theta})_+}$$

and

$$\varphi_N(\psi\mathrm{L}^{s\,\mathtt{d}}_{p\,\theta}, L_q) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{\theta})_+}.$$

This theorem is full analog of the theorem from [3] for the the Nikolskii – Besov and Lizorkin – Triebel type classes associated with system $\psi^{(\mathtt{d})}$.

Note that in [3] was obtained the estimates of Fourier widths for the Nikolskii – Besov and Lizorkin – Triebel type classes w.r.t. $n-$multiple system of Haar wavelets.

These estimates with $N \geq \psi(\gamma, u)$ such that $N \asymp \psi(\gamma, u)$ combined with the inequality (6) easily imply lower bounds for $E^{\gamma}_{u}(\psi\mathrm{F}^{s\,\mathtt{d}}_{p\,\theta}, L_q)$ ($\mathrm{F} \in \{\mathrm{B}, \mathrm{L}\}$) in Theorem 1.

Acknowledgement. Research was financially supported by the Ministry of Education and Sciences, Republic of Kazakhstan, under the grant 5130/GF4. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Bazarkhanov, D.B.: Wavelet approximation and Fourier widths of classes of periodic functions of several variables. I. Proc. Steklov Inst. Math. **269**(1), 2–24 (2010)
2. Bazarkhanov, D.B.: Wavelet approximation and Fourier widths of classes of periodic functions of several variables. II. Anal. Math. **38**(4), 249–289 (2012)
3. Bazarkhanov, D.B.: Hyperbolic cross approximation of some multivariate function classes w. r. t. multiple Haar system on the unit cube. Int. J. Pure Appl. Math. (in press) (2017)
4. Temlyakov, V.N.: Widths of some classes of functions of several variables. Sov. Math. Dokl. **26**, 619–622 (1982)
5. Temlyakov, V.N.: Approximations of functions with bounded mixed derivative. Proc. Steklov Inst. Math. **178**, 1–121 (1989)
6. Temlyakov, V.N.: Estimates for the asymptotic characteristics of classes of functions with bounded mixed derivative or difference. Proc. Steklov Inst. Math. **189**, 161–197 (1990)
7. Temlyakov, V.N.: Approximation of Periodic Functions. Computational Mathematics Analytical Series. Nova Science Publ, Commack (1993)
8. Wojtaszczyk, P.: A mathematical introduction to wavelets. London Mathematical Society Student Texts, vol. 37. CUP, Cambridge (1997)

Hyperbolic Cross Approximation of Some Function Classes with Respect to Multiple Haar System on the Unit Cube

Dauren B. Bazarkhanov(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st.,
Almaty 050010, Kazakhstan
dauren.mirza@gmail.com

Abstract. In this paper we obtain estimates (sharp in order) for hyperbolic cross approximation w.r.t. multiple ($\mathtt{n}$−fold) Haar system $\chi^{(\mathtt{n})}$ of the Nikol'skii – Besov and Lizorkin – Triebel type classes associated with this Haar system in the space $L_r([0,1]^n)$ for a number of relations between the parameters of the classes and the space.

Keywords: Approximation · Hyperbolic cross · Multiple Haar system · The Nikol'skii – Besov and the Lizorkin – Triebel spaces associated with the multiple Haar system

1 Introduction

As usual, let $L_r = L_r([0,1]^n)\,(1 \le r \le \infty,\ 2 \le n \in \mathbb{N})$ be the space of all (equivalence classes of) measurable functions $f : [0,1]^n \to \mathbb{C}$ that are r power integrable (essentially bounded for $r = \infty$) on $[0,1]^n$, endowed with the standard norm

$$\| f \mid L_r \| = \| f \mid L_r([0,1]^n) \| = \left(\int_{[0,1]^n} |\, f(x)\,|^r dx \right)^{1/r} (1 \le r < \infty),$$

$$\| f \mid L_\infty \| = \| f \mid L_\infty([0,1]^n) \| = \text{ess sup}\,\{|\, f(x)\,| : x \in [0,1]^n \}.$$

Let $\Psi = \{\psi_i \,|\, i \in \mathscr{I}\}$ be a countable collection of functions in L_∞ which are orthonormal in L_2 and let $\{J(u) \subset \mathscr{I} \,|\, u \in \mathbb{N}\}$ be such that $J(u) \subset J(u+1)$ and $\#J(u) < \infty$ for all $u \in \mathbb{N}$, $\cup_{u\in\mathbb{N}} J(u) = \mathscr{I}$.

For $f \in L_1$, we consider Fourier sums w. r. t. system Ψ of the form

$$S_u^{\Psi}(f,x) = \sum_{i \in J(u)} \langle f, \psi_i \rangle \psi_i(x),$$

where $\langle f, g\rangle = \int_{[0,1]^n} f(x)\overline{g}(x)dx$ ($\overline{z}$ is the number complex conjugate to $z \in \mathbb{C}$).

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_3

For a set $\mathrm{F} \in L_r$, we denote

$$E_u(\mathrm{F}, \varPsi, L_r) = \sup\{\|f - S_u^{\varPsi}(f,x) \mid L_r\| \mid f \in \mathrm{F}\}. \tag{1}$$

In the present paper, we study quantities (1) in the special case of $\mathtt{n}-$fold Haar system $\chi^{(\mathtt{n})}$ and so called hyperbolic crosses used as $\varPsi$ and $J(u)$, respectively (for definitions see below). Main goal is to give estimates (sharp in order) for hyperbolic cross approximation w. r. t. system $\chi^{(\mathtt{n})}$ of the Nikol'skii – Besov and Lizorkin – Triebel type classes associated with that system (for definition see sect. 2) in the space $L_r([0,1]^n)$ for a number of relations between the parameters of the classes and the space in the style of theorem 4.1 from [2].

First we introduce some notations. Let $n \in \mathbb{N}$, $\mathrm{z}_n = \{1,\ldots,n\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0,+\infty)$.

Let $k \in \mathbb{N}:\ k \le n$. Fix a multi-index $\mathtt{n} = (n_1,\ldots,n_k) \in \mathbb{N}^k$ such that $n_1 + \cdots + n_k = n$ (thus, $\mathtt{n} = n$ if $k=1$, while if $k=n$, then $\mathtt{n} = \overrightarrow{1} = (1,\ldots,1) \in \mathbb{N}^n$) and representation of $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ in the form $x = (x^1,\ldots,x^k)$, where $x^\kappa \in \mathbb{R}^{n_\kappa}$, $\kappa \in \mathrm{z}_k$.

Denote

$$\mathrm{E}^n = \mathrm{E}^n(0) = \{\mathit{0},\mathit{1}\}^n, \quad \mathrm{E}^n(1) = \mathrm{E}^n \setminus \{(\mathit{0},\ldots,\mathit{0})\};$$

$$\varLambda(n,j) = \mathbb{Z}^n \cap [0, 2^j - 1]^n,\ j \in \mathbb{N}_0.$$

and define

$$\chi^{(0)}(t) = \begin{cases} 1, & t \in [0,1); \\ 0, & t \in \mathbb{R} \setminus [0,1), \end{cases}$$

and

$$\chi^{(1)}(t) = \begin{cases} \frac{1}{2}, & t \in [0,\frac{1}{2}); \\ -\frac{1}{2}, & t \in [\frac{1}{2},1); \\ 0, & t \in \mathbb{R} \setminus [0,1); \end{cases}$$

further, multiple ($\mathtt{n}-$fold) Haar system is defined as follows:

$$\chi^{(\mathtt{n})} = \{\chi^\iota_{\alpha\lambda}(x) \ \mid\ \iota \in \mathrm{E}^{\mathtt{n}}(\alpha),\ \lambda \in \varLambda(\mathtt{n},\alpha),\ \alpha \in \mathbb{N}_0^k\},$$

where

$$\chi^\iota_{\alpha\lambda}(x) = \prod_{\kappa=1}^k \chi^{\iota^\kappa}_{\alpha_\kappa \lambda^\kappa}(x^\kappa),$$

and

$$\chi^{\iota^\kappa}_{j\lambda^\kappa}(x^\kappa) = 2^{\frac{jn_\kappa}{2}} \chi^{\iota^\kappa}(2^j x^\kappa - \lambda^\kappa), \quad \chi^{\iota^\kappa}(x^\kappa) = \prod_{\nu \in \mathrm{K}_\kappa} \chi^{(\iota_\nu)}(x_\nu);$$

here

$$\mathrm{E}^{\mathtt{n}}(\alpha) = \mathrm{E}^{n_1}(\mathrm{sign}(\alpha_1)) \otimes \cdots \otimes \mathrm{E}^{n_k}(\mathrm{sign}(\alpha_k)),$$

$$\varLambda(\mathtt{n},\alpha) = \varLambda(n_1,\alpha_1) \times \ldots \times \varLambda(n_k,\alpha_k).$$

It is clear that the system $\chi^{(\mathtt{n})}$ is orthonormal in $L_2([0,1]^n)$. Furthermore, the system $\chi^{(\mathtt{n})}$ is unconditional basis in $L_r([0,1]^n)$ with $1 < r < \infty$: in the case $k=1$ this fact is proved in [13, ch. 8], general case $1 \le k \le n$ follows from here because the system $\chi^{(\mathtt{n})}$ (w.r.t. variable x) is tensor product of the systems $\chi^{(n_\kappa)}$ (w.r.t. variables x^κ), $\kappa \in \mathrm{z}_k$.

Define operators Δ^χ_α $(\alpha \in \mathbb{N}_0^k)$ as follows: for $f \in L_1$, let

$$\Delta^\chi_\alpha(f,x) = \sum_{\iota \in \mathrm{E}^{\mathtt{n}}(\alpha)} \sum_{\lambda \in \Lambda(\mathtt{n},\alpha)} f^\iota_{\alpha\lambda} \chi^\iota_{\alpha\lambda}(x)$$

be its dyadic packet, where

$$f^\iota_{\alpha\lambda} = \int_{[0,1]^n} f(x)\, \chi^\iota_{\alpha\lambda}(x)\, dx.$$

Furthermore, for $f \in L_1$, we define its so called hyperbolic cross Fourier sum w. r. t. system $\chi^{(\mathtt{n})}$ (with fixed $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k_+$) by the formula

$$S^{\chi,\gamma}_u(f,x) = \sum_{\alpha\gamma \le u} \Delta^\chi_\alpha(f,x) \quad (u \in \mathbb{N})$$

(here $\alpha\gamma := \alpha_1\gamma_1 + \cdots + \alpha_k\gamma_k$). In the sequel, we will denote quantity (1) with $S^{\chi,\gamma}_u$ instead of S^Ψ_u simply as $E^\gamma_u(\mathrm{F}, L_r)$.

Finally, we define the Nikol'skii – Besov and Lizorkin – Triebel type function spaces (and classes) associated with system $\chi^{(\mathtt{n})}$ whose approximation properties w.r.t. that system $\chi^{(\mathtt{n})}$ will be studied in the next sections.

Let $1 \le p, q \le \infty$ and let $\ell_q \equiv \ell_q(\mathbb{N}_0^k)$ be the space of complex number sequences $(c_\alpha) = (c_\alpha \,|\, \alpha \in \mathbb{N}_0^k)$ with the finite norm

$$\|(c_\alpha) \,|\, \ell_q\| = \Big(\sum_{\alpha \in \mathbb{N}_0^k} |\, c_\alpha\,|^q \Big)^{1/q} \; (1 \le q < \infty), \quad \|(c_\alpha) \,|\, \ell_\infty\| = \sup_{\alpha \in \mathbb{N}_0^k} |\, c_\alpha\,|;$$

let $\ell_q(L_p) \equiv \ell_q(L_p([0,1]^n))$ (respectively $L_p(\ell_q) \equiv L_p([0,1]^n; \ell_q)$) be the space of function sequences $(g_\alpha(x)) = (g_\alpha(x) \,|\, \alpha \in \mathbb{N}_0^k)$ $(x \in [0,1]^n)$ with the finite norm

$$\|\,(g_\alpha(x)) \,|\, \ell_q(L_p)\,\| = \|\,(\,\|\,g_\alpha \,|\, L_p\,\|) \,|\, \ell_q\,\|$$

(respectively

$$\|\,(g_\alpha(x)) \,|\, L_p(\ell_q)\,\| = \|\,\|\,(g_\alpha(\cdot)) \,|\, \ell_q\,\| \,|\, L_p\,\|).$$

Now we are in position to introduce function spaces (as well as classes) under consideration.

Definition 1. Let $s = (s_1, \dots, s_k) \in \mathbb{R}^k_+$, $1 \le p, q \le \infty$. Then

`i)` the Nikol'skii – Besov type space $\chi B^{s\,\mathtt{n}}_{p\,q} \equiv \chi B^{s\,\mathtt{n}}_{p\,q}([0,1]^n)$ associated with the system $\chi^{(\mathtt{n})}$ consists of all functions $f \in L_p$ for which the norm

$$\|\,f \,|\, \chi B^{s\,\mathtt{n}}_{p\,q}\,\| = \|(2^{\alpha s} \Delta^\chi_\alpha(f,x)) \,|\, \ell_q(L_p)\| \tag{2}$$

is finite;

`ii)` the Lizorkin – Triebel type space $\chi L^{s\,\mathtt{n}}_{p\,q} \equiv \chi L^{s\,\mathtt{n}}_{p\,q}([0,1]^n)$ associated with the system $\chi^{(\mathtt{n})}$ consists of all functions $f \in L_p$ for which the norm

$$\| f \,|\, \chi L^{s\,\mathtt{n}}_{p\,q} \| \;=\; \|(2^{\alpha s} \Delta^{\chi}_{\alpha}(f,x)) \,|\, L_p(\,\ell_q)\| \tag{3}$$

is finite.

Unit balls $\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q} \equiv \chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}([0,1]^n)$ and $\chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q} \equiv \chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q}([0,1]^n)$ of these spaces will be called the Nikol'skii – Besov and Lizorkin – Triebel classes associated with the system $\chi^{(\mathtt{n})}$, respectively.

Remark 1. Note that the approximation properties of the system $\chi^{(\mathtt{n})}$ with $n \geq 2$ (particularly, nonlinear approximation properties: best $N-$term approximations, different Greedy type constructive algorithms etc.) have been studied intensively in last two decades for two special cases: i) $n-$dimensional case where $\mathtt{n} = n$, i.e. $k = 1$; ii) (pure) multiple case where $\mathtt{n} = (1, \ldots, 1)$, i.e. $k = n$ (see, for example, recent monograph [12]).

2 Main Results

For $s = (s_1, \ldots, s_k) \in \mathbb{R}^k_+$ and $1 \leq p, r \leq \infty$ such that $\bar{s}_\kappa \equiv s_\kappa - n_\kappa(\frac{1}{p} - \frac{1}{r})_+ > 0$, $\kappa \in \mathrm{z}_k$, define the following vectors and numbers (here $a_+ = \max\{a, 0\}$ for a number $a \in \mathbb{R}$).

Put $\bar{s} = s - \mathtt{n}(\frac{1}{p} - \frac{1}{r})_+$, $\varsigma_\kappa = \frac{s_\kappa}{n_\kappa}$, $\bar{\varsigma}_\kappa = \frac{\bar{s}_\kappa}{n_\kappa}$, $\kappa \in \mathrm{z}_k$; without loss of generality, assume that $\varsigma \equiv \min\{\, \varsigma_\kappa \mid \kappa \in \mathrm{z}_k \,\} = \varsigma_1 = \ldots = \varsigma_\omega < \varsigma_\kappa$, $\kappa \in \mathrm{z}_k \setminus \mathrm{z}_\omega$ for certain $\omega \in \mathrm{z}_k$; $\bar{\varsigma} = \varsigma - (\frac{1}{p} - \frac{1}{r})_+$. Take a vector $\tilde{\varsigma} = (\tilde{\varsigma}_1, \ldots, \tilde{\varsigma}_k)$ such that $\bar{\varsigma} = \tilde{\varsigma}_1 = \ldots = \tilde{\varsigma}_\omega < \tilde{\varsigma}_\kappa < \bar{\varsigma}_\kappa$, $\kappa = \omega + 1, \ldots, k$, and put $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_k)$ with $\tilde{s}_\kappa = \tilde{\varsigma}_\kappa n_\kappa$ ($\kappa \in \mathrm{z}_k$) and $\gamma = \frac{1}{\bar{\varsigma}}\tilde{s}$.

We will use the symbols $\ll$ and $\asymp$ to show relations between the orders of quantities : for functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ we write $F(u) \ll H(u)$ as $u \to \infty$, if there exists a constant $C = C(F, H) > 0$ such that the inequality $F(u) \leq CH(u)$ holds true for $u \geq u_0 > 0$, and $F(u) \asymp H(u)$, if $F(u) \ll H(u)$ and $H(u) \ll F(u)$, simultaneously.

Below, for $1 \leq p \leq \infty$ put $p_* = \min\{p, 2\}$.

Theorem 1. *Let $1 \leq r \leq p \leq \infty$, $r < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}^k_+$. Then*

$$E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}, L_r\,) \asymp 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{p_*} - \frac{1}{q})_+}.$$

If, in addition, $1 < p < \infty$, then

$$E^{\gamma}_u(\chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q}, L_r\,) \asymp 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{2} - \frac{1}{q})_+}.$$

Moreover,

$$E^{\gamma}_u(\chi \mathrm{L}^{s\,\mathtt{n}}_{1\,q}, L_1\,) \asymp 2^{-\varsigma u} u^{(\omega-1)(1 - \frac{1}{q})_+}.$$

Theorem 2. *Let* $1 \le p < r < \infty$, $1 \le q \le \infty$ *and let* $s \in \mathbb{R}^k_+$ *be such that* $\varsigma > \frac{1}{p} - \frac{1}{r}$. *Then*

$$E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathfrak{n}}_{p\,q}, L_r) \asymp 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u} u^{(\omega-1)(\frac{1}{r} - \frac{1}{q})_+};$$

$$E^{\gamma}_u(\chi \mathrm{L}^{s\,\mathfrak{n}}_{p\,q}, L_r) \asymp 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u}.$$

Theorem 3. *Let* $1 \le p, q \le \infty$ *and let* $s \in \mathbb{R}^k_+$ *be such that* $\varsigma > \frac{1}{p}$. *Then*

$$E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathfrak{n}}_{p\,q}, L_\infty) \asymp 2^{-(\varsigma - \frac{1}{p})u} u^{(\omega-1)(1 - \frac{1}{q})_+}.$$

If, in addition, $p < \infty$, *then*

$$E^{\gamma}_u(\chi \mathrm{L}^{s\,\mathfrak{n}}_{p\,q}, L_\infty) \asymp 2^{-(\varsigma - \frac{1}{p})u} u^{(\omega-1)(1 - \frac{1}{p})}.$$

Remark 2. i) These theorems are analogs of theorem 4.1 from [2] for the Nikol'skii – Besov and Lizorkin – Triebel type classes associated with system $\chi^{(\mathtt{n})}$ and hyperbolic cross approximation w.r.t. that system.

ii) Note that the estimates in Theorems 1 and 2 for quantities $E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}, L_r)$ in special case $\omega = n = k$, $(\frac{1}{p} - \frac{1}{r})_+ < \varsigma < 1$, are proved in theorem 1 from [1] $(q = \infty)$ and in theorem 1 from [7] $(1 \le q < \infty)$.

Below, when proving upper bounds from Theorem 1–3 and estimating the dimensions of the corresponding subspaces, we will systematically making use the following lemma from [2] (for proof see also [3]).

Lemma 1. *Let* $\beta, \gamma \in \mathbb{R}^k_+$ *be such that* $\beta_\nu = \gamma_\nu$ *for* $\nu \in \mathrm{z}_\omega$ *and* $\beta_\nu > \gamma_\nu$ *for* $\nu \in \mathrm{z}_k \setminus \mathrm{z}_\omega$, *and let* $L > 0$. *Then the following relations are valid:*

$$\mathscr{I}^{\beta,\gamma}_L(u) \equiv \sum_{\alpha \in \mathbb{N}^n_0 : \alpha\gamma > u} 2^{-L\alpha\beta} \asymp 2^{-Lu} u^{\omega-1} \quad \text{as} \quad u \to +\infty; \tag{4}$$

$$\mathscr{J}^{\gamma,\beta}_L(u) \equiv \sum_{\alpha \in \mathbb{N}^n_0 : \alpha\beta \le u} 2^{L\alpha\gamma} \asymp 2^{Lu} u^{\omega-1} \quad \text{as} \quad u \to +\infty. \tag{5}$$

3 Proof of Upper Bounds in Theorem 1

First we formulate the Littlewood – Paley – type theorem related to the system $\chi^{(\mathtt{n})}$ and its corollary.

Theorem 4. *Let* $1 < p < \infty$. *Then there exists a constant* $C = C(\mathfrak{n}, p) > 0$ *such that*

$$C^{-1} \| f \,|\, L_p \| \le \| (\varDelta^{\chi}_\alpha(f, x)) \,|\, L_p([0,1]^n; \ell_2) \| \le C \| f \,|\, L_p \|,$$

for all functions $f \in L_p$.

Since the system $\chi^{(\mathtt{n})}$ is an unconditional basis in L_p, Theorem 4 can easily be proved by arguments involving the classical Khinchin inequality for Rademacher functions (see, for instance, [13, ch. 8], where the case $k = 1$ is analyzed).

Corollary 1. *Let* $1 < p < \infty$, $p_* = \min(p, 2)$. *Then, for any function* $f \in L_p$

$$\| f \,|\, L_p \| \le C(\mathtt{n}, p) \,\| \,(\Delta^{\chi}_{\alpha}(f, x)) \,|\, \ell_{p_*}(L_p([0,1]^n)) \,\|.$$

B. We begin with upper estimates for the classes $\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}$.

a) Let, first, $p = r = 1$. Then for $f \in \chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}$, we have

$$\| f - S^{\chi,\gamma}_{u}(f) \,|\, L_1 \| = \| \sum_{\alpha s' > u\varsigma} \Delta^{\chi}_{\alpha}(f) \,|\, L_1 \| \le$$

$$\| \,(\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \,|\, \ell_1(L_1) \,\| = \| \,(2^{-\alpha s} 2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \,|\, \ell_1(L_1) \,\| \equiv \Im_1(u).$$

By the definition (2) of the norm in space $\chi B^{s\,\mathtt{n}}_{p\,q}$, for $q = 1$, we obviously have $(\alpha s \ge \alpha s' > u\varsigma)$

$$\Im_1(u) \le 2^{-\varsigma u} \| \,(2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \,|\, \ell_1(L_1) \,\| \ll 2^{-\varsigma u} \,\| f \,|\, \chi B^{s\,m}_{1\,1} \,\| \le 2^{-\varsigma u}.$$

For $1 < q < \infty$, successively applying the Hölder inequality for series $\big(\| \,(c_j d_j) \,|\, \ell_1(J) \,\| \le \| \,(c_j) \,|\, \ell_a(J) \,\| \cdot \| \,(d_j) \,|\, \ell_b(J) \,\|$ with $1 \le a \le \infty$, $\frac{1}{a} + \frac{1}{b} = 1\big)$ (c $a = q$) and relation (4) and taking into account the definition (2) of the norm in $\chi B^{s\,\mathtt{n}}_{p\,q}$, we obtain (here and below, $q' = \frac{q}{q-1}$)

$$\Im_1(u) \le \left(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s q'} \right)^{1/q'} \| \,(2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \,|\, \ell_q(L_1) \,\| \ll$$

$$\ll (2^{-\varsigma u q'} u^{\omega - 1})^{1/q'} \;\| f \,|\, \chi B^{s\,\mathtt{n}}_{1\,q} \,\| \le 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{q})}.$$

Finally, if $q = \infty$, then, using again (4) and (2), we find

$$\Im_1(u) \le \left(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s} \right) \| \,(2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \,|\, \ell_\infty(L_1) \,\| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)} \,\| f \,|\, \chi B^{s\,\mathtt{n}}_{1\,q} \,\| \le 2^{-\varsigma u} u^{(\omega-1)}.$$

The required upper estimate is proved.

b) Now, let $1 < p < \infty$ and $1 \le r \le p$. By Corollary 1, for $f(x) \in \chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}$ we have

$$\| f - S^{\chi,\gamma}_{u}(f) \,|\, L_r \| = \| \sum_{\alpha s' > u\varsigma} \Delta^{\chi}_{\alpha}(f) \,|\, L_r \| \le$$

$$\le \| \sum_{\alpha s' > u\varsigma} \Delta^{\chi}_{\alpha}(f) \,|\, L_p \| \le \| \,(\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \,|\, \ell_{p_*}(L_p) \,\| \equiv \Im_2(u).$$

For $1 \le q \le p_*$, using Jensen's inequality for series $\big(\|\,(c_j)\,|\,\ell_a(J)\,\| \le \|\,(c_j)\,|\,\ell_b(J)\,\|$ for $0 < b \le a \le \infty\big)$ and the definition (2) of the norm in $\chi B^{s\,\mathtt{n}}_{p\,q}$, we obtain

$$\Im_2(u) \le \|\,(\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_q(L_p)\,\| \le \|\,(2^{-\alpha s}2^{\alpha s}\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_q(L_p)\,\|$$

$$\ll 2^{-\varsigma u}\|\,(2^{\alpha s}\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_q(L_p)\,\| \ll 2^{-\varsigma u}\,\|\,f\,|\,\chi B^{s\,\mathtt{n}}_{p\,q}\,\| \le 2^{-\varsigma u}.$$

For $p_* < q < \infty$, applying the Hölder inequality for series with exponents $a = \frac{q}{p_*}$ and $b = \frac{q}{q-p_*}$ and relation (4) and taking into account (2), we find

$$\Im_2(u) = \|\,(2^{-\alpha s}2^{\alpha s}\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_{p_*}(L_p)\,\| \le \Bigg(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s\frac{qp_*}{q-p_*}}\Bigg)^{\frac{q-p_*}{qp_*}}$$

$$\times\|\,(2^{\alpha s}\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_q(L_p)\,\| \ll 2^{-\varsigma u}u^{(\omega-1)(\frac{1}{p_*}-\frac{1}{q})}\,\|\,f\,|\,\chi B^{s\,\mathtt{n}}_{p\,q}\,\| \le$$

$$\le 2^{-\varsigma u}u^{(\omega-1)(\frac{1}{p_*}-\frac{1}{q})}.$$

If $q = \infty$, then (4) and (2) obviously imply

$$\Im_2(u) = \|\,(2^{-\alpha s}2^{\alpha s}\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_{p_*}(L_p)\,\| \le$$

$$\le \Bigg(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s p_*}\Bigg)^{1/p_*} \times \|\,(2^{\alpha s}\Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma}\,|\,\ell_{\infty}(L_p)\,\| \ll$$

$$\ll 2^{-\varsigma u}u^{(\omega-1)\frac{1}{p_*}}\,\|\,f\,|\,\chi B^{s\,\mathtt{n}}_{p\,\infty}\,\| \le 2^{-\varsigma u}u^{(\omega-1)\frac{1}{p_*}}.$$

Thus, the upper estimate is proved.

c) Let $1 \le r < p = \infty$. In this case $p_* = 2$. We also denote $r^* = \max(r, 2)$. Since $\|\cdot\,|\,L_r\,\| \le \|\cdot\,|\,L_{r^*}\,\| \le \|\cdot\,|\,L_{\infty}\,\|$, we have the elementary embedding $\chi B^{s\,\mathtt{n}}_{\infty\,q} \subset \chi B^{s\,\mathtt{n}}_{r^*\,q}$, $1 \le q \le \infty$, and the inequality

$$E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathtt{n}}_{\infty\,q}, L_r) \le E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathtt{n}}_{r^*\,q}, L_{r^*}).$$

Therefore, it follows from what has been proved in item b) that $(\min(r^*, 2) = 2)$

$$E^{\gamma}_u(\chi \mathrm{B}^{s\,\mathtt{n}}_{\infty\,q}, L_r) \ll 2^{-\varsigma u}u^{(\omega-1)(\frac{1}{2}-\frac{1}{q})_+};$$

i.e., the required upper estimate is proved.

Thus, all the upper estimates of the theorem for the classes $\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}$ are established.

L. Now, we obtain upper estimates for the classes $\chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q}$.

a) Let, first, $p = r = 1$. For $q = 1$, the upper estimate is already obtained, because $\chi \mathrm{L}^{s\,\mathtt{n}}_{1\,1} = \chi \mathrm{B}^{s\,\mathtt{n}}_{1\,1}$, (see the estimates above for the classes $\chi \mathrm{B}^{s\,\mathtt{n}}_{1\,1}$).

Consider the case $1 < q \le \infty$. Let $f \in \chi \mathrm{L}^{s\,\mathtt{n}}_{1\,q}$. Then

$$\|\,f - S^{\chi,\,\gamma}_u(f)\,|\,L_1\,\| = \|\sum_{\alpha s' > u\varsigma} \Delta^{\chi}_{\alpha}(f)\,|\,L_1\,\| \le$$

$$\leq \| \, (2^{-\alpha s} 2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_1(\ell_1) \, \| \equiv \Im_3(u).$$

For $1 < q < \infty$, applying the Hölder inequality for series, relation (4) and the definition (3) of the norm in $\chi L^{s\,\mathtt{n}}_{p\,q}$, we obtain

$$\Im_3(u) \leq \Big(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s q'} \Big)^{1/q'} \| \, (2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_1(\ell_q) \, \| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{q})} \| \, f \, | \, \chi L^{s\,\mathtt{n}}_{1\,q} \, \| \leq 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{q})}.$$

If $q = \infty$, then in a similar way we arrive at

$$\Im_3(u) \leq \sum_{\alpha s' > u\varsigma} 2^{-\alpha s} \| \, (2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_1(\ell_\infty) \, \| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)} \| \, f \, | \, \chi L^{s\,\mathtt{n}}_{1\,\infty} \, \| \leq 2^{-\varsigma u} u^{(\omega-1)}.$$

Thus, the required upper estimate

$$E^{\gamma}_{u}(\chi \mathrm{L}^{s\,\mathtt{n}}_{1q}, L_1) \ll 2^{-\varsigma u} u^{(\omega-1)(1-\frac{1}{q})}$$

is established for $1 \leq q \leq \infty$.

b) Now, let $1 < p < \infty$ and $1 \leq r \leq p$. For $f \in \chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q}$, by Theorem 4 we find

$$\| \, f - S^{\chi,\gamma}_{u}(f) \, | \, L_r \, \| = \| \sum_{\alpha s' > u\varsigma} \Delta^{\chi}_{\alpha}(f) \, | \, L_r \, \| \leq$$

$$\leq \| \sum_{\alpha s' > u\varsigma} \Delta^{\chi}_{\alpha}(f) \, | \, L_p \, \| \asymp \| \, (2^{-\alpha s} 2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_2) \, \| \equiv \Im_4(u).$$

For $1 \leq q \leq 2$, applying Jensen's inequality for series and taking into account (3), we obtain ($\alpha s \geq \alpha s' > u\varsigma$)

$$\Im_4(u) \leq \| \, (2^{-\alpha s} 2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_q) \, \| \leq$$

$$\leq 2^{-\varsigma u} \| \, (2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_q) \, \| \ll 2^{-\varsigma u} \| \, f \, | \, \chi L^{s\,\mathtt{n}}_{p\,q} \, \| \leq 2^{-\varsigma u}.$$

If $2 < q < \infty$, then, successively applying the Hölder inequality for series with exponents $a = \frac{q}{q-2}$ and $b = \frac{q}{2}$, relation (4), and (3), we obtain

$$\Im_4(u) \leq \Big(\sum_{\alpha s' > u\varsigma} 2^{-\alpha s \frac{2q}{q-2}} \Big)^{\frac{q-2}{2q}} \| \, (2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_q) \, \|$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{2}-\frac{1}{q})} \| \, f \, | \, \chi L^{s\,\mathtt{n}}_{p\,q} \, \| \ll 2^{-\varsigma u} u^{(\omega-1)(\frac{1}{2}-\frac{1}{q})}.$$

If $q = \infty$, then, as above, relation (4) combined with (3) yields

$$\Im_4(u) \leq \Big(\sum_{\alpha s' > u\varsigma} 2^{-2\alpha s} \Big)^{1/2} \| \, (2^{\alpha s} \Delta^{\chi}_{\alpha}(f))_{\alpha s' > u\varsigma} \, | \, L_p(\ell_\infty) \, \| \ll$$

$$\ll 2^{-\varsigma u} u^{(\omega-1)\frac{1}{2}} \| \, f \, | \, \chi L^{s\,\mathtt{n}}_{p\,\infty} \, \| \leq 2^{-\varsigma u} u^{(\omega-1)\frac{1}{2}},$$

i.e., the required upper estimate is established.

Thus, all the upper estimates in Theorem 1 for the classes $\chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q}$ are also proved.

4 Proof of Upper Bounds in Theorems 2 and 3

Recall that for $s = (s_1, \ldots, s_k) \in \mathbb{R}^k_+$ and $1 \le p < r \le \infty$, $\bar{s} = s - \mathtt{n}(\frac{1}{p} - \frac{1}{r})$, i.e. $\bar{s}_\kappa = s_\kappa - \frac{1}{p} + \frac{1}{r}$, $\kappa \in \mathrm{z}_k$.

Below we shall use the following three embeddings:

Let $1 \le p < r \le \infty$, $1 \le q \le \infty$; $s \in \mathbb{R}^k_+$ s. t. $\bar{s} \in \mathbb{R}^k_+$. Then

$$\chi B^{s\,\mathtt{n}}_{p\,q} \hookrightarrow \chi B^{\bar{s}\,\mathtt{n}}_{r\,q}. \tag{6}$$

Let $1 \le p < r < \infty$, $1 \le q \le \infty$; $s \in \mathbb{R}^k_+$ s. t. $\bar{s} \in \mathbb{R}^k_+$. Then

$$\chi L^{s\,\mathtt{n}}_{p\,q} \hookrightarrow \chi L^{\bar{s}\,\mathtt{n}}_{r\,1}. \tag{7}$$

Let $1 \le p < r = \infty$, $1 \le q \le \infty$; $s \in \mathbb{R}^k_+$ s. t. $\bar{s} \in \mathbb{R}^k_+$. Then

$$\chi L^{s\,\mathtt{n}}_{p\,q} \hookrightarrow \chi B^{\bar{s}\,\mathtt{n}}_{\infty\,p}. \tag{8}$$

For proof of these embeddings see [5] and Remark 5.1 in [2].

I. We begin with proof of the upper estimates in Theorem 3.

(B) First, consider the class $\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}$. In view of the embedding (6), it suffices to derive an upper estimate for $E^\gamma_u(\chi \mathrm{B}^{s,\mathtt{n}}_{\infty q}, L_\infty)$.

Let $f \in \chi \mathrm{B}^{s,\mathtt{n}}_{\infty q}$. Then, as above, by the Holder inequality for series and by (4), we obtain

$$\| f - S^{\chi,\gamma}_u(f) \,|\, L_\infty \| \le \sum_{\alpha\tilde{s} > u\varsigma} 2^{-\alpha\bar{s}} 2^{\alpha\bar{s}} \| \Delta^\chi_\alpha(f) \,|\, L_\infty \| \le$$

$$\le \| (2^{-\alpha\bar{s}})_{\alpha\tilde{s} > u\varsigma} \,|\, \ell_{q'} \| \| (2^{\alpha\bar{s}} \Delta^\chi_\alpha(f) \,|\, \ell_q(L_\infty) \|$$

$$\ll 2^{-(\varsigma - \frac{1}{p})u} u^{(\omega-1)(1-\frac{1}{q})} \| f \,|\, \chi \mathrm{B}^{s,\mathtt{n}}_{\infty q} \| \le 2^{-(\varsigma - \frac{1}{p})u} u^{(\omega-1)(1-\frac{1}{q})}.$$

Thus, the upper estimate is established.

(L) Consider the class $\chi \mathrm{L}^{s\,\mathtt{n}}_{p\,q}$. According to the embedding (8) and previous estimate (with p instead of q), we get

$$E^\gamma_u(\chi \mathrm{L}^{s\,\mathtt{n}}_{pq}, L_\infty) \ll E^\gamma_u(\chi \mathrm{B}^{\bar{s}\,\mathtt{n}}_{\infty p}, L_\infty) \ll 2^{-(\varsigma - \frac{1}{p})u} u^{(\omega-1)(1-\frac{1}{p})};$$

i.e., the upper estimate is proved. Hence, proof of both upper estimates in Theorem 3 is completed.

II. Let us proceed to upper estimates in Theorem 2. We have $1 \le p < r < \infty$, $1 \le q \le \infty$, $s, \bar{s} \in \mathbb{R}^k_+$.

(B) First, we obtain upper estimates for class $\chi \mathrm{B}^{s\,\mathtt{n}}_{p\,q}$. In view of embedding (6) we get

$$E^\gamma_u(\chi \mathrm{B}^{s\,\mathtt{n}}_{pq}, L_r) \ll E^\gamma_u(\chi \mathrm{B}^{\bar{s}\,\mathtt{n}}_{rq}, L_r).$$

Therefore, we can use the upper estimates for the right-hand side of this inequality from Theorem 3. It is easily seen that these estimates provide the required order if either $1 < r \le 2$ and $1 \le q \le \infty$ or $1 \le q \le 2 < r$.

For $r > 2$ and $q > 2$, these estimates turn out to be rouhger than required. To obtain upper estimates (sharp in order) in that case, we need the following

Theorem 5. *Let $1 \le p < r < \infty$. Then there exists a constant $C(p, r, \mathfrak{n}) > 0$ such that, for $f \in L_p$, the following inequality holds:*

$$\| f \,|\, L_r \| \le C(p, r, \mathfrak{n}) \| (2^{\alpha n(\frac{1}{p} - \frac{1}{r})} \Delta_\alpha^\chi(f)) \,|\, \ell_r(L_p) \|.$$

The proof of Theorem 5 in the case $k = n$ is given in [11]; it remains valid in slightly more general case $1 \le k \le n$.

Let $f \in \chi \mathrm{B}_{pq}^{s\,\mathtt{n}}$. Then, by Theorem 5, we have

$$\| f - S_u^{\chi,\gamma}(f) \,|\, L_r \| \ll \| (2^{\alpha \mathtt{n}(\frac{1}{p} - \frac{1}{r})} \Delta_\alpha^\chi(f))_{\alpha\tilde{s} > u\bar{\varsigma}} \,|\, \ell_r(L_p) \|$$

$$= \| (2^{-\alpha\bar{s}} 2^{\alpha s} \Delta_\alpha^\chi(f))_{\alpha\tilde{s} > u\bar{\varsigma}} \,|\, \ell_r(L_p) \| \equiv \Im_5(u).$$

If $r < q < \infty$, applying the Holder inequality for series, relation (4), we obtain

$$\Im_5(u) \le \left(\sum_{\alpha\tilde{s} > u\bar{\varsigma}} 2^{-\alpha\bar{s}\frac{rq}{q-r}} \right)^{\frac{1}{r} - \frac{1}{q}} \| (2^{\alpha s} \Delta_\alpha^\chi(f))_{\alpha\tilde{s} > u\bar{\varsigma}} \,|\, \ell_q(L_p) \|$$

$$\ll 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u} u^{(\omega - 1)(\frac{1}{r} - \frac{1}{q})} \| f \,|\, \chi \mathrm{B}_{pq}^{s,\mathtt{n}} \| \le 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u} u^{(\omega - 1)(\frac{1}{r} - \frac{1}{q})}.$$

If $2 < q \le r$, using Jensen's inequality for series and taking into account that $\alpha\bar{s} \ge \alpha\tilde{s} > u\bar{\varsigma}$, we get

$$\Im_5(u) \le 2^{-\bar{\varsigma}u} \| (2^{\alpha s} \Delta_\alpha^\chi(f))_{\alpha\tilde{s} > u\bar{\varsigma}} \,|\, \ell_r(L_p) \|$$

$$\le 2^{-\bar{\varsigma}u} \| (2^{\alpha s} \Delta_\alpha^\chi(f)) \,|\, \ell_q(L_p) \| \le 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})}.$$

If $q = \infty$, then again by (4), we have

$$\Im_5(u) \le \left(\sum_{\alpha\tilde{s} > u\bar{\varsigma}} 2^{-\alpha\bar{s}r} \right)^{\frac{1}{r}} \| (2^{\alpha s} \Delta_\alpha^\chi(f))_{\alpha\tilde{s} > u\bar{\varsigma}} \,|\, \ell_\infty(L_p) \|$$

$$\ll 2^{-\bar{\varsigma}u} u^{(\omega - 1)\frac{1}{r}} \| f \,|\, \chi \mathrm{B}_{p\infty}^{s,\mathtt{n}} \| \le 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u} u^{(\omega - 1)\frac{1}{r}}.$$

Thus, all the required upper estimates in Theorem 2 for the class $\chi \mathrm{B}_{p\infty}^{s,\mathtt{n}}$ are found.

(L) Finally, let us obtain the upper estimate for the class $\chi \mathrm{L}_{p\infty}^{s,\mathtt{n}}$. According to the embedding (7) and the upper estimate from Theorem 1, it follows that

$$E_u^\gamma(\chi \mathrm{L}_{pq}^{s\,\mathtt{n}}, L_r) \ll E_u^\gamma(\chi \mathrm{L}_{r1}^{\bar{s}\,\mathtt{n}}, L_r) \ll 2^{-(\varsigma - \frac{1}{p} + \frac{1}{r})u},$$

i.e., the required upper estimate is established.

Thus, proof of both upper estimates in Theorem 2 is completed.

5 Proof of Lower Bounds in Theorems 1 – 3

To obtain lower bounds in Theorems 1 – 3 we need the notion of Fourier width. Recall that $N-$th Fourier width (or, which is the same, orthowidth) of a set $\mathrm{F} \subset L_r$ is defined as

$$\varphi_N(\mathrm{F}, L_r) = \inf_{\{g_j\}_{j=1}^N} \sup_{f\in\mathrm{F}} \| f - \sum_{j=1}^{N} \langle f, g_j\rangle g_j \mid L_r \|,$$

where the infimum is taken over all orthonormal (in L_2) systems $\{g_j\}_{j=1}^N \subset L_\infty$.

By virtue of (5) (with $L = 1$), the dimension $\delta(\gamma, u)$ of the linear span of the set $\{\chi^\iota_{\alpha\lambda} \mid \iota \in \mathrm{E}^{\mathtt{n}}(\alpha),\ \lambda \in \Lambda(\mathtt{n}, \alpha),\ \alpha \in \mathbb{N}_0^k : \alpha\gamma \le u\}$ is of order $2^u u^{\omega-1}$. Hence, choosing $N \ge \delta(\gamma, u)$, we get the inequality

$$E_u^\gamma(\mathrm{F}, L_r) \ge \varphi_N(\mathrm{F}, L_r). \tag{9}$$

In [4], we obtained the estimates for Fourier widths of the classes under consideration. Under the hypotheses of Theorem 1 we have

$$\varphi_N(\chi\mathrm{B}^{s\,\mathtt{n}}_{p\,q}, L_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma} (\log^{\omega-1} N)^{(\frac{1}{p_*}-\frac{1}{q})_+},$$

$$\varphi_N(\chi\mathrm{L}^{s\,\mathtt{n}}_{p\,q}, L_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+} \quad (p > 1),$$

$$\varphi_N(\chi\mathrm{L}^{s\,\mathtt{n}}_{1\,q}, L_1) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma} (\log^{\omega-1} N)^{(1-\frac{1}{q})_+}.$$

Further, under the hypotheses of Theorem 2 we have

$$\varphi_N(\chi\mathrm{B}^{s\,\mathtt{n}}_{p\,q}, L_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma-\frac{1}{p}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+},$$

$$\varphi_N(\chi\mathrm{L}^{s\,\mathtt{n}}_{p\,q}, L_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma-\frac{1}{p}+\frac{1}{r}}.$$

Finally, under the hypotheses of Theorem 3 we have

$$\varphi_N(\chi\mathrm{B}^{s\,\mathtt{n}}_{p\,q}, L_\infty) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma-\frac{1}{p}} (\log^{\omega-1} N)^{1-\frac{1}{q}},$$

$$\varphi_N(\chi\mathrm{L}^{s\,\mathtt{n}}_{p\,q}, L_\infty) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\varsigma-\frac{1}{p}} (\log^{\omega-1} N)^{1-\frac{1}{p}}.$$

(Here $\log \equiv \log_2$.) These estimates with $N \ge \delta(\gamma, u)$ such that $N \asymp \delta(\gamma, u)$ combined with the inequality (9) easily imply lower bounds for $E_u^\gamma(\chi\mathrm{F}^{s\,\mathtt{n}}_{p\,q}, L_r)$ ($\mathrm{F} \in \{\mathrm{B}, \mathrm{L}\}$) in Theorems 1 – 3 matching the upper bounds obtained in the previous sections.

Remark 3. Notion of Fourier width was invented by V.N. Temlyakov [8] in 1982. Remark that extensive literature has been devoted to the estimates sharp in order for Fourier widths of various classes of smooth functions in one and several variables; here we only refer to monographs [9,10] and articles [3,6], where one can also find detailed history of the problem and comprehensive references.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Andrianov, A.V.: Approximation of functions belonging to the classes MH_q^r by Haar polynomials. Math. Notes. **66**(3–4), 261–270 (2000)
2. Bazarkhanov, D.B.: Wavelet approximation and Fourier widths of classes of periodic functions of several variables. I. Proc. Steklov Inst. Math. **269**(1), 2–24 (2010)
3. Bazarkhanov, D.B.: Wavelet approximation and Fourier widths of classes of periodic functions of several variables. II. Anal. Math. **38**(4), 249–289 (2012)
4. Bazarkhanov, D.B.: Fourier widths of some function classes associated with multiple Haar system. AIP Conf. Proc. **1759**, 020110 (2016)
5. Hansen, M., Vybiral, J.: The Jawerth - Franke embedding of spaces with dominating mixed smoothness. Georgian Math. J. **16**(4), 667–682 (2009)
6. Romanyuk, A.S.: Best approximations and widths of classes of periodic functions of several variables. Sb. Math. **199**(1–2), 253–275 (2008)
7. Stasyuk, S.A.: Approximation of certain smoothness classes of periodic functions of several variables by polynomials with regard to the tensor Haar system. Trudy Inst. Mat. i Mekh. UrO RAN. **21**(4), 251–260 (2015)
8. Temlyakov, V.N.: Widths of some classes of functions of several variables. Sov. Math. Dokl. **26**, 619–622 (1982)
9. Temlyakov, V.N.: Approximations of functions with bounded mixed derivative. Proc. Steklov Inst. Math. **178**, 1–121 (1989)
10. Temlyakov, V.N.: Approximation of Periodic Functions. Computational Mathematics Analytical Series. Nova Science Publ, Commack (1993)
11. Temlyakov, V.N.: Some inequalities for multivariate Haar polynomials. East J. Approx. **1**(1), 61–72 (1998)
12. Temlyakov, V.N.: Sparse Approximation with Bases. Advanced Courses in Mathematics. Birkhäuser, CRM Barcelona (2015)
13. Wojtaszczyk, P.: A mathematical introduction to wavelets. London Mathematical Society Student Texts, vol. 37. CUP, Cambridge (1997)

Compactness of Commutators for One Type of Singular Integrals on Generalized Morrey Spaces

Nurzhan Bokayev(✉) and Dauren Matin

L. N. Gumilyov Eurasian National University, 13 K. Munaitpasov st., Astana 010000, Kazakhstan
Bokayev2011@yandex.ru, D.matin@mail.ru

Abstract. This paper is dedicated to a sufficient condition for compactness of Commutators for singular integrals $[b, T]$ in the generalized Morrey space M_p^w.

Keywords: Morrey space · Compactness · Commutators · Singular integrals · Generalized Morrey space · Pre-compactness

First we give some definitions.

Let $1 \le p \le \infty$, w be a measurable non-negative function on $(0, \infty)$. The generalized Morrey space $M_p^w \equiv M_p^w(R^n)$ is defined as a set of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm

$$\|f\|_{M_p^w} \equiv \sup_{x \in R^n} \left\| w(r) \, \|f\|_{L_p(B(x,r))} \right\|_{L_\infty(0,\infty)},$$

where $B(t, r)$ is a ball with center at the point t and of radius r.

The space M_p^w coincides with the known Morrey space M_p^λ at $w(r) = r^{-\lambda}$, where $0 \le \lambda \le \frac{n}{p}$, which, in turn, for $\lambda = 0$ coincides with the space $L_p(R^n)$.

Following the notation of [1,2], we denote by $\Omega_{p\infty}$ the set of all functions which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r) r^{\frac{n}{p}}\|_{L_\infty(0,t)} < \infty, \quad \|w(r)\|_{L_\infty(t,\infty)} < \infty.$$

Note that the space M_p^w is non-trivial, that is, consists not only of functions equivalent to 0 on R^n, if and only if $w \in \Omega_{p\infty}$.

In this paper we consider a singular integral in the following form

$$Tf(x) = \int_{R^n} |x - y|^{-n} f(y) dy,$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_4

It is well-known that the Calderon–Zygmund operator

$$Kf(x) = \int_{R^n} k(x,y)f(y)dy,$$

with a certain kernel $k(x,y)$ satisfies the condition, where

$$|K(x,y)| \le \int_{R^n} \frac{f(y)}{|x-y|^n}dy,$$

(see [3,11]).

The singular integral T plays an important role in the harmonic analysis and theory of operators.

For a function $b \in L_{loc}(R^n)$ by M_b denote multiplier operator $M_b f = bf$, where f is a measurable function. Then a commutator between T and M_b is defined by

$$[b,T] = M_b T - T M_b = \int_{R^n} \frac{[b(x) - b(y)]\, f(y)}{|x-y|^n} dy.$$

The commutators for singular integrals were investigated [4–6].

It is said that the function $b(x) \in L_\infty(R^n)$ belongs to the space $BMO(R^n)$, if

$$\|b\|_* = \sup_{Q \subset R^n} \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx = \sup_{Q \in R^n} M(b,Q) < \infty,$$

where Q is a cube R^n and $b_Q = \frac{1}{|Q|} \int_{R^n} f(y)dy$.

By $VMO(R^n)$ we denote the BMO-closure $C_0^\infty(R^n)$, where $C_0^\infty(R^n)$ is the set of all functions from $C^\infty(R^n)$ with compact support. By $\chi(A)$ denote the characteristic function of the set $B \subset R^n$, and by ${}^cA$ denote the complement of A.

The main purpose of this work is to find sufficient conditions for the compactness of commutators operators $[b,T]$ on the generalized Morrey space $M_p^w(R^n)$.

We note that in the case of the Morrey space this question was investigated in [4]. The following well-known theorem gives necessary and sufficient conditions for the boundedness and compactness for $[b,T]$ on the generalized Morrey spaces $M_p^w(R^n)$.

Theorem A. (see. [6]) Let $1 < p < \infty$, $b \in BMO(R^n)$, let (w_1, w_2) satisfy the following condition

$$\int_r^\infty \frac{ess \inf\limits_{t<s<\infty} w_1(s)dt}{t} \le C w_2(r), \tag{1}$$

where C does not depend on x and r. Let the operator T be bounded on $L_p(R^n)$. Then the operator T is bounded from $M_p^{w_1}(R^n)$ to $M_p^{w_2}(R^n)$.

The following statement is a consequence of Theorem 3.7 of [6].

Theorem B. (see. [6]) Let $1 < p < \infty$, $b \in BMO(R^n)$, let (w_1, w_2) satisfy the condition

$$\int_r^\infty ln\left(e + \frac{t}{r}\right) \frac{ess \inf_{t<s<\infty} w_1(s)dt}{t} \le Cw_2(r). \tag{2}$$

where C does not depend on x and r. Let the operator $[b, T]$ be bounded on $L_p(R^n)$. Moreover, let

$$\|[b, T]\|_{L_p(R^n)} \le \|b\|_* \|f\|_{L_p(R^n)}. \tag{3}$$

Then $[b, T]$ is bounded from $M_p^{w_1}(R^n)$ to $M_p^{w_2}(R^n)$. Moreover, we have

$$\|[b, T]\|_{M_p^{w_2}(R^n)} \le \|b\|_* \|f\|_{M_p^{w_1}(R^n)}. \tag{4}$$

It is well known that the boundedness of such operators on Morrey space $M_p^\lambda(R)$ was considered in [4,5].

The following theorem on sufficient conditions for the precompactness of sets on generalized Morrey spaces was proved in [7,8].

Theorem C. (see. [7,8]) Suppose that $1 \le p \le \infty$ and $w \in \Omega_{p\infty}$. Suppose that a subset S of M_p^w satisfies the following conditions:

$$\sup_{f\in S} \|f\|_{M_p^w} < \infty, \tag{5}$$

$$\lim_{u\to 0} \sup_{f\in S} \|f(\cdot + u) - f(\cdot)\|_{M_p^w} = 0, \tag{6}$$

$$\lim_{r\to\infty} \sup_{f\in S} \left\|f\chi_{{}^c B(0,r)}\right\|_{M_p^w} = 0. \tag{7}$$

Then S is a pre-compact set in $M_p^w(R)$.

Note that for the case of Morrey space $M_p^\lambda (0 < \lambda < 1)$ (i.e., if $w(r) = r^{-\lambda}$) this assertion was proved earlier in [3], and in the case of $\lambda = 0$ it is the known Frechet-Kolmogorov theorem [10]. We note that the pre-compactness of some sets in Banach function spaces were investigated in [9].

Now we give theorem about the compactness of the operators $[b, T]$ on generalized Morrey space $M_p^w(R^n)$.

Theorem 1. *Let $1 < p < \infty$, $b \in VMO(R^n)$, let functions $w_1, w_2 \in \Omega_{p,\infty}$ satisfy the conditions (2) and (3). Then the commutator $[b, T]$ is a compact operator from $M_p^{w_1}$ to $M_p^{w_2}$.*

To prove this theorem we need the following auxiliary assertions.

Lemma 1. Let $n \in N, 1 < p \le \infty, 1 < q < \infty, 0 < n\left(1 - \frac{1}{q}\right), \beta > 0$. Then there exists $C > 0$ depending only on n, p, q such that for some $f \in L_p(B(0, \beta))$ satisfying the condition $suppf \subset \overline{B(0, \beta)}$, and for some $\gamma \ge 2\beta$, $t \in R^n$, $r > 0$

$$\left\|(Tf)\chi_{{}^c B(0,\gamma)}\right\|_{L_q(B(t,r))} \le C\gamma^{-n} \left(min\{\gamma, r\}\right)^{\frac{n}{q}} \|f\|_{L_p B(0,\beta)}. \tag{8}$$

Proof. From the definition of the operator T, we have

$$I = \left\| (Tf)\, \chi^c_{B(0,\gamma)} \right\|_{L_q(B(t,r))}$$

$$= \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{R^n} \frac{f(y)}{|x-y|^n} dy \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left| \int\limits_{B(0,\beta)} \frac{f(y)}{|x-y|^n} dy \right|^q dx \right)^{\frac{1}{q}}. \tag{9}$$

From the fact that $\beta \leq \frac{\gamma}{2}$ for $x \in^c B(0,\gamma), y \in B(0,\beta)$ we have

$$|x-y| \geq |x| - |y| \geq |x| - \beta = \frac{|x|}{2} + \frac{|x|}{2} - \beta \geq \frac{|x|}{2}. \tag{10}$$

From this it follows that

$$I \leq 2^n \left(\int\limits_{^c B(0,\gamma)} \frac{dx}{|x|^{nq}} \right)^{\frac{1}{q}} \int\limits_{B(0,\beta)} |f(y)| dy$$

$$\leq 2^n \left(\int\limits_{\gamma}^{\infty} \rho^{nq-2} d\rho \right)^{\frac{1}{q}} (\upsilon_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))}$$

$$\leq 2^n \left(\delta_n \rho^{nq+n-1} \right)^{\frac{1}{q}} (\upsilon_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))}$$

$$= 2^n \left(\frac{\delta_n}{nq+n} \right)^{\frac{1}{q}} \upsilon_n^{1-\frac{1}{p}} \beta^{n(1-\frac{1}{p})} \gamma^{-n(1-\frac{1}{q})} \|f\|_{L_p(B(0,\beta))}$$

$$\equiv C_1 \gamma^{-n(1-\frac{1}{q})} \|f\|_{L_p(B(0,\beta))}. \tag{11}$$

Next, we consider

$$I \leq 2^n \gamma^{-n} \left(\int\limits_{B(t,r)} dx \right)^{\frac{1}{q}} \int\limits_{B(0,\beta)} |f(y)| dy$$

$$\leq 2^n \gamma^{-n} (\upsilon_n r^n)^{\frac{1}{q}} (\upsilon_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))}$$

$$= C_2 \gamma^{-n} r^{\frac{n}{q}} \|f\|_{L_p(B(0,\beta))}. \tag{12}$$

From inequality (11) and (12) it follows (8), where $C = \max\{C_1, C_2\}$

Lemma 2. Let $n \in N, 1 < p < \infty, 1 < q < \infty, 0 < n\left(1 - \frac{1}{q}\right), \beta > 0$. Then there exists $C > 0$ depending only on n, p, q such that for some $f \in L_p(B(0,\beta))$, $b \in L_\infty(R^n)$, satisfying the condition *supp* $b \subset \overline{B(0,\beta)}$, and for some $\gamma \geq 2\beta$, $t \in R^n$, $r > 0$

$$\left\|([b,T]f)\chi_{^cB(0,\gamma)}\right\|_{L_q(B(t,r))} \leq C\gamma^{-n}\left(min\{\gamma, r\}\right)^{\frac{n}{q}} \|b\|_{L_\infty(R^n)} \|f\|_{L_pB(0,\beta)} \cdot \tag{13}$$

Proof. Let $\gamma > \beta$, *supp* $b \subset B(0,\beta)$ for $x \in^c B(0,\gamma), b(x) = 0$. Then

$$\left\|[b,T]\, f\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))}$$

$$= \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{R^n} \frac{(b(x)-b(y))f(y)}{|x-y|^n} dy\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{R^n} \frac{b(y)f(y)}{|x-y|^n} dy\right|^q dx\right)^{\frac{1}{q}} \leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{B(0,\beta)} \frac{|b(y)|\cdot|f(y)|}{|x-y|^n} dy\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{B(0,\beta)} \frac{|f(y)|}{|x-y|^n} dy\right|^q dx\right)^{\frac{1}{q}} \|b\|_{L_\infty(R^n)}$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{R^n} \frac{|f(y)|}{|x-y|^n} dy\right|^q dx\right)^{\frac{1}{q}} \|b\|_{L_\infty(R^n)} = \left\|(Tf)\,\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))} \|b\|_{L_\infty(R^n)}$$

From this and from Lemma 1 we obtain the inequality (13).

Proof of theorem. To prove Theorem 1 it is sufficient to show that the conditions (5)-(7) of Theorem C hold.

Let F be an arbitrary bounded subset of $M_p^{w_1}$. Since $C_c^\infty(R^n)$ is dense in $VMO(R^n)$, we only need to prove that the set $G = \{[b,T]f : f \in F, b \in C_c^\infty\}$ is pre-compact in $M_p^{w_2}$. By Theorem C, we only need to verify that the conditions (5), (6) and (7) hold uniformly F for $\{b \in C_c^\infty\}$.

Suppose that

$$\|f\|_{M_p^{w_1}} \leq D.$$

Applying Theorem B, we have

$$\|[b,T]f\|_{M_p^{w_2}} \leq C \cdot \|b\|* \sup_{f\in F} \|f\|_{M_p^{w_1}} \leq C \cdot D\|b\|* < \infty.$$

this implies that the condition (5) of Theorem C holds.

Now we prove that condition (7) of Theorem C also holds, i. e.

$$\lim_{\gamma\to\infty}\left\|([b,T]\,f)\chi_{B(0,\gamma)}^{c}\right\|_{M_p^{w2}}=0.$$

It follows from Lemma 2. Indeed

$$\left\|([b,T]\,f)\chi_{B(0,\gamma)}^{c}\right\|_{M_p^{w2}}=$$

$$\sup_{\substack{r>0,\\ x\in R^n}}\left\|w(r)\left\|([b,T]\,f)\chi_{B(0,\gamma)}^{c}\right\|_{L_p(B(x,r))}\right\|_{L_\infty(0,\infty)}$$

$$\le C\gamma^{-n}\,\|b\|_{L_\infty(R^n)}\,\|f\|_{L_pB(0,\beta)}\sup_{\substack{r>0,\\ x\in R^n}}\left\|w_2(r)\,(min\{\gamma,r\})^{\frac{n}{p}}\right\|_{L_\infty(0,\infty)}$$

When $r<l<\gamma$ we have $(min\{\gamma,r\})^{\frac{n}{p}}=r^{\frac{n}{p}}$. By condition $\left\|w_2(r)r^{\frac{n}{p}}\right\|_{L_\infty(l,\infty)}<\infty$.

When $\gamma<t<r$ we have $(min\{\gamma,r\})^{\frac{n}{p}}=\gamma^{\frac{n}{p}}$. By condition $\|w_2(r)\|_{L_\infty(0,t)}<\infty$.

Therefore

$$\lim_{\gamma\to\infty}\left\|([b,T]\,f)\chi_{B(0,\gamma)}^{c}\right\|_{M_p^{w2}}=0.$$

This implies the required condition (7).

Now we prove that condition (6) of Theorem C for the set $[b,T](f)$, $f\in F$ holds, i.e., we show that for any $0<\varepsilon<\frac{1}{2}$ and if $|z|$ is sufficiently small depending only on ε, then for every $f\in F$.

$$\|[(b,Tf)(\cdot+z)]-[b,T]\,f(\cdot)\|_{M_p^{w_2}}\le C\cdot\varepsilon.$$

Let ε be an arbitrary number such that $0<\varepsilon<\frac{1}{2}$. For $|z|\in R^n$ we have that

$$[f,T]f(x+z)-[b,T]f(x)=\int\limits_{R^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^n}dy-\int\limits_{R^n}\frac{[b(x)-b(y)]f(y)}{|x-y|^n}dy$$

$$=\int\limits_{|x-y|\le\frac{|z|}{\varepsilon}}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^n}dy+\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^n}dy$$

$$-\int\limits_{|x-y|\le\frac{|z|}{\varepsilon}}\frac{[b(x)-b(y)]f(y)}{|x-y|^n}dy-\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\frac{[b(x)-b(y)]f(y)}{|x-y|^n}dy$$

$$=-\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\frac{[b(x)-b(y)+b(x+z)-b(x+z)]f(y)}{|x-y|^n}dy+\int\limits_{|x-y|>\frac{|z|}{\varepsilon}}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^n}dy$$

$$+ \int\limits_{|x-y|\le\frac{|z|}{\varepsilon}} \frac{[b(y)-b(x)]f(y)}{|x-y|^n}dy - \int\limits_{|x-y|\le\frac{|z|}{\varepsilon}} \frac{[b(y)-b(x+z)]f(y)}{|x+z-y|^n}dy$$

$$= \int\limits_{|x-y|>\frac{|z|}{\varepsilon}} \frac{[b(x+z)-b(x)]f(y)}{|x-y|^n}dy + \int\limits_{|x-y|>\frac{|z|}{\varepsilon}} \left(\frac{1}{|x-y|^n} - \frac{1}{|x+z-y|^n}\right)([b(x+z)-b(y)]f(y))dy$$

$$+ \int\limits_{|x-y|\le\frac{|z|}{\varepsilon}} \frac{[b(y)-b(x)]f(y)}{|x-y|^n}dy - \int\limits_{|x-y|\le\frac{|z|}{\varepsilon}} \frac{[b(y)-b(x+z)]f(y)}{|x+z-y|^n}dy$$

$$= J_1 + J_2 + J_3 - J_4 \tag{14}$$

Since $b \in C_0^n(R^n)$, we have

$$|b(x) - b(x+z)| \le |\nabla f(x)| \cdot |z| \le C|z|.$$

Then

$$|J_1| \le C|z|T(|f|)(x).$$

By Theorem A

$$||J_1||_{M_p^{w_2}} \le C|z| \parallel T(f) \parallel_{M_p^{w_2}} \le C|z| \parallel f \parallel_{M_p^w} \le CD|z|. \tag{15}$$

For J_2 we have that

$$(b(x+z) - b(y)) \le 2 \parallel b \parallel_\infty \le C.$$

Therefore

$$|J_2| \le C|z| \int\limits_{|x-y|>\frac{|z|}{\varepsilon}} \frac{f(y)}{|x-y|^n}dy \le C_\varepsilon T(|f|)(x).$$

Again by the of Theorem A we get

$$\parallel J_2 \parallel_{M_p^{w_2}} \le c\varepsilon \parallel T(f) \parallel_{M_p^{w_1}} \le c\varepsilon \parallel f \parallel_{M_p^w} \le C \cdot D \cdot \varepsilon.$$

Consequently,

$$|J_3| \le C \int\limits_{|x-y|\le z-\varepsilon^{-1}} \frac{f(y)}{|x-y|^n}dy \le C\varepsilon^{-1}|z| \int\limits_{|x-y|\le\varepsilon^{-1}|z|} \frac{f(y)}{|x-y|^n}dy \le C \cdot \varepsilon^{-1}|z|T(|f|)(x).$$

Thus, we have

$$\parallel J_3 \parallel_{M_p^{w_2}} \le C \cdot \varepsilon^{-1}|z| \parallel T(f) \parallel_{M_p^{w_2}} \le C \cdot \varepsilon^{-1}|z| \parallel f \parallel_{M_p^{w_1}} \le \varepsilon^{-1}|z|. \tag{16}$$

Similarly, using the estimate

$$|b(x+z) - b(y)| \le C|x+z-y|,$$

finally we have

$$|J_4| \leq C \int\limits_{|x-y|\leq\varepsilon^{-1}(y)} |x+z-y|^{-n+1}|b(y)|dy \leq C(\varepsilon^{-1}|z|+|z|)T(|f|)(x+z).$$

Therefore

$$\| J_4 \|_{M_q^{w_2}} \leq C \cdot (\varepsilon^{-1}|z|+|z|) \| f \|_{M_p^{w_1}} \leq C \cdot D \cdot (\varepsilon^{-1}|z|+|z|). \quad (17)$$

Here C does not depend on z and ε. Finally from (14) - (17), taking $|z|$ small enough, we have

$$\|[b,T(f)(\cdot+z)] - [b,T]\,f(\cdot)\|_{M_p^{w_2}} \leq \|J_1\|_{M_p^{w_2}} + \|J_2\|_{M_p^{w_2}} + \|J_3\|_{M_p^{w_2}} + \|J_4\|_{M_p^{w_2}} \leq C \cdot D \cdot \varepsilon$$

i.e., the set $[b,T](f),\ f \in F$ satisfies the condition (6) of Theorem C. Then by Theorem C, the set $[b,T](f),\ f \in F$ is precompact in the $M_p^{w_2}$. This fact completes the proof of the theorem.

Acknowledgements. This work was supported by the National Project 0085/PTSF-14, the grant of the Ministry of Education and Science (Project 2709/GF4).

References

1. Burenkov, V.I.: Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces I. Eurasian Math. J. **3**(3), 11–32 (2012)
2. Burenkov, V.I.: Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces II. Eurasian Math. J. **4**(1), 21–45 (2013)
3. Burenkov, V.I., Bokayev, N.A., Matin, D.T.: On the pre-compactness of a set in the generalized Morrey spaces. AIP Conf. Proc. **3**, 020108 (2016). doi:10.1063/1.4959722
4. Burenkov, V.I., Bokayev, N.A., Matin, D.T.: On the pre-compactness of a set in the generalized Morrey spaces. Vestnyk KarGU. **4**, 18–40 (2016)
5. Chen, Y., Ding, Y.: Compactness of commutators for singular integrals on Morrey spaces. Canad. J. Math. **64**(2), 257–281 (2012)
6. Chen, Y., Ding, Y., Wang, X.: Compactness of commutators of Riesz potential on Morrey space. Potential Anal. **30**(4), 301–313 (2009)
7. Guliyev, V.S.: Generalized weighted Morrey spaces and higher order commutators of sublinear operators. Eurasian Math. J. **3**(3), 33–61 (2012)
8. Otelbaev, M., Cend, L.: To the theorem about the compactess. Sib. Math. J. **13**(4), 817–822 (1972)
9. Soria, F., Weiss, G.: A remark on singular integrals and power weights. Indiana Univ. Math. J. **43**, 187–204 (1994)
10. Stein, E.M., Weiss, G.: Introduction to Fourier Analisis on Euclidean Spaces. Princeton University Press, Princeton (1971)
11. Yosido, K.: Functional Analysis. Springer, Berlin (1978)

On Multipliers from Weighted Sobolev Spaces to Lebesgue Spaces

Leili Kussainova(✉) and Aigul Myrzagaliyeva

Gumilyov Eurasian National University, 2 Satpayev st., Astana 010000, Kazakhstan
leili2006@mail.ru, mir_aigul@mail.ru

Abstract. The aim of the paper is to obtain descriptions of multipliers acting from weighted Sobolev spaces $W^l_{p,\rho}$ to $L_{q,\omega}$. The space $W^l_{p,\rho}$ is defined as the completion of the set C_0^∞ in the following finite norm $\|u;\, W^l_{p,\rho}\| = \|\rho|\nabla_l u|\|_p + \|u\|_p$, where ρ is a weight on R^n; $L_{q,\omega}$ denotes the Lebesgue space.

Keywords: Description of pointwise multiplier · Weighted Sobolev space · Lebesgue space

1 Preliminaries and Notation

Let R^n be the n-dimensional Euclidian space with the norm $|x| = \left(\sum\limits_{i=1}^{n} x_i^2\right)^{1/2}$. Below $L_{p,\omega}$, $1 \le p < \infty$, is the space of all real valued functions with the finite weighted Lebesgue norm

$$\|u\|_{L_{p,\omega}} = \left(\int |u|^p \omega(x)\, dx\right)^{\frac{1}{p}}.$$

Note that $L_p(R^n) = L_{p,\omega}(R^n)$, if $\omega = 1$, $\|\cdot\|_p = \|\cdot\|_{L_p}$.

We denote by L^+_{loc} the space of all a.e. positive and locally integrable functions in R^n. A function v of L^+_{loc} is called a weight.

Let Ω be a bounded domain in R^n. By $C^\infty(\Omega)$, $C_0^\infty(\Omega)$ we denote the space of all infinitely differentiable functions in Ω and the space of functions of $C^\infty(\Omega)$ with compact support supp$f$ in Ω, respectively. When the domain is not indicated in the notation of a space or a norm, then it is assumed to be R^n.

By I^n we denote the family of all cubes Q in the following form

$$Q = Q_h = Q_h(x) = \left\{y \in R^n \colon |y_i - x_i| < \frac{h}{2}, i = 1, \dots, n\right\}.$$

We set $\lambda Q = Q_{\lambda h}(x)$.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_5

By definition $W_p^m(\Omega)$ is the completion of the linear manifold [1]

$$C^\infty W_p^m(\Omega) = \{u \in C^\infty(\Omega)\colon \|u; W_p^m(\Omega)\| = \|\nabla_m u; L_p(\Omega)\| + \|u; L_p(\Omega)\| < \infty\}.$$

Let k, m be integers, $0 \le k < m$, $1 \le p \le q < \infty$. If $p > 1$, $p(m-k) > n$ or $p = 1$, $m - k \ge n$, then

$$W_p^m(Q_1) \hookrightarrow C^k(Q_1),$$

i.e., $u \in W_p^m(Q_1)$ has, up to equivalence, continuous derivatives $D^\alpha u$ of order $|\alpha| = k$, and

$$\sup_{Q_1} |\nabla_k u| \le c\, \|u\|_{W_p^m(Q_1)}. \tag{1}$$

Let $Q_h = Q_h(x)$. By changing variables $y = x + h\xi$, $\xi \in Q_1(0)$ in inequality (1) for $p > 1$, $p(m-k) > n$ or $p = 1$, $m - k \ge n$, $|\alpha| = k$, we have

$$\sup_{Q_h} |D^\alpha u| \le c\, h^{m-k-n/p} \left(\|\nabla_m u\|_{L_p(Q_h)} + h^{-m}\|u\|_{L_p(Q_h)}\right). \tag{2}$$

Let $h(\cdot)$ be a positive locally bounded function in R^n. $B = \{Q(x)\}$ denotes the family of cubes $Q(x) = Q_{h(x)}(x)$, $x \in R^n \setminus e$, where e is a set with measure 0.

Definition 1. A weight ρ satisfies the slow variation condition with respect to the family $B = \{Q(x)\}$, if there exist $b > 1$ such that for a.a. x

$$b^{-1}\rho(x) \le \rho(y) \le b\,\rho(x)$$

for a.a. $y \in Q(x)$.

Example 1. Let Γ be a compact set in R^n, which has no interior points. We set $|x - y|_\infty = \max\limits_{1\le j\le n} |x_j - y_j| \quad (x,\ y \in R^n)$. The function $\rho(x) = \inf\limits_{y\in\Gamma} |x - y|_\infty$ satisfies the slow variation condition with respect to the family $\{(1-\tau)Q_{\rho(x)}(x)\}$, $0 < \tau < 1$. Since ρ is a continuous function, then $\rho(x) = \min\limits_{y\in\Gamma} |x-y|_\infty = 0$ for $x \in \Gamma$ and $\rho \in L^+_{loc}$. Let $h = \rho(x) > 0$, $0 < \varepsilon < 1$. Let us prove that $\Gamma \bigcap \varepsilon Q(x) = \emptyset$. If $y \in \Gamma \bigcap \varepsilon Q(x) \ne \emptyset$, then

$$0 < \rho(x) \le |x - y|_\infty + \inf_{z\in\Gamma} |y - z|_\infty = |x - y|_\infty < \varepsilon\rho(x)/2,$$

which is impossible. Thus, $Q(x) = \bigcup\limits_{0<\varepsilon<1} \varepsilon Q(x) \subset R^n \setminus \Gamma$. Let $y \in (1-\tau)Q(x)$. Then $|z - x|_\infty \ge \rho(x)$ for $z \in \Gamma$, and $|y - x|_\infty = \max\limits_{1\le j\le n} |y_j - x_j| \le (1-\tau)\rho(x)/2$, which imply that $|z - y|_\infty \ge ||z - x|_\infty - |x - y|_\infty| \ge \rho(x)/2$. Hence, $\rho(y) = \inf\limits_{z\in\Gamma} |z - y| \ge \rho(x)/2$. Since $|z - y|_\infty \le |z - x|_\infty + |x - y|_\infty$, then $\rho(y) \le 2\rho(x)$ for $y \in (1-\tau)Q(x)$. By taking $\tau = 1/2$, we get

$$\frac{1}{2}\rho(x) \le \rho(y) \le 2\rho(x),$$

if $y \in \frac{1}{2}Q(x)$.

Example 2. It is easy to show that $\rho(x) = (1 + |x|)^s$, $s > 0$ satisfies the slow variation condition on cubes $Q(x) = Q_{h(x)}(x)$, $h(x) = (1 + |x|)^{-s/lp}$.

By c we denote constants depending only on the assigned numerical parameters, for example, $c = c(l, p, n)$, etc.

2 Main Results

Let X, Y be Banach spaces of functions $u\colon R^n \to R$. We say that a function $\gamma\colon R^n \to R$ is a (pointwise) multiplier acting from X to Y, if

$$Tu = \gamma u \in Y$$

for all $u \in X$. We denote by $M(X \to Y)$ the space of all multipliers for which the operator T is bounded from X to Y. We introduce the norm by

$$\|\gamma;\ M(X \to Y)\| = \|T;\ X \to Y\|$$

for $\gamma \in M(X \to Y)$.

Theorem 1. *Let* $1 < p \le q < \infty$, $lp > n$. *Let* ρ *satisfy the slow variation condition on cubes* $Q(x) = Q_{h(x)}(x)$, *where* $h(x) = \rho(x)^{\mu/l}$, $\mu > 0$. *Assume that*

$$\mathbf{C} = \operatorname*{ess\,sup}_x \rho(x)^{-\mu n/lp} \left(\int_{Q(x)} |\gamma|^q \omega \right)^{1/q} < \infty.$$

Then $\gamma \in M(W^l_{p,\rho^\mu} \to L_{q,\omega})$. *And,*

$$c \operatorname*{ess\,sup}_x \rho(x)^{-\mu n/lp} \left(\int_{\frac{1}{2}Q(x)} |\gamma|^q \omega \right)^{1/q} \le \|\gamma; M(W^l_{p,\rho^\mu} \to L_{q,\omega})\| \le c\,\mathbf{C}.$$

Let $\varGamma$ be a compact manifold in R^n with dimension $\le n - 1$. Let α be a measure on $\varGamma$, $\alpha(\varGamma) < \infty$. Let $L_{q,\alpha}(\varGamma)$ be the space of all continuous functions in $\varGamma$ with the norm

$$\|u; L_{q,\alpha}(\varGamma)\| = \left(\int_\varGamma |u|^q \, d\alpha \right)^{1/q} < \infty.$$

Theorem 2. *Let* $1 < p \le q < \infty$, $lp > n$. *Let* $\gamma \in C(R^n)$. *Let* ρ *satisfy the slow variation condition on cubes* $Q(x) = Q_{h(x)}(x)$, *where* $h(x) = \rho(x)^{\mu/l}$, $\mu > 0$. *If*

$$\mathbf{C} = \operatorname*{ess\,sup}_x \rho(x)^{-\mu n/lp} \left(\int_{Q(x) \bigcap \varGamma} |\gamma|^q \, d\alpha \right)^{1/q} < \infty,$$

then $\gamma \in M(W^l_{p,\rho^\mu} \to L_{q,\alpha}(\varGamma))$. *The norm satisfies the following inequality*

$$\|\gamma; M(W^l_{p,\rho^\mu} \to L_{q,\alpha}(\varGamma))\| \le c\,\mathbf{C}.$$

Let $\sigma(x) = \inf_{y \in \Gamma} |y - x|_\infty$. The function $\sigma(x)$ satisfies the slow variation condition with respect to the family $B = \{Q(x) = Q_{h(x)}(x),\ x \in E\}$, where $E = R^n \setminus \Gamma$, $h(x) = 2^{-1}\sigma(x)$. Let $\widetilde{\rho}(x) = \sigma(x)$, if $0 < \sigma(x) \leq 1$ and $\widetilde{\rho}(x) = 1$, if $\sigma(x) > 1$.

The function $\widetilde{\rho}$ satisfies the slow variation condition with respect to the family $\widetilde{B} = \{\widetilde{Q}(x) = Q_{\widetilde{h}(x)}(x),\ x \in E\}$, $\widetilde{h}(x) = \widetilde{\rho}(x)$:

$$2^{-1}\widetilde{\rho}(x) \leq \widetilde{\rho}(y) \leq 2\widetilde{\rho}(x), \tag{3}$$

if $y \in E \cap \widetilde{Q}(x)$.

Let us prove (3). We have

$$2^{-1}\sigma(x) \leq \sigma(y) \leq 2\sigma(x), \tag{4}$$

if $y \in Q(x)$.

Let $y \in E \cap \widetilde{Q}(x) \subset E \cap Q(x)$, $x \in E$. Then

1. $0 < \sigma(x),\ \sigma(y) \leq 1 \Rightarrow \widetilde{\rho}(x) = \sigma(x),\ \widetilde{\rho}(y) = \sigma(y) \Rightarrow$ (4);
2. $0 < \sigma(x) \leq 1,\ \sigma(y) > 1 \Rightarrow \widetilde{\rho}(x) = \sigma(x) \geq 2^{-1}\sigma(y) > 2^{-1} \Rightarrow$ $2^{-1}\widetilde{\rho}(x) = 2^{-1}\sigma(x) < 1 = \widetilde{\rho}(y) = 2\frac{1}{2} \leq 2\widetilde{\rho}(x) \Rightarrow$ (4);
3. $0 < \sigma(y) \leq 1 < \sigma(x) \Rightarrow \widetilde{\rho}(x) = 1$, $2^{-1}\widetilde{\rho}(x) = 2^{-1} < 2^{-1}\sigma(x) < \sigma(y) = \widetilde{\rho}(y) \leq 1 = \widetilde{\rho}(x) \Rightarrow$ (4);
4. $1 < \sigma(x), \sigma(y) \Rightarrow \widetilde{\rho}(x) = \widetilde{\rho}(y) = 1 \Rightarrow$ (4).

Corollary 1. Let $1 < p \leq q < \infty$, $lp > n$. Let $\gamma \in C(R^n)$ and

$$\mathbf{C} = \operatorname*{ess\,sup}_x \widetilde{\rho}(x)^{-n/p} \left(\int_{\widetilde{Q}(x) \bigcap \Gamma} |\gamma|^q \, d\alpha \right)^{1/q} < \infty.$$

Then $\gamma \in M(W^l_{p,\widetilde{\rho}^l} \to L_{q,\alpha}(\Gamma))$. Here the norm satisfies the following inequality

$$\|\gamma; M(W^l_{p,\widetilde{\rho}^l} \to L_{q,\alpha}(\Gamma))\| \leq c\,\mathbf{C}.$$

To prove the main results we will apply the method of local estimates on cubes $Q(x) \in B$, the essence of which is as following: let $f,\ g$ be finite functions, $g \in L_q$, $f \in L_p$ $(1 \leq p, q < \infty)$ with bounded $\operatorname{supp} g = \operatorname{supp} f = E$. There exists a positive function $S(x)$, $x \in E$, such that

$$\int_{Q(x)} |g|^q \leq c\,(S(x))^q \left(\int_{Q(x)} |f|^p \right)^{q/p} \tag{5}$$

a.e. in E.

Let $\{Q^j,\ j \in J\}$ be a finite-multiple and finite-separable Besicovitch covering extracted from the family of cubes $\{Q(x),\ x \in E\}$, $Q^j = Q(x^j)$. Since the set of indices $J\ (\subset N)$ can be represented as a disjoint union $J = \bigcup_{1\le i\le \kappa_2} J_i$, where Q^j of J_i is pairwise disjoint, then by using (5), we obtain

$$|g|^q = {}_{E}\, |g|^q \le \kappa_2 \max_{1\le i\le\kappa_2} {}_{j\in J_i}\ {}_{Q^j}\, |g|^q \le \kappa_2 \max_{1\le i\le\kappa_2} {}_{j\in J_i} (S(x^j))^q\ \ {}_{Q^j}\, |f|^p \quad {}^{q/p}. \tag{6}$$

Next, we use well known inequalities for sums $(a_j, b_j \ge 0,\ r \ge 1)$:

$$\sum_j a_j^r \le \left(\sum_j a_j\right)^r. \tag{7}$$

Then (6), (7) imply that

$$\int |g|^q \le \kappa_2 \left(\operatorname*{ess\,sup}_{x\in E} S(x)\right)^q \left(\int |f|^p\right)^{q/p} \ (1 \le p \le q < \infty). \tag{8}$$

Proof of Theorem 1. Let $\operatorname{supp} u = E$. By using the embedding inequality (2) for each cube $Q(x)$, we have

$$ {}_{Q(x)}\, |\gamma u|^q \omega(y)\, dy \le \sup_{Q(x)} |u|^q \ {}_{Q(x)}\, |\gamma|^q \omega(y)\, dy \le$$

$$\le c \quad h(x)^{l-n/p}\rho(x)^{-\mu} \quad {}_{Q(x)}\, |\gamma|^q\omega(y)\, dy \ {}^{1/q}\ {}^{q} \quad {}_{Q(x)}\, |\rho^\mu \nabla_l u| + |u| \ {}^{p}\ dy \ {}^{q/p}.$$

By taking $|f|^p = (|\rho^\mu\nabla_l u| + |u|)^p$, $(S(x))^q = \rho(x)^{-\mu nq/lp}\int_{Q(x)} |\gamma|^q\omega$, $|g|^q = |\gamma u|^q\omega$ in (5), and by using (8), we get the upper estimate of $\|\gamma; M(W^l_{p,\rho^\mu} \to L_{q,\omega})\|$.

Next step is to show that

$$\|\gamma; M(W^l_{p,\rho^\mu} \to L_{q,\omega})\| \ge c \operatorname*{ess\,sup}_x \rho(x)^{-\mu n/lp}\left(\int_{\frac12 Q(x)} |\gamma|^q\omega\right)^{1/q}.$$

We take the function $\eta \in C_0^\infty(Q_1)$, $0 \le \eta \le 1$, $\eta = 1$ in $\frac12 Q_1$. Let $u_0(y) = \eta\left(h(x)^{-1}(y-x)\right)$. Then

$$\|\gamma; M(W^l_{p,\rho^\mu} \to L_{q,\omega})\| \ge \frac{\left(\int_{\frac12 Q(x)} |\gamma u_0|^q\omega\right)^{1/q}}{\left(\int_{Q(x)} (|\rho^\mu\nabla_l u_0|^p + |u_0|^p)\, dy\right)^{1/p}} \ge$$

$$\ge c\,\rho(x)^{-n\mu/lp}\left(\int_{\frac12 Q(x)} |\gamma|^q\omega\right)^{1/q}.$$

Proof of Theorem 2. Let $\widetilde{\alpha}$ be a measure in R^n, defined by $\widetilde{\alpha}(e) = \alpha(e \cap \Gamma)$. Let $u \in C_0^\infty$. In each $Q(x)$ we have

$$\int_{Q(x) \bigcap \Gamma} |\gamma u|^q \, d\alpha = \int_{Q(x)} |\gamma u|^q \, d\widetilde{\alpha} \leq c \sup_{\bar{Q}(x)} |u|^q \int_{Q(x) \bigcap \Gamma} |\gamma|^q \, d\alpha.$$

Next we follow the lines of the proof of Theorem 1.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Guzman, M.: Differentiation of integrals in R^n. Mir, Moscow (1978)
2. Maz'ya, V.G., Shaposhnikova, T.O.: Theory of multipliers in spaces of differentiable functions. LSU, Leningrad (1986)

Some New Fourier Multiplier Results of Lizorkin and Hörmander Types

Erlan Nursultanov[1(✉)], Lyazzat Sarybekova[2], and Nazerke Tleukhanova[3]

[1] Kazakhstan Branch of Lomonosov Moscow State University, 11 Munaitpasov str., Astana 010000, Kazakhstan
er-nurs@yandex.ru
[2] Kazakh-British Technical University, 59 Tole bi str., Almaty 050000, Kazakhstan
lsarybekova@yandex.ru
[3] L.N. Gumilyov Eurasian National University, 2 Satpayev str., Astana 010008, Kazakhstan
tleukhanova@rambler.ru

Abstract. This paper is devoted to the study of Fourier series and Fourier transform multipliers and contains introduction, which put some new results into a general frame. In the following Sections several further examples and results are presented and discussed. In Section 2 we present some important results (including the most early papers we know) concerning Fourier series multipliers of particular interest for the investigations. The corresponding result for Fourier transform multipliers can be found in Section 3. In Section 4 we give some applications and in Section 5 we describe shortly the main results: A generalization and sharpening of the Lizorkin theorem concerning Fourier transform multipliers between L_p and L_q. The Fourier series multipliers in the case with a regular system, which is rather general. A generalization and sharpening of the Lizorkin type theorem concerning Fourier series multipliers between L_p and L_q in this general case. A generalization of the Hörmander multiplier theorem for two dimensional Fourier series to the case with a general regular system.

Keywords: Multiplier of Fourier series · Trigonometric polynomial · Fourier transform multipliers · Lizorkin theorem · Hormander theorem · General regular system

1 Definitions and Some Examples of Fourier Multipliers

The problem about Fourier multipliers for Fourier series can be formulated as follows:

Let $1 \leq p \leq q \leq \infty$. It is said that the sequence of complex numbers $\lambda = \{\lambda_k\}_{k\in Z}$ is a *trigonometrical Fourier series multiplier* from $L_p[0,1]$ to $L_q[0,1]$,if

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_6

for every function $f \in L_p[0,1]$ with Fourier series $\sum_{k\in Z} \hat{f}(k)e^{2\pi ikx}$ there exists a function $f_\lambda \in L_q[0,1]$ with a Fourier series which coincides with the series $\sum_{k\in Z} \lambda_k \hat{f}(k)e^{2\pi ikx}$ and an operator T_λ, $T_\lambda f = f_\lambda$, which is a bounded operator from $L_p[0,1]$ to $L_q[0,1]$.

In other words, we consider the multiplier transformation T_λ defined by a sequence of numbers $\lambda = \{\lambda_k\}_{k\in Z}$ using the formal expansions:

$$\text{if} f \approx \sum_{k\in Z} \hat{f}(k)e^{2\pi ikx}, \text{then } T_\lambda f \approx \sum_{k\in Z} \lambda_k \hat{f}(k)e^{2\pi ikx}$$

and ask under which assumptions on λ the operator T_λ is bounded from $L_p[0,1]$ to $L_q[0,1]$.

For $p = 2$ such a characterization is that $\lambda \in l_\infty$, and for $p = 1$ is characterized as being of Fourier coefficients of some finite measure (see e.g. [85], p. 129).

The set m_p^q of all Fourier series multipliers is a normed space with the norm

$$\|\lambda\|_{m_p^q} = \|T_\lambda\|_{L_p\to L_q}.$$

In the case $p = q$, we just write shortly m_p instead of m_p^p.

We can define the space of multipliers m_p for $1 \le p \le \infty$, as the space of all sequences $\{\lambda_k\}_{k\in\ Z}$ such that

$$\|\sum \lambda_k \hat{f}(k)e^{2\pi ikx}\|_p \le c\|f\|_p$$

for any trigonometric polynomial f with a constant $c > 0$ independent of f. The infimum over all such c defines a norm and the space m_p becomes a Banach space and in 1965 A. Figà-Talamanca [16] even proved that this is a dual space for $1 < p \le 2$ and was even able to find its predual.

The following statements hold:

1. $m_2 = l_\infty, m_p = m_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$;
2. $m_p \subset m_q \subset l_\infty$ if $1 \le p \le q \le 2$.

We can also define multipliers for Fourier transform. Let $1 \le p \le q \le \infty$. It is said that the function φ is a *Fourier transform multiplier* from $L_p = L_p(\ R^n)$ to $L_q = L_q(\ R^n)$, briefly $\varphi \in M_p^q$, if there exists $c > 0$ such that, for every function f in the Schwartz space S, the following inequality holds

$$\|T_\varphi(f)\|_{L_q} \le c\,\|f\|_{L_p}\,, \tag{1}$$

where $T_\varphi(f) = F^{-1}\varphi F f$ with F and F^{-1} which are the direct and the inverse Fourier transforms, respectively.

The smallest constant c is the norm of this operator and it is denoted by the symbol $\|T_\varphi\|_{p\to q}$.

In the case $p = q$, we just write shortly M_p instead of M_p^p.

Note that $\|T_\varphi\|_{2\to 2} = \|\varphi\|_\infty$ and if φ is a L_p- multiplier, $1 < p < \infty$, then it is also a $L_{p'}-$ multiplier and $\|T_\varphi\|_{p'\to p'} = \|T_\varphi\|_{p\to p}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

One of the first results related to multipliers of Fourier series in Lebesgue spaces was a result of M. Riesz in 1927 [65], p. 230, (c.f. also [91], p. 266). He proved the boundedness of the partial sum operator S_n: If $1 < p < \infty$, then

$$\|S_n(f)\|_p \le c\|f\|_p,$$

i.e. the characteristic functions of the segment $A = [-n, n] \bigcap Z$ are multipliers in the L_p space. Here the norms of the multipliers are bounded, and the constant c depends only on the parameter p. If A is an arbitrary set, then the constant c depends on the geometric properties of the set, the dimension n and the parameter p.

Some important examples of Fourier multipliers are the functions

$$m_\delta(x) = \begin{cases} \left(1 - |x|^2\right)^\delta, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

In the case $\delta > 0$ the quantities $T_{m_\delta}(f)$ are called the Bochner-Riesz means, and for $\delta = 0$ the operator T_{m_0} is the spherical summation operator, since in this case $m_\delta = \chi_B$, where $B \subset R^n$ is the unit disk. These terms are justified by their close connection with the periodic case, see e.g. [1].

The well-known Carleson-Sjölin theorem [7] from 1972 claims that $m_\delta \in M_p$, $4/3 \le p \le 4$, for all $\delta > 0$. This justifies the appearance of the disk conjecture that in the limit case $\delta = 0$ it holds that $m_0 \in M_p$ for all $4/3 < p < 4$ (see, for instance, [74]). Nevertheless, it turned out that $m_0 \notin L_p$ for $p \neq 2$. This unexpected result was established by C. Fefferman in 1971 [15] for $n \ge 2$. His method of proof is rather universal and applies to arbitrary domains whose boundary has a smooth curved segment. Moreover, V.D. Stepanov [79] (c.f. also [78,80]) gave an example of a periodic multiplier having an analogous property in the one-dimensional case:

$$\gamma(x) = \sum_{n=2}^{\infty} \frac{e^{icn \ln n}}{n^{1/2}(\ln n)^\beta} e^{2\pi inx}, \beta > 1, c > 0.$$

We note that the characteristic function of a polygon is a L_p - multiplier for $1 < p < \infty$ (see e.g. [9,82]). Moreover, in 1981 M. de Guzman [11] posed the following question: for a domain intermediate in a sense between a disk and an ordinary polygon is its characteristic function a multiplier for some $p, 1 < p < \infty$? A.M. Stokolos [81] (see also [82]) examined this question for domains P_θ bounded by a polygon inscribed in a disk and having vertices at the points $A_k = \left(\cos\theta_k^{-1}, \sin\theta_k^{-1}\right)$. In particular, he showed how the curvature of the boundary influences boundedness of the multiplier operator and pointed out how such results can be used in various fields of analysis.

For the case when A is a polygon in Z^n, the constants c in the definition of multipliers were studied, e.g., in works by A. Cordoba [10] and by A.A. Yudin and V.A. Yudin [89].

Moreover, multipliers which are characteristic functions were studied by G. Diestel and L. Grafakos [12] and by L. Grafakos and X. Li [20] (see also [21]).

An important subclass of multipliers M_p is the class of idempotent multipliers, having the form χ_E, where $E \subset \mathbb{R}^n$ is measurable. In particular, V. Lebedev and A. Olevskii [37] showed that a necessary condition for $\chi_E \in M_p$ is that E is equivalent to an open set.

The problem to study multipliers of multiple Fourier series is closely related with estimates of the Dirichlet kernels norms. Some important results in this direction were obtained by A.A. Yudin and V.A. Yudin [89].

As concrete examples of multipliers $\lambda = \{\lambda_{nm}\}_{n,m\in\mathbb{Z}}$ Marcinkiewicz [45] presented already in 1939 the following ones:

$$\frac{m^2}{n^2+m^2}, \frac{n^2}{n^2+m^2}, \frac{|mn|}{n^2+m^2},$$

and informed that in this way a problem posed by Schauder is solved.

In 1975 M. Zafran [90] studied multipliers of "weak type (p,p)". By definition, a complex function φ on $\mathbb{Z}$ is a weak type (p,p) if and only if it is multiplier from L_p space to $L_{p,\infty}$ space, and has a strong type (p,p) if and only if it is multiplier from L_p space to L_p space. One of the achievements of M. Zafran [90] is the proof that for every p such that $1<p<2$, there exists a multiplier φ of the weak type (p,p), which is not the strong type (p,p). He proved the following theorem:

Theorem 1. *Let $1<p<2$ and $1/p+1/p^{'}=1$. Choose $\delta>0$ such that $1/p^{'}<\delta<1/p$ and consider a function*

$$\varphi = \sum_{n=2}^{\infty} \frac{e^{in\log n}}{n^{1/p}(\log n)^{\delta}} e^{inx}.$$

a) Then $\varphi \in m\left(l_p(\mathbb{Z}), l_{p,\infty}(\mathbb{Z})\right)$, but $\varphi \notin m\left(l_p(\mathbb{Z})\right)$;

b) Extend φ on $\mathbb{R}$ periodically (with period 2π). Then $\varphi \in M\left(L_p(\mathbb{R}), L_{p,\infty}(\mathbb{R})\right)$, but $\varphi \notin M\left(L_p(\mathbb{R})\right)$;

c) Let us consider φ as a function on group $\mathbb{R}_d$ ($\mathbb{R}$ with the discrete topology). Then $\varphi \in M\left(L_p(\overline{\mathbb{R}}), L_{p,\infty}(\overline{\mathbb{R}})\right)$, but $\varphi \notin M\left(L_p(\overline{\mathbb{R}})\right)$.

2 Fourier Series Multipliers

Already in 1916 H. Steinhaus [76] discussed some questions which can be regarded as a pertaining to be the problem concerning Fourier multipliers (see also [77]).

Relations between various classes of Fourier multipliers in an orthogonal system were given and inclusions of certain classes of sequences were discussed by W. Orlicz in 1929 [55] (see also [56–58]).

Moreover, already in 1933 S. Kaczmarz [28] (c.f. also [29]) proved some properties of a class of Fourier multipliers in an orthonormal system $\{\varphi_i(t)\}$.

We note further that S. Kaczmarz and J. Marcinkiewicz [30] in 1938 gave conditions under which the sequence of numbers $\lambda = \{\lambda_k\}$ is a $L_p - L_q$- multiplier, and where the expansions are with respect to any uniformly bounded orthonormal system on $[0,1]$ which is complete in $L_1[0,1]$.

Theorem 2. *If $\varphi_i(t)$ are bounded and $\{\varphi_i\}$ is complete in L, then the condition $\{\lambda_n\} \in (L_p, L_q), 1 \leq p < \infty, 1 \leq q \leq \infty$ is equivalent to the following set of conditions*

a) $H(x,t) \cong \sum \lambda_n \varphi_n(t) \int_0^x \varphi_n(u)du \in L_q$ for each fixed x;

b) for each n and each set of numbers $\varepsilon_0, ..., \varepsilon_{n-1}$ such that $\sum_0^{n-1} |\varepsilon_i|^p = 1$ we have

$$\left\{ \int_0^1 \left| \sum_0^{n-1} \varepsilon_i \left[H\left((i+1)/n, t\right) - H\left(i/n, t\right) \right] \right|^q dt \right\}^{1/q} \leq Mn^{-1/p},$$

where M does not depend on $\{\varepsilon_i\}$ and n.

Another early important result from 1939 in the theory of Fourier series is the following one proved by J. Marcinkiewicz [45]:

Theorem 3. *Let $1 < p < \infty$. If a sequence $\lambda = \{\lambda_m\}_{m \in Z}$ is bounded and the sums of differences over dyadic blocks are bounded, that is*

$$F_0(\lambda) = \sup_{m \in N} \left(\sum_{k=2^m}^{2^{m+1}} |\lambda_k - \lambda_{k+1}| + |\lambda_{-k} - \lambda_{-k-1}| \right) + \sup_{m \in Z} |\lambda_m| < \infty, \quad (2)$$

then λ is a Fourier multiplier in L_p and

$$\|\lambda\|_{m_p} \leq cF_0(\lambda).$$

This theorem means that if a sequence $\lambda = \{\lambda_m\}_{m \in Z}$ satisfies (2), then it belongs to m_p.

The corresponding Marcinkiewicz theorem for multiple Fourier series is also known. For simplicity, let us write it in the two-dimensional case. Consider the multiplier transformation T_λ given by a double sequence $\lambda = \{\lambda_{nm}\}_{n,m \in N}$ with the formula

$$T_\lambda f \sim \sum \lambda_{nm} c_{nm} e^{i(nx+my)}$$

and, where

$$f \sim \sum c_{nm} e^{i(nx+my)}.$$

Moreover, we define the dyadic intervals

$$I_k = \{i \in Z : 2^{k-1} \leq |i| < 2^k\}, J_l = \{j \in Z : 2^{l-1} \leq |j| < 2^l\},$$

and denote

$$\Delta_1 \lambda_{nm} = \lambda_{n+1,m} - \lambda_{n,m}, \quad \Delta_2 \lambda_{nm} = \lambda_{n,m+1} - \lambda_{n,m}, \quad \Delta_{1,2} = \Delta_1 \cdot \Delta_2.$$

In the paper [45] J. Marcinkiewicz also proved the following corresponding two-dimensional result (see also [44]):

Theorem 4. *Let $1 < p < \infty$. If for the double sequence $\lambda = \{\lambda_{nm}\}_{n,m\in \mathbb{Z}}$ the following constants are finite*

$$A = \sup_{n,m} |\lambda_{n,m}|, \quad B_1 = \sup_{k,m} \sum_{n\in I_k} |\Delta_1 \lambda_{n,m}|, \quad B_2 = \sup_{m,l} \sum_{m\in J_l} |\Delta_2 \lambda_{n,m}|$$

and

$$B_{1,2} = \sup_{k,l} \sum_{n\in I_k} \sum_{m\in J_l} |\Delta_1 \Delta_2 \lambda_{n,m}|,$$

then the operator T_λ is bounded in the space $L_p([0, 2\pi]^2)$ and

$$\|T_\lambda f\|_p \leq c(A + B_1 + B_2 + B_{1,2})\|f\|_p, \forall f \in L_p.$$

The Hörmander [23] multiplier theorem from 1960 (see also our Theorem 16) was in 1982 proved and applied also for the case with (one-dimensional) Fourier series multipliers by R.E. Edwards [13].

Theorem 5. *Let $1 \leq p \leq q \leq \infty$ and let $1/s = 1/p - 1/q$. Then the following hold:*

1. If $s \leq 2$, then $l^s(\mathbb{Z}) \subset m_p^q$ and $\|\lambda\|_{m_p^q} \leq \|\lambda\|_s$ for $\lambda \in l^s(\mathbb{Z})$;

2. If $1 < p \leq 2 \leq q < \infty$, and if λ is a complex-valued function on $\mathbb{Z}$ such that

$$M \equiv \sup_{n\in \mathbb{Z}} (1 + |n|)^{1/s} |\lambda_n| < \infty, \tag{3}$$

then $\lambda \in m_p^q$ and

$$\|\lambda\|_{m_p^q} \leq c_{pq'} M. \tag{4}$$

Remark 1. In paper [68] we generalize this (Hörmander - Edwards) Theorem 5 to the case with two-dimensional multipliers in a general regular system and also derive a lower estimate in (4).

We say that an orthonormal system $\Phi = \{\varphi_k\}_{k\in \mathbb{N}}$ of functions defined on [0,1] is a *regular system*, if there exists a constant $B > 0$ such that

1) for every closed interval e from $[0, 1]$ and for $k \in \mathbb{N}$ we have that

$$\left| \int_e \varphi_k(x) dx \right| \leq B \min(me, 1/k),$$

2) for every closed interval w from $\mathbb{N}$ (finite arithmetical sequence with step 1) and $t \in (0, 1]$ it yields that

$$\left(\sum_{k\in w} \varphi_k(\cdot) \right)^* (t) \leq B \min(|w|, 1/t), \tag{5}$$

where $\left(\sum_{k\in w} \varphi_k(\cdot)\right)^* (t)$ is the non-increasing rearrangement of the function $\sum_{k\in w} \varphi_k(x)$ and $|w|$ is the number of elements in w. This definition was introduced in [49].

Here and in the sequel $\{a_r^*\}$ the sign * denotes the non-increasing rearrangement of the sequence $\{a_n\}$ (see e.g. [22]).

Let $1 \le p \le q \le \infty$, let $\Phi = \{\varphi_k\}_{k\in N}$ be a regular system, let $f \in L_p[0,1]$ with Fourier series $\sum_{k\in N} \hat{f}(k)\varphi_k(x)$. Let $\lambda = \{\lambda_k\}_{k\in N}$ be a sequence of complex numbers. Let us define the sequence of partial sums $S_n(f,\lambda,x)$ as follows:

$$S_n(f,\lambda,x) = \sum_{k=1}^{n} \lambda_k \hat{f}(k)\varphi_k(x), n \in N.$$

We say that $\lambda = \{\lambda_k\}_{k\in N}$ is a *Fourier series multiplier in the regular system* Φ, from $L_p[0,1]$ to $L_q[0,1]$, i.e. $\lambda \in \mathbf{m}_p^q$ if

$$\|\lambda\|_{\mathbf{m}_p^q} := \sup_{n\in N}\sup_{f\neq 0} \frac{\|S_n(f,\lambda,x)\|_{L_q}}{\|f\|_{L_p}} < \infty.$$

Let us formulate an one-dimensional generalization of Edwards Theorem 5 for multipliers in a general regular system.

Theorem 6. *Let* $1 < p \le 2 \le q < \infty, 1/s = 1/p - 1/q$. *Then, for some constants* $c_1, c_2 > 0$,

$$c_1 \sup_{m\in N} \frac{1}{m^{1-1/s}} \left|\sum_{k=1}^{m} \lambda_k\right| \le \|\lambda\|_{\mathbf{m}_p^q} \le c_2 \sup_{m\in N} \frac{1}{m^{1-1/s}} \sum_{r=1}^{m} \lambda_r^*.$$

Corollary 1. *Let* $1 < p \le 2 \le q < \infty, 1/s = 1/p - 1/q, M_0$ *be the set of all finite subsets from* N. *If*

$$\sup_{Q\in M_0} \frac{1}{|Q|^{1-1/s}} \left|\sum_{m\in Q} \lambda_m\right| < \infty,$$

then

$$\lambda = \{\lambda_k\}_{k\in N} \in \mathbf{m}_p^q$$

and

$$c_1 \sup_{m\in N} \frac{1}{m^{1-1/s}} \left|\sum_{k=1}^{m} \lambda_k\right| \le \|\lambda\|_{\mathbf{m}_p^q} \le c_2 \sup_{Q\in M_0} \frac{1}{|Q|^{1-1/s}} \left|\sum_{k\in Q} \lambda_k\right|. \tag{6}$$

Remark 2. Note that supremum of the right hand side of inequality (6) less than M (see (3)) so Corollary 1 is a sharpening of Edwards Theorem (Theorem 5).

As we see the assumptions in the Marcinkiewicz theorem (Theorem 3) do not depend on p. The problem to find sufficient conditions which are essentially depending on p for multipliers to belong to some m_p was solved by E.D. Nursultanov [48] in 1998. The corresponding problem for Fourier transform multipliers seems still not to be solved (see e.g. [74], p. 130).

Theorem 7. *Let* $1 < p < \infty$, $\lambda = \{\lambda_m\}_{m\in Z^n}$ *be a sequence of real numbers. If*

$$F_p^1(\lambda) = \sup_k \sum_{r=1}^{|Q_k|} \left(\lambda_r^* - \lambda_{r+1}^*\right) \left([A_r] \left(\ln [A_r] + 1\right)^4\right)^{\left|1/p - 1/p'\right|} + \left|\lambda_{|Q_k|}^*\right| < \infty,$$

where

$$Q_k = \{m \in Z^n : 0 \le |m_j| \le k, j = 1, ..., n\},$$

$\{\lambda_r^*\}_{r=1}^{|Q_k|}$ *is the nonincreasing rearrangement (taking into account the sign) of the sequence* $\{\lambda_m\}_{m\in Q_k}$ *and*

$$A_r = \{m \in Q_k : \lambda_m > \lambda_r^*\},$$

then $\lambda \in m_p$ *and* $\|\lambda\|_{m_p} \le c_p F_p^1(\lambda)$.

In 1999 E.D. Nursultanov and N.T. Tleukhanova [52,54] derived lower and upper estimates of the norms of a sequence $\lambda = \{\lambda_k\}_{k\in Z}$ from class of trigonometrical Fourier series multipliers m_p^q, essentially depending of the parameters, in cases when the parameters p and q are separated by the number 2. In particular, they proved the next two theorems (Theorems 8 and 9).

Theorem 8. *Let* $1 < p \le 2 \le q < \infty, 1/s = 1/p - 1/q$, M_0 *be the set of all harmonic intervals in* Z^n. *Then*

$$c_1 \sup_{Q\in M_0} \frac{1}{|Q|^{1-1/s}} \left|\sum_{m\in Q} \lambda_m\right| \le \|\lambda\|_{m_p^q} \le$$

$$\le c_2 \sup_{k\in N^n} \frac{1}{(k_1 \cdots k_n)^{1-1/s}} \sum_{r_1=1}^{k_1} \cdots \sum_{r_n=1}^{k_n} \lambda_{r_1 \ldots r_n}^{*_1 \ldots *_n},$$

where $|Q|$ *denotes the quantity of elements in the set* Q.

Let us formulate the definition of harmonic intervals in Z^n (see [52]): Let $B = \{m_0, m_0 + 1, ..., m_0 + l\}$ be an interval in Z, and let $d \in N, d > l$. A set of the form $I = \bigcup_{k=0}^N [B + kd] = \bigcup_{k=0}^N \{m + kd : m \in B\}$ will be called a harmonic interval in Z. Accordingly, a set of the form $I = I_1 \times ... \times I_n$, where I_i are harmonic intervals in Z, will be called a harmonic interval in Z^n.

Moreover, the sequence $\{\lambda_{r_1 \ldots r_n}^{*_1 \ldots *_n}\}$ denotes the "repeatedly nonincreasing rearrangement" of the sequence $\{\lambda_{s_1 \ldots s_n}\}$ as defined e.g. in ([51], p.341).

In particular, we have the following result:

Corollary 2. *Let* $1 < p \le 2 \le q < \infty, 1/s = 1/p - 1/q$, E *be the set of all finite subsets from* Z^n *and let* M_0 *be the set of all harmonic intervals in* Z^n. *If*

$$\sup_{e\in E} \frac{1}{|e|^{1-1/s}} \left|\sum_{m\in e} \lambda_m\right| < \infty,$$

then

$$\lambda = \{\lambda_k\}_{k\in \mathbb{Z}^n} \in m_p^q$$

and

$$c_1 \sup_{Q\in M_0} \frac{1}{|Q|^{1-1/s}} \left|\sum_{m\in Q} \lambda_m\right| \le \|\lambda\|_{m_p^q} \le c_2 \sup_{e\in E} \frac{1}{|e|^{1-1/s}} \left|\sum_{m\in e} \lambda_m\right|. \tag{7}$$

Remark 3. Note that it is easy to see that the right hand side of inequality (7) is equivalent to condition (4). Hence, Corollary 2 is a generalization of Edwards' Theorem (Theorem 5).

Next we give an example showing that Theorem 8 is strictly better than the statement in Corollary 2.

Example 1. Let us consider the sequence $\lambda = \{\lambda_m\}_{m\in \mathbb{Z}^2}$, where

$$\lambda_m = \begin{cases} (m_1 \cdot m_2)^{-(1/p-1/q)} \ for \ m \in \mathbb{N}^2; \\ 0 \ for \ m \notin \mathbb{N}^2. \end{cases}$$

Some straightforward calculation shows that this sequence does not satisfy (7) but satisfies the assumptions in Theorem 8.

Let us consider a Shapiro sequence $\{\varepsilon_k\}_{k=1}^{\infty}$, which is defined by $\varepsilon_0 = 1,\ \varepsilon_{2k} = \varepsilon_k,\ \varepsilon_{2k+1} = (-1)^k\varepsilon_k,\ k\in \mathbb{N}$ (see [70] and also [5,52]).

The corresponding multiple Shapiro type sequence is defined by: $\varepsilon = \{\varepsilon_k\}_{k\in \mathbb{N}^n}$, where $\varepsilon_k = \varepsilon_{k_1}\cdot\varepsilon_{k_2}\ldots\varepsilon_{k_n}$.

Theorem 9. *Let $1 < p \le q \le 2$ or $2 \le p \le q < \infty$, $\varepsilon = \{\varepsilon_k\}_{k\in \mathbb{N}^n}$ be a Shapiro type sequence, and*

$$\frac{1}{r} = \begin{cases} 1/2 + 1/q \ for \ 2 \le p \le q < \infty; \\ 1/p' + 1/2 \ for \ 1 < p \le q \le 2. \end{cases}$$

Then

$$c_1 \sup_{\substack{s\in \mathbb{Z}^n \\ m\in \mathbb{N}^n}} \frac{1}{(m_1 ... m_n)^{1/r}} \left|\sum_{1\le k\le m} \varepsilon_k\lambda_{ks}\right| \le \|\lambda\|_{m_p^q} \le$$

$$\le c_2 \sup_{m\in \mathbb{N}^n} \frac{1}{(m_1 ... m_n)^{1/r}} \sum_{1\le k\le m} \lambda^{*_1\ldots *_n}_{k_1\ldots k_n}. \tag{8}$$

The upper and lower bounds in Theorem 9 do not depend on the parameter p for $2 \le p \le q < \infty$ and only on the parameter q for $1 < p \le q \le 2$. This means that if these parameters are varied, then the class m_p^q varies with preservation of relation (8).

This assertion solves the problem of finding sufficient conditions for λ to belong to the space of multipliers $m_p = m_p^q$ essentially depending on the parameter p.

The following complementary results were also proved in [52]:

Theorem 10. *Let* $1 < p \le 2 \le q < \infty$, $p' = p/(p-1)$. *Suppose that* $\lambda = \{\lambda_k\}_{k\in Z^n}$ *is a monotone sequence in the extended sense. Then a necessary and sufficient condition for the relation* $\lambda \in m_p^q$ *to hold is that*

$$F(\lambda) = \sup_{m\in N^n} \frac{1}{(m_1\cdots m_n)^{1/p'+1/q}} \left| \sum_{k_1=-m_1}^{m_1} \cdots \sum_{k_n=-m_n}^{m_n} \lambda_{k_1\ldots k_n} \right| < \infty.$$

In this case, $\|\lambda\|_{m_p^q} \sim F(\lambda)$.

By a monotone sequence in the extended sense we mean that there exists a number $c > 0$ such that

$$|\lambda_k| \le \frac{c}{|Q_k|} \left| \sum_{r\in Q_k} \lambda_r \right|$$

for every $k \in Z^n$, where $Q_k = \{r \in Z^n : 0 \le |r_j| \le |k_j|, j = 1, ..., n\}$ (see [52]).

Theorem 11. *Let* $\varepsilon = \{\varepsilon_k\}_{k\in N^n}$ *be the Shapiro type sequence. Suppose that either* $1 < p \le q \le 2$ *or* $2 \le p \le q < \infty$. *If the sequence* $\{\varepsilon_k\lambda_k\}_{k\in N^n}$ *is monotone in the extended sense, then the relation* $\{\lambda_k\}_{k\in N^n} \in m_p^q$ *is equivalent to the inequality*

$$\sup_{m\in N^n} \frac{1}{(m_1 \ldots m_n)^{1/r}} \sum_{k_1=1}^{m_1} \ldots \sum_{k_n=1}^{m_n} |\lambda_{k_1\ldots k_n}| < \infty,$$

where the number $1/r$ *is defined in Theorem 9.*

In 2007 L. Sarybekova and N. Tleukhanova [66] proved the following theorems on Fourier series multiplier:

Theorem 12. *Let* $2 < p \le q < +\infty$ *and let* $\Phi = \{\varphi_k(x)\}_{k=1}^{+\infty}$ *be a regular system. If* $\lambda = \{\lambda_k\}_{k\in Z}$ *satisfies the following conditions*

$$\left(\sum_{k=1}^{+\infty} \frac{(k|\lambda_k - \lambda_{k+1}| \cdot k^{\frac{1}{p}-\frac{1}{q}})^{p'}}{k} \right)^{\frac{1}{p'}} \le B,$$

$$\sup_{k\in N} |\lambda_k| k^{\frac{1}{p}-\frac{1}{q}} \le B,$$

then $\lambda \in \mathbf{m}_p^q$ *and* $\|\lambda\|_{\mathbf{m}_p^q} \le cB$.

Theorem 13. *Let* $1 < p < \infty, 1 \le q_1 < q_0 < +\infty$ *and let* $\Phi = \{\varphi_k(x)\}_{k=1}^{+\infty}$ *be regular system. If* $\lambda = \{\lambda_k\}_{k\in Z}$ *satisfies the following inequalities*

$$\left(\sum_{k=2}^{+\infty} \frac{(k|\lambda_k - \lambda_{k+1}| \cdot (\ln k)^{1+\frac{1}{q_1}-\frac{1}{q_0}})^{q_0'}}{k \ln k} \right)^{\frac{1}{q_0'}} \le B,$$

$$\sup_{k\in N} |\lambda_k| (\ln k)^{\frac{1}{q_1}-\frac{1}{q_0}} \le B,$$

then $\lambda \in \mathbf{m}_{p,q_0}^{p,q_1}$ *and* $\|\lambda\|_{\mathbf{m}_{p,q_0}^{p,q_1}} \le cB$.

Remark 4. The estimation in Theorem 13 is sharp because in [66] it was proved that: Let $1 < p < \infty, 1 \leq q_1 < q_0 < +\infty$. Then, for every $\varepsilon > 0$ there exists sequence $\lambda = \{\lambda_k\}_{k=2}^{+\infty} \in \mathbf{m}_{p,q_0}^{p,q_1}$, such that

$$\sum_{k=2}^{+\infty} \frac{\left[k|\lambda_k - \lambda_{k+1}| \cdot (\ln k)^{1+\frac{1}{q_1}-\frac{1}{q_0}}\right]^{q_0'-\varepsilon}}{k \ln k} = \infty.$$

3 Fourier Transform Multipliers

The Marcinkiewicz multiplier theorem from 1939 for the Fourier transform has the following form (let us formulate for simplicity only the one-dimensional case), see [44]:

Theorem 14. *Let $\varphi : \ R \rightarrow \ R$ be a bounded function of class C^1 on each dyadic set $(-2^{k+1}, -2^k) \cup (2^k, 2^{k+1})$ for $k \in \ Z$. Assume that the derivative φ' of the function φ satisfies the condition*

$$\sup_{k\in \ Z} \left(\int_{-2^{k+1}}^{-2^k} |\varphi'(t)|dt + \int_{2^k}^{2^{k+1}} |\varphi'(t)|dt \right) \leq A < \infty. \tag{9}$$

Then φ is L_p-multiplier for all $1 < p < \infty$ and

$$\|T_\varphi\|_p \leq c \max\left(p, \frac{1}{p-1}\right)^6 (\|\varphi\|_\infty + A).$$

An important analogue of the Marcinkiewicz theorem for Fourier transform multipliers was proved in 1956 by S. Mikhlin [46] in the following form:

Theorem 15. *Let $\varphi(x)$ be a bounded function in $\ R^n$ and assume that*

$$|x|^{|\alpha|} |D^\alpha \varphi(x)| \leq A, (|\alpha| \leq L)$$

for some integer $L > 1$. Then $\varphi \in M_p, 1 < p < \infty$, and

$$\|\varphi\|_{M_p} \leq c\, A,$$

where the constant c depends only on p.

For the classes M_p^q when p and q are separated by the number 2, there is the following well-known theorem by L. Hörmander [23] from 1960 :

Theorem 16. *If $1 < p \leq 2 \leq q < \infty, 1/s = 1/p - 1/q$, then*

$$L_{s\infty} \hookrightarrow M_p^q,$$

where $L_{s,\infty}$ denotes the Marcinkiewicz space equipped with the usual supremum norm.

In this paper L. Hörmander gave a detailed description of the properties of the spaces M_p^q. Moreover, in his paper from 1960 L. Hörmander also gave a further generalization of the result and a simplification of the proof of the Mikhlin theorem (Theorem 15). This result is sometimes called the Hörmander-Mikhlin multiplier theorem, which in the simplest one-dimensional case has the following form:

Theorem 17. *Let* $\varphi: R \to C$ *be a bounded function on* $R \setminus \{0\}$ *and satisfy either the Mikhlin condition*

$$|x\varphi'(x)| \leq A \tag{10}$$

or the weaker Hörmander condition

$$\sup_{R>0} R \int_{R<|x|<2R} |\varphi'(x)|^2 dx \leq A^2 < \infty. \tag{11}$$

Then φ *is a* L_p- *multiplier for all* $p \in (1, \infty)$ *and*

$$\|T_\varphi\|_p \leq c \max\left(p, \frac{1}{p-1}\right)(\|\varphi\|_\infty + A).$$

Moreover, T_φ *is of weak type* $(1,1)$.

This theorem was generalized by W. Littman [41] in 1965 and by J. Peetre [60] in 1966. They weakened the smoothness parameter of the function to an arbitrary number strictly greater than $n/2$ in terms of "fractional derivatives" in the conditions of the theorem.

In 1981 P. Sjögren and P. Sjölin [71] proved that the Mikhlin-Hörmander property (10)–(11) and Littlewood - Paley property are equivalent. Here, the Littlewood - Paley property means that there exists constant c such that $c^{-1}\|f\|_p \leq \| \left(\sum |S_k f|^2\right)^{1/2} \|_p \leq c\|f\|_p$ for all $f \in L_p$.

In 1986 O. Besov [4] made a great improvement of these results and weakened the smoothness condition to the limit value $n/2$. This was done by using what is later on called Besov spaces, which measure fractional smoothness in a perfect way. Moreover, the conditions on the function were put in more general anisotropic terms and the degree of summability replaced 2 to an arbitrary number $q \in (1,2]$. Concerning definition of the spaces at hand see e.g. in [2].

Theorem 18. *Let* $1 < q \leq 2, \varepsilon > 0, s_i \in N, k_i \in N_0, N_0 = N \cup \{0\}$

$$s_i > \frac{|\lambda|}{\lambda_i q} > k_i \geq 0 \quad (i = 1, ..., n).$$

$$\Delta^m(y)\varphi(x) = \sum_{j=0}^{m} (-1)^{m-j} C_m^j \varphi(x+jy), y \in R^n, m \in N_0, \Omega \subset R^n$$

$$\Delta^m(y,\Omega)\varphi(x) = \Delta^m(y)\varphi(x) \quad \texttt{for} \quad [x, x+my] \subset \Omega,$$

$$\Delta^m(y,\Omega)\varphi(x) = 0 \quad for \quad [x, x+my] \not\subset \Omega,$$

$$\Delta_i^m(h)\varphi(x) = \Delta^m(he^i)\varphi(x), \Delta_i^m(h,\Omega)\varphi(x) = \Delta^m(he^i,\Omega)\varphi(x), h \in R^1.$$

Let, for the measurable function $\varphi: R^n \to C$, which has on $R^n \setminus 0$ generalized Sobolev derivatives $D_i^{k_i}\varphi (i = 1, ..., n)$, the following inequalities hold

$$|\varphi(x)| \leq K < \infty, x \in R^n,$$

$$\int_0^{\varepsilon 2^{j\lambda_i}} \|\Delta_i^{s_i - k_i}(t, R^n \setminus \Pi_j) D_i^{k_i}\varphi\|_q t^{-1-\frac{|\lambda|}{\lambda_i q}+k_i} dt \leq K, j \in Z, i = 1, ..., n.$$

Then $\varphi \in M_p$ for every $p \in (1,\infty)$.

In 1986 P.I. Lizorkin [40] reformulated the multiplier theorem of Hörmander-Mikhlin in the following form, which was suitable for his further investigations:

Theorem 19. *Let $\varphi \in W_2^l(Q_1)$ for any integer $l > n/2$, where $y > 0, Q_1 = \{t : t \in R^n, y/2 < \max_{j=1,,n} |t_j| < y\}$ and, moreover,*

$$\|\varphi(t,\cdot)\|_{W_2^l(Q_1)} \leq B.$$

Then $\varphi \in M_p, p \in (1,\infty)$ and

$$\|\varphi\|_{M_p} \leq cB.$$

In the same paper P.I. Lizorkin proved the following theorem concerning Fourier transform multipliers:

Theorem 20. *Let $\varphi_m \in B_{2,1}^r(R^n), r > n/2$, for every $m \in Z$ and*

$$\|\varphi_m(2^m,\cdot)\|_{B_{2,1}^r(R^n)} \leq B,$$

uniformly on m. Then $\varphi \in M_p, p \in (1,\infty)$ and

$$\|\varphi\|_{M_p} \leq cB.$$

Moreover, he derived a "fractional variant" of the Hörmander-Mikhlin theorem (Theorem 17) as a consequence of this result.

Theorem 21. *Let $\varphi_m \in L_2^r(R^n), r > n/2$, for every $m \in Z$ and*

$$\|\varphi_m(2^m,\cdot)\|_{L_2^r(R^n)} \leq B,$$

uniformly on m. Then $\varphi \in M_p, p \in (1,\infty)$ and

$$\|\varphi\|_{M_p} \leq cB.$$

Remark 5. Here, L_2^r denotes the Sobolev-Liouville space of functions f with finite norm

$$\|f\|_{L_p^r(R^n)} = \|F[(1+|x|^2)^{r/2}F^{-1}f]\|_{L_p(R^n)}.$$

It is known that for integers $r = l$, this space coincides with the Sobolev space W_p^l, and if $p = 2$, it coincides with the Besov space $B_{2,2}^r = B_2^r$.

Also in this paper Lizorkin constructed a "semilocalization" ${}^*B^r_{q,\theta}$ of the Besov spaces $B^r_{q,\theta}$. He proved the following theorem:

Theorem 22. *If* $\varphi \in {}^*B^{q/n}_{q,1}, 1 \le q \le 2$, *then* φ *is a multiplier from* L_p *into* L_p *for every* $p, 1 < p < \infty$.

Moreover, in the same paper Lizorkin also conjectured that if $\varphi \in {}^*B^{n/q}_{q,1}, 2 < q < \infty$, then φ is a multiplier from L_p into L_p, where $|1/p - 1/2| < 1/q$.

Observe that in the one-dimensional case Marcinkiewicz Theorem 14 is stronger than the Hörmander-Mikhlin theorem (Theorem 17), i.e., from the condition (11) follows the condition (9). But if you write these statement in higher dimensions, then the criteria of being multiplier in Marcinkiewicz theorem and Hörmander-Mikhlin theorem are not comparable (see e.g. [19], pp. 361–370). In addition, the assumption in the Marcinkiewicz theorem does not guarantee the weak type (1,1) of the mapping T_φ (see [33], p. 161).

Let $E = \{\varepsilon = (\varepsilon_1, ..., \varepsilon_n) : \varepsilon_i = 0 \texttt{ or } \varepsilon_i = 1, i = 1, ..., n\}$ be the set of corners in the unit cube in R^n.

In 1967 P.I. Lizorkin [38] studied the case $1 < p \le q < \infty$ and, in particular, proved the following theorem:

Theorem 23. *Let* $1 < p \le q < \infty$, $A > 0, \beta = \frac{1}{p} - \frac{1}{q}, \varepsilon \in E, |\varepsilon| = \sum_{i=1}^n \varepsilon_i$ *and let* φ *be a continuously differentiable function on* $R^n \setminus \{0\}$ *satisfying the following condition:*

$$\left| \prod_{i=1}^n y_i^{\varepsilon_i + \beta} \frac{\partial^{|\varepsilon|}\varphi}{\partial y_1^{\varepsilon_1} ... \partial y_n^{\varepsilon_n}} \right| \le A.$$

Then $\varphi \in M_p^q$ *and*

$$\|\varphi\|_{M_p^q} \le cA,$$

where $c > 0$ *depends only on* p *and* q.

Finally, we mention that the book of N. Jacob [27] from 2001 contains the following variant of the Mikhlin - Hörmander theorem:

Theorem 24. *Let* m *be a bounded measurable function defined on* R^n, *and define the operator* T_m *as the Fourier multiplier* $T_m = F^{-1}mFu$, *where* Fu *is the Fourier transform of* u. *Suppose that, for some* $k > \frac{1}{2}n$, *the expressions*

$$\sup_{\mu > 0} \mu^{-n} \int \left| m(x) D^\alpha \varphi \left(\frac{x}{\mu} \right) \right|^2$$

are finite for all C^∞*-functions* φ *with compact support in* $R^n \setminus \{0\}$, *and for all* $\alpha \in N^n$ *for which* $|\alpha| \le k$. *Then* T_m *is a mapping from* $L_p(R^n), 1 < p < \infty$, *to itself.*

The proof is based on the Calderón-Zygmund decomposition theorem for L_1-functions.

Moreover, H. Triebel [87] (see also [86] and [88]), P.I. Lizorkin [39], E. Lanconelli [35], D.G. Orlovski [59], E.E. Berniyazov and E.D. Nursultanov [3], Duvan Cardona and Michael Ruzhansky [6] considered Fourier multiplier in the Triebel-Lizorkin F^s_{pq} spaces and Besov B^s_{pq} space, $-\infty < s < \infty, 0 < p \leq \infty, 0 < q \leq \infty$. Note that these two scales of spaces of tempered distribution contain a lot of well-known classical spaces as special cases. Some of them are the following: Hölder-Zygmund, Bessel, Sobolev, Besov and Hardy spaces.

4 Applications

There are a huge number of applications involving Fourier multiplier in the literature and we can only briefly mention a few of them. In our opinion the most important applications are that such multipliers have been crucial for the development of important part of the following areas in mathematics:

A. Harmonic Analysis;
B. Theory of Function spaces;
C. Interpolation Theory;
D. Partial Differential Equations;
E. Numerical Analysis.

The theory of Fourier multipliers is one of the important directions of Harmonic Analysis, see e.g. the books [74,75] and the references given there. Such results are of great importance for several other areas of pure and applied mathematics and there are many challenging still open problems in this direction. We here just mention that Fourier multipliers are operators such as the operator of convolution, the operator of differentiation, the operator of fractional differentiation, the operator of fractional integration, pseudo-differential operators with constant coefficients and many others (see e.g. [85]).

Research concerning Fourier multipliers in the Theory of Function Spaces was initiated very early, e.g. by H. Steinhaus in 1916 [76] , by W. Orlicz [55] in 1929 (see also [56,57]), by S. Kaczmarz in 1933 [28] (see also [30]) and by J. Marcinkiewicz in 1939 [45].

After that many interesting results and applications have been obtained by several authors, see e.g. the books [13,14,34,86] and the references given there.

There are various other aspects of the fascinating theory of Fourier multipliers not mentioned so far, see e.g. the books of E.M. Stein [74], H. Triebel [86] and R. Larsen [36]. The close relation between the theory of function spaces and Fourier multipliers has been crucial also for the development of real Interpolation Theory, see e.g. the book of J. Bergh and J. Löfström [2] and the references given there. We just mention a few other papers related to interpolation theory by J. Peetre (with coauthors) [61], W. Littman (with coauthors) [41] (see also [42]), J. Löfström [43] and M. Carro [8]. We just discuss shortly this investigation of M. Carro:

Let m be a measurable bounded function and let us assume that there exists a bounded functions S so that $m(\xi)S^{it-1}(\xi)$ is a Fourier multiplier on L_p uniformly in $t \in R^1$. Then, using the analytic interpolation theorem of Stein, one can show that necessarily m is a L_p multiplier. M. Carro [8] showed that under the above conditions, it holds that, for every $k \in N$, $m\,(\log S)^k \in M_p$. The technique is based on the Schechter's interpolation method.

Multiplier methods can in particular be used for developing of Partial Differential Equations theory, see e.g. the books [24,25] by L. Hörmander and the references given there (c.f. also [26]) and the very new book by E.J. Straube [83] (c.f. also [84]).

Concerning the important role of Fourier multipliers in Numerical Analysis we refer to the book of S.L. Sobolev [72] (c.f. also [53]) and to paper of L.V. Kantorovich [31].

We can not mention all details in the applications mentioned above and just finish this Section by illustrating the fact that new applications of Fourier multipliers are still found by shortly discussing some fairly new papers.

In 2005 G. Garello and A. Morando [17] proved continuity for a class of pseudodifferential operators with symbols $a(x,\xi)$ which are smooth in the ξ variable and which are in the weighted Sobolev spaces in the space variable. They use mainly the Lizorkin-Marcinckiewicz theorem on continuity of Fourier multipliers to prove this result.

Moreover, by means of a Lizorkin-Marcinkiewicz theorem on the L_p-continuity of Fourier multipliers, the same authors in 2003 introduced a family of L_p-bounded pseudodifferential operators with symbols in the Hörmander classes (see [18]).

In 2003 V. Keyantuo and C. Lizama [32] studied the system

$$\begin{cases} u^{''}(t) - aAu(t) - \alpha A u^{'}(t) = f(t), t \in (0, 2\pi), \\ u(0) = u(2\pi), \\ u^{'}(0) = u^{'}(2\pi) \end{cases}$$

on a Banach space X. The operator A is a closed linear operator on $X, a, \alpha \in R$ and $f \in L^p_{2\pi}(R;\ X)$ or $f \in C^s_{2\pi}(R,\ X)$ with $0 < s < 1$. The authors gave necessary and sufficient conditions in order to obtain existence and uniqueness of periodic solutions in the spaces $L^p_{2\pi}(R;\ X)$ and $f \in C^s_{2\pi}(R,\ X)$ with $0 < s < 1$. They established maximal regularity results for strong solutions. They also studied mild solutions. The techniques involved operator-valued Fourier multiplier theorems.

In 2007 V. Poblete [64] studied some existence and uniqueness results for a second-order integro-differential equation with infinite delay in spaces of 2π -periodic vector-valued functions (Besov spaces). The technique of Fourier multipliers was applied to obtain a characterization of maximal regularity for the problem by introducing the notion of a strong solution in the Besov space $B^s_{pq}(T;\ X)$, where T denotes the one-dimensional torus $R/\ Z$ and X is a Banach space. Compatibility conditions that ensure the existence of a strong B^s_{pq}-periodic solution of the equation are obtained in a resonance case.

In 2007 Y. Morimoto and Ch-J. Xu [47] proved the smoothness of solutions for a non-standard class of the linear

$$Pu := \partial_t u + x \cdot \nabla_y u + \sigma \left(-\tilde{\Delta}_x\right)^{\alpha} u = f,$$

and the semi-linear

$$Pu = F(u)$$

pseudodifferential equations, where

$$\left(-\tilde{\Delta}_x\right)^{\alpha} = \left|\tilde{D}_x\right|^{2\alpha}$$

is a Fourier multiplier with the symbol $|\xi|^{2\alpha}\chi(\xi)+|\xi|^2(1-\chi(\xi))$. An interesting hypoelliptic estimate proved for the linear equation states that there is a gain in smoothness of order $\frac{1}{4}\left(\alpha-\frac{1}{3}\right)$ with respect to the weight $\langle x\rangle = \left(1+|x|^2\right)^{\frac{1}{2}}$.

5 The Main Results

In paper [67] a generalization of the Lizorkin Theorem 23 on Fourier multipliers is proved.

Theorem 25. *Let* $1 < p < q \le \infty, 0 \le \alpha < 1-\frac{1}{p}+\frac{1}{q}, \beta = \alpha+\frac{1}{p}-\frac{1}{q}$. *If a function* $\varphi \in AC^{loc}(\mathbb{R}\setminus\{0\})$ *satisfies the following conditions*

$$\sup_{y\in \mathbb{R}} |y|^{\frac{1}{p}-\frac{1}{q}}|\varphi(y)| \le A$$

and

$$\sup_{t>0} t^{1-\alpha}\left(y^{\beta}\varphi'(y)\right)^{*}(t) \le A,$$

then $\varphi \in M_p^q$ *and*

$$\|\varphi\|_{M_p^q} \le cA,$$

where $c > 0$ *depends only on* p, q *and* α.

Remark 6. In paper [67] also an example is given of a Fourier multiplier which satisfies the assumptions of the generalized theorem but does not satisfy the assumptions of the Lizorkin Theorem 23.

In paper [62] we prove a generalization and sharpening of the Lizorkin Theorem 23 concerning Fourier multipliers between L_p and L_q spaces.

Let $E = \{\varepsilon = (\varepsilon_1,\varepsilon_2) : \varepsilon_i = 0 \;\texttt{ or }\; \varepsilon_i = 1, i = 1,2\}$ be the set of corners in the unit cube in $\mathbb{R}^2$. The measure of Q is denoted by $|Q|$.

Theorem 26. *Let* $p = (p_1,p_2), q = (q_1,q_2), \alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2)$, $1 < p_i < q_i < \infty, 0 \le \alpha_i < 1-\frac{1}{p_i}+\frac{1}{q_i}$ *and* $\beta_i = \alpha_i+\frac{1}{p_i}-\frac{1}{q_i}, i = 1,2$. *If the continuously differentiable function* φ *on* $\mathbb{R}^2\setminus\{0\}$ *satisfies the following condition:*

$$\sup_{y_i\in \mathbb{R}^+}\prod_{i=1}^{2} y_i^{\varepsilon_i+(1-\varepsilon_i)\beta_i-\alpha_i}\left(\prod_{j=1}^{2} t_j^{\varepsilon_j\beta_j}\frac{\partial^{|\varepsilon|}\varphi}{\partial t_1^{\varepsilon_1}\partial t_2^{\varepsilon_2}}\right)^{*_{\varepsilon_1},*_{\varepsilon_2}}(y_1,y_2) \le A, \forall\varepsilon\in E,$$

then $\varphi \in M_p^q$ *and*

$$\|\varphi\|_{M_p^q} \le cA$$

where $c > 0$ *depends only on* p_i, q_i *and* $\alpha_i, |\varepsilon| = \varepsilon_1+\varepsilon_2$.

Remark 7. Theorem 26 is a strict generalization of (the Lizorkin) Theorem 23. In fact, in paper **B** it is also proved that the assumptions in Theorem 26 are strictly weaker than those in Theorem 23, since

$$\sup_{y_i\in R^+}\prod_{i=1}^{2} y_i^{\varepsilon_i+(1-\varepsilon_i)\beta_i-\alpha_i}\left(\prod_{j=1}^{2} t_j^{\varepsilon_j\beta_j}\frac{\partial^{|\varepsilon|}\varphi}{\partial t_1^{\varepsilon_1}\partial t_2^{\varepsilon_2}}\right)^{*_{\varepsilon_1},*_{\varepsilon_2}}(y_1,y_2)\leq$$

$$\leq\prod_{i=1}^{2}2^{\varepsilon_i+(1-\varepsilon_i)\beta_i-\alpha_i}\sup_{y_i\in R}\left|\prod_{i=1}^{2} y_i^{\varepsilon_i+\beta}\frac{\partial^{|\varepsilon|}\varphi}{\partial y_1^{\varepsilon_1}\partial y_2^{\varepsilon_2}}\right|,$$

and there exists a function φ satisfying the assumptions of Theorem 26, but not satisfying the assumptions in Theorem 23, i.e.

$$\sup_{y_i\in R^+}\prod_{i=1}^{2} y_i^{\beta_i-\alpha_i}\varphi(y_1,y_2)<\infty,$$

$$\sup_{y_i\in R^+} y_1^{\beta_1-\alpha_1}y_2^{1-\alpha_2}\left(t_2^{\beta_2}\varphi'_{t_2}(t_1,t_2)\right)^{*_2}(y_2)<\infty,$$

$$\sup_{y_i\in R^+} y_1^{1-\alpha_1}y_2^{\beta_2-\alpha_2}\left(t_1^{\beta_1}\varphi'_{t_1}(t_1,t_2)\right)^{*_1}(y_1)<\infty,$$

$$\sup_{y_i\in R^+}\prod_{i=1}^{2} y_i^{1-\alpha_i}\left(t_1^{\beta_1}t_2^{\beta_2}\varphi''_{t_1,t_2}(t_1,t_2)\right)^{*_1*_2}(y_1,y_2)<\infty,$$

but

$$\sup_{y_i\in R}\left|\prod_{i=1}^{2} y_i^{1+\frac{1}{p_i}-\frac{1}{q_i}}\varphi''_{y_1y_2}\right|=\infty.$$

Some multidimensional Lorentz spaces and an interpolation technique (see [50, 51, 73]) are used as crucial tools in the proofs. The obtained results are discussed in the light of other generalizations of the Lizorkin theorem and some open questions are raised.

Paper [63] deals with the Fourier series multipliers in the more general case with strong regular system. This system is rather general. For example, all trigonometrical systems, the Walsh system and all multiplicative systems with bounded elements are strong regular. A generalization and sharpening of the Lizorkin type theorem concerning Fourier series multipliers between the spaces L_p and L_q is proved.

Theorem 27. *Let $1<p<q<\infty$, $0\leq\alpha<1-\frac{1}{p}+\frac{1}{q}$, and $\beta=\alpha+\frac{1}{p}-\frac{1}{q}$. Let the sequence of complex numbers $\lambda=\{\lambda_k\}_{k\in N}$ satisfy the following conditions:*

$$\sup_{k\in N} k^{\frac{1}{p}-\frac{1}{q}}|\lambda_k|\leq A,$$

$$\sup_{k\in N} k^{1-\alpha}\left(m^{\beta}(\lambda_m-\lambda_{m+1})\right)^{*}(k)\leq A.$$

Then $\lambda \in \mathbf{m}_p^q$ *for each strong regular system, and*

$$\|\lambda\|_{\mathbf{m}_p^q} \le cA,$$

where $c > 0$ *depends only on* p, q *and* α*.*

The following corollary is a genuine generalization of (the Lizorkin) Theorem 23:

Corollary 3. *Let* $1 < p < q < \infty$*,* $A > 0$*. If a sequence of complex numbers* $\lambda = \{\lambda_k\}_{k \in N}$ *satisfies the following conditions:*

$$\sup_{k \in N} k^{\frac{1}{p}-\frac{1}{q}} |\lambda_k| \le A,$$

$$\sup_{k \in N} k^{1+\frac{1}{p}-\frac{1}{q}} |\lambda_k - \lambda_{k+1}| \le A,$$

then $\lambda \in \mathbf{m}_p^q$ *for each strong regular system, and*

$$\|\lambda\|_{\mathbf{m}_p^q} \le cA,$$

where $c > 0$ *depends on* p, q *and* α*.*

Remark 8. In paper [63] we have also proved that there exists a sequence λ satisfying the assumptions of Theorem 27, but not satisfying the assumptions in Corollary 3, i.e. there exists a sequence λ such that

$$\sup_{k \in N} k^{\frac{1}{p}-\frac{1}{q}} |\lambda_k| < \infty, \tag{12}$$

$$\sup_{k \in N} k^{1-\alpha} \left(m^{\beta}(\lambda_m - \lambda_{m+1})\right)^* (k) < \infty, \tag{13}$$

but

$$\sup_{k \in N} k^{1+\frac{1}{p}-\frac{1}{q}} |\lambda_k - \lambda_{k+1}| = \infty. \tag{14}$$

In paper [68] we obtain upper and lower estimates of the norm of Fourier series multipliers in regular systems thus improving the Edwards Theorem 5.

Theorem 28. *Let* $1 < p \le 2 \le q < \infty$*,* $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$ *and* $i = 1, 2$*. Then, for some constants* $c_1, c_2 > 0$*,*

$$c_1 \sup_{m_i \in N} \frac{1}{(m_1 m_2)^{1-1/s}} \left| \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \lambda_{k_1 k_2} \right| \le \|\lambda\|_{\mathbf{m}_p^q} \le$$

$$\le c_2 \sup_{m_i \in N} \frac{1}{(m_1 m_2)^{1-1/s}} \sum_{r_1=1}^{m_1} \sum_{r_2=1}^{m_2} \lambda_{r_1 r_2}^{*_1 *_2}. \tag{15}$$

Note, that the right hand side inequality in (15) of Theorem 28 for the trigonometrical system is more exact than the statement in the Edwards Theorem 5:

Corollary 4. *Let* $1 < p \le 2 \le q < \infty, \frac{1}{s} = \frac{1}{p} - \frac{1}{q}, M_0$ *be the set of all finite subsets from* N^2. *If*

$$\sup_{Q \in M_0} \frac{1}{|Q|^{1-1/s}} \left| \sum_{m \in Q} \lambda_m \right| < \infty,$$

then

$$\lambda = \{\lambda_k\}_{k \in N^2} \in \mathbf{m}_p^q$$

and

$$c_1 \sup_{m_i \in N} \frac{1}{(m_1 m_2)^{1-1/s}} \left| \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \lambda_{k_1 k_2} \right| \le \|\lambda\|_{\mathbf{m}_p^q} \le c_2 \sup_{Q \in M_0} \frac{1}{|Q|^{1-1/s}} \left| \sum_{m \in Q} \lambda_m \right|.$$

Next we note that the inverse statement is not true, i.e. Corollary 4 does not follow from the Edwards Theorem 5.

Example 2. Let consider the sequence $\lambda = \left\{(m_1 m_2)^{-1/s}\right\}_{m_i \in N}, i = 1, 2$. Then

$$\sup_{Q \in M_0} \frac{1}{|Q|^{1-1/s}} \left| \sum_{m \in Q} \lambda_m \right| = \infty.$$

but

$$\sup_{m_i \in N} \frac{1}{(m_1 m_2)^{1-1/s}} \sum_{r_1=1}^{m_1} \sum_{r_2=1}^{m_2} \lambda_{r_1 r_2}^{*_1 *_2} < \infty.$$

By using the obtained results it is possible to assign the classes of the sequence which satisfy condition of criterion of belonging to $\mathbf{m}_p^q$ space.

Theorem 29. *Let* $1 < p \le 2 \le q < \infty$ *and* $1/s = 1/p - 1/q$. *If the sequence* $\{\lambda_{k_1 k_2}\}_{k_i \in N}$ *is generalized-monotonous, then* $\lambda \in \mathbf{m}_p^q$ *if and only if*

$$F(\lambda) = \sup_{m_i \in N} \frac{1}{(m_1 m_2)^{1-1/s}} \left| \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \lambda_{k_1 k_2} \right| < \infty.$$

Moreover, $\|\lambda\|_{m_p^q} \approx F(\lambda)$.

In paper [69] we study the multipliers of multiple Fourier series for a regular system on anisotropic Lorentz spaces.

Let $E = \{\varepsilon = (\varepsilon_1, ..., \varepsilon_m) : \ \varepsilon_i = 0 \quad or \quad \varepsilon_i = 1, \ i = 1, ..., m\}$.

Theorem 30. *Let* $1 < \mathbf{p} = (p_1, ..., p_m) < \mathbf{q} = (q_1, ..., q_m) < \infty,\ 0 < \mathbf{r} = (r_1, ..., r_m) \leq \infty$, $0 < \alpha < 1 - \frac{1}{\mathbf{p}} + \frac{1}{\mathbf{q}}$ *and* $\beta = \alpha + \frac{1}{\mathbf{p}} - \frac{1}{\mathbf{q}}$. *If the sequence of complex numbers* $\lambda = \{\lambda_k\}_{k \in \ N^m}$ *satisfies the following properties for every* $\varepsilon \in E$

$$\sup_{k_i \in \ N} \prod_{i=1}^{m} k_i^{\varepsilon_i - \alpha_i} \left(\prod_{j=1}^{m} s_j^{\beta_j} |\Delta_\varepsilon \lambda_s| \right)^{*_\varepsilon} (k_1, ..., k_m) \leq \mu, \tag{16}$$

where $\Delta_\varepsilon \lambda_s = \Delta_{\varepsilon_1} ... \Delta_{\varepsilon_m} \lambda_{s_1, ..., s_m}$, $\quad f^{*_\varepsilon} = f^{*_1^{\varepsilon_1}, ..., *_m^{\varepsilon_m}}$, *then* $\lambda \in m_{\mathbf{p},\mathbf{r}}^{\mathbf{q},\mathbf{r}}$ *and*

$$\|\lambda\|_{m_{\mathbf{p},\mathbf{r}}^{\mathbf{q},\mathbf{r}}} \leq c\mu,$$

here constant $c > 0$ *depends only on* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *and* α.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Alimov, S.A., Ashurov, R.R., Pulatov, A.K.: Multiple series and Fourier integrals. Itogi Nauki i Tekhniki, Ser. Sovrem. Propl. Mat., Fundam. Napravleniya **42**, 7–104 (1989); English transl. in Commutative Harmonic Analysis, IV. Harmonic Analysis in R^n, Encycl. Math. Sci. **42**, 1–95 (1992)
2. Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Springer, Berlin (1976)
3. Berniyazov, E.E., Nursultanov, E.D. : On Fourier multipliers in Lorentz L_{pq} spaces. Collection of Contemporary Problems of Function Theory and Functional Analysis. Karaganda (1992), pp. 30–38 (in Russian)
4. Besov, O.V.: Hörmander's theorem on Fourier multipliers. Trudy Mat. Inst. Steklov. **173**, 3–13 (1986); English transl. in Proc. Steklov Inst. Math. **173**, 1–12 (1987)
5. Brillhart, J., Carlitz, L.: Note on the Shapiro polynomials. Proc. Amer. Math. Soc. **25**, 114–118 (1970)
6. Cardona, D., Ruzhansky, M.: Multipliers for Besov spaces on graded Lie groups. C.R. Acad. Sci. Paris, Ser. I (2017) (to appear)
7. Carleson, L., Sjölin, P.: Oscillatory integrals and multiplier problem for the disk. Studia Math. **44**, 287–299 (1972)
8. Carro, M.J.: A multiplier theorem using the Schechter's method of interpolation. J. Approx. Theory **120**(2), 283–295 (2003)
9. Córdoba, A.: The multiplier problem for the polygon. Ann. of Math. (2) **105**(3), 581–588 (1977)
10. Córdoba, A.: Geometric Fourier Analysis. Ann. Inst. Fourier (Grenoble) **32**(3), 215–226 (1982)
11. de Guzmán, M.: Real Variable Methods in Fourier Analysis. North-Holland, Amsterdam-New York (1981)
12. Diestel, G., Grafakos, L.: Unboundedness of the ball bilinear multiplier operator. Nagoya Math. J. **185**, 151–159 (2007)

13. Edwards, R. E.: Fourier Series. In: A Modern Introduction, vol. 2, 2nd edn. Springer, Berlin (1982)
14. Edwards, R.E., Gaudry, G.I.: Littlewood-Paley and Multiplier Theory. Springer-Verlag, Berlin-New York (1977)
15. Fefferman, C.: The multiplier problem for the ball. Ann. Math. **2**(94), 330–336 (1971)
16. Figà-Talamanca, A.: Translation invariant operators in L^p. Duke Math. J. **32**, 495–501 (1965)
17. Garello, G., Morando, A.: L_p-boundedness for pseudodifferential operators with non-smooth symbols and applications. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8**(2), 461–503 (2005)
18. Garello, G., Morando, A.: A class of L_p bounded pseudodifferential operators. Progress in Analysis, Vol. I, II (Berlin, : World Sci. Publ. River Edge **2003**, 689–696 (2001)
19. Grafakos, L.: Classical Fourier Analysis, 2nd edn. Springer, New York (2008)
20. Grafakos, L., Li, X.: The bilinear multiplier problem for the disc. in: Harmonic Analysis at Mount Holyoke (South Hadley, MA, 2001), Contemporary Mathematics, vol. 320. American Mathematical Society, Providence, pp. 173–193 (2003)
21. Grafakos, L., Li, X.: The disc as a bilinear multiplier. Amer. J. Math. **128**(1), 91–119 (2006)
22. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge Univ. Press, Cambridge (1934)
23. Hörmander, L.: Estimates for translation invariant operators in L^p spaces. Acta Math. **104**, 93–140 (1960)
24. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vols. I, II, 2nd edn. Springer, Berlin (1990)
25. Hörmander, L. : The Analysis of Linear Partial Differential Operators. vols. III, IV, 2nd edn. Springer, Berlin (1994)
26. Hörmander, L.: The null space of the $\bar{\partial}$-Neumann operator. Ann. Inst. Fourier (Grenoble) **54**(5), 1305–1369 (2004)
27. Jacob, N.: Pseudo Differential Operators and Markov Processes, vol. I. Imperial College Press, London (2001)
28. Kaczmarz, S.: Sur les multiplicateurs des series orthogonales. Studia Math. **4**, 21–26 (1933)
29. Kaczmarz, S., Steinhaus, H.: Theorie der Orthogonalreihen. Monografie Matematyczne 6, Warszawa–Lwów (1935)
30. Kaczmarz, S., Marcinkiewicz,J.: Sur les multiplicateurs des series orthogonales. Studia Math. **7**, 73–81 (1938); Reprinted in: Józef Marcinkiewicz, Collected Papers, PWN, Warsaw (1964)
31. Kantorovich, L.V.: Mathematical methods of organizing and planning production. Manage. Sci. **6**, 366–422 (1960)
32. Keyantuo, V., Lizama, C.: Periodic solutions of second order differential equations in Banach spaces. Math. Z. **253**(3), 489–514 (2006)
33. Kislyakov, S.V.: Fourier coefficients of continuous functions and a class of multipliers. Ann. Inst. Fourier (Grenoble) **38**(2), 147–183 (1988)
34. Kokilashvili, V.M., Meskhi, A., Persson, L.-E.: Weighted Norm Inequalities for Integral Transforms with Product Kernels. Nova Science Publishers, New-York (2009)
35. Lanconelli, E.: Sui moltiplicatori negli spazi si Besov. Boll. Un. Mat. Ital. **1**(4), 767–779 (1968)

36. Larsen, R.: An Introduction to the Theory of Multipliers. Springer, Heidelberg (1971)
37. Lebedev, V., Olevskii, A.: Idempotents of Fourier multiplier algebra. Geom. Funct. Anal. **4**(5), 539–544 (1994)
38. Lizorkin, P.I.: Multipliers of Fourier integrals in the spaces $L_{p,\theta}$. Trudy Mat. Inst. Steklov **89**, 231–248 (1967); English transl. in Proc. Steklov Inst. Math. **89**, 269–290 (1968)
39. Lizorkin, P.I.: Bases and multipliers in the spaces $B_r^{p\theta}(\Pi)$. Trudy Mat. Inst. Steklov **143**, 88–104 (1977); English transl. in Proc. Steklov Inst. Math. **143**, 93–110 (1980)
40. Lizorkin, P.I.: Limit cases of theorems on FL_p-multipliers. Trudy Mat. Inst. Steklov **173**, 164–180 (1986); English transl. in Proc. Steklov Inst. Math. **173**, 177–194 (1987)
41. Littman, W.: Multipliers in L^p and interpolation. Bull. Amer. Math. Soc. **71**, 764–766 (1965)
42. Littman, W., McCarthy, C., Rivière, N.: L^p-multiplier theorems. Studia Math. **30**, 193–217 (1968)
43. Löfström, J.: Some theorems on interpolation spaces with applications to approximations in L_p. Math. Ann. **172**, 176–196 (1967)
44. Maligranda, L.: Józef Marcinkiewicz (1910–1940)—on the Centenary of his Birth. Banach Center Publications (2011)
45. Marcinkiewicz, J.: Sur les multiplicateurs des series de Fourier. Studia Math. **8**, 78–91 (1939); Reprinted in: Józef Marcinkiewicz, Collected Papers, PWN, Warsaw, pp. 501–512 (1964)
46. Mikhlin, S.G. On the multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR (N.S.) **109**, 701–703 (1956) (in Russian)
47. Morimoto, Y., Xu, Ch-J: Hypoellipticity for a class of kinetic equations. J. Math. Kyoto Univ. **47**(1), 129–152 (2007)
48. Nursultanov, E.D.: On multipliers of Fourier series in the trigonometric system. Mat. Zametki **63**(2), 235–247 (1998); English transl. in Math. Notes **63**(1–2), 205–214 (1998)
49. Nursultanov, E.D.: On the coefficients of multiple Fourier series from L_p-spaces. Izv. Ross. Akad. Nauk Ser. Mat. **64**(1), 95–122 (2000) ; English transl. in Izv. Math. **64**(1), 93–120 (2000)
50. Nursultanov, E.D.: Interpolation theorems for anisotropic function spaces and their applications. Dokl. Akad. Nauk **394**(1), 22–25 (2004); English transl. in Dokl. Math. **69**(1), 16–19 (2004)
51. Nursultanov, E.D.: On the application of interpolation methods in the study of the properties of functions of several variables. Mat. Zametki **75**(3), 372–383 (2004); English transl. in Math. Notes **75**(3–4), 341–351 (2004)
52. Nursultanov, E.D., Tleukhanova, N.T.: On multipliers of multiple Fourier series. Tr. Mat. Inst. Steklova **227**, 237–242 (1999); English transl. in Proc. Steklov Inst. Math. **227**(4), 231–236 (1999)
53. Nursultanov, E.D., Tleukhanova, N.T.: Quadrature formulae for classes of functions of low smoothness. Sb. Math. **194**(10), 1559–1584 (2003); English transl. Mat. Sb. **194**(10), 133–160 (2003)
54. Nursultanov, E.D., Tleukhanova, N.T.: Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces. Funct. Anal. Appl. **34**(2), 151–153 (2000)
55. Orlicz, W.: Beiträge zur Theorie der Orthogonalentwicklungen. Studia Math. **1**, 1–39 (1929); Reprinted in: Władysław Orlicz, Collected Papers, PWN, Warszawa, pp. 50–88 (1988)

56. Orlicz, W.: Beiträge zur Theorie der Orthogonalentwicklungen (IV). Studia Math. **5**, 1–14 (1934); Reprinted in: Władysław Orlicz, Collected Papers, PWN, Warszawa, pp. 276–289 (1988)
57. Orlicz, W.: Ein Satz über die Erweiterung von linearen Operationen. Studia Math. **5**, 127–140 (1934); Reprinted in: Władysław Orlicz, Collected Papers, PWN, Warszawa, pp. 309–322 (1988)
58. Orlicz, W.: Collected Papers. II. PWN, Warszawa, Part I (1988)
59. Orlovskii, D.G.: On multipliers in the spaces $B^r_{p,\theta}$. Anal. Math. **5**(3), 207–218 (1979)
60. Peetre, J.: Application de la theorie des espaces d'interpolation dans l'analyse harmonique. Ricerche Mat. **15**, 3–36 (1966)
61. Peetre, J., Thangavelu, S., Wallin, N.-O.: Generalized Fock spaces, interpolation, multipliers, circle geometry. Rev. Mat. Iberoamericana **12**(1), 63–110 (1996)
62. Persson, L.-E., Sarybekova, L., Tleukhanova, N.: Multidimensional generalization of the Lizorkin theorem on Fourier multipliers. Proc. A. Razmadze Inst. **151**, 83–101 (2009)
63. Persson, L.-E., Sarybekova, L., Tleukhanova, N.: A Lizorkin theorem on Fourier series multipliers for strong regular system. In: Analysis for Science, Engineering and Beyond Springer Proceedings in Mathematics, vol. 6, pp 305–317. Springer, Berlin (2012)
64. Poblete, V.: Solutions of second-order integro-differential equations on periodic Besov spaces. Proc. Edinb. Math. Soc. **50**(2), 477–492 (2007)
65. Riesz, M.: Sur les functions conjugueés. Math. Z. **27**, 218–244 (1927)
66. Sarybekova, L., Tleukhanova, N.: Fourier series multipliers by regular system. Eurasian Math. J. **1**, 96–102 (2007) (in Russian)
67. Sarybekova, L., Tararykova, T., Tleukhanova, N.: On a generalization of the Lizorkin theorem on Fourier multipliers. Math. Inequal. Appl. **13**(3), 613–624 (2010)
68. Sarybekova, L.: Hörmander type theorems for Fourier series in regular systems. Research report 8, Department of Mathematics, Luleå University of Technology, Luleå (2010)
69. Sarybekova, L., Tleukhanova, L.N., Ydyrys, A.: The multipliers of multiple trigonometric Fourier series. Open Eng. **6**, 367–371 (2016)
70. Shapiro, H.S.: Extremal problems for polynomials and power series. Ph.D. Thesis. M.I.T., Cambridge (1952)
71. Sjögren, P., Sjölin, P.: Littlewood-Paley decompositios and Fourier multipliers with singularities on certain sets. Ann. Inst. Fourier (Grenoble) **31**(1), 157–175 (1981)
72. Sobolev S.L. (ed.): Theory of cubature formulas and numerical mathematics. In: Papers of the Conference on Differential Equations and Numerical Mathematics (Novosibirsk, 1978). Akademiya Nauk SSSR, Sibirskoe Otdelenie, Institut Matematiki, Novosibirsk (1980) (in Russian)
73. Sparr, G.: Interpolation of several Banach spaces. Ann. Mat. Pura Appl. **99**(4), 247–316 (1974)
74. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
75. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)
76. Steinhaus, H.: Certain properties of trigonometric series and Fourier series. Rosprawy Akademii Umiejętności, Kraków **56**, 175–225 (1916) (in Polish); English transl. in: Hugo Steinhaus, Selected Papers, PWN, Warszawa, pp. 161–204 (1985)

77. Steinhaus, H.: Additive und stetige Funktionaloperationen. Math. Z. **5**(3–4), 186–221 (1919); Reprinted in: Hugo Steinhaus, Selected Papers, PWN, Warszawa, pp. 252–288 (1985)
78. Stepanov, V. D.: On multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR. **258**(5), 1063–1065 (1981); English transl. in: Sov. Math. Dokl. **23**, 645–647 (1981)
79. Stepanov, V.D.: Periodic multipliers of Fourier integrals. Mat. Zametki. **32**(2), 141–150 (1982) (in Russian)
80. Stepanov, V.D.: Multipliers of Fourier integrals generated by characteristic functions of sets. Dokl. Akad. Nauk SSSR **299**(5), 1068–1070 (1988); English transl. in Soviet Math. Dokl. **37**(2), 532–533 (1988)
81. Stokolos, A.M.: On certain classes of maximal and multiplier operators. Abstracts: The First Göteborg Conference in Harmonic Analysis, Göteborg (1990)
82. Stokolos, A.M.: On a problem of M. de Guzman on Fourier multipliers of polygonal domains. Sibirsk. Mat. Zh. **3**6:6, 1392-1398 (1995); English transl. in Siberian Math. J. **36**(6), 1210–1216 (1995)
83. Straube, E.J.: Lectures on the L^2- Sobolev Theory of the $\bar{\partial}$-Neumann Problem. EMS, Zürich (2010)
84. Straube, E.J.: A sufficient condition for global regularity of the $\bar{\partial}$-Neumann operator. Adv. Math. **217**(3), 1072–1095 (2008)
85. Torchinsky, A.: Real-Variable Methods in Harmonic Analysis. Academic Press, Orlando (1986)
86. Triebel, H.: Theory of Function Spaces. Birkhäuser Verlag, Basel (1983)
87. Triebel, H.: Multipliers and unconditional Suhauder bases in Besov spaces. Studia Math. **60**(2), 145–156 (1977)
88. Triebel, H.: Complex interpolation and Fourier multipliers for the spaces F^s_{pq} and B^s_{pq} of Besov -Hardy-Sobolev type: the case $0<p \leq \infty, 0<q \leq \infty$. Math. Z. **176**(4), 495–510 (1981)
89. Yudin, A.A., Yudin, V.A.: Polygonal Dirichlet kernels and growth of Lebesgue constants. Mat. Zametki **37**(2), 220–236 (1985); English transl. in Math. Notes **37**, 124–135 (1985)
90. Zafran, M.: Multiplier transformations of weak type. Ann. Math. **101**(2), 34–44 (1975)
91. Zygmund, A.: Trigonometric Series. vols. I, II. 2nd edn. Cambridge University Press, London (1968)

Four-Dimensional Generalized Difference Matrix and Almost Convergent Double Sequence Spaces

Orhan Tuğ(✉)

Department of Mathematics Education, Ishik University, Ishik Campus, 100 Meter Street, Erbil, Iraq
orhan.tug@ishik.edu.iq, tugorhan27@gmail.com

Abstract. Tuğ and Başar [3] have recently studied the concept of four dimensional generalized difference matrix $B(r,s,t,u)$ and its matrix domain in some double sequence spaces. In this present paper, as a natural continuation of [3], we introduce new almost null and almost convergent double sequence spaces $B(C_f)$ and $B(C_{f_0})$ as the domain of four-dimensional generalized difference matrix $B(r,s,t,u)$ in the spaces C_f and C_{f_0}, respectively. Firstly, we prove that the spaces $B(C_f)$ and $B(C_{f_0})$ of double sequences are Banach spaces under some certain conditions. We give some inclusion relations with some topological properties. Moreover, we determine the $\alpha-$dual, $\beta(bp)-$dual and $\gamma-$dual of the spaces $B(C_f)$. Finally, we characterize the classes of four dimensional matrix mappings defined on the spaces $B(C_f)$ of double sequences.

Keywords: Four-dimensional generalized difference matrix · Matrix domain · Almost convergent double sequence space · Alpha-dual · Beta-dual · Gamma-dual · Matrix transformations

1 Introduction

We denote the set of all complex valued double sequence by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any subspace of Ω is called a double sequence space. A double sequence $x = (x_{mn})$ of complex numbers is called bounded if $\|x\|_\infty = \sup_{m,n\in N} |x_{mn}| < \infty$, where $N = \{0, 1, 2, \ldots\}$. The space of all bounded double sequences is denoted by M_u which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the double sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$ and $l \in C$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then the double sequence x is called convergent in the Pringsheim's sense to the limit point l and we write $p - \lim_{m,n\to\infty} x_{mn} = l$; where C denotes the complex field. The space of all convergent double sequences in Pringsheim's sense is denoted by C_p. Unlike single sequences there are such double sequences which are convergent in Pringsheim's sense but unbounded.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_7

That is, the set $C_p - M_u$ is not empty. Actually, following Boos [4, p. 16], if we define the sequence $x = (x_{mn})$ by

$$x_{mn} = \begin{cases} n\ , m = 0, n \in N; \\ 0\ , m \geq 1, n \in N, \end{cases}$$

then it is obvious that $p-\lim_{m,n\to\infty} x_{mn} = 0$ but $\|x\|_\infty = \sup_{m,n\in N} |x_{mn}| = \infty$, so $x \in C_p - M_u$. Then, we can consider the set C_{bp} of double sequences which are both convergent in Pringsheim's sense and bounded, i.e., $C_{bp} = C_p \cap M_u$. Hardy [6] showed that a sequence in the space C_p is said to be regular convergent if it is a single convergent sequence with respect to each index and denote the space of all such sequences by C_r. Moreover, by C_{bp0} and C_{r0} we denote the spaces of all double sequences converging to 0 contained in the sequence spaces C_{bp} and C_r, respectively. Móricz [8] proved that C_{bp}, C_{bp0}, C_r and C_{r0} are Banach spaces with the norm $\|\cdot\|_\infty$. By L_q we denote the space of absolutely q-summable double sequences corresponding to the space ℓ_q of q-summable single sequences, that is,

$$L_q := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^q < \infty \right\}, \quad (1 \leq q < \infty)$$

which is a Banach space with the norm $\|\cdot\|_q$ defined by Başar and Sever [2]. Zeltser [10] introduced the space L_u as a special case of the space L_q with $q = 1$. Let λ be a double sequence space, converging with respect to some linear convergence rule $\vartheta - \lim : \lambda \to C$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $\vartheta - \sum_{i,j} x_{ij} = \vartheta - \lim_{m,n\to\infty} \sum_{i,j=0}^{m,n} x_{ij}$. For short, throughout the text the summations without limits run from 0 to ∞, for instance $\sum_{i,j} x_{ij}$ means that $\sum_{i,j=0}^{\infty} x_{ij}$.

Here and after, unless stated otherwise we assume that ϑ denotes any of the symbols p, bp or r.

The $\alpha-$dual λ^α, the $\beta(\vartheta)-$dual $\lambda^{\beta(\vartheta)}$ with respect to the $\vartheta-$convergence and the $\gamma-$dual λ^γ of double sequence space λ are respectively defined by

$$\lambda^\alpha := \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl}x_{kl}| < \infty \ \text{ for all } \ x = (x_{kl}) \in \lambda \right\},$$

$$\lambda^{\beta(\vartheta)} := \left\{ a = (a_{kl}) \in \Omega : \vartheta - \sum_{k,l} a_{kl}x_{kl} \ \text{ exists for all } \ x = (x_{kl}) \in \lambda \right\},$$

$$\lambda^\gamma := \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n\in N} \left| \sum_{k,l=0}^{m,n} a_{kl}x_{kl} \right| < \infty \ \text{ for all } \ x = (x_{kl}) \in \lambda \right\}.$$

It is easy to see for any two spaces λ and μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion

$\lambda^{\alpha} \subset \lambda^{\beta(\vartheta)}$ holds while the inclusion $\lambda^{\beta(\vartheta)} \subset \lambda^{\gamma}$ does not hold, since the $\vartheta-$convergence of the double sequence of partial sums of a double series does not imply its boundedness.

Let λ and μ be two double sequence spaces, and let $A = (a_{mnkl})$ be any four-dimensional complex infinite matrix. Then, we say that A defines a matrix mapping from λ into μ and we write $A : \lambda \to \mu$, if for every sequence $x = (x_{kl}) \in \lambda$ the $A-$transform $Ax = \{(Ax)_{mn}\}_{m,n\in N}$ of x exists and it is in μ; where

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl}x_{kl} \ \ \textit{for each } m,n \in N. \tag{1}$$

We define $\vartheta-$summability domain $\lambda_A^{(\vartheta)}$ of A in a space λ of double sequences by

$$\lambda_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl}x_{kl} \right)_{m,n\in N} \textit{exists and is in } \lambda \right\}.$$

We say with the notation (1) that A maps the space λ into the space μ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices, transforming the space λ into the space μ, by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (1) converges in the sense of ϑ for each $m, n \in N$, i.e, $A_{mn} \in \lambda^{\beta(\vartheta)}$ for all $m, n \in N$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l\in N}$ for all $m, n \in N$. We say that a four-dimensional matrix A is $C_\vartheta - \textit{conservative}$ if $C_\vartheta \subset (C_\vartheta)_A$, and is $C_\vartheta - \textit{regular}$ if it is $C_\vartheta - \textit{conservative}$ and

$$\vartheta - \lim Ax = \vartheta - \lim_{m,n\to\infty} (Ax)_{mn} = \vartheta - \lim_{m,n\to\infty} x_{mn}, \ \textit{where } x = (x_{mn}) \in C_\vartheta.$$

Adams [1] defined that the four-dimensional infinite matrix $A = (a_{mnkl})$ is called a triangular matrix if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both. We also say by [1] that a triangular matrix $A = (a_{mnkl})$ is said to be a triangle if $a_{mnmn} \neq 0$ for all $m, n \in N$. Moreover, by referring Cooke [5, Remark (a), p. 22] we can say that every triangle matrix has a unique inverse which is also a triangle.

Let $r, s, t, u \in R\backslash\{0\}$. Then, the four dimensional generalized difference matrix $B(r, s, t, u) = \{b_{mnkl}(r, s, t, u)\}$ is defined by

$$b_{mnkl}(r,s,t,u) := \begin{cases} su \ , (k,l) = (m-1, n-1), \\ st \ , (k,l) = (m-1, n), \\ ru \ , (k,l) = (m, n-1), \\ rt \ , (k,l) = (m,n) \\ 0 \ , \text{otherwise} \end{cases}$$

for all $m, n, k, l \in N$. Therefore, the $B(r, s, t, u)-$transform of a double sequence $x = (x_{mn})$ is given by

$$y_{mn} := \{B(r,s,t,u)x\}_{mn} = \sum_{k,l} b_{mnkl}(r,s,t,u)x_{kl} \tag{2}$$

$$= sux_{m-1,n-1} + stx_{m-1,n} + rux_{m,n-1} + rtx_{mn} \tag{3}$$

for all $m, n \in N$. Thus, we have the inverse $B^{-1}(r,s,t,u) = F(r,s,t,u) = \{f_{mnkl}(r,s,t,u)\}$, as follows:

$$f_{mnkl}(r,s,t,u) := \begin{cases} \frac{(-s/r)^{m-k}(-u/t)^{n-l}}{rt} & , 0 \leq k \leq m,\ 0 \leq l \leq n, \\ 0 & , \text{otherwise} \end{cases}$$

for all $m, n, k, l \in N$. Therefore, we can obtain $x = (x_{mn})$ by applying the inverse matrix $F(r,s,t,u)$ to (2) that

$$x_{mn} = \frac{1}{rt} \sum_{k,l=0}^{m,n} \left(\frac{-s}{r}\right)^{m-k} \left(\frac{-u}{t}\right)^{n-l} y_{kl} \quad \text{for all } m, n \in N. \tag{4}$$

Throughout the paper, we suppose that the terms of double sequence $x = (x_{mn})$ and $y = (y_{mn})$ are connected with the relation (2). If $p-\lim\{B(r,s,t,u)x\}_{mn} = l$, then the sequence $x = (x_{mn})$ is said to be $B(r,s,t,u)$ convergent to l. Note that in the case $r = t = 1$ and $s = u = -1$ for all $m, n \in N$, the four dimensional generalized difference matrix $B(r,s,t,u)$ is reduced to the four dimensional difference matrix $\Delta = B(1,-1,1,-1)$.

Lorentz [7] introduced the concept of almost convergence for single sequence and Móricz and Rhoades [9] extended and studied this concept for double sequence. A double sequence $x = (x_{mn})$ of complex numbers is said to be almost convergent to a generalized limit L if

$$p - \lim_{q,q' \to \infty} \sup_{m,n>0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0.$$

In this case, L is called the f_2-limit of the double sequence x. Throughout the paper, C_f denotes the space of all almost convergent double sequences. It is known that a convergent double sequence needs not be almost convergent. But it is well known that every bounded convergent double sequence is also almost convergent and every almost convergent double sequence is bounded. That is, the inclusion $C_{bp} \subset C_f \subset M_u$ holds, and each inclusion is proper. Moreover, Móricz and Rhoades [9] considered that four-dimensional matrices transforming every almost convergent double sequence into a $bp-$convergent double sequence with the same limit.

2 Some New Spaces of Double Sequences

In this section, we define new double sequence spaces $B(C_f)$ and $B(C_{f_0})$ derived by the domain of four-dimensional generalized difference matrix $B(r,s,t,u)$ in the double sequence spaces C_f and C_{f_0}, respectively. Then we give some topological properties and inclusion relations of those new double sequence spaces.

$$B(C_f) := \{x = (x_{mn}) \in \Omega : \exists L \in C \ni p-$$

$$\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}\{B(r,s,t,u)x\}_{kl}-L\right|=0\Bigg\}.$$

$$B(C_{f_0}) := \{x=(x_{mn})\in \Omega : p-$$

$$\lim_{q,q'\to\infty}\sup_{m,n>0}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}\{B(r,s,t,u)x\}_{kl}\right|=0\Bigg\}.$$

Theorem 1. *The double sequence spaces $B(C_f)$ and $B(C_{f_0})$ are Banach spaces with coordinatewise addition and scalar multiplication, and are linearly norm isomorphic to the spaces C_f and C_{f_0}, respectively, with the norm*

$$\|x\|_{B(C_f)} = \sup_{q,q',m,n\in N}\left|\frac{1}{(q+1)(q'+1)}\sum_{k=m}^{m+q}\sum_{l=n}^{n+q'}\{B(r,s,t,u)x\}_{kl}\right|. \quad (5)$$

Theorem 2. *Let $s=-r, t=-u$. The inclusions $C_f\subset B(C_f)$ and $C_{f_0}\subset B(C_{f_0})$ strictly hold.*

References

1. Adams, C.R.: On non-factorable transformations of double sequences. Proc. Natl. Acad. Sci. USA **19**(5), 564–567 (1933)
2. Başar, F., Sever, Y.: The space L_q of double sequences. Math. J. Okayama Univ. **51**, 149–157 (2009)
3. Başar, F., Tuğ, O.: Four-dimensional generalized difference matrix and some double sequence spaces. AIP Conf. Proc. **1759**, 1 (2016)
4. Boos, J.: Classical and Modern Methods in Summability. Oxford university Press, New York (2000)
5. Cooke, R.C.: Infinite Matrices and Sequence Spaces. Macmillan and Co., Limited, London (1950)
6. Hardy, G.H.: On the convergence of certain multiple series. Math. Proc. Cambridge Philos. Soc. **19**, 86–95 (1916–1919)
7. Lorentz, G.G.: A contribution to the theory of divergent sequences. Acta math. **80**(1), 167–190 (1948)
8. Móricz, F.: Extensions of the spaces c and c_0 from single to double sequence. Acta Math. Hungar. **57**, 129–136 (1991)
9. Moricz, F., Rhoades, B.E.: Almost convergence of double sequences and strong regularity of summability matrices. Math. Proc. Cambridge Philos. Soc. **104**, 283–294 (1988)
10. Zeltser, M.: On conservative matrix methods for double sequence spaces. Acta Math. Hungar. **95**(3), 225–242 (2002)

The Marcinkiewicz Theorem on the Multipliers of Fourier Series for Weighted Lebesgue Spaces

Nazerke Tleukhanova(✉)

L.N. Gumilyov Eurasian National University, 2 Satpayev str.,
Astana 010008, Kazakhstan
tleukhanova@rambler.ru

Abstract. This paper is devoted to the study of Fourier series multipliers. An analog of the Marcinkiewicz theorem on multipliers of Fourier series in weighted Lebesgue spaces is obtained.

Keywords: Multiplier of Fourier series · Weighted Lebesgue spaces · Net spaces · Average of the function

1 Introduction

Let $f(x)$ be a function on $L_p[0,1]$ with Fourier series $\sum\limits_{k\in Z}\hat{f}(k)e^{2\pi ikx}$ by trigonometric system $\{e^{2\pi ikx}\}_{k\in Z}$.

It is said that the sequence of complex numbers $\lambda=\{\lambda_k\}_{k\in Z}$ is a Fourier series multiplier from $L_p[0,1]$ to $L_q[0,1]$, i.e. $\lambda\in M_p^q$, if for every function $f\in L_p[0,1]$ with Fourier series $\sum\limits_{k\in Z}\hat{f}(k)e^{2\pi ikx}$ there exists a function $f_\lambda\in L_q[0,1]$ with a Fourier series which coincides with the series $\sum\limits_{k\in Z}\lambda_k\hat{f}(k)e^{2\pi ikx}$ and an operator $T_\lambda f=f_\lambda$, which is a bounded operator from $L_p[0,1]$ to $L_q[0,1]$.

The set $M_p^q=M\left(L_p\to L_q\right)$ of all Fourier series multipliers is a linear normed space with the norm

$$\|\lambda\|_{M_p^q}=\|T_\lambda\|_{L_p\to L_q}.$$

One of the first result related to multipliers of Fourier series was a result of Zygmund [16], where he proved that the characteristic function χ_I of the segment I from Z is multiplier from $M_p=M\left(L_p\to L_p\right)$, moreover

$$\|\chi_I\|_{M_p}\le c,$$

here c does not depend on the choice of a segment I from Z.

Another early important result from 1939 in the theory of Fourier series is the following one proved by J. Marcinkiewicz [3] (see also [4]):

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_8

Theorem A. *Let* $1 < p < \infty$, $\lambda = \{\lambda_m\}_{m\in Z}$ *be a sequence of complex numbers such that*

$$F_0(\lambda) = \sup_{m\in N}\Big(\sum_{k=2^m}^{2^{m+1}} |\lambda_k - \lambda_{k+1}| + |\lambda_{-k} - \lambda_{-k-1}|\Big) + \sup_{m\in Z} |\lambda_m| < \infty, \quad (1)$$

then $\lambda \in M_p$ *and*

$$\|\lambda\|_{M_p} \le c(p+p')\, F_0(\lambda), \quad (2)$$

where the constant c *does not depend on* λ *and* p.

In paper [8] sufficient conditions which are essentially depending on p for multipliers to belong to class M_p was received. The case when $1 < p \le 2 \le q < \infty$ was studied in [2,6,7], and the case when $1 < p \le q \le 2$ or $2 \le p \le q \le \infty$ can be found in [9,10,12–15].

Let $\mu(x)$ be a locally integrable function with positive values. The weighted Lebesgue space $L_p(\mu)$ is defined by quasinorm:

$$\|f\|_{L_p(\mu)} = \left(\int_0^1 |f(x)\mu(x)|^p dx\right)^{\frac{1}{p}}$$

for $0 < p < \infty$,

$$\|f\|_{L_\infty(\mu)} = \sup_{x\in[0,1]} |f(x)\mu(x)|$$

for $p = \infty$.

In this work our aim is to find conditions for the weights μ and ν such that the Marcinkiewicz theorem for the multipliers space $M\left(L_p(\mu) \to L_p(\nu)\right)$ holds.

Let $1 \le p \le \infty$, $\mu^{-1} \in L_{p'}[0,1]$, then for $f \in L_p(\mu)$ its Fourier coefficients by the trigonometric system are defined by the following formula

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx, \quad k \in Z.$$

It is said that the sequence of complex numbers $\{\lambda_k\}_{k\in Z}$ is a Fourier series multiplier from $L_p(\mu)$ to $L_p(\nu)$ if for an arbitrary $n \in N$ and every function $f \in L_p(\mu)$ the following inequality holds

$$\left\|\sum_{k=-n}^{n} \lambda_k \hat{f}(k) e^{2\pi i k x}\right\|_{L_p(\nu)} \le c\|f\|_{L_p(\mu)},$$

where the constant c does not depend on function $f \in L_p(\mu)$ and parameter n.

The set $M(L_p(\mu) \to L_p(\nu))$ is a normed space with the norm

$$\|\lambda\|_{M(L_p(\mu)\to L_p(\nu))} = \sup_{n\in N} \sup_{\|f\|_{L_p(\mu)}=1} \left\|\sum_{k=-n}^{n} \lambda_k \hat{f}(k) e^{2\pi i k x}\right\|_{L_p(\nu)}.$$

Note that, in the case $\mu(x) = \nu(x) = 1$ we just write shortly M_p instead of $M(L_p(\mu) \to L_p(\nu))$.

Let u, v be positive functions defined on $(0, \infty)$. We can define the functional

$$G_1(u, v; p) = \sup_{f\downarrow} \frac{\left(\int_0^1 \left(u(t)\frac{1}{t}\int_0^t v(s)(1+\ln\frac{t}{s})f(s)ds\right)^p dt\right)^{\frac{1}{p}}}{\left(\int_0^1 f^p(t)dt\right)^{\frac{1}{p}}}, \quad (3)$$

$$G_2(u, v; p) = \sup_{f\downarrow} \frac{\left(\int_0^1 \left(u(t)\int_t^1 v(s)(1+\ln\frac{s}{t})f(s)\frac{ds}{s}\right)^p dt\right)^{\frac{1}{p}}}{\left(\int_0^\infty f^p(t)dt\right)^{\frac{1}{p}}}, \quad (4)$$

here the supremum is taken over all nonincreasing nonnegative functions.

The characterization of functionals in terms of weight functions can be obtained as corollaries from the papers of R. Oinarov [1,11].

The main result of this paper is the following theorem.

Theorem 1. *Let $0 < p < \infty$, let $\mu(x)$ and $v(x)$ satisfy the following conditions*

$$G_1(\nu^*, (\mu^{-1})^*, p) < \infty,$$

$$G_2(\nu^*, (\mu^{-1})^*, p) < \infty.$$

If the sequence of real numbers $\lambda = \{\lambda_k\}_{k\in Z}$ satisfies the conditions (1), then $\lambda \in M\left(L_p(\mu) \to L_p(\nu)\right)$ and

$$\|f\|_{M(L_p(\mu)\to L_p(\nu))} \le cF_0(\lambda)\left(G_1 + G_2\right).$$

2 Auxilary Results

Let f be a linear Lebesgue measurable function. The distribution function of f is defined by

$$m(f, \sigma) = \mu\{x \in X: \ |f(x)| \ge \alpha\}.$$

The function

$$f^*(t) = \inf\{\sigma: \ m(f, \sigma) < t\}$$

is the non-increasing rearrangement of f.

We also define

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt.$$

Let M^* be the set of all measurable subsets of $[0, 1]$ with positive measure. A fixed subset M of the set M^* will be called a net.

Let the net M be given. For a function $f(x)$, defined and integrable on each e from M, we define a function

$$\bar{f}(t, M) = \sup_{\substack{e\in M\\ |e|>t}} \frac{1}{|e|}\left|\int_e f(x)d\mu\right|,$$

here the supremum is taken over all sets $e \in M$ such that $|e| \stackrel{\text{def}}{=} \mu e > t$, $\quad t \in (0, \infty)$. Here and below $|e| := \mu e$. The function $\bar{f}(t, M)$ is called the *average of the function f over the net M*.

Lemma 1. *Let f be a measurable and integrable on the elements of the net M^*. Then we have*

$$\bar{f}(t, M^*) \le f^{**}(t) \le 4\bar{f}(t/3, M^*), \qquad t > 0. \tag{5}$$

Proof. Let $t \in (0, \infty)$, then for an arbitrary set $e \in M$ with measure equals to t and a function $f(x)$ we can define the sets

$$\omega_1 = \{x \in e : f(x) \ge 0\} \text{ and } \omega_2 = \{x \in e : f(x) < 0\}.$$

Then

$$\int_e |f(x)|dx = \int_{\omega_1} f(x)dx - \int_{\omega_2} f(x)dx \le 2\max\left\{\left|\int_{\omega_1} f(x)dx\right|, \left|\int_{\omega_2} f(x)dx\right|\right\}.$$

For definiteness let us assume that

$$\left|\int_{\omega_1} f(x)dx\right| \ge \left|\int_{\omega_2} f(x)dx\right|.$$

There are two possible cases. 1) $|\omega_1| \ge \frac{1}{2}|\omega_2|$; $\quad$ 2) $|\omega_1| < \frac{1}{2}|\omega_2|$.

In the first case

$$|\omega_1| \ge \frac{1}{2}|\omega_2| \ge \frac{|e|}{3} = \frac{t}{3}$$

and

$$\frac{1}{|e|}\int_e |f(x)|dx \le$$

$$\le 2\frac{1}{|e|}\left|\int_{\omega_1} f(x)dx\right| \le 2\frac{1}{|\omega_1|}\left|\int_{\omega_1} f(x)dx\right| \le 2\bar{f}(\frac{t}{3}, M). \tag{6}$$

In the second case $|\omega_1| < \frac{1}{2}|\omega_2|$, i.e. $|\omega_2| > \frac{2|e|}{3} = \frac{2}{3}t$. Then there exist ω_2^1 and ω_2^2 from M such that $|\omega_2^1 \bigcap \omega_2^2| = 0$, $\quad \omega_2^1 \bigcup \omega_2^2 = \omega_2$, $|\omega_2^i| = \frac{|\omega_2|}{2} > \frac{t}{3}$.

Taking into account the sign-definite nature of the function f on ω_2, we obtain

$$\left|\int_{\omega_1} f(x)dx\right| \ge \left|\int_{\omega_2} f(x)dx\right| = \left|\int_{\omega_2^1} f(x)dx\right| + \left|\int_{\omega_2^2} f(x)dx\right| \ge$$

$$\ge 2\min\left(\left|\int_{\omega_2^1} f(x)dx\right|, \left|\int_{\omega_2^2} f(x)dx\right|\right) = 2\left|\int_{\omega_2^{i_0}} f(x)dx\right|.$$

Here $\omega_2^{i_0}$ are the sets, where the minimum is reached.

Now let $\omega = \omega_1 \bigcup \omega_2^{i_0}$, then $|\omega| > \frac{|e|}{3}$ and

$$\left|\int\limits_{\omega} f(x)dx\right| = \left|\int\limits_{\omega_1} f(x)dx + \int\limits_{\omega_2^{i_0}} f(x)dx\right| \geq$$

$$\geq \left|\int\limits_{\omega_1} f(x)dx\right| - \left|\int\limits_{\omega_2^{i_0}} f(x)dx\right| \geq 1/2 \left|\int\limits_{\omega_1} f(x)dx\right|.$$

Consequently,

$$\frac{1}{|e|}\int\limits_{e} |f(x)|dx \leq 2\frac{1}{|e|}\left|\int\limits_{\omega_1} f(x)dx\right| \leq 4\frac{1}{|\omega|}\left|\int\limits_{\omega} f(x)dx\right| \leq 4\bar{f}(t/3, M).$$

Then, according to (6) the proof of the right-hand inequality (5) is complete.

Let us prove the right-hand inequality of (5).

$$\bar{f}(t, M) = \sup_{|e|\geq t} \frac{1}{|e|}\left|\int\limits_{e} f(x)dx\right| \leq \sup_{|e|\geq t} \frac{1}{|e|}\int\limits_{e} |f(x)|dx =$$

$$= \sup_{|e|\geq t} \frac{1}{|e|}\int\limits_{0}^{|e|} f^*(s)ds = \sup_{|e|\geq t} \frac{1}{|e|}\left(\int\limits_{0}^{t} f^*(s)ds + \int\limits_{t}^{|e|} f^*(s)ds\right) \leq$$

$$\leq \sup_{|e|\geq t} \frac{1}{|e|}\left(\int\limits_{0}^{t} f^*(s)ds + (|e| - t)\frac{1}{t}\int\limits_{0}^{t} f^*(s)ds\right) = \frac{1}{t}\int\limits_{0}^{t} f^*(s)ds =$$

$$= \sup_{|e|=t} \frac{1}{|e|}\int\limits_{e} |f(x)|dx = f^{**}(t).$$

Let $0 < p \leq \infty$ and $0 < q \leq \infty$. We say that a function f belongs to the Lorentz space $L_{p,q}[0, 1]$, if f is measurable on $[0, 1]$ and for $0 < q < \infty$

$$\|f\|_{L_{p,q}} = \left(\int_0^1 (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty,$$

and for $q = \infty$

$$\|f\|_{L_{p,\infty}} = \sup_{0<t\leq 1} t^{1/p} f^*(t) < \infty.$$

Lemma 2. *Let $\mu(x)$, $\nu(x)$ be positively locally integrable functions. Then*

$$\left(\int_0^1 \left|\nu(y)\int_0^1 \mu(x)K(x-y)f(x)dx\right|^q dy\right)^{\frac{1}{q}} \leq c\left(\int\limits_0^1\left(\nu^*(s)\int\limits_0^1 \mu^*(t)\Phi(t,s)f^*(t)dt\right)^q ds\right)^{\frac{1}{q}},$$

where

$$\Phi(s,t) = \sup_{|e|\geq s, |\omega|\geq t} \frac{1}{|e|}\frac{1}{|\omega|}\left|\int\limits_e\int\limits_\omega K(x-y)dxdy\right|.$$

Proof. Following ideas from [5], by the Hardy-Littlewood rearrangement inequality, we have

$$\left(\int_0^1 \left|\nu(y)\int_0^1 \mu(x)K(x-y)f(x)f(x)dx\right|^p dy\right)^{1/p}$$

$$\leq \left(\int_0^1\left(\nu^*(s)\left(\int_0^1 \mu(x)K(x-\cdot)f(x)dx\right)^*(s)\right)^p ds\right)^{1/p}$$

$$\leq \left(\int_0^1\left(\nu^*(s)\left(\int_0^1 \mu(x)K(x-\cdot)f(x)dx\right)^{**}(s)\right)^p ds\right)^{1/p}$$

$$= \left(\int_0^1\left(\nu^*(s)\sup_{|\eta_1|=s}\frac{1}{|\eta_1|}\int_{\eta_1}\left|\int_0^1 \mu(x)K(x-y)f(x)dx\right|dy\right)^p ds\right)^{1/p}$$

$$\leq c\left(\int_0^1\left(\nu^*(s)\sup_{|\eta_1|\geq s/3}\left(\frac{1}{|\eta_1|}\left|\int_0^1 \mu(x)\int_{\eta_1}K(x-y)dyf(x)dx\right|\right)\right)^p ds\right)^{1/p}$$

$$\leq c\left(\int_0^1\left(\nu^*(s)\sup_{|\eta_1|\geq s}\left|\int_0^1 \mu(x)\frac{1}{|\eta_1|}\int_{\eta_1}K(x-y)dyf(x)dx\right|\right)^p ds\right)^{1/p},$$

where in the last estimate we used Lemma 1.

We use similar estimates for the inner integral to get

$$\left(\int_0^1 \left|\nu(y)\int_0^1 \mu(x)K(x-y)f(x)f(x)dx\right|^p dy\right)^{1/p}$$

$$\leq c\left(\int_0^1\left(\nu^*(s)\sup_{|\eta_1|\geq s}\int_0^1 \mu^*(t)f^*(t)\sup_{|\eta_2|\geq t/3}\frac{1}{|\eta_1|}\frac{2}{|\eta_2|}\left|\int_{\eta_2}\int_{\eta_1}K(x-y)dydx\right|dt\right)^p ds\right)^{1/p}$$

$$\leq c\left(\int_0^1\left(\nu^*(s)\int_0^1 \mu^*(t)f^*(t)\sup_{|\eta_1|\geq s}\sup_{|\eta_2|\geq t}\frac{1}{|\eta_1|}\frac{1}{|\eta_2|}\left|\int_{\eta_2}\int_{\eta_1}K(x-y)dydx\right|dt\right)^p ds\right)^{1/p}.$$

Lemma 3. *Let* $1 < p < \infty$, $\lambda = \{\lambda_k\}_{k\in N}$ *be a sequence of real numbers satisfying the condition (1), then*

$$\sup_n \sup_{\substack{|\omega|>0\\|e|>0}} \frac{1}{|e|^{\frac{1}{p'}}} \frac{1}{|\omega|^{\frac{1}{p}}} \left| \int_e \int_\omega \sum_{k=-n}^{n} \lambda_k e^{2\pi i k(x-y)} dx dy \right| \le c(p'+p) F_0(\lambda),$$

where the constant c *does not depend on* λ *and* p.

Proof. Let $n \in N$. Let us define $\eta^n = \{\eta_k^n\}_{k\in Z}$

$$\eta_k^n = \begin{cases} \lambda_k, & |k| \le n, \\ 0, & |k| > n, \end{cases}$$

then $F_0(\eta^n) \le 2F_0(\lambda)$. Therefore, from Theorem A we get

$$2c(p+p')F(\lambda) \ge \|\eta^n\|_{M_p} = \|T_{\eta^n}\|_{L_p \to L_p} =$$

$$= \sup_{\|f\|_{L_p}=1} \left\| \sum_{k=-n}^{n} \lambda_k \widehat{f(k)} e^{2\pi i k x} \right\|_{L_p} =$$

$$= \sup_{\|f\|_{L_p}=1} \left\| \int_0^1 f(y) \sum_{k=-n}^{n} \lambda e^{2\pi i k(x-y)} dy \right\|_{L_p}.$$

Let ω and e be arbitrary compacts. Let us assume

$$f(x) = \chi_\omega(x) |\omega|^{-\frac{1}{p}}.$$

Applying Hölder inequality, we obtain

$$2c(p+p')F(\lambda) \ge \left\| \frac{1}{|\omega|^{\frac{1}{p}}} \int_\omega \sum_{k=-n}^{n} \lambda e^{2\pi i k(x-y)} dy \right\|_{L_p} \ge$$

$$\ge \frac{1}{|e|^{\frac{1}{p'}}} \frac{1}{|\omega|^{\frac{1}{p}}} \left| \int_e \int_\omega \sum_{k=-n}^{n} \lambda e^{2\pi i k(x-y)} dy dx \right|.$$

Taking into account the arbitrariness of the choice of n, e and ω, we obtain the required statement.

3 The Proof of Theorem

Let $n \in N$.

$$\left\| \sum_{-n}^{n} \lambda_k \widehat{f}(k) e^{2\pi i k x} \right\|_{L_p(\nu)} =$$

$$= \left\| \int_0^1 f(y) \sum_{k=-n}^{n} \lambda_k e^{2\pi i k(x-y)} dy \right\|_{L_p(\nu)} =$$

$$= \left(\int_0^1 \left(\nu(x) \int_0^1 f(y)\mu(y)\mu^{-1}(y) K_n(x-y) dy \right)^p dx \right)^{\frac{1}{p}}.$$

Applying Lemma 2, we have

$$\left\| \sum_{k=-n}^{n} \lambda_k \widehat{f}(k) e^{2\pi i k x} \right\|_{L_p(\nu)} \leq c \tag{7}$$

$$\leq c \left(\int_0^1 \left(\nu^*(t) \int_0^1 (f\mu)^*(s)(\mu^{-1})^*(s) \varPhi(s,t) ds \right)^p dt \right)^{\frac{1}{p}},$$

where

$$\varPhi(s,t) = \sup_{\substack{|e|=t \\ |\omega|=s}} \frac{1}{|e||\omega|} \left| \int_e \int_\omega \sum_{k=-n}^{n} \lambda_k e^{2\pi i k(x-y)} dy dx \right|.$$

Let $t > s$. Let us assume $\tau = (2 + \ln \frac{t}{s})$.

Using Lemma 3, we receive

$$\varPhi(s,t) \leq c(\tau + \tau') F_0(\lambda) \frac{1}{t^{\frac{1}{\tau}} s^{\frac{1}{\tau'}}} =$$

$$= c(\tau + \tau') F_0(\lambda) \frac{\left(\frac{t}{s}\right)^{\frac{1}{\tau}}}{t} \leq 4c \left(1 + (\ln \frac{t}{s}) \right) \frac{e}{t} \leq$$

$$\leq 2^4 c \frac{(1 + \ln ts)}{t} F_0(\lambda).$$

Similarly, for $s \geq t$ we have

$$\varPhi(s,t) \leq c_1 \frac{1 + \ln \frac{s}{t}}{s} F_0(\lambda),$$

here constant c_1 does not depend on parameters t and s. Substituting these estimates into the inequality (7), we obtain

$$\left\| \sum_{k=-n}^{n} \lambda_k \widehat{f}(k) e^{2\pi i k x} \right\|_{L_p(\nu)} \leq$$

$$\leq c F_0(\lambda) \left(\int_0^1 \left(\nu^*(s) \frac{1}{t} \int_0^t (f\mu)^*(s)(1 + \ln \frac{t}{s})(\mu^{-1})^*(s) ds \right)^p dt \right)^{\frac{1}{p}} +$$

$$+ \left(\int_0^1 \left(\nu^*(s) \int_t^\infty (f\mu)^*(s) \mu^{-1*}(s)(1 + \ln \frac{s}{t}) \frac{ds}{s} \right)^p dt \right)^{\frac{1}{p}} \leq$$

$$\leq c F_0(\lambda) \left(G_1(\nu^*, \mu^{-1*}, p) + G_2(\nu^*, (\mu^{-1})^*, p) \right) \|f\|_{L_p(\mu)}.$$

Acknowledgements. The author was supported by the Ministry of Education and Science of the Republic of Kazakhstan (Grants 5540/GF4 and 3311/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Arendarenko, L.S., Oinarov, R., Persson, L.-E.: On the boundedness of some classes of integral operators in weighted Lebesgue spaces. Eurasian Math. J. **3**(1), 5–17 (2012)
2. Hörmander, L.: Estimates for translation invariant operators in L_p spaces. Acta Math. **104**, 93–140 (1960)
3. Marcinkiewicz, J.: Sur les multiplicateurs des series de Fourier. Studia Math. **8**, 7891 (1939); Reprinted in: Marcinkiewicz, J.: Collected Papers. PWN, Warsaw, pp. 501–512 (1964)
4. Marcinkiewicz, J., Zygmund, A.: Some theorems on ortogonal systems. Fund. math. **28**, 309–335 (1937)
5. Nursultanov, E., Tikhonov, S.: Weighted norm inequalities for convolution and Riesz potential. Potential Anal. **42**(2), 435–456 (2015)
6. Nursultanov, E., Tikhonov, S.: Lower and upper bounds for the norm of multipliers of multpile trigonometric fourier series in lebesgue spaces. Funct. Anal. Appl. **34**(2), 151–153 (2000)
7. Nursultanov, E., Tikhonov, S.: On multipliers of multiple Fourier series. Tr. Mat. Inst. Steklova. **227**, 237–242 (1999); English transl. in Proc. Steklov Inst. Math. **227**: 4, 231–236 (1999)
8. Nursultanov, E.D.: On multipliers of Fourier series in the trigonometric system. Mat. Zametki. **63**(2), 235–247 (1998); English transl. in Math. Notes. **63**(1–2), 205–214 (1998)
9. Nursultanov, E.D.: Concerning the multiplicators of Fourier series in the trigonometric system. Math. Notes **63**(2), 205–214 (1998)
10. Nursultanov, E.D.: A lower bound for a Fourier transform multiplicator from $M(Lp \to Lq)$. Math. Notes **62**(6), 792–796 (1997)
11. Oinarov, R.: Boundedness and compactness in weighted Lebesgue spaces of integral operators with variable integration limits. Siberian Math. J. **52**(6), 1042–1055 (2011)
12. Persson, L.-E., Sarybekova, L., Tleukhanova, N.: Multidimensional generalization of the Lizorkin theorem on Fourier multipliers. Proc. A. Razmadze Inst. **151**, 83–101 (2009)
13. Persson, L.-E., Sarybekova, L., Tleukhanova, N.: A Lizorkin theorem on Fourier series multipliers for strong regular system. In: Analysis for Science. Engineering and Beyond Springer Proceedings in Mathematics, Vol. 6, Berlin, pp 305–317 (2012)
14. Sarybekova, L., Tleukhanova, N.: Fourier series multipliers by regular system. Eurasian Math. Journ. **1**, 96–102 (2007) (in Russian)
15. Sarybekova, L., Tararykova, T., Tleukhanova, N.: On a generalization of the Lizorkin theorem on Fourier multipliers. Math. Inequal. Appl. **13**(3), 613–624 (2010)
16. Zygmund, A.: Trigonometric Series, vols. I, II. 2nd edn,. Cambridge University Press, London, New York (1968)

Part II

Differential Equations and Boundary Value Problems

Periodic Solution of Linear Autonomous Dynamic System

Serikbai A. Aisagaliev(✉) and Zhanat Kh. Zhunussova

Al-Farabi Kazakh National University, 71 al-Farabi avenue, Almaty 050040, Kazakhstan
Serikbai.Aisagaliev@kaznu.kz, zhanat.zhunusova@kaznu.kz

Abstract. A method for the study of periodic solutions of autonomous dynamic systems described by ordinary differential equations with phase and integral constraints is supposed. General problem of periodic solution is formulated in the form of the boundary value problem with constraints. The boundary problem is reduced to the controllability problem of dynamic systems with phase and integral constraints by introducing a fictitious control. Solution of the controllability problem is reduced to a Fredholm integral equation of the first kind. The necessary and sufficient conditions for existence of the periodic solution are obtained and an algorithm for constructing periodic solution to the limit points of minimizing sequences is developed. Scientific novelty of the results consists in a completely new approach to the study of periodic solutions for linear systems focused on the use of modern information technologies is offered. The existence of periodic solution and its construction are solved together.

Keywords: Linear autonomous system · Dynamic system · Periodic solution · Ordinary differential equation · Boundary value problem · Controllability problem · Controllable system

1 Problem Statement

We consider a linear autonomous system

$$\dot{x} = Ax, t \in (-\infty, +\infty), \tag{1}$$

where A is a constant matrix of $n \times n$ order. The problems are set:

Problem 1. Find necessary and sufficiently conditions for existence of T_* periodic solution of system (1).

Problem 2. Find T_* periodic solution of system (1)
Solving these problems is of interest for system (1) of $(n > 4)$ higher order.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_9

We assume, that system (1) has a periodic solution $x_*(t) = x_*(t+T_)$, $t \in I = (-\infty, +\infty)$, where T_* is period. Let $x_*(0) = x_0$ be a value. Then $x_*(T_*) = x_0$.

Since the periodic solution is defined by values of the phase coordinates in the period limits, then for constructing of periodic solution it should be considered the value $t \in I_* = [0, T_*]$.

We represent the matrix $A = A_1 + B_1 P$, where A_1, B, P are matrixes of $n \times n$, $n \times m$, $m \times n$ orders, accordingly. Then the boundary value problem (1) is written in the form

$$\dot{x} = A_1 x + B_1 P x, t \in I_* = [0, T_*], x(0) = x(T_*) = x_0. \tag{2}$$

Linear controllable system corresponding to system (2) has the form (2)

$$\dot{y} = A_1 y + B_1 u(t), t \in I_* = [0, T_*], \tag{3}$$

$$y(0) = y(T_*) = x(0) = x(T_*) = x_0, u(\cdot) \in L_2(I, R^m), \tag{4}$$

where T_* is period, a unknown value. We note, if $u(t) = Px(t)$, $t \in I_*$, then system (3), (4) coincides to the origin (2).

2 Solution of a Linear Controllable System

We assume that the matrixes A_1, B_1 such that the matrix

$$W_*(0, T_*) = \int_0^{T_*} e^{-A_1 t} B_1 B_1^* e^{-A_1^* t} dt \tag{5}$$

of $n \times n$ order is positively defined.

In the case, when the matrix $A_1 = 0$, $P = I_n$, the matrix $B_1 = A$, relation (5) is written as $W_*(0, T_*) = \int_0^{T_*} AA^* dt$. We note, that the matrix $W_*(0, T_*) > 0$ is equivalent to the fact, that the rank of the matrix $\left\| B_1, A_1 B_1, \ldots, A_1^{n-1} B_1 \right\|$ is equal to n.

Theorem 1. *Let $W_*(0, T_*) > 0$ be a matrix. Then control $u(\cdot) \in L_2(I, R^m)$ transfers the trajectory of system (3) from any initial point $y(0) = x_0 \in R^n$ to any finite state $y(T_*) = x_0$ if and only if, when*

$$\begin{gathered} u(t) \in U = \{u(\cdot) \in L_2(I, R^m)/u(t) = v(t) + \lambda_1(t, x_0, x_0) + \\ + N_1(t) z(T_*, v), \forall v, v(\cdot) \in L_2(I, R^m)\}, \end{gathered} \tag{6}$$

where

$$\lambda_1(t_1, x_0, x_0) = B_1^* e^{-A_1^* t} W_*^{-1}(0, T_*) a, a = e^{-A_1 T_*} x_0 - x_0,$$

$$N_1(t) = -B_1^* e^{-A_1^* t} W_*^{-1}(0, T_*) e^{-A_1 T_*}, t \in I_*,$$

the function $z(t, v_*)$, $t \in I_*$ is a solution of the differential equation

$$\dot{z} = A_1 z + B_1 v, z(0) = 0, v(\cdot) \in L_2(I, R^m). \tag{7}$$

The solution of the differential equation (3) corresponding to control $u(t) \in U$ is defined by formula

$$y(t) = z(t, v) + \lambda_2(t, x_0, x_0) + N_2(t)z(T_*, v), t \in I_*, \quad (8)$$

where

$$\lambda_2(t, x_0, x_1) = e^{A_1 t} W_*(t, T_*) W_*^{-1}(0, T_*) x_0 + e^{A_1 t} W_*(0, t) W_*^{-1}(0, T_*) e^{-A_1 T_*} x_0,$$

$$N_2(t) = -e^{A_1 t} W_*(0, t) W_*^{-1}(0, T_*) e^{-A_1 T_*}, W_*(0, T_*) = \int_0^t e^{-A_1 \tau} B_1 B_1^* e^{-A_1^* \tau},$$

$$W_*(t, T_*) = W_*(0, T_*) - W_*(0, t), t \in I_*.$$

Lemma 1. *Let $W_*(0, T_*) > 0$ be a matrix. The boundary value problem (2) is equivalent to the problem*

$$v(t) + T(t)x_0 + N_1(t)z(T_*, v) = Py(t), t \in I_*, x_0 \in R^n, \quad (9)$$

$$\dot{z} = A_1 z + B_1 v(t), z(0) = 0, t \in I_*, v(\cdot) \in L_2(I, R^m), \quad (10)$$

where

$$T(t) = B_1 e^{-A_1^* t} W_*^{-1}(0, T_*)[e^{-A_1 T_*} - I_n],$$

$$y(t) = z(t, v) + C(t)x_0 + N_2(t)z(T_*, v), t \in I, \quad (11)$$

$$C(t) = e^{A_1 t}[W_*(t, T_*) W_*^{-1}(0, T_*) + W_*(0, t) W_*^{-1}(0, T_*) e^{-A_1 T_*}].$$

Proof of the Lemma follows from relations (6)-(10), at $u(t) \in U$, $u(t) = Py(t)$, $t \in I_*$.

3 Necessary and Sufficient Condition for Existence of a Solution of the Boundary Value Problem

Theorem 2. *Let $W_*(0, T_*) > 0$ be a matrix. In order the boundary value problem (2) to have a solution, it is necessary and sufficient that the value $I(v_*, x_{0_*}) = 0$, where $(v_*, x_{0_*}) \in H = L_2(I, R^m) \times R^n$ is a solution of optimization problem*

$$I(v, x_0) = \int_0^{T_*} |v(t) + T(t)x_0 + N_1(t)z(T_*, v) - Py(t)| \to \inf \quad (12)$$

under conditions

$$\dot{z} = A_1 z + B_1 v(t), z(0) = 0, t \in I_*, \quad (13)$$

$$v(\cdot) \in L_2(I_*, R^m), x_0 \in R^n. \quad (14)$$

Proof of the Theorem follows from Theorem 1, Lemma 1 and relations (9)-(11).

Lemma 2. *Suppose $W_*(0,T_*) > 0$ is a matrix, the function*

$$F_*(q,t) = v + T(t)x_0 + N_1(t)z(T_*,v) - Py,$$

where $y = z + C(t)x_0 + N_2(t)z(T_*,v)$, $q = (v,x_0,z,z(T_*)) \in R^m \times R^n \times R^n \times R^n$.

Then the partial derivatives

$$\begin{gathered} F_{*v}(q,t) = 2[v + T(t)x_0 + N_1(t)z(T_*,v) - Py], \\ F_{*x_0}(q,t) = [2T^*(t) + 2C^*(t)P^*][v + T(t)x_0 + N_1(t)z(T_*) - Py], \\ F_{*z}(q,t) = -2P^*(t)[v + T(t)x_0 + N_1(t)z(T_*) - Py], \\ F_{*z(T_*)}(q,t) = [2N_1^*(t) - 2N_2^*(t)P^*][v + T(t)x_0 + N_1(t)z(T_*) - Py]. \end{gathered} \tag{15}$$

Lemma 3. *Let $W_*(0,T_*) > 0$ be a matrix. Then:*
1) functional (12) under conditions (13), (14) is convex
2) derivative $F_{*q}(q,t) = (F_{*v}, F_{*x_0}, F_{*z}, F_{*z(T_*)})$ satisfies to the Lipshitz condition

$$\|F_{*q}(q + \Delta q, t) - F_{*q}(q,t)\| \le M \|\Delta q\|, \forall q, q + \Delta q \in R^{m+4n}.$$

Theorem 3. *Let $W_*(0,T_*) > 0$ be a matrix. Then functional (12), under conditions (13), (14) continuously differentiable by Freshet, gradient of functional*

$$I'(v,x_0) = (I'_v(v,x_0), I'_{x_0}(v,x_0)) \in H = L_2(I_*,R^m) \times R^n$$

in any point $(v,x_0) \in H$ is computed by the formula

$$\begin{gathered} I'_v(v,x_0) = F_{*v}(q(t),t) - B_1^*\psi(t) \in L_2(I_*,R^m), \\ I'_{x_0}(v,x_0) = \int_0^{T_*} F_{*x_0}(q(t),t)dt \in R^n, \end{gathered} \tag{16}$$

where partial derivatives are defined by formula (15), $q(t) = (v(t), x_0, z(t,v), z(T_*,v))$, the function $z(t)$, $t \in I$ is a solution of the differential equation (12), for $v = v(t)$, $t \in I$, and function $\psi(t)$, $t \in I_*$ is a solution of the adjoint system

$$\dot{\psi} = F_{*z}(q(t),t) - A_1^*\psi, \psi(t_1) = -\int_0^{T_*} F_{*z(T_*)}(q(t),t)dt. \tag{17}$$

Moreover, the gradient $I'(v,x_0)$, $(v,x_0) \in H$ satisfies the Lipshitz condition

$$\left\|I'(v^1,x_0^1) - I'(v^2,x_0^2)\right\| \le K_*(\left\|v^1 - v^2\right\|^2 + \left|x_0^1 - x_0^2\right|^2)^{1/2}, \tag{18}$$

where $K_* = const > 0$ is a Lipshitz constant.

It should be noted, that for a linear system with constant coefficients (3) the following statements are valid:
1) rank of the matrix $\left\|B_1, A_1B_1, \ldots, A_1^{n-1}B_1\right\|$ is equal to n;
2) for any $T > 0$, the matrix

$$W_*(0,T) = \int_0^T e^{-A_1t}B_1B_1^*e^{-A_1^*t}dt$$

is positively defined.

Consequently, for any sequence $\{T_i\} \subset R^1$, $0 < T_1 < T_2 < \ldots < T_k < \ldots$, the matrix $W_*(0, T_k) > 0$.

Let $\{T_k\} \subset R^1$, $0 < T_1 < T_2 < \ldots < T_k <$ be a sequence. We construct the sequences

$$\begin{array}{c} v_{n+1}^k(t) = v_n^k(t) - \alpha_n I'_v(v_n^k, x_{0n}^k), \;\; x_{0_{n+1}}^k(t) = x_0^k(t) - \alpha_n I'_{x_0}(v_n^k, x_{0_n}^k), \\ n = 0, 1, 2, \ldots \;\; 0 < \varepsilon_0 \le \alpha_n \le \frac{2}{K_* + 2\varepsilon_1}, \;\; \varepsilon_1 > 0, \end{array} \tag{19}$$

on the base of formulas (6)-(8), where $t \in [0, T_k]$, $I'_v(v^k, x_0^k)$,$I'_{x_0}(v^k, x_0^k)$ are defined by formula (16) by substituting $W_*(0, T_*)$, T_* on $W_*(0, T_k)$, T_k, accordingly.

In other words, we fix a value $T_k > 0$ from sequence $\{T_i\} \subset R^1, 0 < T_1 < T_2 < \ldots < T_k < \ldots$ and compute the Freshet derivative for functional (12), under the conditions (12), (13) by formulas (16)-(18), by substituting T_*, $W_*(0, T_*)$ on T_k, $W_*(0, T_k)$, accordingly. The result is the sequences (19).

Theorem 4. *Let $W_*(0, T_k) > 0$ be a matrix, $\{T_k\} \subset R^1$, $0 < T_1 < T_2 < \ldots < T_k < \ldots$, the sequences $\{v_n^k\}$, $\{x_{0n}^k\}$ are defined by formula (19), the set*

$$\Lambda_k = \{(v, x_0) \in H / I_k(v, x_0) \le I_k(v_0, x_{00})\}$$

is bounded, where functional is defined by

$$I_k(v, x_0) = \int_0^{T_k} |v(t) + T(t)x_0 + N_1(t)z(T_k, v) - Py(t)|dt.$$

Then for any fixed $T_k > 0$ statements are valid:
3) The sequence $\{v_n^k, x_0^{kn}\}$ is minimizing, i.e.

$$\lim_{n\to\infty} I_k(v_n^k, x_0^{kn}) = I_k(v_*^k, x_0^{k*}) = \inf_{(v,x_0)\in\Lambda_k} I_k(v, x_0);$$

4) The sequences $\{v_n^k\}$, $\{x_0^{kn}\}$ are weakly converged to the points $v_n^k \xrightarrow{A;} v_*^k$, $x_{0n}^k \xrightarrow{A;} x_{0*}^k$ at $n \to \infty$, $(v_*^k, x_0^{k*}) \in X_k^*$;
5) The estimation of the convergence rate is valid

$$0 < I_k(v_n^k, x_{0n}^k) - I(v_*^k, x_{0*}^k) \le \tfrac{C_k}{n}, c_k = const > 0, n = 1, 2, \ldots;$$

6) For system (2) to have a periodic solution it is necessary and sufficient, that for some $T_k = T_*$ there exists the value $I_k(v_*^k, x_{0*}^k) = 0$.
7) Periodic solution of system (12) is defined by the formula

$$x_*(t) = y_*(t) = z(t, v_*^k) + C(t)x_{0*}^k + N_2(t)z(T_k, v_*^k), t \in [0, T_k = T_*],$$

where $T_k = T_*$ is a period, $I_k(v_*^k, x_{0*}^k) = 0$.

4 Algorithm for Constructing a Periodic Solution

We can formulate the following algorithm for constructing periodic solution of system (1) based on Theorems 1-4, Lemmas 1-3.

1. We present the matrix A as the sum $A = A_1 + B_1 P$ such that the matrix

$$W_*(0, T_k) = \int_0^{T_k} e^{-A_1 t} B_1 B_1^* e^{-A_1^* t} dt$$

   will be positively defined, where $T_k > 0$ is a number. We note, that in order to $W_*(0, T_k) > 0$ necessary and sufficiently, that the rank of the matrix $\left\| B_1, A_1 B_1, \ldots, A_1^{n-1} B_1 \right\|$ is equal to n.
2. We choose the sequence $\{T_k\} \subset R^1, 0 < T_1 < T_2 < \ldots < T_k < \ldots$. We note, if the rank $\left\| B_1, A_1 B_1, \ldots, A_1^{n-1} B_1 \right\| = n$, then for any $T_k > 0$ the matrix $W_*(0, T_k) > 0$.
3. We solve the optimization problem: minimize the functional

$$I_k(v, x_0) = \int_0^{T_k} \left| v(t) + T(t) x_0 + N_1(t) z(T_k, v) - P y(t) \right|^2 dt \to \inf \tag{20}$$

   under conditions

$$\dot{z} = A_1 z + B_1 v(t), z(0) = 0, t \in [0, T_k] = I, \tag{21}$$

$$v(\cdot) \in L_2(I, R^m), x_0 \in R^n. \tag{22}$$

   We note, that: 1) the value $I_k(v, x_0) \geq 0$, consequently, functional is bounded from below; 2) functional (20) under conditions (21), (22) is convex; 3) to solve optimization problem (20) – (22) we construct the sequences (19). As a result, we find the solution of optimization problem (20)-(22): $(v_*^k, x_{0*}^k) \in \Lambda_k$, $I_k(v_*^k, x_{0*}^k)$ at fixed T_k.
4. We repeat items 1 - 3. Finally, the values $I_k(v_*^k, x_{0*}^k)$, $k = 1, 2, \ldots$. are known. If for value T_{k_*} the value $I_{k_*}(v_*^{k_*}, x_{0*}^{k_*}) = 0$, then $T_{k_*} = T_*$ is a period of the origin periodic solution, and periodic solution

$$x_*(t) = z(t, v_*^{k_*}) + C(t) x_{0*}^{k_*} + N_2(t) z(T_{k_*}, v_*^{k_*}), t \in [0, T_{k_*}] = [0, T_*].$$

5. If the value $I_k(v_*^k, x_{0*}^k) > 0$ for any sequences $\{T_k\} \subset R^1$,$0 < T_1 << T_2 < \ldots < T_k < \ldots$, then the origin system (2) has no any periodic solution.

The results obtained above can be applied for construction of periodic solutions in non-autonomous systems.

We consider a linear non-autonomous system

$$\dot{x} = A(t) x + \mu(t), t \in (-\infty, +\infty), \tag{23}$$

where elements of the matrix $A(t)$ and vector function $\mu(t)$ are periodic functions with period T_* i.e. $A(t) = A(t + T_*)$, $\mu(t) = \mu(t + T_*)$, $\forall t$, $t \in (-\infty, +\infty)$, T_* is the known function.

The questions arise: Does the system (23) have periodic solution with a period equal to T_*? Find periodic solution of (23) with a period T_*.

Let $x_*(t)$ be a periodic solution of system (23) with a period T_* i.e. $x_*(t) = x(t+T_*)$, $\forall t$, $t \in (-\infty, +\infty)$. Then

$$A(t)x_*(t) = A(t+T_*)x_*(t+T_*), \mu(t) = \mu(t+T_*), t \in (-\infty, +\infty).$$

For constructing a periodic solution it is enough to consider a solution of system (23) for values $t \in [0, T_*]$ in view of the invariance of solution by any displacement on t. Let $x_*(0) = x_*(T_*) = x_0$.

By applying the results above, we get:

1) the matrix $A(t) = A_1(t) + B_1(t)P$, where $W_1(0, T_*) = \int_0^{T_*} \Phi(0,t)B_1(t)B_1^*(t)\Phi^*(0,t)dt > 0$;
2) linear controllable system has the form

$$\dot{y} = A_1(t)y + B_1(t)u(t) + \mu(t), t \in I_* = [0, T_*],$$

$$y(0) = y(T_*) = x_*(0) = x_*(T) = x_0, u(\cdot) \in L_2(I, R^m);$$

3) optimization problem is written: minimize the functional

$$I(v, x_0) = \int_0^{T_*} |v(t) + T(t)x_0 + \bar{\mu}(t) + N_1(t)z(T_*, v) - Py(t)|^2 \, dt \to \inf$$

under conditions

$$\dot{z} = A_1(t)z + B_1(t)v(t), z(0) = 0, t \in [0, T_*] = I_*,$$

$$v(\cdot) \in L_2(I_*, R^m), x_0 \in R^n.$$

4) Necessary and sufficient conditions for existence of a periodic solution of system (23) with period T_* is defined by equality $I(v_*, x_{0*}) = 0$, where (v_*, x_{0*}) is a solution of the optimization problem.
5) Optimal solution (v_*, x_{0*}) is defined by constructing the minimizing sequences.

5 Conclusion

A more general problem of periodic solution of the boundary value problem of ordinary differential equations with phase and integral constraints is formulated on the base of a review of scientific research on the periodic solutions of autonomous dynamical systems [1]-[4].

The boundary value problem is reduced to the problem of controllability of dynamic systems with phase and integral constraints by introducing a fictitious boundary control [5]. Solution of the controllability problem is reduced to a Fredholm integral equation of the first kind. The necessary and sufficient conditions for the solvability of the Fredholm integral equation of the first kind are obtained and the general solution of the integral equation is found.

The results of fundamental research on the controllability theory of dynamic systems, as well as new results on the solvability and construction the solution of the Fredholm integral equation of the first kind enable to reduce solutions of the general problem of periodic solution to the special initial problem of optimal control.

The necessary and sufficient condition for the existence of periodic solution of autonomous dynamic system in the form of requirements on a non-negative functional values is obtained.

The algorithm for constructing periodic solution to the limit points of minimizing sequences is developed. The estimation of the convergence rate is obtained.

Scientific novelty of the results consists in a completely new approach to the study of periodic solutions of autonomous dynamical systems, focused on the use of modern information technologies is offered. The existence of periodic solution and its construction are solved together.

A distinctive feature of the proposed method from the known methods of investigation of periodic solutions is that: firstly, the properties of analytic right-hand sides, the differential equations are not required; secondly, there is no need for small parameter system.

References

1. Aisagaliev, S.A.: The stability theory of dynamical systems. Qazaq university, Almaty (2012)
2. Aisagaliev, S.A., Kalimoldaev, M.N., Zhunussova, Zh.Kh.: Immersion principle for boundary value problem of ordinary differential equations. Math. J. **12**:2(44), 5–22 (2012)
3. Petrovsky, I.G.: Lectures on ordinary differential equations theory. Science, M. (1964)
4. Pontryagin, L.S.: Ordinary differential equations. Science, M. (1970)
5. Tihonov, A.N., Vasileva, A.B., Sveshnikova, A.G.: Differential equations. Science, M. (1985)

On Solvability of Third-Order Singular Differential Equation

Raya D. Akhmetkaliyeva(✉)

L.N. Gumilyov Eurasian National University, 2 Satpayev Str.,
Astana 010008, Kazakhstan
akhmetkaliyeva_rd@enu.kz

Abstract. In this paper some new existence and uniqueness results are proved and maximal regularity estimates of solutions of third-order differential equation with unbounded coefficients are given.

Keywords: Differential equation · Differential operator · Non-semibounded operator · Finite function · Separability · Coercive estimate · Closure · Bounded invertibility · Inverse operator · Unbounded domain · Adjoint operator

1 Introduction and Main Results

In this paper we consider questions of the existence and uniqueness of solutions of equation

$$(L + \lambda E)\, y \equiv -p_1(x)\,(p_2(x)y'')' + [q(x) + ir(x) + \lambda]y = f(x), \tag{1}$$

where $\lambda \geq 0$ is a constant, and q and r are given functions, $f \in L_p$. For solution y of (1) we study conditions providing the following estimate:

$$\left\|p_1(x)\,(p_2(x)y'')'\right\|_p^p + \|(q(x) + ir(x) + \lambda)y\|_p^p \leq c\,\|f(x)\|_p^p. \tag{2}$$

The separation of differential expressions was early studied by W.N. Everitt and M. Giertz [6], and they proved some fundamental results. Later on a number of results concerning the property referred to as separation of differential expressions have been obtained by K.Kh. Boimatov [5], M. Otelbaev [9], A. Zettl [10] and A.S. Mohamed [7]. Some very recent results in this direction were presented and proved in [8] and [1]. In this paper we give the solvability results for (1) with unbounded coefficients p_1 and p_2. With respect to other operators the seperation results have been obtained in [2–4].

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_10

Let $1 < p < +\infty$. By $L_p \equiv L_p(R)$, $R = (-\infty, +\infty)$ we denote the space of functions with finite norm

$$\|\varphi\|_p = \left(\int_R |\varphi(x)|^p dx \right)^{\frac{1}{p}}.$$

Definition 1. A function $y(x) \in L_p(R)$ is called a solution of (1), if there is a sequence of three times continuously differentiable functions with compact support $\{y_n\}_{n=1}^{\infty}$ such that $\|y_n - y\|_p \to 0$ and $\|(L + \lambda E)y_n - f\|_p \to 0$ as $n \to \infty$.

By $C_b^{(k)}(R)$ $(k = 1, 2, ...)$ we denote the set of all k times continuously differentiable functions $\varphi(x)$ such that $\sum_{j=0}^{k} \sup_{x \in R} |\varphi^{(j)}(x)|$ is finite. Let $W_\lambda(x) = \dfrac{|q(x) + \lambda + ir(x)|}{p_1(x)p_2(x)}$.

Our main results in this paper are the following Theorems 1 and 2.

Theorem 1. *Assume, that the functions $p_1(x)$, $q(x)$ and $r(x)$ are continuous, $p_2 \in C_{loc}^{(1)}(R)$ and satisfy the conditions*

$$p_1(x) \geq 1, \quad p_2(x) \geq 1, \quad \frac{q(x)}{p_1^2(x)p_2^2(x)} \geq 1, \ r(x) \geq 1, \tag{3}$$

$$c_0^{-1} \leq \frac{p_j(x)}{p_j(\eta)}, \frac{q(x)}{q(\eta)}, \frac{r(x)}{r(\eta)} \leq c_0, \quad j = 1, 2, \quad x, \eta \in R, \quad |x - \eta| \leq 1, \tag{4}$$

$$|p_2'(x)| \leq c_1 p_2(x), \quad x \in R, \tag{5}$$

$$\sup_{x,\eta \in R: |x-\eta| \leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^\alpha |x - \eta|^\beta} < +\infty, \quad 0 < \alpha < \frac{\beta}{3} + 1, \quad \beta \in (0, 1], \ \lambda \geq 0. \tag{6}$$

Then there exists a number $\lambda_0 \geq 0$, such that the equation (1) for all $\lambda \geq \lambda_0$ has a solution y.

In (4), (5) and elsewhere, c_n $(n = 0, 1)$ denotes a fixed constant which, in general, may be different in the various places it is used.

Theorem 2. *Let the functions $q(x)$, $r(x)$ be continuous, $p_1 \in C_{loc}^{(3)}(R)$, $p_2 \in C_{loc}^{(2)}(R)$ and satisfy conditions (3), (4), (6) and*

$$|p_1^{(l)}(x)| \leq c_l p_1(x) \quad (j = \overline{1,3}), \quad |p_2^{(k)}(x)| \leq c p_2(x) \quad (k = 1, 2), \ x \in R. \tag{7}$$

Then the solution of the equation (1) is unique and the estimate (2) holds.

2 Auxiliary Statements

Below we suppose that conditions of Theorem 1 are fulfilled.

Let $\xi_s = \xi_l(x)$ $(s = 0, 1, 2)$ be roots of the equation

$$p_1(x)p_2(x)\xi^3 - r(x) + i(q(x) + \lambda) = 0.$$

From the conditions of Theorem 1 it follows that $0 < arg\ \xi_0 < \pi$ and $\pi < arg\ \xi_j < 2\pi,\ j = 1, 2$.

We introduce the following kernels:

$$M_0(x,\eta,\lambda) = \begin{cases} -\frac{1}{3p_1(x)p_2(x)}\frac{e^{i(x-\eta)\xi_0}}{\xi_0^2}, & -\infty < \eta < x \\ \\ \frac{1}{3p_1(x)p_2(x)}\sum_{l=1}^{2}\frac{e^{i(x-\eta)\xi_l}}{\xi_l^2}, & x < \eta < +\infty, \end{cases} \tag{8}$$

$$M_1(x,\eta,\lambda) = p_1(\eta)p_2(\eta)\left[\frac{q(\eta)+ir(\eta)+\lambda}{p_1(\eta)p_2(\eta)} - \frac{q(x)+ir(x)+\lambda}{p_1(x)p_2(x)}\right]M_0(x,\eta,\lambda)\omega(\eta - x),$$

$$\begin{aligned} M_2(x,\eta,\lambda) = &- \ p_1(\eta)p_2'(\eta)\omega(\eta-x) + 3p_1(\eta)p_2(\eta)\omega_\eta'(\eta-x)\ \ M_{0\eta\eta}''(x,\eta,\lambda)- \\ &- \ 2p_1(\eta)p_2'(\eta)\omega_\eta'(\eta-x) + 3p_1(\eta)p_2(\eta)\omega_{\eta\eta}''(\eta-x)\ \ M_{0\eta}'(x,\eta,\lambda)- \\ &- \ p_1(\eta)p_2'(\eta)\omega_{\eta\eta}''(\eta-x) + p_1(\eta)p_2(\eta)\omega_{\eta\eta\eta}'''(\eta-x)\ \ M_0(x,\eta,\lambda), \end{aligned}$$

and

$$M_3(x,\eta,\lambda) = M_0(x,\eta,\lambda)\omega(\eta - x),$$

where the function $\omega(\eta) \in C_0^\infty(-1, 1)$ is such that

$$\omega(\eta) = \begin{cases} 1, & |\eta| \le 1/2 \\ 0, & |\eta| \ge 1. \end{cases}$$

It is easy to get the following equalities:

$$\left.\frac{\partial^j M_0(x,\eta,\lambda)}{\partial\eta^j}\right|_{x=\eta-0} = \left.\frac{\partial^j M_0(x,\eta,\lambda)}{\partial\eta^j}\right|_{x=\eta+0},\ j = 0, 1, \tag{9}$$

$$\left.\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial\eta^2}\right|_{x=\eta-0} - \left.\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial\eta^2}\right|_{x=\eta+0} = -\frac{1}{p_1(x)p_2(x)}, \tag{10}$$

$$-p_1(x)\left(p_2(x)\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial\eta^2}\right)_\eta' + [q(x) + ir(x) + \lambda]M_0(x,\eta,\lambda) = 0. \tag{11}$$

We define the operators $M_j(\lambda)$, $(j = \overline{1,3})$ by means of the following equalities:

$$(M_j(\lambda)f)(\eta) = \int_R M_j(x,\eta,\lambda)f(x)dx \quad (j = \overline{1,3}).$$

The following statement is well-known (see [8]).

Lemma 1. *Let $1 < p < +\infty$ and let $k(x,\eta)$ be continuous function and*

$$(K\nu)(\eta) = \int\limits_R k(x,\eta)\nu(x)dx.$$

Then

$$\|K\|_{L_p \to L_p} \le \sup_{\eta \in R} \int\limits_R \left[|k(x,\eta)| + |k(\eta,x)|\right] dx.$$

Lemma 2. *Let all of the conditions of Theorem 1 be satisfied. Then the operators $M_j(\lambda)$, $j = \overline{1,3}$, are continuous in the space L_p and the following estimates hold ($\lambda \ge 0$):*

$$\|M_1(\lambda)\|_{L_p \to L_p} \le \frac{c}{b_\lambda^{\beta+3-3\alpha}(\eta)}, \quad \beta \in (0,1], \quad 0 < \alpha < \frac{\beta}{3} + 1, \tag{12}$$

$$\|M_2(\lambda)\|_{L_p \to L_p} \le \frac{c}{b_\lambda(\eta)}, \tag{13}$$

$$\|M_3(\lambda)\|_{L_p \to L_p} \le \frac{c}{p_1(\eta)p_2(\eta)b_\lambda^3(\eta)}, \tag{14}$$

where $b_\lambda(x) = \sqrt[3]{W_\lambda(x)}$.

Proof. Under the assumptions of Theorem 1 for the functions $q(x), r(x)$ and $p_j(x)$ $(j = 1,2)$ there exists a constant $\sigma > 0$ such that $Im\ \xi_1 \ge \sigma$ and $Im\ \xi_l \le -\sigma$ $(l = 1,2)$. Then from (8) we can derive that

$$|M_0(x,\eta,\lambda)| \le \begin{cases} \frac{1}{3p_1(x)p_2(x)} \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \frac{2}{3p_1(x)p_2(x)} \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty \end{cases} \tag{15}$$

and

$$\left|\frac{\partial^j M_0(x,\eta,\lambda)}{\partial \eta^j}\right| \le \begin{cases} \frac{1}{3p_1(x)p_2(x)} \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & -\infty < \eta < x, \\ \frac{2}{3p_1(x)p_2(x)} \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & x < \eta < +\infty, \end{cases} \tag{16}$$

where j = 1, 2. According to our choice, $M_j(x,\eta,\lambda) = 0$ at $|x-\eta| > 1$. Taking into account conditions (3), (4), (5) and (6) of Theorem 1 and (15), (16) for functions $M_j(x,\eta,\lambda)$ $(j = 0,1,2)$ at $|x-\eta| \le 1$, we obtain the following estimates:

$$|M_1(x,\eta,\lambda)| \le \begin{cases} cp_1(\eta)p_2(\eta)|x-\eta|^\beta b_\lambda^{3\alpha-2}(x)\frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{p_1(x)p_2(x)}, & -\infty < \eta < x, \\ cp_1(\eta)p_2(\eta)|x-\eta|^\beta b_\lambda^{3\alpha-2}(x)\frac{e^{\sigma(x-\eta)b_\lambda(x)}}{p_1(x)p_2(x)}, & x < \eta < +\infty, \end{cases} \tag{17}$$

$$|M_2(x,\eta,\lambda)| \le \begin{cases} \frac{p_1(\eta)p_2(\eta)}{p_1(x)p_2(x)} \sum_{k=0}^{2} \tilde{c}_k \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & -\infty < \eta < x, \\ \\ \frac{p_1(\eta)p_2(\eta)}{p_1(x)p_2(x)} \sum_{k=0}^{2} \tilde{c}_k \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & x < \eta < +\infty, \end{cases} \tag{18}$$

and

$$|M_3(x,\eta,\lambda)| \le \begin{cases} \frac{1}{3p_1(x)p_2(x)} \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \frac{2}{3p_1(x)p_2(x)} \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty. \end{cases} \tag{19}$$

We will estimate the norms $\|M_j(\lambda)\|_{L_p\to L_p}$ $(j=\overline{1,3})$ of the operators $M_j(\lambda)$ using Lemma 1 and inequalities (17), (18), (19) and the conditions (3), (4) and (6). Then making the change of variable $\eta - x = \dfrac{1}{\sigma b_\lambda(\eta)} z$, we obtain

$$\|M_1(\lambda)\|_{L_p\to L_p} \le \frac{c\left|\frac{q(\eta)+\lambda+ir(\eta)}{p_1(\eta)p_2(\eta)}\right|^\alpha}{(cb_\lambda(\eta))^{\beta+3}} + \frac{c\left|\frac{q(\eta)+\lambda+ir(\eta)}{p_1(\eta)p_2(\eta)}\right|^\alpha}{(b_\lambda(\eta))^{\beta+3}} = \frac{c}{\left(\frac{|q(\eta)+\lambda+ir(\eta)|}{p_1(\eta)p_2(\eta)}\right)^{\frac{\beta}{3}+1-\alpha}}.$$

(3) implies $\dfrac{|q(\eta)+\lambda+ir(\eta)|}{p_1(\eta)p_2(\eta)} \ge \sqrt{1+\lambda}$. Therefore, from the previous inequality we obtain (12). Inequalities (13) and (14) are proved similarly. The lemma is proved.

Using the definitions of $M_j(\lambda)$ $(j=1,2,3)$ and equalities (9), (10) and (11), we prove the following Lemma.

Lemma 3. *Let the conditions of Theorem 1 be satisfied. Then the following equality holds:*

$$(L+\lambda E)\left[M_3(\lambda)f\right](\eta) = f(\eta) + \left[M_1(\lambda)f\right](\eta) + \left[M_2(\lambda)f\right](\eta). \tag{20}$$

3 Proofs of the Main Results

Proof of Theorem 1. By estimates (12) and (13), there exists a number $\lambda_0 > 0$, such that $\|M_1(\lambda)\|_{L_p\to L_p} + \|M_2(\lambda)\|_{L_p\to L_p} \le 1/2$ for any $\lambda \ge \lambda_0$. Then the operator $G(\lambda) = E + M_1(\lambda) + M_2(\lambda)$ has a bounded inverse $G^{-1}(\lambda)$ in L_p. Let $h = [E + M_1(\lambda) + M_2(\lambda)]\, f$. By (20), we obtain $(L+\lambda E)\,[M_3(\lambda)G^{-1}(\lambda)h](\eta) = h$. So, for all λ: $\lambda \ge \lambda_0$ the function $y = M_3(\lambda)G^{-1}(\lambda)f$ is a solution to equation (1). The proof is complete.

Let the functions p_i, $(i=1,2)$, q, r satisfy the conditions of Theorem 2, and a number p' is such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We denote by $(L+\lambda E)'$ an operator acting in the space $L_{p'}(R)$ and such that $((L+\lambda E)\,y, z) = \left(y, (L+\lambda E)'\,z\right)$, $y \in D(L+\lambda E)$, $z \in D\left((L+\lambda E)'\right)$. It is clear, that

$$(L+\lambda E)'z \equiv \left(p_2(x)\left(p_1(x)z\right)'\right)'' + \left(q(x)+\lambda - ir(x)\right)z.$$

We consider the following differential equation:

$$(L+\lambda E)'z \equiv \left(p_2(x)\left(p_1(x)z\right)'\right)'' + \left(q(x)+\lambda - ir(x)\right)z = g(x), \qquad (21)$$

where $p_j(x) \geq 1$ $j=1,2$ are continuous together with derivatives up to third and second order, respectively, and $q(x)$ and $r(x)$ are continuous real-valued functions, $\lambda \geq 0$, $g(x) \in L_{p'}(R)$.

The following lemma is proved similarly to Theorem 1.

Lemma 4. *Let the continuous functions $q(x)$, $r(x)$ and the functions $p_1 \in C^{(3)}_{loc}(R)$, $p_2 \in C^{(2)}_{loc}(R)$ satisfy the conditions (3), (4), (6) and (7). Then there exists a number $\lambda_1 \geq 0$, such that for all $\lambda \geq \lambda_1$ the equation (21) has a solution.*

Proof of Theorem 2. Lemma 4 implies that the operator $(L+\lambda E)'$ at $\lambda \geq \lambda_1$ has a right inverse defined on the whole $L_{p'}(R)$. So $ker\left((L+\lambda E)'\right)^* = \{0\}$, where $\left((L+\lambda E)'\right)^*$ is an adjoint operator to $(L+\lambda E)'$. It is clear that $\left((L+\lambda E)'\right)^*$ is an extension of the operator $L+\lambda E$, hence we have $ker(L+\lambda E) = \{0\}$, $\forall \lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1)$. Thus, the operator $L+\lambda E$ is a boundedly invertible in the space $L_{p'}(R)$ and by proof of the Theorem 1,

$$(L+\lambda E)^{-1} = M_3(\lambda)G^{-1}(\lambda), \quad \lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1) \qquad (22)$$

Let y be a solution of equation (1), where $\lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1)$. We shall prove the estimate (2). By (22), Lemma 1 and the conditions (3), (4), (5) and (6), we have

$$\left\|(q+\lambda+ir)(L+\lambda E)^{-1}\right\|_{L_p\to L_p} = \left\|(q+\lambda+ir)M_3(\lambda)G^{-1}(\lambda)\right\|_{L_p\to L_p} \leq$$

$$\leq c\sup_{\eta\in R}\int\limits_{\eta-1}^{\eta+1} b_\lambda^3(\eta)b_\lambda^{-2}(x)\exp[-\sigma|x-\eta|b_\lambda(x)]dx \leq$$

$$\leq c_1 \sup_{\eta\in R} b_\lambda(\eta)\int\limits_{\eta-1}^{\eta+1}\exp[-\sigma|x-\eta|b_\lambda(x)]dx < \infty.$$

By (1), we get $\left\|p_1(x)\left(p_2(x)y''\right)'\right\|_p \leq c\left(\|f\|_p + \|y\|_p\right)$. Combining the last two estimates, we obtain (2). The theorem is proved.

Acknowledgements. I thank Professor K.N. Ospanov for several generous advices, which have improved the final quality of this paper.

References

1. Akhmetkaliyeva, R.D., Ospanov, K.N., Persson, L.-E., Wall, P.: Some new results concerning a class of third order differential equations. Appl. Anal. **94**(2), 419–434 (2015)

2. Akhmetkaliyeva, R.D., Ospanov, K.N., Zulkhazhav, A.: Compactness conditions and estimates for the Fredholm radius of the resolvent of the degenerate second order differential operator. AIP Conf. Proc. **1637**(13), 13–17 (2014)
3. Akhmetkaliyeva, R.D., Ospanov, K.N.: Separation and the existence theorem for second order nonlinear differential equation. Electron. J. Qual. Theory. Differ. Equ. **66**, 1–12 (2012)
4. Akhmetkaliyeva, R.D., Ospanov, K.N.: Some inequalities for second order differential operators with unbounded drift. Eurasian Math. J. **6**(2), 63–74 (2015)
5. Boimatov, KKh: Coercive estimates and separation for second order elliptic differential eqations. Dokl. Akad. Nauk SSSR. **301**(5), 1033–1036 (1988)
6. Everitt, W.N., Giertz, M.: Some properties of the domains of certain differential operators. Proc. Lond. Math. Soc. **23**(3), 301–324 (1971)
7. Mohamed, A.S.: Existence and uniqueness of the solution, separation for certain second order elliptic differential equation. Appl. Anal. **76**, 179–185 (2000)
8. Muratbekov, M.B., Muratbekov, M.M., Ospanov, K.N.: Coercive solvability of odd-order differential equations and its applications. Dokl. Math. **82**(3), 1–3 (2010)
9. Otelbaev, M.: On the separation of the elliptic operators. Dokl. Acad. Nauk SSSR. **234**, 540–543 (1977)
10. Zettl, A.: Separation for differential operators and the L_p spaces. Proc. Amer. Math. Soc. **55**, 44–46 (1976)

Solvability of Multipoint-Integral Boundary Value Problem for a Third-Order Differential Equation and Parametrization Method

Anar T. Assanova[1(✉)] and Askarbek E. Imanchiev[2]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin str., Almaty 050010, Kazakhstan
assanova@math.kz

[2] Zhubanov Aktobe Regional State University, 34 A.Moldagulova str., Aktobe 030000, Kazakhstan
imanchiev_ae@mail.ru

Abstract. A multipoint-integral boundary value problem for a third order differential equation with variable coefficients is considered. The questions of the existence of a unique solution of the considered problem and ways of its construction are investigated. The multipoint-integral boundary value problem for the differential equation of third order with variable coefficients is reduced to a multipoint-integral boundary value problem for a system of three differential equations by introducing new functions. To solve the resulting multipoint-integral boundary value problem, a parametrization method is applied. Algorithms of finding the approximate solution to the multipoint-integral boundary value problem for the system of three differential equations are constructed and their convergence is proved. The conditions of the unique solvability of the multipoint-integral boundary value problem for the system of three differential equations are established in the terms of initial data. The results are also formulated relative to the original of the multipoint-integral boundary value problem for the differential equation of third order with variable coefficients. The obtained results are applied to a two-point boundary value problem for the third order ordinary differential equation.

Keywords: Third order differential equation · Multipoint-integral boundary value problem · Multipoint condition · Integral condition · Parametrization method · Cauchy problem for third order differential equation · Algebraic equation · Algorithm · Approximate solution · Unique solvability

1 Statement of Problem

We consider a third-order ordinary differential equation

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_11

$$\frac{d^3z}{dt^3} = A_1(t)\frac{d^2z}{dt^2} + A_2(t)\frac{dz}{dt} + A_3(t)z + f(t), \qquad t \in (0,T), \tag{1}$$

with multipoint and integral conditions

$$\sum_{i=0}^{m}\Big\{\alpha_{i1}\frac{d^2z(t_i)}{dt^2} + \beta_{i1}\frac{dz(t_i)}{dt} + \gamma_{i1}z(t_i)\Big\}+$$
$$+\int_0^T\Big\{K_{11}(\tau)\frac{d^2z(\tau)}{d\tau^2} + K_{12}(\tau)\frac{dz(\tau)}{d\tau} + K_{13}(\tau)z(\tau)\Big\}d\tau = d_1, \tag{2}$$

$$\sum_{i=0}^{m}\Big\{\alpha_{i2}\frac{d^2z(t_i)}{dt^2} + \beta_{i2}\frac{dz(t_i)}{dt} + \gamma_{i2}z(t_i)\Big\}+$$
$$+\int_0^T\Big\{K_{21}(\tau)\frac{d^2z(\tau)}{d\tau^2} + K_{22}(\tau)\frac{dz(\tau)}{d\tau} + K_{23}(\tau)z(\tau)\Big\}d\tau = d_2, \tag{3}$$

$$\sum_{i=0}^{m}\Big\{\alpha_{i3}\frac{d^2z(t_i)}{dt^2} + \beta_{i3}\frac{dz(t_i)}{dt} + \gamma_{i3}z(t_i)\Big\}+$$
$$+\int_0^T\Big\{K_{31}(\tau)\frac{d^2z(\tau)}{d\tau^2} + K_{32}(\tau)\frac{dz(\tau)}{d\tau} + K_{33}(\tau)z(\tau)\Big\}d\tau = d_3. \tag{4}$$

Here $z(t)$ is unknown function, the functions $A_k(t)$, $f(t)$ are continuous on $[0,T]$, $k = 1,2,3$, α_{ij}, β_{ij}, γ_{ij}, d_j are constants, the functions $K_j(t)$ are continuous on $[0,T]$, $i = \overline{0,m}$, $j = 1,2,3$, $0 = t_0 < t_1 < t_2 < ... < t_{m-1} < t_m = T$.

Let $C([0,T],R)$ be a space of continuous functions $z : [0,T] \to R$ on $[0,T]$ with norm $||u||_0 = \max\limits_{t\in[0,T]} |z(t)|$.

The function $z(t) \in C([0,T],R)$, that has derivatives $\frac{dz(t)}{dt} \in C([0,T],R)$, $\frac{d^2z(t)}{dt^2} \in C([0,T],R)$, $\frac{d^3z(t)}{dt^3} \in C([0,T],R)$ is called a solution to problem (1)–(4) if it satisfies the third-order differential equation (1) for all $t \in (0,T)$ and meets the boundary conditions (2), (3) and (4).

Mathematical modeling of various processes in physics, chemistry, biology, technology, ecology, economics and others are leaded to multipoint-integral boundary value problems for differential equations of higher orders with variable coefficients [5,6,15,16,23,24]. The problems of solvability of multipoint-integral boundary value problems remain important for applications because they are directly connected with the theory of splines and interpolations and are used in the theory of multisupport beams. Despite the presence of numerous works, general statements of multipoint-integral problems for ordinary differential equations remain poorly studied up to now. The method of Green's functions proves to be the main method for the investigation and solution of multipoint-integral boundary value problems. This method reflects the specific features of the analyzed boundary value problems. However, the problem of construction of the Green's function is quite complicated due to the complex nature of the investigated object and the absence of the required information about its properties.

One of possible ways of overcoming these difficulties is connected with the development of constructive methods aimed at the investigation and solving multipoint-integral boundary value problems for higher order differential equations without using the fundamental matrix and the Green's function. Thus, in [11], a parametrization method was proposed for the investigation and solving two-point boundary value problems for ordinary differential equations. Parallel with construction of the coefficient criteria for the unique solvability of the investigated problem, parametrization method enables one to propose algorithms for finding the solution of this problem. In [12,13], the parametrization method was applied to multipoint boundary value problem for ordinary differential equations. A family of multipoint boundary value problems for system of differential equations and multipoint nonlocal problem for system of hyperbolic equations were considered in [2,3].

In the present paper we study a questions of the existence and uniqueness of solutions to multipoint-integral boundary value problem for the third-order differential equation (1)–(4) and the methods of finding its approximate solutions. For these purposes, we apply the parameterizations method to solve the problem (1)–(4). Algorithms of finding the approximate solution to the multipoint-integral boundary value problem for the system of three differential equations are constructed and their convergence is proved. The conditions of the unique solvability of the multipoint-integral boundary value problem for the system of three differential equations are established in the terms of initial data. The results are also formulated relative to the original of the multipoint-integral boundary value problem for the differential equation of third order with variable coefficients. The obtained results are applied to a two-point boundary value problem for the third order ordinary differential equation. The efficiency of the proposed approach for solve of the two-point boundary value problems for the third order differential equations can be used in applications. The results can also be used in the study and solving a nonlinear multipoint-integral boundary value problems for the third order differential equations. Some types of problems (1)–(4) were studied in [1,5–10,14–29]. For $K_{ij}(t) = 0$, $i = \overline{1,3}$, $j = \overline{1,3}$, the problem (1)–(4) was considered in [4].

2 Scheme of the Method

We introduce the following notations

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ A_3(t) & A_2(t) & A_1(t) \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ 0 \\ f(t) \end{pmatrix},$$

$$M_i = \begin{pmatrix} \gamma_{i1} & \beta_{i1} & \alpha_{i1} \\ \gamma_{i2} & \beta_{i2} & \alpha_{i2} \\ \gamma_{i3} & \beta_{i3} & \alpha_{i3} \end{pmatrix}, \qquad K(t) = \begin{pmatrix} K_{13}(t) & K_{12}(t) & K_{11}(t) \\ K_{23}(t) & K_{22}(t) & K_{21}(t) \\ K_{33}(t) & K_{32}(t) & K_{31}(t) \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

I is identity matrix of dimension 3.

Problem (1)–(4) can be written in the vector-matrix form

$$\frac{du}{dt} = A(t)u + F(t), \tag{5}$$

$$\sum_{i=0}^{m} M_i u(t_i) + \int_0^T K(\tau)u(\tau)d\tau = d, \tag{6}$$

where $u = (u_1, u_2, u_3)'$, $u_1(t) = z(t)$, $u_2(t) = \frac{dz(t)}{dt}$, $u_3(t) = \frac{d^2z(t)}{dt^2}$.

A continuously differentiable function $u : [0, T] \to R^3$ is called a solution of the multipoint-integral boundary value problem (5), (6) if it satisfies system (5) for all $t \in [0, T]$ and condition (6).

By μ we denote the value of the function $u(t)$ for $t = t_0$. We perform the change $u(t) = \widetilde{u}(t) + \mu$ in problem (5), (6).

Then problem (5), (6) is reduced to the following equivalent problem with a unknown parameter μ:

$$\frac{d\widetilde{u}}{dt} = A(t)\widetilde{u} + A(t)\mu + F(t), \tag{7}$$

$$\widetilde{u}(t_0) = 0, \tag{8}$$

$$M_0\mu + \sum_{i=1}^{m} M_i\widetilde{u}(t_i) + \sum_{i=1}^{m} M_i\mu + \int_0^T K(\tau)\widetilde{u}(\tau)d\tau + \int_0^T K(\tau)d\tau\mu = d. \tag{9}$$

A pair $(\widetilde{u}(t), \mu)$ is called a solution to problem with parameter (7)–(9) if the function $\widetilde{u}(t)$ is continuously differentiable on $[0, T]$ and satisfies the system (7), the initial condition (8) and the multipoint-integral condition (8).

Problems (5)-(6) and (7)-(9) are equivalent. If a vector function $u(t)$ is a solution to the multipoint-integral problem (5), (6), then a pair $(\widetilde{u}(t), \mu)$, where $\widetilde{u}(t) = u(t) - u(t_0)$, $\mu = u(t_0)$, is a solution to problem with parameter (7)-(9). And conversely, if a pair $(\widetilde{u}^*(t), \mu^*)$ is a solution to problem with parameter (7)-(9), then a vector function $u^*(t) = \widetilde{u}^*(t) + \mu^*$ is a solution to the original multipoint-integral problem (5), (6). At fixed μ the problem (7), (8) is a Cauchy problem for system of three differential equations and the relation (9) connects values of function $\widetilde{u}(t)$ with the unknown parameter μ.

A solution of Cauchy problem (7), (8) is equivalent to a Volterra integral equation of the second kind

$$\widetilde{u}(t) = \int_0^t A(\tau)\widetilde{u}(\tau)d\tau + \int_0^t A(\tau)d\tau\mu + \int_0^t F(\tau)d\tau. \tag{10}$$

Substituting the right-hand side of the integral equation (10) instead of the function $\widetilde{u}(\tau)$ at $t = \tau$, and repeating the process ν - time ($\nu = 1, 2, 3, ...$), we get

$$\widetilde{u}(t) = D_\nu(t)\mu + G_\nu(t, \widetilde{u}) + \widetilde{F}_\nu(t), \tag{11}$$

where

$$D_\nu(t) = \int_0^t A(\tau)d\tau + \int_0^t A(\tau)\int_0^\tau A(\tau_1)d\tau_1 d\tau + ...+$$
$$+\int_0^t A(\tau)\int_0^\tau A(\tau_1)...\int_0^{\tau_{\nu-1}} A(\tau_\nu)d\tau_\nu d\tau_{\nu-1}...d\tau_1 d\tau,$$
$$G_\nu(t, \widetilde{u}) = \int_0^t A(\tau)\int_0^\tau A(\tau_1)...\int_0^{\tau_{\nu-1}} A(\tau_\nu)\widetilde{u}(\tau_\nu)d\tau_\nu d\tau_{\nu-1}...d\tau_1 d\tau,$$
$$\widetilde{F}_\nu(t) = \int_0^t F(\tau)d\tau + \int_0^t A(\tau)\int_0^\tau F(\tau_1)d\tau_1 d\tau + ...+$$
$$+\int_0^t A(\tau)\int_0^\tau A(\tau_1)...\int_0^{\tau_{\nu-1}} F(\tau_\nu)d\tau_\nu d\tau_{\nu-1}...d\tau_1 d\tau.$$

From the representation (11) we determine the values of function $\widetilde{u}(t)$ for $t = t_i$, $i = \overline{1, m}$, $t = \tau$, and substitute them into the appropriate expression (9). Then, we obtain

$$\left[M_0 + \sum_{i=1}^m M_i[I + D_\nu(t_i)] + \int_0^T K(\tau)[I + D_\nu(\tau)]d\tau\right]\mu = d - \sum_{i=1}^m M_i\widetilde{F}_\nu(t_i) -$$
$$-\int_0^T K(\tau)\widetilde{F}_\nu(\tau)d\tau - \sum_{i=1}^m M_i G_\nu(t_i, \widetilde{u}) - \int_0^T K(\tau)G_\nu(\tau, \widetilde{u})d\tau. \tag{12}$$

The relation (12) is a linear system of three algebraic equations with respect to the parameter μ.

If the (3×3) matrix $Q_\nu(T) = M_0 + \sum_{i=1}^m M_i[I + D_\nu(t_i)] + \int_0^T K(\tau)[I + D_\nu(\tau)]d\tau$ is invertible for some $\nu \in \mathbf{N}$, then at fixed values $\widetilde{u}$ the parameter μ is uniquely determined from system (12). So, for finding a solution to problem (7)-(9) we have a closed system of equations (10) and (12).

3 Algorithm and Main Result

If the function $\widetilde{u}(t)$ is known, then the parameter μ can be found from the system of algebraic equations (12). Conversely, if the parameter μ is known, then function $\widetilde{u}(t)$ can be found from the Cauchy problem for system of the differential equations (7), (8). Since neither $\widetilde{u}(t)$, nor μ are known, we use the iterative method and find the solution of problem with parameter (7)-(9) in the form of a pair $(\widetilde{u}^*(t), \mu^*)$ as the limit of a sequence $(\widetilde{u}^{(k)}(t), \mu^{(k)})$, $k = 0, 1, 2, ...$, determined according to the following algorithm:

Step 0. Assume that, for chosen $\nu \in \mathbf{N}$ the matrix $Q_\nu(T) : R^3 \to R^3$ is invertible. We use the initial condition (8). We determine the initial approximation in the parameter $\mu^{(0)}$ from the system of algebraic equations $Q_\nu(T)\mu = d - \sum\limits_{i=1}^{m} M_i \widetilde{F}_\nu(t_i)$.

We solve the Cauchy problem (7), (8) for $\mu = \mu^{(0)}$ and find a function $\widetilde{u}^{(0)}(t)$ for all $t \in [0, T]$.

Step 1. Substituting the obtained function $\widetilde{u}^{(0)}(t)$ for $\widetilde{u}(t)$, from the system of algebraic equations (12), we obtain $\mu^{(1)}$. Further, we solve the Cauchy problem (7), (8) for $\mu = \mu^{(1)}$ and find a function $\widetilde{u}^{(1)}(t)$ for all $t \in [0, T]$.

And so on.

Step k. Substituting the obtained function $\widetilde{u}^{(k-1)}(t)$ for $\widetilde{u}(t)$, from the system of algebraic equations (12), we get $\mu^{(k)}$. Solving the Cauchy problem (7), (8) for $\mu = \mu^{(1)}$, we find $\widetilde{u}^{(k)}(t)$ for all $t \in [0, T]$, $k = 0, 1, 2, ...$

Introduce notations

$$a = \max_{t \in [0,T]} ||A(t)|| = \max\Big(1, \max_{t \in [0,T]} \{|A_1(t)| + |A_2(t)| + |A_3(t)|\}\Big),$$

$$\kappa = \max_{t \in [0,T]} ||K(t)|| = \max_{t \in [0,T]} \max_{i=\overline{1,3}} \{|K_{i1}(t)| + |K_{i2}(t)| + |K_{i3}(t)|\}.$$

The following theorem establishes sufficient conditions for the applicability and convergence of the algorithm proposed above, which also guarantee the unique solvability of problem (5), (6).

Theorem 1. *Let the matrix* $Q_\nu(T) : R^3 \to R^3$ *is invertible for some* $\nu \in \mathbf{N}$ *and let the following inequalities be true:*

a) $||[Q_\nu(T)]^{-1}|| \leq \eta_\nu(T)$, *where* $\eta_\nu(T)$ *is a positive constant;*

b) $q_\nu(T) = \eta_\nu(T) \cdot \Big(\sum\limits_{i=1}^{m} ||M_i|| + \kappa T\Big) \max\limits_{i=\overline{1,m}} \Big[e^{at_i} - 1 - \sum\limits_{j=1}^{\nu} \frac{[at_i]^j}{j!}\Big] < 1.$

Then the multipoint-integral boundary value problem (5), (6) has a unique solution.

The proof of Theorem 1 is similar to the proof of Theorem 1 in [3].

By using the parametrization method, we split the procedure of determination of unknown functions into two part:

1) determination of the unknown function $\widetilde{u}(t)$ from the Cauchy problem for system of three differential equations (7), (8);

2) determination of the introduced parameter μ from the system of algebraic equations (12).

Taking into account the notations and the equivalent transition to problem (5), (6), we have

Theorem 2. *Let the matrix* $Q_\nu(T) : R^3 \to R^3$ *is invertible for some* $\nu \in \mathbf{N}$ *and let the inequalities a), b) of Theorem 1 be true.*

Then the multipoint-integral boundary value problem for the third-order differential equation (1)–(4) has a unique solution.

4 Example

We consider the boundary value problem [27]:

$$\frac{d^3 z}{dt^3} = p(t)z + f(t) + r, \qquad t \in (a,b), \tag{13}$$

$$z(a) = \alpha, \tag{14}$$

$$\frac{dz(a)}{dt} = \beta_1, \tag{15}$$

$$\frac{dz(b)}{dt} = \beta_2. \tag{16}$$

Assume that the functions $f(t)$ and $p(t)$ are given, and $p(t) = 0$ for $t \in [a,c) \cup (d,b]$, $a < c < d < b$, the parameter r, α, β_1, and β_2 are constants.

For this problem

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p(t) \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ 0 \\ f(t) + r \end{pmatrix},$$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad K(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{d} = \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{pmatrix},$$

$$\widetilde{D}_\nu(a,t) = \int_a^t A(\tau)d\tau + \int_a^t A(\tau) \int_a^\tau A(\tau_1)d\tau_1 d\tau + ...+$$

$$+ \int_a^t A(\tau) \int_a^\tau A(\tau_1)... \int_a^{\tau_{\nu-1}} A(\tau_\nu)d\tau_\nu d\tau_{\nu-1}...d\tau_1 d\tau, \quad \nu = 1,2,...,$$

$$\delta = \max\Big(1, \max_{t\in[a,b]} |p(t)|\Big).$$

Theorem 3. *Let the* $(3\times 3$ *matrix* $Q_\nu(a,b) = M_0 + M_1[I + \widetilde{D}_\nu(a,b)]$ *is invertible for some* $\nu \in \mathbf{N}$ *and let the following inequalities be true:*

a) $||[\widetilde{Q}_\nu(a,b)]^{-1}|| \leq \widetilde{\eta}_\nu(a,b)$, *where* $\widetilde{\eta}_\nu(a,b)$ *is a positive constant;*

b) $\widetilde{q}_\nu(a,b) = \widetilde{\eta}_\nu(a,b)\cdot\left[e^{\delta(b-a)} - 1 - \sum\limits_{j=1}^{\nu}\frac{[\delta(b-a)]^j}{j!}\right] < 1.$

Then two-point boundary value problem for the third-order differential equation (13)-(16) has a unique solution.

Note, that in the repeated integrals of $\widetilde{D}_\nu(a,t)$ the element of the matrix $A(t)$ is function $p(t)$ which are calculated on the interval $[c,d]$.

Let $p(t) = 1$ for $t \in [c,d]$, $p(t) = 0$ for $t \in [a,c) \cup (d,b]$. In this case, the conditions of Theorem 3 will be formulated only in the terms of numbers a, b, c, d.

We have

Theorem 4. *Let the* $(3\times 3$ *matrix* $Q_1(a,b) = \begin{pmatrix} 1 & b-a & 0 \\ 0 & 1+b-a & 0 \\ 0 & 1 & d-c \end{pmatrix}$ *is invertible and let the following inequalities be true:*

a) $||[\widetilde{Q}_1(a,b)]^{-1}|| \leq \max\Big(\frac{1}{d-c}, 1\Big) + \max\Big(b-a, 1, \frac{1}{d-c}\Big)\frac{1}{1+b-a}$;

b) $\widetilde{q}_\nu(a,b) =$

$$= \left[\max\Big(\frac{1}{d-c}, 1\Big) + \max\Big(b-a, 1, \frac{1}{d-c}\Big)\frac{1}{1+b-a}\right]\cdot\left[e^{(b-a)} - 1 - (b-a)\right] < 1.$$

Then the two-point boundary value problem for the third-order differential equation (13)-(16) has a unique solution.

Acknowledgements. This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 0822/*ΓΦ*4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Agarwal, R.P.: Existence-uniqueness and iterative methods for third-order boundary value problems. J. Comput. Appl. Math. **17**, 271–289 (1987)
2. Asanova, A.T.: About of the solvability of family multi-point boundary value problems for system of differential equations and their applications to nonlocal boundary value problems. Math. J. **13**(3), 38–42 (2013) (in Russian)
3. Asanova, A.T.: Multipoint problem for a system of hyperbolic equations with mixed derivative. J. Math. Sci. **212**(3), 213–233 (2016)
4. Assanova, A.T., Imanchiev, A.E.: On the unique solvability of a multi-point boundary value problem for a third-order differential equation. News of the NAS of the RK. Phys.-mathem ser. **5**(303), 124–132 (2015)
5. Brown, R.C., Krall, A.M.: Adjoints of multipoint-integral boundary value problems. Proc. Am. Math. Soc. **37**, 213–216 (1973)

6. Brown, R.C., Krall, A.M.: Ordinary differential operators under Stiltjes boundary conditions. Trans. Am. Math. Soc. **198**, 73–92 (1974)
7. Cabada, A., Heikkila, S.: Uniqueness, comparison, and existence results for third-order initial-boundary value problems. Comput. Math. Appl. **41**, 607–618 (2001)
8. Cabada, A., Minhos, F., Santos, A.I.: Solvability for a third-order discontinuous fully equation with nonlinear functional boundary conditions. J. Math. Anal. Appl. **322**, 735–748 (2006)
9. Du, Z., Cai, G., Ge, W.: A class a third-order multi-point boundary value problem. Taiwanese J. Math. **9**, 81–94 (2005)
10. Du, Z., Ge, W., Zhou, M.: Singular perturbations for third-order nonlinear multi-point boundary value problem. J. Differ. Equ. **218**, 69–90 (2005)
11. Dzhumabayev, D.S.: Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. USSR Comput. Math. Math. Phys. **29**, 34–46 (1989)
12. Dzhumabaev, D.S., Imanchiev, A.E.: Well-posedness of linear multi-point boundary value problem. Math. J. **5**, 30–38 (2005) (in Russian)
13. Dzhumabaev, D.S., Imanchiev, A.E.: Criteria of existence izolated solution of multi-point boundary value problem for system of ordinary differential equations. Izvestiya NAN RK. Ser. phyz.-matem. **3**, 117–121 (2010) (in Russian)
14. Henderson, J., Tisdale, C.C.: Five-point boundary value problems for third-order differential equations by solution matching. Math. Comput. Model. **42**, 133–137 (2005)
15. Kiguradze, I.T.: Boundary value problems for systems of ordinary differential equations. In: Modern Problems of Mathematics. Advancement Achievements, vol. 30, pp. 3–103. Nauka, Moscow (1987) (in Russian)
16. Krall, A.M.: Differential operators and their adjoints under integral and multiple point boundary conditions. J. Differ. Equ. **4**, 327–336 (1968)
17. Lan, C.: Boundary value problems for second and third order differential equations. J. Differ. Equ. **18**, 258–274 (1975)
18. Li, S.: Positive solutions of nonlinear singular third-order two-point boundary value problem. J. Math. Anal. Appl. **323**, 413–425 (2006)
19. Lin, X., Du, Z.: Uniqueness and existence results for a third-order nonlinear multi-point boundary value problem. Appl. Math. Comput. **205**, 187–196 (2008)
20. Liu, B., Yu, J.: Solvability of multi-point boundary value problems at resonance (II). Appl. Math. Comput. **129**, 119–143 (2002)
21. Minghe, P., Chang, S.K.: Existence and uniqueness of solutions for third-order nonlinear boundary value problems. J. Math. Anal. Appl. **327**, 23–35 (2007)
22. Palamides, A.P., Smyrlis, G.: Positive solutions to a singular third-order three-point boundary value problem with an indefinitely signed Green's functions. Nonlinear Anal. **68**, 2104–2118 (2008)
23. Samoilenko, A.M., Ronto, N.I.: Numerical-Analytic Methods of Investigation of Solutions of Boundary Value Problems. Naukova Dumka, Kyiv (1985) (in Russian)
24. Samoilenko, A.M., Laptinskii, V.N., Kenzhebayev, K.K.: Constructive methods of investigation of periodic and multipoint boundary value problems. In: Proceedings of Institute of Mathematics of NAS of Ukraine, vol. 29, pp. 3-220. Institute of Mathematics of NAS of Ukraine, Kiev (1999) (in Russian)
25. Sun, B., Zhao, J., Yang, P., Ge, W.: Successive iteration and positive solutions for a third-order multipoint generalized right-focal boundary value problem with p-Laplacian. Nonlinear Anal. **70**, 220–230 (2009)

26. Valarmathi, N., Ramanujam, N.: An asymptotic numerical method for singularly perturbed third-order ordinary differential equations of convection-diffusion type. Comput. Math. Appl. **44**, 693–710 (2002)
27. Xie, S., Li, P., Gao, Z., Wang, H.: High order compact finite difference schemes for a system of third order boundary value problem. Appl. Math. Comput. **219**, 2564–2573 (2012)
28. Yao, Q.: Solution and positive solution for a semilinear third-order two-point boundary value problem. Appl. Math. Lett. **17**, 1171–1175 (2004)
29. Yao, Q.: Successive iteration of positive solution for a discontinuous third-order boundary value problem. Comput. Math. Appl. **53**, 741–749 (2007)

On the Solvability of Nonhomogeneous Boundary Value Problem for the Burgers Equation in the Angular Domain and Related Integral Equations

Meiramkul M. Amangaliyeva[1], Muvasharkhan T. Jenaliyev[1(✉)], Minzilya T. Kosmakova[2], and Murat I. Ramazanov[3]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin str., Almaty 050010, Kazakhstan
meiramkul.amangaliyeva@airastana.com, muvasharkhan@gmail.com
[2] Buketov Karaganda State University, 28 Universitetskaya str., Karaganda 100028, Kazakhstan
svetik_mir69@mail.ru
[3] Institute of Applied Mathematics and Buketov Karaganda State University, 28 Universitetskaya str., Karaganda 100028, Kazakhstan
ramamur@mail.ru

Abstract. In this paper we consider the questions of solvability of the nonhomogeneous boundary value problem for the Burgers equation in infinite angular domain. It is reduced to the study of the solvability of a system consisting of two homogeneous integral equations. We prove some lemmas which establish properties of integral operators in weighted space of essentially bounded functions and prove the existence and properties of non-trivial solutions to the system of homogeneous integral equations. On the basis of Lemmas the solvability theorems of the nonhomogeneous boundary value problem for the Burgers equation in infinite angular domain are established.

Keywords: Burgers equation · Boundary value problem · Solvability · Non-trivial solution · Angular domain · Integral equation · Weighted space of essentially bounded functions

Introduction

Researches of Burgers equation has a long history, some of which are given in work [4] and in the books [5] and [9]. In work [4] in the Sobolev classes it is established the existence, uniqueness and regularity of the solution to the Burgers equation in non-cylindrical (non-degenerating) domain that can be transformed into a rectangular domain by the regular replacement of the independent variables. The authors indicate that the development of their results from the work

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_12

[4] for the case of degenerating domain will be considered by them in further.

In this paper we study the solvability of a nonhomogeneous boundary value problem for the Burgers equation in an infinite angular domain. The work consists of five sections. In section 1 it is given preliminary provisions of the transformation in angular domain of the nonhomogeneous boundary value problem for the Burgers equation to the homogeneous boundary value problem for the heat equation. Section 2 is devoted to reducing the homogeneous boundary value problem of heat conduction to a system of two integral equations. The main results on the research of questions of solvability of integral equations are given in section 3 (the case of constant coefficients w_0 and w_1) and in section 5 (in a special case of variable coefficients $w_0(t)$ and $w_1(t)$). Finally, in section 4 it is given theorems on the solvability of the nonhomogeneous boundary value problem for the Burgers equation in the infinite angular domain.

1 Preliminary Provisions and Statement of the Problem

For Burgers equation:

$$w_t + bww_x - a^2 w_{xx} = f(t), b > 0,$$

in which, without limiting the generality, we assume $b = 1,\ f(t) \equiv 0$, and consider in the domain $G = \{x, t :\ 0 < x < t,\ t > 0\}$ the boundary value problem

$$\begin{cases} w_t + ww_x - a^2 w_{xx} = 0, \ \ \{x, t\} \in G, \\ w\big|_{x=0} = w_0(t), \ \ w\big|_{x=t} = w_1(t), \end{cases} \tag{1}$$

where $w_0(t)$, $w_1(t)$ are some given on $(0, \infty)$ functions.

Using the Hopf-Cole transformation

$$w(x, t) = -2a^2 \cdot \frac{u_x(x, t)}{u(x, t)}, \tag{2}$$

boundary value problem (1) is reduced to the following auxiliary homogeneous boundary value problem

$$\begin{cases} u_t - a^2 u_{xx} = 0, \ \{x, t\} \in G, \\ u_x(0, t) + \frac{1}{2a^2} w_0(t) u(0, t) = 0, \\ u_x(t, t) + \frac{1}{2a^2} w_1(t) u(t, t) = 0. \end{cases} \tag{3}$$

Indeed, substituting function (2) into equation (1), we get

$$\frac{\partial}{\partial x} \left[\frac{u_t(x, t) - a^2 u_{xx}(x, t)}{u(x, t)} \right] = 0, \ \{x, t\} \in G, \tag{4}$$

i.e.

$$u_t(x, t) - a^2 u_{xx}(x, t) = c(t) u(x, t), \tag{5}$$

where $c(t)$ is an arbitrary function and without loss of generality we can take $c(t) \equiv 0$.

The inverse transformation to (2) is the following transformation

$$u(x,t) = \exp\left\{-\frac{1}{2a^2}\int\limits_0^x w(\xi,t)d\,\xi + d(t)\right\},\ 0 < x < t,\ t > 0, \tag{6}$$

from which it follows

$$u_x(x,t) = -\frac{w(x,t)}{2a^2}\exp\left\{-\frac{1}{2a^2}\int\limits_0^x w(\xi,t)d\,\xi + d(t)\right\},\ 0 < x < t,\ t > 0, \tag{7}$$

where $d(t)$ is an arbitrary bounded function on $(0,\infty)$. From formulas (2), (4)–(7), we obtain that boundary value problem (3) follows from (1).

Thus we obtain that one solution of the Burgers equation from (1) corresponds to each solution of the equation

$$u_t(x,t) - a^2 u_{xx}(x,t) = 0; \tag{8}$$

conversely, to any solution of the Burgers equation from (1) there is a family of solutions to equation (5), determined by arbitrary functions $d(t)$. Obviously, the elements of this family differ from each other by the exponential factor $\exp\{d(t\}$.

We are interested in the question: whether the boundary value problem (1) has solution? This question is directly related to the existence of a nontrivial solution to the homogeneous boundary value problem (3), the study of which is reduced to the investigation of the solvability of system consisting of two homogeneous integral equations. As our work shows these integral equations have the properties of singular integral equations.

2 Reducing Problem (3) to the Integral Equation

We are looking for solution of the problem (3) as the sum of the simple-layer potentials ([13], 476–479):

$$u(x,t) = \frac{1}{2a\sqrt{\pi}}\int\limits_0^t \frac{1}{(t-\tau)^{1/2}}\left[\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau) + \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\right]d\tau, \tag{9}$$

which satisfies equation (3) for all functions $\nu(t)$ and $\varphi(t)$, are not yet known and should be defined.

We satisfy solution (9) to the boundary conditions from (3). For this, by calculating the derivative with respect to x from (9):

$$\frac{\partial u(x,t)}{\partial x} = -\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu(\tau)d\tau$$
$$-\frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \varphi(\tau)d\tau, \qquad (10)$$

we obtain as a result:

$$\nu(t) - \frac{w_0(t)}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)d\tau}{(t-\tau)^{1/2}} = \left(N_{w_0}(t)\varphi_1\right)(t), \qquad (11)$$

$$\varphi_1(t) - \frac{1-w_1(t)}{2a\sqrt{\pi}} \int\limits_0^t \frac{\varphi_1(\tau)d\tau}{(t-\tau)^{1/2}} = \left(\Phi_{w_1}(t)\nu\right)(t), \qquad (12)$$

where

$$\left(N_{w_0}(t)\varphi_1\right)(t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \left[\frac{\tau}{(t-\tau)^{3/2}} + \frac{w_0(t)}{(t-\tau)^{1/2}}\right] E(t,\tau)\varphi_1(\tau)d\tau, \qquad (13)$$

$$\left(\Phi_{w_1}(t)\nu\right)(t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \left[\frac{\tau}{(t-\tau)^{3/2}} + \frac{1-w_1(t)}{(t-\tau)^{1/2}}\right] E(t,\tau)\nu(\tau)d\tau, \qquad (14)$$

$$E(t,\tau) = \exp\left\{-\frac{t\tau}{4a^2(t-\tau)}\right\}, \quad \varphi_1(t) = \varphi(t)\exp\left\{\frac{t}{4a^2}\right\}. \qquad (15)$$

Thus, the homogeneous boundary problem (3) is reduced to the problem on the solvability for the system of integral equations (11) and (12).

Below we consider various special cases of problem (11)–(15).

3 Solving the System of Integral Equations (11)–(12). The Case of Constant Coefficients w_0 and w_1

In this section we establish some lemmas related to the solvability of integral equations (11) and (12) in various special cases.

3.1 Homogeneous Case of Problem (1)

The following lemma is valid.

Lemma 1. *Let* $w_0(t) \equiv 0$ *and* $w_1(t) \equiv 0$. *Then the system of equations (11) and (12) has only one pair of non-trivial solutions* $\{\nu(t),\ \varphi_1(t)\}$ *up to a common constant factor*

$$\nu(t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{3/2}} E(t,\tau)\varphi_1(\tau)d\tau, \tag{16}$$

$$\varphi_1(t) = \frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{4a} \exp\left\{\frac{t}{4a^2}\right\}\left[1 + erf\left(\frac{\sqrt{t}}{2a}\right)\right].$$

Proof. If $w_0(t) \equiv 0$ $w_1(t) \equiv 0$, then from (11) and (12) we get:

$$\varphi_1(t) - \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\varphi_1(\tau)d\tau}{(t-\tau)^{1/2}} = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{t}{(t-\tau)^{3/2}} E(t,\tau)\nu(\tau)d\tau, \tag{17}$$

and equality (16). Substituting (16) into (17) and following the work [3], we obtain the integral equation

$$\varphi_1(t) - \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\varphi_1(\tau)d\tau}{(t-\tau)^{1/2}} = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{t+\tau}{(t-\tau)^{3/2}} E(t,\tau)\varphi_1(\tau)d\tau. \tag{18}$$

In work [3] it was shown that equation (18) has only one nontrivial solution up to a constant factor. From this the assertion of Lemma 1 follows. □

3.2 Nonhomogeneous Case of Problem (1): $w_0(t) \equiv w_1(t) \equiv 1$.

The following lemma is valid.

Lemma 2. *Let* $w_0(t) \equiv 1$ *and* $w_1(t) \equiv 1$. *Then the system of equations (11) and (12) has only one pair of non-trivial solutions* $\{\nu(t),\ \varphi_1(t)\}$ *up to a common constant factor*

$$\varphi_1(t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{3/2}} E(t,\tau)\nu(\tau)d\tau, \tag{19}$$

$$\nu(t) = \frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{4a} \exp\left\{\frac{t}{4a^2}\right\}\left[1 + erf\left(\frac{\sqrt{t}}{2a}\right)\right].$$

Proof. If $w_0(t) \equiv 1$ and $w_1(t) \equiv 1$, then from (11) and (12) we obtain:

$$\nu(t) - \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)d\tau}{(t-\tau)^{1/2}} = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{t}{(t-\tau)^{3/2}} E(t,\tau)\varphi_1(\tau)d\tau, \tag{20}$$

and equality (19). Substituting (19) into (20) and following the work [3], we get the integral equation

$$\nu(t) - \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)d\tau}{(t-\tau)^{1/2}} = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{t+\tau}{(t-\tau)^{3/2}} E(t,\tau)\nu(\tau)d\tau, \tag{21}$$

which coincides with equation (18). From this the validity of the assertion of Lemma 2 follows. □

3.3 Nonhomogeneous Case of Problem (1): $w_0(t) \equiv w_1(t) \equiv \lambda \equiv 1/2$.

The following lemma is valid.

Lemma 3. *Let $\lambda = 1/2$. Then the system of equations (11) and (12) has solutions of the form*

$$\nu(t) = C_1 \left[\frac{1}{\sqrt{t}} \exp\left\{ \frac{t}{4a^2} \right\} + \frac{\sqrt{\pi}}{2a} \left(1 + erf\left(\frac{\sqrt{t}}{2a} \right) \right) \right] + \frac{C_2}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (2n+1) \exp\left\{ -\frac{(2n+1)^2}{4a^2} t \right\},\ 0 < t < \infty, \tag{22}$$

$$\varphi_1(t) = C_1 \left[\frac{1}{\sqrt{t}} \exp\left\{ \frac{t}{4a^2} \right\} + \frac{\sqrt{\pi}}{2a} \left(1 + erf\left(\frac{\sqrt{t}}{2a} \right) \right) \right] - \frac{C_2}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (2n+1) \exp\left\{ -\frac{(2n+1)^2}{4a^2} t \right\},\ 0 < t < \infty. \tag{23}$$

up to constant factors C_1 and C_2, and

$$\nu(t),\ \varphi_1(t) \in L_\infty\left(\mathbf{R}_+;\, t^{1/2} \exp\left\{ -\frac{t}{4a^2} \right\} \right).$$

Proof. We introduce the notations

$$\omega_+(t) = \nu(t) + \varphi_1(t), \quad \omega_-(t) = \nu(t) - \varphi_1(t),$$

and from equations (11) and (12) we obtain the following equations concerning the functions $\omega_+(t)$ and $\omega_-(t)$:

$$\omega_+(t) - \frac{1}{4a\sqrt{\pi}} \int\limits_0^t \frac{\omega_+(\tau)d\tau}{(t-\tau)^{1/2}} = \frac{1}{4a\sqrt{\pi}} \int\limits_0^t \frac{t+\tau}{(t-\tau)^{3/2}} E(t,\tau)\omega_+(\tau)d\tau, \tag{24}$$

$$\omega_-(t) - \frac{1}{4a\sqrt{\pi}} \int\limits_0^t \frac{\omega_-(\tau)d\tau}{(t-\tau)^{1/2}} = -\frac{1}{4a\sqrt{\pi}} \int\limits_0^t \frac{t+\tau}{(t-\tau)^{3/2}} E(t,\tau)\omega_-(\tau)d\tau, \tag{25}$$

where we remind
$$E(t,\tau) = \exp\left\{-\frac{t\tau}{4a^2(t-\tau)}\right\}.$$

Equation (24) has been solved by us earlier [1, 2, 6–8, 11] and this solution is equal to:

$$\omega_+(t) = \frac{C_1}{\sqrt{t}}\exp\left\{\frac{t}{4a^2}\right\} + \frac{\sqrt{\pi}}{2a}\left[1 + erf\left(\frac{\sqrt{t}}{2a}\right)\right],\ \ t>0,\ \ C_1 = \text{const}. \tag{26}$$

It remains to us to solve equation (25). For this purpose, in equation (25) we hold the following replacements of the independent variables:

$$t = 1/y,\ \ \tau = 1/x,\ \ d\tau = -d\,x/(x^2),\ \ t<\tau<\infty,\ \ y<x<\infty,$$

as a result we get (for $y>0$):

$$\omega_-(1/y) + \frac{1}{2a\sqrt{\pi}}\int\limits_y^\infty \frac{y^{1/2}}{(x-y)^{3/2}x^{1/2}}\cdot\exp\left\{-\frac{1}{4a^2(x-y)}\right\}\omega_-(1/x)d\,x$$
$$-\frac{1}{4a\sqrt{\pi}}\int\limits_y^\infty \frac{y^{1/2}}{(x-y)^{1/2}x^{3/2}}\left[1+\exp\left\{-\frac{1}{4a^2(x-y)}\right\}\right]\omega_-(1/x)d\,x = 0.$$

Hence, for a new unknown function $\psi(y) = y^{-3/2}\omega_-(1/y)$ we obtain:

$$y\cdot\psi(y) - \frac{1}{4a\sqrt{\pi}}\int\limits_0^y \frac{1}{(y-x)^{1/2}}\left[1+\exp\left\{-\frac{1}{4a^2(y-x)}\right\}\right]\psi(x)d\,x$$
$$+\frac{1}{2a\sqrt{\pi}}\int\limits_0^y \frac{1}{(y-x)^{3/2}}\exp\left\{-\frac{1}{4a^2(y-x)}\right\}x\,\psi(x)d\,x = 0,\ \ 0<y<\infty. \tag{27}$$

Applying the Laplace transform to equation (27), we get

$$-\frac{d\varPsi(p)}{d\,p} - \frac{1}{2a\sqrt{p}}\Big(1+\exp\Big(-\frac{2\sqrt{p}}{a}\Big)\Big)\varPsi(p) + \exp\left(-\frac{2\sqrt{p}}{a}\right)\frac{d\varPsi(p)}{dp} = 0,$$

i.e., we have:

$$\frac{d\varPsi(p)}{d\,p} + \frac{1}{2a\sqrt{p}}\cdot\frac{\operatorname{ch}\dfrac{\sqrt{p}}{a}}{\operatorname{sh}\dfrac{\sqrt{p}}{a}}\varPsi(p) = 0. \tag{28}$$

The general solution of the differential equation (28) is determined by the following formula:

$$\varPsi(p) = \frac{C_2}{\operatorname{sh}\dfrac{\sqrt{p}}{a}},\qquad C_2 = \text{const}. \tag{29}$$

To find the original of function (29), we rewrite it as a series:

$$\Psi(p) = 2C_2 \sum_{n=0}^{\infty} \exp\Big\{ - \frac{(2n+1)\sqrt{p}}{a} \Big\}. \tag{30}$$

Applying the inverse Laplace transform to (30), we will have:

$$\psi(y) = \frac{C_2}{a\sqrt{\pi}} \cdot \frac{1}{y^{3/2}} \sum_{n=0}^{\infty} (2n+1) \exp\Big\{ - \frac{(2n+1)^2}{4a^2\, y} \Big\},\ 0 < y < \infty. \tag{31}$$

Returning to the original variables, from equation (31) we obtain the solution of equation (25):

$$\omega_-(t) = \frac{C_2}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (2n+1) \exp\Big\{ - \frac{(2n+1)^2}{4a^2}\, t \Big\},\ 0 < t < \infty. \tag{32}$$

It is obvious that the solutions (26) and (32) belong to the space

$$\omega_+(t),\ \omega_-(t) \in L_\infty \left(\mathbf{R}_+;\ t^{1/2} \exp\left\{ -\frac{t}{4a^2} \right\} \right),$$

and from here the assertion of lemma 3 follows. □

3.4 Nonhomogeneous Case of Problem (1): $w_0(t) \equiv w_1(t) \equiv \lambda \in \mathbf{R}$, $\lambda \neq 1/2\}$

The following lemma is valid.

Lemma 4. *Let the boundary functions $w_0(t)$ and $w_1(t)$ be given as follows: $w_0(t) \equiv w_1(t) \equiv \lambda$, where $\lambda \in \mathbf{R} \setminus \{1/2\}$. Then for the solutions of equations (11) and (12) the following representations take place*

$$\nu(t) = \frac{\lambda\sqrt{\pi}}{a} \exp\left\{ \frac{\lambda^2 t}{4a^2} \right\} + \int_t^{\infty} \frac{d}{d\tau} \left[\frac{\lambda}{2a\sqrt{\pi}} \int_0^{\tau} \frac{(N_\lambda(\theta)\varphi_1)(\theta)}{\sqrt{\tau-\theta}}\, d\theta \right.$$
$$\left. + \ (N_\lambda(\tau)\varphi_1)(\tau) \right] \exp\left\{ -\frac{\lambda^2(\tau - t)}{4a^2} \right\} d\tau, \tag{33}$$

$$\varphi_1(t) = \frac{(1-\lambda)\sqrt{\pi}}{a} \exp\left\{ \frac{(1-\lambda)^2 t}{4a^2} \right\} + \int_t^{\infty} \frac{d}{d\tau} \left[\frac{1-\lambda}{2a\sqrt{\pi}} \int_0^{\tau} \frac{(\Phi_\lambda(\theta)\nu)(\theta)}{\sqrt{\tau-\theta}}\, d\theta \right.$$
$$\left. + \ (\Phi_\lambda(\tau)\nu)(\tau) \right] \exp\left\{ -\frac{(1-\lambda)^2(\tau - t)}{4a^2} \right\} d\tau. \tag{34}$$

Proof. Let λ be a finite real number which is not equal to the number 1/2. Then from (11) and (12) we have:

$$\nu(t) - \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)d\tau}{(t-\tau)^{1/2}} = \left(N_\lambda(t)\varphi_1\right)(t), \tag{35}$$

$$\varphi_1(t) - \frac{1-\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{\varphi_1(\tau)d\tau}{(t-\tau)^{1/2}} = \left(\Phi_\lambda(t)\nu\right)(t), \tag{36}$$

where

$$\left(N_\lambda(t)\varphi_1\right)(t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{\lambda t + (1-\lambda)\tau}{(t-\tau)^{3/2}} E(t,\tau)\varphi_1(\tau)d\tau, \tag{37}$$

$$\left(\Phi_\lambda(t)\nu\right)(t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{(1-\lambda)\,t + \lambda\tau}{(t-\tau)^{3/2}} E(t,\tau)\nu(\tau)d\tau. \tag{38}$$

We rewrite equation (35) in the form:

$$\frac{\lambda}{2a}\left(I_{0+}^{1/2}\nu\right)(t) \equiv \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)}{\sqrt{t-\tau}}\,d\tau = \nu(t) - \left(N_\lambda(t)\varphi_1\right)(t),\ t>0, \tag{39}$$

where the left expression is written by using the operator $I_{0+}^{1/2}$ of Riemann-Liouville fractional integration of the order 1/2 ([12], 38–39, 41–43, 84–86). Considering the right side of (39) temporarily known and applying the operator $\mathscr{D}_{0+}^{1/2}$ Riemann-Liouville fractional differentiation of the order 1/2 [12]:

$$(\mathscr{D}_{0+}^{1/2}\psi)(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int\limits_0^t \frac{\psi(\tau)d\tau}{\sqrt{t-\tau}},$$

we find as a solution to Abel equation of the first kind (39) ([12], 38–39, 50, 96, 105):

$$\frac{\lambda}{2a}\nu(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int\limits_0^t \frac{\nu(\tau)d\tau}{\sqrt{t-\tau}} - \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int\limits_0^t \frac{\left(N_\lambda(t)\varphi_1\right)(t)}{\sqrt{t-\tau}}\,d\tau. \tag{40}$$

Further, differentiating equation (35) once with respect to t, we obtain:

$$\frac{\lambda}{2a\sqrt{\pi}}\frac{d}{dt}\int\limits_0^t \frac{\nu(\tau)}{\sqrt{t-\tau}}d\tau = \nu'(t) - \frac{d}{dt}\left(N_\lambda(t)\varphi_1\right)(t),\ t>0. \tag{41}$$

Now multiplying equation (40) by $\lambda/(2a)$ and adding obtained left and right hand parts of equations (40) and (41), we obtain the differential equation:

$$\nu'(t) - \frac{\lambda^2}{4a^2}\,\nu(t) = \frac{d}{dt}\left[\frac{\lambda}{2a\sqrt{\pi}}\int_0^t \frac{(N_\lambda(\tau)\varphi_1)(\tau)}{\sqrt{t-\tau}}\,d\tau + (N_\lambda(t)\varphi_1)(t)\right], \tag{42}$$

with the finite condition:

$$\lim_{t\to\infty} \nu(t)\exp\left\{-\frac{\lambda^2 t}{4a^2}\right\} = \frac{\lambda\sqrt{\pi}}{a}. \tag{43}$$

The solution of the problem (42)–(43) has the form (33).

Similarly, as for equation (35), for integral equation (36) we have (34).

Proof of lemma 4 is completed by this fact. □

3.5 Properties of the Operator $N_\lambda(t)$ (37)

The following lemma is valid.

Lemma 5. *The operator* $N_\lambda(t)$ *(37) boundedly acts on the function* $\varphi_1(t) = \frac{1}{\sqrt{t}}$ *with a value in the space* $L_\infty(\mathbf{R}_+;\, t^{1/2})$.

Proof. Let $\varphi_1(t) = \frac{1}{\sqrt{t}}$. To solve equation (20), we at first calculate its right-hand side $(N_\lambda(t)\varphi_1)(t)$. We get

$$\begin{aligned}
(N_\lambda(t)\varphi_1)(t) &= \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\lambda t + (1-\lambda)\tau}{\tau^{1/2}(t-\tau)^{3/2}}\exp\ -\frac{t\tau}{4a^2(t-\tau)}\ \ d\tau \\
&= \ z = \frac{t}{\sqrt{t-\tau}},\ \tau = \frac{t(z^2-t)}{z^2},\ t-\tau = \frac{t^2}{z^2},\ d\tau = \frac{2t^2dz}{z^3} \\
&= \frac{1}{2a\sqrt{\pi}}\int_{\sqrt{t}}^\infty \frac{\lambda t + (1-\lambda)\frac{t(z^2-t)}{z^2}\ \ z\,z^3 2t^2}{t^{1/2}(z^2-t)^{1/2}t^3z^3}\exp\ -\frac{t\frac{t(z^2-t)}{z^2}z^2}{4a^2t^2}\ \ dz \\
&= \frac{1}{a\sqrt{\pi t}}\int_{\sqrt{t}}^\infty \frac{z^2-t+\lambda t}{z(z^2-t)^{1/2}}\exp\ -\frac{z^2-t}{4a^2}\ \ dz \\
&= \frac{1}{a\sqrt{\pi}}\int_{\sqrt{t}}^\infty \frac{\frac{z}{\sqrt{t}}^{\ 2} - 1 + \lambda}{\frac{z}{\sqrt{t}}^{\ 2}}\exp\ -t\frac{\frac{z}{\sqrt{t}}^{\ 2}-1}{4a^2}\ \ d\ \ \frac{z}{\sqrt{t}}^{\ 2} - 1^{\ 1/2} \\
&= \ \zeta^2 = \ \frac{z}{\sqrt{t}}^{\ 2} - 1,\ y = \frac{\sqrt{t}\zeta}{2a} \\
&= \frac{1}{\sqrt{t}}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\zeta^2+\lambda}{1+\zeta^2}\exp\ -\frac{t\zeta^2}{4a^2}\ \ d\ \ \frac{\sqrt{t}\zeta}{2a} \\
&= \frac{1}{\sqrt{t}}\ \ 1 - \frac{(1-\lambda)t}{2a^2\sqrt{\pi}}\int_0^\infty \frac{1}{\frac{t}{4a^2}+y^2}\exp\ -y^2\ \ dy\ \ .
\end{aligned}$$

From here, using formula

$$\int_0^\infty \frac{\exp\{-\mu^2 x^2\}}{x^2+\beta^2} dx = \operatorname{erfc}(\beta\mu) \frac{\pi}{2\beta} \exp\{\beta^2\mu^2\}, \ \operatorname{Re}\beta > 0, \ |\arg\mu < \frac{\pi}{4}$$

from ([10], 3.466.1), we obtain

$$(N_\lambda(t)\varphi_1)(t) = \frac{1}{\sqrt{t}} - \frac{(1-\lambda)\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right) \varphi_1(t) = \frac{1}{\sqrt{t}}. \quad (44)$$

Thus, we have

$$(N_\lambda(t)\varphi_1)(t)\|_{\varphi_1(t)=t^{-1/2}} = $$
$$= \frac{1}{\sqrt{t}} - \frac{(1-\lambda)\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right) \in L_\infty(\mathbf{R}_+;\, t^{1/2}),$$

i.e.

$$N_\lambda(t) \in \mathscr{L}\left(L_\infty\left(\mathbf{R}_+,\, t^{1/2}\right)\right).$$

Lemma 5 is proved. □

From the proof of lemma 5 it follows that when $\varphi_1(t) = t^{-1/2}$ equation (20) takes the form:

$$\nu(t) - \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\nu(\tau)d\tau}{(t-\tau)^{1/2}} = \frac{1}{\sqrt{t}} - \frac{(1-\lambda)\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right). \quad (45)$$

3.6 Properties of the Solution of Equation (45)

For the solution of equation (45) the following lemma is valid.

Lemma 6. *The solution of equation (45) can be represented in the form* $\nu(t) = \nu_1(t) + \nu_2(t)$, *and,*

$$\nu_1(t) \in L_\infty\left(\mathbf{R}_+;\, t^{1/2} \exp\left\{-\frac{(\lambda^2+\varepsilon)t}{4a^2}\right\}\right), \ \varepsilon > 0, \quad (46)$$

$$\nu_2(t) \in L_\infty\left(\mathbf{R}_+;\, \exp\left\{-\frac{\lambda^2 t}{4a^2}\right\}\right), \quad (47)$$

$$\nu(t) \in L_\infty\left(\mathbf{R}_+;\, t^{1/2} \exp\left\{-\frac{(\lambda^2+\varepsilon)t}{4a^2}\right\}\right) \oplus L_\infty\left(\mathbf{R}_+;\, \exp\left\{-\frac{\lambda^2 t}{4a^2}\right\}\right). \quad (48)$$

Proof. We will look for the solution of equation (45) as a sum $\nu(t) = \nu_1(t) + \nu_2(t)$, where if $\lambda > 0,\ \lambda \neq 1/2$, then

$$\nu_1(t) = \frac{1}{\sqrt{t}} + \frac{\lambda\sqrt{\pi}}{4a} \exp\left\{\frac{\lambda^2\, t}{4a^2}\right\} \left[1 + erf\left(\frac{\lambda\sqrt{t}}{2a}\right)\right], \tag{49}$$

$$\nu_2(t) = \frac{\lambda\sqrt{\pi}}{2a} \exp\left\{\frac{\lambda^2 t}{4a^2}\right\} + \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{g(\tau)d\tau}{(t-\tau)^{1/2}} + g(t)$$
$$-\frac{\lambda^2}{4a^2} \int\limits_t^\infty \exp\left\{-\frac{\lambda^2(\tau - t)}{4a^2}\right\} \left[\frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^\tau \frac{g(\tau_1)d\tau_1}{(\tau-\tau_1)^{1/2}} + g(\tau)\right] d\tau; \tag{50}$$

if $\lambda < 0$, then

$$\nu_1(t) = \frac{1}{\sqrt{t}} - \frac{-\lambda\sqrt{\pi}}{4a} \exp\left\{\frac{\lambda^2\, t}{4a^2}\right\} erf\left(\frac{-\lambda\sqrt{t}}{2a}\right), \tag{51}$$

$$\nu_2(t) = -\frac{-\lambda\sqrt{\pi}}{a} \exp\left\{\frac{\lambda^2 t}{4a^2}\right\} - \frac{-\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{g(\tau)d\tau}{(t-\tau)^{1/2}} + g(t)$$
$$-\frac{\lambda^2}{4a^2} \int\limits_t^\infty \exp\left\{-\frac{\lambda^2(\tau - t)}{4a^2}\right\} \left[-\frac{-\lambda}{2a\sqrt{\pi}} \int\limits_0^\tau \frac{g(\tau_1)d\tau_1}{(\tau-\tau_1)^{1/2}} + g(\tau)\right] d\tau; \tag{52}$$

are solutions of the following two integral equations:

$$\nu_1(t) - \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu_1(\tau)d\,\tau}{(t-\tau)^{1/2}} = \frac{1}{\sqrt{t}}, \tag{53}$$

$$\nu_2(t) - \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{\nu_2(\tau)d\,\tau}{(t-\tau)^{1/2}} = g(t), \tag{54}$$

where

$$g(t) \equiv -\frac{(1-\lambda)\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right). \tag{55}$$

If $\lambda = 0$, then from (45) we obtain the solution $\nu(t)$:

$$\nu(t) = \frac{1}{\sqrt{t}} - \frac{\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right), \tag{56}$$

which belongs to the class $L_\infty\left(\mathbf{R}_+;\, t^{1/2}\right)$.

Upon receipt of solutions (49), (51) and (50), (52) of equations (53) and (54), respectively, we use the formula (33), and we apply the formula (43) in the form:

$$\lim_{t\to+\infty} \nu_j(t) \exp\left\{-\frac{\lambda^2 t}{4a^2}\right\} = \frac{\lambda\sqrt{\pi}}{2a},\ \ j = 1, 2,\ \ \lambda > 0,\ \lambda \neq 1/2;$$

$$\lim_{t\to+\infty} \nu_1(t)\exp\left\{-\frac{\lambda^2 t}{4a^2}\right\} = 0, \quad \lim_{t\to+\infty} \nu_2(t)\exp\left\{-\frac{\lambda^2 t}{4a^2}\right\} = \frac{\lambda\sqrt{\pi}}{a}, \quad \lambda < 0;$$

i.e.

$$\lim_{t\to+\infty} \nu(t)\exp\left\{-\frac{\lambda^2 t}{4a^2}\right\} = \lim_{t\to+\infty} [\nu_1(t)+\nu_2(t)]\exp\left\{-\frac{\lambda^2 t}{4a^2}\right\} = \frac{\lambda\sqrt{\pi}}{a}. \qquad (57)$$

Let us remind that we have considered the case $\lambda = 1/2$ separately.

Now we consider solution $\nu_2(t)$ (50) and show that the following difference

$$\nu_2(t) - \frac{\lambda\sqrt{\pi}}{2a}\exp\left\{\frac{\lambda^2 t}{4a^2}\right\} = I_1(t) - I_2(t)$$

is a bounded and continuous function on the semiaxis $(0,\infty)$.

For this purpose, we carry out further calculations of the integrals in formula (50) for the solution $\nu_2(t)$. We have

$$\begin{aligned} I_1(t) &= \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{g(\tau)d\tau}{(t-\tau)^{1/2}} \\ &= \frac{\lambda(\lambda-1)}{2a^2\sqrt{\pi}} \int_0^t \frac{\exp\left\{\frac{\tau}{4a^2}\right\}}{(t-\tau)^{1/2}} \int_{\sqrt{\tau}/(2a)}^{\infty} \exp\left\{-z^2\right\} dz\, d\tau \\ &= \frac{\lambda(\lambda-1)}{8a^3\sqrt{\pi}} \int_0^t \frac{\exp\left\{\frac{\tau}{4a^2}\right\}}{(t-\tau)^{1/2}} \int_\tau^\infty \frac{\exp\left\{-\frac{\theta}{4a^2}\right\} d\theta}{\sqrt{\theta}}\, d\tau = I_{11}(t) + I_{12}(t), \end{aligned}$$

where

$$I_{11}(t) = \frac{\lambda(\lambda-1)}{8a^3\sqrt{\pi}} \int_0^t \frac{\exp\left\{-\frac{\theta}{4a^2}\right\}}{\sqrt{\theta}} \int_0^\theta \frac{\exp\left\{\frac{\tau}{4a^2}\right\} d\tau}{(t-\tau)^{1/2}} d\theta,$$

$$I_{12}(t) = \frac{\lambda(\lambda-1)}{8a^3\sqrt{\pi}} \int_t^\infty \frac{\exp\left\{-\frac{\theta}{4a^2}\right\}}{\sqrt{\theta}} \int_0^t \frac{\exp\left\{\frac{\tau}{4a^2}\right\} d\tau}{(t-\tau)^{1/2}} d\theta.$$

Further, for $I_{11}(t)$ we have

$$\begin{aligned} I_{11}(t) &= \frac{\lambda(\lambda-1)}{2a^2\sqrt{\pi}} \exp\left\{\frac{t}{4a^2}\right\} \int_0^t \frac{\exp\left\{-\frac{\theta}{4a^2}\right\}}{\sqrt{\theta}} \int_{\sqrt{t-\theta}/(2a)}^{\sqrt{t}/(2a)} \exp\left\{-\zeta^2\right\} d\zeta\, d\theta \\ &= \frac{\lambda(\lambda-1)}{4a^2} \exp\left\{\frac{t}{4a^2}\right\} \int_0^t \frac{\exp\left\{-\frac{\theta}{4a^2}\right\}}{\sqrt{\theta}} \left[erf\left(\frac{\sqrt{t}}{2a}\right) - erf\left(\frac{\sqrt{t-\theta}}{2a}\right)\right] d\theta. \end{aligned}$$

Note that $I_{11}(t)$ is a bounded and continuous function on the semiaxis $(0,\infty)$. It is enough to prove this assertion for large values t and $t-\theta$; and also for large values t and for small values $t-\theta$. Indeed, since in the first case, the following relationships are valid:

$$erf\left(\frac{\sqrt{t}}{2a}\right) - erf\left(\frac{\sqrt{t-\theta}}{2a}\right) \approx \operatorname{erfc}\left(\frac{\sqrt{t-\theta}}{2a}\right) \approx \frac{2a}{\sqrt{\pi}\sqrt{t-\theta}} \exp\left\{-\frac{t-\theta}{4a^2}\right\},$$

then for $I_{11}(t)$ we get:

$$I_{11}(t) \approx \frac{\lambda(\lambda-1)}{2a\sqrt{\pi}} \int\limits_0^t \frac{d\theta}{\sqrt{\theta(t-\theta)}} = \frac{\lambda(\lambda-1)\sqrt{\pi}}{2a}.$$

Similarly, it is shown that the above statement is true for large t and small $t-\theta$.

Due to the fact that the right side of integral equation (54) (see (55)) is a bounded and continuous function on a semiaxis $(0,\infty)$, then the expression

$$G(t) = \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{g(\tau)d\tau}{(t-\tau)^{1/2}} + g(t)$$

is the bounded and continuous function on a semiaxis $(0,\infty)$. Consequently, the following integral

$$I_2(t) = \frac{\lambda^2}{4a^2} \int\limits_t^\infty \exp\left\{-\frac{\lambda^2(\tau-t)}{4a^2}\right\} \left[\frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^\tau \frac{g(\tau_1)d\tau_1}{(\tau-\tau_1)^{1/2}} + g(\tau)\right] d\tau$$

will also be the bounded and continuous function of t on the semiaxis $(0,\infty)$.

Similar calculations are also valid for solution (52).

This completes the proof of lemma 6. □

3.7 Estimation of the Norm for Operator $N_\lambda(t)$ (37)

We show the the validity of the following lemma.

Lemma 7. *The operator $N_\lambda(t)$ has property of linearity and boundedness in the space $L_\infty\left(\mathbf{R}_+;\, t^{1/2}\right)$, i.e.*

$$N_\lambda(t) \in \mathscr{L}\left(L_\infty\left(\mathbf{R}_+;\, t^{1/2}\right)\right).$$

Proof. We calculate estimation of the norm for the operator $N_\lambda(t)$ in the space $L_\infty\left(\mathbf{R}_+; t^{1/2}\right)$. We have

$$|\left(N_\lambda(t)\varphi_1\right)(t)| \le C_1 \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{|\lambda t + (1-\lambda)\tau|}{\tau^{1/2}(t-\tau)^{3/2}} E(t,\tau)|\sqrt{\tau}\varphi_1(\tau)|d\tau. \tag{58}$$

Let $\|\varphi_1(t)\|_{L_\infty\left(\mathbf{R}_+;\, t^{1/2}\right)} = 1$. Then from (58) we obtain

$$|\left(N_\lambda(t)\varphi_1\right)(t)| \le \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{|\lambda t + (1-\lambda)\tau|}{\tau^{1/2}(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{4a^2(t-\tau)}\right\} d\,\tau$$

$$\left\| z = \frac{t}{\sqrt{t-\tau}},\ \tau = \frac{t(z^2-t)}{z^2},\ t-\tau = \frac{t^2}{z^2},\ d\tau = \frac{2t^2 dz}{z^3} \right\|$$

$$= \frac{1}{2a\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} \frac{\left|\lambda t + (1-\lambda)\frac{t(z^2-t)}{z^2}\right| z\, z^3 2t^2}{t^{1/2}(z^2-t)^{1/2} t^3 z^3} \exp\left\{-\frac{t\frac{t(z^2-t)}{z^2} z^2}{4a^2t^2}\right\} dz$$

$$= \frac{1}{a\sqrt{\pi t}} \int_{\sqrt{t}}^{\infty} \frac{|z^2 - t + \lambda t|}{z(z^2-t)^{1/2}} \exp\left\{-\frac{z^2-t}{4a^2}\right\} d\,z$$

$$= \frac{1}{a\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} \frac{\left|\left(\frac{z}{\sqrt{t}}\right)^2 - 1 + \lambda\right|}{\left(\frac{z}{\sqrt{t}}\right)^2} \exp\left\{-t\frac{\left(\frac{z}{\sqrt{t}}\right)^2 - 1}{4a^2}\right\}$$

$$\times\, d\left[\left(\frac{z}{\sqrt{t}}\right)^2 - 1\right]^{1/2}$$

$$\left\| \zeta^2 = \left(\frac{z}{\sqrt{t}}\right)^2 - 1,\ y = \frac{\sqrt{t}\zeta}{2a} \right\|$$

$$= \frac{1}{\sqrt{t}} \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\left|\zeta^2 + \lambda\right|}{1+\zeta^2} \exp\left\{-\frac{t\zeta^2}{4a^2}\right\} d\left(\frac{\sqrt{t}\zeta}{2a}\right)$$

$$= \frac{1}{\sqrt{t}} \left| 1 + \frac{(\lambda-1)t}{2a^2\sqrt{\pi}} \int_0^{\infty} \frac{1}{\frac{t}{4a^2} + y^2} \exp\left\{-y^2\right\} d\,y \right|.$$

From here using the following formula

$$\int_0^{\infty} \frac{\exp\{-\mu^2 x^2\}}{x^2+\beta^2} d\,x = \operatorname{erfc}(\beta\mu)\frac{\pi}{2\beta}\exp\{\beta^2\mu^2\},\ \operatorname{Re}\beta > 0,\ |\arg\mu < \frac{\pi}{4}.$$

from ([10], 3.466.1), we get

$$|\left(N_\lambda(t)\varphi_1\right)(t)| \le \left| \frac{1}{\sqrt{t}} + \frac{(\lambda-1)\sqrt{\pi}}{2a} \exp\left\{\frac{t}{4a^2}\right\} \operatorname{erfc}\left(\frac{\sqrt{t}}{2a}\right) \right|. \tag{59}$$

It is obvious that function (59) belongs to the class $L_\infty\left(\mathbf{R}_+;\, t^{1/2}\right)$, i.e.

$$N_\lambda(t) \in \mathscr{L}\left(L_\infty\left(\mathbf{R}_+;\, t^{1/2}\right)\right).$$

Lemma 7 is proved. □

4 The Main Result

The assertions of lemmas in section 3 allow us to formulate the following theorems.

Theorem 1. *Let the conditions of Lemma 1 (or Lemma 2) be satisfied. Then the boundary value problem (1) for Burgers equation has only one solution.*

Proof. According to Lemma 1 (or Lemma 2) the solution to boundary problem (3) by the formula (9) can be found up to a constant factor. However, this constant does not affect the solution to problem (1) according to Hopf-Cole transformation (2). Theorem 1 is proved. □

Theorem 2. *Let the conditions of Lemma 3 be satisfied. Then the boundary value problem (1) for Burgers equation has an infinite number of solutions.*

Proof. According to Lemma 3 constants C_1 and C_2 are given independently of each other. Therefore, in the general case, the solution to boundary value problem (3) is defined by two constants C_1 and C_2, which under transformations of the Hopf-Cole (2) do not cancel. From this the statement of Theorem 2 follows. □

Theorem 3. *Let the conditions of Lemma 4 be satisfied. Then the boundary value problem (1) for Burgers equation has only one solution.*

Proof. According to Lemma 4 the solution to boundary problem (3) by the formula (9) can be found up to a constant factor. However, this constant does not affect the solution to problem (1) according to the transformation of the Hopf-Cole (2). Theorem 3 is proved. □

5 Solving the System of Integral Equations (11)–(12). The Case of Variable Coefficients $w_0(t)$ and $w_1(t)$

We assume that the variable coefficients $w_0(t)$ and $w_1(t)$ satisfy the following conditions:

$$\frac{\sqrt{\pi}\, t^{1/2}|w_0(t)|}{2a} < 1,\ t^{1/2}\,|w_0(t)| < 1,$$

$$\frac{\sqrt{\pi}\, t^{1/2}|w_1(t)-1|}{2a} < 1,\ t^{1/2}\,|w_1(t)-1| < 1, \tag{60}$$

$$w_0(t) + w_1(t) \not\equiv 1\ \forall t \in \mathbf{R}_+,\ \lim_{t\to 0+} t^{1/2}\, w_0(t) = \lim_{t\to 0+} t^{1/2}\,[w_1(t) - 1]. \tag{61}$$

We will look for solutions of the integral equations (11)–(12) in a weight class of functions:

$$t^{1/2}\,\nu(t) \in L_\infty(\mathbf{R}_+);\ t^{1/2}\,\varphi_1 \in L_\infty(\mathbf{R}_+),$$

$$\text{i.e. } \nu(t),\ \varphi_1(t) \in L_\infty(\mathbf{R}_+;\, t^{1/2}). \tag{62}$$

We write equations (11)–(12) in the operator form. We have:

$$(I - A)\ \nu = N\varphi_1,\ \ (I - B)\ \varphi_1 = \Phi\nu, \tag{63}$$

where I is the identity operator, and the operators A and B act in the space $L_\infty(\mathbf{R}_+;\, t^{1/2})$, and, the following lemmas 8 and 9 are valid.

Lemma 8. *Let the conditions (60) be satisfied. Then the operators A and B have the properties of linearity and continuity:*

$$A \in \mathscr{L}\left(L_\infty(\mathbf{R}_+;\, t^{1/2})\right),\ B \in \mathscr{L}\left(L_\infty(\mathbf{R}_+;\, t^{1/2})\right), \tag{64}$$

and for their norms the estimates:

$$\|A\| < 1,\ \|B\| < 1 \tag{65}$$

are valid.

Lemma 9. *For the right parts of equations (11)–(12) we have the estimates*

$$(N(t)\varphi_1)(t) < t^{-1/2},\ \ (\Phi(t)\nu)(t) < t^{-1/2}$$
$$\text{for } \varphi_1(t),\, \nu(t) \in L_\infty(\mathbf{R}_+;\, t^{1/2}). \tag{66}$$

Moreover, the operators of the right sides N and Φ from (63), as well as the operators A and B, act in the space $L_\infty(\mathbf{R}_+,\, t^{1/2})$, and have the properties of linearity and continuity:

$$N \in \mathscr{L}\left(L_\infty(\mathbf{R}_+;\, t^{1/2})\right),\ \Phi \in \mathscr{L}\left(L_\infty(\mathbf{R}_+;\, t^{1/2})\right), \tag{67}$$

also for their norms the estimates:

$$\|N\| < 1,\ \|\Phi\| < 1 \tag{68}$$

are valid.

Proof. For this it is sufficient to establish the first estimate from (66). The second estimate from (66) is established similarly. We have

$$(N(t)\varphi_1)(t) < \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \left[\frac{\tau^{1/2}}{(t-\tau)^{3/2}} + \frac{1}{\sqrt{\tau(t-\tau)}}\right] E(t,\,\tau)\,d\tau$$
$$= \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{t}{\sqrt{\tau}(t-\tau)^{3/2}} \exp\left\{-\frac{t\,\tau}{4a^2(t-\tau)}\right\} d\tau = \left\| z = \frac{t}{\sqrt{t-\tau}} \right\|$$
$$= \frac{1}{a\sqrt{\pi}} \int\limits_{\sqrt{t}}^\infty \exp\left\{-\frac{t}{4a^2}\left[\left(\frac{z}{\sqrt{t}}\right)^2 - 1\right]\right\} d\left[\left(\frac{z}{\sqrt{t}}\right)^2 - 1\right]^{1/2}$$
$$\left\|\zeta^2 = \left(\frac{z}{\sqrt{t}}\right)^2 - 1\right\|$$
$$= \frac{1}{a\sqrt{\pi}} \int\limits_0^\infty \exp\left\{-\frac{t\zeta^2}{4a^2}\right\} d\zeta = \frac{1}{\sqrt{t}}\,.$$

From here assertion (66) as well as assertions (67) and (68) of lemma 9 follow immediately. □

From assertions of lemmas 8 and 9 the validity of next lemma follows.

Lemma 10. *Let the conditions of lemmas 8 and 9 be satisfied. Then the operators $I - A$ and $I - B$ are continuously invertible, i.e. we have*

$$\nu(t) = (I - A)^{-1}(N\varphi_1)(t), \ \ \varphi_1(t) = (I - B)^{-1}(\Phi\nu)(t). \tag{69}$$

The system of the integral equations (11)–(12) is split into the following two integral equations

$$\nu(t) = (I - A)^{-1}(N(I - B)^{-1}(\Phi\nu)(t), \tag{70}$$

$$\varphi_1(t) = (I - B)^{-1}(\Phi(I - A)^{-1}(N\varphi_1)(t). \tag{71}$$

Remark 1. The system of the integral equations (70)–(71) has always the trivial solution. However, as it is shown above, at some given constant values of the functions $w_0(t)$ and $w_1(t)$ this system can have together with the trivial solution and non-trivial solutions.

Remark 2. The solvability of the system of the integral equations (70)–(71), and, respectively, and equations (11)–(12) should be investigated further.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan. This research is financially supported by a grants from the Ministry of Science and Education of the Republic of Kazakhstan (Grants No. 0823/GF4 and 1164/GF4).

References

1. Akhmanova, D.M., Dzhenaliev, M.T., Ramazanov, M.I.: On a particular second kind Volterra integral equation with a spectral parameter. Sib. Math. J. **52**, 1 (2011)
2. Akhmanova, D.M., Amangalieva, M.M., Dzhenaliev, M.T., Ramazanov, M.I.: Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. Sib. Math. J. **47**, 2 (2011)
3. Amangalieva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., Ramazanov, M.I.: On one homogeneous problem for the heat equation in an infinite angular domain. Sib. Math. J. **56**, 6 (2015)
4. Benia, Y., Sadallah, B.-K.: Existence of solutions to Burgers equations in domains that can be transformed into rectangles. Electron. J. Differ. Equ. **2016**, 157 (2016)
5. Burgers, J.M.: The nonlinear diffusion equation. In: Asymptotic Solutions and Statistical Problems. D. Reidel Publishing Company, Dordrecht–Holland, Boston (1974)
6. Dzhenaliev, M.T., Ramazanov, M.I.: On the boundary value problem for the spectrally loaded heat conduction operator. Sib. Math. J. **47**, 3 (2006)

7. Dzhenaliev, M.T., Ramazanov, M.I.: On a boundary value problem for a spectrally loaded heat operator. I. Differ. Equ. **43**(4) (2007)
8. Dzhenaliev, M.T., Ramazanov, M.I.: On a boundary value problem for a spectrally loaded heat operator. II. Differ. Equ. **43**(6) (2007)
9. Fursikov, A.V., Vishik, M.I.: Mathematical Problems of Statistical Thermodynamics. Nauka, Moscow (1980) (in Russian)
10. Gradshtein, I.S., Ryzhik, I.M.: Tables of Integrals, Sums, Series and Products. Fizmatgiz, Moscow (1971) (in Russian)
11. Jenaliyev, M.T., Ramazanov, M.I.: The Loaded Equations as Perturbations of Differential Equations. Gylym, Almaty (2010) (in Russian)
12. Kilbas, A.A., Marichev, O.I., Samko, S.G.: Integrals and Derivatives of Fractional Order, and Some of Their Applications. Nauka i Tekhnika, Minsk (1987) (in Russian)
13. Samarskii, A.A., Tikhonov, A.N.: Equations of Mathematical Physics. Nauka, Moscow (1977) (in Russian)

On Algorithm of Finding Solutions of Semiperiodical Boundary Value Problem for Systems of Nonlinear Hyperbolic Equations

Symbat S. Kabdrakhova[1,2](✉)

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
S_Kabdrachova@mail.ru

[2] Al-Farabi Kazakh National University, 71 al-Farabi avenue, Almaty 050010, Kazakhstan

Abstract. In this paper we consider a semi-periodical boundary value problem for a system of nonlinear hyperbolic equations in a rectangular domain. An algorithm for finding of approximate solution to the semi-periodical boundary value problem for the systems of nonlinear hyperbolic equation is offered. Conditions for the convergence of the approximate solutions to the exact solution of the semi-periodical boundary value problem for the system of nonlinear hyperbolic equations are established.

Keywords: Systems of nonlinear hyperbolic equation · Semi-periodical boundary value problem · Modification of Euler method · Family of periodic boundary value problems · Parametrization method · Approximate solution

1 Introduction

The boundary value problems with nonlocal conditions for hyperbolic equations are studied by A.M. Nakhushev [16,17], Yu. A. Mitropolsky and L.B. Urmancheva [15], S.S. Kharibegashivili [13,14], T.I. Kiguradze [8–12], L.S. Pulkina [6] and others. The theory of nonlocal boundary value problems for system of hyperbolic equations with mixed derivatives are also developed in the works of D.S. Dzhumabaev, A.T. Asanova and their disciples. A.T. Asanova [1,2] elaborated a method of introducing functional parameters for a study of nonlocal problems for the system of hyperbolic equations with mixed derivatives. This method is based on the parametrization method proposed by D.S. Dzhumabaev [3] for investigating and solving of boundary value problems for system of ordinary differential equations.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_13

In this paper we consider a semi-periodic boundary value problem for systems of hyperbolic equations. The algorithm for finding of classical solutions to the semi-periodical boundary value problem for the system of hyperbolic equations with mixed derivatives is constructed and the coefficient criterions of unique solvability to considered problem are obtained. The approximate solutions of semi-periodical boundary value problem for the system of hyperbolic equations are constructed by the method of Euler's modification [7] and the method of functional parametrization [1,2].

2 Formulation of the Problem

At the domain $\bar{\Omega} = [0, \omega] \times [0, T]$ we consider the following semi-periodical boundary value problem for the system of hyperbolic equations with two independent variables

$$\frac{\partial^2 u}{\partial x \partial t} = f(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}), \quad (x, t) \in \overline{\Omega}, \quad u \in R^n, \tag{1}$$

$$u(x, 0) = u(x, T), \ x \in [0, \omega] \tag{2}$$

$$u(0, t) = \psi(t), \quad t \in [0, T], \tag{3}$$

where the n vector function $f : \overline{\Omega} \times R^{3n} \to R^n$ is continuous on $\overline{\Omega}$, the n vector function $\psi(t)$ is continuous differentiable on $[0, T]$, and they satisfy compatibility condition: $\psi(0) = \psi(T)$.

We denote by $C(\overline{\Omega}, R^n)$ the space of functions continuous on $\overline{\Omega}$ $u : \overline{\Omega} \to R$ with the norm $\|u(x, \cdot)\|_1 = \max_{t \in [0,T]} ||u(x, t)|| = \max_{t \in [0,T]} \max_{i=\overline{1,n}} |u_i(x, t)|$.

Definition 1. A function $u(x, t) \in C(\overline{\Omega}, R^n)$, having partial derivatives $\frac{\partial u(x,t)}{\partial x} \in \in C(\overline{\Omega}, R^n)$, $\frac{\partial u(x,t)}{\partial t} \in C(\overline{\Omega}, R^n)$, $\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\overline{\Omega}, R^n)$ is called a solution to problem (1)-(3) if it satisfies system (1) for all $(x, t) \in \overline{\Omega}$ and boundary conditions (2), (3).

2.1 A Family of Periodic Boundary Value Problems. Method of Euler's Modification. Reduction to an Equivalent Problem

We introduce new unknown functions $v(x, t) = \frac{\partial u(x,t)}{\partial x}$, $w(x, t) = \frac{\partial u(x,t)}{\partial t}$. Then we reduce problem (1)-(3) to the equivalent problem:

$$\frac{\partial v}{\partial t} = f(x, t, u(x, t), w(x, t), v), \ (x, t) \in \overline{\Omega}, \tag{4}$$

$$v(x, 0) = v(x, T), \ x \in [0, \omega], \tag{5}$$

$$u(x, t) = \psi(t) + \int_0^x v(\xi, t) d\xi, \quad w(x, t) = \dot{\psi}(t) + \int_0^x v_t(\xi, t) d\xi. \tag{6}$$

Definition 2. A triple of functions $\{u(x,t), w(x,t), v(x,t)\}$ continuous in $\bar{\Omega}$ is called a solution of problem (4)-(6), if the function $v(x,t) \in C(\bar{\Omega}, R^n)$ is continuously differentiable in $\bar{\Omega}$ with respect to t and satisfies the family of periodical boundary value problem (4), (5), where functions $u(x,t)$ and $w(x,t)$ are connected with $v(x,t)$, $\dfrac{\partial v(x,t)}{\partial t}$ by functional relations (6).

The problems (1)-(3) and (4)-(6) are equivalent in that sense: if the function $u(x,t)$ is the solution of problem (1)-(3), then the triple of functions $\{u(x,t), w(x,t),$
$v(x,t)\}$ is a solution of problem (4)-(6), and vice versa. If triple of functions $\{u^*(x,t), w^*(x,t), v^*(x,t)\}$ is the solution of problem (4)-(6), then the function $u^*(x,t)$ is the solution of problem (1)-(3).

For the problem (4)-(6) we apply the parametrization method [18]. We take a step $h_1 > 0 : N_1 h_1 = T$, $N_1 = 1, 2, 3, ...$, and make the partition of interval $[0,T) = \bigcup\limits_{r=1}^{N_1} [(r-1)h_1, rh_1)$. By $v_r(x,t)$ denote the restriction of $v(x,t)$ on $[(r-1)h_1, rh_1)$ such that $v_r : [(r-1)h_1, rh_1) \to R^n$ and $v_r(x,t) = v(x,t)$ for all $(x,t) \in \Omega_r = [0,\omega] \times [(r-1)h_1, rh_1)$ and $r = \overline{1, N_1}$.

By $\lambda_r(x)$ we denote the value of $v(x,t)$ under $t = (r-1)h_1$, $r = \overline{1, N_1}$ and make replacement $\widetilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x)$ on each interval $[(r-1)h_1, rh_1)$. Then problem (4), (5) is reduced to equivalent boundary value problem with parameters

$$\frac{\partial \widetilde{v}_r}{\partial t} = f(x,t,u(x,t), w(x,t), \widetilde{v}_r + \lambda_r(x)),\ (x,t) \in \Omega_r, \tag{7}$$

$$\widetilde{v}_r(x,(r-1)h_1) = 0,\ x \in [0,\omega] \tag{8}$$

$$\lambda_1(x) - \lim_{t \to T-0} \widetilde{v}_{N_1}(x,t) - \lambda_{N_1}(x) = 0, \tag{9}$$

$$\lambda_s(x) + \lim_{t \to sh_1 - 0} \widetilde{v}_s(x,t) - \lambda_{s+1}(x) = 0,\ s = \overline{1, N_1 - 1}. \tag{10}$$

At fixed $\lambda_r(x)$ the Cauchy problem (7)-(8) is equivalent to the family of system of Volterra integral equations of the second kind on intervals of length $h_1 > 0$

$$\widetilde{v}_r(x,t) = \int_{(r-1)h_1}^{t} f(x,\tau,u(x,\tau), w(x,\tau), \widetilde{v}_r(x,\tau) + \lambda_r(x)) d\tau, \tag{11}$$

$(x,t) \in \Omega_r$, $r = \overline{1, N_1}$.

Substituting $\widetilde{v}_r(x,\tau)$ in the right-hand side (11) and repeating the process ν times, $(\nu = 1, 2, ...)$, we obtain for the function $\widetilde{v}_r(x,t)$ the following expression

$$\widetilde{v}_r(x,t) = \int_{(r-1)h_1}^{t} f\Big(x, \tau_1, u(x,\tau_1), w(x,\tau_1), \lambda_r(x) + \int_{(r-1)h_1}^{\tau_1} f\Big(x, \tau_2, u(x,\tau_2),$$

$$w(x,\tau_2), \lambda_r(x) + ... + \int_{(r-1)h_1}^{\tau_{\nu-1}} f(x, \tau_\nu, u(x,\tau_\nu), w(x,\tau_\nu), \lambda_r(x) +$$

$$+\widetilde{v}_r(x,\tau_\nu))d\tau_\nu\Big)...d\tau_2\Big)d\tau_1,\ x\in[0,\omega], t\in[(r-1)h_1, rh_1),\ r=\overline{1,N_1}.$$

Determining $\lim\limits_{t\to rh_1-0}\widetilde{v}_r(x,t)$, $r=\overline{1,N_1}$, and substituting their in (10), we obtain the system of nonlinear equations with respect to unknown functional parameters $\lambda_r(x)$:

$$\lambda_1(x)-\lambda_{N_1}(x)-\int_{(N_1-1)h_1}^{N_1h_1} f\ x,\tau_1,u(x,\tau_1),w(x,\tau_1),\lambda_{N_1}(x)+...+$$

$$+\int_{(N_1-1)h_1}^{\tau_{\nu-1}} f\ x,\tau_\nu,u(x,\tau_\nu),w(x,\tau_\nu),\lambda_{N_1}(x)+v_{N_1}(x,\tau_\nu)\ d\tau_\nu...\ d\tau_1=0,$$

$$\lambda_s(x)+\int_{(s-1)h_1}^{sh_1} f\ x,\tau_1,u(x,\tau_1),w(x,\tau_1),\lambda_s(x)+...+$$

$$+\int_{(s-1)h_1}^{\tau_{\nu-1}} f\ x,\tau_\nu,u(x,\tau_\nu),w(x,\tau_\nu),\lambda_s(x)+v_s(x,\tau_\nu)\ d\tau_\nu...\ d\tau_1-\lambda_{s+1}(x)=0,$$

$s=\overline{1,N_1-1}$.

Thus we rewrite the equations in the following form

$$Q_{\nu,h_1}(x,u,w,\lambda,\widetilde{v})=0. \tag{12}$$

Consider a family of periodic boundary value problems

$$\frac{\partial v}{\partial t}=f(x,t,\hat{u}(x,t),\hat{w}(x,t),v),\ (x,t)\in\overline{\Omega}, \tag{13}$$

$$v(x,0)=v(x,T),\ x\in[0,\omega], \tag{14}$$

where the functions $\hat{u}(x,t)$ and $\hat{w}(x,t)$ are known and continuous on $\overline{\Omega}$. For finding the solution to problem (13), (14) we use parametrization method [5].

We denote by $C(\overline{\Omega},h_1,R^{nN_1})$ a space of systems of functions $\widetilde{v}(x,[t])=$ $=(\widetilde{v}_1(x,t),\dots,\widetilde{v}_{N_1}(x,t))'$, where the functions $\widetilde{v}_r(x,t)$ are continuous on Ω_r and have the finite left-hand limit $\lim\limits_{t\to rh_1-0}\widetilde{v}_r(x,t)$, $r=\overline{1,N_1}$, with the norm $\|\widetilde{v}(x,[\cdot])\|_2=\max\limits_{r=\overline{1,N_1}}\sup\limits_{t\in[(r-1)h_1,rh_1)}\|\widetilde{v}_r(x,t)\|$ and denote by $C([0,\omega],R^{nN_1})$ a space of vector functions $\lambda:[0,\omega]\to R^{nN_1}$ continuous on $[0,\omega]$ with the norm $\|\lambda\|_0=\max\limits_{x\in[0,\omega]}\|\lambda(x)\|=\max\limits_{x\in[0,\omega]}\max\limits_{r=\overline{1,N_1}}|\lambda_r(x)|$.

Assume that the parameter $\lambda^{(0)}(x)=(\lambda_1^{(0)}(x),\dots,\lambda_{N_1}^{(0)}(x))'\in R^{nN_1}$ and system functions $\widetilde{v}^{(0)}(x,[t])=\left(\widetilde{v}_1^{(0)}(x,t),\dots,\widetilde{v}_{N_1}^{(0)}(x,t)\right)'$ are known.

Let

$$v^{(0)}(x,t)=\lambda_r^{(0)}(x)+\widetilde{v}_r^{(0)}(x,t),(x,t)\in\Omega_r,\ r=\overline{1,N_1},$$

$$v^{(0)}(x,T)=\lambda_{N_1}^{(0)}(x)+\lim_{t\to T-0}\widetilde{v}_{N_1}^{(0)}(x,t).$$

We take functions $\rho(x) > 0, \widetilde{\rho}(x) > 0,\ \rho_1(x) = \rho(x) + \widetilde{\rho}(x)$, continuous on $[0,\omega]$ and construct sets

$$S\Big(\lambda^{(0)}(x), \rho(x)\Big) =$$

$$= \{\lambda(x) = (\lambda_1(x), \ldots, \lambda_{N_1}(x))' \in C([0,\omega], R^{N_1}) : \|\lambda(x) - \lambda^{(0)}(x)\| < \rho(x)\},$$

$$S\Big(\widetilde{v}^{(0)}(x,[\cdot]), \widetilde{\rho}(x)\Big) =$$

$$= \{(\widetilde{v}_1(x,t), \ldots, \widetilde{v}_{N_1}(x,t))' \in C(\overline{\Omega}, h_1, R^{N_1}) : \|\widetilde{v}(x,[\cdot]) - \widetilde{v}^{(0)}(x,[\cdot])\|_2 < \widetilde{\rho}(x)\},$$

$$S\Big(v^{(0)}(t), \rho_1(x)\Big) = \{v(x,t) \in C(\overline{\Omega}) : \|v(x,\cdot) - v^{(0)}(x,\cdot)\|_1 < \rho_1(x)\},$$

$$G_1^0 = \{(x,t,u,w,v) : (x,t) \in \overline{\Omega},\ u = \hat{u}(x,t),\ w = \hat{w}(x,t), \|v - v^{(0)}(x,\cdot)\|_1 < \rho_1(x)\}.$$

Condition A_0. The function $f(x,t,\hat{u}(x,t),\hat{w}(x,t),v)$ has uniformly continuous partial derivative with respect to v in G_1^0 and $|f_v(x,t,\hat{u}(x,t),\hat{w}(x,t),v)| \le L(x)$, where $L(x)$ is a function continuous on $[0,\omega]$.

We take a system $(\lambda_r^{(0)}(x), \widetilde{v}_r^0(x,t))$, $r = \overline{1,N_1}$, and construct the successive approximations by the following algorithm

Step 1. (a) From equation(12), where $u = \hat{u}(x,t),\quad w = \hat{w}(x,t),\ \widetilde{v} = \widetilde{v}^{(0)}(x,t)$, we define the functional parameter $\lambda^{(1)}(x) = (\lambda_1^{(1)}(x), \ldots, \lambda_{N_1}^{(1)}(x)) \in R^{nN_1}$.

(b) Solving the Cauchy problem (7), (8) for $u = \hat{u}(x,t), w = \hat{w}(x,t), \lambda_r = \lambda_r^{(1)}(x)$, we find $\widetilde{v}_r^{(1)}(x,t),\ t \in [(r-1)h_1, rh_1),\ r = \overline{1,N_1}$.

Step 2. (a) Substituting $\widetilde{v}$ for $\widetilde{v}^{(1)}$ and solving the equation (12), where $u = \hat{u}(x,t)$,

$w = \hat{w}(x,t)$, we define $\lambda^{(2)}(x) \in R^{nN_1}$.

(b) Solving the Cauchy problem (7), (8) for $u = \hat{u}(x,t), w = \hat{w}(x,t), \lambda_r = \lambda_r^{(2)}(x)$, we find the functions $\widetilde{v}_r^{(2)}(x,t)$, $t \in [(r-1)h_1, rh_1)$, $r = \overline{1,N_1}$. And so on.

Step k. (a) Substituting $\widetilde{v}$ for $\widetilde{v}^{(k-1)}$ and solving the equation (12), where $u = \hat{u}(x,t), w = \hat{w}(x,t)$, we define $\lambda^{(k)}(x) \in R^{nN_1}$.

(b) Solving the Cauchy problem (7), (8) for $u = \hat{u}(x,t), w = \hat{w}(x,t), \lambda_r = \lambda_r^{(k)}(x)$,

we find the functions $\widetilde{v}_r^{(k)}(x,t)$, $t \in [(r-1)h_1, rh_1)$, $r = \overline{1,N_1}$, $k = 1,2,....$

Sufficient conditions for the existence of solutions to boundary value problem (7)-(10) are established in the following assertion.

Theorem 1. *Suppose that for some* $\lambda^{(0)}(x), \widetilde{v}^{(0)}(x,[t]), \rho(x), \widetilde{\rho}(x)$ *condition* A_0 *holds and there exists* $h_1 > 0 : N_1 h_1 = T, (N_1 = 1,2,3,...), \nu \in N$, *such that the Jacobi's matrix* $\dfrac{\partial Q_{\nu,h_1}(x,\hat{u},\hat{w},\lambda,\widetilde{v})}{\partial\lambda}$ *is invertible for all* $x \in [0,\omega]$, $\Big(\lambda(x), \widetilde{v}(x,t)\Big) \in$

$\in S\Big(\lambda^{(0)}(x), \rho(x)\Big) \times S\Big(\widetilde{v}^{(0)}(x,[t]), \widetilde{\rho}(x)\Big)$ *and the following inequalities hold:*

1. $\Big\|\Big[\dfrac{\partial Q_{\nu,h_1}\Big(x,\hat{u},\hat{w},\lambda,\widetilde{v}\Big)}{\partial\lambda}\Big]^{-1}\Big\| \le \gamma_\nu(x,h_1)$, *2.* $q_\nu(x,h_1)=\gamma_\nu(x,h_1)\{e^{L(x)h_1}-\sum_{i=0}^{\nu}\frac{1}{i!}\Big(L(x)h_1\Big)^i\}<1,$

3. $\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L(x)h_1\Big)^\nu\|\widetilde{v}^{(1)}(x,[\cdot])-\widetilde{v}^{(0)}(x,[\cdot])\|_2+$
$+\gamma_\nu(x,h_1)\|Q_{\nu,h_1}(x,\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda^{(0)}(x),\widetilde{v}^{(0)}(x,[\cdot]))\|<\rho(x),$

4. $\Big[\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L(x)h_1\Big)^\nu\Big(e^{L(x)h_1}-1\Big)+1\Big]\|\widetilde{v}^{(1)}(x,[\cdot])-\widetilde{v}^{(0)}(x,[\cdot])\|_2$
$<\widetilde{\rho}(x).$

Then sequence of pairs $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,t)), k=1,2,3,\dots,$ *defined by algorithm, belongs to* $S\Big(\lambda^{(0)}(x),\rho(x)\Big)\times S\Big(\widetilde{v}^{(0)}(x,[t]),\widetilde{\rho}(x)\Big)$, *converges to* $(\lambda^*(x),\widetilde{v}^*(x,t))$ *as* $k\to\infty$, *which is the solution of the problem* (7)-(10) *under* $u=\hat{u}(x,t), w=\hat{w}(x,t)$ *and the following inequalities hold:*

$$a)\ \|\lambda^*(x)-\lambda^{(0)}(x)\|\le\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L(x)h_1\Big)^\nu\|\widetilde{v}^{(1)}(x,[\cdot])-\widetilde{v}^{(0)}(x,[\cdot])\|_2+$$

$$+\gamma_\nu(x,h_1)\|Q_{\nu,h_1}(x,\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda^{(0)}(x),\widetilde{v}^{(0)}(x,[\cdot]))\|,$$

$$b)\ \|\widetilde{v}^*(x,[\cdot])-\widetilde{v}^{(0)}(x,[\cdot])\|_2\le\Big[\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L(x)h_1\Big)^\nu\Big(e^{L(x)h_1}-1\Big)+1\Big]\times$$

$$\times\|\widetilde{v}^{(1)}(x,[\cdot])-\widetilde{v}^{(0)}(x,[\cdot])\|_2.$$

Proof. By condition 3) of theorem, operator $Q_{\nu,h_1}(x,\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda(x),\widetilde{v}^{(0)}(x,[\cdot]))$ satisfies in $S\big(\lambda^{(0)}(x),\rho(x)\big)$ all the assumptions of theorem 1 of [4]. Then, for fixed values of $\hat{x}\in[0,\omega]$ there exists a number $\varepsilon_0>0$ satisfying the inequalities
$\varepsilon_0\gamma_\nu(\hat{x},h_1)\le 1/2$ and

$$\frac{\gamma_\nu(\hat{x},h_1)}{1-\varepsilon_0\gamma_\nu(\hat{x},h_1)}\|Q_{\nu,h_1}(\hat{x},\hat{u}(\hat{x},\cdot),\hat{w}(\hat{x},\cdot),\lambda^{(0)}(\hat{x}),\widetilde{v}^{(0)}(\hat{x},[\cdot]))\|<\rho(\hat{x}).$$

From the uniform continuity of f it follows that the Jacobi's matrix $\frac{\partial Q_{\nu,h_1}\Big(\hat{x},\hat{u},\hat{w},\lambda,\widetilde{v}^{(0)}\Big)}{\partial\lambda}$ is uniformly continuous in $S\big(\lambda^{(0)}(\hat{x}),\rho(\hat{x})\big)$ and for $\varepsilon_0>0$ $\delta_0(\varepsilon_0)\in\Big(0,\frac{\rho(\hat{x})}{2}\Big]$ such that, for any $\lambda(\hat{x}),\widetilde{\lambda}(\hat{x})\in S\big(\lambda^{(0)}(\hat{x}),\rho(\hat{x})\big)$ the inequality $\|\lambda(\hat{x})-\widetilde{\lambda}(\hat{x})\|<\delta_0(\varepsilon_0)$ holds. Choosing

$$\alpha\ge\alpha_0=\max(1,\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u}(\hat{x},\cdot),\hat{w}(\hat{x},\cdot),\lambda^{(0)}(\hat{x}),\widetilde{v}^{(0)}(\hat{x},[\cdot]))\|/\delta_0),$$

we construct an iterative process:

$$\lambda^{(1,0)}(\hat{x})=\lambda^{(0)}(\hat{x}),$$

$$\lambda^{(1,m+1)}(\hat{x})=\lambda^{(1,m)}(\hat{x})-\frac{1}{\alpha}\Big(\frac{\partial Q_{\nu,h_1}\Big(\hat{x},\hat{u},\hat{w},\lambda^{(1,m)},\widetilde{v}^{(0)}\Big)}{\partial\lambda}\Big)^{-1}\times$$

$$\times Q_{\nu,h_1}\Big(\hat{x},\hat{u},\hat{w},\lambda^{(1,m)},\widetilde{v}^{(0)}\Big),\ \hat{x}\in[0,\omega], m=1,2,3,\dots. \tag{15}$$

By theorem 1 of [4], the iterative process (15) in the norm $\|\cdot\|_0$ converges to $\lambda^{(1)}(\hat{x})$ which is isolated solutions of the equation $Q_{\nu,h_1}\Big(\hat{x},\hat{u},\hat{w},\lambda,\widetilde{v}^{(0)}\Big)=0$, in the set $S\Big(\lambda^{(0)}(\hat{x}),\rho(\hat{x})\Big)$ and the estimate holds:

$$\|\lambda^{(1)}(\hat{x})-\lambda^{(0)}(\hat{x})\|\le\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(0)},\widetilde{v}^{(0)})\|<\rho(\hat{x}). \tag{16}$$

By virtue of the arbitrariness of $\hat{x}$ this estimate is valid for all $x\in[0,\omega]$. By our assumptions function $\widetilde{v}_r^{(1)}(\hat{x},t)$ is solution of the Cauchy problem (7), (8) for $u(x,t)=\hat{u}(\hat{x},t), w(x,t)=\hat{w}(\hat{x},t),\lambda_r(x)=\lambda_r^{(1)}(\hat{x})$. From the operator's structure $Q_{\nu,h_1}\big(\hat{x},\hat{u},\hat{w},\lambda,\widetilde{v}\big)$ and the equality $Q_{\nu,h_1}\big(\hat{x},\hat{u},\hat{w},\lambda^{(1)},\widetilde{v}^{(0)}\big)=0$ it follows

$$\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(1)},\widetilde{v}^{(1)})\|\le$$

$$\le\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(1)},\widetilde{v}^{(1)})-Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(1)},\widetilde{v}^{(0)})\|\le\gamma_\nu(\hat{x},h_1)\times$$

$$\times\max_{r=\overline{1,N_1}}\Big|\int_{(r-1)h_1}^{rh_1}L(\hat{x})\dots\int_{(r-1)h_1}^{\tau_{\nu-1}}L(\hat{x})\|\overline{v}_r^{(1)}(\hat{x},\tau_\nu)-\widetilde{v}_r^{(0)}(\hat{x},\tau_\nu)\|d\tau_\nu\dots d\tau_1\Big|\le$$

$$\le\gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\|\widetilde{v}^{(1)})(\hat{x},[\cdot])-\widetilde{v}^{(0)})(\hat{x},[\cdot])\|_2. \tag{17}$$

If $\lambda(\hat{x})\in S\Big(\lambda^{(1)}(\hat{x}),\rho^{(1)}(\hat{x})+\hat{\varepsilon}\Big)$, where $\rho^{(1)}(\hat{x})=\|Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(1)},\widetilde{v}^{(1)})\|\times$ $\times\gamma_\nu(\hat{x},h_1)$, the number $\hat{\varepsilon}>0$ satisfies the following inequality

$$\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda^{(0)}(\hat{x}),\widetilde{v}^{(0)}(\hat{x},[\cdot]))\|+\frac{\gamma_\nu(\hat{x},h_1)}{1-q_\nu(\hat{x},h_1)}\times$$

$$\times\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\|\widetilde{v}^{(1)})(\hat{x},[\cdot])-\widetilde{v}^{(0)})(\hat{x},[\cdot])\|_2+\hat{\varepsilon}<\rho(\hat{x})$$

for all $\hat{x}\in[0,\omega]$; then, by inequalities 3, 4 of theorem and (16), (17), we have

$$\|\lambda(\hat{x})-\lambda^{(0)}(\hat{x})\|\le\|\lambda(\hat{x})-\lambda^{(1)}(\hat{x})\|+\|\lambda^{(1)}(\hat{x})-\lambda^{(0)}(\hat{x})\|\le\|\lambda^{(1)}(\hat{x})-\lambda^{(0)}(\hat{x})\|+$$

$$+\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(1)},v^{(1)})\|+\hat{\varepsilon}+\le\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u},\hat{w},\lambda^{(0)},v^{(0)})\|+$$

$$+\gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\ L(\hat{x})h_1\ ^\nu\|v^{(1)})(\hat{x},[\cdot])-v^{(0)})(\hat{x},[\cdot])\|_2+\hat{\varepsilon}<\rho(\hat{x}), \tag{18}$$

i.e., $S(\lambda^{(1)}(\hat{x}),\rho^{(1)}(\hat{x})+\hat{\varepsilon})\subset S(\lambda^{(0)}(\hat{x}),\rho(\hat{x}))$. From the conditions of Theorem it follows that the operator $Q_{\nu,h_1}(\hat{x},\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda(\hat{x}),\widetilde{v}^{(1)}(\hat{x},[\cdot]))$ in $S(\lambda^{(1)}(\hat{x}),$ $\rho_1(\hat{x})+\hat{\varepsilon})$ satisfies all the conditions of Theorem theo1 of [4].

Therefore, the iterative process

$$\lambda^{(2,0)}(\hat{x})=\lambda^{(1)}(\hat{x}),$$

$$\lambda^{(2,m+1)}(\hat{x}) = \lambda^{(2,m)}(\hat{x}) - \frac{1}{\alpha}\Big(\frac{\partial Q_{\nu,h_1}\Big(\hat{x},\hat{u},\hat{w},\lambda^{(2,m)},\widetilde{v}^{(1)}\Big)}{\partial\lambda}\Big)^{-1}\times$$
$$\times Q_{\nu,h_1}\Big(\hat{x},\hat{u},\hat{w},\lambda^{(2,m)},\widetilde{v}^{(1)}\Big),\ \hat{x}\in[0,\omega],\ m=1,2,3,\dots,$$

converges to $\lambda^{(2)}(\hat{x})$ by the norm $\|\cdot\|_0$, where $\lambda^{(2)}(\hat{x})$ is isolated solution to the equation $Q_{\nu,h_1}(\hat{x},\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda(\hat{x}),\widetilde{v}^{(1)}(\hat{x},[\cdot])) = 0$, in the set $S(\lambda^{(1)}(\hat{x}),\rho^{(1)}(\hat{x})+\hat{\varepsilon})$, $\hat{x}\in[0,\omega]$ and

$$\|\lambda^{(2)}(\hat{x})-\lambda^{(1)}(\hat{x})\| \le \gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda^{(1)}(\hat{x}),\widetilde{v}^{(1)}(\hat{x},[\cdot]))\| \le$$
$$\le \gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\|\widetilde{v}^{(1)}(\hat{x},[\cdot])-\widetilde{v}^{(0)}(\hat{x},[\cdot])\|_2.$$

Since $\widetilde{v}_r^{(2)}(\hat{x},t)$ is the solution to Cauchy problem (7), (8) at $u=\hat{u}, w=\hat{w}, \lambda_r(x)=\lambda_r^2(\hat{x})$, we can evaluate the estimate of difference $\widetilde{v}_r^{(2)}(\hat{x},t)-\widetilde{v}_r^{(1)}(\hat{x},t)$:

$$\|\widetilde{v}_r^{(2)}(\hat{x},[\cdot])-\widetilde{v}_r^{(1)}(\hat{x},[\cdot])\|_2 \le \big(e^{L(\hat{x})h_1}-1\big)\|\lambda_r^2(\hat{x})-\lambda_r^1(\hat{x})\| \le$$
$$\le \big(e^{L(\hat{x})h_1}-1\big)\gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\|\widetilde{v}^{(1)}(\hat{x},[\cdot])-\widetilde{v}^{(0)}(\hat{x},[\cdot])\|_2.$$

From the latest estimates, we have

$$\|\widetilde{v}_r^{(2)}(\hat{x},[\cdot])-\widetilde{v}_r^{(1)}(\hat{x},[\cdot])\|_2 \le$$
$$\le \Big(1+\big(e^{L(\hat{x})h_1}-1\big)\gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\Big)\|\widetilde{v}^{(1)}(\hat{x},[\cdot])-\widetilde{v}^{(0)}(\hat{x},[\cdot])\|_2.$$

Assuming that the pair $\Big(\lambda^{(k-1)(\hat{x})},\widetilde{v}^{(k-1)}(\hat{x},[t])\Big) \in S(\lambda^{(0)}(\hat{x}),\rho(\hat{x}) \times S(\widetilde{v}^{(0)}(x,[t]),\widetilde{\rho}(x))$ is defined and the following estimates are established

$$\|\lambda^{(k-1)}(\hat{x})-\lambda^{(k-2)}(\hat{x})\| \le q_\nu(\hat{x},h_1)\|\lambda^{((k-2))}(\hat{x})-\lambda^{(k-3)}(\hat{x})\| \le$$
$$\le [q_\nu(\hat{x},h_1)]^{k-3}\gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\|\widetilde{v}^{(1)}(\hat{x},[\cdot])-\widetilde{v}^{(0)}(\hat{x},[\cdot])\|_2, \quad (19)$$
$$\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda^{(k-1)}(\hat{x}),\widetilde{v}^{(k-1)}(\hat{x},[\cdot]))\| \le$$
$$\le q_\nu(\hat{x},h_1)\|\lambda^{((k-1))}(\hat{x})-\lambda^{(k-2)}(\hat{x})\|, \quad (20)$$

we find k-th approximation by parameter $\lambda^{(k)}(\hat{x})$ from the next equation

$$Q_{\nu,h_1}(\hat{x},\hat{u}(\hat{x},\cdot),\hat{w}(\hat{x},\cdot),\lambda(\hat{x}),\widetilde{v}^{(k-1)}(\hat{x},[\cdot]))=0.$$

Using (19), (20) and equality $Q_{\nu,h_1}(\hat{x},\hat{u}(x,\cdot),\hat{w}(x,\cdot),\lambda^{(k-1)}(\hat{x}),\widetilde{v}^{(k-2)}(\hat{x},[\cdot]))$=0, similarly to (18), we establish the validity of inequality

$$\gamma_\nu(\hat{x},h_1)\|Q_{\nu,h_1}(\hat{x},\hat{u}(\hat{x},\cdot),\hat{w}(\hat{x},\cdot),\lambda^{(k-1)}(\hat{x}),\widetilde{v}^{(k-1)}(\hat{x},[\cdot]))\| \le$$
$$\le [q_\nu(\hat{x},h_1)]^{k-2}\gamma_\nu(\hat{x},h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu\|\widetilde{v}^{(1)}(\hat{x},[\cdot])-\widetilde{v}^{(0)}(\hat{x},[\cdot])\|_2. \quad (21)$$

We take $\rho^{(k-1)}(\hat{x}) = Q_{\nu,h_1}(\hat{x}, \hat{u}(\hat{x},\cdot), \hat{w}(\hat{x},\cdot), \lambda^{(k-1)}(\hat{x}), \widetilde{v}^{(k-1)}(\hat{x},[\cdot]))$ and show that $S(\lambda^{(k-1)}(\hat{x}), \rho^{(k-1)}(\hat{x})+\hat{\varepsilon}) \in S(\lambda^{(0)}(\hat{x}), \rho(\hat{x}))$. Indeed, from the inequalities (19)-(21) and (3) condition of Theorem, we have

$$\|\lambda(\hat{x}) - \lambda^{(0)}(\hat{x})\| \leq \|\lambda(\hat{x}) - \lambda^{(k-1)}(\hat{x})\| + \|\lambda^{(k-1))}(\hat{x}) - \lambda^{(k-2)}(\hat{x})\| +$$

$$+ \ldots + \|\lambda^{(1)}(\hat{x}) - \lambda^{(0)}(\hat{x})\| < \rho^{(k-1)}(\hat{x}) + \lambda^{(k-2)}(\hat{x}) - \lambda^{(k-1)}(\hat{x})\| +$$

$$+ \ldots + \|\lambda^{((1))}(\hat{x}) - \lambda^{(0)}(\hat{x})\| \leq \gamma_\nu(\hat{x}, h_1)\frac{1}{\nu!}\ L(\hat{x})h_1\ ^\nu \|v^{(1)}(\hat{x},[\cdot]) - v^{(0)}(\hat{x},[\cdot])\|_2 \times$$

$$\times\ [q_\nu(\hat{x}, h_1)]^{k-2} + [q_\nu(\hat{x}, h_1)]^{k-1} + \ldots + 1\ \ + \gamma_\nu(\hat{x}, h_1)\|Q_{\nu,h_1}(\hat{x}, \hat{u}(x,\cdot), \hat{w}(x,\cdot),$$

$$\lambda^{(0)}(\hat{x}), v^{(0)}(\hat{x},[\cdot]))\| + \hat{\varepsilon} + \gamma_\nu(\hat{x}, h_1)\frac{1}{\nu!}\ L(\hat{x})h_1\ ^\nu \|v^{(1)}(\hat{x},[\cdot]) - v^{(0)}(\hat{x},[\cdot])\|_2 < \rho(\hat{x}).$$

Since $Q_{\nu,h_1}(\hat{x}, \hat{u}(\hat{x},\cdot), \hat{w}(\hat{x},\cdot), \lambda(\hat{x}), \widetilde{v}^{(k-1)}(\hat{x},[\cdot]))$ satisfies all the conditions of theorem 1 of [4] in the set $S(\lambda^{(k-1)}(\hat{x}), \rho^{(k-1)}(\hat{x})+\hat{\varepsilon})$, then there exists $\lambda^{(k)}(\hat{x}) \in S(\lambda^{(k-1)}(\hat{x}),\ \rho^{(k-1)}(\hat{x}) + \hat{\varepsilon})$ is a solution to the equation

$$Q_{\nu,h_1}(\hat{x}, \hat{u}(x,\cdot), \hat{w}(x,\cdot), \lambda(\hat{x}), \widetilde{v}^{(k-1)}(\hat{x},[\cdot])) = 0$$

and the following estimate holds

$$\|\lambda^{(k)}(\hat{x}) - \lambda^{(k-1)}(\hat{x})\| \leq$$

$$\leq \gamma_\nu(\hat{x}, h_1)\|Q_{\nu,h_1}(\hat{x}, \hat{u}(\hat{x},\cdot), \hat{w}(\hat{x},\cdot), \lambda^{(k-1)}(\hat{x}), \widetilde{v}^{(k-1)}(\hat{x},[\cdot]))\|. \tag{22}$$

In view of the arbitrariness $\hat{x}$ the estimate (22) is valid for all $x \in [0, \omega]$. Taking into account that $\widetilde{v}_r^{(k)}(\hat{x}, t)$ is the solution of Cauchy problem (7), (8) for $u(x,t) = \underline{\hat{u}(\hat{x},t)}$, $w(x,t) = \hat{w}(\hat{x},t), \lambda_r(x) = \lambda_r^{(k)}(\hat{x})$ on $\in [0,\omega] \times [(r-1)$ $h_1, h_1), r == \overline{1, N_1}$ and using inequalities (19), (22), we set the following estimate

$$\|\lambda^{(k)}(\hat{x}) - \lambda^{(k-1)}(\hat{x})\| \leq$$

$$\leq q_\nu(\hat{x}, h_1)\gamma_\nu(\hat{x}, h_1)\frac{1}{\nu!}\big(L(\hat{x})h_1\big)^\nu \|\widetilde{v}^{(1)}(\hat{x},[\cdot]) - \widetilde{v}^{(0)}(\hat{x},[\cdot])\|_2, \tag{23}$$

$$\|\widetilde{v}_r^{(k)}(\hat{x},[\cdot]) - \widetilde{v}_r^{(k-1)}(\hat{x},[\cdot])\|_2 \leq$$

$$\leq q_\nu(\hat{x}, h_1)\|\widetilde{v}^{(k-1)}(\hat{x},[\cdot]) - \widetilde{v}^{(k-2)}(\hat{x},[\cdot])\|,\ k = 2, 3, \ldots \tag{24}$$

where$(\hat{x}, t) \in [0,\omega] \times [(r-1)h_1, h_1),\ r = \overline{1, N_1}$. From the inequalities (23), (24) and $q_\nu(\hat{x}, h_1) < 1$ it follows that the sequence of pairs $\Big(\lambda^{(k)}(\hat{x}), \widetilde{v}^{(k)}(\hat{x},[t])\Big)$ converges to $\Big(\lambda^*(\hat{x}), \widetilde{v}^*(\hat{x},[t])\Big)$ as $k \to \infty$, the solution of the problem (7)-(10), where $u(x,t) = = \hat{u}(\hat{x},t), w(x,t) = \hat{w}(\hat{x},t)$. Passing to the limit as $k \to \infty$ in inequalities (23), (24), we obtain estimates (a) and (b) of Theorem. Theorem 1 is proved.

3 Application Method of Euler's Modification to the System of Nonlinear Hyperbolic Equations

For finding the approximate solution of semi-periodical boundary problem (4)-(6) we use a method of Euler's modification. Divide the interval $[0, \omega]$ with step h_0 on N_0 parts $N_0 h_0 = \omega$.

Vector functions $v^{(0)}(t)$, $\dot{v}^{(0)}(t)$, $u^{(0)}(t)$, $w^{(0)}(t)$ are defined by:

$$v^{(0)}(t) = 0, \quad \dot{v}^{(0)}(t) = 0, \quad u^{(0)}(t) = \psi(t), \quad w^{(0)}(t) = \dot{\psi}(t), \quad t \in [0, T],$$

respectively. Solving a system of periodic boundary value problems

$$\frac{dv^{(1)}}{dt} = f(0, t, u^{(0)}(t), w^{(0)}(t), v^{(1)}), t \in [0, T], \quad v^{(1)}(0) = v^{(1)}(T), \tag{25}$$

we find a vector function $v^{(1)}(t)$. By $v^{(1)}(t)$ and $\dot{v}^{(1)}(t)$ we define the functions:

$$u^{(1)}(t) = \psi(t) + h_0 v^{(1)}(t), \quad w^{(1)}(t) = \dot{\psi}(t) + h_0 \dot{v}^{(1)}(t), \quad t \in [0, T].$$

Assuming that the functions $v^{(i-1)}(t)$, $u^{(i-1)}(t)$, and $w^{(i-1)}(t)$ are known and solving the boundary value problem

$$\frac{dv^{(i)}}{dt} = f((i-1)h_0, t, u^{(i-1)}(t), w^{(i-1)}(t), v^{(i)}), \quad t \in [0, T], \tag{26}$$

$$v^{(i)}(0) = v^{(i)}(T), \quad i = \overline{2, N_0 + 1}, \tag{27}$$

we find the function $v^{(i)}(t)$.

Using $v^{(i)}(t)$ and $\dot{v}^{(i)}(t)$, we define the functions $u^{(i)}(t)$, and $w^{(i)}(t)$ by the following equals

$$u^{(i)}(t) = \psi(t) + h_0 \sum_{j=0}^{i-1} v^{(j)}(t), \; w^{(i)}(t) = \dot{\psi}(t) + h_0 \sum_{j=0}^{i} \dot{v}^{(j)}(t), \quad t \in [0, T], \, i = \overline{1, N_0}.$$

Assuming that the considered periodic boundary value problems for systems of ordinary differential equations (25), (26), (27) have a solution $v^{(i)}(t)$ for all $i = \overline{1, N_0 + 1}$ on the domain $\overline{\Omega}$, we construct the following functions:

$$u_{h_0}(x, t) = \psi(t) + h_0 \sum_{j=0}^{i-1} v^{(j)}(t) + v^{(i)}(t)(x - (i-1)h_0), \tag{28}$$

$$w_{h_0}(x, t) = \dot{\psi}(t) + h_0 \sum_{j=0}^{i-1} \dot{v}^{(j)}(t) + \dot{v}^{(i)}(t)(x - (i-1)h_0), \tag{29}$$

$$v_{h_0}(x, t) = v^{(i+1)}(t) \frac{x - (i-1)h_0}{h_0} + v^{(i)}(t) \frac{ih_0 - x}{h_0}, \, x \in [(i-1)h_0, ih_0). \tag{30}$$

We investigate the periodic boundary value problems for systems of ordinary differential equations (25), (26), (27) by the parametrization method [5]. Take

a step $h_1 > 0 : N_1 h_1 = T$, $N_1 = 1, 2, 3, ...$, and make a partition $[0, T) = \bigcup_{r=1}^{N_1} [(r-1)h_1, rh_1)$. The restriction of $v^{(i)}(t)$ on r - th interval is denoted by $v_r^{(i)}(t) : v_r^{(i)}(t) = v^{(i)}(t)$,
$t \in [(r-1)h_1, rh_1)$. By $\lambda_r^{(i)}$ we denote the value of the function $v^{(i)}(t)$ at the point $t = (r-1)h_1$, $r = \overline{1, N_1}, i = \overline{1, N_0}$, and on each interval $[(r-1)h_1, rh_1)$ make the replacement: $\widetilde{v}_r^{(i)}(t) = v_r^{(i)}(t) - \lambda_r^{(i)}, r = \overline{1, N_1}, i = \overline{1, N_0 + 1}$. We obtain the following multi-point boundary value problem with a parameter

$$\frac{d\widetilde{v}_r^{(i)}}{dt} = f((i-1)h_0, t, u^{(i-1)}(t), w^{(i-1)}(t), \widetilde{v}_r^{(i)} + \lambda_r^{(i)}),$$

$$t \in [(r-1)h_1, rh_1),\ r = \overline{1, N_1}, \tag{31}$$

$$\widetilde{v}_r^{(i)}[(r-1)h_1] = 0, \quad r = \overline{1, N_1}, i = \overline{1, N_0 + 1}, \tag{32}$$

$$\lambda_1^{(i)} - \lim_{t \to T-0} \widetilde{v}_{N_1}^{(i)}(t) - \lambda_{N_1}^{(i)} = 0,\ i = \overline{1, N_0 + 1}, \tag{33}$$

$$\lambda_s^{(i)} + \lim_{t \to sh_1 - 0} \widetilde{v}_s^{(i)}(t) - \lambda_{s+1}^{(i)} = 0,\ s = \overline{1, N_1 - 1},\ i = \overline{1, N_0 + 1}. \tag{34}$$

For fixed values of the parameters $\lambda_r^{(i)}, i = \overline{1, N_0 + 1}$, the Cauchy problem (31), (32) is equivalent to the nonlinear Volterra integral equations

$$\widetilde{v}_r^{(i)}(t) = \int_{(r-1)h_1}^{t} f((i-1)h_0, \tau, u^{(i-1)}(\tau), w^{(i-1)}(\tau), \widetilde{v}_r^{(i)}(\tau) + \lambda_r^{(i)}) d\tau,$$

$$t \in [(r-1)h_1, rh_1),\ r = \overline{1, N_1}, i = \overline{1, N_0 + 1}. \tag{35}$$

For the problem of (33)-(34) the system of nonlinear equations with respect to entered parameters (12) has the form:

$$Q_{\nu, h_1}((i-1)h_0, \psi(\cdot) + h_0 \sum_{j=0}^{i-1} v^{(j)}(\cdot), \dot{\psi}(\cdot) + h_0 \sum_{j=0}^{i-1} \dot{v}^{(j)}(\cdot), \lambda^{(i)}, \widetilde{v}^{(i)}[\cdot]) = 0. \tag{36}$$

Obviously, that if $v^{(i)}(t)$, $i = \overline{1, N_0 + 1}$ is a solution to problem (26), (27), then its corresponding pair $\left(\lambda_r^{(i)}, \widetilde{v}_r^{(i)}(t)\right)$ is a solution to problem (31)-(34). Taking functions $\rho_{h_0}(x) > 0$, $\widetilde{\rho}_{h_0}(x) > 0$, $\rho_{4,h_0}(x) > 0$, and $\rho_{3,h_0}(x) = \widetilde{\rho}_{h_0}(x) + \rho_{h_0}(x)$ continuous on $[0, \omega]$, we construct the sets

$$S\Big(\lambda_{h_0}^{(0)}(x), \rho_{h_0}(x)\Big) = \{\lambda(x) \in C([0, \omega], R^{nN_1}) : \|\lambda(x) - \lambda_{h_0}^{(0)}(x)\|_0 < \rho_{h_0}(x)\},$$

$$S\Big(\widetilde{v}_{h_0}^{(0)}(x, [t]), \tilde{\rho}_{h_0}(x)\Big) = \{\widetilde{v}(x, [t]) \in C(\overline{\Omega}, h_1, R^{nN_1}) : \|\widetilde{v}(x, [\cdot]) - \widetilde{v}_{h_0}^{(0)}(x, [\cdot])\|_2 < \tilde{\rho}_{h_0}(x)\},$$

$$S\Big(u_{h_0}(x, t), \rho_{4,h_0}(x)\Big) = \{u(x, t) \in C(\overline{\Omega}) : \|u(x, \cdot) - u_{h_0}(x, \cdot)\|_1 < \rho_{4,h_0}(x)\},$$

$$S\Big(v_{h_0}(x, t), \rho_{1,h_0}(x)\Big) = \{v(x, t) \in C(\overline{\Omega}) : \|v(x, \cdot) - v_{h_0}(x, \cdot)\|_1 < \rho_{3,h_0}(x)\}$$

$$S\Big(w_{h_0}(x,t), \rho_{4,h_0}(x)\Big) = \{w_{h_0}(x,t) \in C(\overline{\Omega}) : \|w_{h_0}(x,\cdot) - w_{h_0}(x,\cdot)\|_1 < \rho_{4,h_0}(x)\},$$

$$G_2^0(h_0, \rho_{h_0}(x), \widetilde{\rho}_{h_0}(x), \rho_{4,h_0}(x)) = \{(x,t,u,w,v) : (x,t) \in \overline{\Omega}, \|u - u_{h_0}(x,\cdot)\|_1 <$$

$$< \rho_{4,h_0}(x), \quad \|w - w_{h_0}(x,\cdot)\|_1 < \rho_{4,h_0}(x), \quad \|v - v_{h_0}(x,\cdot)\|_1 < \rho_{4,h_0}(x)\}.$$

Condition B. For some $h_0 > 0 : N_0 h_0 = \omega$ the periodic boundary value problems to systems of ordinary differential equations (27)-(29) have solutions $v^{(1)}(t), v^{(2)}(t), \ldots, v^{(N_0)}(t), v^{(N_0+1)}(t)$ and the function $f(x,t,u,w,v)$ has uniformly continuously partial derivatives in u, w, v on the set $G_2^0(h_0, \rho_{h_0}(x), \widetilde{\rho}_{h_0}(x), \rho_{2,h_0}(x))$, satisfying the inequalities $|f_u(x,t,u,w,v)| \le L_1(x), |f_w(x,t,u,w,v)| \le L_2(x), \|f_v(x,t,u,w,v)| \le L_3(x)$, where $L_i(x)$, $i = 1,2,3$ are functions continuous
on $[0,\omega]$. Taking as an initial approximation of triple $\{u_{h_0}(x,t), w_{h_0}(x,t), v_{h_0}(x,t)\}$, we find the solution to problem (4)-(6) as a limit of the sequence of triples $\{u^{(k)}(x,t), w^{(k)}(x,t),\, v^{(k)}(x,t)\}$, $k = 1,2,3...$ determined by the following algorithm.

Step 1. (A) We find the function $v^{(1)}(x,t)$ from the boundary value problem (4),(5) with $u(x,t) = u_{h_0}^{(0)}(x,t), w(x,t) = w_{h_0}^{(0)}(x,t)$. Let $\lambda_r^{(1,0)}(x) = \lambda_{h_0,r}^{(0)}(x) = v_{h_0}(x,(r-1)h_1)$, and let the function $\widetilde{v}_r^{(1,0)}(x,t)$ be a restriction of function $\widetilde{v}_{h_0}^{(0)}(x,t)$ to $r-$ th interval $t \in [(r-1)h_1, rh_1)$, $r = \overline{1, N_1}$.

The following approaches are defined by the algorithm:

Step 1.1 a_1) Parameter $\lambda^{(1,1)}(x)$ is determined from the equation (2) with $\widetilde{v} = \widetilde{v}^{(1,0)}$.

b_1) Solving the Cauchy problem (7), (8) for $\lambda_r(x) = \lambda_r^{(1,1)}(x)$, we find the function $\widetilde{v}_r^{(1,1)}(x,t)$.

Step 1.2 a_1) Substituting the found function $\widetilde{v}_r^{(1,1)}(x,t), r = \overline{1, N_1}$ in the equation (12), we solve it and find $\lambda^{(1,2)}(x)$.

b_1) For $\lambda_r(x) = \lambda_r^{(1,2)}(x)$ we solve the Cauchy problem (6), (7) and find $\widetilde{v}_r^{(1,2)}(x,t)$. And so on.

We obtain system of pairs $\Big(\lambda_r^{(1,m)}(x),\, \widetilde{v}_r^{(1,m)}(x,t)\Big), r = \overline{1, N_1}$ on m - th step of the algorithm. Assume that $\lambda^{(1,m)}(x)$ and $\widetilde{v}^{(1,m)}(x,[t])$ converges to $\lambda^{(1)}(x) \in C([0,\omega], R^{nN_1}), \widetilde{v}^{(1)}(x,[t]) \in C(\overline{\Omega}, h_1, R^{nN_1})$ as $m \to \infty$ by the norm $\|\cdot\|_0$ and $\|\cdot\|_{2,0}$.

Then the function $v^{(1)}(x,t)$ is determined by the equalities

$$v^{(1)}(x,t) = \lambda_r^{(1)}(x) + \widetilde{v}_r^{(1)}(x,t), \quad (x,t) \in \Omega_r,\ r = \overline{1, N_1},$$
$$v^{(1)}(x,T) = \lambda_{N_1}^{(1)}(x) + \lim_{t \to T-0} \widetilde{v}_{N_1}^{(1)}(x,t).$$

(B) By $v^{(1)}(x,t),\, v_t^{(1)}(x,t)$ we find the first approaches of functions $u(x,t)$, $w(x,t)$ from the equalities $u^{(1)}(x,t) = \psi(t) + \int_0^x v^{(1)}(\xi,t)d\xi$? $w^{(1)}(x,t) = \dot{\psi}(t) + \int_0^x v_t^{(1)}(\xi,t)d\xi$.

Step 2. (A) Solving boundary value problem (4),(5) with $u(x,t) = u^{(1)}(x,t)$, $w(x,t) == w^{(1)}(x,t)$, we find the function $v^{(2)}(x,t)$. We take $\lambda_r^{(2,0)}(x) = \lambda_r^{(1)}(x)$,

$\widetilde{v}_r^{(2,0)}(x,t) == \widetilde{v}_r^{(1)}(x,t)$, $r=\overline{1,N_1}$ and we find the solution to an equivalent boundary value problem with parameter from the following algorithm:

Step 2.1 a_1) We determine parameter $\lambda^{(2,1)}(x)$ from the equation (12) for $\widetilde{v}=\widetilde{v}^{(2,0)}$.

b_1) Solving the Cauchy problem (7), (8) with $\lambda_r(x)=\lambda_r^{(2,1)}(x)$, we find function $\widetilde{v}_r^{(2,1)}(x,t), r=\overline{1,N_1}$.

Step 2.2 a_1) Substituting the found functions $\widetilde{v}_r^{(2,1)}(x,t)$, $r=\overline{1,N_1}$ in the equation (12), we solve it and find $\lambda^{(2,2)}(x)$.

b_1) Solving the Cauchy problem (7), (8) for $\lambda_r(x)=\lambda_r^{(2,2)}(x)$, we find function $\widetilde{v}_r^{(2,2)}(x,t)$. So on. We obtain the system of pairs $\Big(\lambda_r^{(2,m)}(x),\widetilde{v}_r^{(2,m)}(x,t)\Big)$, $r=\overline{1,N_1}$, on the $m-$ th step of the algorithm. Assume that $\lambda^{(2,m)}(x)$ and $\widetilde{v}^{(2,m)}(x,[t])$ converges to $\lambda^{(2)}(x)\in C([0,\omega],R^{nN_1}), \widetilde{v}^{(2)}(x,[t])\in C(\overline{\Omega},h_1,R^{nN_1})$ as $m\to\infty$ by the norm $\|\cdot\|_0$ and $\|\cdot\|_{2,0}$.

Then the function $v^{(2)}(x,t)$ is determined by the equalities $v^{(2)}(x,t)=\lambda_r^{(2)}(x)+$ $+\widetilde{v}_r^{(2)}(x,t)$, $(x,t)\in\Omega_r$, $r=\overline{1,N_1}, v^{(2)}(x,T)=\lambda_{N_1}^{(2)}(x)+\lim\limits_{t\to T-0}\widetilde{v}_{N_1}^{(2)}(x,t)$.

(B) By the function $v^{(2)}(x,t)$ we define $u^{(2)}(x,t)=\psi(t)+\int_0^x v^{(2)}(\xi,t)d\xi$ and $w^{(2)}(x,t)=\dot{\psi}(t)+\int_0^x v_t^{(2)}(\xi,t)d\xi$.

Continuing the process, we obtain a system of triples $\{u^{(k)}(x,t)$, $w^{(k)}(x,t),v^{(k)}(x,t)\}, k=1,2,3,...$, on the **k-th** step of the algorithm. We introduce the following notations:

$a_{h_0}^{1,i}(x,h_1)=[L_1(x)+L_2(x)]h_1\max(\|v^{(i)}(\cdot)\|_1,\|\dot{v}^{(i)}(\cdot)\|_1)|x-(i-1)h_0|e^{L_3(x)h_1}$,

$a_{h_0}^{2,i}(x,h_1)=\|f(x,\cdot,\psi(\cdot)+h_0\sum\limits_{j=0}^{i-1}v^{(j)}(\cdot),\dot{\psi}(\cdot)+h_0\sum\limits_{j=0}^{i-1}\dot{v}^{(j)}(\cdot),v_{h_0}((i{-}1)h_0,\cdot))-$

$-f((i{-}1)h_0,\cdot,\psi(\cdot)+h_0\sum\limits_{j=0}^{i-1}v^{(j)}(\cdot)+h_0\sum\limits_{j=0}^{i-1}\dot{v}^{(j)}(\cdot),v_{h_0}((i-1)h_0,\cdot))\|_1e^{L_3(x)h_1}h_1$,

$a_{h_0}^{3,i}(x,h_1)=a_{h_0}^{1,i}(x,h_1)+a_{h_0}^{2,i}(x,h_1)$, $a_{h_0}^{4,i}(x,h_1)=\|Q_{\nu,h_1}(x,\psi(\cdot)+h_0\sum\limits_{j=0}^{i-1}v^{(j)}(\cdot),\dot{\psi}+$

$+h_0\sum\limits_{j=0}^{i-1}\dot{v}^{(j)}(\cdot),\lambda_{h_0}^{(0)}((i-1)h_0),\widetilde{v}_{h_0}^{(0)}((i-1)h_0,[\cdot]))-Q_{\nu,h_1}((i-1)h_0,\psi(\cdot)+h_0\sum\limits_{j=0}^{i-1}v^{(j)}(\cdot)$,

$\dot{\psi}(\cdot)+h_0\sum\limits_{j=0}^{i-1}\dot{v}^{(j)}(\cdot),\lambda_{h_0}^{(0)}((i-1)h_0),\widetilde{v}_{h_0}^{(0)}((i-1)h_0,[\cdot]))\|$, $\quad a_{h_0}^{5,i}(x,h_1)=\|v^{(i+1)}(\cdot)-$

$-v^{(i)}(\cdot)\|_1\Big|\dfrac{x-(i-1)h_0}{h_0}\Big|+a_{h_0}^{6,i}(x,h_1)$, $a_{h_0}^{6,i}(x,h_1)=\max\limits_{r=\overline{1,N_1}}\|v^{(i+1)}[(r-1)h_1]-$

$-v^{(i)}[(r-1)h_1]\|\Big|\dfrac{x-(i-1)h_0}{h_0}\Big|$, $a_{h_0}^{7,i}(x,h_1)=\sum\limits_{j=0}^{\nu-1}\dfrac{1}{j!}(L_3(x)h_1)^j a_{h_0}^{1,i}(x,h_1)+$

$+a_{h_0}^{4,i}(x,h_1)+\dfrac{1}{\nu!}(L_3(x)h_1)^\nu a_{h_0}^{5,i}(x,h_1),a_{h_0}^{8,i}(x,h_1)=\dfrac{1}{\nu!}(L_3(x)h_1)^\nu a_{h_0}^{6,i}(x,h_1)+$
$+a_{h_0}^{7,i}(x,h_1)$, $a_{h_0}^{9,i}(x,h_1)=\Big[L_1(x)+L_2(x)\Big]\max\Big(\|v^{(i)}(\cdot)\|_1,\|\dot{v}^{(i)}(\cdot)\|_1\Big)\times$

$$\times|x-(i-1)h_0|+\|f(x,\cdot,\psi(\cdot)+h_0\sum_{j=0}^{i-1}v^{(j)}(\cdot),\dot{\psi}(\cdot)+h_0\sum_{j=0}^{i-1}\dot{v}^{(j)}(\cdot),v^{(i)}(\cdot))-$$

$$-f((i-1)h_0,\cdot,\psi(\cdot)+h_0\sum_{j=0}^{i-1}v^{(j)}(\cdot),\dot{\psi}(\cdot)+h_0\sum_{j=0}^{i-1}\dot{v}^{(j)}(\cdot),v^{(i)}(\cdot))\|_1+$$

$$+L_3(x)a^5_{h_0}(x,h_1),\ x\in[(i-1)h_0,ih_0),\quad i=\overline{1,N_0}.$$

The next theorem establishes the sufficient conditions of existence the triples $\{u^*(x,t),w^*(x,t),v^*(x,t)\}$ which is the solution to problem (4)-(6) and establishes the estimate of the differences $\|u^*-u_{h_0}(x,\cdot)\|_1$, $\|w^*-w_{h_0}(x,\cdot)\|_1$, $\|v^*-v_{h_0}(x,\cdot)\|_1$.

Theorem 2. *Suppose, for some* $h_0>0:N_0h_0=\omega,h_1>0:N_1h_1=T(N_1=1,2,3,...)$, $\nu\in N$, *the condition B holds and the Jacobi matrix* $\dfrac{\partial Q_{\nu,h_1}(x,u,w,\lambda,\widetilde{v})}{\partial\lambda}$ *is invertible for any* $(x,u(x,t),w(x,t),\lambda(x),\widetilde{v}(x,[t]))\in[0,\omega]\times S\Big(u_{h_0}(x,t),\rho_{2,h_0}(x)\Big)\times S\Big(w_{h_0}(x,t),\rho_{2,h_0}(x)\Big)\times S\big(\lambda^{(0)}_{h_0}(x),\rho_{h_0}(x)\big)\times S\Big(\widetilde{v}^{(0)}_{h_0}(x,[t]),\widetilde{\rho}_{h_0}(x)\Big)$, *and the following inequalities hold:*

1. $\Big\|\Big[\dfrac{\partial Q_{\nu,h_1}(x,u,w,\lambda,\widetilde{v})}{\partial\lambda}\Big]^{-1}\Big\|\le\gamma_\nu(x,h_1)$,
2. $q_\nu(x,h_1)=\gamma_\nu(x,h_1)\{e^{L_3(x)h_1}-\sum\limits_{i=1}^{\nu}\dfrac{1}{i!}\Big(L_3(x)h_1\Big)^i\}<1$,
3. $a_{h_0}(x,h_1)+c^1_{h_0}(x,h_1)B_{h_0}(x,h_1)\exp\Big(\int_0^x c_{h_0}(\xi,h_1)d\xi\Big)<\rho_{h_0}(x)$,
4. $b_{h_0}(x,h_1)+c^1_{h_0}(x,h_1)B_{h_0}(x,h_1)\exp\Big(\int_0^x c_{h_0}(\xi,h_1)d\xi\Big)\Big(e^{L_3(x)h_1}-1\Big)<\widetilde{\rho}_{h_0}(x)$,
5. $B_{h_0}(x,h_1)\exp\Big(\int_0^x c_{h_0}(\xi,h_1)d\xi\Big)<\rho_{4,h_0}(x)$,

where $b^1_{h_0}(x,h_1)=\gamma_\nu(x,h_1)a^7_{h_0}(x,h_1)+[1+\gamma_\nu(x,h_1)\dfrac{1}{\nu!}\Big(L_3(x)h_1\Big)^\nu]a^6_{h_0}(x,h_1)$,

$$b^2_{h_0}(x,h_1)=2a^3_{h_0}(x,h_1)+e^{L_3(x)h_1}-1)\Big[b^1_{h_0}(x,h_1)+a^6_{h_0}(x,h_1)\Big],\ a_{h_0}(x,h_1)=$$

$$=\gamma_\nu(x,h_1)a^8_{h_0}(x,h_1)+\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L_3(x)h_1\Big)^\nu b^2_{h_0}(x,h_1),b_{h_0}(x,h_1)=$$

$$=b^2_{h_0}(x,h_1)\Big[1+\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L_3(x)h_1\Big)^\nu(e^{L_3(x)h_1}-1)\Big],\ d^1_{h_0}(x,h_1)=$$

$$=a_{h_0}(x,h_1)+b_{h_0}(x,h_1),d^2_{h_0}(x,h_1)=a^9_{h_0}(x,h_1)+L_3(x)d_{h_0}(x,h_1),\ c^1_{h_0}(x,h_1)=$$

$$=[L_1(x)+L_2(x)]h_1\Big[1+\frac{\gamma_\nu(x,h_1)}{1-q_\nu(x,h_1)}\frac{1}{\nu!}\Big(L_3(x)h_1\Big)^\nu(e^{L_3(x)h_1}-1)\Big]e^{L_3(x)h_1}\times$$

$$\times\sum_{j=0}^{\nu-1}\frac{1}{j!}\Big(L_3(x)h_1\Big)^j\gamma_\nu(x,h_1),\ c^2_{h_0}(x,h_1)=L_1(x)+L_2(x)+L_3(x)c^1_{h_0}(x,h_1),$$

$$c_{h_0}(x,h_1) = \max\Big(c^1_{h_0}(x,h_1), c^2_{h_0}(x,h_1)\Big),\ B_{h_0}(x,h_1) = \max\Big(\int_0^x d^1_{h_0}(\xi,h_1)d\xi +$$
$$+\int_0^x d^2_{h_0}(\xi,h_1)d\xi\Big) + 3h_0 \cdot \max\Big(\max_{i=\overline{1,N_0+1}} \|v^{(i)}(\cdot)\|_1, \max_{i=\overline{1,N_0+1}} \|\dot{v}^{(i)}(\cdot)\|_1\Big).$$

Here $a^s_{h_0}(x,h_1) = a^{s,i}_{h_0}(x,h_1)$, $s=\overline{1,9}$ *at* $x \in [(i-1)h_0, ih_0)$, $i=\overline{1,N_0}$.

Then problem (4)-(6) *has the isolated solution* $\{u^*(x,t), w^*(x,t), v^*(x,t)\}$ *in* $S\Big(u_{h_0}(x,t), \rho_{2,h_0}(x)\Big) \times S\Big(w_{h_0}(x,t), \rho_{2,h_0}(x)\Big) \times S\Big(v_{h_0}(x,t), \rho_{1,h_0}(x)\Big)$ *and the following estimates are valid:*

$$\|v^* - v_{h_0}(x,\cdot)\|_1 \le$$
$$\le d^1_{h_0}(x,h_1) + c^1_{h_0}(x,h_1)e^{L(x)h_1} B_{h_0}(x,h_1) \exp\Big(\int_0^x c_{h_0}(\xi,h_1)d\xi\Big),$$
$$\max\big(\|u^* - u_{h_0}(x,\cdot)\|_1, \|w^* - w_{h_0}(x,\cdot)\|_1\big) \le B_{h_0}(x,h_1)\exp\Big(\int_0^x c_{h_0}(\xi,h_1)d\xi\Big).$$

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Asanova, A.T., Dzhumabaev, D.S.: Unigue solvability of the boundary value problem for system of hyperbolic equation with data on the characteristics. Comput. Math. Math. Phys. **42**(11), 1609–1621 (2002) (in Russian)
2. Asanova, A.T., Dzhumabaev, D.S.: Unique solvability of nonlocal boundary value problem for systems of hyperbolic equations. Differ. Equ. **39**(10), 1414–1427 (2003) (in Russian)
3. Dzhumabaev, D.S.: Conditions the unique solvability of a linear boundary value problem for ordinary differential equations. Comput. Math. Math. Phys. **29**(1), 56–66 (1989) (in Russian)
4. Dzhumabaev, D.S.: Iterative processes with damping factors and their application. Math. J. **1**(1), 20–30 (2001) (in Russian)
5. Dzhumabaev, D.S., Temesheva, S.M.: The method of parametrization the solution of nonlinear two-point boundary value problems. J. Comput. Math. Math. Phys. **47**(1), 39–63 (2007)
6. Golubeva, N.D., Pulkina, L.S.: On one nonlocal problem with integral conditions. Math. Zametky. **59**(3), 456–458 (1996) (in Russian)
7. Kabdrakhova, S.S.: Modification of Euler method to the solution of semi-periodic boundary value problem for a nonlinear hyperbolic equation. Math. J. **8**(2), 55–62 (2008) (in Russian)
8. Kiguradze, T.I.: Some boundary value problems for systems of linear partial tial differential equations of hyperbolic type. Mem. Differ. Equ. **42**(11), 1–144 (1994)
9. Kiguradze, T.I., Lakshmikantham, V.: On initial-boundary value problems in bounded and unbounded domains for a class of nonlinear hyperbolic equations of the third order. J. Math. Anal. Appl. **324**, 1242–1261 (2006)

10. Kiguradze, I., Kiguradze, T.: On solvability of boundary value problems for higher order nonlinear hyperbolic equations. Nonlin. Anal. **69**, 1914–1933 (2008)
11. Kiguradze, I., Kiguradze, T.: Optimal conditions of solvability of non-local problems for second order ordinary differential equations. Nonlin. Anal. **74**, 757–767 (2011)
12. Kiguradze, T.I., Kusano, T.O.: On well-posedness of initial boundary value problems for linear higher order hyperbolic equations of two variables. Differ. Equ. **39**(1), 173–185 (2003)
13. Kharibegashivli, S.S.: On two-point problem for linear nonstrictly hyperbolic system. Soobsh. AN Gruzia. **93**(3), 553–556 (1979) (in Russian)
14. Kharibegashivli, S.S.: On one boundary problem for second order hyperbolic equation. Doklady RAN. **280**(6), 1313–1316 (1985) (in Russian)
15. Mitropolsky, Ya.A., Urmancheva, L.B.: On two-point problem for hyperbolic system. Ukrainsky Mathem. J. **42**(12), 1657–1663 (1990) (in Russian)
16. Nakhushev, A.M.: On some boundary value problems for hyperbolic equations and mixed equations. Differ. Equ. **5**(1), 44–59 (1969) (in Russian)
17. Nakhushev, A.M.: A technique of statement of well-posed boundary value problems for second order hyperbolic equations on plane. Differ. Equ. **26**(2), 192–195 (1970) (in Russian)
18. Orumbaeva, N.T.: On an algorithm for finding a solution Semi-periodic boundary value problem. Ser. Phys. Math. Izvestiya NAS RK. **1**, 88–94 (2005)

On a Solution of a Nonlinear Semi-periodic Boundary Value Problem for a Differential Equation with Arbitrary Functions

Nurgul T. Orumbayeva[1(✉)] and Gulmira Sabitbekova[2]

[1] Institute of Applied Mathematics, Buketov Karaganda State University, 28 Universitetskaya st., Karaganda 100028, Kazakhstan
orumbayevan@mail.ru

[2] Arkalyk State Pedagogical Institute named after Y. Altynsarin, 17 Auelbekova st., Arkalyk 110300, Kazakhstan
gulmira7629@mail.ru

Abstract. In this paper we consider a nonlinear semi-periodic boundary-value problem for a partial differential equation. By means of a replacement, the nonlinear problem is reduced to a linear semi-periodic boundary-value problem for hyperbolic equations with a mixed derivative. To solve the obtained problem, partitioning by the first variable is made. Further, in the obtained domains, the parametrization method proposed in the works of D.S. Dzhumabaev for solving a two-point boundary value problem for an ordinary differential equation is applied. A new algorithm for finding the solution to the given problem is proposed. Sufficient conditions for the unique solvability of a semi-periodic boundary-value problem with arbitrary functions for a nonlinear partial differential equation are established.

Keywords: Nonlinear equation · Algorithm · Semi-periodic boundary value problem · Parametrization method · Systems of hyperbolic equations · Solvability conditions

1 Introduction

We consider a nonlinear semi-periodic boundary-value problem for a partial differential equation. Earlier in work of G.B. Whitham [4] the equations containing arbitrary parameters of the form

$$\frac{\partial^2 z}{\partial x \partial y} = K \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + s\frac{\partial z}{\partial x} + m\frac{\partial z}{\partial y}$$

are considered. Such equations are encountered in some problems of chemical technology and chromatography. In the paper, by using a replacement, a nonlinear semi-periodic boundary value problem with arbitrary functions is reduced

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_14

to a linear semi-periodic boundary-value problem for hyperbolic equations with a mixed derivative. The obtained problem was investigated in [2,3] by the parametrization method [1]. In this paper, we propose a new approach to solving a linear semi-periodic boundary-value problem, where the partition is made both with respect to the variable y, and the variable x.

2 Formulation of the Problem

On $\Omega = [0, X] \times [0, Y]$ we consider the periodic boundary value problem for nonlinear differential equations with partial derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = k \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + a(x, y) \cdot \frac{\partial z}{\partial x} + f(x, y), \tag{1}$$

$$z(0, y) = \psi(y), \tag{2}$$

$$z(x, 0) = z(x, T), \tag{3}$$

where $k = const$, $\psi(y)$ is given function depending on y, $a(x, y), f(x, y)$ are arbitrary functions depending on x and y.

To solve the problem (1) - (3) $u = e^{kz}$ we make the replacement, then we obtain the linear periodic boundary value problem

$$\frac{\partial^2 u}{\partial x \partial y} = a(x, y) \cdot \frac{\partial u}{\partial y} + k \cdot f(x, y) \cdot u, \tag{4}$$

$$u(0, y) = e^{k\psi(y)}, \tag{5}$$

$$u(x, 0) = z(x, Y), \tag{6}$$

$$z(x, y) = \frac{1}{k} lnu(x, y). \tag{7}$$

3 Main Result

We take numbers $\tau > 0$, $h > 0$ such that $M\tau = X$, $Nh = Y$ and make a partition $[0, X) = \bigcup_{i=1}^{M} [(i-1)\tau, i\tau)$, $[0, Y) = \bigcup_{j=1}^{N} [(j-1)h, jh)$, $M \geq 2$, $N \geq 2$. In this case, the domain Ω is divided on $M \times N$ parts. We denote $u_{ij}(x, y)$ as the restriction of the function $u(x, y)$ to $\Omega_{ij} = [(i-1)\tau, i\tau) \times [(j-1)h, jh)$, $i = \overline{1, M}$, $j = \overline{1, N}$.

To find the solution to the problem, we introduce a new unknown function $v_{ij}(x, y) = \frac{\partial u_{ij}(x,y)}{\partial x}$, $i = \overline{1, M}$, $j = \overline{1, N}$ and we write the problem (4) - (7) in the form

$$\frac{\partial v_{ij}}{\partial y} = A(x, y) v_{ij} + k \cdot f(x, y) \cdot u_{ij}(x, y), \quad (x, y) \in \Omega_{ij}, \tag{8}$$

$$\lim_{y \to sh-0} v_{is}(x, y) = v_{i,s+1}(x, sh), \quad i = \overline{1, M}, \quad s = \overline{1, N-1}, \tag{9}$$

$$v_{i1}(x,0)-\lim_{t\to Y-0}v_{iN}(x,y)=0,\quad i=\overline{1,M},, \tag{10}$$

$$u_{1j}(x,y)=e^{k\psi(y)}+\int_0^x v_{1j}(\xi,y)d\xi,\quad y\in[(j-1)h,jh),\quad j=\overline{1,N}, \tag{11}$$

$$u_{d+1,j}(x,y)=\lim_{x\to d\tau-0}u_{dj}(x,y)+\int_{d\tau}^x v_{d+1,j}(\xi,y)d\xi,\quad d=\overline{1,M-1}, \tag{12}$$

$$z_{ij}(x,y)=\frac{1}{k}lnu_{ij}(x,y). \tag{13}$$

We introduce the notation $\lambda_{ij}(x)=v_{ij}(x,(j-1)h)$ and make a replacement $\tilde{v}_{ij}(x,y)==v_{ij}(x,y)-\lambda_{ij}(x),\quad i=\overline{1,M},\quad j=\overline{1,N}$. Then we obtain a boundary value problem with unknown functions $\lambda_{ij}(x)$:

$$\frac{\partial\tilde{v}_{ij}}{\partial t}=A(x,y)\tilde{v}_{ij}+A(x,y)\lambda_{ij}(x)+k\cdot f(x,y)\cdot u_{ij}(x,y), \tag{14}$$

$$\tilde{v}_{ij}(x,(j-1)h)=0,(x,y)\in\Omega_{ij},\quad i=\overline{1,M},\quad j=\overline{1,N}, \tag{15}$$

$$\lambda_{i1}(x)-\lambda_{iN}(x)-\lim_{y\to Y-0}\tilde{v}_{iN}(x,y)=0,\quad i=\overline{1,M}, \tag{16}$$

$$\lambda_{is}(x)+\lim_{y\to sh-0}\tilde{v}_{is}(x,y)=\lambda_{i,s+1}(x),\quad s=\overline{1,N-1}, \tag{17}$$

$$u_{1j}(x,y)=e^{k\psi(y)}+\int_0^x(\tilde{v}_{1j}(\xi,y)+\lambda_{1j}(\xi))d\xi,y\in[(j-1t)h,jh),j=\overline{1,N}, \tag{18}$$

$$u_{d+1,j}(x,y)=\lim_{x\to d\tau-0}u_{dj}(x,y)+\int_{d\tau}^x(\tilde{v}_{d+1,j}(\xi,y)+\lambda_{d+1,j}(\xi))d\xi, \tag{19}$$

$$z_{ij}(x,y)=\frac{1}{k}lnu_{ij}(x,y), \tag{20}$$

where $d=\overline{1,M-1}$. The last problem is distinguished by the fact that initial conditions have appeared here, that allow us to define $\tilde{v}_{ij}(x,y)$ from the integral equation

$$\tilde{v}_{ij}(x,y)=\int_{(j-1)h}^y A(x,\eta)\tilde{v}_{ij}(x,\eta)d\eta+\lambda_{ij}(x)\int_{(j-1)h}^y A(x,\eta)d\eta+$$

$$+k\int_{(j-1)h}^y f(x,\eta)\cdot u_{ij}(x,\eta)d\eta,\quad i=\overline{1,M},\quad j=\overline{1,N}. \tag{21}$$

Passing to the limit as $y\to jh-0$ on the right-hand side of (21) and substituting into (16), (17), we obtain a system of equations with respect to the parameters $\lambda_{ij}(x)$:

$$Q(x,h)\lambda_i(x)=-F(x,h,u_i)-G(x,h,\tilde{v}_i), \tag{22}$$

where

$$Q(x,h)=$$

$$= \begin{vmatrix} 1 & 0 & \dots & 0 & -[1+\int\limits_{(N-1)h}^{y} A(x,\eta)d\eta] \\ 1+\int\limits_{0}^{y} A(x,\eta)d\eta & -1 & \dots & 0 & 0 \\ 0 & 1+\int\limits_{h}^{y} A(x,\eta)d\eta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1+\int\limits_{(N-2)h}^{y} A(x,\eta)d\eta & -1 \end{vmatrix},$$

$$F(x,h,u_i) = \begin{pmatrix} k-\int\limits_{(N-1)h}^{y} f(x,\eta)u_{iN}(x,\eta)d\eta \\ k\int\limits_{0}^{y} f(x,\eta)u_{i1}(x,\eta)d\eta \\ k\int\limits_{h}^{y} f(x,\eta)u_{i2}(x,\eta)d\eta \\ \dots \quad \dots \quad \dots \\ k\int\limits_{(N-2)h}^{y} f(x,\eta)u_{i,N-1}(x,\eta)d\eta \end{pmatrix},$$

$$G(x,h,\tilde{v}_i) = \begin{pmatrix} -\int\limits_{(N-1)h}^{y} A(x,\eta)\tilde{v}_{iN}(x,\eta)d\eta \\ \int\limits_{0}^{y} A(x,\eta)\tilde{v}_{i1}(x,\eta)d\eta \\ \int\limits_{h}^{y} A(x,\eta)\tilde{v}_{i2}(x,\eta)d\eta \\ \dots \quad \dots \quad \dots \\ \int\limits_{(N-2)h}^{y} A(x,\eta)\tilde{v}_{i,N-1}(x,\eta)d\eta \end{pmatrix}.$$

To find a system of four functions $\{\lambda_{ij}(x), \tilde{v}_{ij}(x,y), u_{ij}(x,y), z_{ij}(x,y)\}$, $i=\overline{1,M}$, $j=\overline{1,N}$, we have a closed system consisting of equations (22), (21), (18), (19), (20).

On $\Omega_{1j} = [0,\tau)\times[(j-1)h, jh)$ taking as the initial approximation $u_{1j}^{(0)}(x,y) = e^{k\psi(y)}$, $j=\overline{1,N}$, we find as the limit of the sequence $\{\lambda_{1j}^{(k)}(x), \tilde{v}_{1j}^{(k)}(x,y)\}$ the first approximations on $\lambda_{1j}(x), \tilde{v}_{1j}(x,y)$ determined by the following algorithm:

Step 1. Assuming the reversibility of the matrix $Q(x,h)$ for all $x\in[0,\tau)$ from the equation (22), where $\tilde{v}_{1j}^{(0)}(x,y)=0$, we find $\lambda_1^{(1)}(x) = (\lambda_{11}^{(1)}(x), ..., \lambda_{1N}^{(1)}(x))'$:

$$\lambda_1^{(1)}(x) = -[Q(x,h)]^{-1}F(x,h,\psi).$$

Substituting the found $\lambda_{1j}^{(1)}(x)$, $j=\overline{1,N}$ into (21), we get:

$$\tilde{v}_{1j}^{(1)}(x,y) = \lambda_{1j}^{(1)}(x)\int\limits_{(j-1)h}^{y} A(x,\eta)d\eta + \int\limits_{(j-1)h}^{y} (\psi(\eta) + f(x,\eta)d\eta.$$

Step 2. From equation (22), where $\tilde{v}_{1j}(x,y) = \tilde{v}_{1j}^{(1)}(x,y)$, we define:

$$\lambda_1^{(2)}(x) = -[Q(x,h)^{-1}\{F(x,h,\psi) + G(x,h,\tilde{v}_1^{(1)})\}.$$

Again using the expression (21), we find the function $\tilde{v}_{1j}^{(2)}(x,y)$:

$$\tilde{v}_{1j}^{(2)}(x,y) = \int_{(j-1)h}^{y} A(x,\eta)\tilde{v}_{1j}^{(1)}(x,\eta)d\eta +$$

$$+\lambda_{1j}^{(2)}(x)\int_{(j-1)h}^{y} A(x,\eta)d\eta + \int_{(j-1)h}^{y} (\psi(\eta) + f(x,\eta))d\eta.$$

At the k-th step we obtain the system of pairs $\{\lambda_{1j}^{(k)}(x), \tilde{v}_{1j}^{(k)}(x,y)\}$, $j = \overline{1,N}$.

We suppose that the solution to problem (14) - (17) being the sequence of systems of pairs $\{\lambda_{1j}^{(k)}(x), \tilde{v}_{1j}^{(k)}(x,y)\}$ is defined and converges as $k \to \infty$ to continuous functions $\lambda_{1j}^{*}(x), \tilde{v}_{1j}^{*}(x,y)$ respectively on $x \in [0,\tau), (x,y) \in \Omega_{1j}$.

The functions $u_{1j}^{*}(x,y), z_{1j}^{*}(x,y)$, $j = \overline{1,N}$ are determined from the relations

$$u_{1j}^{*}(x,y) = \psi(y) + \int_0^x (\tilde{v}_{1j}^{*}(\xi,y) + \lambda_{1j}^{*}(\xi))d\xi, \quad z_{1j}(x,y) = \frac{1}{k} lnu_{1j}(x,y).$$

On $\Omega_{2j} = [\tau, 2\tau) \times [(j-1)h, jh)$ taking as the initial approximation $u_{2j}^{(0)}$ $(x,y) = = u_{1j}^{*}(\tau,y)$, $j = \overline{1,N}$, we find as the limit of the sequence $\{\lambda_{2j}^{(k)}(x), \tilde{v}_{2j}^{(k)}(x,y)\}$ the first approximations with respect to $\lambda_{21j}(x), \tilde{v}_{2j}(x,y)$ by the algorithm proposed above. And so on.

Sufficient conditions for the unique solvability of a semi-periodic boundary problem and the conditions for implementability and convergence of the proposed algorithm for finding the solution to problem (14) - (20) the following theorem is established

Theorem 1. *Let for some* $h > 0 : Nh = Y, N \geq 2$ $(Nn \times Nn)$ *the matrix* $Q(x,h)$ *be invertible for all* $x \in [(i-1)\tau, i\tau)$, $i = \overline{1,M}, M\tau = X$ *and let the following inequalities hold:*

a) $\|[Q(x,h)]^{-1}\| \leq \gamma(x,h)$;

b) $q(x,h) = [1 + \gamma(x,h)\alpha(x)h]\alpha(x)h \leq \mu < 1$.

Then there exists a unique solution of problem (14) - (20) and the following estimates hold:

1) $$\max_{j=\overline{1,N}} \sup_{y\in[(j-1)h,jh)} \|\tilde{v}_{i+1,j}^{*}(x,y) - \tilde{v}_{i+1,j}^{(m)}(x,y)\| + \max_{j=\overline{1,N}} \|\lambda_{i+1,j}^{*}(x) - \lambda_{i+1,j}^{(m)}(x)\| \leq$$

$$\leq \tilde{\theta}(h)[\tilde{q}(h)]^m[1 + (\tau\tilde{\theta}(h))^i] \cdot \max_{y\in[0,Y]} \|e^{k\psi(y)}\|,$$

2) $$\max_{j=\overline{1,N}} \sup_{y\in[(j-1)h,jh)} \|u_{i+1,j}^{*}(x,y) - u_{i+1,j}^{(m)}(x,y)\| \leq$$

$$\leq \int\limits_{i\tau}^{(i+1)\tau} \max_{j=\overline{1,N}} \sup_{y\in[(j-1)h,jh)} \|\tilde{v}^*_{i+1,j}(\xi,y) - \tilde{v}^{(m)}_{i+1,j}(\xi,y)\|d\xi +$$

$$+ \int\limits_{i\tau}^{(i+1)\tau} \max_{j=\overline{1,N}} \|\lambda^*_{i+1,j}(\xi) - \lambda^{(m)}_{i+1,j}(\xi)\|d\xi,$$

$$3) \max_{j=\overline{1,N}} \sup_{y\in[(j-1)h,jh)} \|z^*_{i+1,j}(x,y) - z^{(m)}_{i+1,j}(x,y)\| =$$

$$= \frac{1}{k} ln \max_{j=\overline{1,N}} \sup_{y\in[(j-1)h,jh)} \|u^*_{i+1,j}(x,y) - u^{(m)}_{i+1,j}(x,y)\|,$$

where $\alpha(x) = \max\limits_{y\in[0,Y]} \|A(x,y)\|$, $\tilde{q}(h) = \max\limits_{x\in[(i-1)\tau,i\tau)} q(x,h), \tilde{\theta}(h) = \max\limits_{x\in[(i-1)\tau,i\tau)} \theta(x,h)$, $\theta(x,h) = [1 + \gamma(x,h)\alpha(x)h + \gamma(x,h)q(x,h)]\frac{hk}{1-q(x,h)} \max\limits_{y\in[0,Y]} \|f(x,y)\|$.

From the equivalence of problems (14) - (20) and (8) - (13) it follows that

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then a semi-periodic boundary value problem for a non-linear differential equation with arbitrary functions (1) - (3) has a unique solution* $z^*(x,y)$.

Acknowledgements. This publication is supported by the grant project 1164 / GF 4 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Dzhumabaev, D.S.: Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. Zh. Vychisl. Mat. Mat. Fiz. **29**(1), 50–66 (1989)
2. Orumbayeva, N.T.: On solvability of non-linear semi-periodic boundary-value problem for system of hyperbolic equations. Russ. Math. **60**, 23–37 (2016)
3. Orumbayeva, N.T., Shayakhmetova, B.K.: On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations. In: AIP Conference Proceedings American Institute of Physics. AIP Publishing (2016) doi:10.1063/1.4959735
4. Whitham, G.B.: Linear and Nonlinear Waves. Wiley (1999)

An Analogue of the Schwarz Problem for the Moisil–Teodorescu System

Viktor A. Polunin[1(✉)] and Alexandr P. Soldatov[1,2]

[1] Belgorod State University, 85 Pobeda str., Belgorod 308015, Russian Federation
polunin@bsu.edu.ru, soldatov@bsu.edu.ru

[2] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan

Abstract. An analogue of the Schwarz problem for the Moisil–Teodorescu system is considered in a domain D. It is shown that this problem has Fredholm property in the Hoelder class $C^{\mu}(\overline{D})$. If the domain D is homeomorphic to a ball, then the problem is investigated in detail. In particular its index is equal to -1 in this case.

Keywords: Moisil–Teodorescu system · Schwarz problem · Cauchy type integral · Singular integral equation

Let us consider the Moisil–Teodorescu system [2] in a bounded domain $D \subseteq \mathbb{R}^3$ with smooth boundary Γ for a 4-vector (u_1, v), $v = (v_1, v_2, v_3)$ written in the form

$$\operatorname{div} v = 0, \quad \operatorname{rot} v + \operatorname{grad} u_1 = 0. \tag{1}$$

It is well known that components u_1 and v_i are harmonic functions. An analogue of the Schwarz problem is the following: to find a solution $(u_1, v) \in C(\overline{D})$ of (1) under boundary value conditions

$$u_1^+ = f_1, \quad v^+ n = f_2, \tag{2}$$

where the sign + points out the boundary value, $n = (n_1, n_2, n_3)$ is the external unit normal and $v^+ n$ denotes the inner product.

If the domain D is homeomorphic to a ball, then a general problem of Riemann – Hilbert type is investigated in detail by V.I. Shevchenko [9,10]. Another approach is based on an integral representation of a special type, and it was described in [7]. These results we can apply to the problem (1), (2).

Using Gauss– Ostrogradskii formula, it follows from (1), (2) that

$$\int_{\Gamma} f_2(y) ds_y = 0. \tag{3}$$

So this orthogonality condition is necessary for a solvability of the problem.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_15

Let us introduce a cut as a simply connected smooth surface $R \subseteq \overline{D}$ with a smooth boundary ∂R such that $R \cap \Gamma = \partial R$.

Theorem 1. *Let $\Gamma = \partial D$ belong to the class $C^{1,\nu}$, $0 < \nu < 1$, and there exist m disjoint cuts $R_1, \ldots, R_m$ such that the set*

$$D_R = D \setminus R, \quad R = R_1 \cup \ldots \cup R_m, \tag{4}$$

is a simply connected domain.

Then the dimension of a solution space of the homogeneous problem (1), (2) is equal to m.

Proof. Let (u_1, v) be a solution of homogeneous problem (1), (2). Since the function u_1 is harmonic in the domain D, then $u_1 = 0$ and the second equality (1) becomes $\operatorname{rot} v = 0$. Hence, in simply connected domain D_R the function v can be defined as $\operatorname{grad} w$ of some function w, which is harmonic by virtue of the first equality of (1). It follows from the second equality of (1) that

$$\frac{\partial w^+}{\partial n} = 0. \tag{5}$$

It follows from (4) that boundary values of w on cuts satisfy the relation

$$(w^+ - w^-)\big|_{R_i} = c_i, \quad 1 \le i \le m, \tag{6}$$

with some constants c_i. Nevertheless equalities $c_1 = \ldots = c_m = 0$ indicate that w is univalent function. So it is harmonic in the whole domain D, while in a view of (5) this is possible only if w is a constant. These arguments prove that the space of solutions of the homogeneous problem is finite dimensional space and its dimension doesn't exceed m.

In fact this dimension is equal to m exactly. Indeed it is sufficiently to prove that the problem (5), (6) with additional condition

$$\left(\frac{\partial w}{\partial n}^+ - \frac{\partial w}{\partial n}^-\right)\bigg|_R = 0, \tag{7}$$

is always solvable in the domain D_R. It is easily to establish with help of the Dirichlet integral. Let $W^{1,2}(\hat{D}_R)$ denote a space of all functions φ such that for every Lipschitzian subdomain $D_0 \subseteq D_R$ the restriction $\varphi|_{D_0}$ belongs to the Sobolev space $W^{1,2}(D_0)$. Obviously there exist one-sided boundary values $\varphi^\pm \in L^2(R)$ for elements φ of this space. It is proved by usual way [8] that the minimum of the integral

$$\mathscr{D}(\varphi) = \int_D |\operatorname{grad} \varphi|^2 dx$$

for $\varphi \in W^{1,2}(\hat{D}_R)$ satisfying (6) gives a generalized solution of the problem (5) – (7). In fact this solution is a classical one, it completes the proof.

Let us consider a question on a Fredholm property of the problem (1), (2). This question is solved with the help of an integral operator I which is defined for two vector-valued functions $\varphi = (\varphi_1, \varphi_2) \in C(\Gamma)$ by the formula

$$u_1(x) = \frac{1}{2\pi} \int_\Gamma \frac{(y-x)n(y)}{|y-x|^3} \varphi_1(y) d_2 y, \quad x \in D,$$
$$v(x) = \frac{1}{2\pi} \int_\Gamma \frac{y-x}{|y-x|^3} \varphi_2(y) d_2 y - \frac{1}{2\pi} \int_\Gamma \frac{[y-x, n(y)]}{|y-x|^3} \varphi_1(y) d_2 y,$$

where $d_2 y$ is the a surface element and brackets denote a vector product. Note that the first equality of this formula is the double layer potential for the Laplace operator.

Lemma 1. *The function $(u_1, v) = I\varphi$ is a solution of (1), under assumption $\Gamma \in C^{1,\nu}$ the operator I is bounded $C^\mu(\Gamma) \to C^\mu(\overline{D})$, $0 < \mu < \nu$, and the following boundary value formula*

$$u_1^+(y_0) = \varphi_1(y_0) + \frac{1}{2\pi} \int_\Gamma \frac{(y-y_0)n(y)}{|y-y_0|^3} \varphi_1(y) d_2 y, \quad y_0 \in \Gamma,$$
$$v^+(y_0) = \varphi_2(y_0)n(y_0) + \frac{1}{2\pi} \int_\Gamma \frac{y-y_0}{|y-y_0|^3} \varphi_2(y) d_2 y - \frac{1}{2\pi} \int_\Gamma \frac{[y-y_0, n(y)]}{|y-y_0|^3} \varphi_1(y) d_2 y,$$

holds, where the integrals of the right-hand side of the second equality are singular in the sense of the limits of the integrals over $\Gamma \cap \{|y-y_0| \geq \varepsilon\}$ as $\varepsilon \to 0$.

Proof. The problem (1), (2) can be written in the form

$$M\left(\frac{\partial}{\partial x}\right) u(x) = 0, \quad M(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & 0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & 0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

$$H(y)u^+(y) = (f_1, f_2), \quad H(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & n_2 & n_3 \end{pmatrix}, \tag{8}$$

for the 4-vector $u(x) = (u_1, v_1, v_2, v_3)$. It is known that the matrix-valued function $M^\top(x)/|x|^3$, where $\top$ is the symbol of matrix transformation is the fundamental solution of this system. So for every 4-vector $\psi = (\psi_1, \dots, \psi_4) \in C(\Gamma)$ the generalized Cauchy type integral

$$(I^0\psi)(x) = \frac{1}{2\pi} \int_\Gamma \frac{M^\top(y-x)}{|y-x|^3} M[n(y)]\psi(y) d_2 y, \quad x \notin \Gamma, \tag{9}$$

gives a solution of (8). If ψ satisfies the Hoelder condition and Γ is a surface of Lyapunov type, then there exists the boundary value

$$u^\pm(y_0) = \lim_{x \to y_0, x \in D^\pm} u(x), \quad y_0 \in \Gamma,$$

and the following analogue Plemelj – Sokhotskii formula

$$u^\pm = \pm\psi + u^* \tag{10}$$

is valid, where $u^* = I^*\psi$ is defined by the singular integral

$$(I^*\psi)(y_0) = \frac{1}{2\pi}\int_\Gamma \frac{M^\top(y-y_0)}{|y-y_0|^3} M[n(y)]\psi(y)d_2y.$$

These formulas were obtained by A.V. Bitsadze [1]. As it is shown in [6] under assumption $\Gamma \in C^{1,\nu},\ 0<\nu<1$ the operator I^0 is bounded $C^\mu(\Gamma) \to C^\mu(\overline{D})$, $0<\mu<\nu$.

Putting $\psi = (\varphi_1, \varphi_2 n)$, we can see that $M(n)\psi = (\varphi_2, \varphi_1 n)$ and $M^\top(\xi)M(n)\psi = ((\xi n)\varphi_1, \varphi_2\xi - \varphi_1[\xi, n])$ so we can write

$$I^0(H^\top\varphi) = I\varphi. \tag{11}$$

Substituting this expression into (9), (10), we complete the proof.

Theorem 2. *Under assumption of Theorem 1 the problem (1), (2) has a Fredholm property.*

Proof. Denote by S the operator of the boundary value problem (1), (2). By Lemma 1 the composition of this operator with I gives the formula

$$(SI\varphi)_1 = \varphi_1 + K_{11}\varphi_1, \quad (SI\varphi)_2 = \varphi_2 + K_{21}\varphi_1 + +K_{22}\varphi_2, \tag{12}$$

where K_{ij} are the correspondent integral operators on Γ with weak singularities. As it is proved in [7], these operators are compact in the space $C^\mu(\Gamma),\ 0<\mu<\nu$. By the known Riesz theorem [5] the operator $1+K$ is Fredholm one and its index is equal to zero. In particular the image $\mathrm{im}\,(1+K) = \mathrm{im}\,(SI)$ of this operator in the space $C^\mu(\Gamma)$ has a finite codimension. Since $\mathrm{im}\,S \supseteq \mathrm{im}\,(SI)$ the image of the operator S has the same property. Together with Theorem 1 it follows that the problem (1), (2) has a Fredholm property.

We can solve the problem of calculating of the index only in the case of the domain which is homeomorphic to a ball.

Theorem 3. *Let the boundary $\Gamma = \partial D$ be homeomorphic to a sphere and belong to $C^{2,\nu}$. Then the problem (1), (2) is one-to-one solvable in the class $C^\mu(\overline{D})$ and the condition (3) is necessary and sufficient for its solvability.*

Proof. By Theorem 1 the homogeneous problem has only trivial solution. It remains to show that (3) is sufficient for solvability of the problem. First we establish that the kernel of the operator $1+K$ in (12) is a one-dimensional space. Let $\varphi + K\varphi = 0$, then considered in D function $u = I^0(H^\top\varphi)$ satisfies the homogeneous boundary conditions (2) and therefore $u = 0$.

Let us set $w = I^0(H^\top\varphi)$ in the external domain $D_1 = \mathbb{R}^3 \setminus \overline{D}$. It is obviously

$$w(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty. \tag{13}$$

According to (10) we have also the relation

$$-w^- = 2H^\top\varphi. \tag{14}$$

Let $\Gamma_0 \subseteq \Gamma$ be a simply connected domain with a smooth boundary $\partial\Gamma_0$. Under assumption $\Gamma \in C^{2,\nu}$ there exist linear independent tangent vectors $p, q \in C^{1,\nu}(\Gamma_0)$. They define 2×4 matrix

$$G = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ 0 & q_1 & q_2 & q_3 \end{pmatrix},$$

such that $GH^\top = 0$, where H is defined in (8). From (14) it follows that

$$Gw^- = 0. \tag{15}$$

The matrix G satisfies the well-known complementary condition which is guaranteed the Fredholm solvability of the problem. Let g^{kr} be minor defined by $k-$ and $r-$th columns of the G. Let us consider the vector $s = (s_1, s_2, s_3)$, where $s_1 = g^{12} + g^{34}$, $s_2 = g^{13} - g^{24}$, $s_3 = g^{14} + g^{23}$, is not tangent to Γ. Then the complementary condition is valid if the vector s is not tangent to Γ [10]. In our case $s = [p, q]$ and this condition is fulfilled.

Let us prove that the function w is continuously differentiable in D_1 up to $\Gamma_0 \setminus \partial\Gamma_0$. For this purpose we consider a domain $D_0 \subset D_1$ with smooth boundary $\partial D_0 \in C^{2,\nu}$ such that $\Gamma \cap \partial D_0 = \Gamma_0$. Then the matrix-valued function G can be extended smoothly to Γ_0 such that this continuation G_0 satisfies the complimentary condition on ∂D_0. The function $w_0 = w\big|_{D_0}$ is a solution of the problem

$$G_0\left(w_0\big|_{\partial D_0}\right) = f_0,$$

where $f_0 \in C^\mu(\partial D_0)$ and $f_0 = 0$ on Γ_0. This function can be considered as a weak solution and it belongs to $C^{1,+0}(\overline{D_0'})$ on the basis of the theorem on a local smoothness [4], where the subdomain $D_0' \subseteq D_0$ is such that $\Gamma' = \partial D_0' \cap \partial D_0 \subseteq \Gamma_0$ and $\Gamma' \cap \partial\Gamma_0 = \emptyset$.

Let us write $w = (u_1, v)$, then the boundary condition (15) for the system (1) takes the form

$$v^- p = v^- q = 0 \quad \text{on} \quad \Gamma. \tag{16}$$

Arguing as above we prove $u_1 = 0$ in D_1. In fact by Stokes theorem

$$\int_{\Gamma_0} (\operatorname{rot} v)^-(x) n(x) ds_x = \int_{\partial\Gamma_0} v^-(y) e(y) dy,$$

where $e(y)$ is unite tangent vector to $\partial\Gamma_0$. By virtue of (16) the vector v^- is proportional to n on Γ_0 and hence $v^- e = 0$. Taking into account (15) the above equality takes the form

$$\int_{\Gamma_0} \frac{\partial u_1^-}{\partial n} ds_x = 0.$$

Since the domain $\Gamma_0 \subseteq \Gamma$ is arbitrary it follows that

$$\frac{\partial u_1^-}{\partial n} = 0.$$

The first coordinate u_1 of the vector $I^0(H^\top \varphi)$ is a harmonic function and it vanishes at ∞ so we have $u_1 = 0$. Thus (1) takes the form

$$\operatorname{div} v = 0, \quad \operatorname{rot} v = 0.$$

The domain D_1 is not simply connected but by virtue of (13) we can use the same reasoning and prove that $v = \operatorname{grad} h$ with some harmonic function h vanishing in ∞. Then the boundary condition (16) transforms into

$$\frac{\partial h^-}{\partial p} = \frac{\partial h^-}{\partial q} = 0,$$

so the function h^- is a constant on Γ. There exists a unique harmonic function $h_0 \in C^2(\overline{D_1})$ vanishing at ∞ such that $h_0^- = 1$ on Γ. Hence $h = \lambda h_0$ with some $\lambda \in \mathbb{R}$ and thus $w = (0, \lambda \operatorname{grad} h_0)$. So (15) takes the form

$$-(0, \lambda \operatorname{grad} h_0)^- = 2(\varphi_1, \varphi_2 n)$$

and hence $\varphi_1 = 0$, $\varphi_2 = \lambda\psi$, where

$$\psi = -(\operatorname{grad} h_0)^- \, n = -\frac{\partial h_0^-}{\partial n}. \tag{17}$$

Conversely if φ is a function of such a type, then $I\varphi = I^0(H^\top \varphi) = 0$ and therefore $\varphi + K\varphi = 0$.

Since the operator $1 + K$ has a Fredholm property and $\operatorname{ind}(1 + K) = 0$ the codimension of its image $\operatorname{im}(1+K)$ is equal to 1. In particular the orthogonality condition (3) is necessary and sufficient for solvability of the equation $\varphi + K\varphi = f$ and therefore of the original problem (1), (2).

Acknowledgements. The work was supported by Project 1.7311.2017/BCH Russian Federation and International Project 3492/GF4 Ministry of Education and Science of Kazakhstan Republic. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Bitsadze, A.V.: Spatial analogue of Cauchy type integral and some its applications. Izv. kad. Nauk SSSR math. **17**(6), 525–538 (1953) (in Russian)
2. Bitsadze, A.V.: Foundation of the Theory of Analytic Functions of a Complex Variable. Nauka (1972) (in Russian)
3. Milnor, J.W.: Topology from the Differentiable Viewpoint. Prinston University Press, Princeton (1965)
4. Nazarov, S.A., Plamenevskiy, B.A.: Elliptic Boundary Value Problems in Piece-Wise Smooth Domains. Nauka (1991) (in Russian)
5. Palais, P.: Seminar on the Atiah-Singer Index Theorem. Princton University Press, Prinston (1965)

6. Polunin, V.A., Soldatov, A.P.: Three-dimensional analogue of the Cauchy type integral. Differ. Uravn. **47**(3), 366–375 (2011) (in Russian)
7. Polunin, V.A., Soldatov, A.P.: On the integral representation of solutions for Moisil-Theodorescu system. Vedomosty. BSU. **5**(118), 84–92 (2011) (in Russian)
8. Rektorys, K.: Variational Methods in Mathematical Sciences and Engineering. SNTL, Dordrect, Boston (1980)
9. Shevchenko, V.I.: On the Riemann–Hilbert problem for a holomorphic vector. Doklady kad. Nauk SSSR. **169**(6), 1285–1288 (1966) (in Russian)
10. Shevchenko, V.I.: Some boundary value problems for holomorphic vector. Sborn Math. Phys. Kiev. **8**, 172–187 (1970) (in Russian)

On an Ill-Posed Problem for the Laplace Operator with Data on the Whole Boundary

Berikbol T. Torebek(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
torebek@math.kz

Abstract. In this paper a nonlocal problem for the Poisson equation in a rectangular domain is considered. It is shown that this problem is ill-posed as well as the Cauchy problem for the Laplace equation. The method of spectral expansion in eigenfunctions of the nonlocal problem for equations with deviating argument establishes a criterion of the strong solvability of the considered nonlocal problem. It is shown that the ill-posedness of the nonlocal problem is equivalent to the existence of an isolated point of the continuous spectrum for a nonself-adjoint operator with the deviating argument.

Keywords: Nonlocal problem · Ill-posed problem · Laplace operator · Well-posedness · Equation with deviating argument · Spectral problem · Self-adjoint operator

1 Introduction

As it is known, Hadamard [3] constructed an example showing the instability of the solutions of the Cauchy problem for the Laplace equation. In [1,7] and others, this Cauchy problem is reduced to integral equations of the first kind, and the different methods of regularization of the problem are shown and its conditional well-posedness is installed. In contrast to the presented results, in this paper a new criterion of well-posedness (ill-posedness) of nonlocal boundary value problem for a Poisson equation in rectangular is proved. The principal difference of our work from the work of other authors is the application of spectral problems for equations with deviating argument in the study of ill-posed nonlocal boundary value problems. The present method was first used in [4] for the solution of the Cauchy problem for the two-dimensional Laplace equation. Further, this method was developed in [5,6,9]. Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $Q = \Omega \times (0, 1) \subset R^{n+1}$ be a cylinder. In Q we consider the following problem for the Poisson equation

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_16

$$\mathscr{L}u \equiv -\Delta_{x,t}u\left(x,t\right) = f\left(x,t\right), \left(x,t\right) \in Q, \tag{1}$$

with the Dirichlet condition

$$u\left(x,t\right) = 0,\ t \in \left[0,1\right],\ x \in \partial\Omega, \tag{2}$$

and with nonlocal conditions

$$u\left(x,0\right) - \alpha u\left(x,1\right) = 0, u_t\left(x,0\right) + \alpha u_t\left(x,1\right) = 0, x \in \overline{\Omega}. \tag{3}$$

Here $\Delta_{x,t} = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial t^2}$ is the Laplace operator and α is a real number.

Definition 1. The function $u \in L_2(Q)$ will be called a strong solution of the nonlocal problem (1)-(3), if there exists a sequence of functions $u_n \in C^2\left(\overline{Q}\right)$ satisfying the conditions (2) and (3), such that u_n and $\mathscr{L}u_n$ converge in the norm $L_2\left(Q\right)$, respectively to $u\left(x,t\right)$ and $f\left(x,t\right)$.

Obviously, when $\alpha^2 = 1$ problem (1)-(3) is not Noetherian. Therefore, everywhere in what follows, we assume that $\alpha^2 \neq 1$.

It is known, that Dirichlet-Laplacian eigenvalue problem

$$\begin{cases} -\Delta_x\varphi(x) = \mu\varphi(x),\ x \in \Omega, \\ \varphi(x) = 0,\ x \in \partial\Omega, \end{cases} \tag{4}$$

is self-adjoint and non-negative definite operator in $L_2\left(\Omega\right)$ and it has a discrete spectrum. All eigenvalues of the problem (4) are discrete and non-negative, and the system of eigenfunctions form a complete orthonormal system in $L_2\left(\Omega\right)$.

By μ_k, $k \in N$ we denote all eigenvalues (numbered in decreasing order) and by $\varphi_k\left(x\right), k \in N$ we denote a complete system of all orthonormal eigenfunctions of the Dirichlet-Laplacian eigenvalue problem (4) in $L_2\left(\Omega\right)$.

We construct an example showing that the stability of the solution of problem (1)-(3) is disrupted. By direct calculation, it is not difficult to make sure that the function

$$u_k\left(x,t\right) = \frac{\sinh k\pi t + \alpha \sinh k\pi\left(1-t\right)}{\left(1-\alpha^2\right)k^2}\varphi_k(x),$$

if $\alpha^2 \neq 1$ is a solution of the Laplace equation with the boundary conditions (2) and

$$u_k\left(x,0\right) - \alpha u_k\left(x,1\right) = 0, \frac{\partial u_k}{\partial t}\left(x,0\right) + \alpha\frac{\partial u_k}{\partial t}\left(x,1\right) = \frac{\varphi_k(x)}{k}, x \in \left[0,1\right].$$

It is easy to see that the boundary data tends to zero as $k \to \infty$, but the solution $u_k\left(x,t\right)$ does not tend to zero in any norm. Therefore the solution of problem (1)-(3) is unstable. Therefore, problem (1) - (3) is ill-posed in the Hadamard sense.

2 Some Auxiliary Statements

In this section we present some auxiliary results to prove the main results.

In the future, the following eigenvalue problem for an elliptic equation with deviating argument will play an important role: *find numerical values of* λ *(eigenvalues), under which a problem for the differential equation with a deviating argument*

$$\mathscr{L}u \equiv -\Delta_{x,t} u\,(x,t) = \lambda u\,(x, 1-t)\,, (x,t) \in Q, \tag{5}$$

has nonzero solutions (eigenfunctions) satisfying conditions (2) *and* (3). It is easy to show that the eigenvalue problem (5), (2), (3) is nonself-adjoint. Obviously, the equivalent representation of equation (5) has the form

$$\mathscr{L}Pu = \lambda u, (t,x) \in Q,$$

where $Pu\,(x,t) = u\,(x, 1-t)$ is a unitary operator.

Theorem 1. *If* $\alpha^2 \neq 1$, *then the spectral problem (5), (2), (3) has a system of eigenvectors forming a Riesz basis*

$$u_{km}(x,t) = v_{km}\,(t)\,\varphi_k(x), \tag{6}$$

where $k, m \in N$, $v_{km}(t)$ *are nonzero solutions of the problem*

$$v''_{km}\,(t) - \mu_k v_{km}\,(t) = \lambda_{km} v_{km}\,(1-t)\,, 0 < t < 1, \tag{7}$$

$$v_{km}\,(0) - \alpha v_{km}\,(1) = v'_{km}\,(0) + \alpha v'_{km}\,(1) = 0, \tag{8}$$

and λ_{km} *are eigenvalues of problem (5), (2), (3). In addition, for large* k *the smallest eigenvalue* λ_{k1} *has the asymptotic behavior*

$$\lambda_{k1} = 4\mu_k e^{-\sqrt{\mu_k}}\,(1 + o(1))\,. \tag{9}$$

For the other eigenvalues of problem (5), (2), (3) there is a uniform estimate (separated from zero and goes to infinity).

Lemma 1. *Let* $\alpha^2 \neq 1$. *For each fixed value of the index* k *a system of normalized eigenvectors* $v_{km}\,(t)\,, m = 1, 2, ...$ *of eigenvalue problem (7)-(8), corresponding to the eigenvalues* λ_{km}, *forms a Riesz basis in* $L_2\,(0,1)\,.$
The eigenvalues λ_{km} *are roots of the equation*

$$\frac{\sqrt{\mu_k + \lambda}}{\sqrt{\mu_k - \lambda}} = \coth\frac{\sqrt{\mu_k + \lambda}}{2}\coth\frac{\sqrt{\mu_k - \lambda}}{2}. \tag{10}$$

For each k *the system of eigenvectors of problem (7)-(8) can be obtained from orthonormal basis via bounded invertible transformation* $\mathscr{A}$. *Here the operator* $\mathscr{A}$ *does not depend on the index* k.

Proof. Applying the operator $\frac{d^2}{dt^2} - \mu_k$ to both sides of equation (7), taking into account the nonlocal condition from (8), we obtain a problem for the equation

$$\frac{d^4 v_{km}}{dt^4}(t) - 2\mu_k \frac{d^2 v_{km}}{dt^2}(t) = \left(\lambda_{km}^2 + \mu_k^2\right) v_{km}(t), 0 < t < 1, \tag{11}$$

with nonlocal conditions

$$\begin{cases} u(0) = \alpha u(1), \ u'(0) = -\alpha u'(1), \\ u''(1) - \alpha u''(0) = \left(1 - \alpha^2\right) \mu_k u(1), \\ u'''(1) + \alpha u'''(0) = \left(1 - \alpha^2\right) \mu_k u'(1). \end{cases} \tag{12}$$

It is easy to verify that for $\alpha^2 \neq 1$ the boundary conditions (12) are regular by Birkhoff, and even strongly regular [8]. Then the system of eigenvectors of the spectral problem (11)-(12) forms a Riesz basis [2]. It is easy to notice that the eigenfunctions of problem (11)-(12) are also the eigenfunctions of problem (7). Therefore the system of eigenvectors of the spectral problem (7)-(8) forms a Riesz basis in $L_2(0,1)$.

A system of elements is a Riesz basis if and only if, when this system can be obtained from orthonormal basis via bounded invertible transformation. Further we need the exact form of this transformation. Let us consider an operator acting in $L_2(0,1)$ according to the formula

$$\mathscr{A}\varphi(t) = \varphi(t) - \alpha\varphi(1-t).$$

It is easy to verify that the operator $\mathscr{A}$ is bounded and for $\alpha \neq 1$ is bounded invertible. The inverse operator acts by the formula

$$\mathscr{A}^{-1}\varphi(t) = \frac{1}{1-\alpha^2}\left(\varphi(t) - \alpha\varphi(1-t)\right).$$

Let $v_{km}(t)$ be eigenvectors of problem (11)-(12). We denote $\hat{v}_{km}(t) = \mathscr{A}^{-1} v_{km}(t)$. By direct calculation is not difficult to make sure that for each k the system of functions $\hat{v}_{km}(t)$ are eigenfunctions of an operator defined by the differential expression

$$\ell_k \hat{v} = \frac{d^4 \hat{v}}{dt^4}(t) - 2\mu_k \frac{d^2 \hat{v}}{dt^2}(t), \ 0 < t < 1,$$

and boundary conditions

$$\begin{cases} u(0) = 0, \ u'(0) = 0, \\ u''(1) = \mu_k u(1), \ u'''(1) = \mu_k u'(1). \end{cases}$$

The operator ℓ_k is self-adjoint. Consequently, for each k the system of normalized eigenvectors of the operator ℓ_k forms an orthonormal basis in $L_2(0,1)$. Thus, for each k the system of eigenvectors of problem (11)-(12) can be obtained from orthonormal basis $\hat{v}_{km}(t)$ by bounded invertible transformation $\mathscr{A}$. At the same time, this operator $\mathscr{A}$ does not depend on k.

It is easy to show that a general solution of equation (7) has the form

$$v(t) = c_1 \cosh\sqrt{\mu_k+\lambda}\left(t-\frac{1}{2}\right) + c_2 \sinh\sqrt{\mu_k-\lambda}\left(t-\frac{1}{2}\right),$$

where c_1 and c_2 are some constants. Using the nonlocal conditions (7), we arrive at the system of linear homogeneous equations concerning these constants. As we know, this system has a nontrivial solution if the determinant of system (10) is zero. Thus, for determining the parameter λ we get (10). The proof is complete.

Let

$$\varpi_k(\lambda) = \ln\coth\frac{\sqrt{\mu_k+\lambda}}{2} + \ln\coth\frac{\sqrt{\mu_k-\lambda}}{2} - \ln\sqrt{\frac{\mu_k+\lambda}{\mu_k-\lambda}} = 0. \tag{13}$$

Lemma 2. *There exists a number λ_0 such that for all*

$$0 < \lambda < \lambda_0 < \frac{\mu_k}{4\mu_k+\theta},\ k \geq 1,\ \theta \in (0,\,1),$$

the following statements are true:

1) the function $\varpi_k'(\lambda)$ is of a fixed sign;
2) for the function $\varpi_k''(\lambda)$ we have the inequality $|\lambda\mu_k\varpi_k''(\lambda)| < 1$, $k > 1$.

Proof. By Lemma 1 we have the real eigenvalues of (7) - (8), that is, real roots λ_{km} of equation (10). It is easy to verify that $\lambda_{km} > 0$. Indeed, let us write the asymptotic behavior of the smallest eigenvalues λ_{km} at $k \to \infty$.

Assuming $|\lambda| < 1$ and taking the logarithm of both sides of (10), we obtain (13). By calculating the derivative, we get $\varpi_k'(0) = -\frac{1}{\mu_k}$. Then the required boundary of monotonicity of $\varpi_k(\lambda)$ can be determined from the relation

$$\varpi_k'(\lambda_0) = \varpi_k'(0) + \varpi_k''(\theta\lambda_0)\,\lambda_0 < 0.$$

Here $0 < \lambda_0 < 1$ and $\theta \in (0,1)$ are arbitrary numbers. Thus, for determining λ_0, we have the condition

$$\lambda_0\mu_k\varpi_k''(\theta\lambda_0) < 1. \tag{14}$$

Then the inequality

$$\varpi_k''(\lambda_0\theta) \leq \frac{1}{(\mu_k-\lambda_0\theta)}\,\frac{2+\left(1-e^{-\sqrt{\mu_k-\lambda_0\theta}}\right)^2}{\left(1-e^{-\sqrt{\mu_k-\lambda_0\theta}}\right)^2}$$

is true. Hence

$$\varpi_k''(\lambda_0\theta) < \frac{1}{(\mu_k-\lambda_0\theta)}\,\frac{3-2e^{-\sqrt{\mu_k-\lambda_0\theta}}+e^{-2\sqrt{\mu_k-\lambda_0\theta}}}{\left(1-e^{-\sqrt{\mu_k-\lambda_0\theta}}\right)^2} \tag{15}$$

Further, for large values k from (15) we obtain the validity of the inequality $\varpi_k''(\lambda_0\theta) \leq \frac{4}{\mu_k-\lambda_0\theta}$. Applying the condition (14) to the last inequality, we obtain the desired estimate for λ_0 : $\lambda_0 < \frac{\mu_k}{4\mu_k+\theta}$, $k > 1$, $0 < \theta < 1$. The proof is complete.

Consider now the question of an asymptotic behavior of the eigenvalues of problem (7)-(8) for large k.

Lemma 3. *An asymptotic behavior of eigenvalues of problem (7)-(8), not exceeding* λ_0, *for the large values of* k *has form (9).*

Proof. According to Lemma 2, the monotonic function $f_k(\lambda)$ in the interval $(0,\, \lambda_0)$ can have only one zero. By the Taylor formula we have

$$\varpi_k\left(\lambda\right)=\varpi_k\left(0\right)+\frac{\varpi_k'\left(0\right)}{1!}\lambda+\frac{\varpi_k''(\theta\lambda)}{2!}\lambda^2<0\,,\,0<\theta<1\,.$$

Substituting the calculated values of the function ϖ_k and its derivative ϖ_k', we get

$$\varpi_k\left(\lambda\right)=2\ln\left(\coth\frac{\sqrt{\mu_k}}{2}\right)-\frac{\lambda}{\mu_k}+\varpi_k''\left(\theta\lambda\right)\frac{\lambda^2}{2}.$$

Then the zero of the linear part of the function

$$\mu_k\varpi_k\left(\lambda\right)=2\mu_k\ln\left(\coth\frac{\sqrt{\mu_k}}{2}\right)-\lambda+\frac{\mu_k\lambda^2}{2}\varpi_k''\left(\theta\lambda\right)$$

will be $\lambda_{k1}=2\mu_k\ln\left(\frac{1+e^{-\sqrt{\mu_k}}}{1-e^{-\sqrt{\mu_k}}}\right).$

For sufficiently large values $k\in N$, considering the asymptotic formulas, λ_{k1} can be written as $\lambda_{k1}=4\mu_ke^{-\sqrt{\mu_k}}\left(1+o\left(1\right)\right)$.

Taking into account the result of Lemma 2 on a circle $|\lambda|=4\mu_ke^{-\sqrt{\mu_k}}\left(1+\varepsilon\right)$, where ε is a greatly small positive number, for sufficiently large $k\geq k_0(\varepsilon)$ it is easy to check the validity of the inequality

$$\left|\varpi_k''\left(\theta\lambda\right)\mu_k\right|_{|\lambda|=4\mu_ke^{-\sqrt{\mu_k}}(1+\varepsilon)}\leq C\left|2\mu_k\ln\left|\frac{1+e^{-\sqrt{\mu_k}}}{1-e^{-\sqrt{\mu_k}}}\right|-\lambda\right|_{|\lambda|=4\mu_ke^{-\sqrt{\mu_k}}(1+\varepsilon)}.$$

Then, by Rouche's theorem [10] we have that the quantity of zeros of $\mu_k\varpi_k\left(\lambda\right)$ and its linear part coincide and are inside the circle $|\lambda|=4\mu_ke^{-\sqrt{\mu_k}}\left(1+\varepsilon\right)$. Consequently, the function $\left(k\pi\right)^2\varpi_k\left(\lambda\right)$ for $0<\lambda<\lambda_0$ has one zero, the asymptotic behavior is given by formula (9). The proof is complete.

Proof. (Theorem 1) The system of eigenfunctions $\varphi_k(x), k\in N$ of the Dirichlet-Laplacian problem (4) forms a complete orthonormal system in $L_2\left(\Omega\right)$. By Lemma 1 for each fixed value of k and for $\alpha^2\neq1$ the spectral problem (7) has the system of eigenvectors $v_{km}\left(t\right), m=1,2,...$ forming a Riesz basis in $L_2\left(0,1\right)$. Here the system of eigenvectors $v_{km}(t)$ of the eigenvalue problem (7)-(8) can be obtained from orthonormal basis $\hat{v}_{km}(t)$ by the bounded invertible transformation $\mathscr{A}$, which does not depend on the index k. Therefore, the system (6) also can be obtained from the orthonormal basis $\hat{v}_{km}(t)\varphi_k(x)$ via the bounded invertible transformation $\mathscr{A}$. Consequently, system (6) forms a Riesz basis in $L_2\left(Q\right)$. The proof is complete.

3 Main Results

Theorem 2. *Let $\alpha^2 \neq 1$. A strong solution of the nonlocal problem (1) - (3) exists if and only if $f(x, t)$ satisfies the inequality*

$$\sum_{k=1}^{\infty} \left| \frac{\tilde{f}_{k1}}{\lambda_{k1}} \right|^2 < \infty, \tag{16}$$

where $\tilde{f}_{km} = (f(x, 1-t), w_{km}(x, t))$, the system $w_{km}(x, t)$ is orthogonal to $u_{km}(x,t)$. If condition (16) holds, then a solution of (1)-(3) can be written as

$$u(x,t) = \sum_{k=1}^{\infty} \frac{\tilde{f}_{k1}}{\lambda_{k1}} v_{k1}(t) \varphi_k(x) + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} v_{km}(t) \varphi_k(x). \tag{17}$$

Proof. Let $u(x,t) \in C^2(Q)$ be a solution of problem (1) - (3). Then, by the basicity of its eigenfunctions $u_{km}(x,t)$ of problem (5), (2), (3), the function $u(x,t)$ in $L_2(Q)$ can be expanded in a series [8]

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_{km} u_{km}(x,t), \tag{18}$$

where a_{km} are Fourier coefficients by the system $u_{km}(x,t)$. Rewriting equation (1) in the form

$$LPu = P(\Delta_x u(x,t) + u_{tt}(x,t)) = Pf(x,t), \tag{19}$$

and substituting the solution of form (18) in equation (19) according to representation

$$P\Delta_{x,t} u_{km}(x,t) = \lambda_{km} u_{km}(x,t),$$

we have $a_{km} = \frac{\tilde{f}_{km}}{\lambda_{km}}$ with $\tilde{f}_{km} = (f(x, 1-t), w_{km}(x, t))$.

Thus for solutions $u(x,t)$ we obtain the following explicit representation

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x,t). \tag{20}$$

Note that the representation (20) remains true for any strong solution of problem (1) - (3). We have obtained this representation under the assumption that the solution of the nonlocal problem (1) - (3) exists.

The question naturally arises, for what subset of the functions $f \in L_2(Q)$ there exists a strong solution? To answer this question, we represent formula (20) in the form (17) from which, by Hilbert's and Bessel's inequality, it follows

$$a \sum_{k=1}^{\infty} \left| \frac{\tilde{f}_{k1}}{\lambda_{k1}} \right|^2 + a \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \left| \frac{\tilde{f}_{km}}{\lambda_{km}} \right|^2 \leq \|\mathrm{u}\|^2 \leq b \sum_{k=1}^{\infty} \left| \frac{\tilde{f}_{k1}}{\lambda_{k1}} \right|^2 + b \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \left| \frac{\tilde{f}_{km}}{\lambda_{km}} \right|^2, \tag{21}$$

where $0 < a \leq b < \infty$ are given constants. By Lemma 3 we have $\lambda_{km} \geq \frac{1}{4}$, $m > 1$. Therefore, the right-hand side of equality (21) is bounded only for such $f(x, t)$, when the weighted norm (16) is bounded. This fact completes the proof.

By $\tilde{L}_2(Q)$ we denote a subspace of $L_2(Q)$, spanned by the $\{v_{k1}(t)\varphi_k(x)\}_{k=p+1}^{\infty}$, $p \in N$ and by $\hat{L}_2(Q)$ we denote its orthogonal complement $L_2(Q) = \tilde{L}_2(Q) \oplus \hat{L}_2(Q)$.

Theorem 3. *Let $\alpha^2 \neq 1$. Then for any $f \in \hat{L}_2(Q)$ a solution of problem (1)-(3) exists, is unique and belongs to $\hat{L}_2(Q)$. This solution is stable and has the form*

$$u(x,t) = \sum_{k=1}^{p} \frac{\tilde{f}_{k1}}{\lambda_{k1}} v_{k1}(t) \sin k\pi x + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} v_{km}(t) \sin k\pi x. \qquad (22)$$

Proof. Obviously, the operator $\mathscr{L}$ is invariant in $\hat{L}_2(Q)$. By Theorem 2 for any $f \in \hat{L}_2(Q)$ there exists a unique solution of problem (1)-(3) and it can be represented in the form (22). Therefore, determined infinite-dimensional space $\hat{L}_2(Q)$ is the space of well-posedness of the nonlocal problem (1)-(3). The proof is complete.

Acknowledgements. This research is financially supported by a grant No. 0820/GF4 and by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Arsenin, V.Y., Tikhonov, A.N.: Methods for Solving Ill-Posed Problems. Nauka, Moskow (1979)
2. Bari, N.K.: Biorthogonal systems and bases in Hilbert space. Uch. Zap. Mosk. Gos. Univ. **148**, 69–107 (1951)
3. Hadamard, J.: Lectures on the Cauchy Problem in Linear Differential Equations. Yale University Press, New Haven (1923)
4. Iskakova, U.A., Kal'menov, T.S.: A criterion for the strong solvability of the mixed Cauchy problem for the Laplace equation. Doklady Math. **753**, 370–373 (2007)
5. Iskakova, U.A., Kal'menov, T.S.: Criterion for the strong solvability of the mixed Cauchy problem for the Laplace equation. Diff. Equ. **4510**, 1460–1466 (2009)
6. Iskakova, U.A., Kal'menov, T.S., Sadybekov, M.A.: On a criterion for the solvability of one ill-posed problem for the biharmonic equation. J. Inverse Ill-Posed Probl. (2016). doi:10.1515/jiip-2016-0003
7. Lavrent'ev, M.M.: On a Cauchy problem for the Poisson equation. Izvestya AS USSR ser. Math. **20**(6), 819–842 (1955)
8. Naimark, M.A.: Linear Differential Operators. Part II. Ungar, New York (1968)
9. Torebek, B.T.: A method for solving ill-posed Robin-Cauchy problems for second-order elliptic equations in multi-dimensional cylindrical domains. Electron. J. Diff. Equ. **2016**(252), 1–9 (2016)
10. Titchmarsh, E.: The Theory of Functions. Nauka, Moskow (1980)

Operator Method for Constructing a Solution of a Class of Linear Differential Equations of Fractional Order

Batirkhan Kh. Turmetov[1,2(✉)]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
turmetovbh@mail.ru

[2] Department of Mathematics, Akhmet Yasawi International Kazakh-Turkish University, Turkistan 161200, Kazakhstan

Abstract. In the paper certain method for constructing exact solutions of a class of linear differential equations of fractional order is considered. Algorithms for constructing solutions of the explicit form are developed for homogeneous and inhomogeneous differential equations of fractional order. This method is based on construction of normalized systems associated with fractional differentiation operator. 0 - normalized and f - normalized systems are built concerning to the pair of operators connected with the considered equation. Using 0 - normalized systems, linearly independent solutions of the homogeneous equation are constructed. Similarly, with the help of f - normalized systems partial solutions of the inhomogeneous equation are built in the case, where the right side is a quasi-polynomial, analytic function and an arbitrary function from the class of continuous functions.

Keywords: Riemann–Liouville integral · Riemann-Liouville derivative · Differential equations of fractional order · Homogeneous differential equations · Inhomogeneous differential equations · New method · Normalized systems · Operator method · Constructing exact solutions

1 Introduction

One of the priority areas of research in the theory of differential equations of fractional order is to develop methods for constructing solutions of the explicit form. To date, there are various methods for constructing explicit solutions and a solution of the Cauchy problem for differential equations of fractional order. Such methods include the method of reduction to the integral equation [2,16],

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_17

the method of integral transforms [15], the method of Mikusinski operational calculus [11–14], Adomayn decomposition method [3]. Detailed description of these methods is considered in [10].

In constructing an exact solution of differential equations of fractional order there arise new classes of special functions. In [5,6] properties of the following functions are studied:

$$E_{\alpha,m,l}(z) = \sum_{i=0}^{\infty} c_i z^i, c_0 = 1, c_i = \prod_{k=0}^{i-1} \frac{\Gamma[\alpha(km + l) + 1]}{\Gamma[\alpha(km + l + 1) + 1]}, i \geq 1. \qquad (1)$$

In [7] on the basis of the composition formula of the Riemann-Liouville fractional differentiation operator with the function (1), found in [5], algorithm for constructing a solution of differential equation of the following form was proposed:

$$D^{\alpha} y(t) = \lambda t^{\beta} y(t) + f(t), 0 < t \leq d \leq \infty, \qquad (2)$$

where $\alpha > 0, \lambda \neq 0, \beta \in R$, D^{α} is a differentiation operator of α order in Riemann-Liouville sense, i.e.

$$D^{\alpha} y(t) = \frac{d^m}{dt^m} I^{m-\alpha} y(t), m = [\alpha] + 1, I^{\delta} y(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} y(\tau) d\tau, \delta > 0.$$

Later in [8,9], this algorithm was used to construct exact solutions of some differential equations of fractional order. Moreover, the cases, where $f(t) = 0$ and $f(t)$ is quasi-polynomial, were considered.

In this paper, we propose a new method for constructing a solution of the equation (2). In this case, unlike the work [7], we construct partial solutions of the inhomogeneous equation for a more general class of functions $f(t)$. Note that this method is based on construction of normalized systems with respect to the pair of operators $\left(D^{\alpha}, \lambda t^{\beta}\right)$ (see Section 2). We also note that this method was used in [1,17] for constructing solutions of some linear differential equations of fractional order with constant coefficients.

2 Normalized Systems

In this section we give some information about the normalized systems associated with linear differential operators.

Let L_1 and L_2 be linear operators, acting from a functional space X to X, $L_k X \subset X, k = 1, 2$. Let functions from X be defined in a domain $\Omega \subset R^n$. Let us give the definition of normalized systems [4].

Definition 1. A sequence of functions $\{f_i(x)\}_{i=0}^{\infty}, f_i(x) \in X$ is called $f-$normalized with respect to (L_1, L_2) in Ω, with base $f_0(x)$, if on this domain the equalities $L_1 f_0(x) = f(x)$, $L_1 f_i(x) = L_2 f_{i-1}(x)$, $i \geq 1$ hold.

If $L_2 =$ I is a unit operator, then the system of $f-$normalized functions with respect to (L_1, I) is called $f-$normalized with respect to the operator L_1, i.e. $L_1 f_0(x) = f(x), L_1 f_i(x) = f_{i-1}(x), i \geq 1$.

If $f(x) = 0$, then the system of functions $\{f_i(x)\}$ is called simple normalized.

Main properties of f - normalized systems of functions with respect to the operator (L_1, L_2) in Ω are given in [4]. Consider the main property of the f - normalized system.

Proposition 1. *If the system of functions* $\{f_i(x)\}_{i=0}^{\infty}$ *is* $f-$*normalized with respect to* (L_1, L_2) *in* Ω*, then the functional series* $y(x) = \sum_{i=0}^{\infty} f_i(x), x \in \Omega$ *is a formal solution of the equation:*

$$(L_1 - L_2)y(x) = f(x), x \in \Omega. \tag{3}$$

The following proposition allows us to construct a $f-$normalized system with respect to the pair of operators (L_1, L_2).

Proposition 2. *If for* L_1 *there exists a right inverse operator* L_1^{-1} *i.e.* $L_1 \cdot L_1^{-1} = E$*, where* E *is a unit operator and* $L_1 f_0(x) = f(x)$*, then a system of the functions* $f_i(x) = \left(L_1^{-1} \cdot L_2\right)^i f_0(x), i \geq 1$ *is* $f-$*normalized with respect to the pair of operators* (L_1, L_2) *in* Ω*.*

Proof. Since $L_1 \cdot L_1^{-1} = E$ is a unit operator, then for all $i = 1, 2, ...,$ we have:

$$L_1 f_i(x) = L_1 \left(L_1^{-1} \cdot L_2\right)^i f(x) = L_1 \left(L_1^{-1} \cdot L_2\right) \left(L_1^{-1} \cdot L_2\right)^{i-1} f(x)$$
$$= L_2 \left(L_1^{-1} \cdot L_2\right)^{i-1} f(x) = L_2 f_{i-1}(x).$$

Consequently, $L_1 f_i(x) = L_2 f_{i-1}(x)$, and by assumption of the theorem $L_1 f_0(x) = f(x)$, i.e., the system $f_i(x) = \left(L_1^{-1} \cdot L_2\right)^i f_0(x), i \geq 0$ is $f-$normalized with respect to the pair of operators (L_1, L_2).

3 Properties of the Operators I^α and D^α

Let us give some properties of the operators I^α and D^α. Denote

$$C_\delta[0, d] = \left\{f(t) : \exists \delta \in [0, 1), t^\delta f(t) \in C[0, d]\right\}.$$

The following proposition is well-known [10].

Lemma 1. *Let* $\alpha > 0$*. If* $f(t) \in C_\delta[0, d]$*, then the equality*

$$D^\alpha \left[I^\alpha[f]\right](t) = f(t) \tag{4}$$

holds for all $t \in (0, d]$*. When* $f(t) \in C[0, d]$*, then the equality (4) holds for all* $t \in [0, d]$*.*

Lemma 2. *Let* $\alpha > 0, m = [\alpha] + 1$ *and* $s \in R$*. Then the following equalities are true:*

$$I^\alpha t^s = \frac{\Gamma(s+1)}{\Gamma(s+1+\alpha)} t^{s+\alpha}, s > -1, \tag{5}$$

$$D^\alpha t^s = \frac{\Gamma(s+1)}{\Gamma(s+1-\alpha)} t^{s-\alpha}, s > \alpha - 1. \tag{6}$$

$$D^\alpha t^s = 0, s = \alpha - j, j = 1, 2, ..., m, \tag{7}$$

Corollary 1. *Let $\alpha > 0, m = [\alpha]+1$. Then the equality $D^\alpha y(t) = 0$ holds if and only if $y(t) = \sum\limits_{j=1}^{m} c_j t^{\alpha-j}$, where c_j are arbitrary constants.*

Lemma 3. *Let $\alpha > 0, m = [\alpha] + 1, 0 \le \delta < 1$ and $f(t) \in C_\delta[a, b]$. Then*

1) if $\alpha < \delta$, then $I^\alpha f(t) \in C_{\delta-\alpha}[a, b]$ and the following inequality is true:

$$\|I^\alpha f\|_{C_{\delta-\alpha}[a,b]} \le M\|f\|_{C_\delta[a,b]}, M = \frac{\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)}; \tag{8}$$

2) if $\alpha \ge \delta$, then $I^\alpha f(t) \in C[a, b]$ and the following inequality is true:

$$\|I^\alpha f\|_{C[a,b]} \le M\|f\|_{C_\delta[a,b]}, M = \frac{(b-a)^{\alpha-\delta}\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)}. \tag{9}$$

4 Construction of 0− Normalized Systems

In this section we construct 0 - normalized systems with respect to the pair of operators $\left(D^\alpha, \lambda t^\beta\right)$. To do this, from Proposition 2 it yields that it is necessary to find a solution of the equation $D^\alpha y(t) = 0$ and the right inverse operator for the operator D^α. By assumption of Proposition 1 the right inverse operator to the operator D^α is I^α, and due to (7), linearly independent solutions of the equation $D^\alpha y(t) = 0$ are functions $t^{s_j}, s_j = \alpha - j, j = 1, 2, ..., m$. Furthermore, denote $L_1 = D^\alpha$ and $L_2 = \lambda t^\beta$. Then the equation (2) can be represented in the form (3).

Introduce the following coefficients:

$$C(\alpha+\beta, s, i) = \prod_{k=1}^{i} \frac{\Gamma(k(\alpha+\beta)+s+1)}{\Gamma(k(\alpha+\beta)+s+1+\alpha)}, i \ge 1, C(\alpha+\beta, s, 0) = 1, s \in R.$$

We assume that $C(\alpha+\beta, s, i) \ne 0$. Let $s_j = \alpha - j, j = 1, 2, ..., m, \beta > -\{\alpha\}$ and

$$f_{0,s_j}(t) = \frac{t^{s_j}}{\Gamma(s_j+1)}.$$

Due to (7),

$$L_1 f_{0,s_j}(t) = 0, j = 1, 2, ..., m.$$

Further, consider a system of functions:

$$f_i(t) = \left(I^\alpha \cdot \lambda t^\beta\right)^i f_{0,s_j}(t), i \ge 1. \tag{10}$$

Since $(D^\alpha)^{-1} = I^\alpha$ and $D^\alpha f_{0,s_j}(t) = 0$, then Proposition 2 implies that the system (10) is 0 - normalized with respect to the pair of operators $\left(D^\alpha, \lambda t^\beta\right)$.

We find an explicit form of the system of $f_i(t)$. By definition of the operator I^α and due to (4), we get:

$$f_1(t) = I^{\alpha}[\lambda t^{\beta+s_j}] = \lambda \frac{\Gamma(\beta+s_j+1)}{\Gamma(\beta+s_j+1+\alpha)} t^{\beta+s_j+\alpha}.$$

Similarly,

$$f_2(t) = \lambda \frac{\Gamma(\beta+s_j+1)}{\Gamma(\beta+s_j+1+\alpha)} I^{\alpha}[\lambda t^{2\beta+s_j+\alpha}]$$

$$= \lambda^2 \frac{\Gamma(\beta+s_j+1)}{\Gamma(\beta+s_j+1+\alpha)} \frac{\Gamma(2\beta+s_j+\alpha+1)}{\Gamma\left(2(\alpha+\beta)+s_j+1\right)} t^{2(\alpha+\beta)+s_j}$$

$$= \lambda^2 \frac{\Gamma((\alpha+\beta)+s_j+1-\alpha)}{\Gamma((\alpha+\beta)+s_j+1)} \frac{\Gamma(2(\alpha+\beta)+s_j+1-\alpha)}{\Gamma(2(\alpha+\beta)+s_j+1)} t^{2(\alpha+\beta)+s_j}.$$

In general, using the mathematical induction method, we can get the following equation:

$$f_i(t) = \lambda^i C(\alpha+\beta, s_j, i) t^{i(\alpha+\beta)+s_j}, i \geq 1. \tag{11}$$

Indeed, for some positive integer i the equality (11) holds. Then for $i+1$ we have:

$$f_{i+1}(t) = \left(I^{\alpha} \cdot \lambda t^{\beta}\right)^{i+1} f_{0,s_j}(t) = \left(I^{\alpha} \cdot \lambda t^{\beta}\right)\left(I^{\alpha} \cdot \lambda t^{\beta}\right)^{i} f_{0.s_j}(t) = I^{\alpha}\left[\lambda t^{\beta} f_i(t)\right]$$

$$= \lambda^i C(\alpha+\beta, s_j, i) I^{\alpha}\left[\lambda t^{i(\alpha+\beta)+s_j+\beta}\right]$$

$$= \lambda^{i+1} C(\alpha+\beta, s_j, i) \frac{\Gamma(i(\alpha+\beta)+s_j+\beta+1)}{\Gamma(i(\alpha+\beta)+s_j+\beta+1+\alpha)} t^{i(\alpha+\beta)+s_j+\beta+\alpha}$$

$$= \lambda^{i+1} C(\alpha+\beta, s_j, i) \frac{\Gamma((i+1)(\alpha+\beta)+s_j+1-\alpha)}{\Gamma((i+1)(\alpha+\beta)+s_j+1)} t^{(i+1)(\alpha+\beta)+s_j}$$

$$= \lambda^{i+1} C(\alpha+\beta, s_j, i+1) t^{(i+1)(\alpha+\beta)+s_j}.$$

Therefore, (11) is true and for the case $i+1$. Further, since $s = \alpha - j$, then

$$C(\alpha+\beta, \alpha-j, i) = \prod_{k=1}^{i} \frac{\Gamma(k(\alpha+\beta)+1-j)}{\Gamma(k(\alpha+\beta)+1-j+\alpha)}, i \geq 1. \tag{12}$$

Lemma 4. *Let* $\alpha{>}0, m = [\alpha]+1, \alpha \notin N, s_j = \alpha - j, j = 1, 2, ..., m, \beta > -\{\alpha\}$. *Then for all values* $j = 1, 2, ..., m$ *the system of functions:*

$$f_i(t) = \lambda^i C(\alpha+\beta, s_j, i) t^{(\alpha+\beta)i+s_j}$$

is $0-$*normalized with respect to the pair of operators* $\left(D^{\alpha}, \lambda t^{\beta}\right)$ *in the domain* $t > 0$.

Proof. According to (6), we get $L_1 f_0(t) = D^{\alpha} t^{s_j} = 0$. Let $i \geq 1$. Then

$$L_1 f_i(t) = \lambda^i C(\alpha+\beta, s_j, i) \frac{\Gamma(i(\alpha+\beta)+s_j+1)}{\Gamma(i(\alpha+\beta)+s_j+1-\alpha)} t^{(\alpha+\beta)i+s_j-\alpha}$$

$$= \lambda t^{\beta} \lambda^{i-1} C(\alpha+\beta, s_j, i-1) t^{(\alpha+\beta)(i-1)+s_j+\beta} = \lambda t^{\beta} f_{i-1}(t).$$

Further,

$$L_2 f_{i-1}(t) = \lambda t^{\beta} f_{i-1}(t) = L_1 f_i(t).$$

Consequently, by the definition of the system of functions $f_i(t)$ is 0-normalized with respect to the pair of operators $\left(D^{\alpha}, \lambda t^{\beta}\right)$. Lemma is proved.

Using the main property of normalized systems, we obtain the following statement.

Corollary 2. *Let* $\alpha>0, \alpha \notin N, m = [\alpha]+1, s_j = \alpha - j, j=1,2,...,m, \beta > -\{\alpha\}$, $f(t) = 0$. *Then for all values* $j = 1,2,...,m$ *functions*

$$y_j(t) = \sum_{i=0}^{\infty} \lambda^i C(\alpha+\beta, s_j, i) t^{(\alpha+\beta)i+s_j}$$

satisfy the homogenous equation (2).

Further, we study the coefficients $C(\alpha+\beta, s_j, i)$. Since

$$(\alpha+\beta)k + 1 - j = \alpha\left[\left(1+\frac{\beta}{\alpha}\right)(k-1) + \frac{(\beta-j)}{\alpha} + 1\right] + 1,$$

$$k(\alpha+\beta) + 1 - j + \alpha = \alpha\left[\left(1+\frac{\beta}{\alpha}\right)(k-1) + \frac{\beta-j}{\alpha} + 2\right] + 1$$

replace the index $k-1$ to k, then from (12) it yields that

$$C(\alpha+\beta, \alpha-j, i) = \prod_{k=0}^{i-1} \frac{\Gamma\left[\alpha\left(\left(1+\beta/\alpha\right)k + 1 + (\beta-j)/\alpha\right) + 1\right]}{\Gamma\left[\alpha\left(\left(1+\beta/\alpha\right)k + (\beta-j)/\alpha + 2\right) + 1\right]}, i \geq 1.$$

Consequently, denoting $1+\frac{\beta}{\alpha} = m, 1+\frac{\beta-j}{\alpha} = \ell$ for functions $y_j(t)$ from (12), we get the representation by the functions (1) i.e.

$$y_j(t) = \sum_{i=0}^{\infty} \lambda^i C(\alpha+\beta, \alpha-j, i) t^{(\alpha+\beta)i+\alpha-j} = t^{\alpha-j} E_{\alpha, 1+\beta/\alpha, 1+(\beta-j)/\alpha}\left(\lambda t^{\alpha+\beta}\right),$$

$j = 1,2,...,m$. The representation coincides with representations received in [7] (formulas (19) and (21)).

5 Construction of $f-$ Normalized Systems

Now we turn to construction of a solution of an inhomogeneous differential equation. Let $f(t) \in C[0,d]$. Then by assumption of Lemma 1 for functions $f_0(t) = I^{\alpha} f(t)$ the following equality is true:

$$L_1 f_0(t) = D^{\alpha} I^{\alpha} f(t) = f(t).$$

Consider the system

$$f_i(t) = \left(I^{\alpha} \lambda t^{\beta} I^{\alpha}\right)^{i-1} f_0(t), i = 1,2,... \tag{13}$$

Lemma 5. *Let $\alpha > 0, \alpha \in N, m = [\alpha] + 1, \beta > -\{\alpha\}, f(t) \in C[0, d], d < \infty$. Then the system of functions (13) is $f(t)$ - normalized with respect to the pair of operators $\left(D^\alpha, \lambda t^\beta\right)$ in the domain $t > 0$.*

Proof. Since $f(t) \in C[0, d]$, then $f_0(t) = I^\alpha f(t) \in C[0, d]$. Moreover,

$$|f_0(t)| = |I^\alpha f(t)| \leq \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} |f(\tau)| d\tau \leq \|f\|_{C[0,d]} \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Hence,

$$\|f_0\|_{C[0,d]} \leq \frac{d^\alpha}{\Gamma(\alpha+1)} \|f\|_{C[0,d]}.$$

Further, when $i = 1$ we obtain

$$|f_1(t)| = \left|I^\alpha \left(\lambda t^\beta f_0(t)\right)\right| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} \tau^\beta |f_0(\tau)| d\tau$$

$$\leq \frac{|\lambda| \|f\|_{C[0,d]}}{\Gamma(\alpha)\Gamma(\alpha+1)} \int\limits_0^t (t-\tau)^{\alpha-1} \tau^{\beta+\alpha} d\tau \leq \frac{|\lambda| \|f\|_{C[0,d]}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1+\alpha)} t^{\alpha+\beta+\alpha}.$$

This implies that $f_1(t) \in C[0, d]$ and

$$\|f_1\|_{C[0,d]} \leq \frac{|\lambda|}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1+\alpha)} d^{\alpha+\beta} \|f\|_{C[0,d]}.$$

Analogously, when $i = 2$ we have that

$$|f_2(t)| = \left|I^\alpha \left(\lambda t^\beta f_1(t)\right)\right| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} \tau^\beta |f_1(\tau)| d\tau$$

$$\leq \frac{|\lambda|^2}{\Gamma(\alpha)} \frac{\|f\|_{C[0,d]}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1+\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} \tau^{2(\alpha+\beta)} d\tau$$

$$= \frac{|\lambda|^2 \|f\|_{C[0,d]}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+\alpha+1)} \frac{\Gamma(2(\alpha+\beta)+1)}{\Gamma(2(\alpha+\beta)+\alpha+1)} t^{2(\alpha+\beta)+\alpha}.$$

Moreover,

$$\|f_2\|_{C[0,d]} \leq \frac{|\lambda|^2}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+\alpha+1)} \frac{\Gamma(2(\alpha+\beta)+1) d^{2(\alpha+\beta)+\alpha}}{\Gamma(2(\alpha+\beta)+\alpha+1)} \|f\|_{C[0,b]}.$$

In general, for all $k \geq 1$ we get:

$$|f_k(t)| = \left|\left(I^\alpha \cdot \lambda t^\beta\right) f_{k-1}(t)\right| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} \tau^\beta |f_{k-1}(\tau)| d\tau$$

$$\leq \|f\|_{C[0,b]} \frac{|\lambda|^k}{\Gamma(\alpha+1)} \frac{\Gamma((\alpha+\beta)+1)}{\Gamma((\alpha+\beta)+\alpha+1)} \frac{\Gamma(2(\alpha+\beta)+1)}{\Gamma(2(\alpha+\beta)+\alpha+1)}$$

$$\cdots \frac{\Gamma(k(\alpha+\beta)+1)}{\Gamma(k(\alpha+\beta)+\alpha+1)} t^{k(\alpha+\beta)+\alpha},$$

and for norm the following equality holds:

$$\|f_k\|_{C[0,d]} \leq \frac{|\lambda|^k}{\Gamma(\alpha+1)} \frac{\Gamma((\alpha+\beta)+1)}{\Gamma((\alpha+\beta)+\alpha+1)} \frac{\Gamma(2(\alpha+\beta)+1)}{\Gamma(2(\alpha+\beta)+\alpha+1)}$$

$$\cdots \frac{\Gamma(k(\alpha+\beta)+1)}{\Gamma(k(\alpha+\beta)+\alpha+1)} d^{k(\alpha+\beta)+\alpha} \|f\|_{C[0,b]}.$$

Further, using representation of the coefficients $C(\alpha+\beta, s, i)$, we obtain

$$\frac{\Gamma((\alpha+\beta)+1)}{\Gamma((\alpha+\beta)+\alpha+1)} \frac{\Gamma(2(\alpha+\beta)+1)}{\Gamma(2(\alpha+\beta)+\alpha+1)}$$

$$\cdots \frac{\Gamma(i(\alpha+\beta)+1)}{\Gamma(i(\alpha+\beta)+\alpha+1)} = \prod_{k=1}^{i} \frac{\Gamma(k(\alpha+\beta)+1)}{\Gamma(k(\alpha+\beta)+\alpha+1)} = C(\alpha+\beta, 0, i).$$

Then the last estimation can be rewritten in the form:

$$\|f_k\|_{C[0,d]} \leq \frac{\|f\|_{C[0,b]}}{\Gamma(\alpha+1)} \lambda^k C(\alpha+\beta, 0, k) d^{k(\alpha+\beta)+\alpha}. \tag{14}$$

Thus, if $f(t) \in C[0,d]$ and $f_0(t) = I^\alpha f(t)$, that at every $i = 1, 2, ...$ the system of functions (13) belongs to the class $C[0,d]$, and (14) is true. Moreover,

$$L_1 f_1(t) = D^\alpha I^\alpha f_0(t) = f_0(t),$$

$$L_1 f_i(t) = D^\alpha \left(I^\alpha \cdot \lambda t^\beta\right)^i f_{0,s_j}(t) = D^\alpha I^\alpha \cdot \lambda t^\beta \left(I^\alpha \cdot \lambda t^\beta\right)^{i-1} f_{0,s_j}(t) = \lambda t^\beta f_{i-1}(t)$$
$$= L_2 f_{i-1}(t), i \geq 1.$$

So, in the class of functions $X = C[0,d]$ the following equalities hold:

$$L_1 f_0(t) = f(t), L_1 f_i(t) = L_2 f_{i-1}(t), i \geq 1,$$

i.e. the system (13) is $f-$ normalized with respect to the pair of operators $\left(D^\alpha, \lambda t^\beta\right)$. Lemma is proved.

Theorem 1. *Let $\alpha > 0, \alpha \in N, m = [\alpha]+1, \beta > -\{\alpha\}, f(t) \in C[0,d], d < \infty$. If the functions $f_i(t)$ are defined by the equality (13), then the function*

$$y_f(x) = \sum_{i=0}^{\infty} f_i(t) \tag{15}$$

is a partial solution of (2) from the class $C[0,d]$.

Proof. Estimate the series (15). Due to (14), we have

$$\|y_f\|_{C[0,d]} \le \sum_{i=0}^{\infty} \|f_i(t)\|_{C[0,d]} \le \frac{\|f\|_{C[0,d]} d^\alpha}{\Gamma(\alpha+1)} \quad 1 + \sum_{i=1}^{\infty} |\lambda|^i C(\alpha+\beta, 0, i) d^{i(\alpha+\beta)} \quad .$$

Since

$$C(\alpha+\beta, 0, i) = \prod_{k=1}^{i} \frac{\Gamma(k(\alpha+\beta)+1)}{\Gamma(k(\alpha+\beta)+\alpha+1)} = \prod_{k=1}^{i} \frac{\Gamma[\alpha(m(k-1)+\ell)+1]}{\Gamma[\alpha(m(k-1)+\ell)+1]}$$

$$= \prod_{k=0}^{i-1} \frac{\Gamma[\alpha(mk+\ell)+1]}{\Gamma[\alpha(mk+\ell)+1]},$$

where $m = 1 + \frac{\beta}{\alpha}, \ell = 1 + \frac{\beta}{\alpha}$, then for the function $y_f(t)$:

$$|y_f(t)| \le \frac{\|f\|_{C[0,d]}}{\Gamma(\alpha+1)} t^\alpha E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta}{\alpha}}(|\lambda| t^{\alpha+\beta}),$$

$$\|y_f\|_{C[0,d]} \le \frac{\|f\|_{C[0,d]}}{\Gamma(\alpha+1)} d^\alpha E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta}{\alpha}}(|\lambda| d^{\alpha+\beta}). \tag{16}$$

Since $E_{\alpha,m,\ell}(z)$ is an entire function, then we get convergence of the series (16) in the class of functions $C[0,d]$. Theorem is proved.

Now we study representation of functions (14) for some particular cases of the function $f(t)$.

Lemma 6. *Let $\alpha > 0, \beta > -\{\alpha\}, f(t) = t^\mu, \mu > -1$. Then partial solution of the equation (2) has the form:*

$$y_f(t) = \frac{\Gamma(\mu+1) t^{\alpha+\mu}}{\Gamma(\mu+1+\alpha)} \sum_{k=0}^{\infty} \lambda^k C(\alpha+\beta, \mu+\alpha, k) t^{k(\alpha+\beta)}.$$

Proof. Let $f(t) = t^\mu, \mu > -1$. Then, according to (4), we obtain

$$f_0(t) = I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} t^{\mu+\alpha}.$$

Further,

$$f_1(t) = \left(I^\alpha \cdot \lambda t^\beta\right) f_0(t) = \frac{\lambda \Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} I^\alpha t^{\mu+\alpha+\beta}$$

$$= \frac{\lambda \Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \frac{\Gamma(\mu+\alpha+\beta+1)}{\Gamma(\mu+\beta+2\alpha+1)} t^{\mu+2\alpha+\beta},$$

$$f_2(t) = \left(I^\alpha \cdot \lambda t^\beta\right)^2 f_0(t) = \lambda^2 \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \frac{\Gamma(\mu+\alpha+\beta+1)}{\Gamma(\mu+\beta+2\alpha+1)} I^\alpha t^{\mu+2\alpha+2\beta} =$$

$$= \frac{\lambda^2 \Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \frac{\Gamma(\alpha+\beta+\mu+1)}{\Gamma((\alpha+\beta)+\alpha+\mu+1)} \frac{\Gamma(2(\alpha+\beta)+\mu+1)}{\Gamma(2(\alpha+\beta)+\alpha+\mu+1)} t^{2(\alpha+\beta)+\alpha+\mu}.$$

In general, for every $k \geq 1$ we have

$$f_k(t) = \lambda^k \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \frac{\Gamma(\alpha+\beta+\mu+1)}{\Gamma((\alpha+\beta)+\alpha+\mu+1)}$$

$$\cdots \frac{\Gamma(k(\alpha+\beta)+\mu+1)}{\Gamma(k(\alpha+\beta)+\alpha+\mu+1)} t^{k(\alpha+\beta)+\alpha+\mu}.$$

Consequently,

$$y_f(t) = \sum_{k=0}^{\infty} f_k(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \sum_{k=0}^{\infty} \lambda^k C(\alpha+\beta, \mu+\alpha, k) t^{k(\alpha+\beta)+\mu+\alpha}.$$

Lemma is proved.

This lemma implies the following statement:

Theorem 2. *Let* $\alpha > 0, \beta > -\{\alpha\}, f(t) = \sum_{j=1}^{p} \lambda_j t^{\mu_j}, \mu_j > -1$. *Then a partial solution of the equation (2) has the form:*

$$y_f(t) = \sum_{j=1}^{p} \frac{\lambda_j \Gamma(\mu_j+1) t^{\alpha+\mu_j}}{\Gamma(\mu_j+1+\alpha)} \sum_{k=0}^{\infty} \lambda^k C(\alpha+\beta, \mu_j+\alpha, k) t^{k(\alpha+\beta)}. \tag{17}$$

The representation (17) of a partial solution of the equation (2) coincides with the result of [7] (see Theorem 2, formula (27)).

Now we give an algorithm for construction of partial solutions of the inhomogeneous equation (2) in the case when $f(t)$ is an analytical function.

Theorem 3. *Let* $\alpha > 0, \beta > -\{\alpha\}, f(t)$ *is an analytical function. Then a partial solution of the equation (1) has the form:*

$$y_f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) t^{\alpha+k}}{\Gamma(\alpha+k+1)} y_{k+\alpha}(t),$$

where $y_{k+\alpha}(t)$ *is defined by the equality*

$$y_{k+\alpha}(t) = \sum_{i=0}^{\infty} \lambda^i C(\alpha+\beta, k+\alpha, i) t^{i(\alpha+\beta)} \equiv E_{\alpha, 1+\beta/\alpha, 1+(k+\alpha+\beta)/\alpha}(\lambda t^{\alpha+\beta}).$$

Proof. If $f(t)$ is an analytical function, then it can be represented in the form

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k.$$

Then

$$f_0(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} I^\alpha t^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{k+\alpha} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1+\alpha)} t^{k+\alpha},$$

$$f_1(t) = \left(I^\alpha \cdot \lambda t^\beta\right) f_0(t) = \sum_{k=0}^{\infty} \frac{\lambda f^{(k)}(0)}{\Gamma(\alpha+k+1)} I^\alpha t^{k+\alpha+\beta}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda f^{(k)}(0)}{\Gamma(\alpha+k+1)} \frac{\Gamma(\alpha+\beta+k+1)}{\Gamma((\alpha+\beta)+k+1+\alpha)} t^{k+2\alpha+\beta},$$

$$f_2(t) = \left(I^\alpha \cdot \lambda t^\beta\right) f_1(t) = \sum_{k=0}^{\infty} \frac{\lambda f^{(k)}(0)}{\Gamma(\alpha+k+1)} \frac{\Gamma(\alpha+\beta+k+1)}{\Gamma(2\alpha+\beta+k+1+\alpha)}$$

$$\times \frac{\Gamma(2(\alpha+\beta)+k+1)}{\Gamma(2(\alpha+\beta)+k+1+\alpha)} t^{k+2(\alpha+\beta)+\alpha}.$$

In general, by mathematical induction method we can prove the equality:

$$f_i(t) = \left(I^\alpha \cdot \lambda t^\beta\right)^i f_0(t) = \sum_{k=0}^{\infty} \frac{\lambda^i f^{(k)}(0)}{\Gamma(\alpha+k+1)} \frac{\Gamma(\alpha+\beta+k+1)}{\Gamma(2\alpha+\beta+k+1+\alpha)}$$

$$\cdots \frac{\Gamma(i(\alpha+\beta)+k+1) t^{i(\alpha+\beta)+k+\alpha}}{\Gamma(i(\alpha+\beta)+k+1+\alpha)} \sum_{k=0}^{\infty} \frac{\lambda^i f^{(k)}(0)}{\Gamma(\alpha+k+1)} C(\alpha+\beta, k+\alpha, i) t^{i(\alpha+\beta)+k+\alpha}.$$

Therefore,

$$\sum_{i=0}^{\infty} f_i(t) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^i f^{(k)}(0) C(\alpha+\beta, k+\alpha, i)}{\Gamma(\alpha+k+1)} t^{i(\alpha+\beta)+k+\alpha}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0) t^{k+\alpha}}{\Gamma(\alpha+k+1)} \sum_{i=0}^{\infty} \lambda^i C(\alpha+\beta, k+\alpha, i) t^{i(\alpha+\beta)}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(\alpha+k+1)} t^{k+\alpha} y_{k+\alpha}(t).$$

Theorem is proved.

Theorem 4. *Let $\alpha > 0, \beta = n, n \geq 0, f(t) \in C[0,b], b < \infty$. Then a partial solution of the equation (2) has the form:*

$$y_f(t) = \int_0^t G_{n,\alpha}(t-\tau, \tau, \lambda) f(\tau) d\tau, \tag{18}$$

where $G_{n,\alpha}(u,v,\lambda)$ is defined by the equality:

$$G_{n,\alpha}(u,v,\lambda) = \sum_{i=0}^{\infty} G_{n,\alpha,i}(u,v,\lambda),$$

$$G_{n,\alpha,i}(u,v,\lambda)=\frac{\lambda^i}{\Gamma(\alpha)}\sum_{j_1=0}^{n}\cdots\sum_{j_i=0}^{n}\binom{n}{j_1}...\binom{n}{j_i}C(\alpha,j_1+...+j_i+\alpha,i)$$
$$u^{i\alpha+j_1+...+j_i+\alpha-1}v^{kn-j_1-...-j_i}.$$

Proof. If $f_0(t)=I^\alpha f(t)$, then $L_1f_0(t)=D^\alpha f_0(t)=f(t)$. Let $i=1,\beta=n,n=0,1,....$ Then

$$f_1(t)=\left(I^\alpha\cdot\lambda t^n\right)f_0(t)=\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-\tau)^{\alpha-1}\lambda\tau^nf_0(\tau)d\tau$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-\tau)^{\alpha-1}\lambda\tau^n\int\limits_0^\tau\frac{(\tau-z)^{\alpha-1}}{\Gamma(\alpha)}f(z)dzd\tau=$$

$$=\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^tf(z)\frac{1}{\Gamma(\alpha)}\int\limits_z^t(\tau-z)^{\alpha-1}(t-\tau)^{\alpha-1}\tau^nd\tau dz.$$

Study the integral:

$$I_n=\int\limits_z^t(\tau-z)^{\alpha-1}(t-\tau)^{\alpha-1}\tau^nd\tau.$$

After the change of variables $\tau=z+(t-z)\xi$, we have

$$I_n=\int\limits_z^t(\tau-z)^{\alpha-1}(t-\tau)^{\alpha-1}\tau^nd\tau=(t-z)^{2\alpha-1}\int\limits_0^1(1-\xi)^{\alpha-1}\xi^{\alpha-1}((t-z)\xi+z)^nd\xi$$

$$=(t-z)^{2\alpha-1}\sum_{j=0}^{n}C_n^j(t-z)^jz^{n-j}\int\limits_0^1(1-\xi)^{\alpha-1}\xi^{j+\alpha-1}d\xi$$

$$=\sum_{j=0}^{n}C_n^j\frac{\Gamma(\alpha)\Gamma(j+\alpha)}{\Gamma(j+2\alpha)}(t-z)^{j+2\alpha-1}z^{n-j}.$$

Consequently,

$$f_1(t)=\frac{\lambda}{\Gamma(\alpha)}\sum_{j_1=0}^{n}C_n^{j_1}\frac{\Gamma(j_1+\alpha)}{\Gamma(j_1+2\alpha)}\int\limits_0^t(t-z)^{j_1+2\alpha-1}z^{n-j_1}f(z)dz.$$

Similarly, for $f_2(t)$ we obtain

$$f_2(t)=\left(I^\alpha\cdot\lambda t^n\right)f_1(t)=\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-\tau)^{\alpha-1}\lambda\tau^nf_1(\tau)d\tau$$

$$= \frac{\lambda^2}{\Gamma^2(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} \tau^n \sum_{j_1=0}^{n} C_n^{j_1} \frac{\Gamma(j_1+\alpha)}{\Gamma(j_1+2\alpha)} \int\limits_0^\tau (\tau-z)^{j_1+2\alpha-1} z^{n-j_1} f(z) dz d\tau$$

$$= \frac{\lambda^2}{\Gamma^2}(\alpha) \sum_{j_1=0}^{n} C_n^{j_1} \frac{\Gamma(j_1+\alpha)}{\Gamma(j_1+2\alpha)} \int\limits_0^t z^{n-j_1} f(z) \int\limits_0^\tau (t-\tau)^{\alpha-1} (\tau-z)^{j_1+2\alpha-1} \tau^n d\tau dz.$$

Further,

$$I_{n,2} = \int\limits_0^\tau (t-\tau)^{\alpha-1} (\tau-z)^{j_1+2\alpha-1} \tau^n d\tau$$

$$= (t-z)^{3\alpha+j_1-1} \int\limits_0^1 (1-\xi)^{\alpha-1} \xi^{j+2\alpha-1} ((t-z)\xi + z)^n d\xi =$$

$$= (t-z)^{3\alpha+j_1-1} \sum_{j_2=0}^{n} C_n^{j_2} (t-z)^{j_2} z^{n-j_2} \int\limits_0^1 (1-\xi)^{\alpha-1} \xi^{j_1+j_2+2\alpha-1} d\xi$$

$$= \sum_{j_2=0}^{n} C_n^{j_2} \frac{\Gamma(\alpha)\Gamma(2\alpha+j_1+j_2)}{\Gamma(3\alpha+j_1+j_2)} (t-z)^{3\alpha+j_1+j_2-1} z^{n-j_2}.$$

Thus,

$$f_2(t) = \frac{\lambda^2}{\Gamma(\alpha)} \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} C_n^{j_1} C_n^{j_2} \frac{\Gamma(\alpha+j_1)}{\Gamma(2\alpha+j_1)} \frac{\Gamma(2\alpha+j_1+j_2)}{\Gamma(3\alpha+j_1+j_2)}$$

$$\times \int\limits_0^t (t-z)^{3\alpha+j_1+j_2-1} z^{2n-j_1-j_2} f(z) dz.$$

In general, using the mathematical induction method, we get

$$f_k(t) = \frac{\lambda^k}{\Gamma(\alpha)} \sum_{j_1=0}^{n} ... \sum_{j_k=0}^{n} C_n^{j_1} ... C_n^{j_k} \frac{\Gamma(\alpha+j_1)}{\Gamma(2\alpha+j_1)} ... \frac{\Gamma(k\alpha+j_1+...+j_k)}{\Gamma(k\alpha+j_1+...+j_k+\alpha)}$$

$$\times \int\limits_0^t (t-z)^{k\alpha+j_1+...+j_k-1} z^{2n-j_1-...-j_k} f(z) dz.$$

Since

$$\frac{\Gamma(\alpha+j_1)}{\Gamma(2\alpha+j_1)} ... \frac{\Gamma(k\alpha+j_1+...+j_k)}{\Gamma(k\alpha+j_1+...+j_k+\alpha)} = \prod_{p=1}^{k} \frac{\Gamma(p\alpha+j_1+...+j_p)}{\Gamma(p\alpha+j_1+...+j_p+\alpha)}$$

$$= C(\alpha, j_1+...+j_p, k),$$

denoting

$$G_{n,\alpha,i}(u,v,\lambda)=\frac{\lambda^i}{\Gamma(\alpha)}\sum_{j_1=0}^{n}...\sum_{j_i=0}^{n}C_n^{j_1}...C_n^{j_k}C(\alpha,j_1+...+j_i,i)$$

$$\times u^{i\alpha+j_1+...+j_k+\alpha-1}v^{kn-j_1-...-j_k},$$

$$G_{n,\alpha}(u,v,\lambda)=\sum_{i=0}^{\infty}G_{n,\alpha,i}(u,v,\lambda),$$

for the functions (15), when $\beta=n$, we obtain (18). Theorem is proved.

Example 1. Let $n=0$. Then $j_1=j_2=...=j_k=0$,

$$C(\alpha,0,k)=\prod_{p=1}^{k}\frac{\Gamma(p\alpha)}{\Gamma(p\alpha+\alpha)}=\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\frac{\Gamma(2\alpha)}{\Gamma(3\alpha)}...\frac{\Gamma(k\alpha)}{\Gamma(k\alpha+\alpha)}=\frac{\Gamma(\alpha)}{\Gamma(k\alpha+\alpha)}$$

$$G_{0,\alpha,i}(u,v,\lambda)=\frac{\lambda^i}{\Gamma(k\alpha+\alpha)}u^{i\alpha+\alpha-1},$$

$$G_{0,\alpha}(u,v,\lambda)=\sum_{i=0}^{\infty}\lambda^i\frac{u^{i\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}=t^{\alpha-1}E_{\alpha,\alpha}(\lambda u^\alpha),$$

where $E_{a,\alpha}(\lambda u^\alpha)$ is Mittag-Leffler type function [10]. In this case

$$y_f(t)=\int_0^t G_{n,\alpha}(t-\tau,\tau,\lambda)f(\tau)d\tau=\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)f(\tau)d\tau.$$

The formula is obtained from [10].

Acknowledgements. This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (0819/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Ashurov, R.R., Cabada, A., Turmetov, BKh: Operator method for construction of solutions of linear fractional differential equations with constant coefficients. Fract. Calc. Appl. Anal. **19**, 229–251 (2016)
2. Dzerbashyan, M.M., Nersesyan, A.B.: Fractional derivatives and the Cauchy problem for differential equations of fractional order. Izv. Akad. Nauk Armyan. SSR Ser. Mat. **3**, 33–54 (1995) (in Russian)
3. Hu, Y., Luo, Y., Lu, Z.: Analytical solution of the linear fractional differential equation by Adomian decomposition method. J. Comput. Appl. Math. **215**, 220–229 (2008)

4. Karachik, V.V.: Normalized system of functions with respect to the Laplace operator and its applications. J. Math. Anal. Appl. **287**, 577–592 (2003)
5. Kilbas, A.A., Saigo, M.: Fractional integrals and derivatives of functions of Mittag-Leffler type. Dokl. Akad. Nauk Belarusi. **3**(4), 22–26 (1995)
6. Kilbas, A.A., Saigo, M.: On Mittag-Leffler type functions, fractional calculus operators and solution of integral equations. Integr. Transf. Spec. Funct. **4**, 355–370 (1996)
7. Kilbas, A.A., Saigo, M.: Solution in closed form of a class of linear differential equations of fractional order. Differ. Equ. **33**, 195–204 (1997)
8. Kilbas, A.A., Saigo, M.: The solution of a class of linear differential equations via functions of the Mittag-Leffler type. Differ. Equ. **36**, 193–202 (2000)
9. Kilbas, A.A., Saigo, M.: On Mittag-Leffler type function and applications. Integr. Transf. Spec. Funct. **7**, 97–112 (1998)
10. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, North-Holland (2006)
11. Kim, M.H., Ri, G.C., Hyong-Chol, O.: Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives. Fract. Calc. Appl. Anal. **17**, 79–95 (2014)
12. Luchko, Yu., Gorenflo, R.: An operational method for solving fractional differential equations. Acta Mathematica Vietnamica **24**, 207–234 (1999)
13. Luchko, Yu., Srivastava, H.M.: The exact solution of certain differential equations of fractional order by using operational calculus. Comput. Math. Appl. **29**, 73–85 (1995)
14. Luchko, Yu., Yakubovich, S.B.: An operational method for solving some classes of integro-differential equations. Differ. Equ. **30**, 247–256 (1994)
15. Podlubny, I.: Fractal Differential Equations. Academic Press, San Diego (1999)
16. Pskhu, A.V.: Initial-value problem for a linear ordinary differential equation of noninteger order. Sbornik: Math. **202**, 571–582 (2011)
17. Shinaliyev, K., Turmetov, B.Kh., Umarov, S.: A fractional operator algorithm method for construction of solutions of fractional order differential equations. Fract. Calc. Appl. Anal. **15**, 267–281 (2012)

Part III

Differential and Integral Operators and Spectral Theory

On Degenerate Boundary Conditions for Operator D^4

Azamat M. Akhtyamov[1,2(✉)]

[1] Bashkir State University, 32 Zaki Validi st., Ufa 450076, Russia
Akhtyamov@mail.ru
[2] Institute of Mechanics of Ufa Scientific Center, 71 October ave., Ufa 450054, Russia

Abstract. The common form for degenerate boundary conditions for the operator D^4 (D^n) is found. It is shown that the matrix for coefficients of degenerate boundary conditions has a two diagonal form and the elements for one of the diagonal are units. Operator D^4 whose spectrum fills the entire complex plane are studied, too. Earlier, examples of eigenvalue problems for the differential operator of even order with common boundary conditions (not containing a spectral parameter) whose spectrum fills the entire complex plane were given. However, in connection with this, another question arises whether there are other examples of such operators. In this paper we show that such examples exist. Moreover, all eigenvalue boundary problems for the operator D^4 whose spectrum fills the entire complex plane are described. It is proved that the characteristic determinant is identically equal to zero if and only if the matrix of coefficients of boundary conditions has a two diagonal form. The elements of this matrix for one of the diagonal are units, and the elements of the other diagonal are 1, −1 and an arbitrary constant.

Keywords: Eigenvalue problems · Differential operator of even order · Degenerate boundary conditions · Operator · Spectrum · Characteristic determinant

1 Introduction

Consider the following problem for operator D^4:

$$y^{(4)}(x) = \lambda\, y(x) = s^4\, y(x), \qquad x \in [0, 1] \tag{1}$$

$$U_j(y) = \sum_{k=0}^{n} a_{jk}\, y^{(k-1)}(0) + \sum_{k=0}^{n} a_{j\,k+n}\, y^{(k-1)}(1) = 0, \qquad j, k = 1, 2, 3, 4 \tag{2}$$

It is known [12, P. 26] that if the coefficients of an ordinary linear differential equation are continuous on [0, 1], then for the spectrum of the problem (1), (2)

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_18

the following two possibilities occur: 1) there exists at most a countable number of eigenvalues such that do not have limit points in $\mathbb{C}$; 2) every $\lambda \in \mathbb{C}$ is an eigenvalue.

Direct and inverse problems with nonseparated boundary conditions for case 1) have been fairly well studied (see, for example, [14–16]). The degenerate case 2) has been studied little (The boundary conditions are called degenerate if the characteristic determinant of corresponding eigenvalue problem is constant [11, p. 29]). It is well known, perhaps, only an example for differential operator of any even order for which the spectrum fills the entire complex plane [13] (see also [10]). In this example the boundary conditions (2) have the following form

$$U_j(y) = y^{(j-1)}(0) + (-1)^{j-1}\, y^{(j-1)}(1) = 0, \qquad j = 1, 2, 3, 4. \tag{3}$$

Recently in [1] it is shown that there exist similar differential operators of any odd order. However, in connection with this, another question arises: are there other examples of such operators? In the present paper, for the operator D^4 we find other examples of such operators and describe all boundary value problems for the operator D^4 whose spectrum fills the entire complex plane. The form of degenerate boundary conditions is found, too.

The question of describing all boundary value problems with degenerate boundary conditions is related to a description of all Volterra problems. The problem for operator L is called Volterra problem if inverse operator L^{-1} is Volterra operator (see [5, p. 208]). In the case of nondegenerate boundary conditions for an arbitrary continuous function $q(x)$, the system of eigen-vectors of the operator L is complete in $L_2(0, \pi)$ (see [11, p. 29]). Therefore, Volterra problems are among problems with degenerate boundary conditions.

In [4] it is shown, that all Volterra problems for operator D^2 with common boundary conditions have the form

$$y(0) \mp a\, y(\pi) = 0, \quad y'(0) \pm a\, y'(\pi) = 0, \tag{4}$$

where $a \neq 1$. A similar result is obtained in [3] for Sturm-Liouville problems with differential equation $-y'' + q(x)\, y = \lambda\, y$ and symmetric potential $(q(x) = q(\pi - x))$.

In [9] it is described all degenerate boundary conditions for D^2. In [2] a similar result is obtained for Sturm-Liouville problems (see also [17], where there are given examples, in which it is shown that if the potential $q(x)$ is not symmetric, then the spectrum can not fill the entire complex plane).

In [9, p. 556] - [10] it is shown that there can not exist example for the operators D^2 and D^4 with finite (but not empty) spectrum. In [6] it is shown that the spectrum of common nth order linear differential operators generated by regular boundary conditions is either empty or infinite.

We denote the matrix consisting of the coefficients a_{lk} in the boundary conditions (2) by A and the minor consisting of the i_1th, i_2th, i_3th and i_4th columns of this matrix A by A_{i_1,i_2,i_3,i_4},

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \end{matrix} \right\|. \tag{5}$$

$$A_{i_1,i_2,i_3,i_4} = \begin{vmatrix} a_{1,i_1} & a_{1,i_2} & a_{1,i_3} & a_{1,i_4} \\ a_{2,i_1} & a_{2,i_2} & a_{2,i_3} & a_{2,i_4} \\ a_{3,i_1} & a_{3,i_2} & a_{3,i_3} & a_{3,i_4} \\ a_{4,i_1} & a_{4,i_2} & a_{4,i_3} & a_{4,i_4} \end{vmatrix}. \tag{6}$$

In what follows, we assume that the rank of the matrix A is equal to 4,

$$\operatorname{rank} A = 4. \tag{7}$$

The aim of this paper is to prove the following theorems:

Theorem 1. *Matrix (5) for coefficients of the degenerate boundary conditions (2) has the following form:*

$$A_1 = \left\| \begin{matrix} 1 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_4 \end{matrix} \right\| \tag{8}$$

or

$$A_2 = \left\| \begin{matrix} a_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 1 \end{matrix} \right\|, \tag{9}$$

where a_i $(i = 1, 2, 3, 4)$ are some numbers.

Theorem 2. *The characteristic determinant of problem (1), (2) is identically equal to zero if and only if matrix (5) of coefficients of the boundary conditions (2) has form (8) or (9), where $\{a_i\}$ $(i = 1, 2, 3, 4)$ are one of the following 12 sets:*

$$\begin{array}{l} 1.\quad a_1 = C_1, \quad a_2 = -1, \quad a_3 = C_1^{-1}, \quad a_4 = 1, \\ 2.\quad a_1 = C_2, \quad a_2 = 1, \quad a_3 = C_2^{-1}, \quad a_4 = -1, \\ 3.\quad a_1 = C_3, \quad a_2 = -1, \quad a_3 = 1, \quad a_4 = -1, \\ 4.\quad a_1 = C_4, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = 1, \\ 5.\quad a_1 = -1, \quad a_2 = C_5, \quad a_3 = -1, \quad a_4 = 1, \\ 6.\quad a_1 = -1, \quad a_2 = C_6, \quad a_3 = 1, \quad a_4 = C_6^{-1}, \\ 7.\quad a_1 = 1, \quad a_2 = C_7, \quad a_3 = -1, \quad a_4 = C_7^{-1}, \\ 8.\quad a_1 = 1, \quad a_2 = C_8, \quad a_3 = 1, \quad a_4 = -1, \\ 9.\quad a_1 = -1, \quad a_2 = 1, \quad a_3 = C_9, \quad a_4 = 1, \\ 10.\quad a_1 = 1, \quad a_2 = -1, \quad a_3 = -1, \quad a_4 = -1, \\ 11.\quad a_1 = -1, \quad a_2 = 1, \quad a_3 = -1 \quad a_4 = C_{11}, \\ 12.\quad a_1 = 1, \quad a_2 = -1, \quad a_3 = 1, \quad a_4 = C_{12}, \end{array} \tag{10}$$

where C_j $(j = 1, 2, \ldots, 12)$ are arbitrary constants.

Remark 1. Theorem 1 may be generalized for any order $n \geq 2$. If n is an order of differential equation, then the matrix A for coefficients of the boundary conditions has the following form

$$A = \left\| \begin{array}{cccccccc} a_{11} & a_{12} & \dots & a_{1n} & a_{1\,n+1} & a_{1\,n+2} & \dots & a_{1\,2n} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2\,n+1} & a_{2\,n+2} & \dots & a_{2\,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n\,n+1} & a_{n\,n+2} & \dots & a_{n\,2n} \end{array} \right\|, \tag{11}$$

where $\operatorname{rank} A = n$.

If the matrix A determines degenerate boundary conditions, then it has the forms:

$$A_1 = \left\| \begin{array}{cccccccc} 1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & a_n \end{array} \right\| \tag{12}$$

or

$$A_2 = \left\| \begin{array}{cccccccc} a_1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n & 0 & 0 & \dots & 1 \end{array} \right\|. \tag{13}$$

This statement can be proved similarly to Theorem 1.

Remark 2. The example (3) is a special case of only four solutions (1, 3, 8, 10) from Theorem 2. And the remaining solutions (10) differ from the example (3). Note that all 12 solutions (10) contain an arbitrary constant.

This paper is organized as follows: In Sect. 2 we prove Theorem 1, in Sect. 3 we prove Theorem 2, and in Sect. 4 we give conclusions.

2 The Form of Degenerate Boundary Conditions

In this section we prove Theorem 1 and show that the matrix for coefficients of degenerate boundary conditions has a two diagonal form and the elements for one of the diagonal are units.

Proof. The eigenvalues of the problem (1), (2) are roots of the entire function [12, P. 26] $\Delta(\lambda)$:

$$\Delta(\lambda) = \left| \begin{array}{cccc} U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) \end{array} \right|, \tag{14}$$

where

$$y_1 = \frac{1}{4}\,\exp(s\,x) + \frac{1}{4}\,\exp(-s\,x) + \frac{1}{2}\,\cos(s\,x),$$
$$y_2 = \frac{1}{4\,s}\,\exp(s\,x) - \frac{1}{4\,s}\,\exp(-s\,x) + \frac{1}{2\,s}\,\sin(s\,x),$$
$$y_3 = \frac{1}{4\,s^2}\,\exp(s\,x) + \frac{1}{4\,s^2}\,\exp(-s\,x) - \frac{1}{2\,s^2}\,\cos(s\,x),$$
$$y_4 = \frac{1}{4\,s^3}\,\exp(s\,x) - \frac{1}{4\,s^3}\,\exp(-s\,x) - \frac{1}{2\,s^3}\,\sin(s\,x),$$

are linearly independent solutions of Eq. (1) satisfying the conditions

$$y_j^{(r-1)}(0,\lambda) = \begin{cases} 0 \text{ for } j \neq r, \\ 1 \text{ for } j = r, \end{cases} \qquad j,\, r = 1,\, 2,\, 3,\, 4. \tag{15}$$

By B, B_1 and B_2 denote the following matrixes

$$B = \begin{Vmatrix} y_1(0) & y_1'(0) & y_1''(0) & y_1'''(0) & y_1(1) & y_1'(1) & y_1''(1) & y_1'''(1) \\ y_2(0) & y_2'(0) & y_2''(0) & y_2'''(0) & y_2(1) & y_2'(1) & y_2''(1) & y_2'''(1) \\ y_3(0) & y_3'(0) & y_3''(0) & y_3'''(0) & y_3(1) & y_3'(1) & y_3''(1) & y_3'''(1) \\ y_4(0) & y_4'(0) & y_4''(0) & y_4'''(0) & y_4(1) & y_4'(1) & y_4''(1) & y_4'''(1) \end{Vmatrix}.$$

$$B_1 = \begin{Vmatrix} y_1(0) & y_1'(0) & y_1''(0) & y_1'''(0) \\ y_2(0) & y_2'(0) & y_2''(0) & y_2'''(0) \\ y_3(0) & y_3'(0) & y_3''(0) & y_3'''(0) \\ y_4(0) & y_4'(0) & y_4''(0) & y_4'''(0) \end{Vmatrix}, \quad B_2 = \begin{Vmatrix} y_1(1) & y_1'(1) & y_1''(1) & y_1'''(1) \\ y_2(1) & y_2'(1) & y_2''(1) & y_2'''(1) \\ y_3(1) & y_3'(1) & y_3''(1) & y_3'''(1) \\ y_4(1) & y_4'(1) & y_4''(1) & y_4'''(1) \end{Vmatrix},$$

where

$$y_1(1) = \tfrac{1}{4}\left(e^s + e^{-s} + 2\cos(s)\right), \quad y_1'(1) = \tfrac{1}{4}\,s\big(e^s - e^{-s} - 2\sin(s)\big),$$

$$y_1''(1) = \tfrac{1}{4}\,s^2\left(e^s + e^{-s} - 2\cos(s)\right), \; y_1'''(1) = \tfrac{1}{4}\,s^3\big(e^s - e^{-s} + 2\sin(s)\big),$$

$$y_2(1) = \tfrac{1}{4\,s}\left(e^s - e^{-s} + 2\sin(s)\right), \quad y_2'(1) = \tfrac{1}{4}\left(e^s + e^{-s} + 2\cos(s)\right),$$

$$y_2''(1) = \tfrac{1}{4}\,s\left(e^s - e^{-s} - 2\sin(s)\right), \quad y_2'''(1) = \tfrac{1}{4}\,s^2\left(e^s - e^{-s} - 2\cos(s)\right),$$

$$y_3(1) = \tfrac{1}{4\,s^2}\left(e^s - e^{-s} - 2\cos(s)\right), \; y_3'(1) = \tfrac{1}{4\,s}\left(e^s - e^{-s} + 2\sin(s)\right),$$

$$y_3''(1) = \tfrac{1}{4}\left(e^s + e^{-s} + 2\cos(s)\right), \quad y_3'''(1) = \tfrac{1}{4}\,s\left(e^s - e^{-s} - 2\sin(s)\right),$$

$$y_4(1) = \tfrac{1}{4\,s^3}\left(e^s - e^{-s} - 2\sin(s)\right), \; y_4'(1) = \tfrac{1}{4\,s^2}\left(e^s + e^{-s} - 2\cos(s)\right),$$

$$y_4''(1) = \tfrac{1}{4\,s}\left(e^s - e^{-s} + 2\sin(s)\right), \quad y_4'''(1) = \tfrac{1}{4}\left(e^s + e^{-s} + 2\cos(s)\right),$$

Note that

$$y_{j-1}^{(k-1)}(1,\lambda) \equiv y_j^{(k)}(1,\lambda), \qquad j = 2,3,4,5 \quad k = 1,2,3,4. \tag{16}$$

From (15) and (16) it follows that

$$B = \|B_1, B_2\| = \left\| \begin{matrix} 1\,0\,0\,0 & y_1(1) & y_1'(1) & y_1''(1) & y_1'''(1) \\ 0\,1\,0\,0 & y_2(1) & y_1(1) & y_1'(1) & y_1''(1) \\ 0\,0\,1\,0 & y_3(1) & y_2(1) & y_1(1) & y_1'(1) \\ 0\,0\,0\,1 & y_4(1) & y_3(1) & y_2(1) & y_1(1) \end{matrix} \right\|. \tag{17}$$

Using A and B the determinant (14) represents in the form

$$\Delta(\lambda) \equiv \det(A \cdot B^T).$$

It follows from Cauchy-Binet formula [8, 1.14] that

$$\Delta(\lambda) = \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le 8} A_{i_1,i_2,i_3,i_4}\, B_{i_1,i_2,i_3,i_4} = 0. \tag{18}$$

Here we denote by $B_{i_1,i_2,i_3,i_4} = B_{i_1,i_2,i_3,i_4}(\lambda)$ the minor consisting of the i_1th, i_2th, i_3th and i_4th columns of the matrix B (lines of the matrix B^T).

By $P(s)$ denote $P(s) = A_{1234}\, B_{1234} + A_{5678}\, B_{5678}$. From the Liouville-Ostrogradsky connecting the Wronskian for the solutions of the differential equation and the coefficients in this equation it follows that [7, 17.1] $B_{1234} = \det(B_1) = W(0) = 1$, $B_{5678} = \det(B_2) = W(1) = 1$, and $P(s) = A_{1234} + B_{5678} =$ const.

All other functions $B_{i_1,i_2,i_3,i_4} = B_{i_1,i_2,i_3,i_4}(s)$ (except B_{1234} and B_{5678}) are not constants.

So if $\Delta(\lambda) \equiv C =$ const, then $\Delta(\lambda) - P(s) \equiv 0$ and one of minors A_{1234} or A_{5678} are not equal to zero. Assume the converse. Then all minors A_{i_1,i_2,i_3,i_4} of the matrix are equal to zero. This fact contradicts the condition $\operatorname{rank} A = 4$.

Suppose $A_{1234} \neq 0$. Then the matrix (5) has the following form:

$$A = \left\| \begin{matrix} 1\,0\,0\,0 & a_{15} & a_{16} & a_{17} & a_{18} \\ 0\,1\,0\,0 & a_{25} & a_{26} & a_{27} & a_{28} \\ 0\,0\,1\,0 & a_{35} & a_{36} & a_{37} & a_{38} \\ 0\,0\,0\,1 & a_{45} & a_{46} & a_{47} & a_{48} \end{matrix} \right\|.$$

(In order not to introduce new notations by a_{ij} we denote other coefficients a_{ij} than (5)).

Let us remark that the determinant $B_{2348} = y_1'''(1)$ and any other determinant B_{i_1,i_2,i_3,i_4} are linear independent. Suppose $\Delta(\lambda) \equiv C =$ const, then $\Delta(\lambda) - P(s) \equiv 0$ and $A_{2348} = 0$. From this it follows that

$$A_{2348} = \left| \begin{matrix} 0\,0\,0 & a_{18} \\ 1\,0\,0 & a_{28} \\ 0\,1\,0 & a_{38} \\ 0\,0\,1 & a_{48} \end{matrix} \right| = -a_{18} = 0. \tag{19}$$

Let us show that a_{17} , a_{28} are equal to zero, too. Indeed, $B_{3478} = y_1'(1)\, y_1'''(1) - (y_1''(1))^2$ and any other determinant B_{i_1,i_2,i_3,i_4} are linear independent. Suppose $\Delta(\lambda) \equiv C =$ const, then $\Delta(\lambda) - P(s) \equiv 0$ and $A_{3478} = 0$.

From this it follows that

$$A_{3478} = \begin{vmatrix} 0 & 0 & a_{17} & 0 \\ 0 & 0 & a_{27} & a_{28} \\ 1 & 0 & a_{37} & a_{38} \\ 0 & 1 & a_{47} & a_{48} \end{vmatrix} = a_{17} \cdot a_{28} = 0. \quad (20)$$

In addition, $B_{2347} = -B_{1348} = -y_1''(1)$ and any other determinant B_{i_1,i_2,i_3,i_4} are linear independent. This implies that

$$A_{2347} - A_{1348} = -(a_{17} + a_{28}) = 0. \quad (21)$$

Combining (20) and (21), we get

$$a_{17} = a_{28} = 0.$$

Likewise,

$$a_{16} = a_{27} = a_{38} = 0.$$

Further, $B_{1235} = y_4(1)$ and any other determinant B_{i_1,i_2,i_3,i_4} are linear independent. So if $\Delta(\lambda) - P(s) \equiv 0$, then the minor $A_{1235} = a_{45} = 0$. As before, we have

$$a_{34} = a_{46} = a_{25} = a_{36} = a_{47} = 0.$$

Therefore if $A_{1234} \neq 0$, then the matrix A has the form A_1.

Arguing as above, we see that if $A_{5678} \neq 0$, then the matrix A has the form A_2.

This completes the proof of Theorem 1. □

3 Eigenvalue Boundary Problems for the Operator D^4 Whose Spectrum Fills the Entire Complex Plane

In this section we prove that the characteristic determinant is identically equal to zero if and only if the matrix of coefficients of boundary conditions has a two diagonal form. The elements of this matrix for one of the diagonal are units, and the elements of the other diagonal are numbers (10).

Proof. If $A_{1234} \neq 0$ and $\Delta(\lambda) \equiv 0$ it follows from Theorem 1 that

$$\begin{aligned} 0 \equiv \Delta(\lambda) = \det(A_1 \cdot B^T) = 1 + \tfrac{1}{2}\left(a_1 a_2 + a_1 a_4 + a_2 a_3 + a_3 a_4\right) + a_1 a_2 a_3 a_4 + \\ + \tfrac{1}{4}\left(a_1 a_2 + a_1 a_4 + a_2 a_3 + a_3 a_4 + 2 a_1 a_3 + 2 a_2 a_4\right)\left(e^s + e^{-s}\right) \cos s + \\ + \tfrac{1}{4}\left(a_1 + a_2 + a_3 + a_4 + a_1 a_2 a_3 + \right. \\ \left. + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4\right)\left(e^s + e^{-s} + 2 \cos s\right). \end{aligned} \quad (22)$$

The functions 1, $(e^s + e^{-s})\cos s$, and $(e^s + e^{-s} + 2\cos s)$ are linear independent. So characteristic determinant (22) is identically equal to zero if and only if the coefficients a_1, a_2, a_3, a_4 are solutions of the following system of the equations

$$\begin{aligned} &2 + a_1 a_2 + a_1 a_4 + a_2 a_3 + a_3 a_4 + 2 a_1 a_2 a_3 a_4 = 0, \\ &a_1 a_2 + a_1 a_4 + a_2 a_3 + a_3 a_4 + 2 a_1 a_3 + 2 a_2 a_4 = 0, \\ &a_1 + a_2 + a_3 + a_4 + a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4 = 0. \end{aligned} \tag{23}$$

By direct calculation we find the solutions of the system of the equations (23). This solutions are (10).

If $A_{5678} \neq 0$ and $\Delta(\lambda) \equiv 0$ it follows from Theorem 1 that

$$0 \equiv \Delta(\lambda) = \det(A_2 \cdot B^T). \tag{24}$$

From this we have the system of equations (23) , the solutions of whose are (10).

This concludes the proof of Theorem 2. □

4 Conclusion

In this paper it is shown that the matrix for coefficients of degenerate boundary conditions has a two diagonal form and the elements for one of the diagonal are units. All eigenvalue boundary problems for the operator D^4 whose spectrum fills the entire complex plane are described. It is proved that the characteristic determinant is identically equal to zero if and only if the matrix of coefficients of boundary conditions has a two diagonal form. The elements of this matrix for one of the diagonal are units, and the elements of the other diagonal are numbers (10).

Let us remark that if

$$2 + a_1 a_2 + a_1 a_4 + a_2 a_3 + a_3 a_4 + 2 a_1 a_2 a_3 a_4 = C \neq 0$$

in (23), then solving of the new system of equations reduces to solving a sixth-degree equation, and therefore is no longer analytically. Therefore, we can not write specific expressions for the coefficients in Theorem 1. The system (23) can be solved analytically in view of the fact that the coefficients of odd powers vanish, and therefore the sixth-degree equation reduces to a three-degree equation. So specific expressions for the coefficients are given in Theorem 2.

Acknowledgements. This work was supported by the program *Leading Scientific Schools* (project No. 7461.2016.1) and by the Russian Foundation for Basic Research (projects No. 15-01-01095_a, No. 17-41-020230-r_Bashkortostan).

References

1. Akhtyamov, A.M.: On spectrum for differential operator of odd order. Math. Notes **101**, 5 (2017)
2. Akhtyamov, A.M.: On degenerate boundary conditions in the Sturm-Liouville problem. Differ. Equ. **52**(8), 1085–1087 (2016) https://doi.org/10.1134/S0012266116080140
3. Biyarov, B.N., Dzhumabaev, S.A.: A criterion for the Volterra property of boundary value problems for Sturm-Liouville equations. Math. Notes **56**(1), 751–753 (1994) https://doi.org/10.1007/BF02110567
4. Dezin, A.A.: Spectral characteristics of general boundary-value problems for operator D^2. Math. Notes Acad. Sci. USSR **37**(2), 142–146 (1985). https://doi.org/10.1007/BF01156759
5. Dunford, N., Schwartz, J.T.: Linear Operators, Part III: Spectral Operators. Wiley-Interscience, New York (1971)
6. Kal'menov, TSh, Suragan, D.: Determination of the structure of the spectrum of regular boundary value problems for differential equations by V.A. Il'in's method of anti-priori estimates. Doklady Math. **78**(3), 913–915 (2008)
7. Kamke, E.: Differentialgleichungen: Losungsmethoden und Losungen, I, Gewohnliche Differentialgleichungen. B. G. Teubner, Leipzig (1977). (in German)
8. Lancaster, P.: Theory of Matrices. Academic Press, NY (1969)
9. Lang, P., Locker, J.: Spectral theory of two-point differential operators determined by D^2. I. Spectral properties. J. Math. Anal. Appl. **141**, 538–558 (1989)
10. Locker, J.: Eigenvalues and completeness for regular and simply irregular two-point differential operators. Mem. Am. Math. Soc. **195**, 911 (2008)
11. Marchenko, V.A.: Sturm-Liouville Operators and Applications. Basel, Boston, Shtuttgart, Birkhauser (1986)
12. Naimark, M.A.: Linear Differential Operators. Nauka, Moscow (1969) (in Russian)
13. Sadovnichy, V.A., Kanguzhin, B.E.: On the connection between the spectrum of a differential operator with symmetric coefficients and boundary conditions. Dokl. Akad. Nauk SSSR. **267**(2), 310–313 (1982) (in Russian)
14. Sadovnichii, V.A., Sultanaev, Ya.T, Akhtyamov, A.M.: General inverse sturm-liouville problem with symmetric potential. Azerbaijan J. Math. **5**(2), 96–108 (2015)
15. Sadybekov, M.A., Akhymbek, M.E.: Spectral properties of some non strongly regular boundary value problems for fourth order differential operators. In: 2nd International Conference on Analysis and Applied Mathematics (ICAAM 2014). AIP Conference Proceedings, vol. 1611, pp. 156–160 (2014). https://doi.org/10.1063/1.4893822
16. Shkalikov, A.A.: Boundary problems for ordinary differential equations with parameter in the boundary conditions. J. Sov. Math. **33**(6), 1311–1342 (1986)
17. Zhumabayev, S.A., Kanguzhin, B.E.: On an irregular problem on finite interval. Doklady Arademii nauk Kazakhstana. Seriya Fiz.-math. **1**, 14–18 (1988) (in Russian)

Stability of a Hyperbolic Equation with the Involution

Allaberen Ashyralyev[1,2] and Abdizhahan M. Sarsenbi[2,3(✉)]

[1] Department of Mathematics, Near East University, Nicosia TRNC, Mersin 10, Turkey
aallaberen@gmail.com
[2] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
abzhahan@gmail.com, abzhahan@mail.ru
[3] Department of Mathematical Methods and Modeling, M. Auezov SKS University, Shimkent, Kazakhstan

Abstract. In the present study, the problem of a hyperbolic equation with the involution is investigated. The stability estimates in maximum norm in t for the solution of this problem are established.

Keywords: Hyperbolic equation · Self-adjoint operator · Positive definite operator · Stability estimates · Involution · Stability estimates · Differential equations in Banach space · Operator method · Boundary value problems · Neuman conditions

1 Introduction

Hyperbolic partial differential equations arise in many branches of science and engineering e.g., electromagnetic, electrodynamics, thermodynamics, hydrodynamics, elasticity, fluid dynamics, wave propagation, materials science. The method of operators as a tool for the investigation of the solution of local and nonlocal problems to hyperbolic differential equations in Hilbert and Banach spaces, has been systematically developed by several authors (see, e.g., [2,3,5–9,12,13,16,17,23,24] and the references given therein). The theory of functional-differential equations with the involution has received less attention than functional-differential equations. Moreover, one of the unstudied areas of partial differential equations are parabolic differential and difference equations with the involution (see, e.g., [25]-[1] and the references given therein). For example, in the paper [25], the mixed problem for a parabolic partial differential equation with the involution with respect to t

$$u_t\left(t,x\right)=au_{xx}\left(t,x\right)+bu_{xx}\left(-t,x\right),0<x<l,\ \ -\infty<t<\infty \tag{1}$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_19

with the Dirichlet condition in x was studied. The Fourier method was used to get existence of unbounded solutions and non existence of solution dependent on coefficients a and b. Moreover, in papers [19]-[10], the mixed problem for a first-order partial differential equation with the involution was investigated. The Fourier method was used to find a classical solution of the mixed problem for a first-order differential equation with involution. The application of the Fourier method was substantiated using refined asymptotic formulas obtained for the eigenvalues and eigenfunctions of the corresponding spectral problem. The Fourier series representing the formal solution was transformed using certain techniques, and the possibility of its term-by-term differentiation was proved.

The paper [11] was devoted to the study of first order linear problems with involution and periodic boundary value conditions. First, it was proved a correspondence between a large set of such problems with different involutions to later focus attention to the case of the reflection. Then in different cases, for which a Green's function can be obtained explicitly, it was derived several results in order to obtain information about its sign. More general existence and uniqueness of solution results were established.

In papers [14]-[15], the basis properties of systems of eigenfunctions and associated functions for one kind of generalized spectral problems for a second-order and a first-order ordinary differential operators. In the paper [21], the notion of regularity of boundary conditions for a simplest second-order differential equation with a deviating argument was introduced. The Riesz basis property for a system of root vectors of the corresponding generalized spectral problem with regular boundary conditions (in the sense of the introduced definition) was established. Examples of irregular boundary conditions, to which the theory of Il'in basis property can be applied, were given.

In the paper [22], a nonclassical operator L in $L_2(-1,1)$, generated by the differential expression with shifted argument

$$Lu := -u''(-x), -1 < x < 1 \tag{2}$$

and the boundary conditions

$$\alpha_j u'(-1) + \beta_j u'(1) + \alpha_{j1} u(-1) + \beta_{j1} u(1) = 0, j = 1, 2 \tag{3}$$

was considered. For the spectral problem corresponding to (1), (2), the author introduces a concept of regular boundary conditions (2). In some sense, the definition is similar to that of strong (Birkhoff) regular boundary conditions (2) for second-order ordinary differential equations. The main result of the paper states that a system of eigenfunctions and associated functions of the operator L forms an unconditional basis of the space $L_2(-1,1)$. In the paper [20], the spectral problem for a model second-order differential operator with an involution was considered. The operator is given by the differential expression $Lu = -u^{''}(-x)$ and boundary conditions of general form. A criterion for the basis property of the systems of eigenfunctions of this operator in terms of the coefficients in the boundary conditions was obtained. In the paper [4], the problem of a parabolic equation with the involution was investigated. The stability and coercive stability estimates in Hölder norms in t for the solution of this problem were established. In the present paper, we will study the mixed problem for a hyperbolic equation with the involution

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \left(a(x)u_x\left(t,x\right)\right)_x + \beta\left(a(-x)u_x\left(t,-x\right)\right)_x - \sigma u\left(t,x\right) + f\left(t,x\right), \\ -l < x < l,\ 0 < t < T, \\ u_x\left(t,-l\right) = 0, u_x\left(t,l\right) = 0, 0 \le t \le T, \\ u\left(0,x\right) = \varphi\left(x\right),\ u_t\left(0,x\right) = \psi\left(x\right), -l \le x \le l, \varphi_x\left(-l\right) = \varphi_x\left(l\right) = 0, \end{cases} \tag{4}$$

where $u\left(t,x\right)$ is unknown function, $\varphi\left(x\right), \psi\left(x\right), a\left(x\right)$, and $f\left(t,x\right)$ are sufficiently smooth functions, $a \ge a\left(x\right) = a\left(-x\right) \ge \delta > 0$ and $\sigma > 0$ is a sufficiently large number. The stability estimates in maximum norm in t for the solution of problem (4) are established.

2 Preliminaries. Main Results

To formulate our results, we introduce the Hilbert space $L_2[-l,l]$ of all integrable functions f defined on $[-l,l]$, equipped with the norm

$$\| f \|_{L_2[-l,l]} = \left\{ \int_{-l}^{l} |f(x)|^2 dx \right\}^{\frac{1}{2}}. \tag{5}$$

We introduce the inner product in $L_2[-l,l]$ by the following formula

$$\langle u, v \rangle = \int_{-l}^{l} u(x)v(x)dx. \tag{6}$$

Moreover, $C\left([0,T], H\right)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\|\varphi\|_{C([0,T],H)} = \max_{0 \le t \le T} \|\varphi(t)\|_H. \tag{7}$$

Finally, we introduce a differential operator A^x defined by the formula

$$A^x v(x) = -\left(a(x)v_x(x)\right)_x - \beta\left(a(-x)v_x\left(-x\right)\right)_x + \sigma v\left(x\right) \tag{8}$$

with the domain $D(A^x) = \{u, u_{xx} \in L_2[-l,l] : u_x(-l) = 0, u_x(l) = 0\}$. We can rewrite the problem (4) in the following abstract form as the abstract Cauchy problem for hyperbolic equations

$$v''(t) + Av(t) = f(t) \quad (0 \le t \le T), v(0) = \varphi, v'(0) = \psi \tag{9}$$

in a Hilbert space H with the self -adjoint positive definite operator $A = A^x$ defined by formula (8). Here, $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are respectively, known and unknown abstract functions defined on $(0,T)$ with values in $H = L_2[-l,l]$, $\varphi = \varphi(x), \psi = \psi(x)$ and $a = a(x)$ are given smooth elements of $H = L_2[-l,l]$. The main result of the present paper is the following theorem on stability estimates of (4) in spaces $C([0,T)], L_2[-l,l])$ for the solution of this problem.

Theorem 21. Assume that $\delta - a|\beta| \ge 0, \varphi(x), \varphi_{xx}(x) \in L_2[-l,l]$, $\psi(x), \psi_x(x) \in L_2[-l,l]$, and $f(t,x) \in C^{(1)}([0,T], L_2[-l,l])$. Then for the solution of problem (4) the following stability estimates

$$\max_{0\le t\le T} \|u(t,\cdot)\|_{W_2^1[-l,l]} \le M[\max_{0\le t\le T} \|f(t,\cdot)\|_{L_2[-l,l]} + \| \varphi \|_{W_2^1[-l,l]} + \| \psi \|_{L_2[-l,l]}], \tag{10}$$

$$\max_{0\le t\le T} \|u(t,\cdot)\|_{W_2^2[-l,l]} + \max_{0\le t\le T} \|u_{tt}(t,\cdot)\|_{L_2[-l,l]} \tag{11}$$

$$\le M\left[\max_{0\le t\le T} \|f_t(t,\cdot)\|_{L_2[-l,l]} + \|f(0,\cdot)\|_{L_2[-l,l]} + \|\varphi\|_{W_2^2[-l,l]} + \|\psi\|_{W_2^1[-l,l]}\right] \tag{12}$$

hold, where M does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$. Here, the Sobolev space $W_2^1[-l,l]$ is defined as the set of all functions f defined on $[-l,l]$ such that f and first order derivative function f' are both locally integrable in $L_2[-l,l]$, equipped with the norm

$$\| f \|_{W_2^1[-l,l]} = \left\{ \int_{-l}^{l} |f(x)|^2 dx \right\}^{\frac{1}{2}} + \left\{ \int_{-l}^{l} |f_x(x)|^2 dx \right\}^{\frac{1}{2}}, \tag{13}$$

and the Sobolev space $W_2^2[-l,l]$ is defined as the set of all functions f defined on $[-l,l]$ such that f and second order derivative function f'' are both locally integrable in $L_2[-l,l]$, equipped with the norm

$$\| f \|_{W_2^2[-l,l]} = \left\{ \int_{-l}^{l} |f(x)|^2 dx \right\}^{\frac{1}{2}} + \left\{ \int_{-l}^{l} |f_{xx}(x)|^2 dx \right\}^{\frac{1}{2}}. \tag{14}$$

The proof of Theorem 21 is based on the following abstract Theorem on stability of problem (9) in $C([0,T],H)$ space and on self-adjointness and positive definiteness of the unbounded operator $A=A^x$ defined by formula (8) in $L_2[-l,l]$ space.

Theorem 22. [5]. Suppose that $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ and $f(t)$ are continuously differentiable on $[0,T]$ function. Then there is a unique solution of the problem (9) and the stability inequalities

$$\max_{0\le t\le T}\|v(t)\|_H\le M\left[\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\max_{0\le t\le T}\left\|A^{-1/2}f(t)\right\|_H\right], \qquad (15)$$

$$\max_{0\le t\le T}\left\|A^{1/2}v(t)\right\|_H\le M\left[\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\max_{0\le t\le T}\|f(t)\|_H\right], \qquad (16)$$

$$\max_{0\le t\le T}\|Av(t)\|_H\le M\left[\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\max_{0\le t\le T}\|f'\|_H\right], \quad (17)$$

hold, where M does not depend on $f(t),\ t\in[0,T]$ and φ,ψ.

In the next Section, the self-adjointness and positive definiteness of the operator $A=A^x$ defined by formula (8) in $L_2[-l,l]$ space will be studied.

3 Self-adjointness and Positive Definiteness

Theorem 31. Assume that $\delta-a|\beta|\ge 0$, then the operator $A=A^x$ defined by formula (8) is the self-adjoint and positive definite operator in $L_2[-l,l]$ space with the spectral angle $\varphi(A,H)=0$.

Proof. We will prove the following identity and estimate

$$\langle A^x u,v\rangle=\langle u,A^x v\rangle\,, u,v\in D(A^x), \qquad (18)$$

$$\langle A^x u,u\rangle\ge\sigma\langle u,u\rangle\,, u\in D(A^x). \qquad (19)$$

Applying the definition of the inner product and integrating by part, we get

$$\langle A^x u,v\rangle=-\int_{-l}^{l}(a(x)u_x(x))_x\,v(x)dx-\beta\int_{-l}^{l}(a(-x)u_x(-x))_x\,v(x)dx+\sigma\int_{-l}^{l}u(x)\,v(x)dx \quad (20)$$

$$=-a(l)u_x(l)v(l)+a(-l)u_x(-l)v(-l)+\int_{-l}^{l}a(x)u_x(x)v_x(x)dx \qquad (21)$$

$$+\beta\left[-a(-l)u_x(-l)v(-l)+a(l)u_x(l)v(l)\right]+\beta\int_{-l}^{l}a(-x)u_x(-x)\,v_x(x)dx+\sigma\int_{-l}^{l}u(x)\,v(x)dx. \quad (22)$$

From $u, v \in D(A^x)$ it follows that

$$\langle A^x u, v\rangle = \int_{-l}^{l} a(x)u_x(x)v_x(x)dx + \beta \int_{-l}^{l} a(x)u_x\,(x)\,v_x(-x)dx + \sigma \int_{-l}^{l} u\,(x)\,v(x)dx. \quad (23)$$

In a similar manner one establishes formula

$$\langle u, A^x v\rangle = \int_{-l}^{l} a(x)u_x(x)v_x(x)dx + \beta \int_{-l}^{l} u_x\,(x)\,a(-x)v_x(-x)dx + \sigma \int_{-l}^{l} u\,(x)\,v(x)dx. \quad (24)$$

Therefore, from these formulas and condition $a\,(x) = a\,(-x)$ it follows identity (18). Now, we will prove the estimate (19). Applying the identity (23), we get

$$\langle A^x u, u\rangle = \int_{-l}^{l} a(x)u_x(x)u_x(x)dx + \beta \int_{-l}^{l} u_x\,(x)\,a(-x)u_x(-x)dx + \sigma \int_{-l}^{l} u\,(x)\,u(x)dx \quad (25)$$

$$\geq \sigma\,\langle u, u\rangle + \delta \int_{-l}^{l} u_x(x)u_x(x)dx + \beta\delta \int_{-l}^{l} a(-x)u_x\,(x)\,u_x(-x)dx. \quad (26)$$

Using the Cauchy inequality, we get

$$\int_{-l}^{l} a(-x)u_x\,(x)\,u_x(-x)dx \leq a \left(\int_{-l}^{l} |u_x\,(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_{-l}^{l} |u_x\,(-x)|^2\,dx\right)^{\frac{1}{2}} = a\,\langle u_x, u_x\rangle\,. \quad (27)$$

Since $\beta \geq -\,|\beta|$, we have that

$$\beta \int_{-l}^{l} a(-x)u_x\,(x)\,u_x(-x)dx \geq -\,|\beta|\,a\,\langle u_x, u_x\rangle\,. \quad (28)$$

Then

$$\langle A^x u, u\rangle \geq \sigma\,\langle u, u\rangle + (\delta - |\beta|\,a)\,\langle u_x, u_x\rangle \geq \sigma\,\langle u, u\rangle\,. \quad (29)$$

Theorem 31 is proved.

4 Conclusion

In the present study, the mixed problem (4) for a hyperbolic equation with the involution is investigated. The stability estimates in $C([0,T], L_2[-l,l])$ norm for the solution of this problem are established.

Moreover, applying results of paper [3] and present paper, we can study the nonlocal problem for a hyperbolic equation with the involution

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = (a(x)u_x(t,x))_x + \beta\,(a(-x)u_x(t,-x))_x - \sigma u(t,x) + f(t,x), \\ -l < x < l,\ 0 < t < T, \\ u_x(t,-l) = 0, u_x(t,l) = 0, 0 \le t \le T, \\ u(0,x) = \int\limits_0^T \alpha(\rho)\, u(\rho,x)\, d\rho + \sum\limits_{i=1}^{n} a(\lambda_i)\, u(\lambda_i,x) + \varphi(x), \\ u_t(0,x) = \int\limits_0^T \beta(\rho)\, u_t(\rho,x)\, d\rho + \sum\limits_{i=1}^{n} \beta(\lambda_i)\, u(\lambda_i,x) + \psi(x), \\ -l \le x \le l, \varphi_x(-l) = \varphi_x(l) = 0, \end{cases} \tag{30}$$

where $u(t,x)$ is unknown function, $\varphi(x), \psi(x), a(x)$, and $f(t,x)$ are sufficiently smooth functions, $a \ge a(x) = a(-x) \ge \delta > 0$ and $\sigma > 0$ is a sufficiently large number, and $\alpha(s)$, $\beta(s)$, $a(s)$, $b(s)$ are scalar real-valued continuous functions. Under the assumption

$$\left|1 + \int\limits_0^T \alpha(s)\beta(s)\,ds + \sum_{k=1}^{n} a(\lambda_k) \sum_{k=1}^{n} b(\lambda_k) + \sum_{k=1}^{n} a(\lambda_k) \int\limits_0^T \beta(s)\,ds + \sum_{k=1}^{n} b(\lambda_k) \int\limits_0^T \alpha(s)\,ds\right|$$
$$> \int\limits_0^T (|\alpha(s)| + |\beta(s)|)\,ds + \sum_{k=1}^{n} |a(\lambda_k) + b(\lambda_k)|,$$

stability estimates in maximum norm in t for the solution of problem (30) can be established. Finally, applying the result of the monograph [5], the high order of accuracy two-step difference schemes for the numerical solution of the mixed problem (4) can be presented. Of course, the stability estimates for the solution of these difference schemes have been established without any assumptions about the grid steps.

Acknowledgements. This work was supported by the Committee of Science of the Ministry of Education and Science of the Republic Kazakhstan, project no. 5414/GF4. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Agarwal, R.P., Bohner, M., Li, W.T.: Nonoscillation and oscillation theory for functional differential equations. In: Monographs and Textbooks in Pure and Applied Mathematics, vol. 267. Marcel Dekker, New York, NY, USA (2004)
2. Ashyralyev, A.: Computational Mathematics. Textbook, The South Kazakhstan State University Named after M. Auezovuku Printing House, Chimkent (2013)
3. Ashyralyev, A., Aggez, N.: On the solution of NBVP for multidimensional hyperbolic equations. Sci. World J. 841602 (2014)

4. Ashyralyev, A., Sarsenbi, A.M.: Well-posedness of a parabolic equation with the involution. In: Numerical Functional Analysis and Optimization (2017) (in press)
5. Ashyralyev, A., Sobolevskii, P.E.: New difference schemes for partial differential equations. In: Operator Theory Advances and Applications. Birkhäuser Verlag, Basel, Boston, Berlin (2004)
6. Ashyralyev, A., Yildirim, O.: On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations. Taiwan. J. Math. **14**(1), 165–194 (2010)
7. Ashyralyev, A., Yurtsever, A.: On a nonlocal boundary value problem for semilinear hyperbolic-parabolic equations. Nonlinear Anal.-Theory Methods Appl. **47**(5), Special Issue: SI, 3585–3592 (2001)
8. Ashyraliyev, M.: A note on the stability of the integral-differential equation of the hyperbolic type in a Hilbert space. Numer. Funct. Anal. Optim. **29**(7–8), 750–769 (2008)
9. Ashyraliyev, M.: Generalizations of Wendroff integral inequalities and their discrete analogues. Abstr. Appl. Anal. 768062 (2012). https://doi.org/10.1155/2012/768062
10. Burlutskaya, MSh: Mixed problem for a first-order partial differential equation with involution and periodic boundary conditions. Comput. Math. Math. Phys. **54**(1), 1–10 (2014)
11. Cabada, A., Adrián, F., Tojo, F.: Existence results for a linear equation with reflection, non-constant coefficient and periodic boundary conditions. J. Math. Anal. Appl. **412**(1), 529–546 (2014)
12. Fattorini, H.O.: Second Order Linear Differential Equations in Banach Spaces. Elsevier Science Publishing Company, Norht-Holland (1985)
13. Goldstein, J.A.: Semigroups of Linear Operators and Applications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1985)
14. Kopzhassarova, A.A., Lukashov, A.L., Sarsenbi, A.M.: Spectral properties of non-self-adjoint perturbations for a spectral problem with involution. Abstr. Appl. Anal. **2012**, Article ID 590781 (2012)
15. Kopzhassarova, A., Sarsenbi, A.: Basis properties of eigenfunctions of second-order differential operators with involution. Abstr. Appl. Anal. **2012**, Article ID 576843 (2012)
16. Krein, S.G.: Linear differential equations in Banach space. American Mathematical Society, Providence, R.I. (1971) (Translated from the Russian by J.M. Danskin, Translations of Mathematical Monographs, **29**)
17. Pogorelenko, V., Sobolevskii, P.: The "counter-example" to W. Littman counter-example of Lp-energetic inequality for wave equation. Funct. Differ. Equ. **4**(1–2), 165–173 (1998)
18. Przeworska-Rolewicz, D.: Equations with Transformed Argument. An Algebraic Approach. Published by Amsterdam/Warszawa, Elsevier Scientific Publishing Comp./PWN - Polish Scientific Publishers (1973)
19. Sadybekov, M.A., Sarsenbi, A.M.: Mixed problem for a differential equation with involution under boundary conditions of general form. In: Ashyralyev, A., Lukashov, A. (eds.) AIP Conference Proceedings, vol. 1470, pp. 225–227 (2012)
20. Sadybekov, M.A., Sarsenbi, A.M.: Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution. Differ. Equ. **48**(8), 1112–1118 (2012)
21. Sarsenbi, A.M., Tengaeva, A.A.: On the basis properties of root functions of two generalized eigenvalue problems. Differ. Equ. **48**(2), 306–308 (2012)

22. Sarsenbi, A.M.: Unconditional bases related to a nonclassical second-order differential operator. Differ. Equ. **46**(4), 509–514 (2010)
23. Sobolevskii, P.E., Semenov, S.: On some approach to investigation of singular hyperbolic equations. Dokl. Akad. Nauk SSSR. **270**(1), 555–558 (1983) (Russian)
24. Vasilev, V.V., Krein, S.G., Piskarev, S.: Operator semigroups, cosine operator functions, and linear differential equations. Math. Anal. **28** (Russian); Itogi Nauki i Tekhniki. **204**, 87–202 (1990); Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow (1990) (Translated in J. Sov. Math. **54**(4), 1042–1129 (1991))
25. Wiener, V.: Generalized Solutions of Functional-Differential Equations. World Scientific Pub, River Edge, NJ (1993)

Relatively Bounded Perturbations of Correct Restrictions and Extensions of Linear Operators

Bazarkan N. Biyarov[1(✉)] and Gulnara K. Abdrasheva[2]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin Str., Almaty 050010, Kazakhstan
bbiyarov@gmail.com
[2] L. N. Gumilyov Eurasian National University, 2 Satpayev Str., Astana 010008, Kazakhstan
gulnara.abdrash@gmail.com

Abstract. In this paper we study the spectral properties of relatively bounded correct perturbations of the correct restrictions and extensions. Method for constructing a class of correct perturbations, which spectra coincide with the spectrum of a fixed boundary correct extension, is obtained. Examples illustrating the application of the obtained results are given.

Keywords: Correct restrictions · Correct extensions · Relatively bounded perturbations · Spectral properties · Volterra correct extensions

1 Introduction

Let us present some definitions, notation, and terminology.

In a Hilbert space H, we consider a linear operator L with domain $D(L)$ and range $R(L)$. By the *kernel* of the operator L we mean the set

$$\operatorname{Ker} L = \big\{ f \in D(L) : \; Lf = 0 \big\}.$$

Definition 1. An operator L is called a *restriction* of an operator L_1, and L_1 is called an *extension* of the operator L, briefly $L \subset L_1$, if:
1) $D(L) \subset D(L_1)$,
2) $Lf = L_1 f$ for all f from $D(L)$.

Definition 2. A linear closed operator L_0 in a Hilbert space H is called *minimal* if there exists a bounded inverse operator L_0^{-1} on $R(L_0)$ and $R(L_0) \neq H$.

Definition 3. A linear closed operator $\widehat{L}$ in a Hilbert space H is called *maximal* if $R(\widehat{L}) = H$ and $\operatorname{Ker} \widehat{L} \neq \{0\}$.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_20

Definition 4. A linear closed operator L in a Hilbert space H is called *correct* if there exists a bounded inverse operator L^{-1} defined on all of H.

Definition 5. We say that a correct operator L in a Hilbert space H is a *correct extension* of minimal operator L_0 (*correct restriction* of maximal operator $\widehat{L}$) if $L_0 \subset L$ $(L \subset \widehat{L})$.

Definition 6. We say that a correct operator L in a Hilbert space H is a *boundary correct* extension of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ if L is simultaneously a correct restriction of the maximal operator $\widehat{L}$ and a correct extension of the minimal operator L_0, that is, $L_0 \subset L \subset \widehat{L}$.

Let $\widehat{L}$ be a maximal linear operator in the Hilbert space H, let L be any known correct restriction of $\widehat{L}$, and let K be an arbitrary linear bounded (in H) operator satisfying the following condition:

$$R(K) \subset \operatorname{Ker} \widehat{L}.$$

Then the operator L_K^{-1} defined by the formula (see [5])

$$L_K^{-1} f = L^{-1} f + K f, \tag{1}$$

describes the inverse operators to all possible correct restrictions L_K of $\widehat{L}$, i.e., $L_K \subset \widehat{L}$.

Let L_0 be a minimal operator in the Hilbert space H, let L be any known correct extension of L_0, and let K be a linear bounded operator in H satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $\operatorname{Ker}(L^{-1} + K) = \{0\}$,

then the operator L_K^{-1} defined by formula (1) describes the inverse operators to all possible correct extensions L_K of L_0 (see [5]).

Let L be any known boundary correct extension of L_0, i.e., $L_0 \subset L \subset \widehat{L}$. The existence of at least one boundary correct extension L was proved by Vishik in [7]. Let K be a linear bounded (in H) operator satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $R(K) \subset \operatorname{Ker} \widehat{L}$,

then the operator L_K^{-1} defined by formula (1) describes the inverse operators to all possible boundary correct extensions L_K of L_0 (see [5]).

Definition 7. A bounded operator A in a Hilbert space H is called *quasinilpotent* if its spectral radius is zero, that is, the spectrum consists of the single point zero.

Definition 8. An operator A in a Hilbert space H is called a *Volterra operator* if A is compact and quasinilpotent.

Definition 9. A correct restriction L of a maximal operator $\widehat{L}$ ($L \subset \widehat{L}$), a correct extension L of a minimal operator L_0 ($L_0 \subset L$) or a boundary correct extension L of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ ($L_0 \subset L \subset \widehat{L}$), will be called *Volterra* if the inverse operator L^{-1} is a Volterra operator.

Definition 10. A densely defined closed linear operator A in a Hilbert space H is called *formally normal* if

$$D(A) \subset D(A^*), \quad \|Af\| = \|A^*f\| \quad \text{for all } f \in D(A).$$

Definition 11. A formally normal operator A is called *normal* if

$$D(A) = D(A^*).$$

2 Main Results

Let L_0 be some minimal operator, and let M_0 be another minimal operator related to L_0 by the equation $(L_0u, v) = (u, M_0v)$ for all $u \in D(L_0)$ and $v \in D(M_0)$. Then $\widehat{L} = M_0^*$ and $\widehat{M} = L_0^*$ are maximal operators such that $L_0 \subset \widehat{L}$ and $M_0 \subset \widehat{M}$. The existence of at least one boundary correct extension L was proved by Vishik in [7], that is, $L_0 \subset L \subset \widehat{L}$. In this case, L^* is a boundary correct extension of the minimal operator M_0, that is, $M_0 \subset L^* \subset \widehat{M}$. The inverse operators to all possible correct restrictions L_K of the maximal operator $\widehat{L}$ have the form (1), then $D(L_K)$ is dense in H if and only if $\operatorname{Ker}(I + K^*L^*) = \{0\}$. Thus, it is obvious that any correct extension M_K of M_0 is adjoint of some correct restriction L_K with dense domain, and vice versa [2]. Finally, all possible correct extensions M_K of M_0 have inverses of the form

$$M_K^{-1}f = (L_K^*)^{-1}f = (L^*)^{-1}f + K^*f, \tag{2}$$

where K is an arbitrary bounded linear operator in H with $R(K) \subset \operatorname{Ker}\widehat{L}$ such that $\operatorname{Ker}(I + K^*L^*) = \{0\}$. It is also clear that $R(M_0) \subset \operatorname{Ker} K^*$. In particular, M_K is a boundary correct extension of M_0 if and only if $R(M_0) \subset \operatorname{Ker} K^*$ and $R(K^*) \subset \operatorname{Ker}\widehat{M}$.

Lemma 1. *Let L_K be a densely defined correct restriction of the maximal operator $\widehat{L}$ in a Hilbert space H. Then $D(L^*) = D(L_K^*)$ if and only if $R(K^*) \subset D(L^*) \cap D(L_K^*)$, where L and K are the operators from the representation (1).*

Proof. If $D(L^*) = D(L_K^*)$ then from the representation (1), we easily get

$$R(K^*) \subset D(L^*) \cap D(L_K^*) = D(L^*) = D(L_K^*)$$

Let us prove the converse. If

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

then we obtain

$$(L_K^*)^{-1} f = (L^*)^{-1} f + K^* f = (L^*)^{-1}(I + L^* K^*) f, \tag{3}$$

$$(L^*)^{-1} f = (L_K^*)^{-1} f - K^* f = (L_K^*)^{-1}(I - L_K^* K^*) f, \tag{4}$$

for all f in H. It follows from (3) that $D(L_K^*) \subset D(L^*)$, and from (4) it implies that $D(L^*) \subset D(L_K^*)$. Thus $D(L^*) = D(L_K^*)$. Lemma 1 is proved. □

Lemma 2. *If $R(K^*) \subset D(L^*) \cap D(L_K^*)$ then a bounded operators $I + L^* K^*$ and $I - L_K^* K^*$ from (3) and (4), respectively, have a bounded inverse defined on H.*

Proof. By virtue of the density of the domains of the operators L_K^* and L^* we imply that the operators $I + L^* K^*$ and $I - L_K^* K^*$ are invertible. Since from (3) and (4) we have $\operatorname{Ker}(I + L^* K^*) = \{0\}$ and $\operatorname{Ker}(I - L_K^* K^*) = \{0\}$, respectively. From the representations (3) and (4) we also note that $R(I + L^* K^*) = H$ and $R(I - L_K^* K^*) = H$, since $D(L^*) = D(L_K^*)$. The inverse operators $(I + L^* K^*)^{-1}$ and $(I - L_K^* K^*)^{-1}$ of the closed operators $I + L^* K^*$ and $I - L_K^* K^*$, respectively, are closed. Then the closed operators $(I + L^* K^*)^{-1}$ and $(I - L_K^* K^*)^{-1}$, defined on the whole of H, are bounded. Lemma 2 is proved. □

Under the conditions of Lemma 2 the operators KL and KL_K will be (see [3]) a part of bounded operators $\overline{KL}$ and $\overline{KL_K}$, respectively, where the bar denotes the closure of operators in H. Thus $(I - L_K^* K^*)^{-1} = I + L^* K^*$ and $(I - \overline{KL_K})^{-1} = I + \overline{KL}$.

Next we consider the following statement

Theorem 1. *Let L_K be a densely defined correct restriction of the maximal operator $\widehat{L}$ in a Hilbert space H. If $R(K^*) \subset D(L^*) \cap D(L_K^*)$, where L and K are the operators from the representation (1) then*

1. *The operator $B_K = (I + \overline{KL}) L_K$ is relatively bounded correct perturbations of the correct restriction L_K and the spectra of the operators B_K and L coincide, that is, $\sigma(B_K) = \sigma(L)$;*
2. *The operator L is a quasinilpotent (the Volterra) boundary correct extension of L_0, and B_K is a quasinilpotent correct operator simultaneously;*
3. *If L is an operator with discrete spectrum then the system of root vectors of the operator L is complete (the basis) in H if and only if the system of root vectors of the operator B_K is complete (the basis) in H;*
4. *In particular, when L is a normal operator with discrete spectrum, then the system of root vectors of the operator B_K form a Riesz basis in H.*

Proof. 1. Note that $B_K^{-1} = L_K^{-1}(I - \overline{KL_K})$, and $(I - \overline{KL_K}) L_K^{-1} = L_K^{-1} - K = L^{-1}$. The correctness of the operator B_K is obvious. For bounded operators R and S it is known (see [1]) the property $\sigma(RS) \setminus \{0\} = \sigma(SR) \setminus \{0\}$. Thus, Item 1 is proved.

2. Note that $B_K^{-1} = (I - \overline{KL_K})^{-1} L^{-1} (I - \overline{KL_K})$. It follows easily from Lemmas 1 and 2 that the operators $I - \overline{KL_K}$ and $(I - \overline{KL_K})^{-1}$ are bounded and defined on the whole of H. It is then obvious that the operators L^{-1} and B_K^{-1} are quasinilpotent (the Volterra) simultaneously. Item 2 is proved.
3. From the known facts of functional analysis (see [6]) imply that the system of root vectors of the operators L and B_K are complete (the basis) simultaneously.
4. The system of root vectors of the normal discrete correct operator L form an orthonormal basis in H. Then the system of root vectors of the correct operator B_K form a Riesz basis in H.

Theorem 1 is proved. □

Example 1. In the Hilbert space $L_2(0,1)$ let us consider the minimal operator L_0 generated by the differentiation operator

$$\widehat{L}y = y' = f \;\text{ for all }\; f \in L_2(0,1).$$

Then

$$D(L_0) = \{y \in W_2^1(0,1) :\; y(0) = y(1) = 0\}.$$

The action of the maximum operator $\widehat{M} = L_0^*$ has the form

$$\widehat{M}v = -v' = g \quad \text{for all }\; g \in L_2(0,1).$$

Then

$$D(M_0) = \{v \in W_2^1(0,1) :\; v(0) = v(1) = 0\}.$$

As a fixed boundary correct extension L of L_0 we take the operator acting as the maximal operator $\widehat{L}$ on the domain

$$D(L) = \{y \in D(\widehat{L}) :\; y(0) = 0\}.$$

Then all possible correct restriction L_K of $\widehat{L}$ have the following inverse

$$y = L_K^{-1}f = L^{-1} + Kf = \int_0^x f(t)dt + \int_0^1 f(t)\overline{\sigma(t)}dt,$$

where $\sigma(x) \in L_2(0,1)$ defines the operator K. The domain $D(L_K)$ of L_K is defined as

$$D(L_K) = \{y \in W_2^1(0,1) :\; y(0) = \int_0^1 y'(t)\overline{\sigma(t)}dt\}.$$

Then $D(L_K)$ is not dense in $L_2(0,1)$ if and only if $\sigma(x) \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) = -1$. If we exclude such $\sigma(x)$ from $L_2(0,1)$ then there exists L_K^* which has an inverse of the form

$$v = (L_K^*)^{-1}g = (L_K^{-1})^*g = (L^*)^{-1}g + K^*g \quad \text{for all }\; g \in L_2(0,1).$$

This is a description of inverse operators of all possible correct extensions L_K^* of M_0. Let the condition of Theorem 1 holds. Then $\sigma(x) \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) \neq -1$. Let us construct the following operators

$$\overline{KL}f = -\int_0^1 f(t)\sigma'(t)dt,$$
$$\overline{KL_K}f = -\frac{1}{1+\overline{\sigma(0)}}\int_0^1 f(t)\sigma'(t)dt.$$

Note that

$$L_K^* v = -v'(x) + \frac{\sigma'(x)}{1+\sigma(0)}v(0) = f(x),$$
$$D(L_K^*) = D(L^*) = \{v \in W_2^1(0,1) : \ v(1) = 0\}.$$

Then the operator B_K has the following form

$$B_K u = u'(x) - \int_0^1 u'(t)\overline{\sigma'(t)}dt = f(x),$$
$$D(B_K) = D(L_K) = \{u \in W_2^1(0,1) : \ u(0) = \int_0^1 u'(t)\overline{\sigma(t)}dt\},$$

where $\sigma(x) \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) \neq -1$. By virtue of Theorem 1 B_K is a Volterra correct operator. We know that for a first order differentiation operator there are no Volterra correct restrictions or correct extensions, except the Cauchy problem at some point $x = d$, $0 \leq d \leq 1$. But the operator B_K is neither correct restriction of $\widehat{L}$ nor correct extension of L_0. This Volterra problem is obtained by the perturbation of the differentiation operator itself and the boundary conditions of Cauchy simultaneously.

Example 2. If in Example 1 as a fixed boundary correct operator L we take the operator $\widehat{L}$ with the domain

$$D(L) = \{y \in W_2^1(0,1) : \ y(0) + y(1) = 0\},$$

then L is a normal operator. In this case, the operator B_K has the form

$$B_K y = y'(x) - \int_0^1 y'(t)\overline{\sigma'(t)}dt = f(x),$$
$$D(B_K) = \{y \in W_2^1(0,1) : \ y(0) + y(1) = 2\int_0^1 y'(t)\overline{\sigma(t)}dt\},$$

where $\sigma(x) \in W_2^1(0,1)$, $\sigma(0) + \sigma(1) = 0$, and $\sigma(0) \neq -\frac{1}{2}$. The operator B_K is correct and the system of root vectors form a Riesz basis in $L_2(0,1)$. The eigenvalues of the normal operator L and the correct operator B_K coincide.

Corollary 1. *The results of Theorem 1 are also valid for the operator $B_K^* = L_K^*(I + L^*K^*)$. All four items will take place for a pair of operators B_K^* and L^*.*

Remark 1. The results of Examples 1–2 are also valid for the operator B_K^*.

$$B_K^* v = -\frac{d}{dx}[v(x) - \sigma'(x)\int_0^1 v(t)dt] = f,$$

$$D(B_K^*) = \{v \in L_2(0,1):\ v(x) - \sigma'(x)\int_0^1 v(t)dt \in D(L^*)\},$$

where $\sigma(x) \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) \neq -1$, in the case of Example 1, and $\sigma(x) \in W_2^1(0,1)$, $\sigma(0) + \sigma(1) = 0$, and $\sigma(0) \neq -\frac{1}{2}$, in the case of Example 2. We recall that the conditions $\sigma(0) \neq -1$ and $\sigma(0) \neq -\frac{1}{2}$ provide the density of the domain $D(L_K)$ in H.

Example 3. In the Hilbert space $L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$ with an infinitely smooth boundary $\partial\Omega$, let us consider the minimal L_0 and maximal $\widehat{L}$ operators generated by the Laplace operator

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2}\right). \tag{5}$$

The closure L_0, in the space $L_2(\Omega)$ of the Laplace operator (5) with the domain $C_0^\infty(\Omega)$, is *the minimal operator corresponding to the Laplace operator*. The operator $\widehat{L}$, adjoint to the minimal operator L_0 corresponding to the Laplace operator, is *the maximal operator corresponding to the Laplace operator* (see [4]). Note that

$$D(\widehat{L}) = \{u \in L_2(\Omega):\ \widehat{L}u = -\Delta u \in L_2(\Omega)\}.$$

Denote by L_D the operator, corresponding to the Dirichlet problem with the domain

$$D(L_D) = \{u \in W_2^2(\Omega):\ u|_{\partial\Omega} = 0\}.$$

Then, by virtue of (1), the inverse operators L^{-1} to all possible correct restrictions of the maximal operator $\widehat{L}$ corresponding to the Laplace operator (5) have the following form:

$$u \equiv L^{-1}f = L_D^{-1}f + Kf,$$

where, by virtue of (1), K is an arbitrary linear operator bounded in $L_2(\Omega)$ with

$$R(K) \subset \operatorname{Ker}\widehat{L} = \{u \in L_2(\Omega):\ -\Delta u = 0\}.$$

Then the direct operator L is determined from the following problem:

$$\widehat{L}u = -\Delta u = f, \quad f \in L_2(\Omega),$$

$$D(L) = \{u \in D(\widehat{L}):\ [(I - K\widehat{L})u]|_{\partial\Omega} = 0\},$$

where I is the identity operator in $L_2(\Omega)$. There are no other linear correct restrictions of the operator $\widehat{L}$ (see [2]). The operators $(L^*)^{-1}$, corresponding to the adjoint operators L^*

$$v = (L^*)^{-1}g = L_D^{-1}g + K^*g,$$

describe the inverse operators to all possible correct extensions of L_0 if and only if K satisfies the condition (see [2]):

$$\mathrm{Ker}\,(I + K^* L^*) = \{0\}.$$

Note that the last condition is equivalent to the following: $\overline{D(L)} = L_2(\Omega)$.

We apply Theorem 1 to the particular case when

$$Kf = \omega(x) \iint_{\Omega} f(\xi)\overline{g(\xi)}d\xi, \quad x,\, \xi \in \Omega \subset \mathbb{R}^m,$$

where $\omega(x)$ is a harmonic function from $L_2(\Omega)$, and $g(x) \in L_2(\Omega)$.

$$K^* f = g(x) \iint_{\Omega} f(\xi)\overline{\omega(\xi)}d\xi.$$

From the conditions of Theorem 1 it follows that $g(x) \in W_2^2(\Omega)$, $g(x)\mid_{\partial\Omega} = 0$, and

$$\iint_{\Omega} (\Delta g)(\xi)\overline{\omega(\xi)}d\xi \neq 1.$$

Then

$$B_K u = -\Delta u - \omega(x) \iint_{\Omega} (\Delta u)(\xi)(\Delta \overline{g})(\xi)d\xi = f(x), \quad \text{for all } f \in L_2(\Omega),$$

$$D(B_K) = \left\{ u \in W_2^2(\Omega) : \Big(u(x) + \omega(x) \iint_{\Omega} (\Delta u)(\xi)\overline{g(\xi)}d\xi\Big) \mid_{\partial\Omega} = 0 \right\}$$

We obtained a relatively compact perturbation B_K of L which has the same eigenvalues as the Dirichlet problem L_D. The system of root vectors of B_K forms a Riesz basis in $L_2(\Omega)$. If $\{v_k\}$ are an orthonormal system of eigenfunctions of L (the Dirichlet problem), then the system of eigenvectors $\{u_k\}$ of B_K have the form

$$u_k = (I + \overline{KL})v_k = v_k(x) + \omega(x) \iint_{\Omega} v_k(\xi)(\Delta \overline{g})(\xi)d\xi, \quad k = 1, 2, \ldots$$

Acknowledgements. Research supported by the grant 0825/GF4 and by the target program 0085/PTSF-14 of the Ministry of Education and Science of Republic of Kazakhstan.

References

1. Barnes, B.A.: Common operator properties of the linear operators RS and SR. Proc. Am. Math. Soc. **126**(4), 1055–1061 (1998)
2. Biyarov, B.N.: Spectral properties of correct restrictions and extensions of the Sturm-Liouville operator. Differ. Equ. **30**(12), 1863–1868 (1994) (Translated from Differenisial'nye Uravneniya **30**(12), 2027–2032 (1994))
3. Biyarov, B.N., Abdrasheva, G.K.: Bounded perturbations of the correct restrictions and extensions. In: AIP Conference Proceedings 1759, p. 020116. (2016). https://doi.org/10.1063/1.4959730
4. Hörmander, L.: On the Theory of General Partial Differential Operators. IL, Moscow (1959) (in Russian)
5. Kokebaev, B.K., Otelbaev, M., Shynibekov, A.N.: About expansions and restrictions of operators in Banach space. Uspekhi Mat. Nauk. **37**(4), 116–123 (1982) (in Russian)
6. Taldykin, A.T.: Elements of Applied Functional Analysis. Vysshaya Shkola, Moscow (1982) (in Russian)
7. Vishik, M.I.: On general boundary problems for elliptic differential equations. Tr. Mosk. Matem. Obs. **1**, 187–246 (1952) (in Russian); (English transl., Am. Math. Soc. Trans. II **24**, 107–172 (1963))

Regular Sturm-Liouville Operators with Integral Perturbation of Boundary Condition

Nurlan S. Imanbaev[1] and Makhmud A. Sadybekov[2](✉)

[1] South Kazakhstan State Pedagogical Institute, 16 G.Ilyev str., Shymkent 160012, Kazakhstan
imanbaevnur@mail.ru

[2] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty, 050010, Kazakhstan
sadybekov@math.kz

Abstract. We are studying the issue of stability and instability of the basis property of the system of eigenfunctions and associated functions of the Sturm-Liouville operator with an integral perturbation of one boundary condition. This paper is devoted to a spectral problem for operator with an integral perturbation of boundary conditions, which are regular, but not strongly regular. We assume that the unperturbed problem has system of normalized eigenfunctions and associated functions which forms a Riesz basis. We construct a characteristic determinant of the spectral problem with an integral perturbation of the boundary conditions. The present work is the continuation of authors' researchers on stability (instability) of basis property of root vectors of a differential operator with nonlocal perturbation of one of boundary conditions. The work includes a more detailed exposition of some previous results of authors in this directive, and there are given new results.

Keywords: Sturm-Liouville operators · Regular boundary condition · Not strongly regular boundary condition · Integral perturbation · Spectral problem · Characteristic determinant · Basis property · Eigenfunctions · Associated functions

1 Introduction

A well-known fact is that the system of eigenfunctions of an operator given by formally adjoint differential expressions, with arbitrary self-adjoint boundary conditions providing a discrete spectrum, forms an orthonormal basis in L_2. The question of persisting the basis properties under some (weak in definite sense) perturbation of an original operator has been investigated in many works. For example, the analogous question for the case of a self-adjoint original operator has been investigated in [7,11,13], and for a non-selfadjoint operator in [4,18].

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_21

In [19] the spectral properties of operators of the form $A = T + B$ are analyzed (where B is a non-symmetric operator subordinate to a self-adjoint or normal operator T) and a survey of research in this area is presented.

Spectral theory of non-self-adjoint boundary value problems for ordinary differential equations on a finite interval goes back to the classical works of Birkhoff [1] and Tamarkin [20]. They introduced the concept of regular boundary conditions and investigated asymptotic behavior of eigenvalues and eigenfunctions of such problems.

In $L_2(0,1)$ we consider an operator L_0, generated by the following second order ordinary differential expression:

$$\ell u \equiv -u''(x) + q(x)u(x), \quad q(x) \in C[0,1],\ 0 < x < 1 \tag{1}$$

and the boundary value conditions of the form

$$\begin{cases} U_1(u) = a_{11}u'(0) + a_{12}u'(1) + a_{13}u(0) + a_{14}u(1) = 0, \\ U_2(u) = a_{21}u'(0) + a_{22}u'(1) + a_{23}u(0) + a_{24}u(1) = 0. \end{cases} \tag{2}$$

When the boundary conditions (2) are strongly regular, the results by Dunford [2,3], Mikhailov [14] and Kesel'man [8] provide the Riesz basis property in $L_2(0,1)$ of system of the eigenfunctions and associated functions (EAF) of the problem. In the case when the boundary conditions are regular but not strongly regular, the question on basis property of the system of EAF is not yet completely resolved. When $q(x) \equiv 0$, the problem about basis property of the system of EAF of the problem with general regular boundary conditions has been completely resolved in [10].

2 Statement of the Problem

In the present paper we consider a spectral problem with integral perturbation of one of the boundary conditions (2). By L_1 denote an operator given by the differential expression (1) and by the "perturbed" boundary conditions

$$U_1(u) = 0, \quad U_2(u) = \int_0^1 \overline{p(x)}u(x)dx, \quad p(x) \in L_2(0,1). \tag{3}$$

When the boundary conditions (2) are strongly regular, the results by A.A. Shkalikov [18] provide the Riesz basis property in $L_2(0,1)$ of the system of EAF of the operator L_1.

In the present paper we consider the case when the boundary conditions (2) are not strongly regular.

3 Classes of Not Strongly Regular Boundary Conditions

We introduce the matrix of coefficients of the boundary conditions (2):

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

By $A(ij)$ we denote the matrix composed of the i-th and j-th columns of the matrix A, $A_{ij} = \det A(ij)$.

Let the boundary conditions (2) be regular but not strongly regular. According to [15, p. 73], if the following conditions hold:

$$A_{12} = 0, \quad A_{14} + A_{23} \neq 0, \quad A_{14} + A_{23} = \mp(A_{13} + A_{24}), \tag{4}$$

then the boundary conditions (2) are regular, but not strongly regular boundary conditions.

Makin [12] suggested dividing all regular, but not strongly regular, boundary conditions into four types:

$$\begin{array}{ll} I. \;\; A_{14} = A_{23}, \; A_{34} = 0; & III. \;\; A_{14} \neq A_{23}, \; A_{34} = 0; \\ II. \;\; A_{14} = A_{23}, \; A_{34} \neq 0; & IV. \;\; A_{14} \neq A_{23}, \; A_{34} \neq 0. \end{array}.$$

For example, periodical or antiperiodical boundary conditions form the type I, and can be determined in the following form: $A_{14} = A_{23}$, $A_{34} = 0$. That is, $a_{11} = -a_{12}$, $a_{13} = a_{14} = a_{21} = a_{22} = 0$ and $a_{23} = -a_{24}$. These conditions will be equivalent to matrix A, where the following two options are possible:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

are periodical or

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

are antiperiodical.

And the same boundary conditions with "the lowest coefficients" form the type II. These conditions will be equivalent to matrix A, where the following two options are possible:

$$A = \begin{pmatrix} 1 & \pm 1 & \alpha & 0 \\ 0 & 0 & 1 & \pm 1 \end{pmatrix}, \quad \alpha \neq 0.$$

This case was allocated by Makin [12] as one type of non-strongly regular boundary conditions, when the systems of EAF of the spectral problem

$$L_0(u) \equiv -u''(x) + q(x)u(x) = \lambda u(x), \quad q(x) \in C[0,1], \; 0 < x < 1, \tag{5}$$

with boundary conditions of type II forms a Riesz basis for any potentials $q(x)$.

4 Adjoint Operator L_1^*

Just as in [15, p. 20] we complement the system of forms U_1, U_2 by some forms U_3, U_4 up to the linearly independent system from 4 forms $U_1, \ldots, U_4$. Then there exist linear homogenous forms $V_4, \ldots, V_1$ equal to

$$V_j(v) \equiv \sum_{k=0}^{1} \left[\alpha_{jk}^* v^{(k)}(0) + \beta_{jk}^* v^{(k)}(1) \right], \quad j = \overline{1,4}, \tag{6}$$

for which Lagrange's formula

$$\int_0^1 \ell(y)\overline{v(x)}\mathrm{d}x - \int_0^1 y(x)\overline{\ell^*(v)}dx = \sum_{j=1}^{4} U_j(y)\overline{V_{5-j}(v)} \tag{7}$$

holds. Here

$$\ell^*(v) = -v''(x) + \overline{q(x)}v(x) \tag{8}$$

is an adjoint differential expression.

Therefore, operator L_0^*, adjoint to the operator L_0, is given by the differential expression (8) and boundary value conditions

$$V_1(v) = 0, \quad V_2(v) = 0. \tag{9}$$

Now we define an adjoint operator L_1^*. Using Lagrange's formula (7) for all functions $y \in D(L_1)$ and $v \in D(L_1^*)$, and taking into account the boundary conditions (3), we obtain

$$(L_1 y, v) - (y, \ell^*(v)) = \sum_{j=1}^{4} U_j(y)\overline{V_{5-j}(v)} =$$

$$= \left\{ \int_0^1 \overline{p(x)} y(x) dx \right\} \overline{V_3(v)} + U_3(y)\overline{V_2(v)} + U_4(y)\overline{V_1(v)} = 0.$$

In view of the linear independence of the forms $U_j(y)$ and $V_3(v)$, we see that the operator L_1^* is given by a loaded differential expression

$$L_1^*(v) = -v''(x) + \overline{q(x)}v(x) + p(x)V_3(v) \tag{10}$$

and the boundary conditions (9). It should be noted that, in the case of integral boundary conditions, adjoint operators were first constructed in Krall's paper [9].

5 Characteristic Determinant of a Spectral Problem

We additionally assume that the potential $q(x)$ is chosen in such a way that the unperturbed spectral problem (5) with boundary conditions (2) has the system of EAF generating an unconditional basis in $L_2(0,1)$. Let λ_k^0 be eigenvalues (numbered in decreasing order of their modules) of the operator L_0 of the multiplicity $m_k + 1$ to which the eigenfunctions $y_{k0}^0(x)$ and chains of the adjoint functions $y_{kj}^0(x)$, $j = \overline{1, m_k}$ correspond. Then the biorthogonal system consists of the eigenfunctions $v_{km_k}^0(x)$ and the associated functions $v_{kj}^0(x),\ j = \overline{0, m_k - 1}$ of the operator L_0^* corresponding the eigenvalues $\overline{\lambda_k^0}$. Obviously that the system of EAF $\left\{ v_{kj}^0(x),\ j = \overline{0, m_k},\ k = \overline{1, \infty} \right\}$ of the operator L_0^* forms an unconditional basis in $L_2(0,1)$.

Now we construct a characteristic determinant of the spectral problem. Let $y_1(x,\lambda), y_2(x,\lambda)$ be fundamental solution system of the equation $\ell(y) = \lambda y$ satisfying the conditions $y_j^{(k-1)}(0,\lambda) = \delta_{jk},\ j,k = 1,2$. Here δ_{jk} is the Kronecker symbol. Introducing the general solution by the formula

$$y(x,\lambda) = C_1 y_1(x,\lambda) + C_2 y_2(x,\lambda),$$

and satisfying the boundary conditions (3), we obtain a linear system regarding the coefficients C_1, C_2:

$$C_1 U_1\left(y_1(1,\lambda)\right) + C_2 U_1\left(y_2(1,\lambda)\right) = 0,$$

$$C_1\left(U_2\left(y_1(1,\lambda)\right) - \int_0^1 \overline{p(x)} y_1(x,\lambda)dx\right) + C_2\left(U_2\left(y_2(1,\lambda)\right) - \int_0^1 \overline{p(x)} y_2(x,\lambda)dx\right) = 0,$$

And the determinant of this system is the characteristic determinant of problem (1), (3):

$$\varDelta_1(\lambda) = \begin{vmatrix} U_2\left(y_2(\cdot,\lambda)\right) - \int_0^1 \overline{p(x)} y_2(x,\lambda)dx & U_1\left(y_2(\cdot,\lambda)\right) \\ U_2\left(y_1(\cdot,\lambda)\right) - \int_0^1 \overline{p(x)} y_1(x,\lambda)dx & U_1\left(y_1(\cdot,\lambda)\right) \end{vmatrix}. \tag{11}$$

It is easy to see that the characteristic determinant of unperturbed problem (1), (2) is obtained from (11) by $p(x) = 0$. We denote it by $\varDelta_0(\lambda)$.

We shall express the kernel of the integral perturbation $p(x)$ as a series expansion in the basis $\left\{v_{kj}^0(x),\ j = \overline{0,m_k},\ k = \overline{1,\infty}\right\}$ of the eigenfunctions and associated functions of the unperturbed adjoint operator L_0^*:

$$p(x) = \sum_{k=1}^{\infty}\left[\sum_{j=0}^{m_k} a_{kj} v_{kj}^0(x)\right], \quad a_{kj} = \left(p(x), y_{kj}^0(x)\right)_{L_2(0,1)}. \tag{12}$$

Using (12), we can find a more convenient representation of the determinant $\varDelta_1(\lambda)$. To do this, let us first calculate

$$\int_0^1 \overline{p(x)} y_s(x,\lambda)dx = \sum_{k=1}^{\infty}\left[\sum_{j=0}^{m_k^0} \overline{a_{kj}}\left(y_s(\cdot,\lambda), v_{kj}^0(\cdot)\right)\right],\ s = 1,2. \tag{13}$$

Taking into account the fact that the chains of eigenfunctions and associated functions of the adjoint problem are defined by the formulas:

$$L_0^* v_{km_k^0}^0 = \overline{\lambda_k^0} v_{km_k^0}^0,\ L_0^* v_{kj}^0 = \overline{\lambda_k^0} v_{kj}^0 + v_{kj+1}^0,\ j = 0, m_k^0 - 1,$$

we can easily verify the following sequence of equalities for $j < m_k^0$:

$$\begin{aligned}(\lambda - \lambda_k^0)\left(y_s(\cdot,\lambda), v_{kj}^0(\cdot)\right) &= \left(\lambda y_s(\cdot,\lambda), v_{kj}^0(\cdot)\right) - \left(y_s(\cdot,\lambda), \overline{\lambda_k^0} v_{kj}^0(\cdot)\right) = \\ &= \left(\ell(y_s), v_{kj}^0\right) - \left(y_s, L_0^* v_{kj}^0\right) + \left(y_s, v_{kj+1}^0\right).\end{aligned}$$

Here let us use Lagrange's formula (7) and the boundary condition (9). Then, for all $j = 0, ..., m_k^0 - 1$, we obtain

$$(\lambda - \lambda_k^0)(y_s(x,\lambda), v_{kj}^0(x)) = B_{ks}(j) + (y_s, v_{kj+1}^0),$$

where we denote

$$B_{ks}(j) = U_1\left(y_s\right)\overline{V_4\left(v_{kj}^0\right)} + U_2\left(y_s\right)\overline{V_3\left(v_{kj}^0\right)}. \tag{14}$$

Repeating similar calculations $(m_k^0 - 1 - j)$ times, we can write

$$(y_s(\cdot,\lambda), v_{kj}^0(\cdot)) = \sum_{r=0}^{m_k^0-1-j} \frac{B_{ks}(j+r)}{(\lambda-\lambda_k^0)^{r+1}} + \frac{1}{(\lambda-\lambda_k^0)^{m_k^0-j}}(y_s, v_{k\ m_k^0}^0).$$

Similarly for the eigenfunction $v_{k\ m_k^0}^0$ we get

$$(\lambda - \lambda_k^0)(y_s(\cdot,\lambda), v_{k\ m_k}^0(\cdot)) = B_{ks}(m_k).$$

Combining the last two equalities, we can write

$$(y_s(\cdot,\lambda), v_{kj}^0(\cdot)) = \sum_{r=0}^{m_k^0-j} \frac{B_{ks}(j+r)}{(\lambda-\lambda_k^0)^{r+1}}.$$

Substituting expression (14) into the above formula, after elementary transformations, we obtain

$$(y_s(\cdot,\lambda), v_{kj}^0(\cdot)) = \sum_{r=0}^{m_k-j} \frac{U_1\left(y_s\right)\overline{V_4\left(v_{k\ j+r}^0\right)} + U_2\left(y_s\right)\overline{V_3\left(v_{k\ j+r}^0\right)}}{(\lambda-\lambda_k^0)^{r+1}}. \tag{15}$$

Now, substituting (15) into formula (13), we can write

$$\int_0^1 \overline{p(x)} y_s(x,\lambda)dx = U_1\left(y_s\right) A_1(\lambda) + U_2\left(y_s\right) A_2(\lambda),$$

where we denote

$$A_i(\lambda) = \sum_{k=1}^{\infty}\left[\sum_{j=0}^{m_k^0} \overline{a_{k\ j}} \left(\sum_{r=0}^{m_k^0-j} \frac{V_{5-i}\left(v_{k\ j+r}^0\right)}{(\lambda-\lambda_k^0)^{r+1}}\right)\right]. \tag{16}$$

Using the resulting expressions from (11), we obtain

$$\varDelta_1(\lambda) = \varDelta_0(\lambda) - \sum_{i=1}^{2} A_i(\lambda) \begin{vmatrix} U_i(y_2) & U_1(y_2) \\ U_i(y_1) & U_1(y_1) \end{vmatrix}. \tag{17}$$

It is easy to see that one summand (for $i = 1$) in (17) vanishes (as determinants with identical rows). Therefore,

$$\Delta_1(\lambda) = \Delta_0(\lambda) - \Delta_0(\lambda)A_2(\lambda) = \Delta_0(\lambda)\left(1 - A_2(\lambda)\right). \tag{18}$$

Substituting the value of $A_2(\lambda)$ from (16), we obtain the following representation of the characteristic determinant of the operator L_1:

$$\Delta_1(\lambda) = \Delta_0(\lambda)\left(1 - \sum_{k=1}^{\infty}\left[\sum_{j=0}^{m_k^0}\overline{a_{k\ j}}\left(\sum_{r=0}^{m_k^0-j}\frac{V_3\left(v_{k\ j+r}^0\right)}{\left(\lambda-\lambda_k^0\right)^{r+1}}\right)\right]\right). \tag{19}$$

Let us state the obtained result as a theorem.

Theorem 1. *Let problem (1), (2) possess the eigenvalues λ_k^0 and the EAF generating an unconditional basis in $L_2(0,1)$. Then the characteristic determinant of problem (1), (3) with the perturbed boundary conditions is expressed as (19), where $\Delta_0(\lambda)$ is the characteristic determinant of the unperturbed problem (1), (2); V_3 is the linear homogeneous form arising from the construction of the boundary conditions (9) of the adjoint unperturbed problem; $\{v_{kj}^0\}$ are the EAF of the adjoint unperturbed problem; and a_{kj} are the Fourier coefficients of the biorthogonal expansion (12) of functions $p(x)$ by this system.*

First, it is necessary to see that, in the representation (18) the function $A_2(\lambda)$ can have poles of maximal order m_k^0+1 at the points $\lambda = \lambda_k^0$. However, at these points, the function $\Delta_0(\lambda)$ has zeros of order m_k^0+1. Therefore, the function $\Delta_1(\lambda)$ expressed by formula (19) is an entire analytic function of the variable λ.

Second, it does not follow from (18) that all the zeros λ_k^0 of the characteristic determinant $\Delta_0(\lambda)$ will be zeros of $\Delta_1(\lambda)$, because, at these points, the function $A_2(\lambda)$ can have poles. The same also applies the multiplicity of the eigenvalues. And this fact will depend on the behavior of the coefficients a_{kj} of the expansion (12) of the function $p(x)$.

Note that in our paper [16] formula (19) was obtained for the case of linear ordinary differential operator of n-th order and regular boundary conditions of general form with the integral perturbation of one of its boundary conditions.

6 Particular Cases of the Characteristic Determinant

The simplest form of formula (19) corresponds to the case in which all the eigenvalues of the unperturbed problem (1), (2) are simple:

$$\Delta_1(\lambda) = \Delta_0(\lambda)\left(1 - \sum_{k=1}^{\infty}\overline{a_k}\frac{V_3\left(v_k^0\right)}{\lambda-\lambda_k^0}\right),$$

where $v_k^0(x)$ are eigenfunctions of the adjoint unperturbed problem and a_k are Fourier coefficients of the biorthogonal expansion of the function $p(x)$ in this system.

Another case of the simple form of the characteristic determinant is when $p(x)$ can be expressed as a finite sum in (12). In other words, when there exists a number N such that $a_{kj} = 0$ for all $k > N$. In this case, formula (19) takes the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left(1 - \sum_{k=1}^{N} \left[\sum_{j=0}^{m_k^0} \overline{a_{k\ j}} \left(\sum_{r=0}^{m_k^0 - j} \frac{V_3\left(v_{k\ j+r}^0\right)}{\left(\lambda - \lambda_k^0\right)^{r+1}} \right) \right] \right). \tag{20}$$

For this particular case, it is easy to justify the following statement.

Corollary 1. *Under the assumptions of Theorem 1, for all given numbers: the complex* $\widehat{\lambda}$ *and the natural number* $\widehat{m}$*, there always exists a function* $p(x)$ *such that* $\widehat{\lambda}$ *will be an eigenvalue of problem (1), (3) of multiplicity* $\widehat{m}$*.*

7 On Stability of Basis Property

It is easy to see from the analysis of formula (20) that $\Delta_1(\lambda_k^0) = 0$ for all $k > N$. In other words, for $k > N$, all the eigenvalues λ_k^0 of the unperturbed problem (1), (2) are eigenvalues of the perturbed problem (1), (3). In addition it is not difficult to verify that the multiplicity of the eigenvalues λ_k^0, $k > N$ is also preserved.

Moreover, it follows from the biorthogonality condition for the system of EAF of the adjoint problems that, in this case

$$\int_0^1 \overline{p(x)} y_{kj}^0(x, \lambda) dx = 0, \quad j = 0, \ldots, m_k^0, \quad k > N.$$

Therefore, for $k > N$, the EAF $u_{kj}^0(x)$ of problem (1), (2) satisfy the boundary conditions (3) and, therefore, are the EAF of problem (1), (3). Hence, in this case, the system of EAF of problem (1), (3) differs from the system of EAF of problem (1), (2) (forming an unconditional basis) only by a finite number of first terms. Therefore, the system of EAF of problem (1), (3) also forms the unconditional basis in $L_2(0, 1)$.

In view of the basis property (in $L_2(0, 1)$) of the system of EAF $v_{kj}^0(x)$ of the adjoint unperturbed problem, the set of functions $p(x)$ expressible as the finite series (12) is dense in $L_2(0, 1)$. Thus, we establish the following statement.

Theorem 2. *Let problem (1), (2) possess EAF forming an unconditional basis. Then the set of such functions* $p(x) \in L_2(0, 1)$*, for which the system of EAF of problem (1), (3) forms an unconditional basis in* $L_2(0, 1)$*, is dense in* $L_2(0, 1)$*.*

Note that, in first time in [11], an analog of Theorem 2 was proved for the particular case of the integral perturbation of periodic boundary conditions for the double differentiation operator. In addition, it was proved in [11] that the set of functions $p(x) \in L_2(0, 1)$ such that the system of EAF of problem (1), (3) does not even form a usual basis in $L_2(0, 1)$, is also dense in $L_2(0, 1)$.

Let us now demonstrate the application of formula (19) to the particular cases of a problem with integral perturbation of the boundary condition. Since the case of strongly regular boundary conditions was completely solved in [18], we shall consider an example from regular, but not strongly regular, boundary conditions.

8 Type I. Periodic Boundary Conditions

In this section we consider the problem in case whether the basis property changes under integral perturbations of the boundary condition for the periodic boundary value problem

$$\begin{gathered} L_1 y = -y''(x) + q(x)y(x) = \lambda y(x), \quad 0 < x < 1, \\ U_1(y) \equiv y(0) - y(1) = 0, \;\; U_2(y) \equiv y'(0) - y'(1) = \int_0^1 \overline{p(x)} y(x)dx. \end{gathered} \tag{21}$$

Note that problem (21) possesses specific spectral properties. For $p(x) \equiv 0$ depending on the coefficient $q(x)$, the system of EAF of the problem can either form or not form an unconditional basis in $L_2(0,1)$.

In this case, Lagrange's formula is of the form

$$\begin{gathered} \int_0^1 \ell(y)(x)\overline{v(x)}dx - \int_0^1 y(x)\overline{\ell^*(v)(x)}dx = -\left[y(0) - y(1)\right]\overline{v'(0)} + \\ + \left[y'(0) - y'(1)\right]\overline{v(0)} + y'(1)\left[\overline{v(0)} - \overline{v(1)}\right] - y(1)\left[\overline{v'(0)} - \overline{v'(1)}\right]. \end{gathered}$$

Therefore

$$U_1(y) = y(0) - y(1), \;\; U_2(y) = y'(0) - y'(1), \;\; U_3(y) = y'(1), \;\; U_4(y) = -y(1),$$

and

$$V_4(v) = -v'(0), \;\; V_3(v) = v(0), \;\; V_2(v) = v(0) - v(1), \;\; V_1(v) = v'(0) - v'(1).$$

On the basis of Theorem 1 we get the characteristic determinant of problem (21):

$$\varDelta_1(\lambda) = \varDelta_0(\lambda)\left(1 - \sum_{k=1}^{\infty}\left[\overline{a_{k0}}\left(\frac{v_{k0}^0(0)}{\lambda - \lambda_k^0} + \frac{v_{k1}^0(0)}{(\lambda - \lambda_k^0)^2}\right) + \overline{a_{k1}}\frac{v_{k1}^0(0)}{\lambda - \lambda_k^0}\right]\right), \tag{22}$$

where $\varDelta_0(\lambda)$ is the characteristic determinant of the unperturbed problem (21); $\{v_{k0}^0, v_{k1}^0\}$ are the EAF of the adjoint unperturbed problem; and a_{k0}, a_{k1} are the Fourier coefficients of the biorthogonal expansion of functions $p(x)$ by this system:

$$p(x) = \sum_{k=1}^{\infty}\left\{a_{k0}v_{k0}^0 + a_{k1}v_{k1}^0\right\}.$$

Note that formula (22) obtained from Theorem 1 coincides with the results of [5,17], in which the characteristic determinant was obtained by direct calculation. In these works on the basis of formula (22) there was also obtained the following result:

Theorem 3. *([5, 17]). Let unperturbed (i.e., for $p(x) \equiv 0$) periodic problem (21) have a system of EAF forming a Riesz basis in $L_2(0,1)$, let the eigenvalues of the problem be double, and let root subspaces corresponding to double eigenvalues consist of two eigenfunctions. Then the set of all functions $p \in L_2(0,1)$, for which the system of EAF of problem (21) does not form even an ordinary basis in $L_2(0,1)$, is dense in $L_2(0,1)$.*

9 Type II. Boundary Conditions of Periodical Type

The present section is devoted to the spectral problem of type II:

$$\begin{gathered} L_1 y = -y''(x) + q(x)y(x) = \lambda y(x), \quad 0 < x < 1, \\ U_1(y) \equiv y'(0) - y'(1) + \alpha y(1) = 0, \;\; U_2(y) \equiv y(0) - y(1) = \int_0^1 \overline{p(x)} y(x) dx. \end{gathered} \tag{23}$$

Note that problem (23) possesses specific spectral properties. For $p(x) \equiv 0$ undepending on the coefficient $q(x)$, the system of EAF of the problem forms an unconditional basis in $L_2(0,1)$ [12]. And for any $\alpha \neq 0$ the unperturbed problem

$$L_1 y = \lambda y; \;\; U_1(y) = 0; \;\; U_2(y) = 0$$

has an asymptotically simple spectrum, and the system of its normalized eigenfunctions generates the Riesz basis in $L_2(0,1)$.

In this case, Lagrange's formula is of the form

$$\begin{gathered} \int_0^1 \ell(y)(x)\overline{v(x)}dx - \int_0^1 y(x)\overline{\ell^*(v)(x)}dx = [y'(0) - y'(1) + \alpha y(1)]\,\overline{v(0)} - \\ - [y(0) - y(1)]\,\overline{v'(0)} + y'(1)\left[\overline{v(0)} - \overline{v(1)}\right] - y(1)\left[\overline{v'(0)} - \overline{v'(1)} + \alpha\overline{v(0)}\right]. \end{gathered}$$

Therefore

$$U_1(y) = y'(0) - y'(1) + \alpha y(1), \;\; U_2(y) = y(0) - y(1), \;\; U_3(y) = y'(1), \;\; U_4(y) = -y(1),$$

and

$$V_4(v) = v(0), \;\; V_3(v) = -v'(0), \;\; V_2(v) = v(0) - v(1), \;\; V_1(v) = v'(0) - v'(1) + \alpha v(0).$$

On the basis of Theorem 1 we obtain the characteristic determinant of problem (23):

$$\Delta_1(\lambda) = \Delta_0(\lambda)\left(1 - \sum_{k=1}^{\infty}\left[\overline{a_{k0}}\frac{\left(v_{k0}^0\right)'(0)}{\lambda - \lambda_{k0}^0} + \overline{a_{k1}}\frac{\left(v_{k1}^0\right)'(0)}{\lambda - \lambda_{k1}^0}\right]\right), \tag{24}$$

where $\Delta_0(\lambda)$ is the characteristic determinant of the unperturbed problem (21); v_{k0}^0, v_{k1}^0 are the eigenfunctions of the adjoint unperturbed problem to which the eigenvalues λ_{k0}^0, λ_{k1}^0 correspond; and a_{k0}, a_{k1} are the Fourier coefficients of the biorthogonal expansion of functions $p(x)$ by this system:

$$p(x) = \sum_{k=1}^{\infty}\left\{a_{k0}v_{k0}^0 + a_{k1}v_{k1}^0\right\}.$$

Note that formula (24) obtained from Theorem 1 coincides with the results of [6], in which the characteristic determinant was obtained by direct calculation for the case $q(x) \equiv 0$. In this work on the basis of formula (24) there was also obtained the result of stability of basis property. By the methods of [6] we prove the following theorem:

Theorem 4. *([6]). For any function $p(x) \in W_2^1(0,1)$ the system of eigen and adjoint functions of problem (23) forms a Riesz basis in $L_2(0,1)$.*

10 Types III and IV. Boundary Conditions of Samarskii-Ionkin Type

In the space $L_2(0,1)$, consider the operator L_0 generated by the ordinary differential expression and the boundary conditions

$$\begin{gathered} \ell y = -y''(x) + q(x)y(x), \quad 0 < x < 1, \\ U_1(y) \equiv y'(0) - y'(1) + \alpha y(1) = 0, \;\; U_2(y) \equiv y(0) = 0. \end{gathered} \tag{25}$$

For $\alpha = 0$ these conditions belong to type III, and for $\alpha \neq 0$ they are conditions of type IV.

In the literature, this problem is called the Samarskii-Ionkin problem. Note that problem (25) possesses specific spectral properties. Depending on the coefficient $q(x)$, the system of EAF of the problem can either form or not form an unconditional basis in $L_2(0,1)$. As is shown in [12], the EAF of the problem constitute an unconditional basis in $L_2(0,1)$ only if the eigenvalues of the problem are asymptotically double and the corresponding root subspaces consist of one eigenfunction and one associated function.

Let L_1 be an operator in $L_2(0,1)$ given by the ordinary differential expression and the "perturbed" boundary conditions

$$\begin{gathered} L_1 y = -y''(x) + q(x)y(x), \quad 0 < x < 1, \\ U_1(y) \equiv y'(0) - y'(1) + \alpha y(1) = 0, \;\; U_2(y) \equiv y(0) = \int_0^1 \overline{p(x)} y(x) dx. \end{gathered} \tag{26}$$

In this case, Lagrange's formula is of the form

$$\begin{gathered} \int_0^1 \ell(y)(x)\overline{v(x)}dx - \int_0^1 y(x)\overline{\ell^*(v)(x)}dx = \left[y'(0) - y'(1) + \alpha y(1)\right]\overline{v(0)} - \\ -y(0)\overline{v'(0)} + y'(1)\left[\overline{v(0)} - \overline{v(1)}\right] + y(1)\left[\overline{v'(1)} - \alpha\overline{v(0)}\right]. \end{gathered}$$

Therefore

$$U_1(y) = y'(0) - y'(1) + \alpha y(1), \;\; U_2(y) = y(0), \;\; U_3(y) = y'(1), \;\; U_4(y) = y(1),$$

and

$$V_4(v) = v(0), \;\; V_3(v) = -v'(0), \;\; V_2(v) = v(0) - v(1), \;\; V_1(v) = v'(1) - \alpha v(0).$$

On the basis of Theorem 1 we obtain the characteristic determinant of problem (26): $\Delta_1(\lambda) =$

$$= \Delta_0(\lambda)\left(1 - \sum_{k=1}^{\infty}\left[\overline{a_{k0}}\left(\frac{\left(v_{k0}^0\right)'(0)}{\lambda - \lambda_k^0} + \frac{\left(v_{k1}^0\right)'(0)}{(\lambda - \lambda_k^0)^2}\right) + \overline{a_{k1}}\frac{\left(v_{k1}^0\right)'(0)}{\lambda - \lambda_k^0}\right]\right), \quad (27)$$

where $\Delta_0(\lambda)$ is the characteristic determinant of the unperturbed problem (25); $\{v_{k0}^0, v_{k1}^0\}$ are the EAF of the adjoint unperturbed problem to which the eigenvalues λ_{k0}^0 correspond; and a_{k0}, a_{k1} are the Fourier coefficients of the biorthogonal expansion of functions $p(x)$ by this system:

$$p(x) = \sum_{k=1}^{\infty}\left\{a_{k0}v_{k0}^0 + a_{k1}v_{k1}^0\right\}.$$

Note that the formula obtained from Theorem 1 is new even for the case $q(x) \equiv 0$. On the basis of formula (27) one can obtain asymptotic behavior of the eigenvalues and eigenfunctions of the problem. On the basis of these asymptotics we prove the following theorem.

Theorem 5. *Let the unperturbed problem (25) possess eigenfunctions and associated functions forming an unconditional basis in* $L_2(0,1)$*. Then the set of functions* $p(x) \in L_2(0,1)$*, for which the system of eigenfunctions and associated functions of the perturbed problem (26) does not form an unconditional basis in* $L_2(0,1)$*, is dense in* $L_2(0,1)$*.*

Acknowledgements. The authors expresses his great gratitude to the academicians of NAS RK M. Otelbaev and T.Sh. Kal'menov for consistent support and fruitful discussion of the results. This research is supported by the grant no. 0825/GF4 and target program 0085/PTSF-14 of the Ministry of Education and Science of Republic of Kazakhstan.

References

1. Birkhoff, G.D.: Boundary value and expansion problems of ordinary linear differential equations. Trans. Am. Math. Soc. **9**, 373–395 (1908)
2. Dunford, N.: A survey of the theory of spectral operators. Bull. Am. Math. Soc. **64**, 217–274 (1958)
3. Dunford, N., Schwartz, J.: Linear Operators. Part III, Spectral Operators. Wiley, New York (1971)
4. Il'in, V.A, Kritskov, L.V.: Properties of spectral expansions corresponding to non-self-adjoint differential operators. J. Math. Sci. **116**(5), 3489–3550 (2003)
5. Imanbaev, N.S., Sadybekov, M.A.: On spectral properties of a periodic problem with an integral perturbation of the boundary condition. Eurasian Math. J. **4**(3), 53–62 (2013)
6. Imanbaev, N.S., Sadybekov, M.A.: Stability of basis property of a type of problems on eigenvalues with nonlocal perturbation of boundary conditions. Ufimsk. Mat. Zh. **3**(2), 28–33 (2011)

7. Kerimov, N.B., Mamedov, K.R.: On the Riesz basis property of the root functions in certain regular boundary value problems. Math. Notes **64**(3–4), 483–487 (1998)
8. Kesel'man, G.M.: On the unconditional convergence of eigenfunction expansions of certain differential operators. Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 82–93 (1964)
9. Krall, A.M.: The development of general differential and general differential-boundary systems. Rocky Mt. J. Math. **5**(4), 493–542 (1975)
10. Lang, P., Locker, J.: Spectral theory of two-point differential operators determined by $-D^2$. J. Math. Anal. Appl. **146**(1), 148–191 (1990)
11. Makin, A.S.: On a nonlocal perturbation of a periodic eigenvalue problem. Differ. Equ. **42**(4), 599–602 (2006)
12. Makin, A.S.: On spectral decompositions corresponding to non-self-adjoint Sturm-Liouville operators. Doklady Math. **73**(1), 15–18 (2006)
13. Markus, A.S.: On a expansion of the root vectors of weakly perturbed self-adjoint operator. Dokl. USSR Acad. Sci. **142**(4), 538–541 (1962)
14. Mihailov, V.P.: On Riesz bases in $L_2(0,1)$. Dokl. Akad. Nauk SSSR. **144**(5), 981–984 (1962) (in Russian)
15. Naimark, M.A.: Linear Differential Operators. Nauka, Moscow (1969) (in Russian)
16. Sadybekov, M.A., Imanbaev, N.S.: A regular differential operator with perturbed boundary condition. Math. Notes **101**(5), 878–887 (2017)
17. Sadybekov, M.A., Imanbaev, N.S.: On the basis property of root functions of a periodic problem with an integral perturbation of the boundary condition. Differ. Equ. **48**(6), 896–900 (2012)
18. Shkalikov, A.A.: Basis property of eigenfunctions of ordinary differential operators with integral boundary conditions. Vestnik Moskov. Univ. Ser. I Math. Mech. **6**, 12–21 (1982)
19. Shkalikov, A.A.: Perturbations of self-adjoint and normal operators with discrete spectrum. Russ. Math. Surv. **71**(5), 113–174 (2016)
20. Tamarkin, I.D.: Sur qudques points de In theorie des equations differentiales ordinares of sur generalization de In serie de Fourier. Rend. Cire. Math. Palermo. **34**(2), 345–382 (1912)

A Boundary Condition of the Volume Potential for Strongly Elliptic Differential Equations

Tynysbek Sh. Kal'menov and Bolys Sabitbek(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
kalmenov@math.kz, b.sabitbek@math.kz

Abstract. In this paper we construct a nonlocal integral boundary condition of the volume potential for second order strongly elliptic differential equations, which generalizes previous known results. We also review similar results for polyharmonic operators.

Keywords: Non-local boundary conditions · Volume potential · Polyharmonic operators · Dirichlet boundary conditions · Green function

1 Introduction

Let $\Omega \subset R^d$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$. We consider the second order uniformly strongly elliptic equation

$$D(u) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x),\ x \in \Omega. \quad (1)$$

The functions a_{ij}, b_j and c are real-valued functions which, for convenience, are supposed to be C^∞-functions.

Definition 1. The second order real-valued scalar linear differential operator D is called strongly elliptic in Ω if there exists a smooth function $\gamma(x) > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \gamma(x)|\xi|^2 \quad (2)$$

for all $\xi \in R^d$. If, in addition, $\gamma > 0$ is a constant independent of x and (2) holds for all $x \in \Omega$, then D is called uniformly strongly elliptic.

Dedicated to the 75th anniversary of Academician Mukhtarbay Otelbaev.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_22

Note that strongly elliptic real differential operators are of even order and are properly elliptic.

Definition 2. Let $x \in R^d$ be any chosen point. Then the distribution $E(x, y)$ is called a fundamental solution of the differential operator D (in R^d) if it satisfies the equation

$$D_y(E(x, y)) = \delta(x - y) \tag{3}$$

in the distributional sense, where δ is the Dirac distribution.

As usual, in (3) the notation D_y stands for differentiation with respect to y. For strongly elliptic operators it can be shown with the Green's formula that (3) implies

$$D_x(E(x, y)) = \delta(x - y) \tag{4}$$

for any fixed $y \in R^d$.

For a general differential operator, the existence of a fundamental solution is by no means trivial. However, we have

Lemma 1. *(**Hörmander** [1].) Let D be a uniformly strongly elliptic differential operator of even order with real leading coefficients $a_{ij} \in C^\infty$. Then for every compact domain $\overline{\Omega} \subset R^d$ with $\partial\Omega \in C^\infty$ there exists a local fundamental solution $E(x, y)$ which is a C^∞ function of all variables for $x \neq y$ and $x, y \in \overline{\Omega}$.*

In Section 2 of this paper by using properties of fundamental solutions we construct a correct boundary value problem for the differential equation (1). In Section 3 we review similar results for polyharmonic equations, which hints how to extend results of Section 2 to higher order cases. Throughout this paper we use notations from [2] and [3].

2 Second Order Strongly Elliptic Equations

Let $\Omega_1 \subset ... \subset \Omega_n \subset R^d$ be open bounded domains with boundaries $\partial\Omega_i \in C^\infty, i = 1, ..., n$, respectively. By Hörmander's Lemma there exists a local fundamental solution $E_i(x, y)$ of D for each Ω_i. Consider the following function

$$u(x) = \int_\Omega G(x, y) f(y) dy \tag{5}$$

in $\Omega \subset \Omega_1,\ \partial\Omega \in C^\infty$, where

$$G(x, y) = \sum_{i=1}^{n} \alpha_i E_i(x, y), \quad x, y \in \Omega, \quad \sum_{i=1}^{n} \alpha_i = 1. \tag{6}$$

Here $\alpha_i, i = 1, ..., n$ are numbers. A trivial observation shows that $u(x)$ is a solution of (1) in Ω. The aim of this section is to find a boundary condition such that with this boundary condition the equation (1) has a unique solution in $H^2(\Omega)$, which is $u(x)$.

Theorem 1. *For any $f \in L_2(\Omega)$, (5) is a unique solution of the equation (1) (in $H^2(\Omega)$) with the boundary condition*

$$-\frac{u(x)}{2} + \int_{\partial\Omega} \partial_{\nu y} G(x,y) u(y) dS_y - \tag{7}$$

$$\int_{\partial\Omega} G(x,y)\{\partial_{\nu y} u(y) - \sum_{j=1}^{d} n_j b_j u(y)\} dS_y = 0,\ x \in \partial\Omega,$$

where the conormal derivative is $\partial_{\nu y} = \sum_{k,j=1}^{d} n_j a_{jk} \frac{\partial}{\partial x_k}$ and $n_1, n_2, ..., n_d$ are components of the normal vector on the boundary.

Proof. From (6) it is easy to see that G is a fundamental solution of the operator D in Ω. Therefore,

$$u(x) = \int_{\Omega} G(x,y) f(y) dy$$

is the solution of (1). In addition, the following representation formula can be derived from the generalized second Green's formula in Sobolev spaces as in the classical approach by density and completion arguments [2]

$$u(x) = \int_{\Omega} G(x,y) f(y) dy + \int_{\partial\Omega} \partial_{\nu y} G(x,y) u(y) dS_y - \tag{8}$$

$$\int_{\partial\Omega} G(x,y)\{\partial_{\nu y} u(y) - \sum_{j=1}^{d} n_j b_j u(y)\} dS_y$$

for any $x \in \Omega$. From (5) and (8) it implies that

$$\int_{\partial\Omega} \partial_{\nu y} G(x,y) u(y) dS_y - \int_{\partial\Omega} G(x,y)\{\partial_{\nu y} u(y) - \sum_{j=1}^{d} n_j b_j u(y)\} dS_y = 0$$

for any $x \in \Omega$.

By using the properties of the double and single layer potentials [2] as $x \to \partial\Omega$, we find that

$$-\frac{u(x)}{2} + \int_{\partial\Omega} \partial_{\nu y} G(x,y) u(y) dS_y -$$

$$\int_{\partial\Omega} G(x,y)\{\partial_{\nu y} u(y) - \sum_{j=1}^{d} n_j b_j u(y)\} dS_y = 0,\ x \in \partial\Omega.$$

We have shown that (5) is the solution of the boundary value problem (1) with the boundary condition (7) (in $H^2(\Omega)$). Now let us prove its uniqueness. If the boundary value problem has two solutions u and u_1, then the function $v = u - u_1 \in H^2(\Omega)$ satisfies the homogeneous equation

$$D(v) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial v}{\partial x_j}) + \sum_{i=1}^{d} b_i(x) \frac{\partial v}{\partial x_i} + c(x) v = 0, x \in \Omega, \tag{9}$$

and the boundary condition (7), i.e.

$$-\frac{v(x)}{2}+\int_{\partial\Omega}\partial_{\nu y}G(x,y)v(y)dS_y- \tag{10}$$

$$\int_{\partial\Omega}G(x,y)\{\partial_{\nu y}v(y)-\sum_{j=1}^{d}n_jb_jv(y)\}dS_y=0,\ x\in\partial\Omega.$$

Since $f\equiv 0$ instead of (8) we have the following representation formula

$$v(x)=\int_{\partial\Omega}\partial_{\nu y}G(x,y)v(y)dS_y- \tag{11}$$

$$\int_{\partial\Omega}G(x,y)\{\partial_{\nu y}v(y)-\sum_{j=1}^{d}n_jb_jv(y)\}dS_y$$

for any $x\in\Omega$. As above, by using the properties of the double and single layer potentials as $x\to\partial\Omega$, we obtain

$$\frac{v(x)}{2}+\int_{\partial\Omega}\partial_{\nu y}G(x,y)v(y)dS_y- \tag{12}$$

$$\int_{\partial\Omega}G(x,y)\{\partial_{\nu y}v(y)-\sum_{j=1}^{d}n_jb_jv(y)\}dS_y=0,\ x\in\partial\Omega.$$

Comparing this with (2), we arrive at

$$v(x)=0,\ x\in\partial\Omega. \tag{13}$$

The second order homogeneous strongly elliptic equation (9) with the Dirichlet boundary condition (13) has only trivial solution $v\equiv 0$. This shows that the boundary value problem (1) with the boundary condition (7) has a unique solution in $H^2(\Omega)$. Theorem 1 is proved.

Example 1. Let D be Δ-Laplacian, $n=1$ and $\Omega_1\equiv\Omega$, then

$$\varepsilon_d(x-y):=E_1(x,y)=\begin{cases}\frac{1}{(d-2)s_d}\frac{1}{|x-y|^{d-2}},\ d\geq 3,\\ -\frac{1}{2\pi}log|x-y|,\ d=2,\end{cases}$$

is a fundamental solution of Laplacian in Ω_1, $s_d=\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of the unit sphere in R^d, $d\geq 2$, and $|x-y|$ is the standard Euclidean distance between x and y. In this case instead of (5) we have

$$u(x)=\int_{\Omega}\varepsilon_d(x-y)f(y)dy,\ x\in\Omega, \tag{14}$$

which is a unique solution of

$$-\Delta u(x)=f(x),\quad x\in\Omega, \tag{15}$$

with the boundary condition

$$-\frac{u(x)}{2}+\int_{\partial\Omega}\frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dS_y-\int_{\partial\Omega}\varepsilon_d(x-y)\frac{\partial u(y)}{\partial n_y}dS_y=0,\ x\in\partial\Omega,\quad(16)$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point y on $\partial\Omega$.

For the first time the boundary condition (16) was mentioned (without proof) in Kac's work [4], he called it "the principle of not feeling the boundary" [5]. In [6] T. Sh. Kal'menov and D. Suragan proved the existence of the boundary condition (16) and as byproduct the eigenvalues and eigenfunctions of the Newton potential (14) were calculated in the 2-disk and in the 3-ball.

The boundary value problem (15)-(16) has various interesting extensions and applications (see, for example, [8–13]).

The boundary value problem (15)-(16) can also be generalized for higher degrees of Laplacian [7].

3 Polyharmonic Equations

In this paper we present a result of the paper [7] in a different way, which hints how to extend results of the previous section to higher order cases. Let $\Omega_1\subset ...\subset\Omega_n\subset R^d$ be open bounded domains with boundaries $\partial\Omega_i\in C^\infty, i=1,...,n$, respectively. By Hörmander's Lemma there exists a local fundamental solution $E_i(x,y)$ of the polyharmonic equation

$$(-\Delta_x)^m u(x)=f(x),\quad m=1,2,...,\quad(17)$$

for each Ω_i.

Consider the following function

$$u(x)=\int_\Omega G_{m,d}(x,y)f(y)dy\quad(18)$$

in $\Omega\subset\Omega_1,\ \partial\Omega\in C^\infty$, where

$$G_{m,d}(x,y)=\sum_{i=1}^n\alpha_iE_i(x,y),\quad x,y\in\Omega,\quad\sum_{i=1}^n\alpha_i=1.\quad(19)$$

Here $\alpha_i, i=1,...,n$ are numbers. A trivial observation shows that (18) is a solution of (17) in Ω. The aim of this section is to find a boundary condition such that with this boundary condition the equation (17) has a unique solution in $H^{2m}(\Omega)$, which coincides with (18).

Theorem 2. *For any $f\in L_2(\Omega)$, (18) is a unique solution of the equation (17) (in $H^{2m}(\Omega)$) with the boundary conditions*

$$-\frac{1}{2}(-\Delta_x)^iu(x)+\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j}G_{m,d}(x,y)(-\Delta_y)^{j+i}u(y)dS_y-$$

$$\sum_{j=0}^{m-i-1}\int_{\partial\Omega}(-\Delta_y)^{m-1-j}G_{m,d}(x,y)\frac{\partial}{\partial n_y}(-\Delta_y)^{j+i}u(y)dS_y=0, \tag{20}$$

for $i=0,1,...,m-1$ *and* $x\in\partial\Omega$, *where* $\frac{\partial}{\partial n_y}=n_1\frac{\partial}{\partial y_1}+...+n_n\frac{\partial}{\partial y_n}$ *is the normal derivative on the boundary and* $n_1,...,n_n$ *are the components of the unit normal.*

Proof. By applying the Green's formula for each $x\in\Omega$, we obtain

$$u(x)=\int_{\Omega}G_{m,d}(x,y)f(y)dy=\int_{\Omega}G_{m,d}(x,y)(-\Delta_y)^m u(y)dy=$$

$$\int_{\Omega}(-\Delta_y)G_{m,d}(x,y)(-\Delta_y)^{m-1}u(y)dy+$$

$$\int_{\partial\Omega}\frac{\partial G_{m,d}(x,y)}{\partial n_y}(-\Delta_y)^{m-1}u(y)dS_y-$$

$$\int_{\partial\Omega}G_{m,d}(x,y)\frac{\partial(-\Delta_y)^{m-1}u(y)}{\partial n_y}dS_y=$$

$$\int_{\Omega}(-\Delta_y)^2G_{m,d}(x,y)(-\Delta_y)^{m-2}u(y)dy+$$

$$\int_{\partial\Omega}\frac{\partial(-\Delta)G_{m,d}(x,y)}{\partial n_y}(-\Delta_y)^{m-2}u(y)dS_y-$$

$$\int_{\partial\Omega}(-\Delta)G_{m,d}(x,y)\frac{\partial(-\Delta_y)^{m-2}u(y)}{\partial n_y}dS_y+$$

$$\int_{\partial\Omega}\frac{\partial G_{m,d}(x,y)}{\partial n_y}(-\Delta_y)^{m-1}u(y)dS_y-$$

$$\int_{\partial\Omega}G_{m,d}(x,y)\frac{\partial(-\Delta_y)^{m-1}u(y)}{\partial n_y}dS_y=...=$$

$$u(x)+\sum_{j=0}^{m-1}\int_{\partial\Omega}\frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j}G_{m,d}(x,y)(-\Delta_y)^j u(y)dS_y-$$

$$\sum_{j=0}^{m-1}\int_{\partial\Omega}(-\Delta_y)^{m-1-j}G_{m,d}(x,y)\frac{\partial}{\partial n_y}(-\Delta_y)^j u(y)dS_y,\ x\in\Omega, \tag{21}$$

where $\frac{\partial}{\partial n_y}=n_1\frac{\partial}{\partial y_1}+...+n_n\frac{\partial}{\partial y_n}$ is the normal derivative on the boundary and $n_1,...,n_n$ are the components of the unit normal. This implies the identity

$$\sum_{j=0}^{m-1}\int_{\partial\Omega}\frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j}G_{m,d}(x,y)(-\Delta_y)^j u(y)dS_y-$$

$$\sum_{j=0}^{m-1}\int_{\partial\Omega}(-\Delta_y)^{m-1-j}G_{m,d}(x,y)\frac{\partial}{\partial n_y}(-\Delta_y)^j u(y)dS_y=0,\ x\in\Omega. \tag{22}$$

By using the properties of the double and single layer potentials as $x \to \partial\Omega$, from (22) we obtain

$$-\frac{u(x)}{2} + \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y} (-\Delta_y)^{m-1-j} G_{m,d}(x,y)(-\Delta_y)^j u(y) dS_y -$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} (-\Delta_y)^{m-1-j} G_{m,d}(x,y) \frac{\partial}{\partial n_y} (-\Delta_y)^j u(y) dS_y = 0, \ x \in \partial\Omega. \tag{23}$$

Thus, this relation is one of the boundary conditions of (18).

Let us derive the remaining boundary conditions. To this end, we set

$$(-\Delta_x)^{m-i}(-\Delta_x)^i u(x) = f(x), \ i = 0, 1, ..., m-1, \ m = 1, 2, ..., \tag{24}$$

and carry out similar considerations just as above,

$$(-\Delta_x)^i u(x) = \int_\Omega (-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i}(-\Delta_y)^i u(y) dy =$$

$$\int_\Omega (-\Delta_y)(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y) dy +$$

$$\int_{\partial\Omega} \frac{\partial}{\partial n_y} (-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y) dS_y -$$

$$\int_{\partial\Omega} (-\Delta_y)^i G_{m,d}(x,y) \frac{\partial}{\partial n_y} (-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y) dS_y =$$

$$\int_\Omega (-\Delta_y)^2(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y) dy +$$

$$\int_{\partial\Omega} \frac{\partial}{\partial n_y} (-\Delta_y)(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y) dS_y -$$

$$\int_{\partial\Omega} (-\Delta_y)(-\Delta_y)^i G_{m,d}(x,y) \frac{\partial}{\partial n_y} (-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y) dS_y +$$

$$\int_{\partial\Omega} \frac{\partial}{\partial n_y} (-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y) dS_y -$$

$$\int_{\partial\Omega} (-\Delta_y)^i G_{m,d}(x,y) \frac{\partial}{\partial n_y} (-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y) dS_y =$$

$$... = \int_\Omega (-\Delta_y)^{m-i}(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^i u(y) dy +$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y} (-\Delta_y)^{m-i-1-j}(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^j(-\Delta_y)^i u(y) dS_y -$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} (-\Delta_y)^{m-i-1-j}(-\Delta_y)^i G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y) dS_y =$$

$$(-\Delta_x)^i u(x) +$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-1-j}(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^j(-\Delta_y)^i u(y) dS_y -$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} (-\Delta_y)^{m-i-1-j}(-\Delta_y)^i G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y) dS_y,$$

where $(-\Delta_y)^i G_{m,d}(x,y)$ are fundamental solutions of the polyharmonic equation (24); i.e.,

$$(-\Delta_x)^{m-i}(-\Delta_y)^i G_{m,d}(x,y) = \delta(x-y), \quad i = 0, 1, ..., m-1.$$

From the previous relations, we obtain the identities

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j} G_{m,d}(x,y)(-\Delta_y)^{j+i} u(y) dS_y -$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} (-\Delta_y)^{m-1-j} G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^{j+i} u(y) dS_y = 0$$

for any $x \in \Omega, i = 0, 1, ..., m-1$. By using the properties of the double and single layer potentials as $x \to \partial\Omega$, we find that

$$-\frac{1}{2}(-\Delta_x)^i u(x) +$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j} G_{m,d}(x,y)(-\Delta_y)^{j+i} u(y) dS_y -$$

$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} (-\Delta_y)^{m-1-j} G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^{j+i} u(y) dS_y = 0, \tag{25}$$

for $x \in \partial\Omega, \ i = 0, 1, ..., m-1$. These are all boundary conditions of (18). From this classical approach by density and completion arguments (by passing to the limit), one can readily show that formula (25) remains valid for all $u \in H^{2m}(\Omega)$ [2]. Conversely, let us show that if a function $w \in H^{2m}(\Omega)$ satisfies the equation $(-\Delta)^m w = f$ and the boundary conditions (20), then it coincides with the solution (18). Indeed, otherwise the function

$$v = u - w \in H^{2m}(\Omega),$$

where u is (18), satisfies the homogeneous equation

$$(-\Delta)^m v = 0 \tag{26}$$

and the boundary conditions (20), i.e.

$$I_i(v)(x) := -\frac{1}{2}(-\Delta)^i v(x) +$$
$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j} G_{m,d}(x,y)(-\Delta_y)^{j+i} v(y) dS_y -$$
$$\sum_{j=0}^{m-i-1} \int_{\partial\Omega} (-\Delta_y)^{m-1-j} G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^{j+i} v(y) dS_y = 0, \qquad (27)$$

for any $x \in \partial\Omega$ and $i = 0, 1, ..., m-1$. By applying the Green's formula to the function $v \in H^{2m}(\Omega)$ and by following the lines of the above argument, we obtain

$$0 = \int_{\Omega} (-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-i}(-\Delta_y)^i v(y) dy =$$
$$\int_{\Omega} (-\Delta_y)(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-1} v(y) dy +$$
$$\int_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^i G_{m,d}(x,y)(-\Delta_y)^{m-1} v(y) dS_y -$$
$$\int_{\partial\Omega} (-\Delta_y)^i G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1} v(y) dS_y = ... =$$
$$(-\Delta_x)^i v(x) + \sum_{j=0}^{m-j-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1-j} G_{m,d}(x,y)(-\Delta_y)^{j+i} v(y) dS_y -$$
$$\sum_{j=0}^{m-j-1} \int_{\partial\Omega} (-\Delta_y)^{m-1-j} G_{m,d}(x,y) \frac{\partial}{\partial n_y}(-\Delta_y)^{j+i} v(y) dS_y,$$

for any $x \in \Omega$ and $i = 0, 1, ..., m-1$. By passing to the limit as $x \to \partial\Omega$, hence we obtain the relations

$$(-\Delta_x)^i v(x) \mid_{x \in \partial\Omega} = -I_i(v)(x) \mid_{x \in \partial\Omega} = 0, \; i = 0, 1, ..., m-1, \qquad (28)$$

The uniqueness of the solution of the boundary value problem

$$(-\Delta)^m v = 0,$$
$$(-\Delta)^i v \mid_{x \in \partial\Omega} = 0, i = 0, 1, ..., m-1.$$

implies that $v = u - w \equiv 0, \forall x \in \Omega$, i.e. w coincides with (18). Thus (18) is the unique solution of the boundary value problem (17), (20) in Ω. The proof of Theorem 2 is complete.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Hörmander, L.: The Analysis of Linear Partial Differential Operators. I–IV. Springer, Berlin (1985)
2. Hsiao, G.C., Wendland, W.L.: Boundary Integral Equations. Springer (2008)
3. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge, UK (2000)
4. Kac, M.: On some connections between probability theory and differential and integral equations. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Berkeley and Los Angeles, pp. 189–215 (1951)
5. Kac, M.: Integration in Function Spaces and Some of Its Applications. PISA (1980)
6. Kalmenov, T.Sh., Suragan, D.: To spectral problems for the volume potential. Dok. Math. **80**, 646–649 (2009)
7. Kalmenov, T.Sh., Suragan, D.: Boundary conditions for the volume potential for the polyharmonic equation. Differ. Equ. **48**, 604–608 (2012)
8. Ruzhansky, M., Suragan, D.: On Kac's principle of not feeling the boundary for the Kohn Laplacian on the Heisenberg group. Proc. Am. Math. Soc. **144**(2), 709–721 (2016)
9. Ruzhansky, M., Suragan, D.: Schatten's norm for convolution type integral operator. Russ. Math. Surv. **71**, 157–158 (2016)
10. Ruzhansky, M., Suragan, D.: Isoperimetric inequalities for the logarithmic potential operator. J. Math. Anal. Appl. **434**, 1676–1689 (2016)
11. Ruzhansky, M., Suragan, D.: On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries. Bull. Math. Sci. **6**, 325–334 (2016)
12. Ruzhansky, M., Suragan, D.: Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups. Adv. Math. **308**, 483–528 (2017)
13. Saito, N.: Data analysis and representation on a general domain using eigenfunctions of Laplacian. Appl. Comput. Harmon. Anal. **25**, 68–97 (2008)

Estimates for Root Functions of a Singular Second-Order Differential Operator

Leonid V. Kritskov[1,2(✉)]

[1] Lomonosov Moscow State University, 1-52 Leninskiye gory, Moscow 119991, Russia
kritskov@cs.msu.su

[2] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan

Abstract. Estimates in various Lebesgue spaces $L_s(G)$, $1 \leq s \leq \infty$, are obtained for the root functions of an operator which relates to the differential operation $-u''+p(x)u'+q(x)u$, $x \in G = (a, b)$, with complex-valued singular coefficients. Among these estimates there are also the so-called anti-a priori estimates that link the root functions in the same chain. It is supposed that $p(x)$ and $q(x)$ belong locally to the spaces L_2 and W_2^{-1}, respectively, may have singularities at the end-points of G, and $q(x) = q_1(x) + Q'(x)$ while $Q(x), p(x), Q^2(x)w(x), p^2(x)w(x), q_1(x)w(x)$ are integrable on the whole interval G with $w(x) = (x - a)(b - x)$.

Keywords: Second-order differential operator · Singular coefficients · Potential-distribution · Root functions · L_p-estimates · Anti-a priori estimates

1 Introduction and Main Results

Let $G = (a, b)$ be a finite interval and let L be a general second-order differential operation of the form

$$Lu = -u'' + p(x)u' + q(x)u \tag{1}$$

with complex-valued coefficients $p(x), q(x)$.

We consider a singular case, namely, we suppose that

a) the coefficient $q(x)$ in (1) is a distribution and it could be represented as the sum

$$q(x) = q_1(x) + Q'(x), \tag{2}$$

where $q_1(x), Q^2(x)$ are locally integrable on G;

b) the coefficient $p(x)$ belongs to the space L_2 on each compact subset of G;

c) both coefficients $p(x)$ and $q(x)$ may have singularities at the end-points $x = a$ and $x = b$ which match the conditions

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_23

$$p(x),\ Q(x) \in L_1(G), \tag{3}$$
$$q_1(x)w(x),\ p^2(x)w(x),\ Q^2(x)w(x) \in L_1(G), \tag{4}$$

with $w(x) = (x-a)(b-x)$.

For example, the admissible coefficients in (1) may have the bounds $q_1(x) = O\big((x-a)^{\alpha_1} \log^{\beta_1}(x-a)^{-1}\big)$, $p(x), Q(x) = O\big((x-a)^{\alpha_2} \log^{\beta_2}(x-a)^{-1}\big)$ as $x \to a+0$, where $\alpha_1 > -2$, β_1 is arbitrary or $\alpha_1 = -2$, $\beta_1 > -1$, and $\alpha_2 > -1$, β_2 is arbitrary or $\alpha_2 = -1$, $\beta_2 > -1/2$.

In the present paper we obtain the estimates for the L_s-norms of the root functions (i.e. eigen- and associate functions) of the operation (1) which are considered here as solutions of the respective equations with a spectral parameter.

The root function $u_k(x,\lambda)$ *of the order* $k \geq 0$ that corresponds to the operation (1) and *the eigenvalue* $\lambda \in \mathbb{C}$, is defined as an arbitrary nontrivial solution to the equation

$$Lu_k(x,\lambda) = \lambda u_k(x,\lambda) + \operatorname{sgn} k \cdot u_{k-1}(x,\lambda). \tag{5}$$

If $k = 0$ then $u_0(x,\lambda)$ is called *the eigenfunction* of (1), and if $k \geq 1$ then $u_k(x,\lambda)$ is called *the associated function* which relates to the eigenfunction $u_k(x,\lambda)$.

The equation (5) is understood in the following regularized sense (see [20]). The function $u_k(x,\lambda)$ and its quasi-derivative

$$u_k^{[1]}(x,\lambda) = u_k'(x,\lambda) - Q(x)u_k(x,\lambda) \tag{6}$$

are absolutely continuous in G and the operation L acts on a function $u(x)$ according to the rule

$$Lu = -(u^{[1]})' + (p(x) - Q(x))u^{[1]} + (q_1(x) + p(x)Q(x) - Q^2(x))u. \tag{7}$$

If $Q(x) \equiv 0$ in (2), then the above mentioned definition transforms into the conventional definition of the root functions due to V. Il'in [8], namely, $u_k(x,\lambda)$ is the almost everywhere solution to (5) which is absolutely continuous in G together with its first derivative. We note that the absence of any additional conditions (boundary, general non-local ones etc.) on the functions $u_k(x,\lambda)$ allows to span more general systems that relate solely to the differential operation (1) and consist of the chains of functions $u_0(x,\lambda), u_1(x,\lambda), \dots, u_m(x,\lambda)$ that satisfy (5)–(7) for some set of complex numbers λ.

The main result of this paper gives the following assertion.

Theorem 1. *There exists such constant* $C > 0$ *which depends solely on the order* k *of the root function that, for all* $\lambda \in \mathbb{C}$, $1 \leq s, r \leq \infty$, *the estimates*

$$\|u_k(\cdot,\lambda)\|_s \leq C\big(1 + |\operatorname{Im}\sqrt{\lambda}|\big)^{(1/r)-(1/s)} \|u_k(\cdot,\lambda)\|_r, \tag{8}$$
$$\|u_k(\cdot,\lambda)\|_s \leq C\big(1 + |\sqrt{\lambda}|\big)\big(1 + |\operatorname{Im}\sqrt{\lambda}|\big) \|u_{k+1}(\cdot,\lambda)\|_s \tag{9}$$

hold. Here $\|\cdot\|_s$, $\|\cdot\|_r$ *denote the norms in the Lebesgue spaces* $L_s(G)$ *and* $L_r(G)$, *respectively, and* $\sqrt{\lambda}$ *stands for the value of the square root of* $\lambda \in \mathbb{C}$ *for which the inequality* $\operatorname{Re}\sqrt{\lambda} \geq 0$ *holds.*

The estimates (8)–(9) play a key role in the study of spectral properties of differential operators such as the basis property of root functions, the convergence and equiconvergence of the related spectral expansions. In the regular case when the coefficients in (1) belong to the classes: $q(x) \in L_1(G)$, $p(x) \in W_1^1(G)$, these estimates were proved in [9,11,15,21]. Taking $q(x) = p(x) \equiv 0$, it is easy to show that they are sharp with respect to λ. Estimates for the general nth-order differential operation with integrable coefficients were considered in [10,16].

Note that the estimate (9) that links the norms of "adjacent" root functions in a chain, are essential in the case when the system contains infinitely many associate functions. These estimates were introduced by V. Il'in [6] and called the anti-a priori type estimates.

In the self-adjoint case the estimate (8) for the norms with $s = \infty$, $r = 2$ gives the positive solution to the problem of uniform boundedness of normalized eigenfunctions of any operator which relates to (1) (see further [1,3,5,7,23]).

The interest to the spectral analysis of operators that are generated by the differential operations (1) with singular coefficients is motivated by their applications in the quantum theory (see, e.g., [2]), in particular those that describe short-range and point interactions. Approaches introduced by A. Shkalikov and A. Savchuk [19,20] permitted to study operators that correspond to the operation (1) with arbitrary coefficients in the Sobolev classes with negative derivation order. These studies revived further research in this area [17,18,22].

Operators that correspond to the operation (1) with $p(x) = Q(x) \equiv 0$ and locally integrable coefficient $q_1(x)$ that may have non-integrable singularities on the end-points of G satisfying (4) were studied in [4] (in the self-adjoint case) and in [12,13] (in the non-self-adjoint case).

In the present paper, the estimates of Theorem 1 are obtained through integral representations for regularized by (7) solutions to the equation (5). These representations are constructed in Section 2. Section 3 contains the proof of the estimates (8)–(9).

2 Representation for Root Functions

Let us derive an appropriate integral representation for the root functions $u_k(x, \lambda)$ of the operation (1).

We pass from the root functions $u_k(x, \lambda)$ to the functions

$$\psi_k(x,\lambda) = u_k(x,\lambda)\exp\Big(-\int_a^x [p(\xi) - Q(\xi)]\, d\xi\Big) \tag{10}$$

and redefine their quasi-derivatives by the relation

$$\psi_k^{[1]}(x,\lambda) = \psi_k'(x,\lambda) - P(x)\psi_k(x,\lambda), \tag{11}$$

where the coefficient $P(x) = 2Q(x) - p(x)$ belongs to $L_{2,loc}(G)$.

It follows from the condition (3) that $P(x)$ is integrable over G and, therefore, the estimates[1]

$$C_1\|\psi_k(\cdot,\lambda)\|_s \le \|u_k(\cdot,\lambda)\|_s \le C_2\|\psi_k(\cdot,\lambda)\|_s \tag{12}$$

hold.

Moreover, due to the relation

$$\psi_k^{[1]}(x,\lambda) = u_k^{[1]}(x,\lambda)\exp\Big(-\int_a^x [p(\xi)-Q(\xi)]\,d\xi\Big),$$

the quasi-derivatives (11) are also connected with the quasi-derivatives (6) of the functions $u_k(x,\lambda)$ by the estimates (12).

It follows from (12) that it is sufficient to proof the estimates (8) and (9) for the functions $\psi_k(x,\lambda)$.

For $\psi_k(x,\lambda)$, the equation (5) takes the form

$$-\,(\psi_k^{[1]}(x,\lambda))' + q_0(x)\psi_k(x,\lambda) = \lambda\psi_k(x,\lambda) + \operatorname{sgn} k\cdot\psi_{k-1}(x,\lambda), \tag{13}$$

where

$$q_0(x) = q_1(x) + p(x)Q(x) - Q^2(x). \tag{14}$$

The condition (4) provides that

$$q_0(x)w(x) \in L_1(G). \tag{15}$$

By the standard reasoning [14], the equation (13) yields the relations

$$\begin{aligned}\psi_k(x\pm t,\lambda) &= \psi_k(x,\lambda)\cos\mu t \pm \mu^{-1}\psi_k^{[1]}(x,\lambda)\sin\mu t \pm \\ \pm\int_0^t P(x\pm\tau)\psi_k(x\pm\tau,\lambda)&\cos\mu(t-\tau)\,d\tau + \int_0^t q_0(x\pm\tau)\psi_k(x\pm\tau,\lambda)\times \\ \times\,\mu^{-1}\sin\mu(t-\tau)\,d\tau &- \operatorname{sgn} k\int_0^t \psi_{k-1}(x\pm\tau,\lambda)\mu^{-1}\sin\mu(t-\tau)\,d\tau,\end{aligned} \tag{16}$$

where $x\in G$, $0<t<\min(x-a,b-x)$ and, for brevity, we use the notation $\mu=\sqrt{\lambda}$.

Let us consider the functions $\psi_k(x\pm t,\lambda)$ as the solutions of the integral equation (17) with respect to the variable t. For that purpose, we introduce the operators that act on a function of variable t by the rules:

$$\big[C^{\pm}\chi\big](t) = \pm\int_0^t P(x\pm\tau)\chi(\tau)\cos\mu(t-\tau)\,d\tau, \tag{17}$$

$$\big[S^{\pm}\chi\big](t) = \mu^{-1}\int_0^t q_0(x\pm\tau)\chi(\tau)\sin\mu(t-\tau)\,d\tau, \tag{18}$$

$$\big[S_0\chi\big](t) = -\mu^{-1}\int_0^t \chi(\tau)\sin\mu(t-\tau)\,d\tau. \tag{19}$$

[1] Here and in what follows we denote by $C, C_0, C_1, C_2, \ldots$ any positive constants that do not depend on the parameter λ.

Besides we note that, for any given $x \in G$, the extraintegral terms on the right-hand side of (17):

$$\psi_k(x,\lambda)\cos\mu t \pm \mu^{-1}\psi_k^{[1]}(x,\lambda)\sin\mu t \equiv A_k^{\pm}(t) \tag{20}$$

are the eigenfunctions [2] of the simplest differential operation $L_0 u(t) = -u''(t)$.

Taking into account the accepted notation, we rewrite the relation (17) in the form

$$\psi_k(x\pm t,\lambda) = A_k^{\pm}(t) + T^{\pm}\psi_k(x\pm t,\lambda) + \operatorname{sgn} k\cdot S_0\psi_{k-1}(x\pm t,\lambda), \tag{21}$$

where $T^{\pm} = C^{\pm} + S^{\pm}$.

Thereby, the formal solution to the equation (22) could be written via the Neumann series:

$$\psi_k(x\pm t,\lambda) = (E-T^{\pm})^{-1}A_k^{\pm}(t) + \operatorname{sgn} k\cdot(E-T^{\pm})^{-1}S_0\psi_{k-1}(x\pm t,\lambda). \tag{22}$$

Lemma 1. *The integral operator $T^{\pm}$ satisfies the estimate*

$$\left|\left[T^{\pm}\chi\right](t)\right| \le C\omega(t)\sup_{0\le\tau\le t}|\chi(\tau)\cosh(\operatorname{Im}\mu(t-\tau))|, \tag{23}$$

where $\chi(t)$ is an arbitrary bounded function and $\omega(t)$ is a non-negative non-decreasing function that vanishes as $t\to 0+0$.

Proof. The estimate (24) directly follows from the relations (15), (16), (18) and (19) if we set

$$\omega(t) = \sup_{K\subset G:\ \operatorname{mes} K\le t}\left\{\int_K |P(\xi)|\,d\xi + \int_K |q_0(\xi)|(\xi-a)(b-\xi)\,d\xi\right\}$$

and use the inequalities $|\cos z| \le \cosh(\operatorname{Im} z)$, $|\sin z| \le \cosh(\operatorname{Im} z)$, $|z^{-1}\sin z| \le \cosh(\operatorname{Im} z)$, $z\in\mathbb{C}$, and the inequality $\cosh y_1\cosh y_2 \le \cosh(y_1+y_2)$, $y_1, y_2 \ge 0$.

It is the immediate consequence of the estimate (24) that the equality (23) is correct for all rather small values of t.

Lemma 2. *There exists such $R_0 > 0$ that the functions (10) satisfy the relation (23) for all $t\in(0;R_0]$ and $x\in[a+t,b-t]$. Moreover, the functions $\psi_k(x,\lambda)$ are absolutely continuous on the closed interval $\overline{G}$ and continuously depend on the complex parameter λ.*

If we solve the recurrent relation (23) then, after extracting the main terms in each Neumann series, we obtain the equality

$$\begin{aligned}\psi_k(x\pm t,\lambda) &= A_k^{\pm}(t) + S_0A_{k-1}^{\pm}(t) + \ldots + (S_0)^kA_0^{\pm}(t) + \\ &+\sum_{j=0}^{k}\Big\{\psi_{k-j}(x,\lambda)F_j^{\pm}(t,x,\lambda) \pm \mu^{-1}\psi_{k-j}^{[1]}(x,\lambda)\Phi_j^{\pm}(t,x,\lambda)\Big\},\end{aligned} \tag{24}$$

[2] For brevity, we omit the arguments x and μ in the notation $A_k^{\pm}(t)$.

where

$$F_0^{\pm}(t,x,\lambda) = T^{\pm}(E-T^{\pm})^{-1}\cos\mu t, \quad \Phi_0^{\pm}(t,x,\lambda) = T^{\pm}(E-T^{\pm})^{-1}\sin\mu t, \quad (25)$$

$$F_j^{\pm}(t,x,\lambda) = ((E-T^{\pm})^{-1}S_0)^j(E-T^{\pm})^{-1}\cos\mu t - (S_0)^j\cos\mu t,$$

$$\Phi_j^{\pm}(t,x,\lambda) = ((E-T^{\pm})^{-1}S_0)^j(E-T^{\pm})^{-1}\sin\mu t - (S_0)^j\sin\mu t. \quad (26)$$

Let us introduce the functions

$$\Psi_k^{\pm}(t) = A_k^{\pm}(t) + S_0A_{k-1}^{\pm}(t) + \ldots + (S_0)^kA_0^{\pm}(t) \quad (27)$$

in the right-hand side of (25).

As the function $\varphi(t) = S_0f(t)$ is a solution to the equation $-\varphi''(t) = \mu^2\varphi(t) + f(t)$ while each function (21) is the eigenfunction of the operation L_0, the sequence $\Psi_0^{\pm}(t), \Psi_1^{\pm}(t), \ldots, \Psi_k^{\pm}(t)$ forms the chain of root function for the operation L_0. The estimates (8)–(9) of Theorem 1 for these functions are well-known — this fact will be use hereinafter.

Thereby, the relations (20), (24)–(28) could be summarized in the following assertion.

Lemma 3. *There exists the number $R_0 > 0$ such that, for all $t \in (0, R_0]$ and $x \in [a+t, b-t]$, the representation*

$$\psi_k(x \pm t, \lambda) = \Psi_k^{\pm}(t) + \sum_{j=0}^{k} \psi_{k-j}(x,\lambda)F_j^{\pm}(t,x,\lambda) \pm \mu^{-1}\psi_{k-j}^{[1]}(x,\lambda)\Phi_j^{\pm}(t,x,\lambda) \quad (28)$$

holds, where $\psi_k(x,\lambda)$ and $\Psi_k(t)$ are defined by the relations (10) and (28) respectively, and the coefficients on its right-hand side introduced by (26), (27) satisfy the estimates

$$|F_j(t,x,\lambda)| \le C\omega(t)\left(\frac{2\min\big(t,(1+|\mathrm{Im}\,\mu|)^{-1}\big)}{1+|\mu|}\right)^j\cosh(\mathrm{Im}\,\mu t), \quad (29)$$

$$|\Phi_j(t,x,\lambda)| \le C\omega(t)\min(|\mu|t,1)\left(\frac{2\min\big(t,(1+|\mathrm{Im}\,\mu|)^{-1}\big)}{1+|\mu|}\right)^j\cosh(\mathrm{Im}\,\mu t), \quad (30)$$

here $\omega(t)$ is a non-negative non-decreasing function that vanishes as $t \to 0+0$.

It follows from the estimate (24) that it is sufficient to choose $R_0 > 0$ matching the inequality $\omega(R_0) \le 1/2$.

3 Proof of the Estimates for the Root Functions

By virtue of Lemma 2, it is sufficient to prove the estimates (8), (9) for rather large values of $|\mu|$. In fact, each norm $\|u_k(\cdot,\lambda)\|_s$ is non-zero for all $k \ge 0$ and $\lambda \in \mathbb{C}$ and continuous with respect to λ. Therefore, if $|\mu| \le \mu_0$ then the inequalities

$$0 < C_1 \le \|u_k(\cdot,\lambda)\|_s \le C_2 < \infty$$

hold.

Further we suppose that the inequality $|\mu| \ge \mu_0$ is satisfied with some positive number μ_0. For any $R \in (0, R_0]$, we will obtain the estimates of Theorem 1 over each closed interval $K = K(x) \equiv [x-R, x+R]$, where $x \in [a+R, b-R]$, i.e., taking into account (12), we will prove that, for any $k \ge 0$, the estimates

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\,(1+|\mathrm{Im}\,\mu|)^{(1/r)-(1/s)}\|\psi_k(\cdot,\lambda)\|_{r,K}, \tag{31}$$
$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\,|\mu|(1+|\mathrm{Im}\,\mu|)\|\psi_{k+1}(\cdot,\lambda)\|_{s,K} \tag{32}$$

hold, where $\|\cdot\|_{s,K}$ denotes the norm in the space $L_s(x-R, x+R)$.

The estimates for norms on the whole interval G apparently follow from (32), (33) while the corresponding constants C in them will depend on R (the choice of R is clarified further).

First of all we refine the dependence on R of the constants in the estimates for the functions $\Psi_k^{\pm}(t)$ in the spaces $L_s(0,R)$.

Lemma 4. *For any $k \ge 0$ and all $1 \le s, r \le \infty$, the following estimates hold uniformly with respect to $R \in (0, R_0]$:*

a) if $|\mathrm{Im}\,\mu R| \ge 1$, then

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \le C|\mathrm{Im}\,\mu|^{(1/r)-(1/s)}\|\Psi_k^{\pm}(t)\|_{L_r(0,R)}, \tag{33}$$
$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \le C|\mathrm{Im}\,\mu|\cdot|\mu|\,\|\Psi_{k+1}^{\pm}(t)\|_{L_s(0,R)}, \tag{34}$$

b) if $|\mathrm{Im}\,\mu| \le \nu_0$, then

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \le CR^{(1/s)-(1/r)}\|\Psi_k^{\pm}(t)\|_{L_r(0,R)}, \tag{35}$$

c) if $|\mathrm{Im}\,\mu| \le \nu_0$ and $|\mu R| \ge 1$, then

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \le CR^{-1}|\mu|\,\|\Psi_{k+1}^{\pm}(t)\|_{L_s(0,R)}, \tag{36}$$

d) if $|\mu| \le \mu_0$, then

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \le CR^{-2}\|\Psi_{k+1}^{\pm}(t)\|_{L_s(0,R)}. \tag{37}$$

Proof. Let us make the substitution $t = \xi R$ in the equations for the functions $\Psi_k^{\pm}(t)$. Then $0 < \xi < 1$ and the equalities

$$-\partial_\xi^2\Psi_0^{\pm}(\xi R) = (\mu R)^2\Psi_0^{\pm}(\xi R), \quad -\partial_\xi^2\Psi_k^{\pm}(\xi R) = (\mu R)^2\Psi_k^{\pm}(\xi R) + R^2\Psi_{k-1}^{\pm}(\xi R)$$

are satisfied.

Hence, the estimates

$$\|R^2\Psi_k^{\pm}(\xi R)\|_{L_s(0,1)} \le C(1+|\mathrm{Im}\,\mu R|)(1+|\mu R|)\|\Psi_{k+1}^{\pm}(\xi R)\|_{L_s(0,1)}, \tag{38}$$
$$\|\Psi_k^{\pm}(\xi R)\|_{L_s(0,1)} \le C(1+|\mathrm{Im}\,\mu R|)^{(1/r)-(1/s)}\|\Psi_k^{\pm}(\xi R)\|_{L_r(0,1)} \tag{39}$$

hold with constants C that do not depend on R.

Let us take into account that

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} = R^{1/s}\|\Psi_k^{\pm}(\xi R)\|_{L_s(0,1)}. \tag{40}$$

Thus, the estimate (40) yields directly the estimate (34) if $|\mathrm{Im}\,\mu R| \geq 1$, and the estimate (36) if $|\mathrm{Im}\,\mu| \leq \nu_0$, since in the latter case $|\mathrm{Im}\,\mu R| \leq \nu_0 R_0$ holds.

If $|\mathrm{Im}\,\mu R| \geq 1$ then $|\mu R| \geq 1$, and therefore, the estimate (35) follows from (39) and (41).

The estimate (37) also reformulates (39) since $|\mathrm{Im}\,\mu R| \leq \nu_0 R_0$, and, to obtain (38), it suffices to note that, in this case, $|\mathrm{Im}\,\mu| \leq |\mu| \leq \mu_0$ holds.

The proof of the estimates (32) and (33) is accomplished separately in the following cases: 1) when μ satisfies the condition $|\mathrm{Im}\,\mu| \geq \nu_0 > 0$ (the value of ν_0 is chosen further), and 2) when $|\mathrm{Im}\,\mu| \leq \nu_0$ holds.

The case $|\mathrm{Im}\,\mu| \geq \nu_0$. Suppose the inequality $|\mathrm{Im}\,\mu R| \geq 1$ holds. Using the representation (29) and the estimates (30) and (31) we calculate the norms of the functions $\psi_k(x+t,\lambda)$ and $\psi_k(x-t,\lambda)$ in the space $L_s(0<t<R)$:

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \leq \|\Psi_k^{+}(t)\|_{L_s(0,R)} + \|\Psi_k^{-}(t)\|_{L_s(0,R)} + C\omega(R)|\mathrm{Im}\,\mu|^{-1/s} \times \\ \times \cosh\left(\mathrm{Im}\,\mu R\right)\big) \sum_{j=0}^{k} \big(|\mu|\,|\mathrm{Im}\,\mu|\big)^{-j} \Big\{ |\psi_{k-j}(x,\lambda)| + \frac{|\psi_{k-j}^{[1]}(x,\lambda)|}{|\mu|} \Big\}. \tag{41}$$

It is also clear that one can swap the terms $\|\psi_k(\cdot,\lambda)\|_{s,K}$ and $\|\Psi_k^{+}(t)\|_{L_s(0,R)} + \|\Psi_k^{-}(t)\|_{L_s(0,R)}$ of this estimate.

Now we apply the representation (29) to the half-sum $\frac{1}{2}\big(\psi_k(x+t,\lambda)+\psi_k(x-t,\lambda)\big)$. Since[3]

$$\frac{1}{2}\big(\Psi_k^{+}(t) + \Psi_k^{-}(t)\big) = \psi_k(x,\lambda)\cos\mu t + \sum_{j=1}^{k} \psi_{k-j}(x,\lambda)(S_0)^j \cos\mu t, \tag{42}$$

we obtain the relation

$$\psi_k(x,\lambda)\cos\mu t = \frac{1}{2}\big(\psi_k(x+t,\lambda)+\psi_k(x-t,\lambda)\big) - \\ -\sum_{j=1}^{k} \psi_{k-j}(x,\lambda)(S_0)^j \cos\mu t - \sum_{j=0}^{k} \Big\{ \psi_{k-j}(x,\lambda) \frac{F_j^{+}(t,x,\lambda) + F_j^{-}(t,x,\lambda)}{2} + \\ + \psi_{k-j}^{[1]}(x,\lambda) \frac{\Phi_j^{+}(t,x,\lambda) - \Phi_j^{-}(t,x,\lambda)}{2\mu} \Big\}. \tag{43}$$

Since the condition $|\mathrm{Im}\,\mu R| \geq 1$ provides the inequality

$$\|\cos\mu t\|_{L_s(0,R)} \geq C_0 |\mathrm{Im}\,\mu|^{-1/s} \cosh\left(\mathrm{Im}\,\mu R\right), \tag{44}$$

[3] In the case when $k=0$ the sum $\sum_{j=1}^{k}$ should be omitted.

and the estimates (30), (31) hold, the definition (20) of the operator S_0 and the relation (43) yield the estimate

$$C_0|\mathrm{Im}\,\mu|^{-1/s}\cosh(\mathrm{Im}\,\mu R)|\psi_k(x,\lambda)| \le \|\psi_k(\cdot,\lambda)\|_{s,K} + \\ + C\omega(R)|\mathrm{Im}\,\mu|^{-1/s}\cosh(\mathrm{Im}\,\mu R)\Big\{|\psi_k(x,\lambda)| + \frac{|\psi_k^{[1]}(x,\lambda)|}{|\mu|}\Big\} + \\ + C|\mathrm{Im}\,\mu|^{-1/s}\cosh(\mathrm{Im}\,\mu R)\sum_{j=1}^{k}\big(|\mu|\,|\mathrm{Im}\,\mu|\big)^{-j}\Big\{|\psi_{k-j}(x,\lambda)| + \frac{|\psi_{k-j}^{[1]}(x,\lambda)|}{|\mu|}\Big\}. \quad (45)$$

If we repeat the above reasoning for the half-difference $\frac{1}{2}\big(\psi_k(x+t,\lambda)-\psi_k(x-t,\lambda)\big)$ instead of the half-sum and take into account the equality

$$\frac{1}{2}\big(\Psi_k^+(t)-\Psi_k^-(t)\big) = \mu^{-1}\psi_k^{[1]}(x,\lambda)\sin\mu t + \sum_{j=1}^{k}\mu^{-1}\psi_{k-j}^{[1]}(x,\lambda)(S_0)^j\sin\mu t,$$

instead of (42), we obtain the relation for $\mu^{-1}\psi_k^{[1]}(x,\lambda)\sin\mu t$ which is similar to (43). As the norm $\|\sin\mu t\|_{L_s(0,R)}$ also satisfies the inequality (44), we will actually get the estimate (45), but with $C_0|\mathrm{Im}\,\mu|^{-1/s}\cosh(\mathrm{Im}\,\mu R)|\mu|^{-1}|\psi_k^{[1]}(x,\lambda)|$ on its left-hand side.

Summing these two estimates and choosing R to match the inequality $4C\omega(R)\le C_0$, we finally obtain the estimate

$$|\mathrm{Im}\,\mu|^{-1/s}\cosh(\mathrm{Im}\,\mu R)\Big\{|\psi_k(x,\lambda)| + \frac{|\psi_k^{[1]}(x,\lambda)|}{|\mu|}\Big\} \le C\|\psi_k(\cdot,\lambda)\|_{s,K} + \\ + C|\mathrm{Im}\,\mu|^{-1/s}\cosh(\mathrm{Im}\,\mu R)\sum_{j=1}^{k}\big(|\mu|\,|\mathrm{Im}\,\mu|\big)^{-j}\Big\{|\psi_{k-j}(x,\lambda)| + \frac{|\psi_{k-j}^{[1]}(x,\lambda)|}{|\mu|}\Big\}. \quad (46)$$

Applying the estimate (46) $k+1$ times in the right-hand side of (42), we primarily get the estimate

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le \|\Psi_k^+(t)\|_{L_s(0,R)} + \|\Psi_k^-(t)\|_{L_s(0,R)} + \\ + C\omega(R)\sum_{j=0}^{k}\big(|\mu|\,|\mathrm{Im}\,\mu|\big)^{-j}\|\psi_{k-j}(\cdot,\lambda)\|_{s,K}, \quad (47)$$

which, for rather small R, transforms into the estimate

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\big\{\|\Psi_k^+(t)\|_{L_s(0,R)} + \|\Psi_k^-(t)\|_{L_s(0,R)}\big\} + \\ + C\omega(R)\sum_{j=1}^{k}\big(|\mu|\,|\mathrm{Im}\,\mu|\big)^{-j}\|\psi_{k-j}(\cdot,\lambda)\|_{s,K}.$$

Hence we get

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\big\{\|\Psi_k^+(t)\|_{L_s(0,R)} + \|\Psi_k^-(t)\|_{L_s(0,R)}\big\} + \\ + C\omega(R)\sum_{j=1}^{k}\big(|\mu|\,|\mathrm{Im}\,\mu|\big)^{-j}\big\{\|\Psi_{k-j}^+(t)\|_{L_s(0,R)} + \|\Psi_{k-j}^-(t)\|_{L_s(0,R)}\big\}. \quad (48)$$

The estimate

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \le \|\psi_k(\cdot,\lambda)\|_{s,K} + C\omega(R)\sum_{j=1}^{k}\bigl(|\mu|\,|\mathrm{Im}\,\mu|\bigr)^{-j}\|\psi_{k-j}(\cdot,\lambda)\|_{s,K} \quad (49)$$

could be derived similar to the estimate (47).

Now we turn directly to the anti-a priori estimate (33).

We take into account the anti-a priori estimate (35) for the functions $\Psi_k^{\pm}(t)$ and rewrite the estimate (48) in the form

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\bigl\{\|\Psi_k^{+}(t)\|_{L_s(0,R)} + \|\Psi_k^{-}(t)\|_{L_s(0,R)}\bigr\}. \quad (50)$$

Then, by virtue of the estimates (35) and (50) with $k=0$ and the estimate (49) with $k=1$, we obtain

$$\begin{aligned}\|\psi_0(\cdot,\lambda)\|_{s,K} &\le C\bigl\{\|\Psi_0^{+}(t)\|_{L_s(0,R)} + \|\Psi_0^{-}(t)\|_{L_s(0,R)}\bigr\} \le \\ &\le C|\mathrm{Im}\,\mu|\,|\mu|\,\bigl\{\|\Psi_1^{+}(t)\|_{L_s(0,R)} + \|\Psi_1^{-}(t)\|_{L_s(0,R)}\bigr\} \le \\ &\le C|\mathrm{Im}\,\mu|\,|\mu|\,\|\psi_1(\cdot,\lambda)\|_{s,K} + C\omega(R)\|\psi_0(\cdot,\lambda)\|_{s,K}.\end{aligned}$$

This, for rather small R, implies the anti-a priori estimate (33) with $k=0$.

Now, by virtue of the estimates (35) and (50) with $k=1$, the estimate (49) with $k=2$, and already justified anti-a priori estimate (33) with $k=0$, we obtain

$$\begin{aligned}\|\psi_1(\cdot,\lambda)\|_{s,K} &\le C\bigl\{\|\Psi_1^{+}(t)\|_{L_s(0,R)} + \|\Psi_1^{-}(t)\|_{L_s(0,R)}\bigr\} \le \\ &\le C|\mathrm{Im}\,\mu|\,|\mu|\,\bigl\{\|\Psi_2^{+}(t)\|_{L_s(0,R)} + \|\Psi_2^{-}(t)\|_{L_s(0,R)}\bigr\} \le \\ &\le C|\mathrm{Im}\,\mu|\,|\mu|\,\|\psi_2(\cdot,\lambda)\|_{s,K} + C\omega(R)\|\psi_1(\cdot,\lambda)\|_{s,K},\end{aligned}$$

whence, for rather small R, we again derive the anti-a priori estimate (33), but with $k=1$.

Repeating the reasoning, one can obtain the estimate (33) for all $k \ge 2$.

Let us proceed with proving the estimates for the norms in (32).

For that purpose, it is sufficient to apply the estimates for the norms (34), the estimate (50), and also the estimate

$$\|\Psi_k^{\pm}(t)\|_{L_r(0,R)} \le C\|\psi_k(\cdot,\lambda)\|_{r,K}$$

which follows from (49) and the anti-a priori estimate (33).

Summing up the above reasoning, we note that the constant ν_0 for this case equals R^{-1} where R matches all the mentioned restrictions.

The case $|\mathrm{Im}\,\mu| \le \nu_0$. We once again use the representation (29) and calculate the norms of the functions $\psi_k(x+t,\lambda)$ and $\psi_k(x-t,\lambda)$ in the space $L_s(0<t<R)$. Applying the estimates (30), (31) and the boundedness of the factors $\cosh(\mathrm{Im}\,\mu R)$, we obtain

$$\begin{aligned}\|\psi_k(\cdot,\lambda)\|_{s,K} &\le \|\Psi_k^{+}(t)\|_{L_s(0,R)} + \|\Psi_k^{-}(t)\|_{L_s(0,R)} + \\ &+ C\omega(R)R^{1/s}\sum_{j=0}^{k}\bigl(2|\mu|^{-1}R\bigr)^{j}\Bigl\{|\psi_{k-j}(x,\lambda)| + \frac{|\psi_{k-j}^{[1]}(x,\lambda)|}{|\mu|}\Bigr\}. \quad (51)\end{aligned}$$

Similarly one can also swap the norm $\|\psi_k(\cdot,\lambda)\|_{s,K}$ and the sum of norms $\|\Psi_k^+(t)\|_{L_s(0,R)} + \|\Psi_k^-(t)\|_{L_s(0,R)}$ in this estimate.

Now we take the norms in $L_s(0<t<R)$ of the both sides of the relation (43). As the function $\cos\mu t$ is the eigenfunction of the operation L_0, it follows from (36) that it satisfies the estimate

$$\|\cos\mu t\|_{L_s(0,R)} \geq C_0 R^{1/s}\|\cos\mu t\|_{L_\infty(0,R)} \geq C_0 R^{1/s}\|\cos\operatorname{Re}\mu t\|_{L_\infty(0,R)}.$$

The latter norm on the right-hand side equals 1 if the condition

$$|\mu R| \geq \mu_1 \geq 1 \tag{52}$$

is satisfied and μ_1 is chosen to provide $\operatorname{Re}\mu R \geq \frac{\pi}{2}$. As a result, we obtain the estimate

$$C_0 R^{1/s}|\psi_k(x,\lambda)| \leq \|\psi_k(\cdot,\lambda)\|_{s,K} + C\omega(R)R^{1/s}\Big\{|\psi_k(x,\lambda)| + \frac{|\psi_k^{[1]}(x,\lambda)|}{|\mu|}\Big\} + \\ + CR^{1/s}\sum_{j=1}^{k}\big(|\mu|^{-1}R\big)^j\Big\{|\psi_{k-j}(x,\lambda)| + \frac{|\psi_{k-j}^{[1]}(x,\lambda)|}{|\mu|}\Big\}. \tag{53}$$

Reproducing the same reasoning for the half-difference $\frac{1}{2}\big(\psi_k(x+t,\lambda)-\psi_k(x-t,\lambda)\big)$, we obtain the estimate (53) for $R^{1/s}|\mu|^{-1}|\psi_k^{[1]}(x,\lambda)|$. Combining these two estimates together we conclude that the relation

$$R^{1/s}\Big\{|\psi_k(x,\lambda)| + \frac{|\psi_k^{[1]}(x,\lambda)|}{|\mu|}\Big\} \leq C\|\psi_k(\cdot,\lambda)\|_{s,K} + \\ + CR^{1/s}\sum_{j=1}^{k}\big(|\mu|^{-1}R\big)^j\Big\{|\psi_{k-j}(x,\lambda)| + \frac{|\psi_{k-j}^{[1]}(x,\lambda)|}{|\mu|}\Big\} \tag{54}$$

holds instead of (53).

Applying successively the estimate (54) in the right-hand side of (51) we obtain the estimate

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \leq C\big\{\|\Psi_k^+(t)\|_{L_s(0,R)} + \|\Psi_k^-(t)\|_{L_s(0,R)}\big\} + \\ + C\omega(R)\sum_{j=1}^{k}\big(|\mu|^{-1}R\big)^j\big\{\|\Psi_{k-j}^+(t)\|_{L_s(0,R)} + \|\Psi_{k-j}^-(t)\|_{L_s(0,R)}\big\} \tag{55}$$

which replaces (48), and the estimate

$$\|\Psi_k^{\pm}(t)\|_{L_s(0,R)} \leq C\|\psi_k(\cdot,\lambda)\|_{s,K} + C\omega(R)\sum_{j=1}^{k}\big(|\mu|^{-1}R\big)^j\|\psi_{k-j}(\cdot,\lambda)\|_{s,K} \tag{56}$$

which replaces (49).

Since under the conditions $|\operatorname{Im}\mu| \le \nu_0$ and (52) the anti-a priori estimate (37) holds, one needs now to repeat the reasoning of the first case and obtain the anti-a priori estimate (33) in the form

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\,|\mu| R^{-1} \|\psi_{k+1}(\cdot,\lambda)\|_{s,K}. \tag{57}$$

The proof of the estimate (32) for the norms in the form

$$\|\psi_k(\cdot,\lambda)\|_{s,K} \le C\,R^{(1/s)-(1/r)} \|\psi_k(\cdot,\lambda)\|_{r,K}$$

now comes clear by virtue of the estimates (36), (55)–(56).

Summing up the second case, we note that here μ should satisfy the condition $|\mu| \ge \mu_0$, where $\mu_0 = \mu_1 R^{-1}$ and μ_1 is the constant in (52) while rather small value of R matches all the mentioned restrictions.

Theorem 1 is completely proved.

Let us conclude with one more estimate that follows from the estimates (8), (9), (46) and (54).

Theorem 2. *For any* $R \in (0, R_0]$ *and all* $k \ge 0$, *there exists a constant* $C_k = C_k(R)$ *such that the estimate*

$$\cosh(\operatorname{Im}\mu R)\Big\{|u_k(x,\lambda)| + \frac{|u_k^{[1]}(x,\lambda)|}{|\mu|}\Big\} \le C_k(R)(1+|\operatorname{Im}\mu|)^{1/s}\|u_k(\cdot,\lambda)\|_s$$

holds for all $|\mu R| \ge \mu_1 \ge 1$.

Acknowledgements. This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 0825/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Aigunov, G.A.: On the boundedness problem for the set of orthonormal eigenfunctions for a class of Sturm-Liouville operators with a weight function of bounded variation on a finite interval. Math. Notes. **60**(3), 321–323 (1996)
2. Albeverio, S., Gesztesy, F., Høegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics. AMS Chelsea Publ. Providence, Rhode Island (2004)
3. Aptekarev, A.I., Denisov, S.A., Tulyakov, D.N.: On a problem by Steklov. J. Amer. Math. Soc. **29**, 1117–1165 (2016)
4. Atkinson, F.V., Fulton, C.T.: Asymptotics of Sturm-Liouville eigenvalues for problems on a finite interval with one limit-circle singularity. I. Proc. Royal Soc. Edinburgh **99A**, 51–70 (1984)
5. Gekhtman, M.M., Zagirov, YuM, Yakubov, V.: Ya.: Asymptotic behavior of eigenfunctions of the Sturm-Liouville spectral problem. Funct. Anal. Appl. **17**(3), 221–223 (1983)
6. Il'in, V.A.: Necessary and sufficient conditions for the functions used in spectral expansions to form a basis, and for the equiconvergence of spectral expansions and trigonometric series. I. Differ. Equ. **16**(5), 479–496 (1980)

7. Il'in, V.A., Joó, I.: Uniform estimate of eigenfunctions and upper estimate of the number of eigenvalues of Sturm-Liouville operator with potential from the class L^p. Differ. Equ. **15**(7), 1164–1174 (1979)
8. Il'in, V.A., Kritskov, L.V.: Properties of spectral expansions corresponding to non-self-adjoint differential operators. J. Math. Sci. NY. **116**(5), 3489–3550 (2003)
9. Joó, I.: Upper estimates for the eigenfunctions of the Schrödinger operator. Acta Sci. Math. **44**(1), 87–93 (1982)
10. Kerimov, N.B.: Some properties of the eigenfunctions and associated functions of ordinary differential operators. Sov. Math. Dokl. **34**, 573–575 (1987)
11. Komornik, V.: Lower estimates for the eigenfunctions of the Schrödinger operator. Acta Sci. Math. **44**(1), 95–98 (1982)
12. Kritskov, L.V.: Uniform bound of the order of associated functions and the distribution of eigenvalues of the one-dimentional Schrödinger operator. Differ. Equ. **25**(7), 784–791 (1989)
13. Kritskov, L.V.: On necessary conditions for the basis property in $L_p(G)$ of a system of root functions of the one-dimensional Schrödinger operator. Sov. Math. Dokl. **41**(2), 374–377 (1990)
14. Kritskov, L.V.: Uniform, on the entire axis, convergence of the spectral expansion for Schrödinger operator with a potential-distribution. Differ. Equ. **53**(2), 180–191 (2017)
15. Lomov, I.S.: Properties of eigenfunctions and associated functions of a Sturm-Liouville operator. Differ. Equ. **18**(1), 1206–1213 (1983)
16. Lomov, I.S.: Estimates of eigen- and adjoint functions of ordinary differential operators. Differ. Equ. **21**(5), 903–906 (1985)
17. Mirzoev, K.A., Shkalikov, A.A.: Differential operators of even order with distribution coefficients. Math. Notes. **99**(5), 779–784 (2016)
18. Neyman-zade, M.I., Savchuk, A.M.: Schrödinger operators with singular potentials. Proc. Steklov Inst. Math. **236**, 250–259 (2002)
19. Savchuk, A.M.: On the eigenvalues and eigenfunctions of the Sturm-Liouville operator with a singular potential. Math. Notes. **69**(1), 245–252 (2001)
20. Savchuk, A.M., Shkalikov, A.A.: Sturm-Liouville operators with singular potentials. Math. Notes. **66**(6), 741–753 (1999)
21. Tikhomirov, V.V.: Sharp estimates for regular solutions of the one-dimensional Schrödinger equation with a spectral parameter. Sov. Math. Dokl. **28**, 722–725 (1983)
22. Vladykina, V.E., Shkalikov, A.A.: Asymptotics of the solutions of the Sturm-Liouville equation with singular coefficients. Math. Notes. **98**(6), 891–899 (2015)
23. Yakubov, V.Y.: Boundedness of normalized eigenfunctions of the Sturm–Liouville problem under minimum restrictions on the smoothness of the coefficients. Differ. Equ. **30**(8), 1361–1364 (1994)

A Sturm-Liouville Operator with a Negative Parameter and Its Applications to the Study of Differential Properties of Solutions for a Class of Hyperbolic Type Equations

Mussakan B. Muratbekov[1(✉)], Madi M. Muratbekov[2], and Asijat N. Dadaeva[3]

[1] Taraz State Pedagogical Institute, 62 Tole bi str., Taraz 080000, Kazakhstan
musahan_m@mail.ru

[2] Kazakh University of Economics, Finance and International Trade, 7 Zhubanov str., Astana 010005, Kazakhstan
madi.muratbekov@yandex.kz

[3] K.I.Satpayev Kazakh National Research Technical University, 22a Satpaev str., Almaty 050013, Kazakhstan
dady1262@mail.ru

Abstract. In this work a unique solvability of a class of hyperbolic type partial differential equations with unbounded coefficients is proved in $\mathbb{R}^2$. The estimates of the weight norms of the solution u and its partial derivatives u_x and u_y are derived.

Keywords: Sturm-Liouville operator · Hyperbolic type equations · Existence of a solution · Unique solvability · Estimation of the norm · Forces of a friction · Behavior of the coefficients · Unbounded domain · Fast growing coefficients

1 Introduction

Consider the equation

$$L_\lambda u = u_{xx} - u_{yy} + a(y)u_x + c(y)u + \lambda u = f(x, y), (x, y) \in R^2, \tag{1}$$

where $f \in L_2(R^2)$, $\lambda \geq 0$. Assume that the following conditions hold:

i) $a(y), c(y)$ are the continuous functions: $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$, $y \in R$.

In case of a bounded domain an extensive literature is devoted to the questions of existence, uniqueness and smoothness of solutions of boundary value problems for the hyperbolic type equation (1) (see [1]-[3]). The solvability of

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_24

hyperbolic type equation, in general, depends on the behavior of the coefficients a and c. For example (see [4], p. 108), the solution of the steady state problem

$$u_{tt} = u_{yy} - \alpha u_t \ (0 < y < l, t \in R),\ u(0,t) = \mu_1(t), u(l,t) = \mu_2(t),$$

where the term αu_t, $\alpha > 0$, on the right-hand side of the equation corresponds to the friction which is proportional to speed, may not be unique. Moreover it does not always exist when $\alpha = 0$. Minor terms and the right-hand of the equation (1) characterize forces of a friction which are inherent in any real physical system. Hence, studying the equation (1) is of practical interest.

In the present work we show that the condition $i)$ provides unique solvability and a uniform estimates for the solution in weighted L_2- norms and its first derivatives. These questions have been investigated only in the cases of the elliptic and pseudo-differential equations in [5]-[10]. The problem with periodic conditions with respect to the variable x and in the domain $\Omega = \{(x,y) : -\pi < x < \pi, -\infty < y < \infty\}$ was studied in [11]-[12] for the equation (1). Unlike the case considered in [11]-[12] the spectrum of the differential operator corresponding to (1) is continuous. Generally, in case of unbounded domains with the fast growing coefficients the theory of the differential equations of hyperbolic and mixed type has a rather short history.

By a solution of equation (1) we mean a function $u \in L_2$ for which there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R^2)$ such that $\|u_n - u\|_2 \to 0$, $\|L_\lambda u_n - f\|_2 \to 0$ as $n \to \infty$ ($\|\cdot\|_2$ is the L_2- norm).

The main results of this work are Theorems 1 and 2.

Theorem 1. *Let the condition i) be fulfilled. Then there exists a unique solution u of the equation (1).*

Theorem 2. *Let the condition i) be fulfilled. Then the solution u of the equation (1) satisfies the estimate $\|u_x\|_2 + \|u_y\|_2 + \|c(\cdot)u\|_2 \le C\|f\|_2$, where $C > 0$ is a constant.*

In what follows $c_0, c_1, c_2, ...$ are positive constants, and $\langle \cdot, \cdot \rangle_D$ is a scalar product in $L_2(D)$.

2 Preliminaries

Denote by L_λ the closure in L_2- norm of the differential operator $l_\lambda u = u_{xx} - u_{yy} + a(y)u_x + c(y)u + \lambda u$ defined on the set $C_0^{\infty}(R^2)$. Evidently, l_λ is a closable operator. In what follows in Lemmas 1-10 we will assume that the condition $i)$ holds.

Lemma 1. *Assume that $\lambda \ge 0$. Then the following inequality holds for all $u \in D(L_\lambda)$:*

$$\|L_\lambda u\|_2 \ge c_0\|u\|_2, c_0 = c_0(\delta_0, \delta). \tag{2}$$

Proof. Let $u \in C_0^\infty(R^2)$. Transforming the expressions $\langle L_\lambda u, u\rangle_{R^2}$ and $\langle L_\lambda u, u_x\rangle_{R^2}$, we obtain the following inequalities

$$\frac{1}{2\delta}\|L_\lambda u\|_2^2 \geq \int\limits_{R^2} \left[|u_y|^2 + (\lambda + \delta/2)\,|u|^2\right] dxdy - \int\limits_{R^2} |u_x|^2 dxdy, \tag{3}$$

$$\|L_\lambda u\|_2^2 \geq \delta_0^2 \|u_x\|_2^2. \tag{4}$$

Here we used the $\varepsilon-$ Cauchy inequality, with $\varepsilon = \delta/2$. From (3) and (17) the estimate (2) follows. Since L_λ is the closed operator the estimate (2) holds for all $u \in D(L_\lambda)$. □

Let $-\infty < t < +\infty$, $\Delta_j = (j-1, j+1)$ $(j \in Z)$, and let γ be a constant such that $\gamma a(y) > 0$. Denote by $l_{t,j,\gamma} + \lambda E$ the closure in $L_2(\Delta_j)$ of the differential operator $(l_{t,j.\gamma} + \lambda E)u = -u'' + \left[-t^2 + ita(y) + it\gamma + c(y) + \lambda\right] u$ defined on the set $C_0^2(\overline{\Delta}_j)$ of twice continuously differentiable functions u on $\overline{\Delta}_j$ which satisfy the equalities $u(j-1) = u(j+1) = 0$.

Lemma 2. *Let $\lambda \geq 0$ and let γ be constants such that*

$$\frac{(\delta_0 + |\gamma|)^2}{\sqrt{\delta + \lambda}} - 1 \geq 0. \tag{5}$$

Then for any $u \in D(l_{t,j.\gamma} + \lambda E)$ the following inequalities hold:
a) $\|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)} \geq c_1(\delta) \ \left(\|u'\|_{L_2(\Delta_j)} + \|\sqrt{c+\lambda}u\|_{L_2(\Delta_j)} + \|t\sqrt{|a+\gamma|}u\|_{L_2(\Delta_j)}\right)$;
b) $c_2(\delta)/\sqrt{\delta+\lambda} \ \|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)} \geq \|u\|_{L_2(\Delta_j)}$;

c) $c_3(\delta)/\sqrt[4]{\delta+\lambda} \ \|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)} \geq \|u'\|_{L_2(\Delta_j)}$, *where* $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$, $c_3 = c_3(\delta)$.

Proof. Let $u \in C_0^2(\overline{\Delta}_j)$. Then we have

$$\left|\langle(l_{t,j,\gamma} + \lambda E)u, u\rangle_{\Delta_j}\right| \geq \|u'\|_{L_2(\Delta_j)}^2 + \int\limits_{\Delta_j} [c(y) + \lambda]|u|^2 dy - \left|\int\limits_{\Delta_j} t^2|u|^2 dy\right|.$$

Hence

$$\|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)}\|u\|_{L_2(\Delta_j)} \geq \int\limits_{\Delta_j} |u'|^2 dy - \int\limits_{\Delta_j} t^2|u|^2 dy \tag{6}$$

and

$$\frac{1}{2\delta}\|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)}^2 \geq \frac{1}{2}\int\limits_{\Delta_j} [c(y) + \lambda]\,|u|^2 dy - \int\limits_{\Delta_j} t^2|u|^2 dy. \tag{7}$$

On the other hand, by transforming the expression $\langle (l_{t,j,\gamma} + \lambda E)u, -itu\rangle_{\Delta_j}$, we have

$$\|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)} \geq \|t\sqrt{|a(\cdot) + \gamma|}u\|_{L_2(\Delta_j)}, u \in C_0^2(\overline{\Delta}_j), \tag{8}$$

and

$$\|(l_{t,j,\gamma} + \lambda E)u\|^2_{L_2(\Delta_j)} \geq (\delta_0 + |\gamma|)^2 |t|^2 \|u\|^2_{L_2(\Delta_j)}, u \in C_0^2(\overline{\Delta}_j). \tag{9}$$

Combining (7) with (9), then using condition (18), we obtain

$$c_4(\delta)\|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)} \geq \|\sqrt{c(\cdot) + \lambda}u\|_{L_2(\Delta_j)}. \tag{10}$$

Hence by the condition $i)$ we conclude:

$$\frac{c_4(\delta)}{\sqrt{\delta + \lambda}}\|(l_{t,j,\gamma} + \lambda E)u\|_{L_2(\Delta_j)} \geq \|u\|_{L_2(\Delta_j)}. \tag{11}$$

From inequalities (6), (9) and (11) we obtain the following estimate

$$\frac{c_4(\delta) + 1}{\sqrt{\delta + \lambda}}\|(l_{t,j,\gamma} + \lambda E)u\|^2_{L_2(\Delta_j)} \geq \|u'\|^2_{L_2(\Delta_j)} + t^2 \int_{\Delta_j} \left(\frac{(\delta_0 + |\gamma|)^2}{\sqrt{\delta + \lambda}} - 1\right) |u|^2 dy. \tag{12}$$

The estimate $a)$ follows from inequalities (8), (10) and (12) by using the condition $i)$ and (18). Further, the estimate (11) implies $b)$. Finally it follows from (12) that

$$\frac{c_4(\delta) + 1}{\sqrt{\delta + \lambda}}\|(l_{t,j,\gamma} + \lambda E)u\|^2_{L_2(\Delta_j)} \geq \|u'\|^2_{L_2(\Delta_j)}.$$

This implies the estimate $c)$, which completes the proof of the lemma. □

Lemma 3. *Assume that* $\lambda \geq 0$ *and condition (18) holds. Then the operator* $l_{t,j,\gamma} + \lambda E$ *is invertible, and the inverse operator* $(l_{t,j,\gamma} + \lambda E)^{-1}$ *is defined in all* $L_2(\Delta_j)$, $j \in Z$.

Proof. By estimate $b)$ in Lemma 2 it is enough to prove that $R(l_{t,j,\gamma} + \lambda E) = L_2(\Delta_j)$. Assume the contrary. Then there exists a function $v \in L_2(\Delta_j)$, $v \neq 0$, such that

$$(l_{t,j,\gamma} + \lambda E)^* v := -v'' + \left[-t^2 - ita(y) - it\gamma + c(y) + \lambda\right] v = 0. \tag{13}$$

This implies that $v'' \in L_2(\Delta_j)$. By transforming the expression $\langle (l_{t,j,\gamma} + \lambda E)u, v\rangle_{\Delta_j}$ we have $u'(j+1)\bar{v}(j+1) - u'(j-1)\bar{v}(j-1) = 0$ for any function $u \in D(l_{t,j,\gamma} + \lambda E)$. Therefore $v(j+1) = v(j-1) = 0$, and using these equalities, we can derive the similar to (11) estimate:

$$\|(l_{t,j,\gamma} + \lambda E)^* v\|^2_{L_2(\Delta_j)} \geq c_5\|v\|^2_{L_2(\Delta_j)}. \tag{14}$$

From (13) and (14) we conclude that $v = 0$. □

By $l_{t,\gamma}+\lambda E$ $(-\infty<t<+\infty)$ we denote the closure of the differential expression $(l_{t,\gamma}+\lambda E)u=-u''+\left[-t^2+ita(y)+it\gamma+c(y)+\lambda\right]u$ defined on the set $C_0^\infty(R)$.

Lemma 4. *Let $\lambda\geq 0$ and let condition (5) hold. Then for any $u\in D(l_{t,\gamma}+\lambda E)$ the following estimates hold:*

$$\|(l_{0,\gamma}+\lambda E)u\|_{L_2(R)}\geq\sqrt{\delta+\lambda}\|u\|_{L_2(R)},$$

$$\|(l_{t,\gamma}+\lambda E)u\|_{L_2(R)}\geq|t|(\delta_0+|\gamma|)\|u\|_{L_2(R)},t\neq 0.$$

Lemma 4 is proved by transforming the expression $<(l_{t,\gamma}+\lambda E)u,-itu>$, where $u\in C_0^\infty(R)$.

Let now $\{\varphi_j(y)\}_{j=-\infty}^{+\infty}\subset C_0^\infty(R)$ be a sequence of functions satisfying the conditions $\varphi_j\geq 0$, *supp* $\varphi_j\subseteq\Delta_j$ $(j\in Z)$, $\sum\limits_{j=-\infty}^{+\infty}\varphi_j^2(y)=1$. Introducing the operators $K_{\lambda,\gamma}f=\sum\limits_{j=-\infty}^{+\infty}\varphi_j(l_{t,j,\gamma}+\lambda E)^{-1}\varphi_jf$, $B_{\lambda,\gamma}f=\sum\limits_{j=-\infty}^{+\infty}\varphi_j''(l_{t,j,\gamma}+\lambda E)^{-1}\varphi_jf+2\sum\limits_{j=-\infty}^{+\infty}\varphi_j'd/dy(l_{t,j,\gamma}+\lambda E)^{-1}\varphi_jf$, $f\in L_2(R)$, $\lambda\geq 0$, we can prove that

$$(l_{t,\gamma}+\lambda E)K_{\lambda,\gamma}f=f-B_{\lambda,\gamma}f. \tag{15}$$

Lemma 5. *There exists a number $\lambda_0>0$ such that $\|B_{\lambda,\gamma}\|_{L_2(R)\to L_2(R)}<1$ for all $\lambda\geq\lambda_0$, where γ satisfies condition (18).*

Proof. Let $f\in C_0^\infty(R)$. Since only the functions φ_{j-1}, φ_j, φ_{j+1} can be nonzero on $\overline{\Delta}_j$ $(j\in Z)$ we have

$$\|B_{\lambda,\gamma}f\|_{L_2(R)}^2\leq\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\sum_{k=j-1}^{j+1}\left[\varphi_k''(l_{t,k,\gamma}+\lambda E)^{-1}\varphi_kf+2\varphi_k'\frac{d}{dy}(l_{t,k,\gamma}+\lambda E)^{-1}\varphi_kf\right]\right|^2dy.$$

Hence using the inequality $(a_0+b_0+d_0)^2\leq 3(a_0^2+b_0^2+d_0^2)$ and estimates b), c) in Lemma 2, we obtain $\|B_{\lambda,\gamma}f\|_{L_2(R)}^2\leq c_6\left[(\lambda+\delta)^{-\frac{1}{2}}+(\lambda+\delta)^{-\frac{1}{4}}\right]\|f\|_{L_2(R)}^2$, where the constant c_6 depends on $\max\limits_{j\in Z}\{|\varphi_j'|\}$, $\max\limits_{j\in Z}\{|\varphi_j''|\}$, $c_2(\delta)$, and $c_3(\delta)$. Now choose $\lambda_0=16c_6^4+1-\delta$. This completes the proof. □

In what follows in Lemmas 6-10 λ_0 is a constant as in Lemma 5. From the representation (15) by Lemmas 4 and 5 follows the next lemma.

Lemma 6. *Let us assume that* $\lambda \geq \lambda_0$ *and condition (18) holds. Then the operator* $l_{t,\gamma} + \lambda E$ *is continuously invertible, and for the inverse operator* $(l_{t,\gamma} + \lambda E)^{-1}$ *the following equality holds:*

$$(l_{t,\gamma} + \lambda E)^{-1} = K_{\lambda,\gamma}(E - B_{\lambda,\gamma})^{-1}. \tag{16}$$

Lemma 7. *Let us assume that* $\lambda \geq \lambda_0$, γ *satisfies condition (5), and* $\rho(y)$ *is a continuous function defined on* R. *Then for* $\alpha = 0, 1$ *the following estimate holds:*

$$\|\rho|t|^{\alpha}(l_{t,\gamma} + \lambda E)^{-1}\|^2_{L_2(R)\to L_2(R)} \leq \\ \leq c_7(\lambda) \sup_{j\in Z} \|\rho|t|^{\alpha}\varphi_j(l_{t,j,\gamma} + \lambda E)^{-1}\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)}. \tag{17}$$

Proof. For $f \in C_0^{\infty}(R)$ from representation (16) and by the properties of the functions φ_j $(j \in Z)$, we conclude:

$$\|\rho|t|^{\alpha}(l_{t,\gamma} + \lambda E)^{-1} f\|^2_{L_2(R)} \leq$$

$$\sum_{j=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| \sum_{k=j-1}^{j+1} \left[\rho(y)|t|^{\alpha}\varphi_k(l_{t,k,\gamma} + \lambda E)^{-1}\varphi_k(E - B_{\lambda,\gamma})^{-1} f\right] \right|^2 dy.$$

Hence by the obvious inequality $(a_0 + b_0 + d_0)^2 \leq 3(a_0^2 + b_0^2 + d_0^2)$ and by Lemma 5, we obtain estimate (17). □

The result below follows from Lemma 2 and the estimate (17).

Lemma 8. *Let* $\lambda \geq \lambda_0$ *and let condition (18) hold. Then*
a) $\|\sqrt{c+\lambda}(l_{t,\gamma} + \lambda E)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$;
b) $\|it(l_{t,\gamma} + \lambda E)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$;
c) $\|d/dy(l_{t,\gamma} + \lambda E)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$.

Consider the equation

$$(l_t + \lambda E)u = -u'' + \left[-t^2 + ita(y) + c(y) + \lambda\right] u = f, \tag{18}$$

where $f \in L_2(R)$. The function $u \in L_2(R)$ is called a solution of the equation (18) if there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R)$ such that $\|u_n - u\|_{L_2(R)} \to 0$, $\|(l_t + \lambda E)u_n - f\|_{L_2(R)} \to 0$ as $n \to \infty$. The closure in $L_2(R)$ of the operator $l_t + \lambda E$ is denoted by $l_t + \lambda E$, too.

Lemma 9. *Let us assume that* $\lambda \geq \lambda_0$. *Then the operator* $l_t + \lambda E$, $t \in R$ *is boundedly invertible, and for the inverse operator* $(l_t + \lambda E)^{-1}$ *the equality*

$$(l_t + \lambda E)^{-1} f = (l_{t,\gamma} + \lambda E)^{-1}(E - A_{\lambda,\gamma})^{-1} f, f \in L_2(R), \tag{19}$$

holds for any $t \neq 0$, *where* $\|A_{\lambda,\gamma}\|_{L_2(R)\to L_2(R)} < 1$, *and* γ *satisfies condition (5).*

Proof. First assume that $t \neq 0$. We rewrite the equation $(l_t + \lambda E)u = f$ in the form $v - A_{\lambda,\gamma}v = f$, where $v = (l_{t,\gamma} + \lambda E)u$, $A_{\lambda,\gamma} = it\gamma(l_{t,\gamma} + \lambda E)^{-1}$, and γ satisfies condition (5). From Lemma 4 it follows that $\|A_{\lambda,\gamma}\|_{L_2(R)\to L_2(R)} < 1$. Then there exists the inverse operator $(l_t+\lambda E)^{-1}$, and $u = (l_t+\lambda E)^{-1}f = (l_{t,\gamma}+\lambda E)^{-1}(E-A_{\lambda,\gamma})^{-1}f$, $f \in L_2(R)$. Further, since $l_0+\lambda E$ is a self-adjoint operator [13] (p. 208), then the estimate $\|(l_0 + \lambda E)u\|_{L_2(R)} \geq (\delta + \lambda)\|u\|_{L_2(R)}$ holds for any $u \in D(l_0 + \lambda E)$. These implies that the operator $l_0 + \lambda E$ is boundedly invertible. □

Lemma 8 and the equality (19) imply the following lemma.

Lemma 10. *If* $\lambda \geq \lambda_0$, *then*
a)$\|\sqrt{c+\lambda}(l_t + \lambda E)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$;
b)$\|it(l_t + \lambda E)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$;
c)$\|d/dy(l_t + \lambda E)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$.

We will use also the following well-known lemma [14] (p. 350).

Lemma 11. *Let us assume that* $\theta_0 > 0$ *is a constant and the operator* $L + \theta_0 E$ *is boundedly invertible in* $L_2(R)$, *and the estimate* $\|(L + \theta E)u\|_{L_2(R)} \geq c_8\|u\|_{L_2(R)}, u \in D(L + \theta E)$ *holds for* $\theta \in [0, \theta_0)$. *Then the operator* $L : L_2(R) \to L_2(R)$ *is also boundedly invertible.*

3 Proofs of Theorems

Proof of Theorem 1. Applying the Fourier transform with respect to the variable x from the equation (1), we obtain:

$$(l_t + \lambda E)\tilde{u} = -\tilde{u}'' + \left[-t^2 + ita(y) + c(y) + \lambda\right]\tilde{u} = \tilde{f}, \tag{20}$$

where

$$\tilde{u} = (F_{x\to t}u)(t, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x, y)e^{-itx}dx, \tilde{f} = (F_{x\to t}f)(t, y)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x, y)e^{-itx}dx.$$

If $\lambda \geq \lambda_0$ (λ_0 is a constant as in Lemma 5, then by Lemma 9 there exists a unique solution of equation (20). Then from Lemma 11 it follows that equation (20) is uniquely solvable for all $\lambda \geq 0$ and $\tilde{u} = (l_t + \lambda E)^{-1}\tilde{f}$ is a solution. Therefore by Lemma 1

$$u = F^{-1}_{t\to x}(l_t + \lambda E)^{-1}\tilde{f} \tag{21}$$

is the unique solution of equation (1). □

Proof of Theorem 2. Using the representation (21), we obtain

$$\|u_x\|_2^2 \leq \int_{-\infty}^{+\infty} \|it(l_t+\lambda E)^{-1}\|^2_{L_2(R)\to L_2(R)}\|\tilde{f}\|^2_{L_2(R)}dt \leq$$

$$\sup_{t\in R}\|it(l_t+\lambda E)^{-1}\|^2_{L_2(R)\to L_2(R)}\|f\|_2^2,$$

$$\|u_y\|_2^2 \leq \int_{-\infty}^{+\infty} \left\|\frac{d}{dy}(l_t+\lambda E)^{-1}\right\|^2_{L_2(R)\to L_2(R)}\|\tilde{f}\|^2_{L_2(R)}dt$$

$$\leq \sup_{t\in R}\left\|\frac{d}{dy}(l_t+\lambda E)^{-1}\right\|^2_{L_2(R)\to L_2(R)}\|f\|_2^2,$$

$$\|\sqrt{c(\cdot)+\lambda}u\|_2^2 \leq \int_{-\infty}^{+\infty} \|\sqrt{c+\lambda}(l_t+\lambda E)^{-1}\tilde{f}\|^2_{L_2(R)}dt$$

$$\leq \sup_{t\in R}\|\sqrt{c+\lambda}(l_t+\lambda E)^{-1}\|^2_{L_2(R)\to L_2(R)}\|f\|_2^2.$$

The proof of Theorem 2 follows from these estimates, by taking into account the assertions $a)-c)$ of Lemma 10. □

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Akhieser, N.I., Glasman, I.M.: Theory of the Linear Operators in the Hilbert Space. Nauka, Moscow (1966) (in Russian)
2. Bers, L., John, F., Schechter, M.: Partial Differential Equations. Mir, Moscow (1966) (in Russian)
3. Bitsadze, A.V.: Some Classes of Partial Differential Equations. Gordon and Breach Science Publishers, New-York (1988)
4. Boimatov, K.Kh.: Separability theorems, weighted spaces and their applications. Trudy Mat. Inst. Steklov (Studies in the theory of differentiable functions of several variables and its applications, X). **170**, 37–76 (1984) (in Russian)
5. Muratbekov, M.B.: Separability and estimates for the widths of sets connected with the domain of a nonlinear operator of Schrödinger type. Diff. Equ. **27**(6), 1034–1042 (1991)
6. Muratbekov, M.B.: Discreteness of the spectrum and the distribution of singular numbers (s-numbers) of a class of differential operators of hyperbolic type. Math. J., Almaty. **12**(3), 113–118 (2012) (in Russian)
7. Muratbekov, M.B., Muratbekov, M.M., Abylayeva, A.M.: On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. Elect. J. Qual. Theory Diff. Equ. **64**, 1–10 (2013). www.math.u-szeged.hu/ejqtde/p2337.pdf

8. Muratbekov, M.B., Muratbekov, M.M., Ospanov, K.N.: Coercive solvability of odd-order differential equations and its applications. Dokl. Math. **82**, 909–911 (2010)
9. Kal'menov, T.Sh.: Regular boundary value problems for the wave equation. Diff. Uravn. **17**(6), 1105–1121 (1981) (in Russian)
10. Ospanov, K.N.: Coercive estimates for degenerate elliptic system of equations with spectral applications. Appl. Math. Lett. **24**, 1594–1598 (2011)
11. Ospanov, K.N., Akhmetkaliyeva, R.D.: Separation and the existence theorem for second order nonlinear differential equation. Elect. J. of Qual. Theory Diff. Equ. **66**, 1–12 (2012). http://www.math.u-szeged.hu/ejqtde/p1535.pdf
12. Otelbaev, M.: Coercive estimates and separability theorems for elliptic equations in R^n. Trudy Mat. Inst. Steklov (Studies in the theory of differentiable functions of several variables and its applications, IX). **161**, 195–217 (1983) (in Russian)
13. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. 1: Functional analysis. Academic Press, San Diego (1980)
14. Tikhonov, A.N., Samarskiy, A.A.: Equations of Mathematical Physics. Macmillan, New York (1963)

Remark on a Regularized Trace Formula for m-Laplacian in a Punctured Domain

Gulzat Nalzhupbayeva and Niyaz Tokmagambetov(✉)

Department of Mathematics, Imperial College London, Al-Farabi Kazakh National University, 71 Al-Farabi Ave., Almaty 050040 Kazakhstan, 180 Queen's Gate, London SW7 2AZ, UK
gulzat.nalzhupbayeva@gmail.com, niyaz.tokmagambetov@gmail.com

Abstract. In this paper we extend results on regularized trace formulae which were established in [9,10] for the Laplace and m-Laplace operators in a punctured domain with the fixed iterating order $m \in \mathbb{N}$. By using techniques of Sadovnichii and Lyubishkin [21], the authors in the papers [9,10] described regularized trace formulae in the spatial dimension $d = 2$. In this remark one is to be claimed that the formulae are also valid in the higher spatial dimensions, namely, $2 \leq d \leq 2m$. Also, we give the further discussions on a development of the analysis associated with the operators in punctured domains. This can be done by using so called 'nonharmonic' analysis.

Keywords: Regularization · Trace formulae · Laplacian · m-Laplace operator · Punctured domain · Nonharmonic analysis

1 Introduction

In this note we study a differential operator in a punctured domain. For motivation, we refer to the manuscripts [1,4,5,11,13,14,22,23] and references therein, where different model differential equations in punctured domains or with δ–like potentials are investigated, and some spectral properties, for example, formulae for the resolvents and regularized traces, are also established.

Here, we observe that the results of the paper [9] are also true when the spatial dimension is greater than two.

Let $D \subset \mathbb{R}^d$ be a simply connected domain with the smooth boundary ∂D. Denote by $s = (s_1, \ldots, s_d)$ a fixed point of the domain D. Then we define a punctured domain $D_0 := D \backslash \{s\}$. During this paper, we deal with the differential expression

$$(-\Delta)^m u := \left(- \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \right)^m \tag{1}$$

in a punctured domain D_0.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_25

Since D_0 is not simply connected, we need a special functional space for (1) to define an operator correctly. For this, we introduce the functional class $\mathscr{F}_m$ that can be represented in the following form

$$w(x) = w_0(x) + kG_m(x,s), \tag{2}$$

where k is some constant. The function w_0 is from the functional space $\mathbb{F}_m$ consisting of the functions $v \in \mathrm{H}^{2m}(D)$ such that

$$\left(\frac{\partial}{\partial n}\right)^j v\big|_{\partial D} = 0, \tag{3}$$

for all $j = 0, \ldots, m-1$, where $\frac{\partial}{\partial n}$ is the outer normal derivative. Here H^q stands for the usual Sobolev space with the parameters $(2,q)$, and $G_m(x,s)$ is the Green's function of the Dirichlet problem for the equation (1) in the whole domain D with the boundary conditions (3).

Now, we define a functional for our further investigations. To this, we consider the parallelled

$$\Pi_{s,\delta} = \{x : -\delta \le |x-s| \le \delta\}.$$

Then for the function h from the space $\mathscr{F}_m$ defined as (2) we introduce the following functional

$$\alpha_m(h) = \lim_{\delta \to +0} \int\limits_{\partial \Pi_{s,\delta}} \left[\frac{\partial (-\Delta)^{m-1} h(\xi)}{\partial n_\xi}\right] ds_\xi. \tag{4}$$

Remark 1. We note that the functional (4) is defined for all $d \in \mathbb{N}$. Moreover, the value of α_m from the function $G(x,s)$ exists.

For our convenience, we denote

$$\gamma := \alpha_m(G(\cdot,s)), \;\; \alpha(\cdot) := \frac{1}{\gamma}\alpha_m(\cdot),$$

and

$$\xi^-(w) := \alpha(w), \;\; \xi^+(w) := w_0(s).$$

2 Main Results

In this section we repeat the results of the paper [9]. However, here we formulate them also for the case $d \le 2m$.

Now, we are in a way in the Hilbert space $\mathrm{H}^2(D)$ to introduce an operator associated with the differential equation (1), that is, $(-\Delta)^m u$. We denote by $\mathscr{K}_M$ the operator defined as

$$\mathscr{K}_M u = (-\Delta)^m u,$$

in the punctured domain D_0 for all functions $u \in \mathscr{F}_m$. Assign $\mathscr{K}_m$ as the restriction of the operator $\mathscr{K}_M$ to

$$D(\mathscr{K}_m) = \{u | u \in \mathscr{F}_m,\ \xi^-(u) = 0, \xi^+(u) = 0\}.$$

Discussing as in the works [7,9,10], we get the following statements:

Proposition 1. *Let $d \leq 2m$. Assume that $u, v \in \mathscr{F}_m$. Then, we have*

$$< \mathscr{K}_M u, v >=< u, \mathscr{K}_M v > +\xi^-(u)\xi^+(v) - \xi^-(v)\xi^+(u).$$

Moreover, the operator $\mathscr{K}_\theta$ defined on $\mathscr{F}_m$ by the expression

$$(-\Delta)^m u = f,$$

in the punctured domain D_0 with the condition

$$\theta_1 \xi^-(u) = \theta_2 \xi^+(u) \tag{5}$$

is a self-adjoint extension of $\mathscr{K}_m$ in the functional space $\mathscr{F}_m$. Here $\theta = (\theta_1, \theta_2)$, $\theta_1, \theta_2 \in \mathbb{R}$ with the property $\theta_1^2 + \theta_2^2 \neq 0$.

In the Hilbert space $\mathrm{H}^2(D)$ consider the operator

$$\mathscr{K}_Q u(x) := (-\Delta)^m u(x), \quad x \in D_0 \tag{6}$$

on $u \in \mathscr{F}_m$ with

$$\alpha(u) + \int_D Q(x)((-\Delta)^m u_0)(x) dx = 0, \tag{7}$$

where $Q \in \mathrm{H}^2(D)$. Here we can write

$$\int_D Q(x)((-\Delta)^m u_0)(x) dx =: \langle Q, (-\Delta)^m u_0 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product of $\mathrm{H}^2(D)$.

Now, we consider the operator $\mathscr{K}_Q$ as a perturbation of $\mathscr{K}_0$. Here $\mathscr{K}_0$ stands for the Dirichlet problem for m–Laplace operator in the whole domain D. Then, we assume that $\{\mu_n\}_{n=1}^\infty$ are the eigenvalues of $\mathscr{K}_Q$ ordered in the increasing order of their absolute values taking into account the multiplicities, and suppose that $\{\lambda_n\}_{n=1}^\infty$ are the eigenvalues of $\mathscr{K}_0$ ordered in the increasing order by taking into account their multiplicities.

Theorem 1. *Let the spatial dimension $d \leq 2m$. Suppose that $p, \epsilon > 0$ are fixed numbers. Assume that $Q \in D(\mathscr{K}_0^m)$, $\mathscr{K}_0^{m-1} Q \in \mathrm{H}^p(\Pi_{s,\epsilon})$, and $Q(s) \neq -1$. Then, we have the following regularized trace formula*

$$\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \frac{\widetilde{Q}(s)}{1 + Q(s)}. \tag{8}$$

Here $\widetilde{Q}(s) = -\lim_{x \to s} \mathscr{K}_0^{m-1} Q(x)$.

The proof of Theorem 1 follows directly from the proofs of the main theorems of the papers [9,21].

3 Further Discussions

Finally, we note that Proposition 1 implies the following corollary, which gives a way to find out self–adjoint operators from the class of operators $\{\mathscr{K}_Q : Q \in \mathrm{H}^2(D)\}$, namely:

Corollary 1. *Suppose that $\theta_1 \neq 0$ and $Q(x) = -\mu G_m(x,s)$ with $\mu = \theta_2/\theta_1$. Then the operator $\mathscr{K}_Q$ is self–adjoint with the parameter (θ_1, θ_2) in the space $\mathscr{F}_m$:*

$$\mathscr{K}_{-\mu G_m} \sim \mathscr{K}_{(1,\mu)} = \mathscr{K}_{(\theta_1,\theta_2)}.$$

Thus, we observe that the class of operators given by the equation (6) and condition (7) has a huge number of self–adjoint operators in a punctured domain. One can be started a 'nonharmonic' analysis connected with the singular, in the above sense, operators. Note, that the nonharmonic analysis is developed in the works [2,3,12,15,17] with applications given in [16]. Also, the reader is referred to [20] and the monograph [19], where the analysis on the torus was developed by Ruzhansky and Turunen. For more general setting of the nonharmonic analysis, see for instance [8,18].

Acknowledgements. This research is partially supported by a grant from the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. 0773/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Albeverio, S., Fenstad, J.F., Hoegh-Krohn, R.J., Lindstrom, T.L.: Nonstandard Methods in Stochastic Analysis and Mathematical Physics [Russian translation]. Mir, Moscow (1980)
2. Delgado, J., Ruzhansky, M.: Fourier multipliers, symbols and nuclearity on compact manifolds. J. Anal. Math. https://arxiv.org/abs/1404.6479 (To appear)
3. Delgado, J., Ruzhansky, M., Tokmagambetov, N.: Schatten classes, nuclearity and nonharmonic analysis on compact manifolds with boundary. J. Math. Pures Appl. (2016). https://doi.org/10.1016/j.matpur.2016.10.005
4. Golovaty, YuD, Man'ko, S.S.: Solvable models for the Schrödinger operators with δ-like potentials. Ukrain. Math. Bull. **6**, 169–203 (2009)
5. Goloshchapova, N.I., Zastavnyi, V.P., Malamud, M.M.: Positive definite functions and spectral properties of the Schrödinger operator with point interactions. Math. Notes. **90**, 149–154 (2011)
6. Kanguzhin, B.E., Aniyarov, A.A.: Well-posed problems for the Laplace operator in a punctured disk. Math. Notes. **89**, 819–830 (2011)
7. Kanguzhin, B.E., Nurakhmetov, D.B., Tokmagambetov, N.E.: Laplace operator with δ-like potentials. Russian Math. (Iz. VUZ). **58**, 6–12 (2014)
8. Kanguzhin, B.E., Ruzhansky, M.V., Tokmagambetov, N.E.: On convolutions in Hilbert spaces. Funct. Anal. Appl. (To appear)
9. Kanguzhin, B.E., Tokmagambetov, N.E.: On regularized trace formulas for a well-posed perturbation of the m-Laplace operator. Diff. Equ. **51**, 1583–1588 (2015)

10. Kanguzhin, B.E., Tokmagambetov, N.E.: A regularized trace formula for a well-perturbed Laplace operator. Dokl. Math. **91**, 1–4 (2015)
11. Kanguzhin, B.E., Tokmagambetov, N.E.: Resolvents of well-posed problems for finite-rank perturbations of the polyharmonic operator in a punctured domain. Sib. Math. J. **57**, 265–273 (2016)
12. Kanguzhin, B., Tokmagambetov, N., Tulenov, K.: Pseudo-differential operators generated by a non-local boundary valuem problem. Complex Var. Elliptic Equ. **60**, 107–117 (2015)
13. Kostenko, A.S., Malamud, M.M.: 1D Schrödinger operators with local point interactions on a discrete set. J. Diff. Equ. **249**, 253–304 (2010)
14. Pavlov, B.S.: The theory of extensions and explicitly-soluble models. Russ. Math. Surv. **42**, 127–168 (1987)
15. Ruzhansky, M., Tokmagambetov, N.: Nonharmonic analysis of boundary value problems. Int. Math. Res. Not. IMRN. **2016**, 3548–3615 (2016)
16. Ruzhansky, M., Tokmagambetov, N.: Very weak solutions of wave equation for Landau Hamiltonian with irregular electromagnetic field. Lett. Math. Phys. (2016). https://doi.org/10.1007/s11005-016-0919-6
17. Ruzhansky, M., Tokmagambetov, N.: Nonharmonic analysis of boundary value problems without WZ condition. Math. Model. Nat. Phenom. **12**, 115–140 (2017)
18. Ruzhansky, M., Tokmagambetov, N.: Convolution, Fourier analysis, and distributions generated by Riesz bases. https://arxiv.org/abs/1611.03791
19. Ruzhansky, M., Turunen, V.: Pseudo-Differential Operators and Symmetries. Background analysis and advanced topics. Birkhäuser Verlag, Basel (2010)
20. Ruzhansky, M., Turunen, V.: Quantization of pseudo-differential operators on the torus. J. Fourier Anal. Appl. **16**, 943–982 (2010)
21. Sadovnichii, V.A., Lyubishkin, V.A.: Finite-dimensional perturbations of discrete operators and trace formulas. (Russian). Funktsional. Anal. i Prilozhen. **20**, 55–65 (1986)
22. Savchuk, A.M., Shkalikov, A.A.: Sturm-Liouville operators with distribution potentials. Moscow Math. Soc. **64**, 143–192 (2003)
23. Zubok, D.A., Popov, IYu.: Two physical applications of the Laplace operator perturbed on a set of measure zero. Theoret. Math. Phys. **119**, 629–639 (1999)

Internal Boundary Value Problems for the Laplace Operator with Singularity Propagation

Daulet B. Nurakhmetov(✉) and Almir A. Aniyarov

S. Seifullin Kazakh agrotechnical university, 62 Zhenis avenue, Astana 010011, Kazakhstan
dauletkaznu@gmail.com, aniyarov.a@gmail.com

Abstract. We consider well-posedness issues of problems of the Laplace operator in the unit circle with two internal points. For boundary value problems, one of the main issues is the well-posedness of the problem. When the problem is considered in a non-simply-connected domain, there usually appear additional conditions depending on the features of the domain under consideration. If for the well-posedness of the problem, in addition to the boundary conditions, one requires to take into account the internal communications of the domain, then such problems are called internal boundary value problems. For such problems there is written out a class of functions in which there exist such kinds of well-posed problems. A constructive method for constructing solutions to such problems is developed. As an illustration, examples are considered.

Keywords: Differential operator · Internal boundary value problems · Non-simply-connected domain · Well-posed problems · Green function · Dirichlet problem · Laplace operator

1 Introduction

In the theory of differential operators, one of the key questions is the well-posedness of the operator. At the end of the 20th century, the question of the correct perturbation of differential operators in a simply connected domain was actively researched [8,10,12]. In these problems, an essential important point was the development of a mathematical apparatus, which would allow us to obtain correct differential operators perturbing the original correct differential operator in a simply connected domain.

At that time, there was simultaneously developing a theory of explicitly solvable models associated with the problems of quantum mechanics, the physics of solids [2,9]. These problems were considered in non-simply-connected domains [5]. The Sturm-Liouville problem with delta-shaped potential

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_26

$$\ell(y) = -y'' + \delta\left(x - \pi/2\right) y(x) = \lambda y, \quad x \in (0, \pi), \tag{1}$$

with the Dirichlet conditions

$$y\left(0\right) = y\left(\pi\right) = 0 \tag{2}$$

is transformed to the problem (this is known from [4])

$$L(y) = -\left(y' - u(x)y\right)' - u(x)\left(y' - u(x)y\right) - u^2(x)y, \tag{3}$$

where

$$u\left(x\right) = \begin{cases} 1, & if \;\; x \geq \frac{\pi}{2} \\ 0, & if \;\; x < \frac{\pi}{2} \end{cases}$$

$$D\left(L\right) = \left\{ y \in AC \left| \begin{array}{c} y' \in AC\left[0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right] \\ y'\left(\frac{\pi}{2} + 0\right) - y'\left(\frac{\pi}{2} - 0\right) = y\left(\frac{\pi}{2}\right), \quad y\left(0\right) = y\left(\pi\right) = 0 \end{array} \right. \right\}. \tag{4}$$

The regularized trace of problem (1) - (2) was calculated in [24,25]. A further generalization of the Sturm-Liouville problem with singular coefficients was studied in papers [26]. In the case of the Schrodinger operator $T+Q$, where $T = -\Delta$ (or $T = (-\Delta)^n$) is Laplace operator, Q is the operator of multiplication by the generalized function, an abstract generalization of the following result is given: if the sequence of functions q_n converges to q in the space of multipliers, then the sequence of operators $-\Delta + q_n$ converges to $-\Delta + q$ in the sense of uniform resolvent convergence and there is a convergence of spectra [21].

In the early 2000s there was a question about the existence of well-posed problems in a multiply connected domain. In Kazakhstan, this issue is addressed by the scientific school of professor B.E. Kanguzhin. In [14,15] there are written out all well-posed solvable problems and an explicit form of the resolvent for ordinary differential operators in a multiply connected domain. The regularized trace of two-fold differentiation operator in a multiply connected domain with a correct perturbation is calculated in [1]. In [13] all well-posed solvable problems for the Laplace operator in the unit circle with one punctured point are written out. An explicit form of the resolvent is obtained in [3]. In [17] self-adjoint extension of the Laplace operator in a punctured circle is described. In [18] regularized trace formula of correctly perturbed Laplace operator in a circle with one punctured point is calculated. The properties of the Green's function for such problems are researched in [16]. In [6,19], for a polyharmonic differential Laplace operator with a one punctured point, there is written out the resolvent formula of well-posed solvable problems. In [11] properties of the Green's function for the Dirichlet problem of polygarmonic equation in a sphere are researched. In [23] a representation of the Green's function of the classical Neumann problem for the Poisson equation in the unit ball of any dimension is obtained.

The main purpose of this work is a description of all well-posed solvable problems for the Laplace operator in the unit circle with two punctured points. Note that this paper is a continuation of the work [13].

2 Statement of the problem

Let $\Omega = \{x^2+y^2 < 1\}$, $\Omega_0 = \Omega \backslash \{(x_0, y_0), (x_1, y_1)\}$, $(x_1-x_0)^2+(y_1-y_0)^2 \geq 4\delta^2$, δ be sufficiently small positive number, and let $(x_0, y_0), (x_1, y_1)$ be internally fixed field points of Ω. In this paper we consider the internal boundary value problem for an non-homogeneous Laplace equation in the punctured area of Ω_0

$$\Delta W(x,y) = f\,(x,y) \tag{5}$$

with the boundary condition

$$W(x,y)|_{\partial\Omega} = \langle \Delta W\,(x,y)\,, \sigma(x,y)\rangle \tag{6}$$

and the internal boundary conditions

$$\begin{gathered}
\frac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial W(x_i+\delta,\eta)}{\partial\xi}-\frac{\partial W(x_i-\delta,\eta)}{\partial\xi}\right]d\eta+\\
+\frac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial W(\xi,y_i+\delta)}{\partial\eta}-\frac{\partial W(\xi,y_i-\delta)}{\partial\eta}\right]d\xi=\langle \Delta W\,(x,y)\,, \sigma_i(x,y)\rangle\,,\\
\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[W\,(x_i-\delta,\eta)-W\,(x_i+\delta,\eta)\right]d\eta=\langle \Delta W\,(x,y)\,, \sigma_{i+2}(x,y)\rangle\,,\\
\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[W\,(\xi,y_i-\delta)-W\,(\xi,y_i+\delta)\right]d\xi=\langle \Delta W\,(x,y)\,, \sigma_{i+4}(x,y)\rangle\,,\quad i=0,1,
\end{gathered} \tag{7}$$

where $\sigma_{i+k}\,(\xi,\eta) \in L_2\,(\Omega)\,,\ i = 0,1,\ k = 0,2,4, \langle\cdot,\cdot\rangle$ is a scalar product in $L_2(\Omega)$.

Let $\mathscr{D}$ be a set of all functions

$$h\,(x,y) = h_1\,(x,y) + \alpha_0 G\,(x,y,x_0,y_0) + \alpha_1 G\,(x,y,x_1,y_1)\,, (x,y) \in \Omega_0$$

$\alpha_i \in R, i = 0,1, h_1 \in \mathrm{D} = \left\{h_1 \in W_2^2\,(\Omega)\,,\, h_1|_{\partial\Omega} = 0\right\}$. Here and below $G(x,y,\xi,\eta)$ is the Green's function of the Dirichlet problem for the Laplace operator in Ω. The properties of these types of operators are studied in [13,20].

It is convenient to introduce the class of functions $\widetilde{W_2^1}\,(\Omega_0)$, which consists of the function $h(x,y) \in \mathscr{D}$, that in a neighborhood of (x_i, y_i) have the following behavior:

$$\begin{gathered}
\sup_{0<\delta<\delta_i}\ \sup_{y_i-\delta<\eta<y_i+\delta}\delta\left(\left|\frac{\partial h(x_i+\delta,\eta)}{\partial\xi}\right|+|h(x_i+\delta,\eta)|+\right.\\
\left.+\left|\frac{\partial h(x_i-\delta,\eta)}{\partial\xi}\right|+|h(x_i-\delta,\eta)|\right)\leq C,
\end{gathered} \tag{8}$$

$$\begin{gathered}
\sup_{0<\delta<\delta_i}\ \sup_{x_i-\delta<\xi<x_i+\delta}\delta\left(\left|\frac{\partial h(\xi,y_i+\delta)}{\partial\eta}\right|+|h(\xi,y_i+\delta)|+\right.\\
\left.+\left|\frac{\partial h(\xi,y_i-\delta)}{\partial\eta}\right|+|h(\xi,y_i-\delta)|\right)\leq C,
\end{gathered} \tag{9}$$

($C - const$, δ is a sufficiently small positive number) and there are limits

$$\alpha_i(h) = \frac{1}{2} \lim_{\delta \to +0} \int\limits_{y_i-\delta}^{y_i+\delta} \left[\frac{\partial h(x_i+\delta,\eta)}{\partial \xi} - \frac{\partial h(x_i-\delta,\eta)}{\partial \xi} \right] d\eta +$$

$$+ \frac{1}{2} \lim_{\delta \to +0} \int\limits_{x_i-\delta}^{x_i+\delta} \left[\frac{\partial h(\xi,y_i+\delta)}{\partial \eta} - \frac{\partial h(\xi,y_i-\delta)}{\partial \eta} \right] d\xi < \infty,$$

$$\beta_i(h) = \lim_{\delta \to +0} \int\limits_{y_i-\delta}^{y_i+\delta} \left[h(x_i-\delta,\eta) - h(x_i+\delta,\eta) \right] d\eta < \infty,$$

$$\gamma_i(h) = \lim_{\delta \to +0} \int\limits_{x_i-\delta}^{x_i+\delta} \left[h(\xi,y_i-\delta) - h(\xi,y_i+\delta) \right] d\xi < \infty, \quad i = 0,1. \tag{10}$$

3 Auxiliary Statements

We introduce a new function by the formula

$$I(x,y) = \int\int\limits_{\Omega} G(x,y,\xi,\eta)\, \Delta_{\xi,\eta} h(\xi,\eta)\, d\xi d\eta, \tag{11}$$

where $G(x,y,\xi,\eta)$ is the Green's function of the Dirichlet problem in Ω, $\Delta_{\xi,\eta} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$ is a Laplace operator with relative to variables ξ, η.

From [7] it is known that

$$G(x,y,\xi,\eta)\big|_{(x,y)\in\partial\Omega,(\xi,\eta)\in\Omega} = 0 \tag{12}$$

for the Dirichlet problem in Ω. Then the function $I(x,y)$ has properties:

$$\Delta_{x,y} I(x,y) = \Delta_{x,y} h(x,y),\ (x,y) \in \Omega \tag{13}$$

$$I(x,y)\big|_{\partial\Omega} = 0 \tag{14}$$

On the other hand, remembering the Green's formula $\int\int\limits_{\Omega} \Delta u v dx dy = \int\int\limits_{\Omega} u \Delta v dx dy - \int\limits_{\partial\Omega} \left(u \frac{\partial v}{\partial \bar{n}} - \frac{\partial u}{\partial \bar{n}} v \right) ds$, function $I(x,y)$ can be rewritten in the form

$$\begin{aligned} I(x,y) &= \int\int\limits_{\Omega} G(x,y,\xi,\eta)\, \Delta_{\xi,\eta} h(\xi,\eta)\, d\xi d\eta = \\ &= \int\int\limits_{\Omega} \Delta_{\xi,\eta} G(x,y,\xi,\eta)\, h(\xi,\eta)\, d\xi d\eta + \\ &+ \int\limits_{\partial\Omega} \left(G(x,y,\xi,\eta) \frac{\partial h}{\partial \overline{n}_{\xi,\eta}} - \frac{\partial G(x,y,\xi,\eta)}{\partial \overline{n}_{\xi,\eta}} h(\xi,\eta) \right) ds_{\xi,\eta}, \end{aligned} \tag{15}$$

where $\overline{n}_{\xi,\eta}$ is an outer normal to the circle $\partial\Omega$ in point (ξ,η). Note that $G(x,y,\xi,\eta) = 0$ at $(\xi,\eta) \in \partial\Omega$, because $G(x,y,\xi,\eta) = G(\xi,\eta,x,y)$ and (12) holds. Similarly, by the symmetry of the Green's function $G(x,y,\xi,\eta)$ with respect to pairs (x,y) and (ξ,η), we have the equality

$$\Delta_{\xi,\eta} G(x,y,\xi,\eta) = \delta_{\Omega}((x,y),(\xi,\eta)), \tag{16}$$

where $\delta_{\Omega}((x,y),(\xi,\eta))$ is the Dirac delta function in the domain Ω.

From (15) and (16) it follows

$$I(x,y) = h(x,y) - \int\limits_{\partial\Omega} \frac{\partial G(x,y,\xi,\eta)}{\partial \overline{n}_{\xi,\eta}} h(\xi,\eta)\, ds_{\xi,\eta} \tag{17}$$

From [13] the following theorem is known

Theorem 1. *Function*

$$W(x,y) = \int\int\limits_{\Omega} G(x,y,\xi,\eta) f(\xi,\eta)\, d\xi d\eta + h(x,y) - I(x,y) \tag{18}$$

is the solution of the following problem:

$$\begin{cases} \Delta W(x,y) = f(x,y),\ (x,y) \in \Omega \\ W(x,y)|_{\partial\Omega} = h(x,y)|_{\partial\Omega}, \end{cases} \tag{19}$$

where $h(x,y)$ is an arbitrary sufficiently smooth function.

Also in [13] all well-posed solvable problems for the Poisson equation in the domain Ω are written out. Our purpose is to generalize this result in the domain Ω_0.

It is required to describe all well-posed internal boundary value problems for the Poisson equation in the class $\tilde{W}_2^1(\Omega_0)$.

We take an arbitrary function $h(x,y)$ from $\mathscr{D}$ and consider the function $I(x,y)$ as follows

$$I(x,y) = \lim_{\delta\to+0} \int\limits_{\Omega_\delta} G(x,y,\xi,\eta)\, \Delta_{\xi,\eta} h(\xi,\eta)\, d\xi d\eta,$$

where $\Omega_\delta = \Omega \backslash \{\Pi_\delta(M_0), \Pi_\delta(M_1)\}$, δ is a sufficiently small positive number. $\Pi_\delta(M_i) = \{(\xi,\eta) : x_i - \delta \le \xi \le x_i + \delta, y_i - \delta \le \eta \le y_i + \delta\},\ i = 0,\ 1$ (x_i, y_i) are coordinates of a point M_i.

Then from (15) and (16) similarly to equality (17) it follows

$$\begin{aligned} I(x,y) &= h(x,y) - \int\limits_{\partial\Omega} \tfrac{\partial G(x,y,\xi,\eta)}{\partial \overline{n}_{\xi,\eta}} h(\xi,\eta)\, ds_{\xi,\eta} - \\ &- \lim_{\delta\to+0} \sum_{i=0}^{1} \int\limits_{\partial\Pi_\delta(M_i)} \Big(G(x,y,\xi,\eta) \tfrac{\partial h(\xi,\eta)}{\partial \overline{n}_{\xi,\eta}} - \tfrac{\partial G(x,y,\xi,\eta)}{\partial \overline{n}_{\xi,\eta}} h(\xi,\eta) \Big)\, ds_{\xi,\eta} \end{aligned} \tag{20}$$

for $(x, y) \neq M_i$. Write out the limit in the neighborhood of the point M_0.

$$\lim_{\delta\to+0}\int\limits_{\partial\Pi_\delta(M_0)}\left(G(x,y,\xi,\eta)\,\tfrac{\partial h(\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}-\tfrac{\partial G(x,y,\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}h(\xi,\eta)\right)ds_{\xi,\eta}=$$

$$=\lim_{\delta\to+0}\int\limits_{y_0-\delta}^{y_0+\delta}\tfrac{G(x,y,x_0+\delta,\eta)-G(x,y,x_0,y_0)}{\delta}\cdot\delta\cdot\tfrac{\partial h(x_0+\delta,\eta)}{\partial\xi}d\eta+$$

$$+\lim_{\delta\to+0}\int\limits_{y_0-\delta}^{y_0+\delta}\tfrac{G(x,y,x_0,y_0)-G(x,y,x_0-\delta,\eta)}{\delta}\cdot\delta\cdot\tfrac{\partial h(x_0-\delta,\eta)}{\partial\xi}d\eta+$$

$$+G(x,y,x_0,y_0)\lim_{\delta\to+0}\int\limits_{y_0-\delta}^{y_0+\delta}\left[\tfrac{\partial h(x_0+\delta,\eta)}{\partial\xi}-\tfrac{\partial h(x_0-\delta,\eta)}{\partial\xi}\right]d\eta+$$

$$+\lim_{\delta\to+0}\int\limits_{x_0-\delta}^{x_0+\delta}\tfrac{G(x,y,x_0,y_0)-G(x,y,\xi,y_0-\delta)}{\delta}\cdot\delta\cdot\tfrac{\partial h(\xi,y_0-\delta)}{\partial\eta}d\xi+$$

$$+\lim_{\delta\to+0}\int\limits_{x_0-\delta}^{x_0+\delta}\tfrac{G(x,y,\xi,y_0+\delta)-G(x,y,x_0,y_0)}{\delta}\cdot\delta\cdot\tfrac{\partial h(\xi,y_0+\delta)}{\partial\eta}d\xi+$$

$$+G(x,y,x_0,y_0)\lim_{\delta\to+0}\int\limits_{x_0-\delta}^{x_0+\delta}\left[\tfrac{\partial h(\xi,y_0+\delta)}{\partial\eta}-\tfrac{\partial h(\xi,y_0-\delta)}{\partial\eta}\right]d\xi+$$

$$+\lim_{\delta\to+0}\int\limits_{y_0-\delta}^{y_0+\delta}\frac{\frac{\partial G(x,y,x_0,y_0)}{\partial\xi}-\frac{\partial G(x,y,x_0+\delta,\eta)}{\partial\xi}}{\delta}\cdot\delta\cdot h(x_0+\delta,\eta)\,d\eta+$$

$$+\lim_{\delta\to+0}\int\limits_{y_0-\delta}^{y_0+\delta}\frac{\frac{\partial G(x,y,x_0-\delta,\eta)}{\partial\xi}-\frac{\partial G(x,y,x_0,y_0)}{\partial\xi}}{\delta}\cdot\delta\cdot h(x_0-\delta,\eta)\,d\eta+$$

$$+\frac{\partial G(x,y,x_0,y_0)}{\partial\xi}\lim_{\delta\to+0}\int\limits_{y_0-\delta}^{y_0+\delta}\left[h(x_0-\delta,\eta)-h(x_0+\delta,\eta)\right]d\eta+$$

$$+\lim_{\delta\to+0}\int\limits_{x_0-\delta}^{x_0+\delta}\frac{\frac{\partial G(x,y,\xi,y_0-\delta)}{\partial\eta}-\frac{\partial G(x,y,x_0,y_0)}{\partial\eta}}{\delta}\cdot\delta\cdot h(\xi,y_0-\delta)\,d\xi+$$

$$+\lim_{\delta\to+0}\int\limits_{x_0-\delta}^{x_0+\delta}\frac{\frac{\partial G(x,y,x_0,y_0)}{\partial\eta}-\frac{\partial G(x,y,\xi,y_0+\delta)}{\partial\eta}}{\delta}\cdot\delta\cdot h(\xi,y_0+\delta)\,d\xi+$$

$$+\frac{\partial G(x,y,x_0,y_0)}{\partial\eta}\lim_{\delta\to+0}\int\limits_{x_0-\delta}^{x_0+\delta}\left[h(\xi,y_0-\delta)-h(\xi,y_0+\delta)\right]d\xi.$$

Since by assumption for the function $h(\xi,\eta)$ there exists δ_0 and $C>0$ such that the relations (8), (9) hold, then the limit relation is true

$$\lim_{\delta\to+0}\int\limits_{\partial\Pi_\delta(M_0)}\left(G(x,y,\xi,\eta)\frac{\partial h(\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}-\frac{\partial G(x,y,\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}h(\xi,\eta)\right)ds_{\xi,\eta}=$$

$$=\alpha_0(h)\,G(x,y,x_0,y_0)+\beta_0(h)\frac{\partial G(x,y,x_0,y_0)}{\partial\xi}+\gamma_0(h)\frac{\partial G(x,y,x_0,y_0)}{\partial\eta}.$$

Similarly, in the neighborhood of the point M_1.

$$\lim_{\delta\to+0}\int\limits_{\partial\Pi_\delta(M_1)}\left(G(x,y,\xi,\eta)\frac{\partial h(\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}-\frac{\partial G(x,y,\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}h(\xi,\eta)\right)ds_{\xi,\eta}=$$

$$=\alpha_1(h)\,G(x,y,x_1,y_1)+\beta_1(h)\frac{\partial G(x,y,x_1,y_1)}{\partial\xi}+\gamma_1(h)\frac{\partial G(x,y,x_1,y_1)}{\partial\eta},$$

where $\alpha_i(h),\ \beta_i(h),\ \gamma_i(h),\quad (i=0,1)$ are defined by formulas (10).

It is taken into account that the function $G(x,y,\xi,\eta)$ for $(x,y)\neq(\xi,\eta)$ is a sufficiently smooth function. Then we get

$$I(x,y)=h(x,y)-\int\limits_{\partial\Omega}\tfrac{\partial G(x,y,\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}h(\xi,\eta)\,ds_{\xi,\eta}-$$

$$-\sum_{i=0}^{1}\left(\alpha_i(h)\,G(x,y,x_i,y_i)+\beta_i(h)\tfrac{\partial G(x,y,x_i,y_i)}{\partial\xi}+\gamma_i(h)\tfrac{\partial G(x,y,x_i,y_i)}{\partial\eta}\right).$$

Hence

$$h(x,y)-I(x,y)=\int\limits_{\partial\Omega}\tfrac{\partial G(x,y,\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}h(\xi,\eta)\,ds_{\xi,\eta}+$$

$$+\sum_{i=0}^{1}\left(\alpha_i(h)\,G(x,y,x_i,y_i)+\beta_i(h)\tfrac{\partial G(x,y,x_i,y_i)}{\partial\xi}+\gamma_i(h)\tfrac{\partial G(x,y,x_i,y_i)}{\partial\eta}\right).$$

Then the analogous function (18)

$$\begin{aligned}W(x,y)=\int\int\limits_{\Omega}G(x,y,\xi,\eta)\,f(\xi,\eta)\,d\xi d\eta+\int\limits_{\partial\Omega}\tfrac{\partial G(x,y,\xi,\eta)}{\partial\overline{n}_{\xi,\eta}}h(\xi,\eta)\,ds_{\xi,\eta}+\\+\sum_{i=0}^{1}\left(\alpha_i(h)\,G(x,y,x_i,y_i)+\beta_i(h)\tfrac{\partial G(x,y,x_i,y_i)}{\partial\xi}+\gamma_i(h)\tfrac{\partial G(x,y,x_i,y_i)}{\partial\eta}\right)\end{aligned}\tag{21}$$

gives the solution of the non-homogeneous Laplace equation in the punctured domain Ω_0. We formulate the result in the form of a separate statement.

Theorem 2. *A boundary value problem for the non-homogeneous Laplace equation in the punctured domain* Ω_0

$$\Delta W(x,y)=f(x,y)\tag{22}$$

with boundary conditions

$$W(x,y)|_{\partial\Omega} = h(x,y)|_{\partial\Omega},$$

$$\frac{1}{2}\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial W(x_i+\delta,y)}{\partial x}-\frac{\partial W(x_i-\delta,y)}{\partial x}\right]dy+$$
$$+\frac{1}{2}\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial W(x,y_i+\delta)}{\partial y}-\frac{\partial W(x,y_i-\delta)}{\partial y}\right]dx=\alpha_i(h),$$

$$\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[W(x_i-\delta,y)-W(x_i+\delta,y)\right]dy=\beta_i(h),$$

$$\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[W(x,y_i-\delta)-W(x,y_i+\delta)\right]dx=\gamma_i(h),\quad (i=0,1) \tag{23}$$

for any right-hand side of $f(x,y)\in L_2(\Omega)$ has a unique solution $W(x,y)$ from $\tilde{W}_2^1(\Omega_0)$, and it is given by formula (21). We have four internal boundary conditions in (23).

To prove Theorem 2, we give the following lemmas.

Lemma 1. *For any continuously differentiable function $F(x,y)$ the following equalities hold:*

$$\begin{aligned}&\frac{1}{2}\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial F(x_i+\delta,y)}{\partial x}-\frac{\partial F(x_i-\delta,y)}{\partial x}\right]dy+\\&+\frac{1}{2}\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial F(x,y_i+\delta)}{\partial y}-\frac{\partial F(x,y_i-\delta)}{\partial y}\right]dx=0\end{aligned} \tag{24}$$

$$\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[F(x_i-\delta,y)-F(x_i+\delta,y)\right]dy=0 \tag{25}$$

$$\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[F(x,y_i-\delta)-F(x,y_i+\delta)\right]dx=0. \tag{26}$$

Lemma 2. *For $G(x,y,x_i,y_i)$, $(i=0,1)$ the following equalities hold:*

$$\begin{aligned}&\frac{1}{2}\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial G(x_i+\delta,y,x_i,y_i)}{\partial x}-\frac{\partial G(x_i-\delta,y,x_i,y_i)}{\partial x}\right]dy+\\&+\frac{1}{2}\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial G(x,y_i+\delta,x_i,y_i)}{\partial y}-\frac{\partial G(x,y_i-\delta,x_i,y_i)}{\partial y}\right]dx=1\end{aligned} \tag{27}$$

$$\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[G\left(x_i-\delta,y,x_i,y_i\right)-G\left(x_i+\delta,y,x_i,y_i\right)\right]dy=0 \tag{28}$$

$$\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[G\left(x,y_i-\delta,x_i,y_i\right)-G\left(x,y_i+\delta,x_i,y_i\right)\right]dx=0. \tag{29}$$

Lemma 3. *For* $\frac{\partial G(x,y,x_i,y_i)}{\partial\xi}$, $(i=0,1)$ *the following equalities hold:*

$$\begin{gathered}\tfrac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\tfrac{\partial^2G(x_i+\delta,y,x_i,y_i)}{\partial\xi\partial x}-\tfrac{\partial^2G(x_i-\delta,y,x_i,y_i)}{\partial\xi\partial x}\right]dy+\\ +\tfrac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\tfrac{\partial^2G(x,y_i+\delta,x_i,y_i)}{\partial\xi\partial y}-\tfrac{\partial^2G(x,y_i-\delta,x_i,y_i)}{\partial\xi\partial y}\right]dx=0\end{gathered} \tag{30}$$

$$\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial G\left(x_i-\delta,y,x_i,y_i\right)}{\partial\xi}-\frac{\partial G\left(x_i+\delta,y,x_i,y_i\right)}{\partial\xi}\right]dy=1 \tag{31}$$

$$\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial G\left(x,y_i-\delta,x_i,y_i\right)}{\partial\xi}-\frac{\partial G\left(x,y_i+\delta,x_i,y_i\right)}{\partial\xi}\right]dx=0. \tag{32}$$

Lemma 4. *For* $\frac{\partial G(x,y,x_i,y_i)}{\partial\eta}$, $(i=0,1)$ *the following equalities hold:*

$$\begin{gathered}\tfrac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\tfrac{\partial^2G(x_i+\delta,y,x_i,y_i)}{\partial\eta\partial x}-\tfrac{\partial^2G(x_i-\delta,y,x_i,y_i)}{\partial\eta\partial x}\right]dy+\\ +\tfrac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\tfrac{\partial^2G(x,y_i+\delta,x_i,y_i)}{\partial\eta\partial y}-\tfrac{\partial^2G(x,y_i-\delta,x_i,y_i)}{\partial\eta\partial y}\right]dx=0\end{gathered} \tag{33}$$

$$\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial G\left(x_i-\delta,y,x_i,y_i\right)}{\partial\eta}-\frac{\partial G\left(x_i+\delta,y,x_i,y_i\right)}{\partial\eta}\right]dy=0 \tag{34}$$

$$\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial G\left(x,y_i-\delta,x_i,y_i\right)}{\partial\eta}-\frac{\partial G\left(x,y_i+\delta,x_i,y_i\right)}{\partial\eta}\right]dx=1. \tag{35}$$

Lemmas 1-4 are proved similarly as in [17].

Proof. We show that for $W\left(x,y\right)$ defined by the formula (21) the following equation (22) is true. The validity of (22) follows from the fact that the relations (16) are true, and also that $\Delta_{x,y}G\left(x,y,x_i,y_i\right)=0$, $\Delta_{x,y}\frac{\partial G(x,y,x_i,y_i)}{\partial\xi}=0$, and $\varDelta_{x,y}\frac{\partial G(x,y,x_i,y_i)}{\partial \eta} = 0$ for $(x,y) \neq (x_i,y_i)$, $(i=0,1)$. Verify that for $W(x,y)$ the first relation from (23) is true. Let $(x,y) \in \partial\varOmega$. Then from the properties of the Green's function $G(P,Q) = 0$, $P \in \partial\varOmega$, $Q \in \varOmega$ and relation $\frac{\partial}{\partial \bar{n}_{\xi,\eta}} G(x,y,\xi,\eta)\Big|_{(x,y)\in\partial\varOmega,(\xi,\eta)\in\partial\varOmega} = \delta_{\partial\varOmega}((x,y),(\xi,\eta))$ from [20] and $G(x,y,x_i,y_i) = 0$, $\frac{\partial G(x,y,x_i,y_i)}{\partial \xi} = 0$, $\frac{\partial G(x,y,x_i,y_i)}{\partial \eta} = 0$ for $(x,y) \in \partial\varOmega$ it follows the required first boundary relation from (23).

Denote by

$$F(x,y) = \int\int\limits_{\varOmega} G(x,y,\xi,\eta) f(\xi,\eta)\, d\xi d\eta + \int\limits_{\partial\varOmega} \frac{\partial G(x,y,\xi,\eta)}{\partial \bar{n}_{\xi,\eta}} h(\xi,\eta)\, ds_{\xi,\eta}, \quad (36)$$

then $F(x,y)$ satisfies the limit relations of Lemma 1. Indeed, $F(x,y)$ is a continuously differentiable function. Since the first term on the right-hand side of (36) is a solution of the Dirichlet problem for the non-homogeneous Laplace equation, then it can be differentiated twice. Consequently, the conditions of Lemma 1 for the first term on the right-hand side of (36) are satisfied. As (x,y) changes in the neighborhood (x_0,y_0), and (ξ,η) is on the external border, that is $(x,y) \neq (\xi,\eta)$, then $\frac{\partial G(x,y,\xi,\eta)}{\partial \bar{n}_{\xi,\eta}}$ function is continuously differentiable. The conditions of Lemma 1 for the second term on the right-hand side of (36) are satisfied.

Now we show the implementation of the second internal boundary condition from (23)

$$\frac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial W(x_i+\delta,y)}{\partial x} - \frac{\partial W(x_i-\delta,y)}{\partial x}\right] dy +$$

$$+\frac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial W(x,y_i+\delta)}{\partial y} - \frac{\partial W(x,y_i-\delta)}{\partial y}\right] dx =$$

$$= \frac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial F(x_i+\delta,y)}{\partial x} - \frac{\partial F(x_i-\delta,y)}{\partial x}\right] dy +$$

$$+\frac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial F(x,y_i+\delta)}{\partial y} - \frac{\partial F(x,y_i-\delta)}{\partial y}\right] dx +$$

$$+\frac{1}{2}\sum_{k=0}^{1}\alpha_k(h)\lim_{\delta\to+0}\left\{\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial G(x_i+\delta,y,x_k,y_k)}{\partial x} - \frac{\partial G(x_i-\delta,y,x_k,y_k)}{\partial x}\right] dy +\right.$$

$$\left.+\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial G(x,y_i+\delta,x_k,y_k)}{\partial y} - \frac{\partial G(x,y_i-\delta,x_k,y_k)}{\partial y}\right] dx\right\} +$$

$$+\frac{1}{2}\sum_{k=0}^{1}\beta_k(h)\lim_{\delta\to+0}\left\{\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial^2 G(x_i+\delta,y,x_k,y_k)}{\partial\xi\partial x}-\frac{\partial^2 G(x_i-\delta,y,x_k,y_k)}{\partial\xi\partial x}\right]dy+\right.$$

$$\left.+\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial^2 G(x,y_i+\delta,x_k,y_k)}{\partial\xi\partial y}-\frac{\partial^2 G(x,y_i-\delta,x_k,y_k)}{\partial\xi\partial y}\right]dx\right\}+$$

$$+\frac{1}{2}\sum_{k=0}^{1}\gamma_k(h)\lim_{\delta\to+0}\left\{\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial^2 G(x_i+\delta,y,x_k,y_k)}{\partial\eta\partial x}-\frac{\partial^2 G(x_i-\delta,y,x_k,y_k)}{\partial\eta\partial x}\right]dy+\right.$$

$$\left.+\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial^2 G(x,y_i+\delta,x_k,y_k)}{\partial\eta\partial y}-\frac{\partial^2 G(x,y_i-\delta,x_k,y_k)}{\partial\eta\partial y}\right]dx\right\},\quad (i=0,1).$$

Respectively using the limit equalities (24), (27), (30), (33) from Lemmas 1, 2, 3 and 4, we obtain the required second equality in (23), then

$$\frac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial W(x_i+\delta,y)}{\partial x}-\frac{\partial W(x_i-\delta,y)}{\partial x}\right]dy+$$
$$+\frac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial W(x,y_i+\delta)}{\partial y}-\frac{\partial W(x,y_i-\delta)}{\partial y}\right]dx=\alpha_i(h).$$

Since the above function $F(x,y)$ is equal

$$F(x,y)=\int\int\limits_{\Omega}G(x,y,\xi,\eta)\,f(\xi,\eta)\,d\xi d\eta+\int\limits_{\partial\Omega}\frac{\partial G(x,y,\xi,\eta)}{\partial\bar{n}_{\xi,\eta}}h(\xi,\eta)\,ds_{\xi,\eta},$$

then $F(x,y)$ satisfies the limit relations of Lemma 1. This statement has been proved in the above proof of the second boundary condition from (23). Let us check implementation of the third internal boundary condition from (23)

$$\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[W(x_i-\delta,y)-W(x_i+\delta,y)\right]dy=$$

$$=\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[F(x_i-\delta,y)-F(x_i+\delta,y)\right]dy+$$

$$+\sum_{k=0}^{1}\alpha_k\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[G(x_i-\delta,y,x_k,y_k)-G(x_i+\delta,y,x_k,y_k)\right]dy+$$

$$+\sum_{k=0}^{1}\beta_k\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial G(x_i-\delta,y,x_k,y_k)}{\partial\xi}-\frac{\partial G(x_i+\delta,y,x_k,y_k)}{\partial\xi}\right]dy+$$

$$+\sum_{k=0}^{1}\gamma_k\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial G(x_i-\delta,y,x_k,y_k)}{\partial\eta}-\frac{\partial G(x_i+\delta,y,x_k,y_k)}{\partial\eta}\right]dy,\quad (i=0,1).$$

Respectively using the limit equalities (25), (28), (31), (34) from Lemmas 1, 2, 3 and 4, we obtain the required third equality in (23), then

$$\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[W\left(x_i-\delta,y\right)-W\left(x_i+\delta,y\right)\right]dy=\beta_i\left(h\right).$$

Since the above function $F\left(x,y\right)$ is equal

$$F\left(x,y\right)=\int\int\limits_{\Omega}G\left(x,y,\xi,\eta\right)f\left(\xi,\eta\right)d\xi d\eta+\int\limits_{\partial\Omega}\frac{\partial G\left(x,y,\xi,\eta\right)}{\partial\bar{n}_{\xi,\eta}}h\left(\xi,\eta\right)ds_{\xi,\eta},$$

then $F\left(x,y\right)$ satisfies the limit relations of Lemma 1. This statement has been proved in the above proof of the second boundary condition from (23). Let us check implementation of the fourth internal boundary condition from (23)

$$\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[W\left(x,y_i-\delta\right)-W\left(x,y_i+\delta\right)\right]dx=$$

$$=\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[F\left(x,y_i-\delta\right)-F\left(x,y_i+\delta\right)\right]dx+$$

$$+\sum_{k=0}^{1}\alpha_k\left(h\right)\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[G\left(x,y_i-\delta,x_k,y_k\right)-G\left(x,y_i+\delta,x_k,y_k\right)\right]dx+$$

$$+\sum_{k=0}^{1}\beta_k\left(h\right)\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial G(x,y_i-\delta,x_k,y_k)}{\partial\xi}-\frac{\partial G(x,y_i+\delta,x_k,y_k)}{\partial\xi}\right]dx+$$

$$+\sum_{k=0}^{1}\gamma_k\left(h\right)\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial G(x,y_i-\delta,x_k,y_k)}{\partial\eta}-\frac{\partial G(x,y_i+\delta,x_k,y_k)}{\partial\eta}\right]dx,\quad\left(i=0,1\right).$$

Respectively using the limit equalities (26), (29), (32), (35) from Lemmas 1, 2, 3 and 4, we obtain the required fourth equality in (23), then

$$\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[W\left(x,y_i-\delta\right)-W\left(x,y_i+\delta\right)\right]dx=\gamma_i\left(h\right).$$

Problem (22) - (23) in the punctured domain has a unique solution at $f\left(x,y\right)\equiv 0$, $h\left(x,y\right)\equiv 0$. This follows from the theorem on the removable singularity of a harmonic function [22]. ◻

4 Main result

Further, we consider operators K (may be nonlinear), mapping the elements of the space $L_2\left(\Omega\right)$ into the elements of space $\tilde{W}_2^1\left(\Omega_0\right)$ and continuous in the sense of L_2:

If the sequence of norms $\|f_j\|_{L_2(\Omega)}$ tends to zero as $j \to \infty$, then $\|Kf_j\|_{L_2(\Omega)}$ also tends to zero as $j \to \infty$.

In this case we will say that K is continuous in the sense of L_2, which maps $L_2(\Omega)$ into $\tilde{W}_2^1(\Omega_0)$. And we write $h = K(f)$.

Now we show how, using Theorem 2, we can obtain new boundary well-posed solvable problems for the non-homogeneous Laplace solution in the punctured domain Ω_0. For this it is sufficient that the function $h(x,y) \in \mathscr{D}$ continuously depends on the function $f(x,y) \in L_2(\Omega)$.

Suppose that there exists operator K continuous in the sense of L_2, mapping $f(x,y) \in L_2(\Omega)$ into $h(x,y) \in \mathscr{D}$. Let $h = K(f)$. Then problem (22) - (23) takes the form

$$\Delta W(x,y) = f(x,y),\ (x,y) \in \Omega_0 \tag{37}$$

$$W(x,y)|_{\partial\Omega} - K(\Delta W)|_{\partial\Omega} = 0,$$

$$\frac{1}{2}\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial W(x_i+\delta,y)}{\partial x} - \frac{\partial W(x_i-\delta,y)}{\partial x}\right]dy+$$

$$+\frac{1}{2}\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial W(x,y_i+\delta)}{\partial y} - \frac{\partial W(x,y_i-\delta)}{\partial y}\right]dx = \alpha_i(K(\Delta W)),$$

$$\lim_{\delta\to+0}\int_{y_i-\delta}^{y_i+\delta}\left[W(x_i-\delta,y) - W(x_i+\delta,y)\right]dy = \beta_i(K(\Delta W)),$$

$$\lim_{\delta\to+0}\int_{x_i-\delta}^{x_i+\delta}\left[W(x,y_i-\delta) - W(x,y_i+\delta)\right]dx = \gamma_i(K(\Delta W)),\ \ (i=0,1). \tag{38}$$

The conditions (38) imposed on the function $W(x,y)$ can be interpreted as additional conditions in order that equation (37) for any right-hand side of $f(x,y) \in L_2(\Omega)$ has a unique solution. Thus, problem (37) - (38) represents a well-posed solvable problem with an internally "boundary" condition of the form (38).

We also need the concept of a stable solution in the sense of L_2: The solution of problem (37) - (38) will be called stable in the sense of L_2, if from the fact that the sequence of norms of the right-hand sides of $\|f_j\|_{L_2(\Omega)}$ tends to zero as $j \to \infty$ it follows that the sequence of norms of solution $\|W_j\|_{L_2(\Omega)}$ tends to zero. So, it is fair

Theorem 3. *For any operator K continuous in the sense of L_2 mapping the space $\{f\} \in L_2(\Omega)$ into the set of smooth functions $h(x,y) \in \mathscr{D}$ the problem (37) - (38) has a unique stable solution $W(x,y) \in \tilde{W}_2^1(\Omega_0)$ in the sense of L_2 for all right-hand sides of f from $L_2(\Omega)$*

Now we prove the converse statement.

Theorem 4. *If equation (37) for all right-hand sides of f from $L_2(\Omega)$ with some additional conditions has a unique stable solution $W(x,y) \in \tilde{W}_2^1(\Omega_0)$ in the sense of L_2. Then there exists an operator K continuous in the sense of L_2, mapping the space $\{f\} \in L_2(\Omega_0)$ into the set of smooth functions $\{h\} \in \mathscr{D}$, such that the additional condition is equivalent to the condition of the form (38) with the operator K.*

Proof. Suppose that equation (37) with some additional conditions is uniquely solvable for any right-hand side of $f(x,y) \in L_2(\Omega)$. The corresponding unique solution we denote by $W(x,y,f)$. Introduce the function $u(x,y,f) = \int\int\limits_{\Omega} G(x,y,\xi,\eta) f(\xi,\eta)\, d\xi d\eta$ and make up the difference

$$v(x,y) = W(x,y,f) - u(x,y,f). \tag{39}$$

It is clear that $v(x,y)$ is a solution of the homogeneous equation $\Delta v = 0$ and is uniquely determined by $f \in L_2(\Omega)$. Thus, to any element of $f \in L_2(\Omega)$ there corresponds a single function v, which is a sufficiently smooth function and is a harmonic function. **By K denote an operator, putting each $f \in L_2(\Omega)$ in accordance with $v \in \tilde{W}_2^1(\Omega_0)$, that is** $v = K(f)$. Consider an entirely new function according to the formula

$$\begin{aligned} w(x,y) = u(x,y,f) + \int\limits_{\partial\Omega} \tfrac{\partial G(x,y,\xi,\eta)}{\partial \overline{n}_{\xi,\eta}} v(\xi,\eta)\, ds_{\xi,\eta} + \\ + \sum_{i=0}^{1} \Big(\alpha_i(\nu) G(x,y,x_i,y_i) + \beta_i(\nu) \tfrac{\partial G(x,y,x_i,y_i)}{\partial \xi} + \gamma_i(\nu) \tfrac{\partial G(x,y,x_i,y_i)}{\partial \eta} \Big), \end{aligned} \tag{40}$$

where

$$\alpha_i(\nu) = \tfrac{1}{2} \lim_{\delta \to +0} \int\limits_{y_i-\delta}^{y_i+\delta} \left[\tfrac{\partial v(x_i+\delta,\eta)}{\partial \xi} - \tfrac{\partial v(x_i-\delta,\eta)}{\partial \xi} \right] d\eta +$$

$$+ \tfrac{1}{2} \lim_{\delta \to +0} \int\limits_{x_i-\delta}^{x_i+\delta} \left[\tfrac{\partial v(\xi,y_i-\delta)}{\partial \eta} - \tfrac{\partial v(\xi,y_i+\delta)}{\partial \eta} \right] d\xi,$$

$$\beta_i(\nu) = \lim_{\delta \to +0} \int\limits_{y_i-\delta}^{y_i+\delta} \left[v(x_i-\delta,\eta) - v(x_i+\delta,\eta) \right] d\eta,$$

$$\gamma_i(\nu) = \lim_{\delta \to +0} \int\limits_{x_i-\delta}^{x_i+\delta} \left[v(\xi,y_i-\delta) - v(\xi,y_i+\delta) \right] d\xi.$$

Formula (40) is similar to formula (21). In this case, the function $v(x,y)$ plays the role of $h(x,y)$. Consequently, the above arguments from Theorem 2 show that

$$\Delta w(x,y) = f(x,y) \tag{41}$$

$$w(x,y)|_{\partial\Omega} = v(x,y)|_{\partial\Omega},$$

$$\frac{1}{2}\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[\frac{\partial w(x_i+\delta,y)}{\partial x}-\frac{\partial w(x_i-\delta,y)}{\partial x}\right]dy+$$

$$+\frac{1}{2}\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[\frac{\partial w(x,y_i+\delta)}{\partial y}-\frac{\partial w(x,y_i-\delta)}{\partial y}\right]dx=\alpha_i(\nu)$$

$$\lim_{\delta\to+0}\int\limits_{y_i-\delta}^{y_i+\delta}\left[w(x_i-\delta,y)-w(x_i+\delta,y)\right]dy=\beta_i(\nu)\,,\tag{42}$$

$$\lim_{\delta\to+0}\int\limits_{x_i-\delta}^{x_i+\delta}\left[w(x,y_i-\delta)-w(x,y_i+\delta)\right]dx=\gamma_i(\nu)\,,\quad(i=0,1)\,,$$

where $v(x,y)=K(f)$ or $v(x,y)=K(\Delta w)$.

On the other hand, from the representation (39) it follows that $W(x,y,f)=u(x,y,f)+v(x,y)$ also satisfies the relation (41). Therefore, from the uniqueness theorem it follows that $W(x,y,f)=w(x,y)$. Consequently, the additional conditions for unique solvability have the form (42). □

Note that by the Riesz theorem [27] on the general form of a bounded linear functional $h=K(\Delta W)$ in a Hilbert space $L_2(\Omega)$ takes the form (6)-(7).

The resulting explicit solution of the problem makes it possible to analyze the behavior of an analytic solution in the neighborhood of singular points and to perform a comparative numerical analysis. For clarity in this section, we will look at a few examples.

Example 1. Let $f(x,y)=0$, $M_0=\left(\frac{1}{2},\frac{1}{4}\right)$, $M_1=\left(\frac{1}{2},-\frac{1}{2}\right)$, $\alpha_0(h)=2$, $\alpha_1(h)=1$, $h(x,y)|_{\partial\Omega}=0$, $\beta_i(h)=0$, $\gamma_i(h)=0$, $(i=0,1)$. (fig 1. a)

Example 2. Let $f(x,y)=0$, $M_0=\left(-\frac{1}{2},-\frac{1}{2}\right)$, $M_1=\left(\frac{1}{2},-\frac{1}{2}\right)$, $\alpha_0(h)=1$, $\alpha_1(h)=-2$, $h(x,y)|_{\partial\Omega}=0$, $\beta_i(h)=0$, $\gamma_i(h)=0$, $(i=0,1)$. (fig 1. b)

a) b)

Fig. 1. a) Two jumps are positive, b) one jump is positive and the other one is negative

Acknowledgements. Research supported by the target program 0085/PTSF-14 of the Ministry of Education and Science of Republic of Kazakhstan.

References

1. Akhymbek, M.E., Nurakhmetov, D.B.: The first regularized trace for the two-fold differentiation operator in a punctured segment. Sib. Elektron. Mat. Izv. **11**, 626–633 (2014)
2. Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics. Springer (1988)
3. Aniyarov, A.A.: Resolvent well posed inner boundary problems for Helmholtz equation in the punctured region. In: The 4th Congress of the Turkic World Mathematical Society (TWMS) Baku, Azerbaijan. 165 (2011)
4. Atkinson, F.V.: Discrete and Continuous Boundary Value Problems. Academic Press, New York (1964)
5. Berezin, F.A., Faddeev, L.D.: Remark on the Schrödinger equation with singular potential. (Russian). Dokl. Akad. Nauk SSSR. **137**, 1011–1014 (1961)
6. Berikhanova, G.E., Kanguzhin, B.E.: Resolvent of finite-dimensional perturbed of the correct problems for the biharmonic operator. Ufimsk. Mat. Zh. **2**(1), 17–34 (2010)
7. Bitsadze, A.V.: Equations of Mathematical Physics. Nauka, Moscow (1976). [in Russian]
8. Dezin, A.A.: General Questions of the Theory of Boundary Value Problems. Nauka, Moscow (1980). [in Russian]
9. Gladwell, G.h.L.: lnverse Problems in Vibration. Kluwer Academic Publishers, New York (2005)
10. Kal'menov, TSh: Regular boundary value problems for the wave equation. Differ. Uravn. **17**(6), 1105–1121 (1981)
11. Kalmenov, TSh, Koshanov, B.D., Nemchenko, M.Y.: Green function representation for the Dirichlet problem of the polyharmonic equation in a sphere. Complex Var. Ell. Equ. **53**, 177–183 (2008)
12. Kal'menov, TSh, Otelbaev, M.O.: Regular boundary value problems for the Lavrent'evBicadze equation. Differ. Uravn. **17**(5), 873–885 (1981)
13. Kanguzhin, B.E., Aniyarov, A.A.: Well-Posed problems for the Laplace operator in a punctured disk. Math. Notes. **89**(6), 819–829 (2011)
14. Kanguzhin, B.E., Nurakhmetov, D.B.: Correct boundary value problems for 2-order in nonhomogeneous differential equation with variable coeffcients. Bull. KazNU, Ser. Math., Mech., Inf. **1**(64), 9–26 (2010)
15. Kanguzhin, B.E., Nurakhmetov, D.B.: Boundary value problems for 2nd order nonhomogeneous differential equations with variable coefficients. J. Xinjiang Univ. (Nat. Sci. Edn.) **28**(1), 46–56 (2011)
16. Kanguzhin, B.E., Nurakhmetov, D.B., Tokmagambetov, N.E.: On green function's properties. Int. J. Math. Anal. **15**(7), 747–753 (2013)
17. Kanguzhin, B.E., Nurakhmetov, D.B., Tokmagambetov, N.E.: Laplace operator with δ-like potentials. Russ. Math. **58**(2), 6–12 (2014)
18. Kanguzhin, B.E., Tokmagambetov, N.E.: A regularized trace formula for a well-perturbed Laplace operator. Dokl. Akademii Nauk. **460**(1), 7–10 (2015)
19. Kanguzhin, B.E., Tokmagambetov, N.E.: Resolvents of well-posed problems for finite-rank perturbations of the polyharmonic operator in a punctured domain. Sibirsk. Mat. Zh. **57**(2), 338–349 (2016)

20. Kurant, R., Gilbert, D.: Methods of Mathematical Physics, vol. 1 (1933)
21. Neiman-Zade, M.I., Shkalikov, A.A.: Schrödinger operators with singular potentials from spaces of multipliers. Math. Notes. **66**(5), 599–607 (1999)
22. Oleinik, O.A.: Lectures on Partial Differential Equations. Binom, Moscow (2005). [in Russian]
23. Sadybekov, M.A., Torebek, B.T., Turmetov, BKh: Representation of Green's function of the Neumann problem for a multidimensional ball. Complex Var. Ell. Equ. **61**(1), 104–123 (2016)
24. Savchuk, A.M.: The first-order regularized trace of the Sturm-Liouville operator with δ-potential. Russ. Math. Surveys. **55**(6), 1168–1169 (2000)
25. Savchuk, A.M., Shkalikov, A.A.: Trace formula for Sturm-Liouville operators with singular potentials. Math. Notes. **69**(3), 377–390 (2001)
26. Savchuk, A.M., Shkalikov, A.A.: On the eigenvalues of the Sturm-Liouville operator with potentials from Sobolev spaces. Math. Notes. **80**(5), 814–832 (2006)
27. Riesz, F., Sz.-Nagy, B.: Functional Analysis. Dover Publications (1990)

Spectral Properties of Degenerate High-Order Differential Operator

Kordan N. Ospanov(✉)

L.N. Gumilyov Eurasian National University, 2 Satpaev st., Astana 010008, Kazakhstan
kordan.ospanov@gmail.com

Abstract. In this paper we investigate a singular high-order differential operator with rapidly growing intermediate coefficients. We give sufficient conditions for complete continuity of its resolvent in the space $L_2(-\infty, +\infty)$. Furthermore, we show that this resolvent belongs to the Schatten class σ_p, $1 < p < \infty$, and give the uniform estimate for the resolvent norm.

Keywords: Singular differential operator · Separable operator · Discrete spectrum · Resolvent · Schatten class · Hilbert-Schmidt operator · Nuclear operator

1 Introduction

We consider the following linear differential operator with variable coefficients

$$L_0 y = y^{(2n)} + a_1 y^{(2n-1)} + a_2 y^{(2n-2)} + ... + a_{2n} y$$

defined on the set $C_0^{(2n)}(R)$, $R = (-\infty, +\infty)$ of $2n$-times continuously differentiable functions with compact support. We denote by L its closure in the space $L_2 = L_2(R)$.

There are questions: when the spectrum of L is discrete and its resolvent belongs to Schatten class under some conditions for the coefficients? These questions are important goals of the spectral theory of operators. In present paper, we study above questions for the singular nonselfadjoint differential operator L. The singularity of the operator L is that its coefficients are unbounded on R.

Series of practical problems lead to study the linear operator L in an unbounded domain. It is well-known that a representative of these operators is the Sturm-Liouville operator. This is the fundamental operator of quantum mechanics. A large number of papers was devoted to the questions of self-adjoint and spectral properties of the Sturm-Liouville operator. We note that in the general case, the operator L with unbounded coefficients can not lead to self-adjoint

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_27

form. The singular higher-order operator L began to be studied in the first half of the twentieth century. Powerful methods for constructing the asymptotics of the eigenfunctions, the study of the Green's functions and computing the index of the defect have been developed in [1–5]. However, the results of these works were obtained only in the case, where the operator L is self-adjoint and intermediate coefficients a_s ($s = 1, 2, \dots, 2n-1$) are nearly constant or at infinity they are controlled above by some degree of $|a_{2n}|$. We are not impose similar restrictions on a_s ($s = 1, 2, \dots, 2n-1$) in this work, so previous methods not useful for this case.

We give results on the compactness of the resolvent L^{-1} of L, as well as estimates for its singular numbers. Our results (Theorem 1, Theorem 2 and its corollaries) extend some results of [4,6] on the differential operators with rapidly growing intermediate coefficients.

For the Sturm-Liouville operator (the case $n = 1$), the problems of compactness of the resolvent and when it belongs to the Schatten class are studied in a lot of works (see [6,7]) and the references therein. These problems for other differential operators were considered in [6,8–11,14].

2 Auxillary Statements

First, we give sufficient conditions of continuous invertibility of L obtained in [12]. Let

$$\alpha_{g,h_l}(t) = \left[\int_0^t |g(s)|^2 ds\right]^{1/2} \left[\int_t^{+\infty} \theta^{2(l-1)}|h(\theta)|^{-2} d\theta\right]^{1/2} (t > 0),$$

$$\beta_{g,h_l}(\tau) = \left[\int_\tau^0 |g(\nu)|^2 d\nu\right]^{1/2} \left[\int_{-\infty}^{\tau} \xi^{2(l-1)}|h(\xi)|^{-2} d\xi\right]^{1/2} (\tau < 0),$$

$$\gamma_{g,h_l} = \max\left(\sup_{t>0} \alpha_{g,h_l}(t), \sup_{\tau<0} \beta_{g,h_l}(\tau)\right) (l = 1, 2, \dots, 2n-1),$$

where g and h are given functions. Let $C^{(k)}(R)$ ($k \in N$) be the set of all bounded functions with bounded continuous derivatives up to order k. Let $C^{(k)}_{loc}(R)$ denote the set of all f such that $\psi f \in C_0^{(k)}(R)$ for all $\psi \in C_0^{(k)}(R)$ ($k \in N$).

Lemma 1 [12]. Let the coefficients $a_j \in C^{(2n-j)}_{loc}(R)$ ($j = 1, 2, \dots, 2n$) satisfy the following conditions:

$$|a_1| \geq 1, \gamma_{1,\left(\sqrt{|a_1|}\right)_{2n-1}} < \infty, \gamma_{a_k,(a_1)_{k-1}} < +\infty \ (k = 2, 3, \dots, 2n). \quad (1)$$

Then L is invertible bounded , and the inverse L^{-1} is defined on entire space L_2 and for any $y \in D(L)$ the following estimate holds:

$$\left\| \sqrt{|a_1|} y^{(2n-1)} \right\|_2 + \sum_{j=1}^{2n-2} \left\| a_j y^{(2n-j)} \right\|_2 \leq c_1 \left\| Ly \right\|_2 ,$$

where $\| \cdot \|_2$ is the norm in L_2.

Lemma 2 [12]. Let $a_j \in C_{loc}^{(2n-j)}(R)$ $(j = 1, 2, ..., 2n)$ satisfy the conditions (1). If there is a constant $c > 1$ such that

$$c^{-1} \leq \frac{a_1(x)}{a_1(\eta)} \leq c \tag{2}$$

for all $x, \eta \in R: \ |x - \eta| \leq 1$ holds, then for $y \in D(L)$ the following estimate holds:

$$\left\| y^{(2n)} \right\|_2 + \sum_{j=1}^{2n} \left\| a_j y^{(2n-j)} \right\|_2 \leq c_2 \left\| Ly \right\|_2 \tag{3}$$

Equation (3) is called a separability estimate, and if (3) holds, then L is called a separable operator in L_2. Lemma 2 shows that under the conditions (1) and (2) the inverse L^{-1} is bounded from L_2 to the following weighted Sobolev space

$$W_{2,\,a}^{2n}(R) = \left\{ y \in L_2 : \ \left\| y^{(2n)} \right\|_2 + \sum_{j=1}^{2n} \left\| a_j y^{(2n-j)} \right\|_2 < +\infty \right\}.$$

3 Main Results

We give some conditions under which the resolvent L^{-1} has some spectral and approximation properties.

Theorem 1. *Let $a_j \in C_{loc}^{(2n-j)}(R)$ $(j = 1, 2, ..., 2n)$ satisfy (1) and (2). If*

$$\lim_{|x| \to +\infty} |a_{2n}(x)| = +\infty \tag{4}$$

or

$$\lim_{t \to +\infty} \alpha_{1,\,(a_s)_{2n-s}}(t) = 0, \lim_{\tau \to -\infty} \beta_{1,\,(a_s)_{2n-s}}(\tau) = 0 \tag{5}$$

hold for at least one s of $s = 1, 2, ..., 2n-1$, then L^{-1} is completely continuous operator in the space L_2.

To prove this theorem we consider the set $M = \{y \in D(L) : \|Ly\|_2 \leq 1\}$. By Lemma 2, M is bounded in $W_{2,\,a}^{2n}(R)$. Using Theorem 3 in [13] (Chapter IX), we obtain the desired result.

Under the conditions of this theorem the self-adjoint positive definite operator $\left(L^{-1}\right)^* \left(L^{-1}\right)$ is also completely continuous. The existence of the operator $\left(L^{-1}\right)^*$ conjugate to the operator L^{-1} follows from the smoothness of the coefficients a_j $(j = 1, 2, ..., 2n)$.

We number the eigenvalues of the operator $\sqrt{(L^{-1})^{*}(L^{-1})}$ in decreasing order in accordance with their multiplicity and denote by $s_k\left(L^{-1}\right)$. They are called s-numbers of the operator L^{-1}. It is known that

$$s_k\left(L^{-1}\right)=\inf_{K\in T_{k-1}}\left\|L^{-1}-K\right\|_{L_2\to L_2},\quad k=1,\ 2,\ ...,$$

where T_{k-1} is a set of operators, whose ranks do not exceed $k-1$. If

$$\sum_{k=1}^{\infty}\left[s_k\left(L^{-1}\right)\right]^p<\infty,$$

then we say that the operator L^{-1} belongs to the class σ_p. In this case L^{-1} is called a finite type operator. Operators of the class σ_2 are called Hilbert-Schmidt operators, and σ_1 is a class of nuclear operators. We denote by $W_{2,\,a}^{2n}(R)$ the Banach space with norm

$$\left\|y^{(2n)}\right\|_2+\sum_{j=1}^{2n}\left\|a_j y^{(2n-j)}\right\|_2.$$

Under the conditions of Lemma 2, the operator L^{-1} is bounded from L_2 to $W_{2,\,a}^{2n}(R)$, in additional, if the conditions (4) or (5) are fulfilled, then it is completely continuous.

In the next statement we find sufficient conditions such that the resolvent L^{-1} of a singular high-order differential operator L belongs to Schatten class σ_p, $1\le p<\infty$, and we give the uniform estimate of the resolvent norm.

Theorem 2. *Suppose that $a_j\in C_{loc}^{(2n-j)}(R)$ $(j=1,\ 2,\ ...,\ 2n)$ satisfy the conditions (1), (2) and (4). Let $4n\theta>1$ and*

$$\int_{-\infty}^{+\infty}\frac{dx}{|a_{2n}(x)|^{\theta\,-\,1/4n}}<\infty.$$

Then $L^{-1}\in\sigma_\theta$ and

$$\left\{\sum_{k=1}^{\infty}\left[s_k\left(L^{-1}\right)\right]^{\theta}\right\}^{1/\theta}\le C\left\|\,|a_{2n}(\cdot)|^{\theta-1/4n}\,\right\|_{L_1(R)}^{1/\theta}.$$

This statement implies the following:

Corollary 1. Let $a_j\in C_{loc}^{(2n-j)}(R)$ $(j=1,\ 2,\ ...,\ 2n)$ satisfy the conditions (1), (2) and (4). Then L^{-1} is a nuclear operator, if

$$\int_{-\infty}^{+\infty}\frac{dx}{|a_{2n}(x)|^{1\,-\,1/4n}}<\infty.$$

For the trace of the operator L^{-1} the following estimate holds:

$$\sum_{k=1}^{\infty} s_k\left(L^{-1}\right) \le C \left\| \, |a_{2n}(\cdot)|^{-1+1/4n} \right\|_{L_1(R)}.$$

Corollary 2. Let $a_j \in C_{loc}^{(2n-j)}(R)$ $(j = 1, 2, ..., 2n)$ satisfy the conditions (1), (2) and

$$\lim_{|x|\to+\infty} |a_{2n}(x)| = +\infty.$$

Then L^{-1} is a Hilbert-Schmidt operator, if

$$\int_{-\infty}^{+\infty} \frac{dx}{|a_{2n}(x)|^{2-1/4n}} < \infty.$$

For the Hilbert-Schmidt norm of the operator L^{-1} the following estimate holds:

$$\left\{\sum_{k=1}^{+\infty} \left[s_k\left(L^{-1}\right)\right]^2\right\}^{1/2} \le C \left\| \, |a_{2n}(\cdot)|^{-2+1/4n} \right\|_{L_1(R)}^{1/2}.$$

Example. We consider the following minimal closed operator

$$l_0 y = y^{(4)} + (1+x^2)^5 y^{(3)} + (x^4-6)y' + (3x^2+5)y$$

in the space L_2. Since the coefficients are fast growing, intermediate terms y''' and y' in the expression $l_0 y$ does not obey to the operator $ly = y^4 + (3x^2+5)y$ in the operational sense. Therefore, l_0 is similar to L, which we considered above.

A simple verification shows that all conditions of Theorem 1 are satisfied. So, there is a bounded inverse l_0^{-1} of l_0 and it is completely continuous in the space L_2. It is easy to check that

$$\int_{-\infty}^{+\infty} \frac{dx}{(3x^2+5)^{\,1-1/8}} < \infty.$$

By Corollary 1, l_0^{-1} is the nuclear operator, and the following estimate holds:

$$\sum_{k=1}^{\infty} s_k\left(L^{-1}\right) \le C \int_{-\infty}^{+\infty} \frac{dx}{(3x^2+5)^{1-1/8}}.$$

Acknowledgements. This work is partially supported by project 5132/GF4 of the Science Committee of Ministry of Education and Science of the Republic of Kazakhstan and by L.N. Gumilyov Eurasian National University Research Fund. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Boimatov, K.k.: Separability theorems, weighted spaces and their applications. Proc. Steklov Inst. Math. **170**, 37–76 (1984)
2. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)
3. Fedoryuk, M.V.: Asymptotic Analysis: Linear Ordinary Differential Equations. Springer, Berlin (1993)
4. Glazman, I.M.: Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators. Israel Program for Scientific Translation, Jerusalem (1965)
5. Muratbekov, M.B.: Separability of an operator of mixed type and the completeness of its root vectors. Differ. Equ. **27**, 1517–1526 (1991)
6. Muratbekov, M.B., Otelbaev, M.: Smoothness and approximation properties for solutions of a class of nonlinear equations of Schrodinger type. Sov. Math. (Iz. VUZ). **33**, 68–74 (1989)
7. Mynbaev, K.T., Otelbaev, M.: Weighted function spaces and the spectrum of differential operators. Nauka, Moscow (1988) (in Russian)
8. Naimark, M.A.: Linear Differential Operators, vol. I. II. Ungar, New York (1968)
9. Ospanov, K.: Coercive estimates for a degenerate elliptic system of equations with spectral applications. Appl. Math. Lett. **24**, 1594–1598 (2011)
10. Ospanov, K.N.: Qualitative and approximate characteristics of solutions of Beltrami-type systems. Complex Var. Elliptic Equ. **60**, 1005–1014 (2015)
11. Ospanov, K.N.: Nonlocal estimates for solutions of a singular higher order differential equation. In: AIP Conference Proceedings. vol. 1759, 020033 (2016). https://doi.org/10.1063/1.4959647
12. Ospanov, K.N., Akhmetkaliyeva, R.D.: Separation and the existence theorem for second order nonlinear differential equation. Electron. J. Qual. Th. Differ. Equ. **66**, 1–12 (2012)
13. Otelbaev, M.: Imbedding theorems for spaces and their applications in the study of the spectrum of the schrodinger operator. Proc. Steklov Inst. Math. **4**, 281–321 (1981)
14. Otelbaev, M.: Estimates for the spectr of the Sturm-Liouville operator. Gylym, Alma-Ata (1991) (in Russian)

On Compactness of Resolvent of a First Order Singular Differential Operator in Bounded Vector-Valued Function Space

Myrzagali N. Ospanov(✉)

L.N. Gumilyov Eurasian National University, 2 Satpaev st., Astana 010008, Kazakhstan
myrzan66@mail.ru

Abstract. In this paper we give sufficient conditions for complete continuity of resolvent of a differential operator corresponding to a system of first order singular differential equations. Using coercive estimates for the solution of the above differential equation, we obtain the main result.

Keywords: Resolvent of a differential operator · Singular differential equation · Parameterization method · Diagonal dominance · Vector-valued function space · Compactness

1 Introduction

It is well known that if $A(t)$ is the $(n \times n)$-matrix with continuous elements, and $-\infty < a < b < \infty$, then the resolvent of the linear differential operator

$$L = \frac{d}{dt} - A(t)I_n,$$

in the space $C([a,b], R^n)$ is compact, where $R = (-\infty, +\infty)$ and I_n is an identity $(n \times n)-$matrix.

There is natural question whether the result also holds for the case $a = -\infty$ and $b = +\infty$? In this paper we study the operator L in continuous and bounded vector-valued function space $C_b(R, R^n)$, where $A(t) = (a_{i,j}(t))_{i,j=1}^n$ is the $(n \times n)-$ matrix with continuous, in general, not bounded elements.

The compactness is one of main problems in the theory of bounded linear operators in a Banach space. For differential operator usually it is considered the compactness of its resolvent. The compactness of the resolvent allows to apply the approximate methods for solving the corresponding differential equation $LV = F$. The compactness of the resolvent of a self-adjoint Dirac operator was investigated by Dzhumabaev [1]. Dzhumabaev [2] has studied the more general self-adjoint system of differential equations. With respect to the compactness

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_28

of resolvents of the singular elliptic operators we refer to [3,4,7–10] and the references therein. But these works are devoted to the Hilbert case. There is a growing interest to the study of non-selfadjoint operators and the case of non-Hilbert space.

First, we give sufficient conditions for the existence and uniqueness of a bounded and continuous solution $V = (V_1,\, V_2,\, ...,\, V_n)$ of the equation $LV = F$. Under some conditions we prove that the solution V has bounded derivative and satisfies the following estimate:

$$\left\| \frac{dV}{dt} \right\|_1 + \|AV\|_1 \le C_0 \|F\|_1 ,$$

where $\|V(\cdot)\|_1 = \sup\limits_{t\in R} \|V(t)\| = \sup\limits_{t\in R} \max\limits_{i=\overline{1,n}} |V_i(t)|$.

Using this result, we prove that the inverse L^{-1} is compact.

2 Auxillary Statements

We denote by $C_b(R,\, R^n)$ the set of all continuous and bounded vector-valued functions on R. We consider the following system of differential equations

$$\frac{dV}{dt} = A(t)V + F(t),\, t \in R, \tag{1}$$

where $V = (V_1(t),\, V_2(t),\, ...,\, V_n(t))$. We assume that elements of the $(n \times n)-$ matrix $A(t) = (a_{i,j}(t))_{i,j=1}^n$ and the vector-valued function $F(t)$ are continuous. We only study the solution of (8), which satisfies the following condition

$$V(t) \in C_b(R,\, R^n). \tag{2}$$

A continuously differentiable function $V(t) \in C_b(R,\, R^n)$ is called a solution of the problem (8), (2), if it satisfies the system (8) for all $t \in R$.

Let the matrix $A(t)$ satisfy the following conditions:

$i)$ the diagonal dominance holds by rows and a continuous function $\theta(t) \ge \theta_0 > 0$, i.e.,

$|a_{ii}(t)| \ge \sum\limits_{j\ne i} |a_{ij}(t)| + \theta(t) \quad (i = \overline{1,n})$, where $\theta(t) \ge \theta_0 > 0$;

$ii)$ $\theta(t) \ge \eta\, |a_{ii}(t)| \quad (i = \overline{1,n}), \quad 0 < \eta < 1$.

Let $a_{ii}(t) < 0$ $(i = \overline{1,n_1})$ and $a_{jj}(t) > 0$ $(j = \overline{n_1+1,n}\ )$, and $T > 0$. We consider the following auxiliary problem

$$\frac{dv}{dt} = A(t)v + F(t),\, t \in (-T,\, T), \tag{3}$$

$$P_{(1)}v(-T) = 0, \qquad P_{(2)}v(T) = 0, \tag{4}$$

where $P_{(1)} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}$, $P_{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-n_1} \end{pmatrix}$ are $(n \times n)$-matrices.

A continuously differentiable function $v(t)$ in $[-T,\, T]$ is called a solution of the problem (3), (4), if it satisfies the system (3) and the condition (4).

Lemma 1. *Let the condition i) hold. Then, for each $T > 0$ the problem (3), (4) has a unique solution $v_T(t)$ and for v_T the following estimate holds:*

$$\max_{t\in[-T;T]} \|v_T(t)\| \leq \max_{t\in[-T;T]} \left\| \frac{F(t)}{\theta(t)} \right\| = \max_{t\in[-T;T]} \max_{i=\overline{1,n}} \left| \frac{F_i(t)}{\theta(t)} \right|. \tag{5}$$

Using the parameterization method (see [5]) and Hadamard's lemma [6], one can prove this Lemma.

Theorem 1. *Let conditions i) and ii) be fulfilled, and let columns of the matrix $\frac{A(t)}{\theta(t)}$ and the vector-valued function $\frac{F(t)}{\theta(t)}$ belong to $C_b(R, R^n)$. Then the problem (8), (2) has a unique solution $V^*(t)$ and*

$$\|V^*(\cdot)\|_1 \leq \left\| \frac{F(\cdot)}{\theta(\cdot)} \right\|_1 \tag{6}$$

holds.

Proof. By Lemma 1, for each $T > 0$ there is a unique solution $V_T^*(t)$ of the problem (3), (4). By estimate (5) and the condition $\frac{F(t)}{\theta(t)} \in C_b(R, R^n)$, we obtain that the sequence $\{V_T^*(t)\}$ is bounded uniformly with respect to T. Then using the standard diagonal method, we can get a subsequence which converges to the solution $V^*(t)$ of the equation (8) for all $t \in R$:

$$\lim_{T'\to\infty} V_{T'}^*(t) = V^*(t) \ \ \forall t \in R.$$

Moreover, here the convergence is uniform with respect to $[-T, T]$. Passing to the limit as $T \to \infty$ in (5), we obtain (6).

Next, we prove that the solution of (8), (2) is unique. Suppose that $\overline{V}(t)$ and $\overline{\overline{V}}(t)$ are its solutions. Then $\Delta\tilde{V}(t) = \overline{V}(t) - \overline{\overline{V}}(t)$ is a solution in $C_b(R, R^n)$ of the following system of differential equations:

$$\frac{d\Delta\tilde{V}_i}{dt} = a_{ii}(t)\Delta\tilde{V}_i + \sum_{j\neq i} a_{ij}(t)\Delta\tilde{V}_j, \ \ i = \overline{1, n}, \ \ t \in R. \tag{7}$$

Since (7) satisfies conditions of the theorem, we can prove the existence of the solution of (7) as above. Thez homogeneous system corresponding to (7) is

$$\frac{d\Delta\tilde{V}_i}{dt} = a_{ii}(t)\Delta\tilde{V}_i, \ \ i = \overline{1, n}, \ \ t \in R.$$

It is well known that if $|a(t)| \geq \gamma > 0$ is a continuous function in R, and $\tilde{z}$ is the bounded solution of the equation $\frac{dz}{dt} = a(t)z$, then $\tilde{z} = 0$. Hence $\Delta\tilde{V}(t) = \overline{V}(t) - \overline{\overline{V}}(t) = 0$ and $\overline{V}(t) = \overline{\overline{V}}(t)$. The theorem is proved.

Theorem 2. *Suppose the conditions i) and ii) are fulfilled, and*
a) columns of the matrix $\frac{A(t)}{\theta(t)}$ *and vector-valued function* $F(t)$ *belong in* $C_b(R,\, R^n)$*;*
b) there exists the constant $c>1$ *such that*

$$c^{-1} \le \frac{\theta(t)}{\theta(\bar{t})} \le c$$

holds for $t,\ \bar{t} \in R$ *with* $|t-\bar{t}| \le d$*.*

Then the solution $V(t) \in C_b(R,\, R^n)$ *of (8) satisfies the following estimate:*

$$\left\| \frac{dV(\cdot)}{dt} \right\|_1 \le M \left\| \frac{A(\cdot)}{\theta(\cdot)} \right\|_1 + \|F(\cdot)\|_1 \,, \tag{8}$$

where

$$M = \frac{c+1}{cd} \left\| \frac{F(\cdot)}{\theta(\cdot)} \right\|_1 + \|F(\cdot)\|_1 \,.$$

Proof. Let

$$\tilde{\theta}(t) = \frac{1}{d} \int_t^{t+d} \theta(\tau) d\tau.$$

We put $\omega_T(t) = \tilde{\theta}(t) v_T(t)$, where $v_T(t)$ is a solution of (3), (4). The equality $\frac{dv_T(t)}{dt} = A(t)v_T(t) + F(t), t \in [-T, T]$ implies

$$\frac{d\omega_T(t)}{dt} = \frac{1}{d}[\theta(t+d) - \theta(t)] v_T(t) + \tilde{\theta}(t) A(t) v_T(t) + \tilde{\theta}(t) F(t)\,.$$

So, the function $\omega_T(t)$ is a solution of the following problem

$$\frac{d\omega}{dt} = A(t)\omega + \tilde{F}(t), t \in (-T, T), \tag{9}$$

$$P_{(1)}\omega(-T) = 0, P_{(2)}\omega(T) = 0\,, \tag{10}$$

where $\tilde{F}(t) = \frac{1}{d}[\theta(t+d) - \theta(t)] v_T(t) + \tilde{\theta}(t) F(t)$. It is clear that

$$\max_{t\in[-T,T]} \left\| \frac{\tilde{F}(t)}{\theta(t)} \right\| \le \max_{t\in[-T,T]} \frac{1}{d} \left\| \frac{[\theta(t+d) - \theta(t)] v_T(t)}{\theta(t)} \right\| + \max_{t\in[-T,T]} \left\| \frac{\tilde{\theta}(t) F(t)}{\theta(t)} \right\| \le$$

$$\le \frac{c+1}{d} \max_{t\in[-T,T]} \left\| \frac{F(t)}{\theta(t)} \right\| + c \max_{t\in[-T,T]} \|F(t)\| < \infty.$$

Thus $\frac{\tilde{F}(t)}{\theta(t)} \in C_b(R,\, R^n)$. Then, by Lemma 1, there exists the unique solution $\omega_T(t)$ of problem (9), (10) and the following estimate holds:

$$\max_{t\in[-T,T]} \|\omega_T(t)\| \le M',$$

where $M' = (c+1)/d \max_{t\in[-T,T]} \|F(t)/\theta(t)\| + c \max_{t\in[-T,T]} \|F(t)\|$. We replace T with T' and take the limit as $T' \to \infty$ in the last inequality. Then, taking into account the equality $\lim_{T'\to\infty} \omega_{T'}(t) = \tilde{\theta}(t)v^*(t)$ $(t \in R)$ (the convergence is uniform with respect to compact $[-T,T]$), we obtain $\max_{t\in[-T,T]} \|\omega^*(t)\| = \max_{t\in[-T,T]} \left\|\tilde{\theta}(t)v^*(t)\right\| \leq M'$. This implies $\sup_{t\in R} \|\omega^*(t)\| = \sup_{t\in R} \left\|\tilde{\theta}(t)v^*(t)\right\| \leq M'$. Furthermore,

$$\sup_{t\in R} \|A(t)V(t)\| = \sup_{t\in R} \left\|\frac{A(t)}{\theta(t)}\frac{\theta(t)}{\tilde{\theta}(t)}\tilde{\theta}(t)V(t)\right\| \leq \\ \leq \sup_{t\in R}\left|\frac{\theta(t)}{\tilde{\theta}(t)}\right| \sup_{t\in R}\left\|\frac{A(t)}{\theta(t)}\right\| \sup_{t\in R}\left\|\tilde{\theta}(t)V(t)\right\| \leq \frac{M}{c}\sup_{t\in R}\left\|\frac{A(t)}{\theta(t)}\right\|. \tag{11}$$

From (8) it follows that

$$\sup_{t\in R}\left\|\frac{dV(t)}{dt}\right\| \leq \frac{1}{c}\sup_{t\in R}\left\|\frac{A(t)}{\theta(t)}\right\|\sup_{t\in R}\|\omega^*(t)\| + \sup_{t\in R}\|F(t)\|.$$

By this estimate and condition a) and $\omega(t) \in C_b(R,\, R^n)$, we obtain $\frac{dV(t)}{dt} \in C_b(R,\, R^n)$ and the inequality (8). The theorem is proved.

Equations (8) and (11) imply the following estimate:

$$\left\|\frac{dV}{dt}\right\|_1 + \|AV\|_1 \leq C_0\|F\|_1. \tag{12}$$

3 The Main Result and Its Proof

Consider the differential operator $L = \frac{d}{dt} - A(t)I_n$ in continuous and bounded vector-valued function space $C_b(R,R^n)$, where $A(t) = (a_{i,j}(t))_{i,j=1}^n$ is the $(n\times n)$-matrix with continuous elements.

Theorem 3. *Let the matrix $A(t)$ satisfy the following conditions:*
i) $\sum_{j\neq i}|a_{ij}(t)| + \theta(t) \leq |a_{ii}(t)| \leq \frac{1}{\eta}\theta(t)$ $(i = \overline{1,n})$, where $\theta(t) \geq \theta_0 > 0$ is continuous, and $0 < \eta < 1$;
ii) there exist the constant $c > 1$ such that

$$c^{-1} \leq \frac{|\theta(t)|}{|\theta(\bar{t})|} \leq c$$

holds for $t,\ \bar{t} \in R$ with $|t-\bar{t}| \leq 1$;
iii) $\underline{\lim}_{|t|\to+\infty} \theta(t) = +\infty$.
Then the resolvent L^{-1} of the operator L is a completely continuous in the space $C_b(R,R^n)$.

Proof. Let $C^{(1)}_{b,\,A}(R, R^n)$ be the space of all functions that are bounded and have a bounded and continuous derivative, with finite norm

$$\|V\|_{1,\,A} = \left\|\frac{dV}{dt}\right\|_1 + \|AV\|_1 .$$

By Theorem 2, there exists the inverse L^{-1} of the operator L, which is defined on the whole $C_b(R, R^n)$ and is continuous from $C_b(R, R^n)$ to the space $C^{(1)}_{b,\,A}(R, R^n)$, and $\left\|L^{-1}F\right\|_{1,\,A} \le C_0\ \|F\|_1$ holds.

We show that the set $T = \left\{U \in C_b(R, R^n) :\ \|U\|_{1,\,A} \le C_0\ \|F\|_1\right\}$ is compact in $C_b(R, R^n)$. Let $\alpha > 0$, then, according to the condition ii), there is $n \in N$ such that

$$\|U\|_{1,\,n} = \sup_{t\in R\setminus[-N,\,N]} \|U(t)\| \le \frac{\alpha}{2},\ \ \forall U(t) \in T. \tag{13}$$

Let $\varphi_n(t) \in C^{(1)}_0(-n-1,\, n+1)$ satisfy $0 \le \varphi_n(t) \le 1$ and $\varphi_n(t) = 1$, $\forall t \in [-n,\, n]$. We consider the set $T_{\varphi_n} = \{U\varphi_n :\ U \in T\}$. By (13) T_{φ_n} is the α-net for T. On the other hand, T_{φ_n} is a subset of $C^{(1)}_{b,\,A}([-n,\, n], R^n) \bigcap C_0([-n-1,\, n+1], R^n)$. Then by Hausdorff theorem (see [7], Ch. 1) T is compact. The theorem is proved.

Acknowledgements. This work is partially supported by project 5132/GF4 of the Science Committee of Ministry of Education and Science of the Republic of Kazakhstan and by L.N. Gumilyov Eurasian National University Research Fund. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Dzhumabaev, D.S.: Method of parametrization for the decision of boundary value problems for ordinary differential equations. Vestnik AN KazSSR **1**, 48–52 (1988)
2. Dzhumabaev, D.S.: Approximation of a bounded solution of the ordinary differential equation by solutions of the two-point boundary value problems. Comput. Math. Math. Phys. **30**, 388–404 (1990)
3. Kantorovich, L.V., Akilov, G.P.: Functional Analysis. Pergamon Press, Oxford (1982)
4. Martynov, V.V.: Conditions of discreteness and continuousness of the spectra in the case of self-adjoint system of differential equations. DAN SSSR **165**, 996–999 (1965)
5. Mynbaev, K.T., Otelbaev, M.: Weighted Function Spaces and the Spectrum of Differential Operators. Nauka, Moscow (1988) (in Russian)
6. Oinarov, R., Otelbaev, M.: Discreteness criterion for the spectrum of the total of the general Sturm-Liouville operator and the associated embedding theorems. Differ. Equ. **24**(2), 402–408 (1988)
7. Ospanov, K.N.: Qualitative and approximate characteristics of solutions of Beltrami-type systems. Complex Var. Elliptic Equ. **60**, 1005–1014 (2015)
8. Ospanov, K.N., Akhmetkaliyeva, R.D.: Some inequalities for second order differential operator with unbounded drift. Eurasian Math. J. **6**, 63–74 (2015)

9. Ospanov, M.N.: Coercive estimates for solutions of one singular equation with the third-order partial derivative. AIP Conf. Proc. **1611**, 37–42 (2014)
10. Ospanov, M.N.: The maximal regularity of the hyperbolic system. AIP Conf. Proc. **1676**, 020037 (2015)

Some Reverse Hölder Type Inequalities Involving $(k, s)-$Riemann-Liouville Fractional Integrals

Muharrem Tomar[1], Praveen Agarwal[2(✉)], Shilpi Jain[3], and Gradimir V. Milovanović[4]

[1] Faculty of Arts and Sciences, Department of Mathematics, Ordu University, 52200 Ordu, Turkey
muharremtomar@gmail.com
[2] Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India
goyal.praveen2011@gmail.com
[3] Department of Mathematics, Poornima College of Engineering, Jaipur 303029, India
shilpijain1310@gmail.com
[4] Full member of the Serbian Academy of Sciences and Arts, 36 Kneza Mihaila, 11000 Belgrad, Serbia
gvm@mi.sanu.ac.rs

Abstract. In this paper, we aim to present the improved version of the reverse Hölder type inequalities by taking $(k, s)-$Riemann-Liouville fractional integrals. Furthermore, we also discuss some applications of Theorem 1 using some types of fractional integrals.

Keywords: $(k, s)-$Riemann-Liouville fractional integrals · Holder inequality · Reverse Holder inequality

1 Introduction

Fractional integral inequalities involving $(k, s)-$ type integrals attract the attentions of many researchers due their diverse applications see, for examples, [1–4]. In [5], Farid *et al.* an integral inequality obtained by Mitrinovic and Pecaric was generalized to measure space as follows.

Theorem 1. *Let* $(\Omega_1, \Sigma_1, \mu_1)$*,*$(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* $\sigma-$*finite measures and let* $f_i : \Omega_2 \to \mathbb{R}$*,* $i = 1, 2, 3, 4$ *be non-negative functions. Let* g *be the function having representation*

$$g(x) = \int_{\Omega_1} k(x, t) f(t) d\mu_1(t),$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_29

where $k : \Omega_2 \times \Omega_1 \to \mathbb{R}$ is a general non-negative kernel and $f : \Omega_1 \to \mathbb{R}$ is real-valued function, and μ_2 is a non-decreasing function. If p, q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then

$$\int_{\Omega_2} f_1(x) f_2(x) g(x) d\mu_2(x) \leq C \left(\int_{\Omega_2} f_3(x) g(x) d\mu_2(x) \right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_4(x) g(x) d\mu_2(x) \right)^{\frac{1}{q}}, \tag{1}$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_a^b k(x,t) f_1(x) f_2(x) d\mu_2(x) \right) \left(\int_a^b k(x,t) f_3(x) d\mu_2(x) \right)^{\frac{-1}{p}} \left(\int_a^b k(x,t) f_4(x) d\mu_2(x) \right)^{\frac{-1}{q}} \right\}. \tag{2}$$

The following definitions and results are also required.

2 Preliminaries

Recently fractional integral inequalities are considered to be an important tool of applied mathematics and their many applications described by a number of researchers. As well as, the theory of fractional calculus is used in solving differential, integral and integro-differential equations and also in various other problems involving special functions [6–8].

We begin by recalling the well-known results.

1. The Pochhammer k-symbol $(x)_{n,k}$ and the k-gamma function Γ_k are defined as follows (see [9]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) \quad (n \in \mathbb{N};\ k > 0) \tag{3}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!\, k^n\, (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad \left(k > 0;\ x \in \mathbb{C}\backslash k\mathbb{Z}_0^-\right), \tag{4}$$

where $k\mathbb{Z}_0^- := \left\{kn : n \in \mathbb{Z}_0^-\right\}$. It is noted that the case $k = 1$ of equation ((3)) and equation ((4)) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt \quad (\Re(x) > 0). \tag{5}$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties: (i) $\Gamma_k(x+k) = x\,\Gamma_k(x)$; (ii) $\Gamma_k(k) = 1$; (iii) $\Gamma_k(x)$ is logarithmically convex. It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\frac{x}{k}\right) \quad (\Re(x) > 0;\ k > 0)\,. \tag{6}$$

2. Mubeen and Habibullah [10] introduced k-fractional integral of the Riemann-Liouville type of order α as follows:

$$_kJ_a^{\alpha}\left[f(t)\right] = \frac{1}{\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)\,d\tau, (\alpha > 0, x > 0, k > 0)\,, \tag{7}$$

which, upon setting $k = 1$, is seen to yield the classical Riemann-Liouville fractional integral of order α:

$$J_a^{\alpha}\left\{f(t)\right\} := {}_1J_a^{\alpha}\left\{f(t)\right\} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau \quad (\alpha > 0;\ t > a)\,. \tag{8}$$

3. Sarikaya *et al.* [11] presented (k,s)-fractional integral of the Riemann-Liouville type of order α, which is a generalization of the k-fractional integral (7), defined as follows:

$$^s_kJ_a^{\alpha}\left[f(t)\right] := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s f(\tau)\,d\tau,\ \tau \in [a,b]\,, \tag{9}$$

where $k > 0, s \in \mathbb{R}\backslash\left\{-1\right\}$ and which, upon setting $s = 0$, immediately reduces to the k-integral (7).

4. In [11], the following results have been obtained. For f be continuous on $[a,b],\ k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then,

$$^s_kJ_a^{\alpha}\left[{}^s_kJ_a^{\beta} f(t)\right] = {}^s_kJ_a^{\alpha+\beta} f(t) = {}^s_kJ_a^{\beta}\left[{}^s_kJ_a^{\alpha} f(t)\right], \tag{10}$$

and

$$^s_kJ_a^{\alpha}\left[\left(x^{s+1} - a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+\beta)}\left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1},$$

for all $\alpha, \beta > 0,\ x \in [a,b]$ and Γ_k denotes the $k-$gamma function.

5. Also, in [12], Akkurt *et al.* introduced $(k,H)-$fractional integral. Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by

$$\begin{aligned}&\left({}_kJ_{a^+,h}^{\alpha} f\right)(x) \\ &:= \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left[h(x) - h(t)\right]^{\frac{\alpha}{k}-1} h'(t) f(t)\,dt,\ k > 0\ , \Re(\alpha) > 0\end{aligned} \tag{11}$$

$$\left({}_kJ^{\alpha}_{b^-,h}f\right)(x) \tag{12}$$
$$:= \frac{1}{k\Gamma_k(\alpha)}\int_x^b [h(x)-h(t)]^{\frac{\alpha}{k}-1}h'(t)f(t)dt,\ k>0\ ,\Re(\alpha)>0.$$

Recently, Tomar and Agarwal [13] obtained following results for $(k,s)-$fractional integrals.

Theorem 2 (Hölder Inequality for (k,s)-fractional integrals). *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, for all $t>0,\ k>0,\ \alpha>0, s\in\mathbb{R}-\{-1\}$,*

$${}^s_kJ^{\alpha}_a\mid fg\,(t)| \leq \left[{}^s_kJ^{\alpha}_a\,|f\,(t)|^p\right]^{\frac{1}{p}}\left[{}^s_kJ^{\alpha}_a\,|g\,(t)|^q\right]^{\frac{1}{q}}. \tag{13}$$

Lemma 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions and $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,k>0$ and $s\in\mathbb{R}-\{-1\}$, such that for $t\in[a,b]$, ${}^s_kJ^{\alpha}_af^p(t)<\infty$, ${}^s_kJ^{\alpha}_ag^q(t)<\infty$. If*

$$0\leq m\leq\frac{f(\tau)}{g(\tau)}\leq M<\infty,\tau\in[a,b], \tag{14}$$

then the inequality

$$\left[{}^s_kJ^{\alpha}_af(t)\right]^{\frac{1}{p}}\left[{}^s_kJ^{\alpha}_ag(t)\right]^{\frac{1}{q}}\leq\left(\frac{M}{m}\right)^{\frac{1}{pq}}\ {}^s_kJ^{\alpha}_a\left[f^{\frac{1}{p}}(t)g^{\frac{1}{q}}(t)\right] \tag{15}$$

holds.

Lemma 2. *Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions $\alpha,k>0$ and $s\in\mathbb{R}-\{-1\}$, such that for $t\in[a,b]$, ${}^s_kJ^{\alpha}_af^p(t)<\infty$, ${}^s_kJ^{\alpha}_ag^q(t)<\infty$. If*

$$0\leq m\leq\frac{f^p(\tau)}{g^q(\tau)}\leq M<\infty,\tau\in[a,b], \tag{16}$$

then we have

$$\left[{}^s_kJ^{\alpha}_af^p(t)\right]^{\frac{1}{p}}\left[{}^s_kJ^{\alpha}_ag^q(t)\right]^{\frac{1}{q}}\leq\left(\frac{M}{m}\right)^{\frac{1}{pq}}\ {}^s_kJ^{\alpha}_a\left(f(t)g(t)\right), \tag{17}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by this work, we establish in this paper some new extensions of the reverse Hölder type inequalities by taking $(k,s)-$Riemann-Liouville fractional integrals.

3 Reverse Hölder Type Inequalites

In this section we prove our main results (Theorems 3 and 4).

Theorem 3. *Let $f(x)$ and $g(x)$ be integrable functions and let $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequality holds*

$$ {}_k^sJ_a^\alpha \left|fg(t)\right| \geq {}_k^sJ_a^\alpha \left|f^p(t)\right|^{\frac{1}{p}} \ {}_k^sJ_a^\alpha \left|f^q(t)\right|^{\frac{1}{q}}. \tag{18} $$

Proof. Set $c = \frac{1}{p}$, $q = -pd$. Then we have $d = \frac{c}{c-1}$. By the Hölder inequality for $(k,s)-$fractional integrals, we have

$$ \begin{aligned} {}_k^sJ_a^\alpha \left|f^p(t)\right| &= {}_k^sJ_a^\alpha \left|fg(t)\right|^p \left|g^{-p}(t)\right| \\ &\leq \left[{}_k^sJ_a^\alpha \left|fg(t)\right|^{pc}\right]^{\frac{1}{c}} \left[{}_k^sJ_a^\alpha \left|g(t)\right|^{-pd}\right]^{\frac{1}{d}} \\ &= \left[{}_k^sJ_a^\alpha \left|fg(t)\right|\right]^{\frac{1}{c}} \left[{}_k^sJ_a^\alpha \left|g(t)\right|^{q}\right]^{1-p}. \end{aligned} \tag{19} $$

In equation (19), multiplying both sides by $\left({}_k^sJ_a^\alpha \left|g^q(t)\right|\right)^{p-1}$, we obtain

$$ \begin{aligned} &{}_k^sJ_a^\alpha \left|f^p(t)\right| \left({}_k^sJ_a^\alpha \left|g^q(t)\right|\right)^{p-1} \\ &\leq \left[{}_k^sJ_a^\alpha \left|fg(t)\right|\right]^{p}. \end{aligned} \tag{20} $$

Inequality (20) implies inequality

$$ {}_k^sJ_a^\alpha \left|fg(t)\right| \geq {}_k^sJ_a^\alpha \left|f^p(t)\right|^{\frac{1}{p}} \ {}_k^sJ_a^\alpha \left|f^q(t)\right|^{\frac{1}{q}} \tag{21} $$

which completes this theorem.

Theorem 4. *Suppose $p, q, l > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{l} = 1$. If f, g and h are positive functions such that*

i.) $0 < m \leq \frac{f^{\frac{p}{s}}}{g^{\frac{g}{s}}} \leq M < \infty$ for some $l > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$,
ii.) $0 < m \leq \frac{(fg)^s}{h^r} \leq M < \infty$,

then

$$ \begin{aligned} &\left({}_k^sJ_a^\alpha f^p(t)\right)^{\frac{1}{p}} \left({}_k^sJ_a^\alpha f^q(t)\right)^{\frac{1}{q}} \left({}_k^sJ_a^\alpha f^r(t)\right)^{\frac{1}{r}} \\ &\leq \left(\frac{M}{m}\right)^{\frac{1}{sr}+\frac{pq}{s^3}} \ {}_k^sJ_a^\alpha (fgh)(t). \end{aligned} \tag{22} $$

Proof. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ for some $s > 0$. Thus, $\frac{s}{p} + \frac{s}{q} = 1$ and $\frac{1}{s} + \frac{1}{r} = 1$. If we use *ii* and Lemma 2 for $H = fg$ and h, then we get

$$ \left({}_k^sJ_a^\alpha H^s(t)\right)^{\frac{1}{s}} \left({}_k^sJ_a^\alpha h^r(t)\right)^{\frac{1}{r}} \leq \left(\frac{M}{m}\right)^{\frac{1}{sr}} \left({}_k^sJ_a^\alpha (Hh)(t)\right) \tag{23} $$

which is equivalent to

$$\left({}_k^sJ_a^\alpha[f^s(t)g^s(t)]\right)^{\frac{1}{s}}\left({}_k^sJ_a^\alpha h^r(t)\right)^{\frac{1}{r}} \leq \left(\frac{M}{m}\right)^{\frac{1}{sr}}\left({}_k^sJ_a^\alpha(fgh)(t)\right). \tag{24}$$

Now, using i and the fact that $\frac{s}{p}+\frac{s}{q}=1$, and applying Lemma 2 to f^s and g^s, we also have

$$\left({}_k^sJ_a^\alpha f^p(t)\right)^{\frac{s}{p}}\left({}_k^sJ_a^\alpha g^q(t)\right)^{\frac{s}{q}} \leq \left(\frac{M}{m}\right)^{\frac{pq}{s^2}}\left({}_k^sJ_a^\alpha f^s(t)g^s(t)\right) \tag{25}$$

which is equivalent to

$$\left({}_k^sJ_a^\alpha f^p(t)\right)^{\frac{1}{p}}\left({}_k^sJ_a^\alpha g^q(t)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{pq}{s^3}}\left({}_k^sJ_a^\alpha f^s(t)g^s(t)\right)^{\frac{1}{s}}. \tag{26}$$

Combining equations (24) and (26), we obtain desired inequality equation (22), which is complete the proof.

4 Applications for Some Types Fractional Integrals

Here in this section, we discuss some applications of Theorem 1 in the terms of Theorems 5-7 and Corollary 1-5.

Theorem 5. *Let p, q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and let f be continuous on $[a,b]$, $k>0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then*

$$\int_a^b f_1(x)f_2(x){}_k^sJ_a^\alpha f(x)dx \tag{27}$$
$$\leq C\left(\int_a^b f_3(x){}_k^sJ_a^\alpha f(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_4(x){}_k^sJ_a^\alpha f(x)dx\right)^{\frac{1}{q}},$$

where

$$C=\sup_{t\in[a,b]}\left\{\left(\int_a^b\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_1(x)f_2(x)dx\right)\right. \tag{28}$$
$$\left.\left(\int_a^b\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_3(x)dx\right)^{\frac{-1}{p}}\left(\int_a^b\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}f_4(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Proof. In Theorem 1, if we take $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(t)=dt, d\mu_2(x)=dx$ and the kernel

$$k(x,t)=\begin{cases}\frac{(s+1)^{1-\frac{\alpha}{k}}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s}{k\Gamma_k(\alpha)} & \text{if } a\leq t\leq x\\ 0 & \text{if } x<t\leq b,\end{cases}$$

then $g(x)$ becomes ${}_k^sJ_a^\alpha f(t)$ and so we get desired inequality (27). This completes the proof of Theorem 5.

Corollary 1. *In Theorem 5, if we take $s = 0$, then we get*

$$\int_a^b f_1(x) f_2(x)_k J_a^\alpha f(x) dx \tag{29}$$
$$\leq C \left(\int_a^b f_3(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_4(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1(x) f_2(x) dx \right) \right. \tag{30}$$
$$\left. \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_3(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_4(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Remark 1. In Corollary 1, $\alpha = k = 1$, Theorem 1 reduces to Theorem 3.1 in [5].

Corollary 2. *In Theorem 5, if we take $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get*

$$\int_a^b f_1(x) f_2(x)_k^s J_a^\alpha f(x) dx \tag{31}$$
$$\leq C \left(\int_a^b f_1^p(x)_k^s J_a^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x)_k^s J_a^\alpha f(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} f_1(x) f_2(x) dx \right) \right. \tag{32}$$
$$\left. \left(\int_a^b \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} f_1^p(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Corollary 3. *In Corollary 2, if we take $s = 0$, then we get*

$$\int_a^b f_1(x) f_2(x)_k J_a^\alpha f(x) dx \tag{33}$$
$$\leq C \left(\int_a^b f_1^p(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x)_k J_a^\alpha f(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1(x) f_2(x) dx \right) \right. \tag{34}$$
$$\left. \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_1^p(x) dx \right)^{\frac{-1}{p}} \left(\int_a^b (x-t)^{\frac{\alpha}{k}-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Remark 2. In Corollary 3, $\alpha = k = 1$, Corollary 3 reduces to Corollary 3.2 in [5].

Theorem 6. *Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). Also, let p,q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and let f be continuous on $[a,b]$, $k>0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then*

$$\int_a^b f_1(x)f_2(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx \tag{35}$$
$$\leq C\left(\int_a^b f_3(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_4(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_1(x)f_2(x)dx\right)\right.$$
$$\times\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_3(x)dx\right)^{\frac{-1}{p}}$$
$$\left.\left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f_4(x)dx\right)^{\frac{-1}{q}}\right\}. \tag{36}$$

Proof. Applying Theorem 1 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(t)=dt, d\mu_2(x)=dx$ and the kernel

$$k(x,t)=\begin{cases}\frac{(h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)}{k\Gamma_k(\alpha)} & \text{if } a\leq t\leq x\\ 0 & \text{if } x<t\leq b,\end{cases}$$

then $g(x)$ becomes $\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)$ and so we get desired inequality (35). This completes the proof of Theorem 6.

Corollary 4. *In Theorem 6, setting $f_3(x)=f_1^p(x)$ and $f_4(x)=f_2^q(x)$, we get*

$$\int_a^b f_1(x)f_2(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx \tag{37}$$
$$\leq C\left(\int_a^b f_1^p(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^b f_2^q(x)\left({}_kJ^{\alpha}_{a^+,h}f\right)(x)dx\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1^p(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{38}$$

Theorem 7. *Under the assumptions of Theorem 6, we have*

$$\int_a^b f_1(x) f_2(x) \left({}_kJ^{\alpha}_{b^-,h} f \right)(x) dx \tag{39}$$
$$\leq C \left(\int_a^b f_3(x) \left({}_kJ^{\alpha}_{b^-,h} f \right)(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_4(x) \left({}_kJ^{\alpha}_{b^-,h} f \right)(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_3(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f_4(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{40}$$

Proof. In contrast to Theorem 6, if we take the kernel

$$k(x,t) = \begin{cases} \frac{(h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t)}{k\Gamma_k(\alpha)} & \text{if } x \leq t \leq b \\ 0 & \text{if } a < t \leq x, \end{cases}$$

we obtain desired inequality.

Corollary 5. *In Theorem 7, setting $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, we get*

$$\int_a^b f_1(x) f_2(x) \left({}_kJ^{\alpha}_{b^-,h} f \right)(x) dx \tag{41}$$
$$\leq C \left(\int_a^b f_1^p(x) \left({}_kJ^{\alpha}_{b^-,h} f \right)(x) dx \right)^{\frac{1}{p}} \left(\int_a^b f_2^q(x) \left({}_kJ^{\alpha}_{b^-,h} f \right)(x) dx \right)^{\frac{1}{q}},$$

where

$$C = \sup_{t\in[a,b]} \left\{ \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k}-1} h'(t) f_1(x) f_2(x) dx \right) \right.$$
$$\times \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k}-1} h'(t) f_1^p(x) dx \right)^{\frac{-1}{p}}$$
$$\left. \left(\int_a^b (h(x) - h(t))^{\frac{\alpha}{k}-1} h'(t) f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}. \tag{42}$$

References

1. Agarwal, P.: Some inequalities involving Hadamard-type k-fractional integral operators. Math. Methods Appl. Sci. (2017)
2. Agarwal, P., Jleli, M., Tomar, M.: Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. J. Inequal. Appl. **2017**(1) (2017)
3. Agarwal, P., Tariboon, J., Ntouyas, S.K.: Some generalized Riemann-Liouville k-fractional integral inequalities. J. Inequal. Appl. **2016**(1), 1–13 (2016)
4. Ntouyas, S.K., Agarwal, P., Tariboon, J.: On Pólya-Szegö and Chebyshev types inequalities involving the Riemann-Liouville fractional integral operators. J. Math. Inequal. **10**(2), 491–504 (2017)
5. Farid, G., Iqbal, S., Pecaric, J.: On generalization of an integral inequality and its applications. Cogent. Math. **2**(1), 1066528 (2015)
6. Anastassiou, G.A.: Fractional Differentiation Inequalities. Springer Science, LLC (2009)
7. Set, E., Tomar, M., Sarikaya, M.Z.: On generalized Grüss type inequalities via $k-$Riemann-Liouville fractional integral. Appl. Math. Comput. **269**, 29–34 (2015)
8. Katugampola, U.N.: New approach generalized fractional integral. Appl. Math. Comput. **218**(3), 860–865 (2011)
9. Diaz, R., Pariguan, E.: On hypergeometric functions and Pochhammer k-symbol. Divulgaciones Matemáticas **15**, 179–192 (2007)
10. Mubeen, S., Habibullah, G.M.: $k-$Fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(2), 89–94 (2012)
11. Sarikaya, M.Z., Dahmani, Z., Kiris, M.E., Ahmad, F.: $(k, s)-$Riemann-Liouville fractional integral and applications. Hacet. J. Math. Stat. (in press)
12. Akkurt, A., Yildirim, M.E., Yildirim, H.: On some integral inequalities for $(k; h)-$Riemann-Liouville fractional integral. https://www.researchgate.net/publication/289245260
13. Tomar, M., Agarwal, P.: Some generalized integral inequalities involving $(k, s)-$Riemann-Liouville fractional integrals. submitted

Volterra Type Integral Equation with Super-Singular Kernels

Nusrat Rajabov(✉)

Tajik National University, Dushanbe, Tajikistan
nusrat38@mail.ru

Abstract. In this work we suggest a new method for investigating the model Volterra type integral equation with super-singularity, the kernel of which consists of a composition of polynomial functions with super-singularity and functions with super-singular points. The problem of investigating this type of integral equation for $n = 2m$ is reduced to m Volterra type integral equation for $n = 2$, and for $n = 2m + 1$ it is reduced to m Volterra integral equation for $n = 2$ and one integral equation for $n = 1$.

Keywords: Volterra type integral equation · Super-singular kernels · Asumptotic behavior · Explicit solution · Representation manifold solution · Characteristic integral equation · n-order degenerate ordinary differential equation

Let $\Gamma = \{x : a < x < b\}$ be a set of points on real axis and consider an integral equation

$$\varphi(x) + \int_a^x \left[\sum_{j=1}^{n} A_j(\omega_a^\alpha(t) - \omega_a^\alpha(x))^{j-1}\right] \frac{\varphi(t)}{(t-a)^\alpha} dt = f(x), \qquad (1)$$

where $A_j (1 \leq j \leq n)$ is given constants, $f(x)$ is a given function in $\overline{\Gamma}$ and $\varphi(x)$ to be found , $\omega_a^\alpha(x) = \left[(\alpha - 1)(x-a)^{\alpha-1}\right]^{-1}$, $\alpha = const > 1$.

The works [1–7] are dedicated to the problem of investigating integral equations of type (1) with kernels $K(x,t) = \sum_{j=1}^{n} A_j \ln^{j-1}\left(\frac{x-a}{t-a}\right)(t-a)^{-1}$. **Monograph [1] and the case $n = 2$ of the work [8] are devoted to the problem investigation integral equation (1) for** $n = 1$.
The solution of the integral equation (1) we will seak in the class of function $\varphi(x) \in C\left(\overline{\Gamma}\right)$ vanishing at the singular point t $= a$, i.e

$$\varphi(x) = o\left[(x-a)^{\gamma_1}\right], \ \ \gamma_1 > n(\alpha - 1) \quad \text{at} \quad x \to a.$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_30

Assume that, the solution of the equation (I) is the function $\varphi(x) \in C^{(n)}(\Gamma)$. Besides let the function $f(x) \in C^{(n)}(\Gamma)$ in equation (I). Then differentiating integral equation (I) n times and every time multiplying by $(x-a)^{\alpha}$, we obtain the following nth order degenerate ordinary differential equation

$$\begin{aligned}&(D_x^{\alpha})^n\varphi(x) + A_1(D_x^{\alpha})^{n-1}\varphi(x) + A_2(D_x^{\alpha})^{n-2}\varphi(x)\\&+2!A_3(D_x^{\alpha})^{n-3}\varphi(x) + \ldots (n-1)!A_n\varphi(x) = (D_x^{\alpha})^n f(x),\end{aligned} \tag{2}$$

where $D_x^{\alpha} = (x-a)^{\alpha}\frac{d}{dx}$.

The homogeneous differential equation (2) is corresponding to the following characteristic equation

$$\lambda^n + A_1\lambda^{n-1} + A_2\lambda^{n-2} + 2!A_3\lambda^{n-3} + 3!A_4\lambda^{n-4} + \cdots + (n-1)!A_n = 0 \ . \tag{3}$$

To the investigation problem of different cases $(n=1,\ n=2)$ for the integral equation (1) the papers [1,8] are devoted. The case of equation (1) when the parameters $A_j(1 \le j \le n)$ are such that the roots of the characteristic equation (3) are real, different and positive is investigated in [8]. In this case we have the following confirmation

Theorem 1. *Let in the integral equation (1) the parameters $A_j(1 \le j \le n)$ be such that the roots of the characteristic equation (3) $\lambda_j(1 \le j \le n)$ are real, different and positive, let the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior*
$f(x) = o\left[e^{-\lambda\omega_a^{\alpha}(x)}(x-a)^{\gamma}\right]$, $\gamma > \alpha - 1$ at $x \to a$,
where $\lambda = max(\lambda_1, \lambda_2, \ldots\ldots\ldots\ \lambda_n)$. Then integral equation (1) in the class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ is always solvability and its solution is given by the formula

$$\begin{aligned}&\varphi(x) = \textstyle\sum_{j=1}^{n} C_j exp\left[-\lambda_j\omega_a^{\alpha}(x)\right] + f(x)\\&+\tfrac{1}{\Delta_0}\int_a^x \left\{\textstyle\sum_{j=1}^{n}(-1)^{n+j}\Delta_{jn}\lambda_j^n exp\left[\lambda_j\left(\omega_a^{\alpha}(t) - \omega_a^{\alpha}(x)\right)\right]\right\}\tfrac{f(t)}{(t-a)^{\alpha}}dt\\&\equiv \textstyle\sum_{j=1}^{n} C_j exp\left[-\lambda_j\omega_a^{\alpha}(x)\right] + K_{\alpha}(f),\end{aligned} \tag{4}$$

where Δ_0 is a Vandermond determinant, Δ_{jn} is minor of $(n-1)-$order, which is obtained from Δ_0 by dividing $n-th$ lines and $j-th$ column, $C_j\ (1 \le j \le n)$ are arbitrary constants.

But in investigating other cases for the roots of the characteristic equation (3) and obtaining the manifold of solutions to equation (1) there arise great difficulties of analytical character.

In this connection we offer here a method for representing the manifold of solution of integral equation (1) for $n = 1$ and $n = 2$. This theory was constructed in [1] for $n = 1$ and in [8] for $n = 2$. In [8] other possible cases were investigated for $n = 2$. Depending on the cases $n = 2m$ and $n = 2m+1$ we give a representation of the manifold of solutions to equation (1) with respect to m of second order algebraic equations.

Here a new method is offered. Let $n = 2m$. In this case a general solution of the integral equation (1) is represented by the solutions m of integral equation of the type

$$T^{\alpha}_{p_j,q_j}(\varphi) = f(x), \tag{5}$$

where

$$T^{\alpha}_{p_j,q_j}(\varphi) \equiv \varphi(x) + \int_a^x [p_j + q_j(\omega^{\alpha}_a(t) - \omega^{\alpha}_a(x))] \frac{\varphi(t)}{(t-a)^{\alpha}} dt \ ,$$

and when $n = 2m + 1$ the general solution of (1) we represent by the solution m of the integral equation of the type (5) and by one solution to the integral equation

$$\Pi^{\alpha}_{\lambda}(\varphi) = \ g(x), \tag{6}$$

where

$$\Pi^{\alpha}_{\lambda}(\varphi) \equiv \varphi(x) + \lambda \int_a^x \frac{\varphi(t)}{(t-a)^{\alpha}} dt.$$

The respective theory is constructed in [1,8].

Let $n = 2m$ in integral equation (1). Then we represent the integral equation in the form

$$\prod_{j=1}^{m} T^{\alpha}_{p_j,q_j}(\varphi) = f(x), \tag{7}$$

where $p_j, \ q_j \ (1 \le j \le m)$ are constants , which are the coefficients of the following characteristic equation

$$\left(\lambda^{(j)}\right)^2 + p_j\lambda^{(j)} + q_j = 0(1 \le j \le m). \tag{8}$$

Later on we denote the roots of the characteristic equation (6) by

$$\lambda^{(j)}_k (k = 1, 2, \ 1 \le j \le m).$$

We can represent the integral equation (1) in the form (5) when the roots of the characteristic equation (6) are connected with the parameters $A_j (1 \le j \le n)$ of equation (1) by

$$\begin{aligned}
&A_1 = -\sum_{j=1}^{m}\left(\lambda^{(j)}_1 + \lambda^{(j)}_2\right),\\
&A_2 = \sum^{m}_{\substack{j,k=1\\ j\neq k}}\left(\lambda^{(k)}_1\lambda^{(j)}_2\right) + \sum^{m}_{\substack{j,k=1\\ j\neq k}}\left(\lambda^{(k)}_1\lambda^{(j)}_1\right) + \sum^{m}_{\substack{j,k=1\\ j\neq k}}\left(\lambda^{(k)}_2\lambda^{(j)}_2\right),\\
&2!A_3 = \sum^{m}_{\substack{j,k,\ s=1\\ j\neq k\neq s}}\left(\lambda^{(k)}_1\lambda^{(j)}_2\lambda^{(s)}_1\right) + \sum^{m}_{\substack{j,k,\ s=1\\ j\neq k\neq s}}\left(\lambda^{(k)}_2\lambda^{(j)}_1\lambda^{(s)}_2\right),\\
&\ldots, (n-1)!A_n = \prod_{j=1}^{m}\lambda^{(j)}_1\lambda^{(j)}_2.
\end{aligned} \tag{9}$$

Equation (5) is written as the system

$$\begin{array}{l} \Psi_m(x) = T^{\alpha}_{p_{m},q_{m}}(\varphi), \\ \Psi_{m-1}(x) = T^{\alpha}_{p_{m-1},q_{m-1}}(\Psi_m), \\ \Psi_{m-2}(x) = T^{\alpha}_{p_{m-2},q_{m-2}}(\Psi_{m-1}), \\ \ldots\ldots, \\ \Psi_2(x) = T^{\alpha}_{p_{2},q_{2}}(\Psi_3), T^{\alpha}_{p_{1},q_{1}}(\Psi_2) = f(x). \end{array} \tag{10}$$

So, in this case the problem of finding the general solution of the integral equation (1) is reduced to the problem of finding the solution of the system (6) of the Volterra integral equations.

In particular if the roots of the characteristic equation (6) are real, equal and negative and the constants p_j $(1 \le j \le m)$ satisfy the following inequities

$$|p_m| > |p_{m-1}| > |p_{m-2}| \ldots\ldots > |p_1|, \tag{11}$$

and the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o\left[\exp\left(\frac{p_m}{2}\ \omega_a^{\alpha}(x)\right)(x-a)^{\gamma_3}\right], \ \gamma_3 > n(\alpha - 1), \ \text{ at } \ x \to a, \tag{12}$$

then the solution of the integral equation (1) is given by the formula

$$\varphi(x) = \prod_{j=1}^{m}\left(T^{\alpha C_1^{m-j+1},\ C_2^{m-j+1}}_{p_{m-j+1},\ q_{m-j+1}}\right)^{-1}(f), \tag{13}$$

where $C_1^{m-j+1},\ C_2^{m-j+1}$ $(1 \le j \le m)$ are arbitrary constants

$$\begin{array}{l} \left(T^{\alpha C_1^{m-j+1},\ C_2^{m-j+1}}_{p_{m-j+1},\ q_{m-j+1}}\right)^{-1}(f) \\ = exp\left[\frac{p_{m-j+1}}{2}\omega_a^{\alpha}(x)\right]\left[C_1^{m-j+1} + \omega_a^{\alpha}(x)\, C_2^{m-j+1}\right] + f(x) \\ - \int_a^x exp\left[\frac{p_{m-j+1}}{2}\left(\omega_a^{\alpha}(x) - \omega_a^{\alpha}(t)\right)\right]\left[p_{m-j+1} + \frac{(p_{m-j+1})^2}{4}\ \left(\omega_a^{\alpha}(x) - \omega_a^{\alpha}(t)\right)\right]\frac{f(t)}{(t-a)^{\alpha}}dt. \end{array}$$

So, we have proved the following confirmation

Theorem 1. (**Main Theorem**). *Let in integral equation (1) $n = 2m$, let parameters $A_j (1 \le j \le n)$ be connected with the coefficients of the algebraic equation (8) given by formula (9). Moreover, let the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (12) and let in (8) the parameters p_j $(1 \le j \le m)$ satisfy conditions (11). Then the integral equations (1) in the class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in the point $x = a$ is always solvable, and its general solution contains $2m$ arbitrary constants and is given by formula (13), where C_k^{m-j+1} $(k = 1, 2,\ 1 \le j \le m)$ are arbitrary constants.*

Remark 1. The representation of the manifold of solution of the integral equation (1) in form (5) for the case $n = 2m$ gives the possibility to write the general solution in dependence of the roots of the characteristic equation (5). When $n = 2m + 1$, we represent equation (1) in the form

$$\prod_{j=1}^{m} T^{\alpha}_{p_{j},q_{j}} \quad (\Psi) = f\,(x), \tag{14}$$

where

$$\Psi\,(x) = \Pi^{\alpha}_{p_{m+1}}\,(\varphi)\,, \tag{15}$$

$$\Pi^{\alpha}_{p_{m+1}}\,(\varphi) \equiv \varphi\,(x) + p_{m+1} \int_a^x \frac{\varphi\,(t)}{(t-a)^{\alpha}} dt,$$

p_j, q_j $(1 \le j \le m)$ are the coefficients of the algebraic equations

$$\left(\mu^{(j)}\right)^2 + p_j \mu^{(j)} + q_j = 0(1 \le j \le m). \tag{16}$$

In this case equation (14) is represented in the form

$$\Psi\,(x) + \int_a^x \left[\sum_{j=1}^{2m} B_j (\omega^{\alpha}_a\,(t) - \omega^{\alpha}_a\,(x)\,)^{j-1}\right] \frac{\Psi\,(t)}{t-a} dt = f\,(x)\,, \tag{17}$$

where the parameters $B_j (1 \le j \le 2m)$ are connected with the roots of the algebraic equations (16) defined by the formula

$$\begin{aligned}
&B_1 = -\textstyle\sum_{j=1}^{m}\left(\mu_1^{(j)} + \mu_2^{(j)}\right),\\
&B_2 = \textstyle\sum_{\substack{j,\,k\,=\,1\\ j\neq k}}^{m}\left(\mu_1^{(k)}\mu_2^{(j)}\right) + \textstyle\sum_{\substack{j,\,k\,=\,1\\ j\neq k}}^{m}\left(\mu_1^{(k)}\mu_1^{(j)}\right) + \textstyle\sum_{\substack{j,\,k\,=\,1\\ j\neq k}}^{m}\left(\mu_2^{(k)}\mu_2^{(j)}\right),\\
&2!\;B_3 = \textstyle\sum_{\substack{j,\,k,\;\;s\,=\,1\\ j\neq k\neq s}}^{m}\left(\mu_1^{(k)}\mu_2^{(j)}\mu_1^{(s)}\right) + \textstyle\sum_{\substack{j,\,k,\;\;s\,=\,1\\ j\neq k\neq s}}^{m}\left(\mu_2^{(k)}\mu_1^{(j)}\mu_2^{(s)}\right),\\
&\ldots..,\\
&(n-1)!B_n = \textstyle\prod_{j=1}^{m}\mu_1^{(j)}\mu_2^{(j)},
\end{aligned} \tag{18}$$

where $\mu_1^{(j)}$, $\mu_2^{(j)} (1 \le j \le m)$ are the roots of the algebraic equations (16). Substituting $\Psi\,(x)$ from (15) into formula (17) and taking into account the equation

$$\int_a^x (\omega^{\alpha}_a\,(t) - \omega^{\alpha}_a\,(x))^{j-1} \left[\int_a^t \frac{\varphi\,(\tau)}{(t-a)^{\alpha}} d\tau\right] \frac{\mathrm{dt}}{(t-a)^{\alpha}}$$
$$= \frac{1}{\mathrm{j}} \int_a^x (\omega^{\alpha}_a\,(t) - \omega^{\alpha}_a\,(x))^{j} \frac{\varphi\,(t)}{(t-a)^{\alpha}} dt,$$

we obtain

$$\varphi\,(x) + \int_a^x \left[\sum_{j=1}^{2m+1} A_j (\omega^{\alpha}_a\,(t) - \omega^{\alpha}_a\,(x))^{j-1}\right] \frac{\varphi\,(t)}{(t-a)^{\alpha}} dt = f\,(x)\,, \tag{19}$$

where

$$\begin{aligned}
A_1 &= p_{m+1} + B_1,\\
A_2 &= B_2 + B_1 p_{m+1},\\
A_3 &= B_3 + \tfrac{B_2 p_{m+1}}{2},\\
A_4 &= B_4 + \tfrac{B_3 p_{m+1}}{3},\\
&\ldots\ldots\ldots\ldots,\\
A_{2m} &= B_{2m} + \tfrac{B_{2m-1}\cdot p_{m+1}}{2m-1},\\
A_{2m+1} &= \tfrac{B_{2m}\cdot p_{m+1}}{2m}.
\end{aligned}$$

Substituting into these equations B_j $(1 \le j \le 2m)$ from formula (18), we have

$$\begin{aligned}
&A_1 = p_{m+1} - \textstyle\sum_{j=1}^{m}\left(\mu_1^{(j)} + \mu_2^{(j)}\right),\\
&A_2 = \textstyle\sum_{\substack{j,k=1\\ j\neq k}}^{m}\left(\mu_1^{(k)}\mu_2^{(j)}\right) + \sum_{\substack{j,k=1\\ j\neq k}}^{m}\left(\mu_1^{(k)}\mu_1^{(j)}\right) + \sum_{\substack{j,k=1\\ j\neq k}}^{m}\left(\mu_2^{(k)}\mu_2^{(j)}\right)\\
&-\textstyle\sum_{j=1}^{m}\left(\mu_1^{(j)} + \mu_2^{(j)}\right)p_{m+1},\\
&2!\ A_3 = \textstyle\sum_{\substack{j,k,\ s=1\\ j\neq k\neq s}}^{m}\left(\mu_1^{(k)}\mu_2^{(j)}\mu_1^{(s)}\right) + \sum_{\substack{j,k,\ s=1\\ j\neq k\neq s}}^{m}\left(\mu_2^{(k)}\mu_1^{(j)}\mu_2^{(s)}\right)\\
&+p_{m+1}[\textstyle\sum_{\substack{j,k=1\\ j\neq k}}^{m}\left(\mu_1^{(k)}\mu_2^{(j)}\right) + \sum_{\substack{j,k=1\\ j\neq k}}^{m}\left(\mu_1^{(k)}\mu_1^{(j)}\right) + \sum_{\substack{j,k=1\\ j\neq k}}^{m}\left(\mu_2^{(k)}\mu_2^{(j)}\right)],
\end{aligned} \tag{20}$$

where $\mu_1^{(j)}, \mu_2^{(j)}$ are the roots of the algebraic equations (16).

So, we proof the following confirmation

Theorem 2. (**Main Theorem**). *Let the parameters $A_j (1 \le j \le n)$ in integral equation (1) be connected with the roots of the characteristic equation (16) and the number p_{m+1} given by formula (20). Then the problem of finding the solution of the integral equation (1) for $n = 2m+1$, or the integral equation (19) reduces to the problem for finding the solution of the integral equation*

$$\prod_{j=1}^{m} T^{\alpha}_{p_{j},q_{j}}\, \Pi^{\alpha}_{m+1}\left((\varphi)\right) = f\left(x\right). \tag{21}$$

By introducing in the integral equation (13) the new unknown functions

$$\begin{aligned}
&\Psi_{m+1}\left(x\right) = \Pi^{\alpha}_{m+1}\left(\varphi\right),\\
&\Psi_{m}\left(x\right) = T^{\alpha}_{p_{m},q_{m}}\left(\Psi_{m+1}\right),\\
&\Psi_{m-1}\left(x\right) = T^{\alpha}_{p_{m-1},q_{m-1}}\left(\Psi_{m}\right),\\
&\Psi_{m-2}\left(x\right) = T^{\alpha}_{p_{m-2},q_{m-2}}\left(\Psi_{m-1}\right),\\
&\ldots\ldots,\\
&\Psi_{2}\left(x\right) = T^{\alpha}_{p_{2},q_{2}}\left(\Psi_{1}\right),
\end{aligned} \tag{22}$$

$$T^{\alpha}_{p_{1},q_{1}}\left(\Psi_{1}\right) = f\left(x\right), \tag{23}$$

we reduce the problem of finding of the general solution of this integral equation to the solution m of integral equation of type (5) and one integral equation of the type (6).

In particular case, if all the roots of the characteristic equations (16) are real, equal, negative and

$$|p_{m+1}| > |p_m| > |p_{m-1}| > |p_{m-2}| \ldots\ldots.. > |p_1|, \tag{24}$$

function $f(x) \in C\left(\overline{\Gamma}\right)$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o\left[exp\left(\frac{p_{m+1}}{2}\omega_a^\alpha(x)\right)(x-a)^{\gamma_3}\right], \gamma_3 > |p_{m+1}|, \text{ at } x \to a, \tag{25}$$

then the solution of the integral equation (1) for $n = 2m+1$ is given by the formula

$$\varphi(x) == \left(\Pi_{p_{m+1}}^{\alpha,\ C_{m+1}}\right)^{-1}[\prod_{j=1}^{m}\left(T_{p_{m-j+1},\ q_{m-j+1}}^{\alpha, C_1^{m-j+1},\ C_2^{m-j+1}}\right)^{-1}(f)]\ , \tag{26}$$

where $C_1^{m-j+1},\ C_2^{m-j+1}\,(1 \le j \le m)$, C_{m+1} are arbitrary constants,

$$\begin{array}{l}\left(\Pi_{p_{m+1}}^{\alpha, C_{m+1}}\right)^{-1}(\omega) \equiv exp\left[p_{m+1}\omega_a^\alpha(x)\right]C_{m+1} + \omega(x) \\ -p_{m+1}\int_a^x exp\left[p_{m+1}\left(\omega_a^\alpha(x) - \omega_a^\alpha(t)\right)\right]\frac{\omega(t)}{(t-a)^\alpha}dt.\end{array} \tag{27}$$

So, we have proved the following confirmation

Theorem 3. (**Main Theorem**). *Let $n = 2m+1$ in integral equation (1), let the parameters $A_j(1 \le j \le n)$ be connected with coefficients of the algebraic equation (16) by formula (20). Moreover, let the roots of the characteristic equation (16) be real, equal and positive, and let the function $f(x) \in C\left(\overline{\Gamma}\right)$, $f(a) = 0$ with asymptotic behavior (25) and let in (16) parameters $p_j\ (1 \le j \le m)$, p_{m+1} satisfy conditions (24). Then integral equations (1) in the class of function $\varphi(x) \in C\left(\overline{\Gamma}\right)$, vanishing at the point $x = a$ are solvable, and its general solution contains $2m+1$ arbitrary constants and is given by formula (26), where $C_k^{m-j+1}\,(k = 1, 2,\ 1 \le j \le m)$, C_{m+1} are arbitrary constants.*

References

1. Rajabov, N.: About new class of Volterra type integral equation with boundary singularity in kernels. Adv. Appl. Math. Approx. Theory **41**, 41–60 (2013). Springer Proceedings in Mathematics and Statistics
2. Rajabov, N.: On the of a class of Volterra-type integral equations with a fixed boundary singularity in the kernel. Dokladi Math. **90**(1), 1–5 (2014)
3. Rajabov, N.: A new method for investigating a new class of the Volterra type integral equation with a boundary singularity in the kernel. In: Abstract book of The 3^{rd} Abu Dhabi University Annual International Conference "Mathematical Science and its Applications", pp. 69–72. Abu Dhabi University, Abu Dhabi (2014)
4. Rajabov, N.: A new method for investigating a new class of the Volterra type integral equation with a boundary singularity in the kernel. Int. J. Math. Comput. **4**, 410–424 (2015)

5. Rajabov, N.: integral representation of manifold solution for new class of the Volterra type integral equation with a boundary singularity in the case, when kernel contain logarithmic singularity and its power. J. Math. Syst. Sci. **6**, 23–37 (2016)
6. Rajabov, N.: New method for Volterra type integral equations with boundary singular point. New Trends in Analysis and Interdisciplinary Applications, pp. 109–115. Springer International Publishing (2017)
7. Rajabov, N.: About one class of the model super-singular integral equation, extension the one-dimensional Volterra type integral equation with left boundary super-singular point in kernel. In: Proceedings VII International Scientific Conference "Problems of Differential Equations. Analysis and Algebra", pp. 202–205. Aktobe University, Aktobe (2015)
8. Rajabov, N.: To theory one class of Volterra type super-singular integral equation. Natural science series vol. 1, pp. 3–9. Herald Tajik National University (2017)

Isoperimetric Inequalities for Some Integral Operators Arising in Potential Theory

Michael Ruzhansky[1] and Durvudkhan Suragan[2(✉)]

[1] Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom
m.ruzhansky@imperial.ac.uk
[2] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
suragan@math.kz

Abstract. In this paper we review our previous isoperimetric results for the logarithmic potential and Newton potential operators. The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce *a priori* bounds for spectral invariants of operators on arbitrary domains. We demonstrate these in explicit examples.

Keywords: Logarithmic potential operator · Newton potential operator · Geometric extremum problem · Schatten p-norm · Rayleigh-Faber-Krahn inequality · Polya inequality · Luttinger type inequality

1 Introduction

In a bounded domain of the Euclidean space $\Omega \subset \mathbb{R}^d,\ d \geq 2$, it is well known that the solution to the Laplacian equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1}$$

is given by the Newton potential formula (or the logarithmic potential formula when $d = 2$)

$$u(x) = \int_\Omega \varepsilon_d(|x-y|) f(y) dy,\ x \in \Omega, \tag{2}$$

for suitable functions f with $suppf \subset \Omega$. Here

$$\varepsilon_d(|x-y|) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x-y|},\ d = 2, \\ \frac{1}{(d-2)s_d} \frac{1}{|x-y|^{d-2}},\ d \geq 3, \end{cases} \tag{3}$$

is the fundamental solution to $-\Delta$ and $s_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of the unit sphere in $\mathbb{R}^d$.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_31

An interesting question having several important applications is what boundary conditions can be put on u on the (Lipschitz) boundary $\partial\Omega$ so that equation (1) complemented by this boundary condition would have the solution in Ω still given by the same formula (2), with the same kernel ε_d given by (3). It turns out that the answer to this question is the integral boundary condition

$$-\frac{1}{2}u(x)+\int_{\partial\Omega}\frac{\partial\varepsilon_d(|x-y|)}{\partial n_y}u(y)dS_y-\int_{\partial\Omega}\varepsilon_d(|x-y|)\frac{\partial u(y)}{\partial n_y}dS_y=0,\ x\in\partial\Omega, \quad (4)$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point y on $\partial\Omega$. A converse question to the one above would be to determine the trace of the Newton potential (2) on the boundary surface $\partial\Omega$, and one can use the potential theory to show that it has to be given by (4).

The boundary condition (4) appeared in M. Kac's work [1], where he called it "the principle of not feeling the boundary". This was further expanded in Kac's book [2] with several further applications to the spectral theory and the asymptotics of the Weyl's eigenvalue counting function. Independently in [3] T.Sh. Kal'menov and the second author proved the existence of the boundary condition (4) and as byproduct the eigenvalues and eigenfunctions of the Newton potential (2) were calculated in the 2-disk and in the 3-ball. In general, the boundary value problem (1)-(4) has various interesting properties and applications (see, for example, [1–7]). The boundary value problem (1)-(4) can also be generalised for higher degrees of the Laplacian, see [8,9]. In the present paper we consider spectral problems of inverse operators to the nonlocal Laplacian (1)-(4), namely the logarithmic potential operator on $L^2(\Omega)$ defined by

$$\mathscr{L}_\Omega f(x):=\int_\Omega \frac{1}{2\pi}\ln\frac{1}{|x-y|}f(y)dy,\quad f\in L^2(\Omega),\quad \Omega\subset\mathbb{R}^2, \quad (5)$$

and the Newton potential operator on $L^2(\Omega)$ defined by

$$\mathscr{N}_\Omega f(x):=\int_\Omega \frac{1}{(d-2)s_d}\frac{1}{|x-y|^{d-2}}f(y)dy,\quad f\in L^2(\Omega),\quad \Omega\subset\mathbb{R}^d,\ d\geq 3. \quad (6)$$

Spectral properties of the logarithmic and the Newton potential operator have been considered in many papers (see, e.g. [4,9–15]). In this paper we are interested in isoperimetric inequalities of these operators, that is also, in isoperimetric inequalities of the nonlocal Laplacian (1)-(4). A recent general review of isoperimetric inequalities for the Dirichlet, Neumann and other Laplacians was made by Benguria, Linde and Loewe in [16]. In addition to [16], we refer G. Pólya and G. Szegö [17], Bandle [18] and Henrot [19] for historic remarks on isoperimetric inequalities, namely the Rayleigh-Faber-Krahn inequality and the Luttinger inequality.

We review an analogue of the Luttinger inequality for the Newton potential operator $\mathscr{N}_\Omega$ and provide related explicit examples. It is a particular case of our previous result with G. Rozenblum in [20] for the Newton potential (see also [21–23] for a non-self adjoint operators). In Section 3 we present:

- Luttinger type inequality for $\mathscr{N}_\Omega$: The d-ball is a maximizer of the Schatten p-norm of the Newton potential operator among all domains of a given measure in $\mathbb{R}^d$, $d \geq 3$, for all integer $\frac{d}{2} < p < \infty$.

In Section 2, we review the following facts for the logarithmic potential from [24]:

- Rayleigh-Faber-Krahn inequality: The disc is a minimizer of the characteristic number of the logarithmic potential $\mathscr{L}_\Omega$ with the smallest modulus among all domains of a given measure.
- Pólya inequality: The equilateral triangle is a minimizer of the first characteristic number of the logarithmic potential $\mathscr{L}_\Omega$ with the smallest modulus among all triangles of a given area.
- Luttinger type inequality for $\mathscr{L}_\Omega$: The disc is a maximizer of the Schatten p-norm of the logarithmic potential operator among all domains of a given measure in $\mathbb{R}^2$, for all integer $2 \leq p < \infty$.
- Luttinger type inequality for $\mathscr{L}_\Omega$ in triangles: The equilateral triangle is a maximizer of the Schatten p-norm of the logarithmic potential operator among all triangles of a given area in $\mathbb{R}^2$, for all integer $2 \leq p < \infty$.

2 Isoperimetric Inequalities for $\mathscr{L}_\Omega$ and Examples

In this section we review our results for the logarithmic potential from [24] and for the Newton potential [20]. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. We consider the logarithmic potential operator on $L^2(\Omega)$ defined by

$$\mathscr{L}_\Omega f(x) := \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y) dy, \quad f \in L^2(\Omega), \tag{7}$$

where ln is the natural logarithm and $|x-y|$ is the standard Euclidean distance between x and y. Clearly, $\mathscr{L}_\Omega$ is compact and self-adjoint. Therefore, all of its eigenvalues and characteristic numbers are discrete and real. We recall that the characteristic numbers are inverses of the eigenvalues. The characteristic numbers of $\mathscr{L}_\Omega$ may be enumerated in ascending order of their modulus,

$$|\mu_1(\Omega)| \leq |\mu_2(\Omega)| \leq ...,$$

where $\mu_i(\Omega)$ is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_1, u_2, ...$, so that for each characteristic number μ_i there is a unique corresponding (normalized) eigenfunction u_i,

$$u_i = \mu_i(\Omega) \mathscr{L}_\Omega u_i, \quad i = 1, 2, ...$$

It is known, see for example [3], that the equation

$$u(x) = \mathscr{L}_\Omega f(x) = \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y) dy$$

is equivalent to the equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{8}$$

with the nonlocal integral boundary condition

$$-\frac{1}{2}u(x) + \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{1}{2\pi} \ln \frac{1}{|x-y|} u(y) dS_y - \int_{\partial\Omega} \frac{1}{2\pi} \ln \frac{1}{|x-y|} \frac{\partial u(y)}{\partial n_y} dS_y = 0, \; x \in \partial\Omega, \tag{9}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point y on the boundary $\partial\Omega$, which is assumed piecewise C^1 here.

Let H be a separable Hilbert space. By $\mathscr{S}^\infty(H)$ we denote the space of compact operators $P : H \to H$. Recall that the singular numbers $\{s_n\}$ of $P \in \mathscr{S}^\infty(H)$ are the eigenvalues of the positive operator $(P^*P)^{1/2}$ (see [25]). The Schatten p-classes are defined as

$$\mathscr{S}^p(H) := \{P \in \mathscr{S}^\infty(H) : \{s_n\} \in \ell^p\}, \quad 1 \leq p < \infty.$$

In $\mathscr{S}^p(H)$ the Schatten p-norm of the operator P is defined by

$$\|P\|_p := \left(\sum_{n=1}^{\infty} s_n^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \tag{10}$$

For $p = \infty$, we can set

$$\|P\|_\infty := \|P\|$$

to be the operator norm of P on H. As outlined in the introduction, we assume that $\Omega \subset \mathbb{R}^2$ is an open bounded set and we consider the logarithmic potential operator on $L^2(\Omega)$ of the form

$$\mathscr{L}_\Omega f(x) = \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y) dy, \quad f \in L^2(\Omega). \tag{11}$$

It is known that $\mathscr{L}_\Omega$ is a Hilbert-Schmidt operator. By $|\Omega|$ we will denote the Lebesque measure of Ω.

Theorem 1. *Let D be a disc centred at the origin. Then*

$$\|\mathscr{L}_\Omega\|_p \leq \|\mathscr{L}_D\|_p \tag{12}$$

for any integer $2 \leq p \leq \infty$ and any bounded open domain Ω with $|\Omega| = |D|$.

Let us give several examples calculating explicitly values of the right hand side of (3) for different values of p.

Example 1. Let $D \equiv U$ be the unit disc. Then by Theorem 1 we have

$$\|\mathscr{L}_\Omega\|_p \leq \|\mathscr{L}_U\|_p = \left(\sum_{m=1}^{\infty} \frac{3}{j_{0,m}^{2p}} + \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{2}{j_{l,m}^{2p}} \right)^{\frac{1}{p}}, \tag{13}$$

for any integer $2 \leq p < \infty$ and any bounded open domain Ω with $|\Omega| = |U|$. Here j_{km} denotes the m^{th} positive zero of the Bessel function J_k of the first kind of order k.

The right hand sight of the formula (21) can be confirmed by a direct calculation of the logarithmic potential eigenvalues in the unit disc, see Theorem 3.1 in [10].

Example 2. Let $D \equiv U$ be the unit disc. Then by Theorem 1 we have

$$\|\mathscr{L}_\Omega\| \leq \|\mathscr{L}_U\| = \frac{1}{j_{01}^2} \tag{14}$$

for any bounded open domain Ω with $|\Omega| = |U|$. Here $\|\cdot\|$ is the operator norm on the space L^2.

From Corollary 3.2 in [10] we calculate explicitly the operator norm in the right hand sight of (23).

In Theorem 1 when $p = \infty$, the following analogue of the Rayleigh-Faber-Krahn theorem for the integral operator $\mathscr{L}_\Omega$ is used.

Theorem 2. *The disc D is a minimizer of the characteristic number of the logarithmic potential $\mathscr{L}_\Omega$ with the smallest modulus among all domains of a given measure, i.e.*

$$0 < |\mu_1(D)| \leq |\mu_1(\Omega)|$$

for an arbitrary bounded open domain $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |D|$.

In Landkof [26] the positivity of the operator $\mathscr{L}_\Omega$ is proved in domains $\overline{\Omega} \subset U$, where U is the unit disc. In general, $\mathscr{L}_\Omega$ is not a positive operator. For any bounded open domain Ω the logarithmic potential operator $\mathscr{L}_\Omega$ can have at most one negative eigenvalue, see Troutman [13] (see also Kac [12]).

In other words Theorem 2 says that the operator norm of $\mathscr{L}_\Omega$ is maximized in a disc among all Euclidean bounded open domains of a given area.

It follows from the properties of the kernel that the Schatten p-norm of the operator $\mathscr{L}_\Omega$ is finite when $p > 1$, see e.g. the criteria for Schatten classes in terms of the regularity of the kernel in [27]. Our techniques do not allow us to prove Theorem 1 for $1 < p < 2$. In view of the Dirichlet Laplacian case, it seems reasonable to conjecture that the Schatten p-norm is still maximized on the disc also for $1 < p < 2$. However, In Section 3 by using different method we prove such conjecture for the Newton potential operator, see also [20].

We can ask the same question of maximizing the Schatten p-norms in the class of polygons with a given number n of sides. We denote by $\mathscr{P}_n$ the class of plane polygons with n edges. We would like to identify the maximizer for Schatten p-norms of the logarithmic potential $\mathscr{L}_\Omega$ in $\mathscr{P}_n$. According to the Dirichlet Laplacian case, it is natural to conjecture that it is the n-regular polygon. Currently, we have proved this only for $n = 3$:

Theorem 3. *The equilateral triangle centred at the origin has the largest Schatten p-norm of the operator $\mathscr{L}_\Omega$ for any integer $2 \leq p \leq \infty$ among all triangles of a given area. More precisely, if $\triangle$ is the equilateral triangle centred at the origin, we have*

$$\|\mathscr{L}_\Omega\|_p \leq \|\mathscr{L}_\triangle\|_p \tag{15}$$

for any integer $2 \leq p \leq \infty$ and any bounded open triangle Ω with $|\Omega| = |\triangle|$.

When $p = \infty$, Theorem 3 implies the following analogue of the Pólya theorem [28] for the operator $\mathscr{L}_\Omega$.

Theorem 4. *The equilateral triangle $\triangle$ centred at the origin is a minimizer of the first characteristic number of the logarithmic potential $\mathscr{L}_\Omega$ among all triangles of a given area, i.e.*

$$0 < |\mu_1(\triangle)| \leq |\mu_1(\Omega)|$$

for any triangle $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |\triangle|$.

In other words Theorem 4 says that the operator norm of $\mathscr{L}_\Omega$ is maximized in an equilateral triangle among all triangles of a given area.

3 The Newton Potential

Let $\Omega \subset \mathbb{R}^d,\ d \geq 3$ be an open bounded set. We consider the Newton potential operator $\mathscr{N}_\Omega : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\mathscr{N}_\Omega f(x) := \int_\Omega \varepsilon_d(|x-y|) f(y) dy, \quad f \in L^2(\Omega), \tag{16}$$

where $\varepsilon_d(|x-y|) = \frac{1}{(d-2)s_d} \frac{1}{|x-y|^{d-2}},\ d \geq 3.$

Since ε_d is positive, real and symmetric function, $\mathscr{N}_\Omega$ is a positive self-adjoint operator. Therefore, all of its eigenvalues and characteristic numbers are positive real numbers. We recall that the characteristic numbers are inverses of the eigenvalues. The characteristic numbers of $\mathscr{N}_\Omega$ may be enumerated in ascending order

$$0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots,$$

where $\mu_i(\Omega)$ is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_1, u_2, ...,$ so that for each characteristic number μ_i there is a unique corresponding (normalized) eigenfunction u_i,

$$u_i = \mu_i(\Omega) \mathscr{N}_\Omega u_i, \quad i = 1, 2, \ldots.$$

This spectral problem has various interesting properties and applications (see [1] and [5], for example). In particular, one can prove that in the unit ball its spectrum contains the spectrum of the corresponding Dirichlet Laplacian by using an explicit calculation (cf. [9]).

Kac [1] proved that

$$1 = \lim_{\delta \to 0} \sum_{j=1}^{\infty} \frac{1}{1+\mu_j \delta} u_j(y) \int_{\Omega} u_j(x)dx, \quad y \in \Omega, \tag{17}$$

where $\mu_j, j = 1, 2, ...,$ and $u_j, j = 1, 2, ...$ are the characteristic numbers and the corresponding normalized eigenfunctions of the Newton potential operator (16), respectively. The purely analytic fact (17) expresses that the expansion of 1 in a series of orthonormal functions u_j is summable to 1 for every $y \in \Omega$. In [29] Kac gave asymptotic formulae for the characteristic numbers in $\mathbb{R}^d, d \geq 3$. In this section we discuss some other pure analytic facts for the Newton potential. It should be noted that similar results are already known for the Dirichlet Laplacian.

By using the Feynman-Kac formula and spherical rearrangement, Luttinger proved that the ball Ω^* is the maximizer of the partition function of the Dirichlet Laplacian among all domains of the same volume as Ω^* for all positive values of time [30], i.e.

$$Z_{\Omega}^{\mathscr{D}}(t) := \sum_{i=1}^{\infty} \exp(-t\lambda_i^{\mathscr{D}}(\Omega)) \leq Z_{\Omega^*}^{\mathscr{D}}(t) := \sum_{i=1}^{\infty} \exp(-t\lambda_i^{\mathscr{D}}(\Omega^*)), \quad |\Omega| = |\Omega^*|, \forall t > 0,$$

where $\lambda_i^{\mathscr{D}}(\Omega), i = 1, 2, ...$ are the eigenvalues of the Dirichlet Laplacian $\Delta_{\Omega}^{\mathscr{D}}$ in Ω.

The partition function and the Schatten norms are related:

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_p^p = \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-1} Z_{\Omega}^{\mathscr{D}}(t)dt,$$

where Γ is the gamma function. Hence it easily follows that

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_p \leq \|\Delta_{\Omega^*}^{\mathscr{D}}\|_p, \ |\Omega| = |\Omega^*|, \tag{18}$$

when $p > d/2$, $\Omega \subset \mathbb{R}^d$. Here the Schatten p-norm of the Dirichlet Laplacian is defined by

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_p := \left(\sum_{i=1}^{\infty} \frac{1}{[\lambda_i^{\mathscr{D}}]^p} \right)^p, \quad d/2 < p < \infty.$$

The right hand side of the inequality (18) gives the exact upper bound of the Schatten p-norm and it can be calculated explicitly.

Example 3. Let U be the unit disk, then

$$\|\Delta_U^{\mathscr{D}}\|_2^2 = 0.0493....$$

Therefore, from (18) we have

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_2^2 \leq 0.0493..., \ |\Omega| = |U|.$$

This inequality is better than the inequality conjectured in [31]

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_p^p \leq \frac{\Gamma(p-\frac{d}{2})}{\Gamma(p)} \frac{Vol|\Omega|^{\frac{2p}{d}}}{(4\pi)^{\frac{d}{2}}}, \ p > \frac{d}{2}, \tag{19}$$

which implies that

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_2^2 \leq 0.7853...,$$

when $|\Omega| = |U|$.

However, it is important to note that in (19) p is an arbitrary real number greater than $\frac{d}{2}$.

The condition $p > d/2$ in (18) is necessary to absolute convergence of series, but in case $p \leq d/2$ one may use regularization process to get an absolute convergent series.

Example 4. ([32]). In $\Omega \subset \mathbb{R}^2$ the sum

$$\|\Delta_{\Omega}^{\mathscr{D}}\|_1 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\mathscr{D}}(\Omega)} = \infty, \;\; \Omega \subset \mathbb{R}^2.$$

However, using the following regularisation, we find that if $U \equiv \Omega \subset R^2$ is the unit disk, then

$$\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k^{\mathscr{D}}(U)} - \frac{1}{4k} \right) = -0.3557....$$

As usual by $|\Omega|$ we will denote the Lebesque measure of Ω [33–37].

Theorem 5. *Let B be a ball centred at the origin, $d \geq 3$. Then*

$$\|\mathscr{N}_{\Omega}\|_p \leq \|\mathscr{N}_B\|_p \tag{20}$$

for any integer $\frac{d}{2} < p \leq \infty$ and an arbitrary bounded open domain Ω with $|\Omega| = |B|$.

Let us give some examples:

Example 5. Let $B \equiv U$ be the unit 3-ball. Then by Theorem 5 we have

$$\|\mathscr{N}_{\Omega}\|_p \leq \|\mathscr{N}_U\|_p = \left(\sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \frac{2l+1}{j_{l-\frac{1}{2},m}^{2p}} \right)^{\frac{1}{p}}, \tag{21}$$

for any real $2 \leq p < \infty$ and any bounded open domain Ω with $|\Omega| = |U|$. Here j_{km} denotes the m^{th} positive zero of the Bessel function J_k of the first kind of order k. The right hand side of the formula (21) can be confirmed by a direct calculation of the characteristic numbers of the Newton potential in the unit 3-ball, see Theorem 4.1 in [10] (cf. [3]).

Example 6. For the Hilbert-Schmidt norm we have

$$\|\mathscr{N}_{\Omega}\|_2 \leq \|\mathscr{N}_U\|_2 = \sqrt{\frac{7}{48}}, \tag{22}$$

for any bounded open domain Ω with $|\Omega| = |U|$, where $B \equiv U$ is the unit 3-ball. Here, when $p = 2$, we have calculated the value on the right hand side of the inequality (22) by using the polar representation. We omit the routine technical calculation.

Example 7. When $p = \infty$ by Theorem 5, we have

$$\|\mathscr{N}_\Omega\|_{op} \leq \|\mathscr{N}_B\|_{op} = \frac{4}{\pi^2} \tag{23}$$

for any domain Ω with $|\Omega| = |B|$, where $\Omega^* \equiv B$ is the unit ball. Here $\|\cdot\|_{op}$ is the operator norm of the Newton potential on the space L^2.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Kac, M.: On some connections between probability theory and differential and integral equations. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 189–215. University of California Press, Berkeley and Los Angeles (1951)
2. Kac, M.: On some probabilistic aspects of classical analysis. Am. Math. Monthly **77**, 586–597 (1970)
3. Kal'menov, TSH, Suragan, D.: To spectral problems for the volume potential. Doklady Math. **80**, 646–649 (2009)
4. Birman, M.Š., Solomjak, M.Z.: Estimates for the singular numbers of integral operators. Uspehi Mat. Nauk. **32**,**1**(193), 17–84 (1977)
5. Saito, N.: Data analysis and representation on a general domain using eigenfunctions of Laplacian. Appl. Comput. Harmon. Anal. **25**(1), 68–97 (2008)
6. Ruzhansky, M., Suragan, D.: On Kac's principle of not feeling the boundary for the Kohn Laplacian on the Heisenberg group. Proc. AMS. **144**(2), 709–721 (2016)
7. Ruzhansky, M., Suragan, D.: Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups. Adv. Math. **308**, 483–528 (2017)
8. Kal'menov, TSH, Suragan, D.: Boundary conditions for the volume potential for the polyharmonic equation. Differential Equations. **48**, 604–608 (2012)
9. Kalmenov, TSH, Suragan, D.: A boundary condition and spectral problems for the Newton potentials. Op. Theory Adv. Appl. **216**, 187–210 (2011)
10. Anderson, J.M., Khavinson, D., Lomonosov, V.: Spectral properties of some integral operators arising in potential theory. Quart. J. Math. Oxford Ser. (2) **43**(172), 387–407 (1992)
11. Arazy, J., Khavinson, D.: Spectral estimates of Cauchy's transform in $L^2(\Omega)$. Integral Equ. Op. Theory **15**(6), 901–919 (1992)
12. Kac, M.: Integration in function spaces and some of its applications. Lezioni Fermiane. [Fermi Lectures]. Accademia Nazionale dei Lincei, Pisa (1980)
13. Troutman, J.L.: The logarithmic potential operator. Illinois J. Math. **11**, 365–374 (1967)
14. Troutman, J.L.: The logarithmic eigenvalues of plane sets. Illinois J. Math. **13**, 95–107 (1969)
15. Zoalroshd, S.: A note on isoperimetric inequalities for logarithmic potentials. J. Math. Anal. Appl. **437**(2), 1152–1158 (2016)
16. Benguria, R.D., Linde, H., Loewe, B.: Isoperimetric inequalities for eigenvalues of the Laplacian and the Schrödinger operator. Bull. Math. Sci. **2**(1), 1–56 (2012)

17. Pólya, G.,Szegö, G.: Isoperimetric inequalities in mathematical physics. Ann. Math. Stud. **27**. Princeton University Press, Princeton, N.J. (1951)
18. Bandle, C.: Isoperimetric inequalities and applications. In: Pitman (Advanced Publishing Program), **7** of Monographs and Studies in Mathematics. Boston, Mass, London (1980)
19. Henrot, A.: Extremum problems for eigenvalues of elliptic operators. In: Frontiers in Mathematics. Birkhäuser Verlag, Basel (2006)
20. Rozenblum, G., Ruzhansky, M., Suragan, D.: Isoperimetric inequalities for Schatten norms of Riesz potentials. J. Funct. Anal. **271**, 224–239 (2016)
21. Ruzhansky, M., Suragan, D.: Schatten's norm for convolution type integral operator. Russ. Math. Surv. **71**, 157–158 (2016)
22. Ruzhansky, M., Suragan, D.: On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries. Bull. Math. Sci. **6**, 325–334 (2016)
23. Kassymov, A., Suragan, D.: Some spectral geometry inequalities for generalized heat potential operators. Complex Anal. Op. Theory (2016). https://doi.org/10.1007/s11785-016-0605-9
24. Ruzhansky, M., Suragan, D.: Isoperimetric inequalities for the logarithmic potential operator. J. Math. Anal. Appl. **434**, 1676–1689 (2016)
25. Gohberg, I.C., Kreĭn, M.G.: Introduction to the Theory of Linear Nonselfadjoint Operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, vol. 18. AMS, Providence, R.I. (1969)
26. Landkof, N.S.: Foundations of modern potential theory. Translated from the Russian by Doohovskoy A.P., Die Grundlehren der mathematischen Wissenschaften, Band, p. 180. Springer, New York-Heidelberg (1972)
27. Delgado, J., Ruzhansky, M.: Schatten classes on compact manifolds: kernel conditions. J. Funct. Anal. **267**(3), 772–798 (2014)
28. Pólya, G.: On the characteristic frequencies of a symmetric membrane. Math. Z. **63**, 331–337 (1955)
29. Kac, M.: An application of probability theory to the study of Laplaces equation. Ann. Soc. Polon. Math. **25**, 122–130 (1953)
30. Luttinger, J.M.: Generalized isoperimetric inequalities. Proc. Nat. Acad. Sci. U.S.A. **70**, 1005–1006 (1973)
31. Harrell, E.M., Hermib, L.: Differential inequalities for Riesz means and Weyl-type bounds for eigenvalues. J. Funct. Anal. **254**, 3173–3191 (2008)
32. Dostanic, M.R.: Regularized trace of the inverse of the Dirichlet Laplacian. Comm. Pure Appl. Math. **64**(8), 1148–1164 (2011)
33. Banuelos, R., Latala, R., Mendez-Hernandez, P.J.: A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes. Proc. Am. Math. Soc. **129**(10), 2997–3008 (2001)
34. Dostanic, M.R.: The asymptotic behavior of the singular values of the convolution operators with kernels whose Fourier transform are rational. J. Math. Anal. Appl. **295**(2), 496–500 (2012)
35. Kac, M.: Distribution of eigenvalues of certain integral operators. Mich. Math. J. **3**, 141–148 (1956)
36. Kac, M.: Aspects probabilistes de la Theorie du Potentiel. Les Presses de la Universite de Montreal (1970)
37. Lieb, E.H., Loss, M.: Analysis. In: Graduate Studies in Mathematics, vol. 14, 2nd edn. AMS, Providence, RI (2001)

Initial-Boundary Value Problem for a Heat Equation with not Strongly Regular Boundary Conditions

Makhmud A. Sadybekov(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
sadybekov@math.kz

Abstract. We consider a problem on finding a solution of an initial-boundary value problem for a heat equation with regular, but not strongly regular boundary conditions. It is shown that in the case of the potential parity $q(x) = q(1-x)$ the researched class of problems can always be reduced to a sequential solution of two analogous problems, but with strongly regular boundary conditions. Herewith the proof does not depend on whether the system of eigen- and associated functions of a corresponding spectral problem for an ordinary differential equation arising in applying the Fourier method forms a basis. The suggested way of the problem solution can be applied for constructing as classical, and for various types of generalized solutions. The solution method earlier suggested by the author is modernized. Due to this fact input data of the problem do not require an additional smoothness.

Keywords: Initial-boundary value problem · Heat equation · Regular boundary conditions · Not strongly regular boundary conditions · Spectral problem · Eigenfunctions · Associated functions · Fourier method · Basis

1 Introduction

A solution of initial-boundary value problems for partial equations (which describe a great amount of physical, chemical, biological and other processes) by the Fourier method leads to a question of possibility of expansion of an initial function in series by eigen- and associated functions of some boundary value problem for an ordinary differential equation. This question is the basis of the spectral theory of differential operators.

Let $\Omega = \{(x,t): \ 0 < x < 1, \ 0 < t < T\}$ be a plane rectangular domain. Consider a problem on finding a solution of the heat equation

$$u_t(x,t) - u_{xx}(x,t) + q(x)u(x,t) = f(x,t), \tag{1}$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_32

satisfying the initial condition

$$u(x,0) = \varphi(x),\ 0 \le x \le 1 \tag{2}$$

and the boundary conditions of the general type

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0, \\ c_1 u_x(0,t) + d_1 u_x(1,t) + c_0 u(0,t) + d_0 u(1,t) = 0. \end{cases} \tag{3}$$

The coefficients $a_k, b_k, c_k, d_k,\ k = 0,1$ of the boundary condition (3) are, generally speaking, complex numbers.

The problems of a parabolic type with two-point boundary conditions of the general type (3) were earlier investigated in works of N.I. Ionkin and E.I. Moissev [1].

Applying the Fourier method for solving problem (1)-(3) leads to the following problem: under which conditions the arbitrary initial function $\varphi(x)$ is expanded in a convergent series by eigenfunctions or by eigen- and associated functions of an operator given by the differential expression

$$l(y) = -y''(x) + q(x)y(x),\ 0 < x < 1 \tag{4}$$

and the boundary conditions

$$\begin{cases} a_1 y'(0) + b_1 y'(1) + a_0 y(0) + b_0 y(1) = 0, \\ c_1 y'(0) + d_1 y'(1) + c_0 y(0) + d_0 y(1) = 0. \end{cases} \tag{5}$$

In the case, when the boundary conditions (5) are strongly regular, the Riesz basis property in L_2 of the system of eigen- and associated functions of the problem follows from the results of V.P. Mikhailov [2] and G.M. Kesselmann [3]. Basing on this fact, in [1] under assumption of the strong regularity of conditions (5) the solution of problem (1)-(3) is constructed by the method of separation of variables, its uniqueness and stable dependence on initial data in various norms are proved.

But in the case, when boundary conditions are regular but not strongly regular, the question on the basis property of the system of eigen- and associated functions is not solved by the end yet. In this connection we note the work of A.S. Makin [4], where it is distinguished one type of not strongly regular boundary conditions under which the system of eigen- and associated functions of the problem forms a Riesz basis for any potentials $q(x)$. In a more simple case $q(x) \equiv 0$ the question on the basis property of the system of eigen- and associated functions of a problem with not strongly regular boundary conditions is fully solved in [5].

The most sufficient contribution to the spectral theory of non-selfadjoint differential operators is the cycle of works of V.A. Il'in, fully enough cited in the review [6]. Particularly, he established necessary and sufficient conditions of the unconditional basis property of the system of eigen- and associated functions of the differential second order operator (4) regardless of type of boundary conditions. If the system of root functions of problem (4)-(5) satisfies conditions of the theorem of V.A. Il'in, then problem (1)-(3) can be solved by the Fourier method.

The well-posedness of problem (1)-(3) in the particular case $q(x) \equiv 0$ under not strongly regular boundary conditions

$$u_x(0,t) - u_x(1,t) = 0, u(0,t) = 0, \tag{6}$$

known in the literature as the boundary conditions of Samarskii-Ionkin, for which the system of eigen- and (specially chosen) associated functions of the corresponding problem (4)-(5) forms a Riesz basis, was established in [7].

Another particular case of problem (1)-(3) was investigated in [8] by the method of separation of variables in the case when $q(x) \equiv 0$, and boundary conditions have the form

$$u_x(0,t) - u_x(1,t) + \alpha u(1,t) = 0,\ u(0,t) = 0,\ \alpha \neq 0.$$

The boundary conditions of this problem are not strongly regular and the system of eigen- and associated functions of the problem does not form a basis.

We must also note that the Abel means of spectral expansions satisfy the heat equation and a solution of the problem of type (1)-(3) can be obtained in the form of a reduction series, if these expansions are summed by the Abel method with brackets. It is also known that the system of root functions of an ordinary differential operator of an arbitrary order with regular boundary conditions forms a block-basis [9]. A lot of works are published in this direction, but in this paper we will not focus on them.

In the present paper we identify a new property of the not strongly regular boundary conditions (1)-(3) which allows in case $q(x) = q(1-x)$ to reduce an arbitrary problem for the heat equation with any regular, but not strongly regular boundary conditions to a sequential solution of two analogous problem, but already with strongly regular boundary conditions. Herewith the proof does not depend on whether the system of eigen- and associated functions of the corresponding spectral problem (4)-(5) for an ordinary differential equation arising in applying the Fourier method forms a basis. The suggested way of the problem solution can be applied for constructing as classical, and for various types of generalized solutions. Therefore we will not focus on concrete conditions of smoothness existing in the formulating the problem of functions.

2 Not Strongly Regular Boundary Conditions

The boundary conditions (5) are regular ([10], p. 73), if one of the following three conditions holds:

$$a_1 d_1 - b_1 c_1 \neq 0;$$

$$a_1 d_1 - b_1 c_1 = 0,\ \ |a_1| + |b_1| > 0,\ \ a_1 d_0 + b_1 c_0 \neq 0;$$

$$a_1 = b_1 = c_1 = d_1 = 0, \;\; a_0 d_0 - b_0 c_0 \neq 0.$$

The regular boundary conditions are strongly regular in the first and third cases, and in the second case under the additional condition:

$$a_1 c_0 + b_1 d_0 \neq [a_1 d_0 + b_1 c_0].$$

Let us distinguish a class of regular but not strongly regular boundary conditions in the form convenient for us.

Lemma 1. *If the boundary conditions (5) are regular, but not strongly regular, then the boundary conditions (3) can always be reduced to the form*

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0, \\ \qquad\qquad\qquad\qquad\quad c_0 u(0,t) + d_0 u(1,t) = 0, \end{cases} |a_1| + |b_1| > 0 \tag{7}$$

of one of the following four types:

$$\begin{array}{ll} I. & a_1 + b_1 = 0,\, c_0 - d_0 \neq 0; \\ II. & a_1 - b_1 = 0,\, c_0 + d_0 \neq 0; \\ III. & c_0 - d_0 = 0,\, a_1 + b_1 \neq 0; \\ IV. & c_0 + d_0 = 0,\, a_1 - b_1 \neq 0. \end{array} \tag{8}$$

Proof. According to ([10], p. 73), if the boundary conditions (5) are regular, but not strongly regular, then $c_1 = d_1 = 0$ and

$$b_1 c_0 + a_1 d_0 \neq 0, \tag{9}$$

$$a_1 c_0 + b_1 d_0 = \pm[a_1 d_0 + b_1 c_0]. \tag{10}$$

In its turn, condition (10) can be written in the form

$$(a_1 \pm b_1)(c_0 \pm d_0) = 0,$$

that is, even one of the equalities of condition (8) holds. If one of this equalities holds, condition (9) provides the fulfillment of the corresponding inequality from (8). The lemma is proved.

It is necessary to note that the regularity and strong regularity of boundary conditions do not depend on coefficients of a differential operator and on coefficients in boundary conditions. The first variant of the lemma for $q(x) \equiv 0$ was firstly formulated in our paper [14]. By virtue of the remark mentioned above, this result remains true and for an equation with lower-order coefficients. It was used for solving inverse problems for the heat equation (1).

Further we will consider only the boundary conditions of type (7) satisfying one of conditions (8).

For applying the suggested method we need the parity of a function of the potential:

$$q(x) = q(1-x). \tag{11}$$

In what follows we will assume that this condition is fulfilled.

3 Reduction of the Problem to a Sequential Solution of Two Initial-Boundary Value Problems with Non-homogeneous Boundary Conditions of the Sturm Type

Let us consider the even $C(x,t)$ and odd $S(x,t)$ parts of the function $u(x,t)$ with respect to x: $u(x,t) = C(x,t) + S(x,t)$, where

$$2C(x,t) = u(x,t) + u(1-x,t);\ 2S(x,t) = u(x,t) - u(1-x,t). \tag{12}$$

Herewith for all $(x,t) \in \Omega$ the equation holds

$$\begin{array}{ll} C(x,t) = C(1-x,t), & S(x,t) = -S(1-x,t), \\ C_x(x,t) = -C_x(1-x,t), & S_x(x,t) = S_x(1-x,t). \end{array} \tag{13}$$

It is obvious that for constructing a solution of $u(x,t)$ it is sufficient to determine the functions $C(x,t)$ and $S(x,t)$ on "a half" of Ω, i.e., in the subdomain

$$\Omega_0 = \{(x,t):\ 0 < 2x < 1,\ 0 < t < T\}.$$

Taking into account requirements (11), it is easy to make sure that the functions $C(x,t)$ and $S(x,t)$ are solutions of the heat equations in Ω_0:

$$C_t(x,t) = C_{xx}(x,t) - q(x)C(x,t) + f_0(x,t), \tag{14}$$

$$S_t(x,t) = S_{xx}(x,t) - q(x)S(x,t) + f_1(x,t), \tag{15}$$

and satisfy the initial conditions

$$C(x,0) = \varphi_0(x),\ 0 \le 2x \le 1, \tag{16}$$

$$S(x,0) = \varphi_1(x),\ 0 \le 2x \le 1. \tag{17}$$

Here it is denoted

$$2f_0(x,t) = f(x,t) + f(1-x,t),\ 2f_1(x,t) = f(x,t) - f(1-x,t),$$

$$2\varphi_0(x) = \varphi(x) + \varphi(1-x),\ 2\varphi_1(x) = \varphi(x) - \varphi(1-x).$$

Now for the functions $C(x,t)$ and $S(x,t)$ find boundary conditions on the boundary of the domain Ω_0. By satisfying the function $u(x,t) = C(x,t) + S(x,t)$ to the boundary conditions (7), taking into account (13), we get:

$$\begin{aligned} (a_1 - b_1)C_x(0,t) + (a_1 + b_1)S_x(0,t) + (a_0 + b_0)C(0,t) \\ +(a_0 - b_0)S(0,t) = 0, \\ (c_0 + d_0)C(0,t) + (c_0 - d_0)S(0,t) = 0. \end{aligned} \tag{18}$$

Under fulfillment of each of conditions (8) of regularity, but not strongly regularity of the boundary conditions, one of "the main" coefficients of relation

(18) always turns to zero. Using this property, for each of types (8) we obtain for $C(x,t)$ and $S(x,t)$ the following boundary conditions on the left-hand boundary of Ω_0:

I. If $a_1 + b_1 = 0,\ c_0 - d_0 \neq 0$, then

$$(a_1 - b_1)(c_0 - d_0)C_x(0,t) - (a_0 d_0 - b_0 c_0)C(0,t) = 0, \tag{19}$$

$$S(0,t) = \frac{(c_0 + d_0)}{(c_0 - d_0)}C(0,t). \tag{20}$$

II. If $a_1 - b_1 = 0,\ c_0 + d_0 \neq 0$, then

$$(a_1 + b_1)(c_0 + d_0)S_x(0,t) + (a_0 d_0 - b_0 c_0)S(0,t) = 0, \tag{21}$$

$$C(0,t) = \frac{(c_0 - d_0)}{(c_0 + d_0)}S(0,t). \tag{22}$$

III. If $c_0 - d_0 = 0,\ a_1 + b_1 \neq 0$, then

$$C(0,t) = 0, \tag{23}$$

$$(a_1 + b_1)S_x(0,t) + (a_0 - b_0)S(0,t) = -(a_1 - b_1)C_x(0,t). \tag{24}$$

IV. If $c_0 + d_0 = 0,\ a_1 - b_1 \neq 0$, then

$$S(0,t) = 0, \tag{25}$$

$$(a_1 - b_1)C_x(0,t) + (a_0 + b_0)C(0,t) = -(a_1 + b_1)S_x(0,t). \tag{26}$$

Additionally, from relations (13) we get the boundary conditions

$$C_x(\frac{1}{2},t) = 0, \tag{27}$$

$$S(\frac{1}{2},t) = 0 \tag{28}$$

on the right-hand boundary of the domain Ω_0.

Consequently, each of types (8) of the not strongly regular boundary value problems is reduced to the sequential solution of two boundary value problems:

Problem I. *In Ω_0 find a solution $C(x,t)$ of equation (14) satisfying the initial condition (16) and the boundary conditions (19), (27). Using the obtained $C(x,t)$, in Ω_0 find a solution $S(x,t)$ of equation (15) satisfying the initial condition (17) and the boundary conditions (20), (28).*

Problem II. *In Ω_0 find a solution $S(x,t)$ of equation (15) satisfying the initial condition (17) and the boundary conditions (21), (28). Using the obtained $S(x,t)$, in Ω_0 find a solution $C(x,t)$ of equation (14) satisfying the initial condition (16) and the boundary conditions (22), (27).*

Problem III. *In Ω_0 find a solution $C(x,t)$ of equation (14) satisfying the initial condition (16) and the boundary conditions (23), (27). Using the obtained $C(x,t)$, in Ω_0 find a solution $S(x,t)$ of equation (15) satisfying the initial condition (17) and the boundary conditions (24), (28).*

Problem IV. *In Ω_0 find a solution $S(x,t)$ of equation (15) satisfying the initial condition (17) and the boundary conditions (25), (28). Using the obtained $S(x,t)$, in Ω_0 find a solution $C(x,t)$ of equation (14) satisfying the initial condition (16) and the boundary conditions (26), (27).*

It is easy to see that all new boundary conditions obtained on the boundary of Ω_0 for the functions $C(x,t)$ and $S(x,t)$ will be separated. Therefore they are the boundary conditions of the Sturm type and consequently, are strongly regular.

Thus, it is proved.

Theorem 1. *A solution of problem (1)–(3) in case of regular, but not strongly regular conditions for $q(x) = q(1-x)$ can be always equivalently reduced to the sequential solution of two boundary value problems with the strongly regular conditions of the Sturm type.*

By the solutions of the boundary problems found in Ω_0 a solution of problem (1)-(3) is constructed by the formula

$$u(x,t) = \begin{cases} C(x,t) + S(x,t), & 2x \leq 1 \\ C(1-x,t) + S(1-x,t), & 2x \geq 1. \end{cases}$$

Herewith the smoothness of the obtained solution in the whole domain Ω is provided by conditions (13).

By using this theorem, the existence of the solution of problem (1)-(3), its uniqueness and stable dependence on the initial data in the various considered classes of solutions can be obtained from theorems for corresponding problems with strongly regular boundary conditions.

Particularly, the Samarskii-Ionkin problem (1), (2), (6) is reduced to the sequential solution of two boundary value problems. First of all, in Ω_0 we solve equation (14) with the initial condition (16) and the homogeneous boundary conditions of Neumann

$$C_x(0,t) = 0, \;\; C_x(\frac{1}{2},t) = 0.$$

Further, using the found value $C(x,t)$, in Ω_0 we find a solution of equation (15) satisfying the initial condition (17) and the inhomogeneous boundary conditions of Dirichlet

$$S(0,t) = C(0,t), \;\; S(\frac{1}{2},t) = 0.$$

The method of separating solutions onto the even and odd parts is not new. Earlier it was successfully applied by, for example, E.I. Moissev in [11] for solving one non-classical boundary value problem of type of the Tricomi generalized problem for an equation of elliptic-hyperbolic type; it was applied by T.Sh. Kal'menov in [12] for constructing the system of eigenfunctions of a boundary value problem with displacement for a wave equation; and it was applied in [13] for proving the well-posedness of the boundary value problem of Bitsadze for a multidimensional wave equation on a half of a characteristic cone.

This method was worked out by us and firstly applied in [14] in solving inverse problems for the heat equation (1). In [14] we considered a class of problems modeling the process of determining the temperature and density of heat sources given through initial and finite temperature. Their mathematical statements involve inverse problems for the heat equation, in which solving the equation, we have to find the unknown right-hand side depending only on the space variable. We proved the existence and uniqueness of classical solutions to the problem. We solved the problem independently on whether the corresponding spectral problem (for the operator of multiple differentiation with not strongly regular boundary conditions) has a basis of generalized eigenfunctions.

Later, other problems were solved by this method. For example, in [14,15] we considered one family of problems simulating the determination of target components and density of sources from given values of the initial and final states. The mathematical statement of these problems leads to the inverse problem for the diffusion equation, where it is required to find not only a solution of the problem, but also its right-hand side that depends only on a spatial variable. One of specific features of the considered problems is that the system of eigenfunctions of the multiple differentiation operator subject to boundary conditions of the initial problem does not have the basis property. The other specific feature of the considered problems is that an unknown function is simultaneously present both in the right-hand side of the equation and in conditions of the initial and final redefinition. We proved the unique existence of a generalized solution to the mentioned problem.

This method also allowed constructing new stable difference schemes for solving heat problems. In [16] we proposed a new method of solving non-local problems for the heat equation with finite difference method. The main important feature of these problems is their non-self-adjointness. This non-self-adjointness causes major difficulties in their analytical and numerical solving. The problems, which boundary conditions do not possess strong regularity, are less studied. The scope of study of the paper justifies possibility of building a stable difference scheme with weights for above-mentioned type of problems.

We need to note once more that by this method problem (1)-(3) can be solved independently on whether the corresponding spectral problem (4), (5) has the basis property of root functions.

However, the main difficulty is that the second of two obtained boundary value problems is a problem with non-homogeneous boundary conditions. For its solution it is required that the input data of the problem have such smoothness that traces of the solution of the first of two problems (for example, $S_x(0,t)$) have higher smoothness. For this we have to require the smoothness of the input data which will be higher than it is required in similar problems.

For correcting such difficulty, in the next section we suggest modification of the method under which problems with homogeneous boundary conditions are obtained.

4 Reduction of the Problem to a Sequential Solution of Two Initial-Boundary Value Problems with Homogeneous Boundary Conditions of the Sturm Type

The method of solution, consisting in reducing the initial problem to a sequential solution of two initial-boundary value problems with homogeneous boundary conditions of the Sturm type with respect to a spatial variable, will be formulated separately for each of types mentioned in Lemma 1.

4.1 Reduction of the Problem of Type I to a Sequential Solution of Two Problems with Homogeneous Boundary Conditions of the Sturm Type

Consider a problem of type I. Since $a_1 + b_1 = 0$, and herewith $|a_1| + |b_1| > 0$, then without loss of generality we can assume $a_1 = -b_1 = 1$. Since $c_0 - d_0 \neq 0$, then without loss of generality we can assume $c_0 - d_0 = -1$. To simplify writing (omitting additional indexes) we denote $c_0 = c$. Then $d_0 = 1 + c$.

Therefore the problem of type I can be formulated in the form:

In $\Omega = \{(x,t) : \ 0 < x < 1, \ 0 < t < T\}$ *find a solution* $u(x,t)$ *of the heat equation* (1) *satisfying the initial condition* (2) *and boundary conditions of type I:*

$$\begin{cases} u_x(0,t) - u_x(1,t) + au(0,t) + bu(1,t) = 0, \\ \qquad\qquad\qquad cu(0,t) + (1+c)\,u(1,t) = 0. \end{cases} \tag{29}$$

Here the coefficients a, b, c of the boundary condition are arbitrary complex numbers.

To solve the problem we introduce the auxiliary functions:

$$v(x,t) = [u(x,t) + u(1-x,t)]/2, \tag{30}$$

$$w(x,t) = u(x,t) - [1 - (1+2c)(2x-1)]\,v(x,t). \tag{31}$$

Note that if the solution has been searched in the form of the sum of even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (see section 3), then now in a variant suggested by us:

- the function $v(x,t)$ is even on the interval $0 < x < 1$, and is the even part of the function $u(x,t)$;
- and the function $w(x,t)$ is not the odd part of the function $u(x,t)$, though it is the odd function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}[u(x,t) - u(1-x,t)] + (1+2c)(2x-1)\,v(x,t), \tag{32}$$

that is, in the form of the sum of the odd part $\frac{1}{2}[u(x,t) - u(1-x,t)]$ of the function $u(x,t)$ and of the summand $(1+2c)(2x-1)\,v(x,t)$, which (it is easy to verify) is also the odd function on the whole interval $0 < x < 1$.

From (31) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + [1 - (1+2c)(2x-1)]\, v(x,t)\,. \tag{33}$$

Thus, if in the previous variant the solution is represented in the form of the sum of even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (33) the first summand is even on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $1 + 2c \neq 0$.

It is easy to make sure that in virtue of the symmetry conditions (11) the functions $v(x,t)$ and $w(x,t)$ are solutions of the heat equations, satisfy the initial and homogeneous boundary conditions in Ω.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$v_t(x,t) - v_{xx}(x,t) + q(x)\, v(x,t) = f_0(x,t)\,, \tag{34}$$

$$v(x,0) = \varphi_0(x)\,, \quad 0 \le x \le 1, \tag{35}$$

$$v_x(0,t) + [a(1+c) - bc]\, v(0,t) = 0, \quad 0 \le t \le T, \tag{36}$$

$$v_x(1,t) - [a(1+c) - bc]\, v(1,t) = 0, \quad 0 \le t \le T. \tag{37}$$

Here we use the notations

$$f_0(x,t) = \frac{1}{2}[f(x,t) + f(1-x,t)]\,, \qquad \varphi_0(x) = \frac{1}{2}[\varphi(x) + \varphi(1-x)]\,. \tag{38}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$w_t(x,t) - w_{xx}(x,t) + q(x)\, w(x,t) = f_1(x,t), \tag{39}$$

$$w(x,0) = \varphi_1(x)\,, \quad 0 \le x \le 1, \tag{40}$$

$$w(0,t) = 0, \quad 0 \le t \le T, \tag{41}$$

$$w(1,t) = 0, \quad 0 \le t \le T. \tag{42}$$

Here we use the notations

$$f_1(x,t) = f(x,t) - [1 - (1+2c)(2x-1)]\, f_0(x,t) - 4(1+2c)\, v_x(x,t), \tag{43}$$

$$\varphi_1(x) = \varphi(x) - [1 - (1+2c)(2x-1)]\, \varphi_0(x)\,. \tag{44}$$

By direct checking from (38) and (44) it is easy to make sure that if the initial data $\varphi(x)$ of problem (1), (2), (29) satisfy necessary (classical and well-known) consistency conditions, then the initial data $\varphi_0(x)$ and $\varphi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type I (1), (2), (29) is reduced to the sequential solution of two problems *with homogeneous* boundary conditions of the Sturm type with respect to the spatial variable:

- At first for the function $v(x,t)$ we solve the initial-boundary value problem (34) – (37) with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable;
- Then, using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (39) – (42) with the homogeneous boundary conditions of the Sturm type (in this particular case they are the Dirichlet conditions) with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type I (1), (2), (29) in classical and generalized senses follows from the well-known theorems on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this main result at once for all the four types of not strongly regular boundary conditions at the end of the section.

4.2 Reduction of the Problem of Type II to a Sequential Solution of Two Problems with Homogeneous Boundary Conditions of the Sturm Type

Consider a problem of type II. Since $a_1 - b_1 = 0$, and herewith $|a_1| + |b_1| > 0$, then without loss of generality we can assume $a_1 = b_1 = 1$. Since $c_0 + d_0 \neq 0$, then without loss of generality we can assume $c_0 + d_0 = 1$. To simplify writing (omitting additional indexes) we denote $c_0 = c$. Then $d_0 = 1 - c$.

Therefore the problem of type I can be formulated in the form:

In $\Omega = \{(x,t) : \ 0 < x < 1, \ 0 < t < T\}$ find a solution $u(x,t)$ of the heat equation (1) satisfying the initial condition (2) and boundary conditions of type II:

$$\begin{cases} u_x(0,t) + u_x(1,t) + au(0,t) + bu(1,t) = 0, \\ cu(0,t) + (1-c)\,u(1,t) = 0. \end{cases} \tag{45}$$

Here the coefficients a, b, c of the boundary condition are arbitrary complex numbers.

We introduce the auxiliary functions:

$$v(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right], \tag{46}$$

$$w(x,t) = u(x,t) - \left[1 - (1-2c)(2x-1)\right] v(x,t). \tag{47}$$

Note that if the solution has been searched in the form of the sum of even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (see section 3), then in a new variant suggested by us:

- the function $v(x,t)$ is odd on the interval $0 < x < 1$, and is the odd part of the function $u(x,t)$;
- and the function $w(x,t)$ is not the even part of the function $u(x,t)$, though it is the even function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}[u(x,t) + u(1-x,t)] + (1-2c)(2x-1)v(x,t), \qquad (48)$$

that is, in the form of the sum of the even part $\frac{1}{2}[u(x,t) - u(1-x,t)]$ of the function $u(x,t)$ and the summand $(1-2c)(2x-1)v(x,t)$, which (it is easy to verify) is also the even function on the interval $0 < x < 1$.

From (47) it is easy to see that we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + [1 - (1-2c)(2x-1)]v(x,t). \qquad (49)$$

Thus if in the previous variant of the method the solution is represented in the form of the sum of the even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (49) the first summand is even on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $1 - 2c \neq 0$.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$v_t(x,t) - v_{xx}(x,t) + q(x)v(x,t) = f_0(x,t), \qquad (50)$$

$$v(x,0) = \varphi_0(x), \quad 0 \le x \le 1, \qquad (51)$$

$$v_x(0,t) + [a(1-c) - bc]v(0,t) = 0, \quad 0 \le t \le T, \qquad (52)$$

$$v_x(1,t) - [a(1-c) - bc]v(1,t) = 0, \quad 0 \le t \le T. \qquad (53)$$

Here we use the notations

$$f_0(x,t) = \frac{1}{2}[f(x,t) - f(1-x,t)], \quad \varphi_0(x) = \frac{1}{2}[\varphi(x) - \varphi(1-x)]. \qquad (54)$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$w_t(x,t) - w_{xx}(x,t) + q(x)w(x,t) = f_1(x,t), \qquad (55)$$

$$w(x,0) = \varphi_1(x), \quad 0 \le x \le 1, \qquad (56)$$

$$w(0,t) = 0, \quad 0 \le t \le T, \qquad (57)$$

$$w(1,t) = 0, \quad 0 \le t \le T. \qquad (58)$$

Here we use the notations

$$f_1(x,t) = f(x,t) - [1 - (1-2c)(2x-1)]f_0(x,t) + 4(1-2c)v_x(x,t), \qquad (59)$$

$$\varphi_1(x) = \varphi(x) - [1 - (1-2c)(2x-1)]\varphi_0(x). \qquad (60)$$

By direct checking from (54) and (60) it is easy to make sure that if the initial data $\varphi(x)$ of problem (1), (2), (45) satisfy necessary consistency conditions, then the initial data $\varphi_0(x)$ and $\varphi_1(x)$ also satisfy the consistency conditions of their corresponding problems.

Thus the solution of the problem of type II (1), (2), (45) is reduced to the sequential solution of two problems *with homogeneous* boundary conditions of the Sturm type with respect to a spatial variable:

- At first for the function $v(x,t)$ we solve the initial-boundary value problem (50)–(53) with the homogeneous boundary conditions of the Sturm type (in this case they are the Dirichlet conditions) with respect to the spatial variable;
- Then, using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (55)–(58) with the homogeneous boundary conditions of the Sturm type (in this case with conditions of the Dirichlet problem) with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type II (1), (2), (45) in classical and generalized senses follows from the well-known theorems on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this main result at once for all the four types of not strongly regular boundary conditions at the end of the section.

4.3 Reduction of the Problem of Type III to a Sequential Solution of Two Problems with Homogeneous Boundary Conditions of the Sturm Type

Consider a problem of type III. Since $c_0 + d_0 = 0$, and herewith $|c_0| + |d_0| > 0$, then without loss of generality we can assume $c_0 = -d_0 = 1$. Since $a_1 - b_1 \neq 0$, then without loss of generality we can assume $a_1 - b_1 = -1$. To simplify writing (omitting additional indexes) we denote $a_1 = c$. Then $b_1 = 1 + c$.

Therefore the problem of type III can be formulated in the form:

In $\Omega = \{(x,t) :\ 0 < x < 1,\ 0 < t < T\}$ find a solution $u(x,t)$ of the heat equation (1) *satisfying the initial condition* (2) *and the boundary condition of type III:*

$$\begin{cases} cu_x(0,t) + (1+c)\,u_x(1,t) + au(0,t) = 0, \\ u(0,t) - u(1,t) = 0. \end{cases} \tag{61}$$

Here the coefficients a, b, c of the boundary condition are arbitrary complex numbers.

We introduce the auxiliary functions:

$$v(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right], \tag{62}$$

$$w(x,t) = u(x,t) - \left[1 - (1+2c)(2x-1)\right] v(x,t)\,. \tag{63}$$

Note that if the solution has been searched in the form of a sum of even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (section 3), then in a variant suggested by us:

- the function $v(x,t)$ is odd on the interval $0<x<1$, and is the odd part of the function $u(x,t)$;
- and the function $w(x,t)$ is not the even part of the function $u(x,t)$, though it is the even function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t)=\frac{1}{2}\left[u(x,t)+u(1-x,t)\right]-(1+2c)(2x-1)\,v(x,t)\,, \tag{64}$$

that is, in the form of the sum of the even part $\frac{1}{2}\left[u(x,t)+u(1-x,t)\right]$ of the function $u(x,t)$ and the summand $-(1+2c)(2x-1)\,v(x,t)$, which (it is easy to verify) is also the even function on the interval $0<x<1$.

From (63) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t)=w(x,t)+\left[1-(1+2c)(2x-1)\right]v(x,t)\,. \tag{65}$$

Thus if in the previous variant of the method the solution is represented in the form of the sum of the even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (65) the first summand is even on the interval $0<x<1$, and the second summand is neither even, nor odd for $(1+2c)\neq 0$.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$v_t(x,t)-v_{xx}(x,t)+q(x)\,v(x,t)=f_0(x,t)\,, \tag{66}$$

$$v(x,0)=\varphi_0(x)\,,\quad 0\le x\le 1, \tag{67}$$

$$v(0,t)=0,\quad 0\le t\le T, \tag{68}$$

$$v(1,t)=0,\quad 0\le t\le T. \tag{69}$$

Here we use the notations

$$f_0(x,t)=\frac{1}{2}\left[f(x,t)-f(1-x,t)\right],\qquad \varphi_0(x)=\frac{1}{2}\left[\varphi(x)-\varphi(1-x)\right]. \tag{70}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$w_t(x,t)-w_{xx}(x,t)+q(x)\,w(x,t)=f_1(x,t), \tag{71}$$

$$w(x,0)=\varphi_1(x)\,,\quad 0\le x\le 1, \tag{72}$$

$$w_x(0,t)-aw(0,t)=0,\quad 0\le t\le T, \tag{73}$$

$$w_x(1,t)+aw(1,t)=0,\quad 0\le t\le T. \tag{74}$$

Here we use the notations

$$f_1(x,t)=f(x,t)-\left[1-(1+2c)(2x-1)\right]f_0(x,t)-4(1+2c)v_x(x,t), \tag{75}$$

$$\varphi_1(x) = \varphi(x) - [1 - (1+2c)(2x-1)]\,\varphi_0(x). \tag{76}$$

By direct checking from (70) and (76) it is easy to make sure that if the initial data $\varphi(x)$ of problem (1), (2), (61) satisfy necessary consistency conditions, then the initial data $\varphi_0(x)$ and $\varphi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type III (1), (2), (61) is reduced to the sequential solution of two problems with homogeneous boundary conditions of the Sturm type with respect to a spatial variable:

- At first for the function $v(x,t)$ we solve the initial-boundary value problem (66)–(69) with the homogeneous boundary conditions of the Sturm type (in this case with conditions of the Dirichlet problem) with respect to the spatial variable;
- Then, using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (71)–(74) with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type III (1), (2), (61) in classical and generalized senses follows from the well-known theorems on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this main result at once for all the four types of not strongly regular conditions at the end of the section.

4.4 Reduction of the Problem of Type IV to a Sequential Solution of Two Problems with Homogeneous Boundary Conditions of the Sturm Type

Consider a problem of type IV. Since $c_0 - d_0 = 0$, and herewith $|c_0| + |d_0| > 0$, then without loss of generality we can assume $c_0 = d_0 = 1$. Since $a_1 + b_1 \neq 0$, then without loss of generality we can assume $a_1 + b_1 = 1$. To simplify writing (omitting additional indexes) we denote $a_1 = c$. Then $b_1 = 1 - c$.

Therefore the problem of type IV can be formulated in the form:

In $\Omega = \{(x,t),\ 0 < x < 1,\ 0 < t < T\}$ find a solution $u(x,t)$ of the heat equation (1) *satisfying the initial condition* (2) *and the boundary conditions of type IV:*

$$\begin{cases} cu_x(0,t) + (1-c)\,u_x(1,t) + au(0,t) = 0, \\ u(0,t) + u(1,t) = 0. \end{cases} \tag{77}$$

Here the coefficients a, b, c of the boundary condition are arbitrary complex numbers.

We introduce the auxiliary functions:

$$v(x,t) = \frac{1}{2}\left[u(x,t) + u(1-x,t)\right], \tag{78}$$

$$w(x,t) = u(x,t) - [1 - (1-2c)(2x-1)]\,v(x,t). \tag{79}$$

Note that if the solution has been searched in the form of the sum of the even and odd parts $u(x,t) = C(x,t) + S(x,t)$ in the initial version of the method (section 3), then in the variant suggested by us:

- the function $v(x,t)$ is even on the interval $0 < x < 1$, and is the even part of the function $u(x,t)$;
- and the function $w(x,t)$ is not the odd part of the function $u(x,t)$, though it is the odd function.

The last follows from the fact that $w(x,t)$ can be represented in the form

$$w(x,t) = \frac{1}{2}\left[u(x,t) - u(1-x,t)\right] - (1-2c)(2x-1)\,v(x,t)\,, \tag{80}$$

that is, in the form of the sum of the odd part $\frac{1}{2}\left[u(x,t) - u(1-x,t)\right]$ of the function $u(x,t)$ and the summand $-(1-2c)(2x-1)\,v(x,t)$, which (it is easy to verify) is also the odd function on the interval $0 < x < 1$.

From (79) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of the initial problem can be reestablished by the formula

$$u(x,t) = w(x,t) + \left[1 - (1-2c)(2x-1)\right] v(x,t)\,. \tag{81}$$

Thus if in the previous variant of the method the solution is represented in the form of the sum of the even and odd parts of the solution, then in the new variant suggested by us it is not quite so. In representation (81) the first summand is odd on the interval $0 < x < 1$, and the second summand is neither even, nor odd for $(1-2c) \neq 0$.

For the function $v(x,t)$ we obtain the initial-boundary value problem which we need to solve first:

$$v_t(x,t) - v_{xx}(x,t) + q(x)\,v(x,t) = f_0(x,t)\,, \tag{82}$$

$$v(x,0) = \varphi_0(x)\,, \quad 0 \le x \le 1, \tag{83}$$

$$v(0,t) = 0, \quad 0 \le t \le T, \tag{84}$$

$$v(1,t) = 0, \quad 0 \le t \le T. \tag{85}$$

Here we use the notations

$$f_0(x,t) = \frac{1}{2}\left[f(x,t) + f(1-x,t)\right], \qquad \varphi_0(x) = \frac{1}{2}\left[\varphi(x) + \varphi(1-x)\right]. \tag{86}$$

Having the solution $v(x,t)$ of this problem, for the function $w(x,t)$ we get the initial-boundary value problem which we need to solve second:

$$w_t(x,t) - w_{xx}(x,t) + q(x)\,w(x,t) = f_1(x,t), \tag{87}$$

$$w(x,0) = \varphi_1(x)\,, \quad 0 \le x \le 1, \tag{88}$$

$$w_x(0,t) + aw(0,t) = 0, \quad 0 \le t \le T, \tag{89}$$

$$w_x(1,t) - aw(1,t) = 0, \quad 0 \le t \le T. \tag{90}$$

Here we use the notations

$$f_1(x,t) = f(x,t) - [1 - (1-2c)(2x-1)] f_0(x,t) - 4(1-2c)v_x(x,t), \tag{91}$$

$$\varphi_1(x) = \varphi(x) - [1 - (1-2c)(2x-1)] \varphi_0(x). \tag{92}$$

By direct checking from (86) and (92) it is easy to make sure that if the initial data $\varphi(x)$ of problem (1), (2), (77) satisfy necessary consistency conditions, then the initial data $\varphi_0(x)$ and $\varphi_1(x)$ also satisfy the necessary consistency conditions of their corresponding problems.

Thus the solution of the problem of type IV (1), (2), (77) is reduced to the sequential solution of two problems with homogeneous boundary conditions of the Sturm type with respect to a spatial variable:

- At first for the function $v(x,t)$ we solve the initial-boundary value problem (82)–(85) with the homogeneous boundary conditions of the Sturm type (in this case with boundary conditions of Dirichlet) with respect to the spatial variable;
- Then using the obtained value $v(x,t)$, for the function $w(x,t)$ we solve the initial-boundary value problem (87)–(90) with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable.

Therefore the main result on the existence and uniqueness of the solution of the problem of type IV (1), (2), (77) in classical and generalized senses follows from the well-known theorems on corresponding solvability of boundary value problems with conditions of the Sturm type. We will formulate this result as well as the results of sections 4.1, 4.2, 4.3 at once for all the four types of not strongly regular boundary conditions in the next subsection.

4.5 Formulation of the Main Result on Solvability of the Heat Equation with not Strongly Regular Boundary Conditions

For completeness of exposition we formulate the considered problem:

In $\Omega = \{(x,t),\ 0 < x < 1,\ 0 < t < T\}$ consider a problem on finding the solution $u(x,t)$ of the heat equation

$$u_t(x,t) - u_{xx}(x,t) + q(x)u(x,t) = f(x,t), \tag{1}$$

satisfying the initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1 \tag{2}$$

and the not strongly regular boundary conditions of the general form

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0, \\ \qquad\qquad\qquad\qquad\qquad c_0 u(0,t) + d_0 u(1,t) = 0. \end{cases} \tag{7}$$

The coefficients $a_k,\ b_k,\ c_k,\ d_k,\ \ (k=0,1)$ of the boundary condition (7) are arbitrary complex numbers, and $\varphi\,(x)$ and $f\,(x,t)$ are given functions.

We consider the boundary conditions which are regular, but not strongly regular, that is, cases when one of the conditions holds:

$$\begin{array}{ll} I. & a_1+b_1=0,\, c_0-d_0\neq 0;\\ II. & a_1-b_1=0,\, c_0+d_0\neq 0;\\ III. & c_0-d_0=0,\, a_1+b_1\neq 0;\\ IV. & c_0+d_0=0,\, a_1-b_1\neq 0. \end{array} \tag{8}$$

As shown in sections 4.1–4.4, the solution of the problem with the not strongly regular boundary conditions of all the four types has been reduced to the sequential solution of two problems with the homogeneous boundary conditions of the Sturm type with respect to the spatial variable. Herewith one of these problems has the Dirichlet boundary conditions with respect to the spatial variable, that is, it is a classical first initial-boundary value problem.

On the basis of this fact, using the known results from [1], now we can easily formulate a theorem on well-posedness of the general problem with the not strongly regular boundary conditions with respect to the spatial variable.

We formulate the main result in the form of two theorems.

Theorem 2. *Let $q\,(x) = q\,(1-x)$ and let one of conditions (8) hold. That is, the boundary conditions (7) are regular, but not strongly regular. If $\varphi\,(x) \in C^2[0,1]$, $f(x,t) \in C^2\left(\overline{\Omega}\right)$ and the functions $\varphi\,(x)\,, f(x,t)$ satisfy the boundary conditions (7), then there exists a unique classical solution $u(x,t) \in C^{2,1}_{x,t}\left(\overline{\Omega}\right)$ of problem (1), (2), (7).*

Theorem 3. *Let $q\,(x) = q\,(1-x)$ and let one of conditions (8) hold. That is, the boundary conditions (7) are regular, but not strongly regular. If $\varphi\,(x) \in W_2^2(0,1)$ and satisfies the boundary conditions (7), then for any $f\,(x,t) \in L_2(\Omega)$ there exists a unique generalized solution $u(x,t) \in W_2^{2,1}\,(\Omega)$ of problem (1), (2), (7).*

Thus in the present paper the method suggested by us in [11] has been modified such way that it can be applied to an equation with lower-order coefficients. Herewith the solution of problem (1), (2), (7) in case of regular, but not strongly regular conditions can be always equivalently reduced to the sequential solution of two problems with the strongly regular homogeneous boundary conditions of the Sturm type. Thus we have no need in having the estimate of traces of solutions of boundary value problems with boundary conditions of the Sturm type with respect to the of spatial variable.

Note that by this method, problem (1), (2), (7) can be solved regardless whether a corresponding spectral problem for an operator of multiple differentiation with the not strongly regular boundary conditions has the basis property of root functions.

It is easy to see that the suggested method can be used in solving a wide range of problems for equations of the form

$$A(t)u(x,t) = u_{xx}(x,t) - q(x)u(x,t) + f(x,t)$$

with the operator coefficient $A(t)$ and with the regular, but not strongly regular boundary conditions (7). For example, initial-boundary value problems for hyperbolic equations belong to the problems of such type [17].

Acknowledgements. The author expresses his great gratitude to the academician of RAS E.I. Moiseev and the academician of NAS RK T.Sh. Kal'menov for consistent support and fruitful discussion of the results. This research is supported by the grant no. 0824/GF4 and target program 0085/PTSF-14 of the Ministry of Education and Science of Republic of Kazakhstan.

References

1. Akhymbek, M.E., Sadybekov, M.A.: On a difference scheme for nonlocal heat transfer boundary-value problem. AIP Conf. Proc. **1759**, 020032 (2016)
2. Ionkin, N.I., Moiseev, E.I.: A problem for a heat equation with two-point boundary conditions. Differ. Uravn. **15**(7), 1284–1295 (1979)
3. Ionkin, N.I.: Solution of boundary value problem in heat conduction theory with nonlocl boundary conditions. Differ. Uravn. **13**(2), 294–304 (1977)
4. Il'in, V.A., Kritskov, L.V.: Properties of spectral expansions corresponding to non-self-adjoint differential operators. J. Math. Sci. **116**(5), 3489–3550 (2003)
5. Kalmenov, T.S.: On multidimensional regular boundary value problems for the wave equation. (Russian). Izv. Akad. Nauk Kazakh. SSR. Ser. Fiz.-Mat. **3**, 18–25 (1982)
6. Kalmenov, T.S.: Spectrum of a boundary-value problem with translation for the wave-equation. Diff. Equ. **19**(1), 64–66 (1983)
7. Kesel'man, G.M.: On the unconditional convergence of eigenfunction expansions of certain differential operators. Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 82–93 (1964)
8. Lang, P., Locker, J.: Spectral theory of two-point differential operators determined by $-D^2$. J. Math. Anal. And Appl. **146**(1), 148–191 (1990)
9. Makin, A.S.: On spectral decompositions corresponding to non-self-adjoint Sturm-Liouville operators. Dokl. Math. **73**(1), 15–18 (2006)
10. Mihailov, V.P.: On Riesz bases in $L_2(0,1)$. Dokl. Akad. Nauk SSSR. **144**(5), 981–984 (1962)
11. Moiseev, E.I.: Uniqueness of the solution of a nonclassical boundary-value problem. Differ. Uravn. **28**(7), 890–900 (1992)
12. Mokin, AYu.: On a family of initial-boundary value problems for the heat equation. Diff. Equ. **45**(1), 126–141 (2009)
13. Naimark, M.A.: Linear Differential Operators. Nauka, Moscow (1969) (in Russian)
14. Orazov, I., Sadybekov, M.A.: On a class of problems of determining the temperature and density of heat sources given initial and final temperature. Sib. Math. J. **53**(1), 146–151 (2012)
15. Orazov, I., Sadybekov, M.A.: On an inverse problem of mathematical modeling of the extraction process of polydisperse porous materials. AIP Conf. Proc. **1676**, 020005 (2015)
16. Orazov, I., Sadybekov, M.A.: One-dimensional diffusion problem with not strengthened regular boundary conditions. AIP Conf. Proc. **1690**, 040007 (2015)
17. Shkalikov, A.A.: Bases formed by eigenfunctions of ordinary differential-operators with integral boundary-conditions. Vestnik Moskovskogo universiteta. Seriya 1: Matematika. Mekhanika **6**, 41–51 (1982)

On S-Number Inequalities of Triangular Cylinders for the Heat Operator

Tynysbek Kal'menov, Aidyn Kassymov(✉), and Durvudkhan Suragan

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
kalmenov@math.kz, kassymov@math.kz, suragan@math.kz

Abstract. In this paper we prove that the first s-number of the Cauchy-Dirichlet heat operator is minimized in the equilateral cylinder among all Euclidean triangular cylindric domains of a given volume as well as we obtain spectral geometric inequalities of the Cauchy-Dirichlet-Neumann heat operator in the right and equilateral triangular cylinder. It is also established that maximum of the second s-number of the Cauchy-Neumann heat operator is reached by the equilateral triangular cylinder among all triangular cylinders of given volume. In addition, we prove that the second s-number of the Cauchy-Neumann heat operator is maximized in the circular cylinder among all cylindrical Lipschitz domains of fixed volume.

Keywords: Isoperimetric inequalities · Non-selfadjoint operator · S-Number · Polya inequality · Heat operator

1 Introduction

In G. Pólya's work [6] he proves that the first eigenvalue of the Dirichlet Laplacian is minimized in the equilateral triangle among all triangles of given area. Our aim (see, e.g. [3]) is to extend those similar known results of the self-adjoint operators to non-self adjoint operators. Thus, first of all, we prove a Pólya type inequality for the Cauchy-Dirichlet heat operator, that is, the first s-number of the Cauchy-Dirichlet heat operator is minimized in the equilateral triangular cylinder among all triangular cylinders of given volume.

In [11] the author proves certain (isoperimetric) eigenvalue inequalities for the mixed Dirichlet-Neumann Laplacian operator in the right and equilateral triangles. As many other isoperimetric inequalities these inequalities have physical interpretation. Note that one can also think about eigenvalues as related to the time dependent survival probability of the Brownian motion on a triangle, reflecting on the Neumann boundary, and dying on the Dirichlet part (see [11]). In this context, it is clear that enlarging the Dirichlet part leads to a shorter survival time. It is also reasonable, that having the Dirichlet condition on one long

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_33

side gives a larger chance of dying, than having a shorter Dirichlet side. In this paper we generalize these inequalities for the mixed Cauchy-Dirichlet-Neumann heat operator in the right and equilateral triangular cylinders.

R. S. Laugersen and B. Siudeja [4] proved that the first nonzero Neumann Laplacian eigenvalue is shown to be maximal for the equilateral triangle among all triangles of given area. Below we also obtain a version of R. S. Laugersen and B. Siudeja's inequality for the Cauchy-Neumann heat operator in the triangular cylinders.

The Szegö-Weinberger inequality (see [2,12,14]) shows that the second eigenvalue of the Laplacian with the Neumann boundary condition is maximized in a ball among all Lipschitz domains in R^d, $d \geq 2$, of the same measure. In this paper analogue of Szegö-Weinberger inequality is also proved for the heat operator. That is, we prove that the second s-number of the Cauchy-Neumann heat operator is maximized in the circular cylinder among all Euclidean cylindric Lipschitz domains of a given volume. Spectral isoperimetric inequalities have been mainly studied for the Laplacian related operators, for instance, for the p-Laplacians and bi-Laplacians. However, there also many papers on this subject for other type compact operators. For instance, in the recent work [7] the authors proved Rayleigh-Faber-Krahn type inequality and Hong-Krahn-Szegö type inequality for the Riesz potential (see also [8–10]). All these works were for self-adjoint operators. As mentioned our main goal is to extend those known isoperimetric inequalities for non-self-adjoint operators (see, e.g. [3]). Summarizing our main results of the present paper, we prove the following facts:

- The first s-number of the Cauchy-Dirichlet heat operator is minimized in the equilateral triangular cylinder among all triangular cylinders of given area.
- Isoperimetric inequalities of s-numbers for the mixed Cauchy-Dirichlet-Neumann heat operator in the right and equilateral triangular cylinders.
- The second s-number of the Cauchy-Neumann heat operator is maximized in the equilateral triangular cylinder among all triangular cylinders with given area.
- The second s-number is maximized in the circular cylinder among all cylindric Lipschtiz domains of the same volume.

In Section 2 we give preliminary discussions and fix notations. In Section 3 we present main results of this paper and their proofs. In Section 4 we give some geometric extremum results on the circular cylinder.

2 Preliminaries

Let $D = \Omega \times (0,1)$ be cylindrical domain, where $\Omega \subset R^2$ is a triangle. We consider (see, for example, [13]) the Cauchy-Dirichlet and Cauchy-Neumann heat operators $\diamondsuit_D,\ \diamondsuit_N : L^2(D) \to L^2(D)$ respectively, by the formulae

$$\diamondsuit_D u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \end{cases} \tag{1}$$

and

$$\diamondsuit_N u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \end{cases} \tag{2}$$

Here $\partial\Omega$ is the boundary of Ω and $\frac{\partial}{\partial n}$ is the normal derivative on the boundary. The operators $\diamondsuit_D$ and $\diamondsuit_N$ are compact, but these are non-selfadjoint operators in $L^2(D)$. Adjoint operators $\diamondsuit_D^*$ and $\diamondsuit_N^*$ to the operators $\diamondsuit_D$ and $\diamondsuit_N$ can be presented as

$$\diamondsuit_D^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,1) = 0, \ \ x \in \Omega, \\ v(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \end{cases} \tag{3}$$

and

$$\diamondsuit_N^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,1) = 0, \ \ x \in \Omega, \\ \frac{\partial \Delta_x v(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1). \end{cases} \tag{4}$$

Recall that if A is a compact operator, then the eigenvalues of the operator $(A^*A)^{1/2}$, where A^* is the adjoint operator to A, are called s-numbers of the operator A (see, e.g. [1]).

A direct calculation gives that those operators $\diamondsuit_D^*\diamondsuit_D$ and $\diamondsuit_N^*\diamondsuit_N$ have the following formulae

$$\diamondsuit_D^*\diamondsuit_D u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \\ \Delta_x u(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \end{cases} \tag{5}$$

and

$$\diamondsuit_N^*\diamondsuit_N u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1). \end{cases} \tag{6}$$

Let $D_\triangle = \triangle \times (0,1)$ be a cylindrical domain, where $\triangle \subset R^2$ is a right triangle with the sides of length $L \geq M \geq S$ (that is, with the boundary $\partial\triangle = \{L, M, S\}$). We also denote by $L, \ M, \ S$ sides of the right triangle with respect to their lengths (cf. [11]). We consider the heat operator with the Cauchy-Dirichlet-Neumann problem $\diamondsuit_\triangle : L^2(D_\triangle) \to L^2(D_\triangle)$ in the form

$$\diamondsuit_\triangle u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \;\; x \in \triangle, \\ u(x,t) = 0, \;\; x \in D \subset \{L, M, S\}, \;\; \forall t \in (0,1) \\ \frac{\partial u(x,t)}{\partial n} = 0, \;\; x \in \partial\triangle \backslash D, \;\; \forall t \in (0,1). \end{cases} \tag{7}$$

Here $D \in \{L, M, S\}$ means D is one of the sides, where we set the Dirichlet condition. Its adjoint operator $\diamondsuit^*_\triangle$ can be presented as

$$\diamondsuit^*_\triangle v(x,t) := \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ u(x,1) = 0, \;\; x \in \triangle, \\ \Delta v(x,t) = 0, \;\; x \in D \subset \{L, M, S\}, \;\; \forall t \in (0,1) \\ \frac{\partial \Delta v(x,t)}{\partial n} = 0, \;\; x \in \partial\triangle \backslash D, \;\; \forall t \in (0,1). \end{cases} \tag{8}$$

A direct calculation gives that the operator $\diamondsuit^*_\triangle \diamondsuit_\triangle$ has the following formula

$$\diamondsuit^*_\triangle \diamondsuit_\triangle u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \;\; x \in \triangle, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \;\; x \in \triangle, \\ u(x,t) = 0, \;\; x \in D \subset \{L, M, S\}, \;\; \forall t \in (0,1), \\ \frac{\partial u(x,t)}{\partial n} = 0, \;\; x \in \partial\triangle \backslash D, \;\; \forall t \in (0,1), \\ \Delta_x u(x,t) = 0, \;\; x \in D \subset \{L, M, S\}, \;\; \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \;\; x \in \partial\triangle \backslash D, \;\; \forall t \in (0,1). \end{cases} \tag{9}$$

Let s_1^N and s_2^N be the first and second s-numbers of the Cauchy-Neumann problem, respectively. Let s_1^{side} be first s-number of the spectral problem with the Dirichlet condition to this side. That is, s_1^{SL} would correspond to the Dirichlet conditions imposed on the shortest and longest sides. s_1^D is the first s-number of the Cauchy-Dirichlet heat operator. Thus, we will use these notations in the following sections.

3 Main Results and Their Proofs

We denote an equilateral triangular cylinder by $C_\triangle = \Omega^* \times (0,1)$, where $\Omega^* \subset R^2$ is an equilateral triangle. Here and after $|\cdot|$ is measure of a domain.

Let us introduce the operators T_D, $L_D : L^2(\Omega) \to L^2(\Omega)$ respectively, by the formulas

$$T_D z(x) = \begin{cases} -\Delta z(x) = \mu z(x), \\ z(x) = 0, \;\; x \in \partial\Omega. \end{cases} \tag{10}$$

and

$$L_D z(x) = \begin{cases} \Delta^2 z(x) = \lambda z(x), \\ z(x) = 0, \;\; x \in \partial\Omega, \\ \Delta z(x) = 0, \;\; x \in \partial\Omega. \end{cases} \tag{11}$$

Lemma 1. *The first eigenvalue of the operator L_D is minimized in the equilateral triangle among all triangles with given area.*

Proof. Pólya's theorem [6] for the operator T_D says that the equilateral triangle is minimizer of the first Dirichlet Laplacian eigenvalue among all triangles Ω of the same area with $|\Omega^*| = |\Omega|$. It is easy to see that

$$T_D^2 z(x) = \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0, \ \ x \in \partial\Omega, \\ \Delta z(x) = 0, \ \ x \in \partial\Omega. \end{cases} \tag{12}$$

That is, $T_D^2 = L_D$ and $\mu^2 = \lambda$. Thus, we establish $\lambda_1(\Omega^*) = \mu_1^2(\Omega^*) \leq \mu_1^2(\Omega) = \lambda_1(\Omega)$. Thus, $\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$.

Theorem 1. *The first s-number of the operator $\diamondsuit_D$ is minimized in the equilateral triangular cylinder among all triangular cylinders of given volume, that is,*

$$s_1^D(C_\triangle) \leq s_1^D(D),$$

with $|D| = |C_\triangle|$.

Proof. Let u be a nonnegative, measurable function on R^2, and let V be a line through the origin of R^2. Choose an orthogonal coordinate system in R^2 such that the x^1-axis is perpendicular to $V = x^2$.

Recall that a nonnegative, measurable function $u^\star(x|V)$, $x = (x^1, x^2)$, on R^2 is called a Steiner symmetrization with respect to V of the function $u(x)$, if $u^\star(x^1, x^2)$ is a symmetric decreasing rearrangement with respect to x^1 of $u(x^1, x^2)$ for each fixed x^2. The Steiner symmetrization (with respect to the x^1-axis) $\Omega^\star$ of a measurable set Ω is defined in the following way: if we write $x = (x^1, y)$ with $y \in R^2$, and let $\Omega_z = \{x^1 : (x^1, y) \in \Omega\}$, then

$$\Omega^\star := \{(x^1, y) \in R \times R : x^1 \in \Omega_y^*\},$$

where Ω_y^* is a symmetric rearrangement of Ω_y.

The domain $D = \{(x,t)|x \in \Omega \subset R^2, \ t \in (0,1)\}$ is a cylindrical domain and we can have $u(x,t) = X(x)\varphi(t)$, so that $u_1(x,t) = X_1(x)\varphi_1(t)$ is the first eigenfunction of the operator $\diamondsuit_D^*\diamondsuit_D$, where $\varphi_1(t)$ and $X_1(x)$ are the first eigenfunctions of variables t and x, respectively. Therefore, we have

$$-\varphi_1^{''}(t)X_1(x) + \varphi_1(t)\Delta^2 X_1(x) = s_1^D \varphi_1(t)X_1(x). \tag{13}$$

By the variational principle for the operator $\diamondsuit_D^*\diamondsuit_D$ and after a straightforward calculation, we obtain

$$s_1^D(D) = \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^1 \varphi_1^2(t)dt \int_\Omega (\Delta X_1(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \int_0^1 \varphi_1^2(t)dt \int_\Omega (-\mu_1(\Omega)X_1(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \mu_1^2(\Omega)\int_0^1 \varphi_1^2(t)dt \int_\Omega (X_1(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \lambda_1(\Omega)\int_0^1 \varphi_1^2(t)dt \int_\Omega (X_1(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_1^2(x)dx},$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the operator L_D given by the formula (11).

For each non-negative function $X \in L^2(\Omega)$, we obtain

$$\int_\Omega |X_1(x)|^2 dx = \int_{\Omega^*} |X_1^*(x)|^2 dx \quad \text{with } |\Omega^*| = |\Omega|. \tag{14}$$

By applying Lemma 1 and (14), we get

$$s_1^D(D) = \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\Omega X_1^2(x)dx + \lambda_1(\Omega)\int_0^1 \varphi_1^2(t)dt \int_\Omega (X_1(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} X_1^2(x)dx}$$

$$\geq \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx + \lambda_1(\Omega^*)\int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx}{\int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx + \int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} X_1^*(x)(\lambda_1(\Omega^*)X_1^*(x))dx}{\int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx + \int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} X_1^*(x)\Delta^2 X_1^*(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_{\Omega^*} (X_1^*(x))^2 dx}$$

$$= \frac{-\int_0^1 \int_{\Omega^*} \frac{\partial^2 u_1^*(x,t)}{\partial t^2} dxdt + \int_0^1 \int_{\Omega^*} u_1^*(x,t)\Delta_x^2 u_1^*(x,t)dxdt}{\int_0^1 \int_{\Omega^*} (u_1^*(x,t))^2 dxdt}$$

$$\geq \inf_{z(x,t)\neq 0} \frac{-\int_0^1 \int_{\Omega^*} z_t(x,t)z(x,t)dxdt + \int_0^1 \int_{\Omega^*} z(x,t)\Delta_x^2 z(x,t)dxdt}{\int_0^1 \int_{\Omega^*} z^2(x,t)dxdt} = s_1^D(C_\triangle).$$

The proof is complete.

Now let us introduce operators $T_\triangle, L_\triangle : L^2(\triangle) \to L^2(\triangle)$, respectively, by

$$T_\triangle z(x) = \begin{cases} -\Delta z(x) = \beta z(x), \\ z(x) = 0, \quad x \in D \subset \{L, M, S\}, \\ \frac{\partial z(x)}{\partial n} = 0, \quad \partial\triangle \backslash D, \end{cases} \tag{15}$$

and

$$L_\triangle z(x) = \begin{cases} \Delta^2 z(x) = \eta z(x), \\ z(x) = 0, \quad x \in D \subset \{L, M, S\}, \\ \frac{\partial z(x)}{\partial n} = 0, \quad x \in \partial\triangle \backslash D, \\ \Delta z(x) = 0, \quad x \in D \subset \{L, M, S\}, \\ \frac{\partial \Delta z(x)}{\partial n} = 0, \quad x \in \partial\triangle \backslash D. \end{cases} \tag{16}$$

Lemma 2. *For the operator $L_\triangle$ and any right triangle $\triangle$ with the smallest angle α with $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$ we have*

$$0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_2^N < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1^D. \tag{17}$$

When $\alpha = \frac{\pi}{6}$, $\eta_1^M = \eta_2^N$, and for $\alpha = \frac{\pi}{4}$ (right isosceles triangle) we have $S = M$ and $\eta_2^N < \eta_1^L$. All other inequalities stay sharp in these cases. For arbitrary triangle

$$\min\{\eta_1^S, \eta_1^M, \eta_1^L\} < \eta_2^N \le \eta_1^{SM} \le \eta_1^{SL} \le \eta_1^{ML} \tag{18}$$

for any length of sides. However, it is possible that $\eta_2^N > \eta_1^L$ (for any small perturbation of the equilateral triangle) or $\eta_2^N < \eta_1^M$ (for the right triangle with $\alpha < \frac{\pi}{6}$).

Proof. It is easy to see that $L_\triangle = T_\triangle^2$. It means that $\eta = \beta^2$. From [11] for any right triangle with smallest angle $\alpha \in (\frac{\pi}{6}, \frac{\pi}{4})$ we have the following inequalities

$$0 = \beta_1^N < \beta_1^S < \beta_1^M < \beta_2^N < \beta_1^L < \beta_1^{SM} < \beta_1^{SL} < \beta_1^{ML} < \beta_1^D.$$

Using this fact, we obtain

$$0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_2^N < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1^D.$$

When $\alpha = \frac{\pi}{6}$, we have $\beta_1^M = \beta_2^N$, then $\eta_1^M = \eta_2^N$ and for $\alpha = \frac{\pi}{4}$ (right isosceles triangle) we have $S = M$ and $\beta_2^N < \beta_1^L$, then $\eta_2^N < \eta_1^L$. For arbitrary triangle

$$\min\{\beta_1^S, \beta_1^M, \beta_1^L\} < \beta_2^N \le \beta_1^{SM} \le \beta_1^{SL} \le \beta_1^{ML}$$

for any length of sides. Moreover,

$$\min\{\eta_1^S, \eta_1^M, \eta_1^L\} < \eta_2^N \le \eta_1^{SM} \le \eta_1^{SL} \le \eta_1^{ML}.$$

However, it is possible that $\beta_2^N > \beta_1^L$ in the case $\eta_2^N > \eta_1^L$ (for any small perturbation of the equilateral triangle) or $\beta_2^N < \beta_1^M$, after that $\eta_2^N < \eta_1^M$ (for right triangle with $\alpha < \frac{\pi}{6}$). This completes the proof.

Theorem 2. *For any right triangular cylinder $D_\triangle$, with the smallest angle α with $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$,*

$$\frac{\pi^2}{4} = s_1^N < s_1^S < s_1^M < s_2^N < s_1^L < s_1^{SM} < s_1^{SL} < s_1^{ML} < s_1. \tag{19}$$

When $\alpha = \frac{\pi}{6}$, $s_1^M = s_2^N$, and for $\alpha = \frac{\pi}{4}$ (right isosceles triangular cylinder) we have $S = M$ and $s_2^N < s_1^L$. All other inequalities stay sharp in these cases. For arbitrary triangular cylinder

$$\min\{s_1^S, s_1^M, s_1^L\} < s_2^N \le s_1^{SM} \le s_1^{SL} \le s_1^{ML} \tag{20}$$

for any length of sides. However, it is possible that $s_2^N > s_1^L$ (for any small perturbation of the equilateral triangular cylinder) or $s_2^N < s_1^M$ (for right triangular cylinders with $\alpha < \frac{\pi}{6}$).

Proof. Let us prove first step of the inequality (19). To do it we solve the following problem by Fourier's method:

$$\diamondsuit^*_\triangle \diamondsuit_\triangle u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ \ x \in \triangle, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \ \ x \in \triangle, \\ u(x,t) = 0, \ \ x \in D \subset \{L, M, S\}, \ \forall t \in (0,1), \\ \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\triangle \backslash D, \ \ \forall t \in (0,1), \\ \Delta_x u(x,t) = 0, \ \ x \in D \subset \{L, M, S\}, \ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ \ x \in \partial\triangle \backslash D, \ \forall t \in (0,1). \end{cases} \tag{21}$$

Thus, we arrive at the spectral problems for $\varphi(t)$ and $X(x)$ separately, i.e

$$\begin{cases} \Delta^2 X(x) = \beta^2(\triangle) X(x), \ \ x \in \triangle, \\ X(x) = 0, \ \ x \in D \subset \{L, M, S\}, \ \forall t \in (0,1), \\ \frac{\partial X(x)}{\partial n} = 0, \ \ x \in \partial\triangle \backslash D, \\ \Delta X(x) = 0, \ \ x \in D \subset \{L, M, S\}, \\ \frac{\partial \Delta X(x)}{\partial n} = 0, \ \ x \in \partial\triangle \backslash D, \end{cases} \tag{22}$$

and

$$\begin{cases} \varphi^{''}(t) + (s - \beta^2)\varphi(t) = 0, \ t \in (0,1), \\ \varphi(0) = 0, \\ \varphi^{'}(1) + \beta(\triangle)\varphi(1) = 0. \end{cases} \tag{23}$$

It also gives that

$$\tan\sqrt{s - \beta^2} = -\frac{\sqrt{s-\beta^2}}{\beta}. \tag{24}$$

We have (see, [11]) $0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_2^N < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1^D$ and

$$\tan\sqrt{s(\beta) - \beta^2} = -\frac{\sqrt{s(\beta)-\beta^2}}{\beta}. \tag{25}$$

It is easy to see that,

$$s'(\beta) = \frac{2s(\beta)\cos^2\sqrt{s-\beta^2}}{\beta^2 + \beta\cos^2\sqrt{s-\beta^2}}. \tag{26}$$

The s-numbers and β are positive, then

$$s_1'(\beta) > 0. \tag{27}$$

It means the function $s(\eta)$ is monotonically increasing. If $\beta_1^N = 0$ from (23) we take $s_1^N = \frac{\pi^2}{4}$ and [11] and from Lemma 2 we take $0 = \eta_1^N < \eta_1^S < \eta_1^M < \eta_1^L < \eta_1^{SM} < \eta_1^{SL} < \eta_1^{ML} < \eta_1$, and thus get

$$\frac{\pi^2}{4} = s_1^N < s_1^S < s_1^M < s_1^L < s_1^{SM} < s_1^{SL} < s_1^{ML} < s_1. \tag{28}$$

Let us prove the second part of inequality (19) $s_1^M < s_2^N < s_1^L$ and from Lemma 2 we get $\eta_1^M < \eta_2^N < \eta_1^L$. The operator $\diamondsuit_\triangle^* \diamondsuit_\triangle$ is a self-adjoint and compact operator. Hence, we have complete orthonormal system in $L^2(D_\triangle)$, therefore,

$$\int_{D_\triangle} u_i u_j dxdt = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Now using Lemma 2, we obtain

$$s_1^M = \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\triangle (X_1^M(x))^2 dx + \int_0^1 \varphi_1^2(t)dt \int_\triangle X_1^M \Delta^2 X_1^M(x)dx}{\int_0^1 \int_\triangle (u_1^M(x,t))^2 dxdt}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt + \int_\triangle X_1^M \Delta^2 X_1^M(x)dx}{\int_0^1 \int_\triangle (u_1^M(x,t))^2 dxdt} = \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt + \eta_1^M}{\int_0^1 \int_\triangle (u_1^M(x,t))^2 dxdt}$$

$$< \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\triangle (X_2^N(x)dx)^2 + \int_0^1 \varphi_1(t)\varphi_1(t)dt \int_\triangle X_2^N(x)\Delta^2 X_2^N(x)dx}{\int_0^1 \int_\triangle (u_2^N(x,t))^2 dxdt}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt + \int_\triangle \eta_2^N (X_2^N(x))^2 dx}{\int_0^1 \int_\triangle (u_2^N(x,t))^2 dxdt}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt + \eta_2^N}{\int_0^1 \int_\triangle (u_2^N(x,t))^2 dxdt} = s_2^N < \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt + \eta_1^L}{\int_0^1 \int_\triangle (u_2^L(x,t))^2 dxdt}$$

$$= \frac{-\int_0^1 \varphi_1^{''}(t)\varphi_1(t)dt \int_\triangle (X_1^L(x)dx)^2 + \int_0^1 \varphi_1(t)\varphi_1(t)dt \int_\triangle X_1^L(x)\Delta^2 X_1^L(x)dx}{\int_0^1 \int_\triangle (u_2^L(x,t))^2 dxdt} = s_1^L.$$

The rest of equalities and inequalities imply from the monotonicity property (27).

Theorem 3. *For all triangular cylinders the second s-number of the Cauchy-Neumann heat operator (2) satisfies*

$$s_2^N(\Omega) \leq (2.78978609910027)^2 + \left(\frac{4\pi^2}{3\sqrt{3}}\right)^2,$$

and equality if only if the triangular cylinder coincides with the equilateral triangular cylinder $\Omega^* \times (0,1)$, that is, $|\Omega| = |\Omega^*|$.

Proof. By using the fact that s-numbers are monotonically increasing (see (27)) and the main result of [4], we obtain

$$s_2^N(\Omega) \leq s_2^N\left(\frac{4\pi^2}{3\sqrt{3}}\right). \tag{29}$$

A straightforward calculation in (25) gives

$$s_2^N(\Omega) \leq s_2^N(\frac{4\pi^2}{3\sqrt{3}}) \cong (2.78978609910027)^2 + \left(\frac{4\pi^2}{3\sqrt{3}}\right)^2. \tag{30}$$

4 On Szegö-Weinberger Type Inequality

Let $D = \Omega \times (0,1)$ be a cylindrical domain, where $\Omega \subset R^d$ is a simply-connected Lipschitz set with smooth boundary $\partial\Omega$. We consider the heat operator with the Cauchy-Neumann problem $\diamondsuit : L^2(D) \to L^2(D)$ in the form

$$\diamondsuit u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1). \end{cases} \tag{31}$$

The operator $\diamondsuit$ is a non-selfadjoint operators in $L^2(D)$. Adjoint operator $\diamondsuit^*$ to operator $\diamondsuit$ is

$$\diamondsuit^* v(x,t) := \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,1) = 0, \ \ x \in \Omega, \\ \frac{\partial v(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1). \end{cases} \tag{32}$$

A direct calculation gives that the operator $\diamondsuit^*\diamondsuit$ has the following formula

$$\diamondsuit^*\diamondsuit u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,1). \end{cases} \tag{33}$$

We consider a (circular) cylinder $C = B \times (0,1)$, where $B \subset R^d$ is an open ball. Let Ω be a simply-connected Lipschitz set with smooth boundary $\partial\Omega$ with $|B| = |\Omega|$, where $|\Omega|$ is the Lebesgue measure of the domain Ω.

Let us introduce the operators $T, L : L^2(\Omega) \to L^2(\Omega)$, respectively, by

$$Tz(x) := \begin{cases} -\Delta z(x) = \mu z(x), \\ \frac{\partial z(x)}{\partial n} = 0, \ \ x \in \partial\Omega, \end{cases} \tag{34}$$

and

$$Lz(x) := \begin{cases} \Delta^2 z(x) = \lambda z(x), \\ \frac{\partial z(x)}{\partial n} = 0, \ \ x \in \partial\Omega, \\ \frac{\partial \Delta z(x)}{\partial n} = 0, \ \ x \in \partial\Omega. \end{cases} \tag{35}$$

Lemma 3. *The second eigenvalue of the operator L is maximized in the ball B among all Lipschitz domains Ω of the same measure with $|\Omega| = |B|$.*

Proof. The Szegö-Weinberger inequality is valid for the Neumann Laplacian, that is, the ball is a maximizer of the second eigenvalue of the operator T among all Lipschitz domains Ω with $|B| = |\Omega|$. A straightforward calculation from (34) gives that

$$T^2 z(x) := \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0, \ \ x \in \partial\Omega, \\ \frac{\partial \Delta z(x)}{\partial n} = 0, \ \ x \in \partial\Omega. \end{cases} \tag{36}$$

Thus, $T^2 = L$ and $\mu^2 = \lambda$. Now using the Szegö-Weinberger inequality, we establish $\lambda_2(B) = \mu_2^2(B) \geq \mu_2^2(\Omega) = \lambda_2(\Omega)$, i.e. $\lambda_2(B) \geq \lambda_2(\Omega)$.

Theorem 4. *The second s-number of the operator $\diamondsuit$ is maximized in the circular cylinder C among all cylindric Lipschitz domains of a given measure, that is,*

$$s_2^N(C) \geq s_2^N(D),$$

for all D with $|D| = |C|$.

Proof. Recall that $D = \Omega \times (0, 1)$ is a bounded measurable set in R^{d+1}. Its symmetric rearrangement $C = B \times (0, 1)$ is the circular cylinder with the measure equal to the (Lebesgue) measure of D, i.e. $|D| = |C|$. Let u be a nonnegative measurable function in D, such that all its positive level sets have finite measure. With the definition of the symmetric-decreasing rearrangement of u we can use the layer-cake decomposition [5], which expresses a nonnegative function u in terms of its level with respect to the space variable x sets as

$$u(x, t) = \int_0^\infty \chi_{\{u(x,t)>z\}} dz, \ \ \forall t \in (0, 1), \tag{37}$$

where χ is the characteristic function of the domain. The function

$$u^*(x, t) = \int_0^\infty \chi_{\{u(x,t)>z\}^*} dz, \ \ \forall t \in (0, 1), \tag{38}$$

is called the (radially) symmetric-decreasing rearrangement of a nonnegative measurable function u.

Consider the following spectral problem

$$\diamondsuit^* \diamondsuit u = su,$$

$$\diamondsuit^* \diamondsuit u(x, t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x, t) = s^N u(x, t), \\ u(x, 0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x, t)|_{t=1} = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ \forall t \in (0, 1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega. \end{cases} \tag{39}$$

We can set $u(x, t) = X(x)\varphi(t)$ and $u_2(x, t) = X_2(x)\varphi_1(t)$ is the second eigenfunction of the operator $\diamondsuit^* \diamondsuit$, where $\varphi_1(t)$ and $X_2(x)$ are the first and second eigenfunctions with respect to variables t and x. Consequently, we have

$$-\varphi_1^{''}(t) X_2(x) + \varphi_1(t) \Delta^2 X_2(x) = s_2^N \varphi_1(t) X_2(x). \tag{40}$$

Now by the variational principle for the self-adjoint compact positive operator $\diamondsuit^*\diamondsuit$, we get

$$s_2^N(D) = \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \int_0^1 \varphi_1^2(t)dt \int_\Omega X_2(x)\Delta^2 X_2(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \int_0^1 \varphi_1^2(t)dt \int_\Omega \lambda_2(\Omega)(X_2(x))^2dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_\Omega X_2^2(x)dx + \lambda_2(\Omega) \int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega X_2^2(x)dx},$$

where $\lambda_2(\Omega)$ is the second eigenvalue of the operator L. For each non-negative function $X \in L^2(\Omega)$, we have

$$\int_\Omega |X_1(x)|^2dx = \int_B |X_1^*(x)|^2dx, \quad \text{with } |\Omega| = |B|. \tag{41}$$

By applying Lemma 3 and (41), we get

$$s_2^N(D) = \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_\Omega (X_2(x))^2dx + \lambda_2(\Omega) \int_0^1 \varphi_2^2(t)dt \int_\Omega (X_2(x))^2dx}{\int_0^1 \varphi_1^2(t)dt \int_\Omega (X_2(x))^2dx}$$

$$\leq \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2dx + \lambda_2(B) \int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}{\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2dx + \int_0^1 \varphi_1^2(t)dt \int_B X_2^*(x)(\lambda_2(B)X_2^*(x))dx}{\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}$$

$$= \frac{-\int_0^1 \varphi_1''(t)\varphi_1(t)dt \int_B (X_2^*(x))^2dx + \int_0^1 \varphi_1^2(t)dt \int_B X_2^*(x)\Delta^2 X_2^*(x)dx}{\int_0^1 \varphi_1^2(t)dt \int_B (X_2^*(x))^2dx}$$

$$= \frac{-\int_0^1\int_B u_2^*(x,t)\frac{\partial^2 u_2^*(x,t)}{\partial t^2}dxdt + \int_0^1\int_B u_2^*(x,t)\Delta_x^2 u_2^*(x,t)dxdt}{\int_0^1\int_B (u_2^*(x,t))^2dxdt}$$

$$\leq \sup_{\nu(x,t)\neq 0} \frac{-\int_0^1\int_B \nu(x,t)\frac{\partial^2\nu(x,t)}{\partial t^2}dxdt + \int_0^1\int_B \nu(x,t)\Delta_x^2\nu(x,t)dxdt}{\int_0^1\int_B \nu^2(x,t)dxdt} = s_2^N(C).$$

The proof is complete.

Remark 1. The norm of the operator $\diamondsuit^{-1}$ is equal to $\frac{2}{\pi}$ in any circular cylinder C with $|C| = |D|$, i.e. $\|\diamondsuit^{-1}\|_C = \|\diamondsuit^{-1}\|_D = \frac{2}{\pi}$.

Proof. Let us consider the following spectral problem by Fourier's method

$$\diamondsuit^*\diamondsuit u(x,t) := \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = s^N u(x,t), \\ u(x,0) = 0, \;\; x \in \Omega(B), \\ \frac{\partial u(x,t)}{\partial t}|_{t=1} - \Delta_x u(x,t)|_{t=1} = 0, \;\; x \in \Omega(B), \\ \frac{\partial u(x,t)}{\partial n} = 0, \;\; x \in \partial\Omega(\partial B), \;\; \forall t \in (0,1), \\ \frac{\partial \Delta_x u(x,t)}{\partial n} = 0, \;\; x \in \partial\Omega(\partial B). \end{cases} \tag{42}$$

Thus, we obtain two spectral problems with respect to variables x and t:

$$\begin{cases} \Delta^2 X(x) = \mu^2 X(x), \\ \frac{\partial X(x)}{\partial n} = 0, \ \ x \in \partial\Omega(\partial B), \ \ t \in (0,1), \\ \frac{\partial \Delta X(x)}{\partial n} = 0, \ \ x \in \partial\Omega(\partial B), \ \ t \in (0,1). \end{cases} \tag{43}$$

and

$$\begin{cases} \varphi^{''}(t) + (s - \mu^2)\varphi(t) = 0, \ t \in (0,1), \\ \varphi(0) = 0, \\ \varphi^{'}(1) + \mu(\Omega)\varphi(1) = 0. \end{cases} \tag{44}$$

We have $\mu_1 = 0$ in any simply-connected bounded domain. Thus, we substitute it to the second spectral problem (44) and we establish $s_1 = \frac{\pi^2}{4}$. According to [1], $\frac{1}{\sqrt{s_1}}$ gives the norm of the operator. Thus, we arrive at

$$||\diamondsuit^{-1}||_C = ||\diamondsuit^{-1}||_D = \frac{2}{\pi}.$$

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Gohberg, I., Krein, M.: Introduction to the Theory of Linear Nonselfadjoint Operators. American Mathematical Society (1988)
2. Henrot, A.: Extremum Problems for Eigenvalues of Elliptic Operators. Birkhauser Verlag, Basel (2006)
3. Kassymov, A., Suragan, D.: Some spectral geometry inequalities for generalized heat potential operators. Complex Anal. Oper. Theory 1–15 (2016). doi:10.1007/s11785-016-0605-9
4. Laugesen, R.S., Siudeja, B.A.: Maximizing Neumann fundamental tones of triangles. J. Math. Phys. **50**, 112903 (2009)
5. Lieb, E.H., Loss, M.: Analysis. Volume 14 of Graduate Studies in Mathematics. AMS, Providence, RI (2001)
6. Pólya, G.: On the characteristic frequencies of a symmetric membrane. Math. Z. **63**, 331–337 (1955)
7. Rozenblum, G., Ruzhansky, M., Suragan, D.: Isoperimetric inequalities for Schatten norms of Riesz potentials. J. Funct. Anal. **271**, 224–239 (2016)
8. Ruzhansky, M., Suragan, D.: Isoperimetric inequalities for the logarithmic potential operator. J. Math. Anal. Appl. **434**, 1676–1689 (2016)
9. Ruzhansky, M., Suragan, D.: Schatten's norm for convolution type integral operator. Russ. Math. Surv. **71**, 157–158 (2016)
10. Ruzhansky, M., Suragan, D.: On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries. Bull. Math. Sci. **6**, 325–334 (2016)
11. Siudeja, B.: On mixed Dirichlet-Neumann eigenvalues of triangles. Proc. Am. Math. Soc. **144**, 2479–2493 (2016)

12. Szegö, G.: Inequalities for certain eigenvalues of a membrane of given area. J. Rational Mech. Anal. **3**, 343–356 (1954)
13. Vladimirov, V.S.: Equations of Mathematical Physics. Moscow (1996)
14. Weinberger, H.F.: An isoperimetric inequality for the N-dimensional free membrane problem. J. Rational Mech. Anal. **3**, 633–636 (1956)

On Some Equations on Non-smooth Manifolds: Canonical Domains and Model Operators

Vladimir B. Vasilyev(✉)

Chair of Differential Equations, Belgorod National Research University, 14/1 Studencheskaya st., Belgorod 308007, Russia
vbv57@inbox.ru

Abstract. We describe different aspects of the theory of pseudo-differential equations on manifolds with non-smooth boundaries. Using a concept of special factorization for an elliptic symbol we consider distinct variants of this approach including asymptotic and discrete situations.

Keywords: Pseudo-differential equation · Wave factorization · Solvability · Asymptotical expansion · Discrete equation

1 Introduction

From the 90s the author develops the theory of boundary value problems based on two principles [10]. These are a local principle and factorizability principle at a boundary point like [2,5,9]. The first principle was known earlier and it also was known as a freezing coefficients principle. Usually the second name corresponds to partial differential equations theory but the first name was introduced for multidimensional singular integral equations and more general for pseudo-differential equations. Main difference between differential and pseudo-differential operator is the following. A differential operator $\mathscr{D}$ has a local property i.e. if one takes two smooth functions φ, ψ with non-intersecting supports and compose the operator $\psi \cdot \mathscr{D} \cdot \varphi$, then it leads to a zero operator. For a pseudo-differential operator $\mathscr{P}$ this property does not hold and we obtain for $\psi \cdot \mathscr{P} \cdot \varphi$ a compact operator only. This case permits to obtain rough properties for pseudo-differential equations and related boundary value problems namely Fredholm properties only in comparison with differential operators and boundary value problems, where one has as a rule results on existence and uniqueness.

There are a lot of approaches to construct such a theory (see for example papers [4,7,8]). I have written many times [12,14] what is difference between this consideration and others, it is choice of distinct key principles. In any case one needs to declare an invertibility of so-called local representatives of an initial pseudo-differential operator to describe its Fredholm properties.

Local principle and factorizability was first introduced in papers of I.B. Simonenko [9] (for multidimensional singular integral operators in Lebesgue L_p-spaces) and M.I. Vishik – G.I. Eskin [2] (for pseudo-differential operators in

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_34

Sobolev – Slobodetskii H^s-spaces). For manifolds with a smooth boundary one uses an idea of “rectification of a boundary“, and the problem reduces to a half-space case, for which a factorizability principle holds immediately because under localization at a boundary point and applying the Fourier transform we obtain well known one-dimensional classical Riemann boundary value problem for upper and lower complex half-planes with a multidimensional parameter. This approach does not work if a boundary has at least one singular point like a conical point. One needs here other considerations and approaches.

2 Domains and Operators

Our main goal is to describe possible solvability conditions for the pseudo-differential equation

$$(Au)(x) = f(x), \ x \in D,$$

where D is manifold with a boundary, A is pseudo-differential operator with the symbol $A(x,\xi)$.

Such operators are defined locally by the formula

$$u(x) \longmapsto \int\limits_{\mathbf{R}^m} \int\limits_{\mathbf{R}^m} A(x,\xi)\tilde{u}(y)e^{-i(x-y)\cdot\xi} d\xi dy, \tag{1}$$

if D is a smooth compact manifold, because one can use “freezing coefficients principle”, or in other words “local principle”. For manifold with a smooth boundary we need new local formula for defining the operator A: more precisely in inner points of D we use the formula 1, but in boundary points we need another formula

$$u(x) \longmapsto \int\limits_{\mathbf{R}^m_+} \int\limits_{\mathbf{R}^m} A(x,\xi)u(y)e^{-i(x-y)\cdot\xi} d\xi dy. \tag{2}$$

For invertibility of such operator (2) with symbol $A(\cdot,\xi)$ non-depending on spatial variable x one can apply the theory of classical Riemann boundary problem for upper and lower complex half-planes with a parameter ξ'. This step was systematically studied in the book [2]. But if the boundary ∂D has at least one conical point, this approach is not effective.

The conical point at the boundary is a such point, for which its neighborhood is diffeomorphic to the cone

$$C^a_+ = \{x \in \mathbf{R}^m : \ x_m > a|x'|, \ x' = (x_1, ..., x_{m-1}), \ a > 0\},$$

hence the local definition for pseudo-differential operator near the conical point is the following

$$u(x) \longmapsto \int\limits_{C^a_+} \int\limits_{\mathbf{R}^m} A(x,\xi)u(y)e^{-i(x-y)\cdot\xi} d\xi dy. \tag{3}$$

We consider the operator 1 in the Sobolev – Slobodetskii space $H^s(\mathbf{R}^m)$ with norm

$$||u||_s^2 = \int\limits_{\mathbf{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi,$$

where $\tilde{u}(\xi)$ denotes the Fourier transform for u, and introduce the following class of symbols non-depending on spatial variable x: $\exists c_1, c_2 > 0$, such that

$$c_1 \leq |A(\xi)(1+|\xi|)^{-\alpha}| \leq c_2, \ \xi \in \mathbf{R}^m. \tag{4}$$

The number $\alpha \in \mathbf{R}$ we call the order of pseudo-differential operator A.

It is well-known that pseudo-differential operator with symbol $A(\xi)$ satisfying 3, is linear bounded operator acting from $H^s(\mathbf{R}^m)$ into $H^{s-\alpha}(\mathbf{R}^m)$ [2].

We are interested in studying invertibility operator 3 in corresponding Sobolev – Slobodetskii spaces. By definition, $H^s(C_+^a)$ consists of distributions from $H^s(\mathbf{R}^m)$ with support in C_+^a. The norm in the space $H^s(C_+^a)$ is induced by the norm $H^s(\mathbf{R}^m)$. We associate such operator with corresponding equation

$$(Au_+)(x) = f(x), \ x \in C_+^a, \tag{5}$$

where right-hand side f is chosen from the space $H_0^{s-\alpha}(C_+^a)$.

$H_0^s(C_+^a)$ is the space of distributions $S'(C_+^a)$, which admit continuation on $H^s(\mathbf{R}^m)$. The norm in $H_0^s(C_+^a)$ is defined by

$$||f||_s^+ = \inf ||lf||_s,$$

where *infimum* is chosen for all possible continuations l.

3 Complex Variables and Wave Factorization

Below we will consider the symbols $A(\xi)$ satisfying the condition 4.

Definition 1. Wave factorization of symbol $A(\xi)$ is called its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined everywhere without may be the points $\{\xi \in \mathbf{R}^m : |\xi'|^2 = a^2\xi_m^2\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C}{}_+^a), T(\overset{*}{C}{}_-^a)$ respectively, which satisfy the estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leq c_1(1+|\xi|+|\tau|)^{\pm æ},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \leq c_2(1+|\xi|+|\tau|)^{\pm(\alpha - æ)}, \ \forall \tau \in \overset{*}{C}{}_+^a.$$

The number æ is called *index of wave factorization.*

Here $\overset{*}{C}{}^a_+$ is conjugate cone to C^a_+, and $\overset{*}{C}{}^a_- = -\overset{*}{C}{}^a_+$.

Example 1. Let

$$A = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_m^2} + k^2, \quad k \in \mathbf{R} \setminus \{\mathbf{0}\},$$

and then according to some properties of the Fourier transform the symbol of this operator has the form

$$A(\xi) = \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2 + k^2.$$

The following equality is the wave factorization of the Helmholtz operator. We will write it as

$$\xi_m^2 + |\xi'|^2 + k^2 =$$
$$\left(\sqrt{a^2+1}\,\xi_m + \sqrt{a^2\xi_m^2 - |\xi'|^2 - k^2}\right)\left(\sqrt{a^2+1}\,\xi_m - \sqrt{a^2\xi_m^2 - |\xi'|^2 - k^2}\right)$$

meaning for $\sqrt{a^2\xi_m^2 - |\xi'|^2 - k^2}$ the boundary value

$$\sqrt{a^2(\xi_m + i0)^2 - |\xi'|^2 - k^2}.$$

4 Pseudo-differential Equations and Solvability

To describe a solvability picture for a model elliptic pseudo differential equation (5) in 2-dimensional cone $C^a_+ = \{x \in \mathbf{R}^2 : x_2 > a|x_1|, a > 0\}$ the author earlier considered a special singular integral operator [10]

$$(K_a u)(x) = \frac{a}{2\pi^2} \lim_{\tau \to 0+} \int\limits_{\mathbf{R}^2} \frac{u(y)dy}{(x_1 - y_1)^2 - a^2(x_2 - y_2 + i\tau)^2}. \tag{6}$$

This operator served a conical singularity in the general theory of boundary value problems for elliptic pseudo differential equations on manifolds with a non-smooth boundary. The operator K_a is a convolution operator, and the parameter a is a size of an angle, $x_2 > a|x_1|, a = \cot\alpha$.

One of author's main result [10] is the following (we formulate it for $m = 2$ for simplicity)

Theorem 1. *If elliptic symbol $A(\xi)$ admits wave factorization with respect to the cone C^a_+ and $|æ - s| < 1/2$, then the equation (5) has a unique solution*

$$\widetilde{u}(\xi) = A^{-1}_{\neq}(\xi)(K_a\widetilde{lv})(\xi),$$

where lv is an arbitrary continuation of v on the whole $H^s(\mathbf{R}^2)$.

A priori estimate holds

$$||u_+||_s \leq c||f||^+_{s-\alpha}.$$

Below we will mention other possible situations.

4.1 Boundary Value Problems

If $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$, then one has the formula for a general solution of the equation (5), and this formula contains a certain number of arbitrary functions from corresponding Sobolev–Slobodetskii spaces [10]. To obtain the uniqueness theorem one needs to add some complementary conditions as a rule these are boundary conditions.

Some classical variants are considered in [10], some new constructions are described in [15].

4.2 Equations with Potentials

It is possible that $æ - s = n + \delta, -n \in \mathbf{N}, |\delta| < 1/2$, then the equation (5) is over-determined so that one needs to add some unknowns. According to the special representation for a solution of the equation (5) these unknowns should have a potential like form [10].

5 Asymptotical Variants

For $|æ - s| < 1/2$ one has the existence and uniqueness theorem [10]

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(K_a \widetilde{lv})(\xi),$$

where lv is an arbitrary continuation of v on the whole $H^s(\mathbf{R}^2)$.

5.1 Preliminaries

The formula (6) can be treated as a convolution of the distribution

$$K_a(\xi) = \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2}$$

with a basic function $u(\xi)$. If so it is interesting to study behavior of the operator (6) for limit cases $(a = 0, a = +\infty)$ from convolution point of view.

Let $S(\mathbf{R}^2)$ be the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions, then $S'(\mathbf{R}^2)$ is a corresponding space of distributions over $S(\mathbf{R}^2)$.

When $a \to +\infty$ one obtains [11] the following limit distribution

$$\lim_{a\to\infty} \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{i}{2\pi} \mathscr{P} \frac{1}{\xi_1} \otimes \delta\left(\xi_2\right), \tag{7}$$

where the notation for distribution $\mathscr{P}$ is taken from V.S. Vladimirov's books [16, 17], and $\otimes$ denotes the direct product of distributions. Here δ denotes one-dimensional Dirac mass-function which acts on $\varphi \in S(\mathbf{R})$ in the following way

$$(\delta, \varphi) = \varphi(0),$$

and the distribution $\mathscr{P}\frac{1}{x}$ is defined by the formula

$$(\mathscr{P}\frac{1}{x},\varphi) = v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(x)dx}{x} \equiv \lim_{\varepsilon\to 0+} \left(\int\limits_{-\infty}^{-\varepsilon} + \int\limits_{\varepsilon}^{+\infty} \right) \frac{\varphi(x)dx}{x}.$$

Our main goal in this paper is obtaining an asymptotical expansion for the two-dimensional distribution

$$K_a(\xi_1,\xi_2) \equiv \frac{a}{2\pi^2}\frac{1}{\xi_1^2 - a^2\xi_2^2}$$

with respect to small a^{-1}. It is defined by the corresponding formula $\forall\varphi \in S(\mathbf{R}^2)$

$$(K_a,\varphi) = \frac{a}{2\pi^2}\int\limits_{\mathbf{R}^2} \frac{\varphi(\xi_1,\xi_2)d\xi}{\xi_1^2 - a^2\xi_2^2}. \tag{8}$$

5.2 Asymptotical Representation for a Solution

Below we denote $lv \equiv V$.

Theorem 2. *If the symbol $A(\xi)$ admits a wave factorization with respect to the cone C_+^a and $|æ - s| < 1/2$, then the equation 1 has a unique solution in the space $H^s(C_+^a)$, and for a large a it can be represented in the form*

$$\widetilde{u}(\xi) = \frac{i}{2\pi} A_{\neq}^{-1}(\xi) v.p. \int\limits_{-\infty}^{+\infty} \frac{(A_{=}^{-1}\widetilde{V})(\eta_1,\xi_2)d\eta_1}{\xi_1 - \eta_1} +$$

$$A_{\neq}^{-1}(\xi)\sum_{m,n} c_{m,n}(a) \int\limits_{-\infty}^{+\infty} (\xi_1-\eta_1)^m (A_{=}^{-1}\widetilde{V})^{(n)}_{\xi_2}(\eta_1,\xi_2)d\eta_1 \tag{9}$$

assuming $\widetilde{V} \in S(\mathbf{R}^2)$, $A_{=}^{-1}\widetilde{V}$ means the function $A_{=}^{-1}(\xi)\widetilde{V}(\xi)$.

Proof. Let $\varphi \in S(\mathbf{R}^2)$.

A formal using the Maclaurin formula for the first integral in 4 will lead to the following result

$$(K_a,\varphi) = \frac{1}{2\pi^2}\sum_{k=0}^{\infty}\frac{b^k}{k!}\int\limits_{-N}^{+N} \varphi^{(k)}_{\xi_2}(\xi_1,0)\left(\int\limits_{-N}^{+N}\frac{t^k dt}{\xi_1^2 - t^2}\right)d\xi_1, \tag{10}$$

and we need to give a certain sense for the expression in brackets.

Let us denote

$$T_{k,N}(\xi_1) \equiv \int\limits_{-N}^{+N} \frac{t^k dt}{\xi_1^2 - t^2}$$

and reproduce some calculations.

First $T_{k,N}(\xi_1) \equiv 0, \forall k = 2n-1, n \in \mathbf{N}$. So the non-trivial case is $k = 2n, n \in \mathbf{N}$. Let us remind $T_{0,\infty}(\xi_1) = \pi i 2^{-1}\xi_1^{-1}$ [11,12]. For other cases we can calculate this integral. We have the following

$k = 2$,

$$T_{2,N}(\xi_1) = -2N - 2^{-1}\xi_1^{-1}\ln\frac{N-\xi_1}{N+\xi_1} + \pi i 2^{-1}\xi_1^{-1};$$

$k = 4$,

$$T_{4,N}(\xi_1) = -2/3N^3 - 2\xi_1^2 N - 2^{-1}\xi_1^3\ln\frac{N-\xi_1}{N+\xi_1} + \pi i 2^{-1}\xi_1^3;$$

$k = 6$,

$$T_{6,N}(\xi_1) = -2/5N^5 - 2/3\xi_1^2 N^3 - 2\xi_1^5 N - 2^{-1}\xi_1^5\ln\frac{N-\xi_1}{N+\xi_1} + \pi i 2^{-1}\xi_1^5,$$

and so on. One can easily write all expressions for arbitrary $T_{2n,N}(\xi_1)$.

In general one can write

$$T_{2n,N}(\xi_1) = P_{2n-1}(N,\xi_1) - 2^{-1}\xi_1^{2n-1}\ln\frac{N-\xi_1}{N+\xi_1} + \pi i 2^{-1}\xi_1^{2n-1},$$

where $P_{2n-1}(N,\xi_1)$ is a certain polynomial of order $2n-1$ on variables N,ξ_1.

Therefore instead of the formula (10) we can write

$$(K_a,\varphi) = \frac{i}{2\pi}(\mathscr{P}\frac{1}{\xi_1}\otimes\delta(\xi_2),\varphi) + \tag{11}$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}\int_{-N}^{+N}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\left(P_{2n-1}(N,\xi_1) - 2^{-1}\xi_1^{2n-1}\ln\frac{N-\xi_1}{N+\xi_1} + \pi i 2^{-1}\xi_1^{2n-1}\right)d\xi_1.$$

Let us describe the polynomial $P_{2n-1}(N,\xi_1)$ more precisely. Obviously

$$P_{2n-1}(N,\xi_1) = c_{2n-1}N^{2n-1} + c_{2n-3}N^{2n-3}\xi_1^2 + \cdots + c_1 N\xi_1^{2n-1}.$$

Further we rewrite the equality (11) in the following form

$$(K_a,\varphi) = \frac{i}{2\pi}(\mathscr{P}\frac{1}{\xi_1}\otimes\delta(\xi_2),\varphi) +$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}\sum_{k=1}^{n}c_{2k-1}N^{2k-1}\int_{-N}^{+N}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\xi_1^{2k-1}d\xi_1 -$$

$$\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}\int_{-N}^{+N}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\xi_1^{2n-1}\ln\frac{N-\xi_1}{N+\xi_1}d\xi_1 + \frac{i}{4\pi}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}\int_{-N}^{+N}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\xi_1^{2n-1}d\xi_1.$$

We will start from two last summands. The second summand does not play any role because

$$\lim_{N\to+\infty}\ln\frac{N-\xi_1}{N+\xi_1}=0.$$

The third summand we will represent according to lemma 1 (see below) taking into account that we can pass to the limit under $N\to+\infty$

$$\frac{i}{4\pi}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}(\widetilde{\delta^{(2n-1)}}(\xi_1)\otimes\delta^{(2n)}(\xi_2),\varphi).$$

For the first summand we consider separately the case $Nb\sim 1(N\to\infty, b\to 0)$. In other words we consider a special limit to justify the decomposition. Then

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}\sum_{k=1}^{n}c_{2k-1}N^{2k-1}\int\limits_{-N}^{+N}\varphi^{(2n)}_{\xi_2}(\xi_1,0)\xi_1^{2k-1}d\xi_1\sim$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}c_{2k-1}b^{2n-2k+1}\int\limits_{-\infty}^{+\infty}\varphi^{(2n)}_{\xi_2}(\xi_1,0)\xi_1^{2k-1}d\xi_1.$$

Therefore

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}c_{2k-1}b^{2n-2k+1}\int\limits_{-\infty}^{+\infty}\varphi^{(2n)}_{\xi_2}(\xi_1,0)\xi_1^{2k-1}d\xi_1=$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}c_{2k-1}b^{2n-2k+1}(\widetilde{\delta^{(2k-1)}}(\xi_1)\otimes\delta^{(2n)}(\xi_2),\varphi).$$

One can note if desirable

$$c_{2k-1}=-2(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}).$$

Further details one can find in [13]. △

6 Discrete Variants

6.1 Discrete Functions and Operators: Preliminaries and Examples

Given function u_d of a discrete variable $\tilde{x}\in\mathbf{Z}^m$ we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi)\equiv\widetilde{u}_d(\xi)=\sum_{\tilde{x}\in\mathbf{Z}^m}e^{i\tilde{x}\cdot\xi}u_d(\tilde{x}),\quad \xi\in\mathbf{T}^m,$$

where $\mathbf{T}^m=[-\pi,\pi]^m$ and partial sums are taken over cubes

$$Q_N=\{\tilde{x}\in\mathbf{Z}^m:\tilde{x}=(\tilde{x}_1,\cdots,\tilde{x}_m),\max_{1\le k\le m}|\tilde{x}_k|\le N\}.$$

One can define some discrete operators for such functions u_d.

Example 2. If $K(x), x \in \mathbf{R}^m \setminus \{0\}$ is a Calderon–Zygmund kernel, then the corresponding operator is defined by the formula

$$(K_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in \mathbf{Z}^m, \tilde{y} \neq \tilde{x}} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}), \quad \tilde{x} \in \mathbf{Z}^m.$$

Example 3. If a first order finite difference of a discrete variable $\tilde{x}_k$ is defined by

$$\delta_k u_d(\tilde{x}) = u_d(\tilde{x}_k + 1) - u_d(\tilde{x}_k),$$

then the discrete Laplacian is

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} \left(u_d(\tilde{x}_k + 2) - 2u_d(\tilde{x}_k + 1) + u_d(\tilde{x}_k) \right),$$

and its discrete Fourier transform is the function

$$(F_d \Delta_d u_d)(\xi) = \sum_{k=1}^{m} (e^{i\xi_k} - 1)^2.$$

Let $D \subset \mathbf{R}^m$ be a sharp convex cone, $D_d \equiv D \cap \mathbf{Z}^m$, and let $L_2(D_d)$ be a space of functions of discrete variable defined on D_d, and let $A(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \mathbf{Z}^m$. We consider the following types of operators

$$(A_d u_d)(\tilde{x}) = \int_{\mathbf{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \widetilde{A}_d(\xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d, \tag{12}$$

and introduce the function

$$\widetilde{A}_d(\xi) = \sum_{\tilde{x} \in \mathbf{Z}^m} e^{i\tilde{x} \cdot \xi} A(\tilde{x}), \quad \xi \in \mathbf{T}^m.$$

Definition 2. The function $\widetilde{A}_d(\xi)$ is called a symbol of the operator A_d, and this symbol is called an elliptic symbol if $\tilde{A}_d(\xi) \neq 0, \forall \xi \in \mathbf{T}^m$.

Remark 1. If $D = \mathbf{R}^m$, then an ellipticity is necessary and sufficient condition for the operator A_d to be invertible in the space $L_2(\mathbf{Z}^m)$.

Remark 2. One can define a general pseudo-differential operator with symbol $\tilde{A}(\tilde{x}, \xi)$ depending on a spatial discrete variable $\tilde{x}$ by the similar formula

$$(A_d u_d)(\tilde{x}) = \int_{\mathbf{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \widetilde{A}(\tilde{x}, \xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

but taking into account a local principle [5], the main aim in this situation is describing invertibility conditions for model operators like (12) in canonical domains D_d.

Below we will refine the lattice $\mathbf{Z}^m$ and introduce more convenient space scale.

6.2 Discrete Sobolev–Slobodetskii Spaces

We consider here refined lattice $h\mathbf{Z}^m, h > 0$, and define corresponding discrete Fourier transform. If a function of a discrete variable is defined on a lattice $h\mathbf{Z}^m$, then its discrete Fourier transform can be introduced by the formula

$$(\tilde{u}_d)(\xi) = \sum_{\tilde{x} \in h\mathbf{Z}^m} u_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbf{T}^m,$$

where $\hbar = h^{-1}$.

Let $H^s(h\mathbf{Z}^m)$ denote the space of functions of a discrete variable for which

$$||u_d||_s^2 \equiv \int\limits_{\hbar\mathbf{T}^m} |\widetilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(h)(\xi)|)^s d\xi < +\infty,$$

where

$$\sigma_{\Delta_d}(h)(\xi) = h^{-2} \sum_{k=1}^{m} (e^{ih\xi_k} - 1)^2, \quad \xi \in \hbar\mathbf{T}^m.$$

6.3 Solvability for Discrete Equations

6.3.1 Conical Case and Periodic Bochner Kernel Let D be a sharp convex cone, and let $\overset{*}{D}$ be a conjugate cone for D, i.e.,

$$\overset{*}{D} = \{x \in \mathbf{R}^m : x \cdot y > 0,\ y \in D\}.$$

Let $T(\overset{*}{D}) \subset \mathbf{C}^m$ be a set of the type $\mathbf{T}^m + i\ \overset{*}{D}$. For $\mathbf{T}^m \equiv \mathbf{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\overset{*}{D}$ ([1,16,17]). We introduce the function

$$B_d(z) = \sum_{\tilde{x} \in D_d} e^{i\tilde{x}\cdot z}, \quad z = \xi + i\tau, \quad \xi \in \mathbf{T}^m, \quad \tau \in \overset{*}{D},$$

and define the operator

$$(B_d u)(\xi) = \lim_{\tau \to 0} \int\limits_{\mathbf{T}^m} B_d(z - \eta) u_d(\eta) d\eta.$$

Lemma 1. *For arbitrary $u_d \in L_2(\mathbf{Z}^m)$, the following property*

$$F_d P_{D_d} u_d = B_d F_d u_d$$

holds.

Let us define the subspace $A(\mathbf{T}^m) \subset L_2(\mathbf{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(\overset{*}{D})$ and satisfy the condition

$$\sup_{\tau \in \overset{*}{D}} \int_{\mathbf{T}^m} |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty.$$

In other words, the space $A(\mathbf{T}^m) \subset L_2(\mathbf{T}^m)$ consists of boundary values of holomorphic in $T(\overset{*}{D})$ functions.

Let us denote

$$B(\mathbf{T}^m) = L_2(\mathbf{T}^m) \ominus A(\mathbf{T}^m),$$

so that $B(\mathbf{T}^m)$ is a direct complement of $A(\mathbf{T}^m)$ in $L_2(\mathbf{T}^m)$.

6.3.2 A Jump Problem We formulate the problem in the following way: finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\mathbf{T}^m), \Phi^{-} \in B(\mathbf{T}^m)$, such that

$$\Phi^{+}(\xi) - \Phi^{-}(\xi) = g(\xi), \quad \xi \in \mathbf{T}^m, \tag{13}$$

where $g(\xi) \in L_2(\mathbf{T}^m)$ is given.

Lemma 2. *The operator $B_d : L_2(\mathbf{T}^m) \to A(\mathbf{T}^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(\mathbf{T}^m)$.*

Theorem 3. *The jump problem has unique solution for arbitrary right-hand side from $L_2(\mathbf{T}^m)$.*

Example 4. If $m = 2$ and D is the first quadrant in a plane, then a solution of a jump problem is given by formulas

$$\Phi^{+}(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau \to 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cot \frac{\xi_1 + i\tau_1 - t_1}{2} \cot \frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1, t_2) dt_1 dt_2$$

$$\Phi^{-}(\xi) = \Phi^{+}(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in D.$$

6.3.3 A General Statement It looks as follows. Finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\mathbf{T}^m), \Phi^{-} \in B(\mathbf{T}^m)$, such that

$$\Phi^{+}(\xi) = G(\xi)\Phi^{-}(\xi) + g(\xi), \quad \xi \in \mathbf{T}^m, \tag{14}$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$, we have the jump problem (3).

Like classical studies [3,6], we want to use a special representation for an elliptic symbol to solve the problem (4).

6.3.4 Periodic Wave Factorization Let us denote by $H^s(D_d)$ a subspace of $H^s(\mathbf{Z}^m)$ consisting of functions of discrete variable $\tilde{x}$ for which their supports belong to $\overline{D_d}$, and $\widetilde{H}^s(D_d), \widetilde{H}^s(\mathbf{Z}^m)$ are their Fourier images.

Lemma 3. *For $|s| < 1/2$, the operator B_d is a bounded projector $\widetilde{H}^s(\mathbf{Z}^m) \to \widetilde{H}^s(D_d)$, and a jump problem has unique solution $\Phi^+ \in \widetilde{H}^s(D_d), \Phi^- \in \widetilde{H}^s(\mathbf{Z}^m \setminus D_d)$ for arbitrary $g \in \widetilde{H}^s(\mathbf{Z}^m)$.*

Definition 3. Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form

$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi),$$

where the factors $A_{\neq}^{\pm 1}(\xi), A_{=}^{\pm 1}(\xi)$ admit bounded holomorphic continuation into domains $T(\pm \overset{*}{D})$.

Theorem 4. *If $|s| < 1/2$ and the elliptic symbol $\tilde{A}_d(\xi) \in S_\alpha(\mathbf{T}^m)$ admits periodic wave factorization, then the operator A_d is invertible in the space $H^s(D_d)$.*

7 Conclusion

As it was shown all aspects of this problem of solving the equation (5) are closely related and use similar ideas and methods. Author hopes that in future it will be possible to unit these considerations in a general theory of elliptic pseudo-differential equations on manifolds with non-smooth boundaries.

References

1. Bochner, S., Martin, W.T.: Functions of Several Complex Variables. Publ. Foreign Lit, Moscow (1951). (Russian)
2. Eskin, G.I.: Boundary Value Problems for Elliptic Pseudodifferential Equations. AMS, Providence (1981)
3. Gakhov, F.D.: Boundary Problems. Nauka, Moscow (1977). (Russian)
4. Melrose, R.B.: Pseudodifferential operators, corners and singular limits. In: I. Satake (ed.) Proceeding of the International Congress of Mathematicians, vol. 1, pp. 217–234, Kyoto, Japan, 21–29 August 1990. Springer, New York (1991)
5. Milkhin, S.G.: Prößdorf S. Singular Integral Operators. Akademie-Verlag, Berlin (1986)
6. Muskhelishvili, N.I.: Singular Integral Equations. Nauka, Moscow (1968). (Russian)
7. Nazarov, S.A., Plamenevsky, B.A.: Elliptic problems in domains with piecewise smooth boundaries. Walter de Gruyter (1994)
8. Schulze, B.-W.: Pseudo-Differential Boundary Value Problems, Conical Singularities, and Asymptotics. Berlin (1994)
9. Simonenko, I.B.: Local method in the theory of translation invariant operators and their envelopings. Rostov on Don (2007). (Russian)

10. Vasil'ev, V.B.: Wave Factorization of Elliptic Symbols: Theory and Applications. Kluwer Academic Publishers, Introduction to the Boundary Value Problems in Non-smooth Domains (2000)
11. Vasilyev, V.B.: Asymptotical analysis of singularities for pseudo differential equations in canonical non-smooth domains. Integral Methods in Science and Engineering. In: Constanda, C., Harris, P.J. (eds.) Computational and Analytic Aspects, pp. 379–390. Birkhäuser, Boston (2011)
12. Vasilyev, V.B.: Pseudo differential equations on manifolds with non-smooth boundaries. Differential and difference equations and applications. Springer Proc. Math. Stat. **47**, 625–637 (2013)
13. Vasilyev, A.V.: On asymptotical expansions for certain singular integrals: 2-dimensional case, pp. 1–9. ArXiv: 1512.08237
14. Vasilyev, V.B.: Multipliers of Fourier Integrals, Pseudodifferential Equations, The Wave Factorization. Boundary Value Problems. Editorial URSS, Moscow (2010). (Russian)
15. Vasilyev, V.B.: Fourier Multipliers, Pseudo Differential Equations, Wave Factorization. Boundary Value Problems, Moscow (2010). (Russian)
16. Vladimirov, V.S.: Methods of Functions Theory of Several Complex Variables. Moscow (1964). (Russian)
17. Vladimirov, V.S.: Distributions in Mathematical Physics. Moscow (1979). (Russian)

Part IV

Mathematical Methods in Physical Sciences

Electromagnetic Field and Constriction Resistance of the Ring-Shaped Contact

Stanislav N. Kharin[1,2], Merey M. Sarsengeldin[1,2(✉)], and Samat A. Kassabek[1,2,3]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
staskharin@yahoo.com, merey@email.ru, kassabek@gmail.com
[2] Kazakh-British Technical University, 59 Tole Bi st., Almaty 050000, Kazakhstan
[3] Suleyman Demirel University, Kaskelen 040900, Kazakhstan

Abstract. The mathematical model describing the axisymmetric electromagnetic field and the constriction resistance of the semispace with AC electrical current passing through a ring-shaped contact is presented. It is based on the system of the Maxwell equations with the special boundary conditions. The analytical formulas for the electric and magnetic fields are obtained. The asymptotic expression for the constriction resistance is found and the corresponding expression for the DC current may be derived from this general expression as a special case. Comparison of this expression with the well known classical formula shows very good approximation.

Keywords: Electromagnetic field · Constriction resistance · Ring-shaped contact · Skin-effect · Maxwell equations

1 Introduction

A ring-shaped electrical contact appears in special types of circuit breakers and fuses [1,2], in a hollow liquid metal bridge at contact opening at great opening velocities [3] and many others contact systems. It is very important also at the modeling of skin effect when the real circle contact spot is replaced by a ring [4]. However the information about electromagnetic field and contact resistance in this case is known in an approximate form only [5]. This paper is an attempt to construct a mathematical model of a ring-shaped contact.

2 Electromagnetic Field

Let us consider two conductors occupied the semi spaces $D_1(-\infty < z < \infty, 0 < r < \infty)$ and $D_1(-\infty < z < \infty, 0 < r < \infty)$ with the contact on the ring-shape $D_0 = (z = 0, \quad r_1 < r < r_2)$. The plane $z = 0$ is a plane of the symmetry, thus we consider the electromagnetic field in the domain D_2 only caused by the

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*, Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_35

current $I(t) = i_0 e^{iwt}$ entering the ring-shape D_0. The axisymmetric component of the electrical field $E_z = (r, z, t)$ satisfies the equation

$$\frac{\mu_0 \mu}{\rho} \frac{\partial E_z}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial E_z}{\partial r}) + \frac{\partial^2 E_z}{\partial z^2}, \tag{1}$$

where μ is the relative magnetic permittivity, μ_0 is the permeability of vacuum, and the boundary conditions

$$E_z(r, 0, t) = \begin{cases} 0, & 0 \leq r < r_1 \\ \frac{\rho I(t)}{\pi(r_2^2 - r_1^2)}, & r_1 \leq r \leq r_2 \\ 0, & r_2 \leq r < \infty \end{cases} \qquad \frac{\partial E_z(\infty, z, t)}{\partial r} = \frac{\partial E_z(z, \infty, t)}{\partial z} = 0. \tag{2}$$

There is no initial condition because it is supposed that the AC current is passing through the contact for all time.The electrical and magnetic fields can be represented in the form

$$\bar{E}(r, z, t) = e^{iwt} \bar{E}(r, z), \qquad \bar{H}(r, z, t) = e^{iwt} \bar{H}(r, z).$$

Applying the Hankel transform

$$\hat{E}(s, z) = \int_0^\infty E_z(r, z) J_0(sr) r \, dr$$

to the equation (1) and conditions (2), we get

$$\frac{\partial^2 \hat{E}_z(s, z)}{\partial z^2} - (s^2 + k^2) \hat{E}_z(s, z) = 0 \tag{3}$$

$$\hat{E}_z(s, 0) = \frac{i_0 \rho}{\pi(r_2^2 - r_1^2)} \left[\frac{r_2}{s} J_1(sr_2) - \frac{r_1}{s} J_1(sr_1) \right], \tag{4}$$

where $k^2 = \frac{\mu_0 \mu \cdot iw}{\rho}$. The solution of the problem (3)-(4) can be found in the form

$$\hat{E}_z(s, z) = \frac{i_0 \rho}{\pi(r_2^2 - r_1^2)} \left[\frac{r_2}{s} J_1(sr_2) - \frac{r_1}{s} J_1(sr_1) \right] e^{-\sqrt{s^2 + k^2} z}. \tag{5}$$

Using the inverse Hankel transform, we get

$$E_z(r, z) = \frac{i_0 \rho}{\pi(r_2^2 - r_1^2)} \int_0^\infty \left[\frac{r_2}{s} J_1(sr_2) - \frac{r_1}{s} J_1(sr_1) \right] e^{-\sqrt{s^2 + k^2} z} J_0(sr) s \, ds \tag{6}$$

The magnetic field can be found from the Maxwell equation

$$rot \bar{H} = \frac{1}{\rho} \bar{E}$$

For axial z-component it means that

$$\frac{1}{r}\frac{\partial(rH_\varphi)}{\partial r} = \frac{1}{\rho}E_z \tag{7}$$

Substituting here the expression (6) and integrating with respect to r on the interval $[0, r]$, we get

$$H_\varphi(r,z) = \frac{i_0\rho}{\pi(r_2^2 - r_1^2)} \int_0^\infty \left[\frac{r_2}{s}J_1(sr_2) - \frac{r_1}{s}J_1(sr_1)\right] e^{-\sqrt{s^2+k^2}z} J_0(sr)s\,ds \tag{8}$$

The radial component of the electrical field E_r is defined from the Maxwell equation

$$rot\bar{E} = -\mu_0\mu\frac{\partial \bar{H}}{\partial t}$$

Taking the φ-component of this equation and using the obtained expressions for E_z and H_φ, we get

$$H_\varphi(r,z) = \frac{i_0\rho}{\pi(r_2^2 - r_1^2)} \int_0^\infty \left[\frac{r_2}{s}J_1(sr_2) - \frac{r_1}{s}J_1(sr_1)\right] e^{-\sqrt{s^2+k^2}z} J_0(sr)\sqrt{s^2+k^2}s\,ds \tag{9}$$

3 Constriction Resistance

The constriction resistance can be found using the expression [6]

$$\int_0^\infty [\bar{E}\bar{H}]\Big|_{z=0} = RJ^2(t) \tag{10}$$

Integration of this expression is very difficult problem, thus we try to find asymptotic formulas for the constriction resistance R.
Let us consider first the case when $r >> r_2$, all the more $r >> r_1$.The integral (6) can be calculated if we us the asymptotic formulas for $J_1(sr_2)$ and $J_1(sr_1)$.Then we can reduce the expression for E_z to form

$$E_z(r,z) = -\frac{i_0\rho}{2\pi}\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{z^2+r^2}}e^{-k\sqrt{z^2+r^2}}\right)$$

Using this formula, we can find $H_\varphi(r,z)$ and $E_r(r,z)$:

$$H_\varphi(r,z) = \frac{i_0\rho}{2k\pi r}\frac{\partial}{\partial z}\left[e^{-k\sqrt{z^2+r^2}} - e^{-kz}\right] = \frac{i_0}{2k\pi r}\left[-\frac{kz}{\sqrt{z^2+r^2}}e^{-k\sqrt{z^2+r^2}} + ke^{-kz}\right]$$

$$E_r(r,z) = \frac{i_0\rho}{2\pi}\left\{-\frac{\partial}{\partial z}\frac{1}{\sqrt{z^2+r^2}}e^{-k\sqrt{z^2+r^2}} + \frac{k}{r}\left[e^{-k\sqrt{z^2+r^2}} - e^{-kz}\right]\right\} =$$

$$= -\frac{i_0\rho}{2\pi}\left\{\frac{k}{r}e^{-kz} + e^{-k\sqrt{z^2+r^2}}\left[\frac{kr}{z^2+r^2} - \frac{k}{r} + \frac{r}{(z^2+r^2)^{3/2}}\right]\right\} \tag{11}$$

Substituting these expressions in the formula (10) and replacing the upper limit of integration ∞ for the radius of conductor r_3 which is sufficiently greater than the radius of the contact spot r, we get

$$R = \frac{\rho}{2\pi r_2}e^{-kr_2} + \frac{k\rho}{2\pi}\int_{r_2}^{\infty}\frac{dr}{r} - \frac{k\rho}{2\pi}\int_{r_2}^{\infty}\frac{e^{-kz}\,dr}{r} = \frac{\rho}{2\pi r_2}e^{-kr_2} + \frac{k\rho}{2\pi}\ln\frac{r_3}{r_2} - \frac{k\rho}{2\pi}Ei(-kr_2) \tag{12}$$

where $Ei(kr_2)$ is the integral exponential function.

Similarly for the condition $r << r_1$ we get

$$E_z(r,z) = -\frac{i_0\rho}{\pi(r_2^2 - r_1^2)}\left[\frac{\partial}{\partial z}e^{-k\sqrt{z^2+r_2^2}} - \frac{\partial}{\partial z}e^{-k\sqrt{z^2+r_1^2}}\right]$$

$$E_r(r,z) = \frac{i_0\rho r k}{2\pi(r_2^2 - r_1^2)}\left[e^{-k\sqrt{z^2+r_2^2}} - e^{-k\sqrt{z^2+r_1^2}}\right]$$

$$H_\varphi(r,z) = \frac{i_0 r}{2\pi k(r_2^2 - r_1^2)}\left[\frac{\partial}{\partial z}e^{-k\sqrt{z^2+r_2^2}} - \frac{\partial}{\partial z}e^{-k\sqrt{z^2+r_1^2}}\right]$$

However at this condition $R = 0$ since $H_\varphi(r,0) = 0$

4 Direct Current

Let us consider the special case of the direct current. For this case $w = 0$ and the fields $E_r(r,0)$ and $H_\varphi(r,0)$ can be obtained from the expressions (8), (9) if we put $k = 0$:

$$E_r(r,0) = \frac{i_0\rho}{\pi(r_2^2 - r_1^2)}\int_0^\infty [r_2 J_1(sr_2) - r_1 J_1(sr_1)]\,\frac{J_1(sr)}{s}\,ds \tag{13}$$

$$H_\varphi(r,0) = \frac{i_0}{2\pi(r_2^2 - r_1^2)}\int_0^\infty \left[\frac{r_2}{s}J_1(sr_2) - \frac{r_1}{s}J_1(sr_1)\right]J_1(sr)s\,ds =$$

$$= \frac{i_0}{2\pi(r_2^2 - r_1^2)}\begin{cases}0, & 0 \le r < r_1\\ r(1 - r_1^2/r^2), & r_1 \le r \le r_2\\ (r_2^2 - r_1^2)/r, & r > r_2\end{cases} \tag{14}$$

Substituting the expression (13), (14) into (10), we get the expression for the constriction resistance

$$R = \frac{2\rho}{\pi(r_2^2 - r_1^2)} \left[\frac{4}{3\pi} r_2^3 + \frac{4}{3\pi} r_1^3 - r_2^2 r_1 F\left(\frac{1}{2}, \frac{1}{2}, 2, \frac{r_1^2}{r_2^2}\right) \right], \tag{15}$$

where F is the hypergeometric function.

It is interesting to note that if we put in (15) $r_1 = 0$ this expression should give the formula for the traditional circle contact spot of the radius r_2. We obtain in this case

$$R = \frac{8\rho}{3\pi^2 r_2} \approx 0.27 \frac{\rho}{r_2}.$$

That is very good approximation for the well-known formula for the construction resistance of the semispace

$$R = \frac{\rho}{4r_2} = 0.25 \frac{\rho}{r_2}.$$

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Kharin, S.N.: Mathematical models of heat and mass transfer in electrical contacts. In: Proceedings of 61th IEEE Holm Conference on Electrical Contacts, IEEE, San Diego, Holm Award, pp. 1–21 (2015)
2. Kharin, S.N.: Mathematical model of a hollow bridge at contact opening. In: Proceedings of the 28th International Conference on Electrical Contacts, Edinburgh, UK, pp. 139–145 (2016)
3. Landau, L.D., Lifshitz, E.M.: Electrodynamics of Continuous Media. GIFML, Moscow (1959)
4. Namitokov, K.K.: Electroerosion Phenomena. Energy, Moscow (1978)
5. Slade, P.: Electrical Contacts. Principles and Applications, 2nd edn. CRC Press (2014)
6. Timsit, R.S.: Temperature distribution in a constriction heated by a high frequency electrical current. In: Proceedings of 28th International Conference on Electrical Contacts. Edinburgh, UK, pp. 3–8 (2016)

Inverse Problem for 1D Pseudo-parabolic Equation

Khonatbek Khompysh(✉)

Al-Farabi Kazakh National University, 71 al-Farabi av., Almaty 050038, Kazakhstan
konat_k@mail.ru

Abstract. In this work we consider an inverse problem of finding a coefficient of right hand side of pseudo-parabolic equation. By successive approximation method the existence and uniqueness of a strong solution are proved. Under the integral overdetermination condition, which has important applications in various areas of applied science and engineering.

Keywords: Inverse problem · Pseudoparabolic equation · Kelvin-Voigt fluids · Successive approximation method · Strong solution · A priori estimate · Integral overdetermination condition

1 Introduction

In this work we consider an inverse problem of identifying the coefficient of right hand side of pseudo-parabolic equation from the integral overdetermination condition, which has important applications in various areas of applied science and engineering.

Statement of the problem. Consider the pseudo-parabolic equation

$$u_t - \nu\, u_{xx} - \chi\, u_{xxt} = f\left(t\right) g\left(x,t\right), \qquad (x,t) \in Q_T \tag{1}$$

with initial condition

$$u\left(x,0\right) = u_0\left(x\right), \quad x \in \left(0,l\right), \tag{2}$$

boundary conditions

$$u\left(0,t\right) = u\left(l,t\right) = 0, \ \ t \in [0,T], \tag{3}$$

and with additional condition in the integral form

$$\int_0^l u\left(x,t\right)\omega\left(x\right) dx = e\left(t\right), \tag{4}$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_36

where $u_0(x)$, $\omega(x)$, $e(t)$, and $g(x,t)$ are given functions, while $u(x,t)$ and $f(t)$ are two unknown functions to be determined.

Equations of this type arise in many areas of mechanics and physics. Such equations are encountered, for example, these equations are included in a model for Kelvin-Voight fluids [3,6,7]. The unique solvability of the inverse problem for Kelvin-Voight equation with overdetermination condition over time variable is considered in [2]. The existence and uniqueness of the strong solution of the direct problem (1)-(3) are proved in [12]. For a discussion of existence and uniqueness results of inverse problems for parabolic and pseudo-parabolic equations see [1,2,4,5,8] (see also the references cited in them).

We understood the strong solution to (1)-(4) as follow.

Definition 1. A pair of functions $(u(x,t), f(t))$ is called a strong solution to problem (1)-(4), if

$$(u(x,t),\, f(t)) \in L_\infty\left(0,T;\overset{\circ}{W}{}_2^1(0,l)\right) \cap W_2^1\left(0,T;W_2^1(0,l)\right) \times L_2(0,T)$$

and (3), (4) hold, and for any $\varphi(x,t) \in L_\infty\left(0,T;\overset{\circ}{W}{}_2^1(0,l)\right) \cap W_2^1 \left(0,T;W_2^1(0,l)\right)$ the integral identity

$$\int_0^T\int_0^l \{u_t\varphi + \nu u_x\varphi_x + \chi u_{xt}\varphi_x\}\,dxdt = \int_0^T\int_0^l f(t)\,g(x,t)\,\varphi(x,t)\,dxdt. \tag{5}$$

Assume that the given functions in the problem (1)-(4) satisfy the following conditions:

$$\begin{aligned} &u_0(x) \in \overset{\circ}{W}{}_2^1(0,l), \quad \omega(x) \in W_2^1(0,l), \\ &e(t) \in W_2^1(0,T), \quad g(x,t) \in L_\infty(0,T;L_2(0,l)), \\ &g_0(t) \equiv \int_0^l \omega(x)g(x,t)\,dx \geq k_0 > 0, \qquad \forall t \in [0,T]. \end{aligned} \tag{6}$$

2 Main Results

Lemma 1. *If (6) holds, then the function $f(t)$ can be determined by the explicit formula*

$$f(t) = \frac{1}{g_0(t)}\left[e'(t) + \int_0^l (\nu u_x\omega_x + \chi u_{xt}\omega_x)\,dx\right] \tag{7}$$

and the inverse problem (1)-(4) is equivalent to the problem (1)-(3) and (7).

Proof. Multiply (1) by $\omega(x)$ and integrate by x from 0 to l using 4. In consequence, by the last condition in (6), we get the (7). The main result in this work is the following theorem.

Theorem 1. *Assume that the condition (6) holds and*

$$\max_{t\in[0,T]} \|g\|^2 \left(\frac{1}{\nu}+\frac{1}{\chi}\right)\frac{l^2\left\|\omega_x\right\|^2}{2k_0^2}<1. \tag{8}$$

Then there exists a unique strong solution $(u\,(x,t)\,,\,f(t))$ *of the inverse problem (1)-(4).*

Proof. We use the successive approximation method. Let us take $u^0=0$ as start approximate. We construct the sequence $\{(u^m, f^m)\}$ as follow: At fist determine the function $f^m\,(t)$ by the relation

$$f^m\,(t)=\frac{1}{g_0(t)}\left[e'\,(t)+\int\limits_0^l\left(\nu u_x^{m-1}\omega_x+\chi u_{xt}^{m-1}\omega_x\right)dx\right], \tag{9}$$

and substituting $f^m\,(t)$ into the right hand side of the following equation, we determine a function $u^m\,(x,t)$ as a strong solution of the following direct problem with given right hand side $F\,(x,t)=g\,(x,t)\,f^m\,(t)$:

$$u_t^m-\nu u_{xx}^m-\chi u_{xxt}^m=f^m\,(t)\,g\,(x,t)\,,\qquad (x,t)\in Q_T, \tag{10}$$

$$u^m\,(x,0)=u_0\,(x)\,,\quad x\in[0,l]\,, \tag{11}$$

$$u^m\,(0,t)=0,\ \ u^m\,(l,t)=0,\ \ t\in[0,T]\,. \tag{12}$$

By [1], the direct problem (10)-(12) for every m has a unique strong solution. Hence, the sequence $\{(u^m, f^m)\}$ is well constructed. If we now prove that the sequence $\{(u^m, f^m)\}$ is a Cauchy sequence in $V_2\,(Q_T)\times L_2\,(0,T)$, then it follows from completeness of the space $V_2\,(Q_T)\times L_2\,(0,T)$ that the sequence $\{(u^m, f^m)\}$ has a limit $\{(\upsilon, f)\}$ as $m\to\infty$, and the limit is a strong solution of (1)-(4), here $V_2\equiv L_\infty\left(0,T;\overset{\circ}{W}{}_2^1\,(0,l)\right)\cap W_2^1\left(0,T;W_2^1\,(0,l)\right)$.

Let us introduce the notation

$$U^m=u^m-u^{m-1},\quad F^m=f^m-f^{m-1}.$$

Then we get from (9)

$$F^m\,(t)=\frac{1}{g_0(t)}\left[\int\limits_0^l\left(\nu U_x^{m-1}\omega_x+\chi U_{xt}^{m-1}\omega_x\right)dx\right] \tag{13}$$

and from (10) -(12) the following initial-boundary value problem

$$U_t^m - \nu U_{xx}^m - \chi U_{xxt}^m = F^m(t)\, g(x,t), \qquad (x,t) \in Q_T, \tag{14}$$

$$U^m(x,0) = 0, \quad x \in [0,l], \tag{15}$$

$$U^m(0,t) = 0, \ \ U^m(l,t) = 0, \quad t \in [0,T]. \tag{16}$$

We estimate (13) by Hölder inequality

$$|F^m(t)| = \frac{1}{|g_0|} \int_0^l \nu U_x^{m-1}\omega_x + \chi U_{xt}^{m-1}\omega_x \ dx \ \le \frac{\|\omega_x\|}{k_0} \ \nu \ U_x^{m-1} \ + \chi \ U_{xt}^{m-1} \ .$$

If one squares both sides of the last inequality and integrates the resulting expressions over τ from 0 to t, then

$$\int_0^t |F^m|^2 d\tau \le \frac{\|\omega_x\|^2}{k_0^2} \int_0^t \left(\nu \left\|U_x^{m-1}\right\|^2 + \chi \left\|U_{xt}^{m-1}\right\|^2\right) d\tau. \tag{17}$$

Now we multiply Eq. (14) by U^m and integrate over $[0,l]$:

$$\frac{1}{2}\frac{d}{dt}\left(\|U^m\|^2 + \chi \|U_x^m\|^2\right) + \nu \|U_x^m\|^2 = \int_\Omega F^m g U^m dx. \tag{18}$$

Using the inequality (Poincare inequality)

$$\|u\| \le \frac{l}{\sqrt{2}} \|u_x\|, \quad \forall u \in \overset{\circ}{W}{}_2^1(0,l),$$

and Cauchy inequality, we estimate right hand side as follows

$$\int_\Omega F^m g U^m dx \ \le \|U^m\| \cdot |F^m| \, \|g\| \le \frac{l}{\sqrt{2}} \|U_x^m\| \, |F^m| \, \|g\| \le \frac{\nu}{2} \|U_x^m\|^2 + \frac{l^2}{4\nu} |F^m|^2 \|g\|^2.$$

Substituting this inequality into (18) and multiplying both sides of the result by 2, and integrating over τ from 0 to t, we obtain

$$\max_{t\in[0,T]} \left(\|U^m\|^2 + \chi \|U_x^m\|^2\right) + \nu \int_0^t \|U_x^m\|^2 d\tau \le \frac{l^2}{2\nu} \max_{t\in[0,T]} \|g\|^2 \int_0^t |F^m|^2 d\tau. \tag{19}$$

Multiplying (14) by U_t^m and integrating over x from 0 to l, we obtain

$$\frac{\nu}{2}\frac{d}{dt} \|U_x^m\|^2 + \|U_t^m\|^2 + \chi \|U_{xt}^m\|^2 = \int_\Omega F^m g U_t^m dx.$$

Estimate the right hand side by Hölder, Poincare and Cauchy inequalities as above. Then we have

$$\nu \max_{t\in[0,T]} \|U_x^m\|^2 + 2\int_0^t \|U_t^m\|^2\,d\tau + \chi\int_0^t \|U_{xt}^m\|^2\,d\tau \le \frac{l^2}{2\chi}\max_{t\in[0,T]}\|g\|^2\int_0^t |F^m|^2\,d\tau. \tag{20}$$

Adding the inequalities (19) and (20), we obtain the estimate

$$\int_0^t \left(\nu\|U_x^m\|^2 + \chi\|U_{xt}^m\|^2\right)d\tau \le \frac{l^2}{2}\max_{t\in[0,T]}\|g\|^2\left(\frac{1}{\nu}+\frac{1}{\chi}\right)\int_0^t |F^m|^2\,d\tau \tag{21}$$

Combining (17) and (21) for every $m = 1, 2, \ldots$, we get

$$\|F^m\|^2_{L_2(0,T)} \le \mu\left\|F^{m-1}\right\|^2_{L_2(0,T)}, \quad \|U^m\|^2_{V_2(Q_T)} \le \mu\left\|U^{m-1}\right\|^2_{V_2(Q_T)}, \tag{22}$$

where $\mu \equiv \max_{t\in[0,T]}\|g\|^2\left(\frac{1}{\nu}+\frac{1}{\chi}\right)\frac{l^2\|\omega_x\|^2}{2k_0^2}$ and by assumption (8) $\mu < 1$.

Hence the convergency of infinite decreasing geometrical progression implies that $\{(u^m, f^m)\}$ is the Cauchy sequence in $V_2(Q_T)\times L_2(0,T)$. By virtue of the above arguments, there exists the unique pair of functions (u, f) in $V_2(Q_T)\times L_2(0,T)$, such that

$$u^m(x,t) \to u(x,t) \quad \text{in} \quad V_2(Q_T),$$
$$f^m(t) \to f(t) \quad \text{in} \quad L_2(0,T)$$

as $m \to \infty$.

Using these information, take the limit of (9)-(12) as $m \to \infty$. Then by convergence of $\{(u^m, f^m)\}$ and by Lemma 1, we see that the pair of limit functions (u, f) is the strong solution of the inverse problem of (1)-(4). Proof of the Theorem is complete.

Acknowledgements. This work was partially supported by the Grant No. N1781/GF4 of MES RK, Kazakhstan.

References

1. Ablabekov, B.C.: Inverse Problems for Pseudo-parabolic Equations. IlIM, Bishkek (2001)
2. Abylkairov, U.U., Khompysh, Kh: An inverse problem of identifying the coefficient in Kelvin-Voight equations. Appl. Math. Sci. J. Theor. Appl. **9**, 5079–5089 (2015)
3. Abylkairov, U.U., Khompysh, Kh: ε-approximation of the equations of heat convection for the Kelvin-Voight fluids. AIP Conf. Proc. **1676**, 020059 (2015). doi:10.1063/1.4930485

4. Asanov, A., Atamanov, E.R.: Nonclassical and Inverse Problems for Pseudo-parabolic Equations, Tokyo (1997)
5. Fedorov, V.E., Urasaeva, A.V.: An inverse problem for linear Sobolev type equation. J. Inverse Ill-Posed Prob. **12**(4), 387–395 (2004)
6. Oskolkov, A.P.: Certain model nonstationary systems in the theory of non-Newtonian fuids. Tr. Mat. Inst. Stek. **127**, 32–57 (1975)
7. Oskolkov, A.P.: Nonlocal problems for equations of Kelvin-Voight fluids. Zap. Nauch. Sem. LOMI. **197**, 120–158 (1992)
8. Prilepko, A.I., Orlovsky, D.G., Vasin, I.A.: Methods for Solving Inverse Problems in Mathematical Physics. Marcel Dekker, New York (2000)

Solution of an Inverse Two Phase Spherical Stefan Test Problem

Merey M. Sarsengeldin[1(✉)], Targyn A. Nauryz[2], Nazerke T. Bizhigitova[3], and Alibek M. Orinbasar[3]

[1] Kazakh British Technical University, 59 Tole Bi St., Almaty 050000, Kazakhstan
merey@email.ru
[2] Institute of Mathematics and Mathematical Modelling, 125 Pushkin St., Almaty 050010, Kazakhstan
targyn.nauryz@gmail.com
[3] Suleyman Demirel University, Kaskelen 040900, Kazakhstan
merey@email.ru

Abstract. The purpose of this study is to test the elaborated theory for inverse two phase spherical Stefan problem by using Integral Error Function and check effectiveness of the suggested solution form for engineering purposes. It was shown that by collocation method we can achieve small error which doesn't exceed 8 percent for three points, which substantially eases calculations. Investigation of such problems enables one to analyse diverse electric contact phenomena.

Keywords: Two-phase Stefan problem · Free boundary value problems · Inverse problem · Flux conditions · Test problem

1 Introduction

This study is the continuation of previous studies [1–4] and an attempt to obtain exact solution and develop an easy way for engineers to find approximate temperature distribution function with relatively small error and to find heat flux function. Nowadays, Stefan type problems are widely used for modelling electric contact phenomena [5–9] and it looks expedient and important to conduct research studies in this field as from theoretical as well as from practical point of view.

Due to the Holm's ideal sphere ($b < 10^{-4}m$) spherical Stefan problem nicely fits as a mathematical model for describing electric contact phenomena with small electric currents.

2 Problem Statement

The contact spot in mathematical model is given by the spherical domain with radius b. $P(t)$ is heat flux which passes through liquid zone $b(t) < r < \alpha(t)$ and then through the solid zone $\alpha(t) < r < \infty$.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_37

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_1}{\partial r} \right), \quad b < r < \alpha(t) \, , \tag{1}$$

$$\frac{\partial \theta_2}{\partial t} = a_2^2 \left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_2}{\partial r} \right), \quad \alpha(t) < r < \infty \tag{2}$$

with initial condition

$$\theta_1(b, 0) = 0, \tag{3}$$

$$\theta_2(r, 0) = f(r), \tag{4}$$

$$f(r) = \theta_m + r - b, \quad f(b) = \theta_m \tag{5}$$

and boundary condition

$$r = b: \quad -\lambda_1 \frac{\partial \theta_1(b, t)}{\partial r} = P(t), \tag{6}$$

$$r = \alpha(t): \quad \theta_1(\alpha(t), t) = \theta_m, \tag{7}$$

$$\theta_2(\alpha(t), t) = \theta_m, \tag{8}$$

the Stefan's condition

$$-\lambda_1 \frac{\partial \theta_1(\alpha(t), t)}{\partial r} = -\lambda_2 \frac{\partial \theta_2(\alpha(t), t)}{\alpha r} + L\gamma \frac{d\alpha}{dt} \, , \tag{9}$$

as well as the condition at the infinity

$$\theta_2(\infty, t) = 0 \, , \tag{10}$$

where θ_1 and θ_2 is an unknown heat functions, $P(t)$ is an unknown heat flux coming from electric arc of radius b. θ_m is a melting temperature of electrical contact material, $f(r)$ is a given function, $\lambda_1, \lambda_2, a_1, a_2$ and $L\gamma$ are constants.

3 Problem Solution

We can expend the initial and boundary functions in Maclaurin series as

$$P(t) = \sum_{n=0}^{\infty} P_n t^{n/2}, \quad f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (r - b)^n$$

and $\alpha(t) = b + \alpha\sqrt{t}$. The solution of (1)-(10) we represent in the new form of series

$$\theta_1(r, t) = \frac{1}{r} \sum_{n=0}^{\infty} (2a_1\sqrt{t})^n \left[A_n i^n erfc \frac{r - b}{2a_1\sqrt{t}} + B_n i^n erfc \frac{b - r}{2a_1\sqrt{t}} \right],$$

$$\theta_2(r, t) = \frac{1}{r} \sum_{n=0}^{\infty} (2a_2\sqrt{t})^n \left[C_n i^n erfc \frac{r - b}{2a_2\sqrt{t}} + D_n i^n erfc \frac{b - r}{2a_2\sqrt{t}} \right],$$

where coefficients A_n, B_n, C_n and D_n have to be found and as the same analogy of previous study we determined

$$D_n = \frac{1}{2} f^{(n)}(b), \quad n = 0, 1, 2, ..., \tag{11}$$

$$C_0 = \frac{1}{2}\theta_m(2b-1), \tag{12}$$

$$C_1 = \frac{\theta_m \alpha}{2a_2 ierfc\frac{\alpha}{2a_2}} - \frac{ierfc\frac{-\alpha}{2a_2}}{2ierfc\frac{\alpha}{2a_2}}, \tag{13}$$

$$C_n = -\frac{1}{2} f^{(n)}(b) \frac{i^n erfc\frac{-\alpha}{2a_2}}{i^n erfc\frac{\alpha}{2a_2}}, \quad n = 2, 3, 4, ..., \tag{14}$$

$$B_0 = A_0 \exp\left(\frac{\alpha^2}{2(a_1)^2}\right) - \frac{L\gamma a_1 \sqrt{\pi}\alpha b}{2\lambda_1} \exp\left(\frac{\alpha}{2a_1}\right)^2, \tag{15}$$

$$A_0 = \frac{\theta_m b}{1 + exp\left(\frac{\alpha}{\sqrt{2}a_1}\right)^2} - \frac{L\gamma a_1 \sqrt{\pi}\alpha b}{2\lambda_1\left(1 + \exp\left(\frac{\alpha}{\sqrt{2}a_1}\right)^2\right)} \exp\left(\frac{\alpha}{2a_1}\right)^2, \tag{16}$$

$$A_1 = \frac{L\gamma\alpha^2 ierfc\frac{-\alpha}{2a_1}}{2\lambda_1\left(ierfc\frac{\alpha}{2a_1} + ierfc\frac{-\alpha}{2a_1}\right)} - \frac{ierfc\frac{-\alpha}{2a_2}}{2ierfc\frac{\alpha}{2a_2}} \frac{\lambda_2 ierfc\frac{-\alpha}{2a_1}}{\lambda_1\left(ierfc\frac{\alpha}{2a_1} + ierfc\frac{-\alpha}{2a_1}\right)} - \frac{\lambda_2 ierfc\frac{-\alpha}{2a_1}}{2\lambda_1\left(ierfc\frac{\alpha}{2a_1} + ierfc\frac{-\alpha}{2a_1}\right)}, \tag{17}$$

$$B_1 = A_1 + \frac{\lambda_2}{\lambda_1}\frac{ierfc\frac{-\alpha}{2a_2}}{2ierfc\frac{\alpha}{2a_2}} + \frac{\lambda_2}{2\lambda_1} - \frac{L\gamma\alpha^2}{2\lambda_1}, \tag{18}$$

$$A_n = -B_n \frac{i^n erfc\frac{-\alpha}{2a_1}}{i^n erfc\frac{\alpha}{2a_1}}, \quad n = 2, 3, 4, ... \tag{19}$$

From Stefan's condition we get expression

$$-\lambda_1(2a_1)^{n-1}\left[-A_n i^{n-1} erfc\frac{\alpha}{2a_1} + B_n i^{n-1} erfc\frac{-\alpha}{2a_1}\right] = -\lambda_2(2a_2)^{n-1}\left[-C_n i^{n-1} erfc\frac{\alpha}{2a_2} + D_n i^{n-1} erfc\frac{-\alpha}{2a_2}\right], \quad n = 2, 3, 4, ... \tag{20}$$

$$P_n = \frac{\lambda_1}{b}(2a_1)^n (A_{n+1} - B_{n+1}) i^n erfc0, \quad n = 0, 1, 2, 3, ... \tag{21}$$

Thus coefficients A_0, B_0 are found in (15),(16) and A_1 and B_1 are found in (17),(18), A_n, B_n can be determined in (19),(20) when $n = 2, 3, 4, ..., C_0, C_n, D_n$ from (11),(12) and (14), P_n from (21).

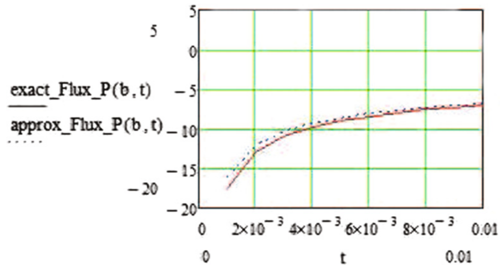


Fig. 1. Exact and approximate values of flux function at small t

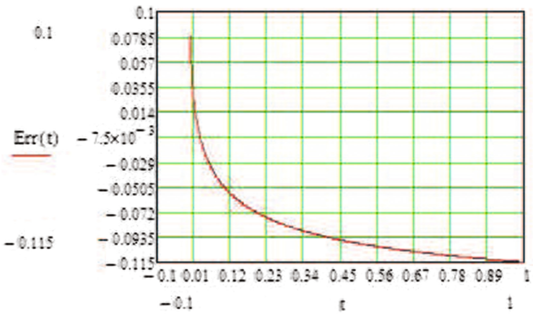


Fig. 2. Relative error

Test result We use Mathcad 15 for calculations and get following exact values $A_0 = 0.43, A_1 = -2,576, A_2 = 0, B_0 = -0.43, B_1 = 0.429$ and $B_2 = 0$, then approximate values $A_0 = 0.579, A_1 = -3.004, A_2 = 7.582 \times 10^{-15}, B_0 = -0.183, B_1 = 0.5$ and $B_2 = 0$. At last we have exact and approximate values of heat flux function as in Fig. 1 and relative error function which reaches maximum value 7.9 percent as in Fig. 2.

4 Conclusion

As a result of the current approach we receive the approximation function, convergent to the exact solution. The heat flux $P(t)$ is determined from expression (21) in electric contacts on the base of two phase spherical inverse Stefan problem. Temperatures θ_1, θ_2 are found by determining coefficients A_n, B_n, C_n and D_n from equations (11),(12),(19) and (20).

Acknowledgements. The authors would like to thank Prof. S. N. Kharin (IMMM and KBTU, Kazakhstan) for his valuable comments and suggestions which were helpful in improving the paper. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Friedman, A.: Free boundary problems in biology. Phil. Trans. R. Soc. **A373**, 20140368 (2015)
2. Holm, R.: Electrical Contacts. Springer (1981)
3. Kavokin, A.A., Nauryz, T.A., Nazerke, B.T.: Exact solution of two phase spherical Stefan problem with two free boundaries. AIP Conf. Proc. **1759**, 020117 (2016)
4. Kharin, S.N., Nouri, H., Davies, T.: The mathematical models of welding dynamics in closed and switching electrical contacts. In: Proceedings of 49th IEEE Holm Conference on Electrical Contacts, pp. 107–123 Washington, USA (2003)
5. Kharin, S.N., Sarsengeldin, M.M., Kassabek, S.A.: The mathematical models of electromagnetic field dynamics and heat transfer in closed electrical contacts including Thomson effects. AIP Conf. Proc. **1759**, 020072 (2016). https://doi.org/10.1063/1.4959686
6. Kharin, S.N., Sarsengeldin, M.M.: Influence of contact materials on phenomena in a short electrical arc. In: Kharin, S., Salam, I., Ahmed, K. (eds.) Key Engineering Materials. Trans Tech Publication. **510-511**, pp. 321–329 (2012)
7. Sarsengeldin, M.M., Kharin, S.N.: Method of the integral error functions for the solution of the one- and two-phase Stefan problems and its application. Filomat **31**, 4 (2017). https://doi.org/10.2298/FIL1704017S
8. Sarsengeldin, M.M., Nauryz, T.A., Erdogan, S.E.: Exact solution of an inverse spherical two phase Stefan problem. arXiv:1703.04946 [math-ph] (2017)
9. Rubinstein, L.I.: The Stefan problem. Trans Math. Monogr. **27**, AMS, Providence, RI (1971)

Generically Semistable Linear Quiver Sheaves

Alexander H.W. Schmitt(✉)

Freie Universität Berlin, Institut Für Mathematik, Arnimallee 3, 14195 Berlin, Germany
alexander.schmitt@fu-berlin.de

Abstract. We present an alternative proof for the classification of semistable representations of a linear quiver and of a circular quiver with three vertices and briefly discuss the meaning of this result for the study of quiver sheaves.

Keywords: Quiver · Representation · Stability parameter · Semiinvariant · Twisted quiver sheaf · Generically semistable/totally unstable · Boundedness · Variation of moduli spaces

1 Introduction

The theory of quiver sheaves over a smooth projective variety has been developed by Álvarez-Cónsul and García-Prada [2–4], Gothen and King [19], the author [29,31,32], and many others. In this note, we will focus on the notion of slope semistability.[1] If n is the number of vertices of the quiver, the notion of slope semistability depends on an n-tuple $\underline{\kappa}$ of positive real numbers and an n-tuple $\underline{\chi}$ of real numbers. If X is a curve and one fixes the ranks and degrees of the vector bundles under consideration, then one may attach to each stability parameter $(\underline{\kappa}, \underline{\chi})$ a moduli space. It is a fundamental problem to understand the variation of the moduli spaces with the stability parameter. This involves the question for which stability parameters the moduli space is non-empty. Explicit examples were studied in [5,9,25] . In those papers, $\underline{\kappa}$ has the fixed value $(1, ..., 1)$ and only $\underline{\chi}$ varies. General properties of the chamber decomposition and related boundedness questions were discussed in [6]. An important step was made in the papers [33,35]. The main result of these papers is that, in the circumstances outlined above, there are only finitely many distinct notions of slope semistability. This implies that there are only finitely many distinct moduli spaces.

[1] In the case that X is a curve, the notion of slope semistability is the "right" one for constructing moduli spaces.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_38

Let us fix a polarized smooth projective variety $(X, \mathscr{O}_X(1))$ over the complex numbers, a quiver $Q = (V, A, t, h)$, and a tuple of line bundles $\underline{M} = (M_a, a \in A)$ on X.[2] We let $\mathbb{K} := \mathbb{C}(X)$ be the function field of X. Recall that an $\underline{M}$-twisted Q-sheaf is a tuple $\mathscr{R} = (\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ in which $\mathscr{E}_v$ is a coherent $\mathscr{O}_X$-module, $v \in V$, and $\varphi_a \colon M_a \otimes \mathscr{E}_{t(a)} \longrightarrow \mathscr{E}_{h(a)}$ is a twisted homomorphism, $a \in A$. The tuple $(\underline{r}, \underline{d})$ with $\underline{r} = (\mathrm{rk}(\mathscr{E}_v), v \in V)$ and $\underline{d} = (\deg(\mathscr{E}_v), v \in V)$ is the *type* of $\mathscr{R}$. We may restrict $\mathscr{R}$ to the generic point η of X. Then, $M_a \otimes_{\mathscr{O}_X} \mathbb{K}$ is isomorphic to the constant sheaf $\mathbb{K}$, and we choose isomorphisms $M_a \otimes_{\mathscr{O}_X} \mathbb{K} \cong \mathbb{K}$. Next, $W_v := \mathscr{E}_v \otimes_{\mathscr{O}_X} \mathbb{K}$ is a $\mathbb{K}$-vector space of dimension $\mathrm{rk}(\mathscr{E}_v)$, $v \in V$, and, using the above trivializations, we get the linear maps $f_a := \varphi_{a|\{\eta\}} \colon W_{t(a)} \longrightarrow W_{h(a)}$, $a \in A$. In other words, $R = (W_v, v \in V, f_a, a \in A)$ is a $\mathbb{K}$-representation of Q. With the help of this construction and King's theory of semistability for $\mathbb{K}$-representations [23], we may divide the class of $\underline{M}$-twisted Q-sheaves into two subclasses:

- the class of **generically totally unstable** ones, i.e., those for which there is no non-zero stability parameter with respect to which the $\mathbb{K}$-representation R of Q is semistable, and
- the class of **generically semistable** ones, that is, the class of $\underline{M}$-twisted Q-sheaves for which the associated $\mathbb{K}$-representation R of Q is semistable with respect to some non-zero stability parameter.

For the first class, one may use the theory of the instability flag of Ramanan and Ramanathan [26] in order to get bounds for the stability parameters $\underline{\chi}$ ([35], compare [30,33]). In addition, we were able to reduce the case of generically semistable $\underline{M}$-twisted Q-sheaves to the case of $\underline{M}$-twisted Q-sheaves which are generically totally unstable in a different sense and obtain bounds on the stability parameters $\underline{\chi}$ in this case, as well. Applying Harder–Narasimhan filtrations of quiver sheaves for the stability parameter $\underline{\kappa}$, an argument introduced in [18], and the analysis of the chamber structure from [6], we obtained the finiteness result explained above.

There is another interesting aspect in the above distinction. The region of stability parameters $(\underline{\kappa}, \underline{\chi})$ for which generically totally unstable $(\underline{\kappa}, \underline{\chi})$-slope semistable quiver sheaves may exist is a priori bounded, by the results discussed before. If there are slope semistable quiver sheaves which are generically semistable with respect to the stability parameter, say, $\underline{\eta}$, then the region of stability parameters may stretch out to infinity in the direction specified by $\underline{\eta}$. This gives rise to an interesting question on representations of quivers over a field, namely, for which quiver representations does there exist a stability parameter $\underline{\chi} \neq 0$, such that the representation is $\underline{\chi}$-semistable in the sense of King?

In order to deal with this question, let us briefly review the basic set-up for studying the classification of quiver representations. Fix a field $\mathbb{K}$ and a quiver $Q = (V, A, t, h)$, let $\underline{r} = (r_v \in \mathbb{N}, v \in V)$ be a dimension vector, and set

$$R := \mathrm{Rep}_{\underline{r}}(Q) := \bigoplus_{a \in A} \mathrm{Hom}_{\mathbb{K}-\mathrm{vs}}(\mathbb{K}^{r_{t(a)}}, \mathbb{K}^{r_{h(a)}}),$$

[2] In general, one may allow vector bundles of arbitrary rank.

$$G := \mathrm{GL}_{\underline{r}}(V) := \bigtimes_{v\in V} \mathrm{GL}_{r_v}(\mathbb{K}), \quad S := \mathrm{SL}_{\underline{r}}(V) := \bigtimes_{v\in V} \mathrm{SL}_{r_v}(\mathbb{K}).$$

The group G acts in a natural way on the vector space R, and the set of G-orbits in R corresponds in a natural way to the set of isomorphy classes of representations of Q with dimension vector $\underline{r}$. In order to study the question just raised, we need to look at the action of S on R, more precisely at the invariant ring $\mathbb{K}[R]^S$. For $r = (f_a, a \in A) \in R$, there exists a non-trivial stability parameter $\underline{\chi} \neq 0$, such that r is $\underline{\chi}$-semistable if and only if there exists a non-zero S-invariant function $F \in \mathbb{K}[R]^S$ with $F(r) \neq 0$. So, in order to answer the above question, we may try to compute the invariant ring $\mathbb{K}[R]^S$. By work of Schofield and Van den Bergh [36,37], Derksen and Weyman [11], and Domokos and Zubkov [13], the ring $\mathbb{K}[R]^S$ is generated by determinants (see Remark 6). This implies a satisfactory answer to our question although it still may be difficult to make it explicit for a concretely given quiver.

Let us turn to a basic example, the quiver[3]

$$A_{n+1}: \quad n \xrightarrow{a_n} n-1 \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_2} 1 \xrightarrow{a_1} 0.$$

This case was studied by Koike [24]. Koike's paper is a predecessor to the work of Schofield, Van den Bergh, Derksen, Weyman, Domokos, and Zubkov. His result implies that, for $r = (f_a, a \in A) \in R$, there exists a non-trivial stability parameter $\underline{\chi} \neq 0$, such that r is $\underline{\chi}$-semistable if and only if there are indices $n \geq m > l \geq 0$, such that $f_{a_{l+1}} \circ \cdots \circ f_{a_n} : \mathbb{C}^{r_{a_n}} \longrightarrow \mathbb{C}^{r_{a_l}}$ is an isomorphism. Koike's proof is based on an analysis of the representation of $\mathrm{SL}_{r_1}(\mathbb{C}) \times \mathrm{SL}_{r_2}(\mathbb{C})$ on $\mathrm{Mat}_{r_1,r_2}(\mathbb{C})$, $r_1, r_2 \in \mathbb{N}$.

In this note, we will provide a different proof based on the Hilbert–Mumford criterion. It works well for linear quivers, because we know the decomposition into indecomposable objects in this case. The technique is similar to the one used by Abeasis and Del Fra [1] in the investigation of degenerations of orbits in $\mathrm{Rep}_{\underline{r}}(A_{n+1})$. In our set-up, we need less detailed information than Abeasis and Del Fra. On a representation space of a quiver, the Hilbert–Mumford criterion for semistability holds over an arbitrary field (see Remark 1, v). So, an advantage of our method is that we get a characterization of semistable quiver representations over any field. Recall that, in the applications we have in mind, the field $\mathbb{K}$ will be the field of rational functions on an algebraic variety X. Linear quiver sheaves play an important role in the theory of Higgs bundles (see, e.g., [5,16]).

Koike also deals with the case of circular quivers. For circular quivers with three vertices, it is possible to work out everything by hand. Let us point out that quiver sheaves associated with circular quivers appeared in the form of cyclotomic or cyclic Higgs bundles in work of Simpson [38], Collier [10], and García-Prada and Ramanan [17].

---
[3] We choose this labeling of the vertices in order to comply with the standard notation for holomorphic chains.

2 Quiver Representations

A quiver Q is a quadruple (V, A, t, h) in which V and A are finite sets, called the *set of vertices* and the *set of arrows* respectively, and $t, h\colon A \longrightarrow V$ are maps, associating with an arrow $a \in A$ its *tail* $t(a)$ and its *head* $h(a)$, respectively. The *dual quiver* is $Q^\vee = (V, A, t^\vee, h^\vee)$ with $t^\vee = h$ and $h^\vee = t$, i.e., $Q^\vee$ is obtained from Q by reversing all arrows.

Obviously, a quiver is the same as a category with a finite set of objects and finite morphism sets. Given an abelian category $\underline{\mathrm{A}}$, one may study the functors from Q to $\underline{\mathrm{A}}$. These form again an abelian category. The classical case is when $\underline{\mathrm{A}}$ is the category $\underline{\mathrm{Vect}}_{\mathbb{K}}$ of finite dimensional vector spaces over a field $\mathbb{K}$. A theorem of Gabriel's [15] (compare [7], Theorem II.3.7) asserts that, for any finite dimensional algebra $\mathscr{A}$ over an algebraically closed field $\mathbb{K}$, there is a unique quiver Q, such that the category of representations of $\mathscr{A}$ is equivalent to the category of representations of Q in $\underline{\mathrm{Vect}}_{\mathbb{K}}$, obeying certain relations.

The classification of quiver representations reduces to the classification of indecomposable representations. Another famous result of Gabriel's ([14], 1.2, Satz, [28], Chap. 8) characterizes quivers of finite representation type, i.e., quivers which admit only finitely many isomorphy classes of indecomposable representations. In addition, it provides a classification of the indecomposable representations. This should be a valuable result for studying quiver sheaves.

In this section, we will briefly review the formalism of quiver representations and the parameter dependent theory of semistability, due to King [23]. Since stable representations are indecomposable, this is an important tool for partially understanding indecomposable representations for wild quivers. For applications to quiver sheaves, it will be important to characterize those quiver representations for which there exists a **non-zero** stability parameter with respect to which they are semistable in the sense of King. For circular and linear quivers, these characterizations were obtained by Koike ([24], Theorem 1 and 2). In fact, Koike determined generators for the whole ring of semi-invariants. Here, we will give different proofs for circular quivers with three vertices and A_{n+1}-quivers (compare Sect. 2.5).

2.1 Representations

Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*representation* of Q is a tuple $R = (W_v, v \in V, f_a, a \in A)$ in which W_v is a finite dimensional $\mathbb{K}$-vector space, $v \in V$, and $f_a\colon W_{t(a)} \longrightarrow W_{h(a)}$ is a $\mathbb{K}$-linear map, $a \in A$. The tuple $\underline{\dim}(R) := (\dim_{\mathbb{K}}(W_v), v \in V)$ is the *dimension vector* of R. A $\mathbb{K}$-*subrepresentation* of R is a collection $(U_v, v \in V)$ in which U_v is a linear subspace of W_v, such that $f_a(U_{t(a)}) \subset U_{h(a)}$ is satisfied, $a \in A$. It is *non-trivial* (*proper*), if there is an index $v_0 \in V$ with $U_{v_0} \neq 0$ ($U_{v_0} \neq W_{v_0}$). A *quotient* $\mathbb{K}$-representation of R consists of a tuple $(q_v\colon W_v \longrightarrow Q_v, v \in V)$ of surjective $\mathbb{K}$-linear maps, such that $(\ker(q_v), v \in V)$ is a $\mathbb{K}$-subrepresentation. We will often denote it in the form $(Q_v, v \in V)$. It is *non-*

zero (*proper*), if $(\ker(q_v), v \in V)$ is proper (non-zero). The *dual* $\mathbb{K}$-representation is $R^\vee := (W_v^\vee, v \in V, f_a^\vee \colon W_{t(a)}^\vee \longrightarrow W_{h(a)}^\vee)$. It is a $\mathbb{K}$-representation of the dual quiver $Q^\vee$ of Q.

2.2 Semistability

Semistability of $\mathbb{K}$-representations depends on a tuple $\underline{\chi} = (\chi_v, v \in V)$ of real numbers. For a collection $(T_v, v \in V)$ of finite dimensional $\mathbb{K}$-vector spaces which are not all trivial, we define the $\underline{\chi}$-*dimension* as $\dim_{\underline{\chi}}(T_v, v \in V) := \sum_{v \in V} \chi_v \cdot \dim_{\mathbb{K}}(T_v)$ and the $\underline{\chi}$-*slope* as

$$\mu_{\underline{\chi}}(T_v, v \in V) := \frac{\dim_{\underline{\chi}}(T_v, v \in V)}{\sum_{v \in V} \dim_{\mathbb{K}}(T_v)}.$$

We say that a $\mathbb{K}$-representation $R = (W_v, v \in V, f_a, a \in A)$ of Q is $\underline{\chi}$-*(semi)stable*, if

$$\mu_{\underline{\chi}}(U_v, v \in V)(\leq)\mu_{\underline{\chi}}(W_v, v \in V)$$

holds true for any non-trivial, proper $\mathbb{K}$-subrepresentation $(U_v, v \in V)$ of R.

Remark 1. i) Note that any $\mathbb{K}$-representation of Q is semistable with respect to the parameter $0 \in \mathbb{R}^n$. It will be 0-stable if and only if it is simple, i.e., does not have a non-zero, proper $\mathbb{K}$-subrepresentation.

ii) One readily checks that, for a stability parameter $\underline{\chi} = (\chi_v, v \in V) \in \mathbb{R}^n$, a real number c, and $\underline{\chi}^c := (\chi_v + c, v \in V)$, a $\mathbb{K}$-representation $R = (W_v, v \in V, f_a, a \in A)$ is $\underline{\chi}$-(semi)stable if and only it is $\underline{\chi}^c$-(semi)stable. One may use this observation to assume without loss of generality that

$$\sum_{v \in V} \chi_v \cdot \dim_{\mathbb{K}}(W_v) = 0.$$

Under this assumption, R is $\underline{\chi}$-(semi)stable if and only if

$$\sum_{v \in V} \chi_v \cdot \dim_{\mathbb{K}}(U_v)(\leq)0$$

holds for every non-zero, proper $\mathbb{K}$-subrepresentation $(U_v, v \in V)$ of R.

iii) Since the dimension of $\mathbb{K}$-vector spaces behaves additively on short exact sequences, one sees that a $\mathbb{K}$-representation $R = (W_v, v \in V, f_a, a \in A)$ is $\underline{\chi}$-(semi)stable if and only if

$$\mu_{\underline{\chi}}(W_v, v \in V)(\leq)\mu_{\underline{\chi}}(Q_v, v \in V)$$

holds true for any non-trivial and proper quotient $\mathbb{K}$-representation $(q_v \colon W_v \longrightarrow Q_v, v \in V)$ of R.

iv) Part iii) implies that a $\mathbb{K}$-representation R of Q is (semi)stable with respect to the parameter $\underline{\chi} = (\chi_v, v \in V)$ if and only if the dual $\mathbb{K}$-representation $R^\vee$ of $Q^\vee$ is (semi)stable with respect to the parameter $-\underline{\chi} = (-\chi_v, v \in V)$.

v) There exists a Harder–Narasimhan filtration for $\mathbb{K}$-representations with respect to the stability parameter $\underline{\chi}$ ([21], Theorem 2.5). As in [22], Theorem 1.3.7, one uses it to show that a $\mathbb{K}$-representation of Q is $\underline{\chi}$-semistable over $\mathbb{K}$ if and only if it is $\underline{\chi}$-semistable over the algebraic closure $\overline{\mathbb{K}}$. This is not necessarily true for $\underline{\chi}$-stability (see Example 1).

vi) For a tuple $(T_v, v \in V)$ of $\mathbb{K}$-vector spaces, we define the *support* as the set $\operatorname{Supp}(T_v, v \in V) := \{\, v \in V \mid T_v \neq 0 \,\}$. Likewise, we define the *support* of a tuple $\underline{\chi} = (\chi_v, v \in V)$ of real numbers as $\operatorname{Supp}(\underline{\chi}) := \{\, v \in V \mid \chi_v \neq 0 \,\}$. When studying (semi)stability of a $\mathbb{K}$-representation $(W_v, v \in V, f_a, a \in A)$, it clearly suffices to restrict to stability parameters $\underline{\chi}$ with $\operatorname{Supp}(\underline{\chi}) \subset \operatorname{Supp}(W_v, v \in V)$.

vii) Let $R = (W_v, v \in V, f_a, a \in A)$ be a $\mathbb{K}$-representation. Define

$$N := \Big\{\, \underline{\psi} = (\psi_v, v \in V) \in \mathbb{R}^{\#V} \;\Big|\; \sum_{v \in V} \psi_v \cdot \dim_{\mathbb{K}}(W_v) = 0 \,\Big\} \qquad (1)$$

and

$$S := \big\{\, \underline{\omega} \in N \mid R \text{ is} \underline{\omega}\text{-semistable} \,\big\}.$$

Note that S is a cone in N, i.e., for $\underline{\chi}^1, \underline{\chi}^2 \in S$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$, the tuple $\lambda_1 \cdot \underline{\chi}^1 + \lambda_2 \cdot \underline{\chi}^2$ also lies in S.

viii) Fix a tuple $\underline{r} = (r_v, v \in V)$ of non-negative integers. A *test configuration* is a tuple $\underline{e} = (e_v, v \in V)$ of integers with $0 \leq e_v \leq r_v$, $v \in V$, and $0 < \sum_{v \in V} e_v < \sum_{v \in V} r_v$. With a test configuration $\underline{e}$, we associate the *wall*

$$\mathscr{W}_{\underline{e}} := \Big\{\, \underline{\omega} = (\omega_v, v \in V) \;\Big|\; \sum_{v \in V} \omega_v \cdot e_v = 0 \,\Big\}.$$

These walls induce a decomposition

$$N = \bigsqcup_{v \in V} \mathscr{C}_i$$

into locally closed subsets, called *chambers*, such that, for $i = 1, \ldots, w$, $\underline{\chi}, \underline{\chi}' \in \mathscr{C}_i$, a $\mathbb{K}$-representation R of Q is $\underline{\chi}$-(semi)stable if and only if it is $\underline{\chi}'$-(semi)stable. Note that every chamber contains integral vectors. This means studying (semi)stability with respect to real stability parameters amounts to the same as studying it with respect to integral stability parameters. A result of Derksen and Weyman ([12], Theorem 5.1, see also [27], Theorem 3) explains how to compute this decomposition in terms of the quiver Q.

The last result we would like to emphasize is the existence of Jordan–Hölder filtrations.

Proposition 1. *For every stability parameter* $\underline{\chi} = (\chi_v, v \in V) \in N$ *and every* $\underline{\chi}$*-semistable* $\mathbb{K}$*-representation* $R = (W_v, v \in V, f_a, a \in A)$*, there exists a filtration*

$$0 =: R^0 \subsetneq R^1 \subsetneq \cdots \subsetneq R^s \subsetneq R^{s+1} := R$$

by $\mathbb{K}$*-subrepresentations, such that*

a) $\dim_{\underline{\chi}}(R^i) = 0$*,* $i = 0, ..., s+1$*,*
b) R^{i+1}/R^i *is* $\underline{\chi}$*-stable,* $i = 0, ..., s$*.*

Furthermore, the isomorphism class of

$$\bigoplus_{i=0}^{s} R^{i+1}/R^i \tag{2}$$

depends only on R *and* $\underline{\chi}$*.*

Proof. See [21], Proposition 2.7.

The filtration in Proposition 1 is the *Jordan–Hölder filtration* of R (with respect to $\underline{\chi}$), and the representation in (2) is denoted by $\mathrm{gr}_{\underline{\chi}}(R)$ and called the *associated graded* $\mathbb{K}$*-representation* of R (with respect to $\underline{\chi}$).

Remark 2. i) Note that $\mathrm{gr}_{\underline{\chi}}(R)$ is $\underline{\chi}$-semistable.

ii) If $\mathbb{K}$ is not algebraically closed, the Jordan–Hölder filtration and the associated graded object may change when passing to an algebraic extension $\mathbb{L}$ of $\mathbb{K}$ (see the following example).

Example 1. We look at the quiver Using the normalization from Remark 1, ii), we see that 0 is the only relevant stability parameter for $\mathbb{K}$-representations of Q. A $\mathbb{K}$-representation of Q is a finite dimensional $\mathbb{K}$-vector space W together with an endomorphism $f\colon W \longrightarrow W$. If $\dim_{\mathbb{K}}(W) = 2$, then a one dimensional $\mathbb{K}$-subrepresentation is spanned by a non-zero eigenvector of f. So, we may take $\mathbb{K} = \mathbb{R}$ and an endomorphism $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with characteristic polynomial $x^2 + 1$. Then, $(\mathbb{R}^2, f)$ is 0-stable (Remark 1, i), but its extension to $\mathbb{C}$ isn't. The Jordan–Hölder filtration over $\mathbb{R}$ is $0 \subset \mathbb{R}^2$, and, over $\mathbb{C}$, the Jordan–Hölder filtrations are of the form $0 \subset \langle v \rangle \subset \mathbb{R}^2$, v being a non-zero eigenvector of f.

A $\mathbb{K}$-representation R is $\underline{\chi}$*-polystable*, if it is $\underline{\chi}$-semistable and isomorphic to $\mathrm{gr}_{\underline{\chi}}(R)$. We say that a $\mathbb{K}$-representation R is *totally unstable*, if it is unstable with respect to any non-zero stability parameter $\underline{\chi} \in \mathbb{R}^{\#V}$. We will present the classification of totally unstable $\mathbb{K}$-representations of certain quivers in the section after next.

2.3 The GIT Set-Up

Fix a tuple $\underline{r} = (r_v, v \in V)$ of non-negative integers and set

$$U := \mathrm{Rep}_{\underline{r}}(Q) := \bigoplus_{v \in V} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{r_{t(a)}}, \mathbb{K}^{r_{h(a)}}).$$

It parameterizes $\mathbb{K}$-representations of Q with dimension vector $\underline{r}$. Every $\mathbb{K}$-representation of Q with dimension vector $\underline{r}$ is isomorphic to one in $\mathrm{Rep}_{\underline{r}}(Q)$. Next, introduce the reductive affine algebraic group

$$G := \mathrm{GL}_{\underline{r}}(\mathbb{K}) := \bigtimes_{v \in V} \mathrm{GL}_{r_v}(\mathbb{K}).$$

It acts on $\mathrm{Rep}_{\underline{r}}(Q)$ via

$$\begin{aligned} \mathrm{GL}_{\underline{r}}(\mathbb{K}) \times \mathrm{Rep}_{\underline{r}}(Q) &\longrightarrow \mathrm{Rep}_{\underline{r}}(Q) \\ (g,u) = \big((B_v, v \in V), (f_a, a \in A)\big) &\longmapsto \big(B_{h(a)} \circ f_a \circ B_{t(a)}^{-1}, a \in A\big). \end{aligned}$$

Note that two $\mathbb{K}$-representations in $\mathrm{Rep}_{\underline{r}}(Q)$ are isomorphic if and only they lie in the same $\mathrm{GL}_{\underline{r}}(\mathbb{K})$-orbit. Let

$$S := \mathbb{K}[U]^G = \Big\{\, f \in \mathbb{K}[U] \,\Big|\, \forall g \in G,\ u \in U : f(g \cdot u) = f(u) \,\Big\}.$$

The affine algebraic variety

$$U /\!\!/ G := \mathrm{Spec}(S)$$

parameterizes semisimple, i.e., 0-polystable, $\mathbb{K}$-representations of Q with dimension vector $\underline{r}$.

Remark 3. Suppose $\underline{\chi} \in \mathbb{R}^{\#V} \setminus \{0\}$ is a non-trivial stability parameter and $R = (W_v, v \in V, f_a, a \in A)$ is a $\underline{\chi}$-semistable $\mathbb{K}$-representation. Let

$$0 =: R^0 \subsetneq R^1 \subsetneq \cdots \subsetneq R^s \subsetneq R^{s+1} := R$$

be the Jordan–Hölder filtration of R with respect to $\underline{\chi}$ and write $R^j = (W_v^j, v \in V)$, $j = 1, ..., s$. We may pick bases $\{\, w_{v1}, ..., w_{vd_v} \,\}$ for W_v, such that

$$\langle\, w_{v1}, ..., w_{v\, \dim_{\mathbb{K}}(W_v^j)} \,\rangle = W_v^j, \quad j = 1, ..., s,\ v \in V.$$

Using these bases, R and $\mathrm{gr}_{\underline{\chi}}(R)$ define points u and u' in $\mathrm{Rep}_{\underline{r}}(A_{n+1})$, $\underline{r}$ being the dimension vector of R. We also choose integral weights $\gamma_1 < \cdots < \gamma_s < \gamma_{s+1}$. One may use these data to define a one parameter subgroup

$$\lambda \colon \mathbb{G}_m(\mathbb{K}) \longrightarrow \mathrm{GL}_{\underline{r}}(\mathbb{K})$$

with

$$\lim_{t \to \infty} \lambda(t) \cdot u = u'.$$

This shows that u' lies in the closure of the orbit of u.

Let $\chi\colon \mathrm{GL}_{\underline{r}}(\mathbb{K}) \longrightarrow \mathbb{G}_m(\mathbb{K})$ be a character. A *χ-semi-invariant* is a regular function

$$f\colon U \longrightarrow \mathbb{K}$$

with the property

$$\forall g \in G,\ u \in U: \quad f(g\cdot u) = \chi(g)\cdot f(u).$$

Remark 4. Let f be a χ-semi-invariant. The open subset

$$U_f := \{\, u \in U \mid f(u) \neq 0 \,\}$$

is invariant under the G-action on U.

We let

$$\mathbb{K}[U]^{\chi} := \Big\{\, f \in \mathbb{K}[U] \,\Big|\, f \text{is a} \chi\text{-semiinvariant} \,\Big\}$$

be the sub vector space of χ-semi-invariant functions. The ring

$$\mathbb{K}[U]_{\chi} := \bigoplus_{d=0}^{\infty} \mathbb{K}[U]^{\chi^d}$$

is a finitely generated $\mathbb{K}$-algebra. We set

$$U/\!\!/_{\chi} G := \mathrm{Proj}(\mathbb{K}[U]_{\chi}).$$

Since $\mathbb{K}[U]^{\chi^0} = \mathbb{K}[U]^G$, we have a projective morphism

$$\pi_{\chi}\colon U/\!\!/_{\chi} G \longrightarrow U/\!\!/G.$$

Remark 5. i) The natural rational map $U \dashrightarrow U/\!\!/_{\chi} G$ is defined in a point $u \in U$ if and only if there exist a positive integer $d > 0$ and a χ^d-semi-invariant $f \in \mathbb{K}[U]^{\chi^d}$ with $f(u) \neq 0$. This is the GIT notion of semistability defined with respect to the linearization of the G-action in $\mathscr{O}_U$ defined by the character χ (see [23], Sect. 2).

ii) Let $\underline{\chi} = (\chi_v, v \in V) \in \mathbb{Z}^{\#V}$ be a tuple of integers. It yields the character

$$\begin{aligned} \chi\colon\ \mathrm{GL}_{\underline{r}}(\mathbb{K}) &\longrightarrow \mathbb{G}_m(\mathbb{K}) \\ (B_v, v \in V) &\longmapsto \prod_{v\in V} \det(B_v)^{\chi_v}. \end{aligned} \tag{3}$$

Theorem 1. ([23], **Theorem 5.1**) Let $\underline{\chi} = (\chi_v, v \in V) \in (N \cap \mathbb{Z}^{\#V})$ be a tuple of integers and let $\chi\colon \mathrm{GL}_{\underline{r}}(\mathbb{K}) \longrightarrow \mathbb{G}_m(\mathbb{K})$ be the character from (3). Then, for a point $u = (f_a, a \in A) \in \mathrm{Rep}_{\underline{r}}(Q)$, the $\mathbb{K}$-representation $R = (\mathbb{K}^{r_v}, v \in V, f_a, a \in A)$ is $\underline{\chi}$-semistable if and only if there exist a positive integer $d > 0$ and a χ^{-d}-semi-invariant[4] $f \in \mathbb{K}[U]^{\chi^{-d}}$ with $f(u) \neq 0$.

[4] Take note of the sign.

In view of Remark 1, viii), this gives the GIT interpretation of the notion of semistability introduced before.

Remark 6. i) An obvious way to construct semi-invariants is the following: For a given dimension vector $\underline{r} = (r_v, v \in V)$, look for tuples $\underline{\lambda} = (\lambda_v, v \in V)$, $\underline{\lambda}' = (\lambda_v, v \in V)$ of non-negative integers with $\sum_{v\in V} \lambda_v \cdot r_v = \sum_{v \in V} \lambda'_v \cdot r_v$ and paths in Q, including those of length zero, connecting points in $\mathrm{Supp}(\underline{\lambda})$ to points $\mathrm{Supp}(\underline{\lambda}')$. Given these data, we may assign to every $\mathbb{K}$-representation $R = (W_v, v \in V, f_a, a \in A)$ of Q a linear map

$$\bigoplus_{v\in V} W_v^{\oplus \lambda_v} \longrightarrow \bigoplus_{v \in V} W_v^{\oplus \lambda'_v}.$$

Mapping R to the determinant of this linear map, gives a semi-invariant function $\mathrm{Rep}_{\underline{r}}(Q) \longrightarrow \mathbb{K}$. It was shown by Schofield and Van den Bergh [36,37], Derksen and Weyman [11], and Domokos and Zubkov [13] that, for a quiver without oriented cycles and an infinite field $\mathbb{K}$, all semi-invariants may be generated from semi-invariants of the above shape. This implies that semistability can be characterized by basic linear algebra conditions, stating that at least one linear map from a certain list must be an isomorphism.

ii) Set

$$\mathrm{SL}_{\underline{r}}(\mathbb{K}) := \bigtimes_{v \in V} \mathrm{SL}_{r_v}(\mathbb{K}).$$

The ring

$$\mathbb{K}\big[\mathrm{Rep}_{\underline{r}}(Q)\big]^{\mathrm{SL}_{\underline{r}}(\mathbb{K})}$$

consists of all semi-invariants. Koike determined in [24] generators for this ring for circular quivers and quivers of type A_{n+1}.

2.4 Quivers of Type a

Let $n \geq 1$ be a natural number. We write A_{n+1} for the quiver

$$n \xrightarrow{\ a_n\ } n-1 \xrightarrow{\ a_{n-1}\ } \cdots \xrightarrow{\ a_2\ } 1 \xrightarrow{\ a_1\ } 0. \tag{4}$$

For a $\mathbb{K}$-representation $(W_i, i = 0, ..., n, f_{a_i}, i = 1, ..., n)$ of A_{n+1}, we set $f_i := f_{a_i}$, $i = 1, ..., n$.

Lemma 1. *Let $R = (W_i, i = 0, ..., n, f_i, i = 1, ..., n)$ be a $\mathbb{K}$-representation of A_{n+1} and $\underline{\chi} = (\chi_i, i = 0, ..., n) \in N$ a stability parameter, such that R is $\underline{\chi}$-semistable. Set $\mathfrak{S} := \mathrm{Supp}(W_i, i = 0, ..., n) \cap \mathrm{Supp}(\underline{\chi})$, $m := \min(\mathfrak{S})$ and $M := \max(\mathfrak{S})$. Then, $\chi_m \leq 0 \leq \chi_M$.*

Proof. The tuple $(U_i, i = 0, ..., n)$ with $U_i = W_i$, $i = 0, ..., m$, and $U_i = 0$, $i = m + 1, ..., n$, is a $\mathbb{K}$-subrepresentation. It satisfies

$$\sum_{i=0}^{n} \chi_i \cdot \dim_{\mathbb{K}}(U_i) = \chi_m \cdot \dim_{\mathbb{K}}(W_m).$$

By our conventions, $\dim_{\mathbb{K}}(W_m) > 0$. The tuple $(V_i, i = 0, ..., n)$ with $V_i = W_i$, $i = 0, ..., M-1$, and $V_i = 0$, $i = M, ..., n$, is also a $\mathbb{K}$-subrepresentation. We find

$$\sum_{i=0}^{n} \chi_i \cdot \dim_{\mathbb{K}}(V_i) = -\chi_M \cdot \dim_{\mathbb{K}}(W_M).$$

Again, our set-up grants $\dim_{\mathbb{K}}(W_M) > 0$.

Our considerations regarding semistability of $\mathbb{K}$-representations of A_{n+1} will rely on the classification of indecomposable $\mathbb{K}$-representations. Let us briefly recall it.[5] For $j = 0, ..., n$, we let $\mathbb{I}_{jj}$ be the $\mathbb{K}$-representation $(W_i, i = 0, ..., n, f_i, i = 1, ..., n)$ with $W_j = \mathbb{K}$ and $W_i = 0$, $i \in \{0, ..., n\} \setminus \{j\}$. Moreover, for $0 \le j < k \le n$, we have the $\mathbb{K}$-representation $\mathbb{I}_{jk} = (W_i, i = 0, ..., n, f_i, i = 1, ..., n)$ with $W_i = \mathbb{K}$, $i = j, ..., k$, $W_i = 0$, $i \in \{0, ..., n\} \setminus \{j, ..., k\}$, and $f_i = \mathrm{id}_{\mathbb{K}}$, $i = j+1, ..., k$. Graphically, we have

$$\mathbb{I}_{jj} : 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{K} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0,$$

$$\mathbb{I}_{jk} : 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{\mathrm{id}_{\mathbb{K}}} \cdots \xrightarrow{\mathrm{id}_{\mathbb{K}}} \mathbb{K} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$

Proposition 2. *Every $\mathbb{K}$-representation of A_{n+1} is isomorphic to a direct sum of $\mathbb{K}$-representations of the form $\mathbb{I}_{jk}$, $0 \le j \le k \le n$.*

Proof. For $n = 0, 1$, one knows the result from basic Linear Algebra. We proceed by induction on n. Let $R = (W_i, i = 0, ..., n, f_i, i = 1, ..., n)$ be a $\mathbb{K}$-representation of A_{n+1}. We form the $\mathbb{K}$-subrepresentation $S = (U_i, i = 0, ..., n)$ of R, by setting $U_0 := 0$, $U_i := \ker(f_1 \circ \cdots \circ f_i)$, $i = 1, ..., n$. Since $U_0 = 0$, we may view S as a $\mathbb{K}$-representation of A_n and apply the induction hypothesis to it. To conclude, we will construct a direct complement T to S which is a direct sum of indecomposable $\mathbb{K}$-representations from the above list.

Set $c_i := \mathrm{codim}_{W_i}(U_i)$, $i = 0, ..., n$, and $c_{n+1} := 0$. Note that $c_i \le c_{i-1}$, $i = 1, ..., n$. In fact, let $i \in \{1, ..., n\}$ and $X_i \subset W_i$ be a direct complement to U_i. Then, $X_i \cong f_i(X_i)$ and $f_i(X_i) \cap U_{i-1} = 0$. Next, let $n \ge j_1 > \cdots > j_r \ge 0$ be the indices where the codimension jumps, i.e., $c_{j_\rho+1} < c_{j_\rho}$, $\rho = 1, ..., r$. Choose vectors $u^1_1, ..., u^1_{d_1} \in W_{j_1}$ which form the basis of a direct complement X_{j_1} of U_{j_1}. Define $I^{1,\nu} = (I^{1,\nu}_i, i = 0, ..., j_1)$ with $I^{1,\nu}_{j_1} := \langle u^1_\nu \rangle$ and $I^{1,\nu}_i = \langle (f_{i+1} \circ \cdots \circ f_{j_1})(u^1_\nu) \rangle$, $\nu = 0, ..., j_1 - 1$. Next, choose vectors $u^2_{d_1+1}, ..., u^2_{d_2} \in W_{j_2}$ which form together with the vectors $(f_{j_2-1} \circ \cdots \circ f_{j_1})(u^1_1), ..., (f_{j_2-1} \circ \cdots \circ f_{j_1})(u^1_{d_1})$ the basis of a direct complement X_{j_2} for U_{j_2}. Then, we may define $I^{2,\nu}$, $\nu = d_1 + 1, ..., d_2$, similarly as before and iterate the construction. Now,[6]

$$\bigoplus_{\rho=1}^{r} \bigoplus_{d_{\rho-1}+1}^{d_\rho} I^{\rho,\nu}$$

[5] The general classification scheme for indecomposable representations of a quiver of finite representation type is given by Gabriel's theorem (see [28], Chap. 8).
[6] Set $d_0 := 0$.

clearly is a $\mathbb{K}$-subrepresentation of R and a direct complement to S. Finally, note that, if $j_r = 0$, then $I^{0,\nu} \cong \mathbb{I}_{00}$, $\nu = d_{r-1}+1,...,d_r$, and that, otherwise, $I^{\rho,\nu} \cong \mathbb{I}_{0j_\rho}$, $\nu = d_{\rho-1}+1,...,d_\rho$, $\rho = 1,...,r$.

Remark 7. i) For $0 \le j < k \le n$, the $\mathbb{K}$-subrepresentations of $\mathbb{I}_{jk}$ are $\mathbb{I}_{jl}$, $l = j,...,k$.

ii) For $l = 1,...,n$, define the stability parameter $\underline{\chi}^l = (\chi^l_i, i = 0,...,n)$ by $\chi_l := 1$, $\chi_{l-1} := -1$, and $\chi_i = 0$, $i \in \{0,...,n\} \setminus \{l-1,l\}$. The $\mathbb{K}$-representation $\mathbb{I}_{jk}$ is semistable with respect to the stability parameters $\underline{\chi}^l$, $l = j+1,...,k$, $0 \le j < k \le n$.

iii) Let $R = (W_i, i = 0,...,n, f_i, i = 1,...,n)$ be a $\mathbb{K}$-representation of A_{n+1} and $\underline{\chi} = (\chi_i, i = 0,...,n) \in N$ a stability parameter, such that R is $\underline{\chi}$-semistable. If $\mathbb{I}_{jj}$ occurs in the decomposition of R into indecomposables, then $\chi_j = 0$, $j = 0,...,n$. If $\mathbb{I}_{jk}$ occurs in the decomposition of R into indecomposables, then $\sum_{i=j}^{k} \chi_i = 0$, $0 \le j < k \le n$. These conditions follow, because the respective indecomposable $\mathbb{K}$-representation is both a $\mathbb{K}$-subrepresentation and a quotient $\mathbb{K}$-representation of R (compare Remark 1, iii).

Lemma 2. *Fix $0 \le j < k \le n$, and let $\underline{\chi} = (\chi_i, i = 0,...,n)$ be a stability parameter with $\sum_{i=j}^{k} \chi_i = 0$ and $\mathrm{Supp}(\underline{\chi}) \subset \{j,...,k\}$. Then, $\mathbb{I}_{jk}$ is $\underline{\chi}$-semistable if and only if $\underline{\chi}$ is a linear combination of the $\underline{\chi}^l$, $l = j+1,...,k$, with non-negative coefficients.*

Proof. If $\underline{\chi}$ is a linear combination of the $\underline{\chi}^l$, $l = j+1,...,k$, with non-negative coefficients, then Remarks 7, ii), and 1, vii), show that $\mathbb{I}_{jk}$ is $\underline{\chi}$-semistable.

Suppose conversely that $\mathbb{I}_{jk}$ is $\underline{\chi}$-semistable. We apply induction on $k-j$. For $k-j = 1$, there is nothing to show. According to Remark 7, i), $\mathbb{I}_{jj}$ is a $\mathbb{K}$-subrepresentation, so that $\chi_j \le 0$. Set $\underline{\chi}' = (\chi'_i, i = 0,...,n) := \underline{\chi} + \chi_j \cdot \underline{\chi}^j$. In order to apply induction, we need to show that $\mathbb{I}_{j+1k}$ is $\underline{\chi}'$-semistable. For $l = j+1,...,k$, we find

$$\sum_{i=j+1}^{l} \chi'_i = \left(\sum_{i=j+2}^{l} \chi_i\right) + (\chi_{j+1} + \chi_j) = \sum_{i=j}^{l} \chi_i \ge 0.$$

So, the $\mathbb{K}$-subrepresentation $\mathbb{I}_{j+1l}$ is not destabilizing, $l = j+1,...,k$.

Remark 8. i) Using the decomposition of a $\mathbb{K}$-representation R into indecomposables, Lemma 2 enables one to determine all stability parameters $\underline{\chi}$ with respect to which R is semistable.

ii) A $\mathbb{K}$-representation which is stable for some stability parameter $\underline{\chi}$ is indecomposable. This follows from Remark 1, iii), because a direct summand is both a $\mathbb{K}$-subrepresentation and a quotient $\mathbb{K}$-representation. For $j = 0,...,n$, the $\mathbb{K}$-representation $\mathbb{I}_{jj}$ is 0-stable, by Remark 1, i). Now, let $0 \le j < k \le n$ and $\underline{\chi} \in \mathbb{R}^{\#V}$ be a stability parameter with $\sum_{i=j}^{k} \chi_i = 0$. Using the description of the

$\mathbb{K}$-subrepresentations of $\mathbb{I}_{jk}$ (Remark 7, i), we see that the following conditions are equivalent:

- $\mathbb{I}_{jk}$ is $\underline{\chi}$-stable.
- $\sum_{i=j}^{l} \chi_i < 0$, $l = j, ..., k-1$.
- $\sum_{i=l}^{k} \chi_i > 0$, $l = j+1, ..., k$.

Lemma 3. *Let $0 \le j \le k \le n$, $0 \le l \le m \le n$, and let $\underline{\chi} \in \mathbb{R}^{\#V} \setminus \{0\}$ be a stability parameter with $\sum_{i=j}^{k} \chi_i = 0 = \sum_{i=l}^{m} \chi_i$. Suppose that $\mathbb{I}_{jk}$ and $\mathbb{I}_{lm}$ are both $\underline{\chi}$-stable.*[7] *Then, one of the following conditions is satisfied:*

- *$j = l$ and $k = m$.*
- *$k < l$.*
- *$m < j$.*
- *$j < l$ and $m < k$.*
- *$l < j$ and $k < m$.*

Proof. Suppose that none of the above conditions is satisfied. Then, after exchanging the roles of (j,k) and (l,m), if necessary, we have $j \le l \le k \le m$ and $j < l$ or $k < m$. Let us assume $j < l$. Then, by Remark 8, ii),[8]

$$0 < \sum_{i=l}^{k} \chi_i \le 0.$$

This is clearly impossible. A similar argument applies if $j = l$ and $k < m$.

The following result gives a characterization of the totally unstable $\mathbb{K}$-representations of A_{n+1}.

Proposition 3. *Let $R = (W_i, i = 0, ..., n, f_i, i = 1, ..., n)$ be a $\mathbb{K}$-representation of A_{n+1}. Then, there exists a stability parameter $\underline{\chi} \in N \setminus \{0\}$ for which R is semistable if and only if there are indices $0 \le j < k \le n$, such that $f_{j+1} \circ \cdots \circ f_k \colon W_k \longrightarrow W_j$ is an isomorphism.*

Proof. Suppose there exist indices $0 \le j < k \le n$, such that $f_{j+1} \circ \cdots \circ f_k \colon W_k \longrightarrow W_j$ is an isomorphism. Then, R is semistable with respect to the stability parameter $\underline{\chi} = (\chi_i, i = 0, ..., n)$ with $\chi_j = -1$, $\chi_k = 1$, and $\chi_i = 0$, $i \in \{0, ..., n\} \setminus \{j, k\}$. In fact, let $(U_i, i = 0, ..., n)$ be a $\mathbb{K}$-subrepresentation. Then, $\dim_{\mathbb{K}}(U_k) \le \dim_{\mathbb{K}}(U_j)$, i.e.,

$$\sum_{i=0}^{n} \chi_i \cdot \dim_{\mathbb{K}}(U_i) = \dim_{\mathbb{K}}(U_k) - \dim_{\mathbb{K}}(U_j) \le 0.$$

[7] Then, $\mathbb{I}_{jk} \oplus \mathbb{I}_{lm}$ is $\underline{\chi}$-semistable.
[8] The second inequality is an equality if and only if $k = m$.

Pick a non-trivial stability parameter $\underline{\chi} \in N\backslash\{0\}$, such that R is $\underline{\chi}$-semistable, and let $\text{gr}_{\underline{\chi}}(R) = (W'_i, i = 0, ..., n, f'_i, i = 1, ..., n)$ be the associated graded $\mathbb{K}$-representation of R with respect to $\underline{\chi}$. We contend that it is sufficient to prove the assertion for $\text{gr}_{\underline{\chi}}(W)$ which is also $\underline{\chi}$-semistable. To this end, we may choose bases for the W_i, $i = 1, ..., n$, which are compatible with the Jordan–Hölder filtration, so that they induce bases for the vector spaces W'_i, $i = 1, ..., n$. So, we may think of R and $\text{gr}_{\underline{\chi}}(R)$ as points of $\text{Rep}_{\underline{r}}(A_{n+1})$, $\underline{r} = (\dim_{\mathbb{K}}(W_0), ..., \dim_{\mathbb{K}}(W_n))$ (compare Remark 3). Suppose that $0 \le j < k \le n$ are indices, such that $f'_{j+1} \circ \cdots \circ f'_k : W_k \longrightarrow W_j$ is an isomorphism. Then, necessarily $r_j = r_k$, and we may study the function

$$\mathfrak{d}_{jk} : \text{Rep}_{\underline{r}}(A_{n+1}) \longrightarrow \mathbb{K} \tag{5}$$

$$(h_i, i = 1, ..., n) \longmapsto \det(h_{j+1} \circ \cdots \circ h_k). \tag{6}$$

It is a semi-invariant for the $\text{GL}_{\underline{r}}(\mathbb{K})$-action with respect to the character

$$\chi_{jk} : \ \text{GL}_{\underline{r}}(\mathbb{K}) \longrightarrow \mathbb{G}_m(\mathbb{K})$$
$$(B_i, i = 0, ..., n) \longmapsto \det(B_j) \cdot \det(B_k)^{-1}.$$

The set where $\mathfrak{d}_{jk}$ does not vanish is open and $\text{GL}_{\underline{r}}(\mathbb{K})$-invariant. By Remark 3, $\text{gr}_{\underline{\chi}}(R)$ is contained in the closure of the $\text{GL}_{\underline{r}}(\mathbb{K})$-orbit of R. Therefore, $\mathfrak{d}_{jk}$ does not vanish in R. This means that $f_{j+1} \circ \cdots \circ f_k : W_k \longrightarrow W_j$ is an isomorphism.

By the above discussion, we may suppose that R is $\underline{\chi}$-polystable. By Remark 8, ii), $\underline{\chi}$-stable $\mathbb{K}$-representations are indecomposable. This means that the direct sum decomposition of R into $\underline{\chi}$-stable $\mathbb{K}$-representations is the decomposition into indecomposable $\mathbb{K}$-representations. If $\mathbb{I}_{jj}$ appears in the decomposition, then $\chi_j = 0$, $j = 0, ..., n$. Since we assume $\underline{\chi}$ to be non-trivial, there must be indices $0 \le j < k \le n$, such that $\mathbb{I}_{jk}$ occurs in the decomposition. Pick $0 \le j_0 < k_0 \le n$, such that $\mathbb{I}_{j_0 k_0}$ is a direct summand of R and $k_0 - j_0$ takes the minimal value among the index pairs (j, k) with $0 \le j < k \le n$ and $\mathbb{I}_{jk}$ appearing in the direct sum decomposition. We assert that $f_{j_0+1} \circ \cdots \circ f_{k_0}$ is an isomorphism. In fact, this follows from Lemma 3 and the fact that R is $\underline{\chi}$-polystable.

Remark 9. The argument given at the end of the above proof shows, more precisely, that, if $\mathbb{I}_{jk}$ appears in the decomposition of $\text{gr}_{\underline{\chi}}(R)$, then $f_{j+1} \circ \cdots \circ f_k : W_k \longrightarrow W_j$ is an isomorphism.

Example 2. Let $R = (W_i, i = 0, ..., n, f_i, i = 1, ..., n)$ be a $\mathbb{K}$-representation of A_{n+1} with $\dim_{\mathbb{K}}(W_j) \neq \dim_{\mathbb{K}}(W_k)$, for $0 \le j < k \le n$. Then, R is totally unstable.

Remark 10. i) Given a quiver Q and a dimension vector $\underline{r}$, one may ask for which stability parameters $\underline{\chi}$ there do exist $\underline{\chi}$-semistable $\mathbb{K}$-representations of Q with dimension vector $\underline{r}$. The results mentioned in Remark 6 and the cone decomposition of N discussed in Remark 1, viii), both give a general answer to this question. I have not checked, if these methods can be applied to classify totally unstable representations. (Note that the quoted results all start from a given dimension vector. The answer we are looking for should not mention the dimension vector explicitly but rather imply conditions for it being eligible as the dimension vector of a semistable representation of the quiver.)

ii) The above reasoning can, in principle, be carried out for all quivers of finite representation type. However, already for other orientations of the graph

the conditions may become much more complicated to state. They can be extracted from [24], Theorem 3.

iii) Note that for every stability parameter $\underline{\chi} \in N$ there exists — up to isomorphy — at most one $\underline{\chi}$-polystable $\mathbb{K}$-representation of dimension vector $\underline{r}$. So, a moduli space for $\underline{\chi}$-semistable $\mathbb{K}$-representations of A_{n+1} of dimension vector $\underline{r}$ is either empty or consists of just one point.

2.5 The Invariant Ring

We would like to recover Koike's computation of the invariant ring $\mathbb{K}[U]^S$. We will apply the following classical result of Hilbert.

Theorem 2. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, S a reductive algebraic group, U a finite dimensional $\mathbb{K}$-vector space, and $\rho\colon S \longrightarrow \mathrm{GL}(U)$ a rational representation. Suppose that $f_1, ..., f_s \in \mathbb{K}[U]^S$ are invariants whose common vanishing locus is the set of nullforms. Then, the invariant ring $\mathbb{K}[U]^S$ equals the integral closure of $\mathbb{K}[f_1, ..., f_s]$ in the field of invariants $Q(\mathbb{K}[U]^S)$.*

Proof. [20], §4, [39], Theorem 4.6.1.

Let n be a positive integer, $\underline{r}$ a dimension vector, and $\mathbb{K}$ a field. As before, we define $U := \mathrm{Rep}_{\underline{r}}(A_{n+1})$, $G := \mathop{\times}\limits_{i=0}^{n} \mathrm{GL}_{r_i}(\mathbb{K})$. and $S := \mathop{\times}\limits_{i=0}^{n} \mathrm{SL}_{r_i}(\mathbb{K})$. For $0 \le l < m \le n$ with $r_l = r_m$, $r_\nu \ge r_m$, $\nu = l+1, ..., m-1$, let $\mathfrak{d}_{lm}\colon U \longrightarrow \mathbb{K}$ be the invariant defined in (5). We say that $\mathfrak{d}_{lm}$ is a *basic invariant*, if

$$\forall \nu \in \{\, l+1, ..., m-1 \,\}: \quad r_\nu > r_m.$$

Remark 11. Let $0 \le l_i < m_i \le n$, $i = 1, 2$, be such that $\mathfrak{d}_{l_1 m_1}$ and $\mathfrak{d}_{l_2 m_2}$ are distinct basic invariants. If $\#(\{\, l_1, ..., m_1 \,\} \cap \{\, l_2, ..., m_2 \,\}) \ge 2$, then $l_1 < l_2 < m_2 < m_1$ or $l_2 < l_1 < m_1 < m_2$.

Proposition 4. (Koike) *Assume that $\mathbb{K}$ is infinite. Then, the basic invariants are algebraically independent.*

Before we start with the proof, let us compare G-orbits and S-orbits in U. For $0 \le l < m \le n$ and $\lambda_i \in \mathbb{K}^*$, $i = l+1, ..., m$, define the indecomposable representation

$$\mathbb{I}_{lm}(\lambda_{l+1}, ..., \lambda_m): \quad \mathbb{K} \xrightarrow{\lambda_m \cdot \mathrm{id}_{\mathbb{K}}} \mathbb{K} \xrightarrow{\lambda_{m-1} \cdot \mathrm{id}_{\mathbb{K}}} \cdots \xrightarrow{\lambda_{l+2} \cdot \mathrm{id}_{\mathbb{K}}} \mathbb{K} \xrightarrow{\lambda_{l+1} \cdot \mathrm{id}_{\mathbb{K}}} \mathbb{K}.$$

We assume that $\mathbb{K}$ admits r-th roots of unity, for $r \in \{ r_0, ..., r_n \}$. This implies that

$$\begin{gathered}\mathbb{G}_n(\mathbb{K})^{\times(n+1)} \times S \longrightarrow G \\ \big((z_i, i=0,...,n), (B_i, i=0,...,n)\big) \longmapsto (z_i \cdot B_i, i=0,...,n)\end{gathered}$$

is surjective. In view of Proposition 2, we see that every S-orbit contains a representation of the form

$$\bigoplus_{i=1}^{s} \mathbb{I}_{l_i m_i}(\lambda_{l_i+1}, ..., \lambda_{m_i})^{\oplus \nu_i}, \tag{7}$$

satisfying $(l_i, m_i) \neq (l_j, m_j)$, $1 \leq i < j \leq s$, and $\nu_i > 0$, $i = 1, ..., s$.

Remark 12. For fixed values of s, and (l_i, m_i), ν_i, $i = 1, ..., s$, there might be distinct tuples $(\lambda_1, ..., \lambda_s)$, $(\lambda'_1, ..., \lambda'_s)$ which define, via (7), representations in the same S-orbit. This applies, for example, if λ_i and λ'_i differ by an r_i-th root of unity, $i = 1, ..., s$. More identifications will appear in the proof of Theorem 3.

Proof. (*of Proposition* 4) We will prove by induction on k that any set of k basic invariants is algebraically independent. For $k = 1$, there is nothing to show.

For the induction step, let us suppose we are given $0 \leq l_i < m_i \leq n$, such that $\mathfrak{d}_{l_i m_i}$ is a basic invariant, $i = 1, ..., k+1$. We choose the indexing in such a way that

$$m_{k+1} - l_{k+1} = \mu := \min\{ m_i - l_i \,|\, i = 1, ..., k+1 \}$$

and

$$m_{k+1} = \max\big\{ m_i \,|\, i \in \{ 1, ..., k+1 \} : m_i - l_i = \mu \big\}.$$

Set $I_i := \mathfrak{d}_{l_i m_i}$, $i = 1, ..., k+1$, and assume that $s \in \mathbb{K}[x_1, ..., x_{k+1}] \setminus \{0\}$ is a polynomial with

$$s(I_1, ..., I_{k+1}) = 0.$$

We write

$$s = a_0 \cdot x_{k+1}^t + \cdots + a_{t-1} \cdot x_{k+1} + a_t \quad \text{with} \quad a_j \in \mathbb{K}[t_1, ..., t_k], \quad j = 0, ..., t.$$

By the induction hypothesis, $A_j := a_j(I_1, ..., I_k) \neq 0$, for those $j \in \{ 0, ..., t \}$ with $a_j \neq 0$. Since we are working over an infinite field, we may pick a representation $r = (f_a, a \in A) \in U$ with $A_j(r) \neq 0$, for those $j \in \{ 0, ..., t \}$ with $a_j \neq 0$. Define

$$\Upsilon := \Big\{ i \in \{ 1, ..., k \} \,\Big|\, \mathfrak{d}_{l_i m_i}(r) \neq 0 \wedge \#\big(\{ l_i, ..., m_i \} \cap \{ l_{k+1}, ..., m_{k+1} \}\big) \geq 2 \Big\}.$$

By Remark11, this set is ordered by the relation "$\prec$" with

$$\forall (l_1, m_1), (l_2, m_2) \in \mathbb{N} \times \mathbb{N} : \ (l_1, m_1) \prec (l_2, m_2) \quad :\Longleftrightarrow \quad l_2 < l_1 \leq m_1 < m_2.$$

Let us first assume $\Upsilon = \varnothing$. For $\lambda \in \mathbb{K}$, we define $\widetilde{r}^\lambda = (f_i^\lambda, i = 0, ..., n)$, such that $f_i^\lambda = f_i$, $i \in \{ 0, ..., n \} \setminus \{ l_{k+1}+1, ..., m_{k+1} \}$, and $f_{l_{k+1}+1}^\lambda \circ \cdots \circ f_{m_{k+1}}^\lambda = \lambda \cdot \mathrm{id}_{\mathbb{K}^{r_{m_{k+1}}}}$. Then,

$$I_i(\widetilde{r}^\lambda) = I_i(r), \quad i = 1, ..., k, \quad \text{and} \quad I_{k+1}(\widetilde{r}^\lambda) = \lambda^{r_{m_{k+1}}}.$$

This is clearly impossible.

Next, we look at the case $\Upsilon \neq \varnothing$. Let i_0 be the index, such that (l_{i_0}, m_{i_0}) is minimal with respect to "$\prec$" among the pairs (l_i, m_i), $i \in \Upsilon$. Without loss of generality, we may assume $i_0 = k$. We may also suppose that r has the form specified in (7). For $\lambda \in \mathbb{K}$, we define $\widetilde{r}^\lambda = (f_i^\lambda, i = 0, ..., n)$, such that $f_i^\lambda = f_i$, $i \in \{0, ..., n\} \setminus \{l_{k+1}+1, ..., m_{k+1}\}$, and $f^\lambda_{l_{k+1}+1} \circ \cdots \circ f^\lambda_{m_{k+1}}$ is given by the block matrix

$$\left(\begin{array}{c|c} (f_{l_{k+1}+1} \circ \cdots \circ f_{m_{k+1}})_{|\mathbb{K}^{m_k}} & 0 \\ \hline 0 & \lambda \cdot \mathrm{id}_{\mathbb{K}^{r_{m_{k+1}} - r_{m_k}}} \end{array}\right).$$

Then, $I_i(\widetilde{r}^\lambda) = I_i(r)$, $i = 1, ..., k$, and

$$I_{k+1}(\widetilde{r}^\lambda) = \lambda^{r_{m_{k+1}} - r_{m_k}} \cdot \mathrm{Det}((f_{l_{k+1}+1} \circ \cdots \circ f_{m_{k+1}})_{|\mathbb{K}^{m_k}}).$$

Again, this is impossible.

Theorem 3. (Koike) *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Then, the invariant ring $\mathbb{K}[U]^S$ is (freely) generated by the basic invariants.*

Proof. Proposition 3 implies that the basic invariants cut out the set of nullforms. By Theorem 2, it remains to show that the field $Q(\mathbb{K}[U]^S)$ is generated by the basic invariants, too. Let $B \subset \mathbb{K}[U]^S$ be the $\mathbb{K}$-subalgebra generated by the basic invariants. We know that

$$\mathrm{Spec}\big(\mathbb{K}[U]^S\big) \longrightarrow \mathrm{Spec}(B)$$

is a finite morphism. Since we are working in characteristic zero, Zariski's main theorem ([8], AG 18.2, Theorem) implies that it suffices to verify that this morphism is (generically) one-to-one. This amounts to showing that the basic invariants separate closed S-orbits.

So, we need to understand what the closed S-orbits are. Let us look at a representation r as in (7). In the obvious labeling, there is a basis

$$e_a^{b,c}, \quad a = l_b, ..., m_b, \quad c = 1, ..., \nu_b, \quad b = 1, ..., s,$$

for $\mathbb{K}^{\sum_{i=0}^n r_i}$, corresponding to the presentation of r in the form (7). Suppose we have two distinct indices $i, j \in \{1, ..., s\}$ with

$$l_j \leq l_i \leq m_j \leq m_i.$$

(Note that $l_j < l_i$ or $m_j < m_i$.) Let $\lambda\colon \mathbb{G}_m(\mathbb{K}) \longrightarrow S$ be the one parameter subgroup which acts on $e_a^{i,c}$ with weight $-\nu_j$, $a = l_i, ..., m_j$, $c = 1, ..., \nu_i$, on $e_a^{j,c}$ with weight ν_i, $a = l_i, ..., m_j$, $c = 1, ..., \nu_i$, and on all other basis vectors with weight zero. Let

$$r_\infty := \lim_{z \to \infty} \lambda(z) \cdot r.$$

Then, we obtain r_∞ from r by replacing

$$\mathbb{I}_{l_i m_i}(\lambda_{l_i+1}, ..., \lambda_{m_i})^{\oplus \nu_i} \oplus \mathbb{I}_{l_j m_j}(\lambda^j_{l_j+1}, ..., \lambda^j_{m_j})^{\oplus \nu_j}$$

by

$$\mathbb{I}_{(m_j+1)m_i}(\lambda_{m_j+2},...,\lambda_{m_i})^{\oplus\nu_i}\oplus\mathbb{I}_{l_i m_j}(\lambda_{l_i+1},...,\lambda_{m_j})^{\oplus(\nu_i+\nu_j)}\oplus\mathbb{I}_{l_j(l_i-1)}(\lambda_{l_j+1},...,\lambda_{l_i-1})^{\oplus\nu_j},$$

and leaving all the other summands unchanged.

Now, let us assume that r is as in (7) and that the S-orbit of r is closed. Set

$$\Pi := \{\,(l_i,m_i)\,|\,i=1,...,s\,\}.$$

By our previous argument, we have, for $1\le i<j\le s$, $l_i\le m_i<l_j\le m_j$, $l_j\le m_j<l_i\le m_i$, $(l_i,m_i)\prec(l_j,m_j)$, or $(l_j,m_j)\prec(l_i,m_i)$. It is now clear that the basic invariants, evaluated at r, determine Π. Then, $\nu_1,...,\nu_s$ are also determined.

Applying induction on s, we will prove that $\lambda_1,...,\lambda_n$ can be recovered from the basic invariants evaluated at r.[9] For $s=1$, the assertion is trivial.

For the induction step, we may assume without loss of generality that Π has a unique maximal element with respect to "$\prec$". We choose the labeling in such a way that (l_{p+1},m_{p+1}) is minimal with respect to "$\prec$" among the elements of Π. Let $i_0\in\{\,1,...,s\,\}$ be an index with $(l_{p+1},m_{p+1})\prec(l_{i_0},m_{i_0})$ and $\neg((l_{i_0},m_{i_0})\prec(l_i,m_i)\prec(l_{p+1},m_{p+1}))$, $i=1,...,s$.[10] We may choose the labeling in such a way that $(l_{i_0},m_{i_0})\prec(l_i,m_i)$, $i=i_0+1,...,s$, and $l_{s+1}\le m_{s+1}<\cdots<l_{i_0+1}\le m_{i_0+1}$. If there is an index $j\in\{\,i_0+1,...,s\,\}$, such that $l_j-m_{j+1}\ge 2$, then we may directly apply the induction hypothesis. If $l_j-m_{j+1}=1$, $j=i_0+1,...,s$, the induction hypothesis shows that all λ_i are determined, except for, possibly, λ_j with $j\in\{\,l_i\,|\,i=i_0+1,...,s+1\,\}\cup\{m_{i_0+1}+1\}$. Let $s\in S$ be the element which multiplies $e_a^{b,c}$ by $\lambda_{l_{s+1}}$, $a=l_{s+1},...,m_{s+1}$, $c=1,...,\nu_b$, $b\in B':=\{\,1,...,i_0\,|\,(l_{s+1},m_{s+1})\prec(l_i,m_i)\,\}$, and $e_a^{s+1,c}$ by μ, $a=l_{s+1},...,m_{s+1}$, $c=1,...,\nu_{s+1}$. Here, $\mu\in\mathbb{K}$ satisfies $\mu^{\nu_{s+1}}=(\lambda_{l_{s+1}})^{-\sum_{b\in B'}\nu_b}$. If we apply s to r, we get r' with $\lambda'_{l_{s+1}}=1$, $\lambda'_{l_s}=\mu^{-1}\cdot\lambda_{l_s}$, and the remaining numbers unchanged. Continuing this way, we see that we may assume without loss of generality $\lambda_{l_j}=1$, $j=i_0+1,...,s+1$. We recover $\lambda_{m_{i_0+1}+1}$ from $\mathfrak{d}_{(l_{s+1}-1)(m_{i_0+1}+1)}(r)$.[11]

2.6 Circular Quivers

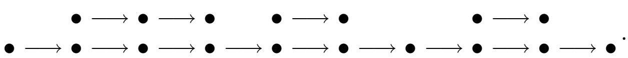
Let $n\ge 2$ be a natural number. In this section, we will work with the quiver $\widetilde{A}_n=(\{\,1,...,n\,\},A,t,h)$ whose arrow set is $A=\{\,a_1,...,a_n\,\}$, $t(a_i)=i$, and

$$h(a_i)=\overline{i+1}:=\begin{cases} i+1,\ \text{for}\, i=1,...,n-1\\ 1,\ \text{for}\, i=n\end{cases},$$

$i=1,...,n$.

[9] There might be different sets of numbers giving isomorphic representations. This will become clear in the proof.

[10] As an example, look at the picture



.

[11] Bear in mind Remark 12.

Remark 13. Note that the dual quiver of a circular quiver is also a circular quiver.

For a $\mathbb{K}$-representation $(W_i, f_{a_i}, i = 1, ..., n)$ of a circular quiver $\widetilde{A}_n$, we will write f_i instead of f_{a_i}, $i = 1, ..., n$, and denote it as $(W_i, f_i, i = 1, ..., n)$. We start this section with two observations which work for any $n \in \mathbb{N}$.

Example 3. Let $(W_i, f_i, i = 1, ..., n)$ be a $\mathbb{K}$-representation of $\widetilde{A}_n$, such that there exist indices $1 \leq j < k \leq w$, such that $f_{k-1} \circ \cdots \circ f_j \colon W_j \longrightarrow W_k$ is an isomorphism. Then, $(W_i, f_i, i = 1, ..., n)$ is semistable with respect to the stability parameter $\underline{\chi} = (\chi_i, i = 1, ..., n)$ with $\chi_j = 1$, $\chi_k = -1$, and $\chi_i = 0$, $i = 1, ..., n \setminus \{ j, k \}$. To check this, let $(U_i, i = 1, ..., n)$ be a $\mathbb{K}$-subrepresentation. Then, $\dim_{\mathbb{K}}(U_j) \leq \dim_{\mathbb{K}}(U_k)$, i.e.,

$$\sum_{v \in V} \chi_i \cdot \dim_{\mathbb{K}}(U_i) = \dim_{\mathbb{K}}(U_j) - \dim_{\mathbb{K}}(U_k) \leq 0.$$

If $\dim_{\mathbb{K}}(W_j) = \dim_{\mathbb{K}}(W_k) > 1$ and $\mathbb{K}$ is algebraically closed, then $(W_i, f_i, i = 1, ..., n)$ is not $\underline{\chi}$-stable. In fact, let $v \in W_k$ be a non-zero eigenvector of $f_{j-1} \circ \cdots \circ f_1 \circ f_n \circ \cdots \circ f_j \colon W_j \longrightarrow W_j$, $U_j := \langle v \rangle$, $U_i := (f_{i-1} \circ \cdots \circ f_1 \circ f_n \circ \cdots \circ f_j)(U_j)$, $i = 1, ..., n \setminus \{j\}$. This is a $\mathbb{K}$-subrepresentation with $\dim_{\mathbb{K}}(U_j) = \dim_{\mathbb{K}}(U_k) = 1$.

Lemma 4. *Let $(W_i, f_i, i = 1, ..., n)$ be a $\mathbb{K}$-representation of the circular quiver $\widetilde{A}_n$, $r_i := \dim_{\mathbb{K}}(W_i)$, $i = 1, ..., n$, and $\underline{\chi} = (\chi_i, i = 1, ..., n)$ a non-zero stability parameter with $\sum_{v \in V} \chi_i \cdot r_i = 0$, such that $(W_i, f_i, i = 1, ..., n)$ is $\underline{\chi}$-semistable.*

i) *Assume that $1 \leq j \leq k \leq w$ are indices with $\chi_j > 0$ and $\chi_i \geq 0$, $i = j + 1, ..., k$. Then, $f_k \circ \cdots \circ f_j \colon W_j \longrightarrow W_{\overline{k+1}}$ is injective.*

ii) *If $1 \leq j \leq k \leq w$ are indices with $\chi_k < 0$ and $\chi_i \leq 0$, $i = j, ..., k-1$, then $f_{k-1} \circ \cdots \circ f_{\overline{j-1}} \colon W_{\overline{j-1}} \longrightarrow W_k$ is surjective.*

Proof. i) Assume that $V := \ker(f_k \circ \cdots \circ f_j)$ is a non-zero subspace of W_j. Then, $(U_i, i = 1, ..., n)$ with $U_i = 0$, $i \notin \{ j, ..., k \}$, and $U_j := V$, $U_{j+l} := (f_{j+l-1} \circ \cdots \circ f_j)(V)$, $l = 1, ..., k - j$, is a non-trivial proper $\mathbb{K}$-subrepresentation which desemistabilizes $(W_i, f_i, i = 1, ..., n)$ with respect to $\underline{\chi}$.

ii) This statement follows from i), by passing to the dual $\mathbb{K}$-representation (see Remark 1, iv). Concretely, we define $U_k := (f_{k-1} \circ \cdots \circ f_{\overline{j-1}})(W_{j-1})$, $U_{k-i} := (f_{k-1} \circ \cdots \circ f_{k-i})^{-1}(U_k)$, $i = 1, ..., k - j$, and $U_i := W_i$, $i \notin \{ j, ..., k \}$. Then, $(U_i, i = 1, ..., n)$ is a $\mathbb{K}$-subrepresentation which desemistabilizes $(W_i, f_i, i = 1, ..., n)$ with respect to $\underline{\chi}$, if U_k is a proper subspace of W_k. This can be most easily seen by looking at the corresponding quotient $\mathbb{K}$-representation (cf. Remark 1, iii).

Proposition 5. *Let $\underline{r} = (r_1, r_2, r_3)$ be a dimension vector with $r_i > 0$, $i = 1, 2, 3$, $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$ a stability parameter with $\chi_1 \cdot r_1 + \chi_2 \cdot r_2 + \chi_3 \cdot r_3 = 0$, $\chi_3 < 0$, $\chi_1 \geq 0$,*[12] *and $(W_i, f_i, i = 1, 2, 3)$ a $\mathbb{K}$-representation of $\widetilde{A}_3$ with*

[12] For any non-trivial stability parameter, this holds true for a suitable cyclic relabeling of the vertices.

$\dim_{\mathbb{K}}(W_i) = r_i$, $i = 1, 2, 3$. *Then, the* $\mathbb{K}$*-representation* $(W_i, f_i, i = 1, 2, 3)$ *is* $\underline{\chi}$*-semistable if and only if one of the following conditions is satisfied:*

a) $r_1 = r_2 = r_3$, *and* f_1 *and* f_2 *are isomorphisms.*
b) $r_1 = r_3$, $f_2 \circ f_1$ *is an isomorphism, and* $\underline{\chi} = \lambda \cdot (1, 0, -1)$, *for some positive real number* λ.
c) $r_2 = r_3$, f_2 *is an isomorphism, and* $\underline{\chi} = \lambda \cdot (0, 1, -1)$, *for some positive real number* λ.

Proof. Let us first check that the stated conditions are sufficient. We start with Case a). Note that $\chi_2 + \chi_3 \le 0$. For any $\mathbb{K}$-subrepresentation (U_1, U_2, U_3), we have $\dim_{\mathbb{K}}(U_1) \le \dim_{\mathbb{K}}(U_2) \le \dim_{\mathbb{K}}(U_3)$, so that

$$\begin{aligned} &\chi_1 \cdot \dim_{\mathbb{K}}(U_1) + \chi_2 \cdot \dim_{\mathbb{K}}(U_2) + \chi_3 \cdot \dim_{\mathbb{K}}(U_3) \\ =\ &\chi_2 \cdot \big(\dim_{\mathbb{K}}(U_2) - \dim_{\mathbb{K}}(U_1)\big) + \chi_3 \cdot \big(\dim_{\mathbb{K}}(U_3) - \dim_{\mathbb{K}}(U_1)\big) \\ \le\ &\chi_3 \cdot \big(\dim_{\mathbb{K}}(U_3) - \dim_{\mathbb{K}}(U_2)\big) \quad \le \quad 0. \end{aligned}$$

Cases b) and c) are included in Example 3.

Let $(W_i, f_i, i = 1, 2, 3)$ be a $\underline{\chi}$-semistable $\mathbb{K}$-representation of $\widetilde{A}_3$. Note that $\chi_3 < 0$ implies that f_2 is surjective, by Lemma 4.

If f_1 is not injective, then $\chi_1 = 0$, for the same reason. But then, $\chi_2 > 0$ and f_2 must be injective, i.e., f_2 has to be an isomorphism, and we have Case c).

Next, suppose f_1 is injective. If $\chi_2 < 0$, then f_1 must be surjective, i.e., an isomorphism. Thus, $r_1 = r_2$ and $\chi_1 + \chi_2 > 0$. This implies that $f_2 \circ f_1$ is injective. Indeed, a non-zero element $v \in \ker(f_2 \circ f_1)$ would provide us with the $\mathbb{K}$-subrepresentation $(v, f_1(v), 0)$. Its dimension vector is $(1, 1, 0)$, so it would be destabilizing. Since f_1 is an isomorphism, f_2 has to be injective. Since this linear map is also surjective, it is an isomorphism, and we are in Case a).

We have already seen that $\chi_2 > 0$ implies that f_2 is an isomorphism. If $\chi_1 > 0$, then $\chi_2 + \chi_3 < 0$. This forces f_1 to be surjective. If it were not surjective, $(0, W_2/f_1(W_1), W_3/(f_2 \circ f_1)(W_1))$ would be a quotient $\mathbb{K}$-representation with dimension vector $(0, r_2 - r_1, r_2 - r_1)$, i.e., it would be destabilizing. So, $\chi_2 > 0$ leads to Case a) or Case c).

It remains to look at the case $\chi_2 = 0$. The same argument as before shows that $f_2 \circ f_1$ is injective. If it were not surjective, then $(0, W_2/f_1(W_1), W_3/(f_2 \circ f_1)(W_1))$ would again be a destabilizing $\mathbb{K}$-quotient representation. This leads to Case a) or Case b).

Example 4. Proposition 5 shows that every $\mathbb{K}$-representation $(W_i, f, i = 1, 2, 3)$ of $\widetilde{A}_3$ with $r_i \ne r_j$, $1 \le i < j \le 3$, is totally unstable.

We refer to [24] for more general results, including the analog of Proposition 5 for arbitrary n.

3 Quiver Sheaves

Let $(X, \mathscr{O}_X(1))$ be a polarized smooth projective variety over the complex numbers. The general formalism of quiver sheaves on X has been surveyed in [6].

We will stick to the notation in that paper.[13] In this section, we will assume that all twisting sheaves are line bundles, that is, in $\underline{M} = (M_a, a \in A)$, we have $\mathrm{rk}(M_a) = 1$, $a \in A$ (see Remark 14, iii). We let $\mathbb{K} := \mathbb{C}(X)$ be the function field of the variety X. The restriction of a quiver sheaf $\mathscr{R}$ to the generic point of the variety X is a representation of the quiver in the category of $\mathbb{K}$-vector spaces. In this way, the theory of quiver sheaves becomes related to the theory of quiver representations. In [5], this observation was used to get an understanding of the shape of parameter regions of semistable A_3-sheaves.[14] In this note, we will slightly expand this method.

For the following, let $Q = (V, A, t, h)$ be a quiver, $\underline{M} = (M_a, a \in A)$ a collection of line bundles on X, $\underline{r} = (r_v, v \in V)$ a tuple of positive integers, and $\underline{d} = (d_v, v \in V)$ a tuple of integers. We say that an $\underline{M}$-twisted Q-sheaf $\mathscr{R} = (\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ has *type* $(\underline{r}, \underline{d})$, if $\mathrm{rk}(\mathscr{E}_v) = r_v$, $\deg(\mathscr{E}_v) = d_v$, $v \in V$.

3.1 The Generic Representation of a Quiver Sheaf

An $\underline{M}$-twisted Q-sheaf $\mathscr{R} = (\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ can be restricted to the generic point of X. This restriction will be denoted by $R = (W_v, v \in V, f_a, a \in A)$. It is a $\mathbb{K}$-representation of Q. It makes sense to study its stability properties. So, let $\underline{\alpha} = (\alpha_v, v \in V) \in \mathbb{R}^{\#V}$ be a stability parameter. We say that $\mathscr{R}$ is *generically* $\underline{\alpha}$*-semistable*, if R is $\underline{\alpha}$-semistable.[15]

Remark 14. i) Assume that $\underline{\alpha}$ satisfies $\sum_{v \in V} \alpha_v \cdot \mathrm{rk}(\mathscr{E}_v) = \sum_{v \in V} \alpha_v \cdot \dim_{\mathbb{K}}(W_v) = 0$. Then, an $\underline{M}$-twisted Q-sheaf $\mathscr{R} = (\mathscr{E}_v, v \in V, \varphi_a, a \in A)$ is generically $\underline{\alpha}$-semistable if and only if, for every Q-subsheaf $(\mathscr{F}_v, v \in V)$, we have $\sum_{v \in V} \alpha_v \cdot \mathrm{rk}(\mathscr{F}_v) \leq 0$. This is because any $\mathbb{K}$-subrepresentation of R can be extended to a Q-subsheaf $(\mathscr{F}_v, v \in V)$ of $\mathscr{R}$.[16]

ii) Any $\underline{M}$-twisted Q-sheaf is generically 0-semistable, by Remark 1, i).

iii) If we allow twisting sheaves of higher rank, then the restriction of a Q-sheaf $\mathscr{R}$ to the generic point of X will be a $\mathbb{K}$-representation of the quiver $Q(\underline{M})$ in which the arrow a is replaced by $\mathrm{rk}(M_a)$ copies of it, $a \in A$. Note that the analysis of totally unstable $\mathbb{K}$-representations of the quiver Q and the quiver $Q(\underline{M})$ may be quite different. For example, look at $\bullet \xrightarrow{a} \bullet$ and $\mathrm{rk}(M_a) = 2$.

It is now clear how the subdivision of the family of semistable quiver sheaves into two classes described in the introduction works. The results of Sect. 2 make this subdivision explicit for the quivers A_{n+1}, $n \in \mathbb{N}$, and $\widetilde{A}_3$. The reader should compare this with [5]. From the detailed investigation of the stability parameters,

[13] An exception is the case of the quiver A_{n+1}. Here, we use the labeling of the vertices according to [3].

[14] In that work, only the stability parameter $\underline{\chi}$ was used, more precisely, $\kappa_i := 1$, $i = 0, 1, 2$.

[15] We do not look at stability, because stability is not well-behaved on non-algebraically closed fields.

[16] The extension is unique, if we require that $\mathscr{E}_v/\mathscr{F}_v$ be torsion free, $v \in V$.

we, thus, obtain important information on the shape of the region of stability parameters for which semistable quiver sheaves might exist. The reader may consult [5,25], for specific examples, [6] for generalities on the chamber decomposition of the region of stability parameters, and [33,35], for general results on boundedness of stability parameters and semistable quiver sheaves.

References

1. Abeasis, S., Del Fra, A.: Degenerations for the representations of an equioriented quiver of type A_m. Boll. Un. Mat. Ital. Suppl. **2**, 157–171 (1980)
2. Álvarez-Cónsul, L.: Some results on the moduli spaces of quiver bundles. Geom. Dedicata. **139**, 99–120 (2009)
3. Álvarez-Cónsul, L., García-Prada, O.: Dimensional reduction, $\mathrm{SL}(2,\mathbb{C})$-equivariant bundles and stable holomorphic chains. Internat. J. Math. **12**, 159–201 (2001)
4. Álvarez-Cónsul, L., García-Prada, O.: Hitchin-Kobayashi correspondence, quivers, and vortices. Comm. Math. Phys. **238**, 1–33 (2003)
5. Álvarez-Cónsul, L., García-Prada, O., Schmitt, A. H. W.: On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces. IMRP Int. Math. Res. Pap. Art. ID 73597 (2006)
6. Andreas, B., Schmitt, A. H. W.: Vector bundles in algebraic geometry and mathematical physics, to appear in Space–Time–Matter. Analytic and Geometric Structures. de Gruyter
7. Assem, I., Simson, D., Skowroński, A.: Elements of the representation theory of associative algebras. 1: Techniques of representation theory. London Mathematical Society Student Texts **65**. Cambridge University Press, Cambridge (2006)
8. Borel, A.: Linear algebraic groups. 2-nd edt. Graduate Texts in Mathematics **126**. Springer-Verlag, New York (1991)
9. Bradlow, S.B., García-Prada, O., Gothen, P.B.: Moduli spaces of holomorphic triples over compact Riemann surfaces. Math. Ann. **328**, 299–351 (2004)
10. Collier, B.: Finite order automorphisms of Higgs bundles: theory and application. www.math. illinois.edu/∼collier3/pdfs/MyThesis.pdf (2016)
11. Derksen, H., Weyman, J.: Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients. J. Amer. Math. Soc. **13**, 467–79 (2000)
12. Derksen, H., Weyman, J.: The combinatorics of quiver representations. Ann. Inst. Fourier. **61**, 1061–131 (2011)
13. Domokos, M., Zubkov, A.N.: Semi-invariants of quivers as determinants. Transform. Groups **6**, 9–24 (2001)
14. Gabriel, P.: Unzerlegbare Darstellungen I. Manuscripta Math. **6**, 71–103 (1972); correction, ibid. **6**, 309 (1972)
15. Gabriel, P.: Indecomposable representations II. Symposia Mathematica XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 81–104. Academic Press, London (1973)
16. García-Prada, O., Heinloth, J.: The $y-$genus of the moduli space of PGL_n-Higgs bundles on a curve (for degree coprime to n). Duke Math. J. **162**, 2731–2749 (2013)
17. García-Prada, O., Ramanan, S.: Involutions and higher order automorphisms of Higgs bundle moduli spaces. arXiv:1605.05143
18. Gómez, T.L., Langer, A., Schmitt, A., Sols, I.: Moduli spaces for principal bundles in arbitrary characteristic. Adv. Math. **219**, 1177–245 (2008)

19. Gothen, P.B., King, A.D.: Homological algebra of twisted quiver bundles. J. London Math. Soc. **2**(71), 85–99 (2005)
20. Hilbert, D.: Über die vollen Invariantensysteme. Math. Ann. **42**, 313–373 (1893)
21. Hille, L., de la Peña, J.A.: Stable representations of quivers. J. Pure Appl. Algebra. **172**, 205–224 (2002)
22. Huybrechts, D., Lehn, M.: The geometry of moduli spaces of sheaves. 2–nd edt. Cambridge University Press, Cambridge (2010)
23. King, A.D.: Moduli of representations of finite-dimensional algebras. Quart. J. Math. Oxford **2**(45), 515–30 (1994)
24. Koike, K.: Relative invariants of the polynomial rings over type A_r, $\widetilde{A}_r$-quivers. Adv. Math. **86**, 235–262 (1991)
25. Laudin, A., Schmitt, A.H.W.: Recent results on quiver sheaves. Cent. Eur. J. Math. **10**, 1246–1279 (2012)
26. Ramanan, S., Ramanathan, A.: Some remarks on the instability flag. Tohoku Math. J. **2**(36), 269–291 (1984)
27. Ressayre, N.: GIT-cones and quivers. Math. Z. **270**, 263–275 (2012)
28. Schiffler, R.: Quiver representations. CMS Books in Mathematics. Springer, Cham (2014)
29. Schmitt, A.H.W.: Moduli problems of sheaves associated with oriented trees. Algebras and Representation Theory. **6**, 1–32 (2003)
30. Schmitt, A.H.W.: Global boundedness for decorated sheaves. Int. Math. Res. Not. **2004**(68), 3637–3671 (2004)
31. Schmitt, A.H.W.: Moduli for decorated tuples of sheaves and representation spaces for quivers. Proc. Indian Acad. Sci. Math. Sci. **115**, 15–49 (2005)
32. Schmitt, A.H.W.: A remark on semistability of quiver bundles. Eurasian Math. J. **3**, 110–138 (2012)
33. Schmitt, A.H.W.: Global boundedness for semistable decorated principal bundles with special regard to quiver sheaves. J. Ramanujan Math. Soc. **28A**, 443–490 (2013)
34. Schmitt, A.H.W.: Semistability and instability in products and applications. In: Proceedings of Symposia in Pure Mathematics **93**, pp. 201–214. American Mathematical Society, Providence (2014)
35. Schmitt, A.H.W.: Stability parameters for quiver sheaves. Preprint (2017)
36. Schofield, A.: Semi-invariants of quivers. J. London Math. Soc. **2**(43), 385–395 (1991)
37. Schofield, A., van den Bergh, M.: Semi-invariants of quivers for arbitrary dimension vectors. Indag. Math. (N.S.). **12**, 125–138 (2001)
38. Simpson, C.: Katz's middle convolution algorithm. Pure Appl. Math. Q. (special issue in honor of Friedrich Hirzebruch, Part 1) **5**, 781–852 (2009)
39. Sturmfels, B.: Algorithms in invariant theory, 2nd edn. Texts and Monographs in Symbolic Computation. Springer, Vienna (2008)

On Construction of the Force Function in the Presence of Random Perturbations

Marat I. Tleubergenov[1(✉)] and Darkhan T. Azhymbaev[2]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
marat207@mail.ru
[2] Zhubanov Aktobe Regional State University, 263 Zhubanov brothers st., Aktobe 030000, Kazakhstan
darkhan70@gmail.com

Abstract. The force function is constructed for the given properties of motion, independent from velocities. Previously the stochastic Ito equation is built for a given integral manifold by quasi-inversion method. Further, the equivalent equation of Lagrangian structure is built according to stochastic Ito equation, and then the force function is defined by Lagrange's function.

Keywords: Inverse problems · Stochastic Ito equation · Integral manifold · Force function

1 Statement of the Problem

On the basis of a given set

$$\Lambda(t) : \lambda(x,t) = 0, \quad \lambda \in R^m \quad x \in R^n, \quad \lambda \in C_{xt}^{22} \tag{1}$$

it is necessary to construct a generalized force function $U = U\,(x, \dot{x}, t)$ so that $\Lambda\,(t)$ is the integral manifold of the stochastic equations of the Lagrange structure

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right) - \frac{\partial L}{\partial x_\nu} = \sigma'_{\nu j}(x,\dot{x},t)\dot{\xi}_j, \left(\nu = \overline{1,n}, \quad j = \overline{1,r}\right). \tag{2}$$

Here, the system of random processes with independent increments $\{\xi_1(t,\omega), \ldots, \xi_r(t,\omega)\}$, as in [16], can be represented as the sum of the processes: $\xi = = \xi_0 + \int c(y)P^0(t,dy)$, where $\xi = (\xi_1(t,\omega), \ldots, \xi_r(t,\omega))^T$ is vector process with independent increments, $\xi_0 = (\xi_{10}(t,\omega), \ldots, \xi_{r0}(t,\omega))^T$ is vector Wiener process; P^0 is Poisson process; $P^0(t,dy)$ is the number of jumps of the process P^0 in the interval $[0,\ t]$ into the set dy; $c(y)$ is vector function mapping the space of R^{2n} into the space of values R^r of the process $\xi(t)$for any t.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_39

The problem posed above is one of the inverse problems of differential systems. The theory of inverse problems for differential systems and its general methods go back to works [2,3] and have been further developed in [1,7–10] for deterministic systems described by ordinary differential equations (ODE). Thus, the set of ODE with given integral curve was constructed in [2]. The cited paper plays a fundamental role in the formation and development of the theory of inverse problems in the dynamics of systems described by ODE. The statement and classification of the inverse problems for differential equations and their solutions in the class of ODE are discussed in [1,3,7–10].

The solving of inverse problems of differential systems (problems of construction of set of the differential equations according to a given integral manifold) is based on two methods: the Erugin method and the quasi-inversion method. Firstly, the Erugin method (the method of introduction of auxiliary Erugin function) provides necessary and sufficient conditions in order that given set is integral manifold [2,3]. And, secondly, the quasi-inversion method, developed in [8,9], allows to write out the common solution of the functional-algebraic equation, to which the problem of construction of set of the differential equations for a given integral manifold is reduced.

However, the increasing requirements to the accuracy of description and serviceability of material systems lead to the situation in which numerous observed phenomena cannot be explained on the basis of the analysis of deterministic processes. Thus, in particular, the probability laws should be used for the simulation of the behavior of actual systems.

Thus, the problem of generalization of the methods used for the solution of inverse problems for differential systems to the class of stochastic differential equations seems to be quite urgent [6,13].

Stochastic differential equations of the Itô-type are used to describe various models of mechanical systems taking into account the action of external random forces and important for numerous applications, e.g., the motion of artificial satellites under the action of gravity and aerodynamic forces [12], the fluctuation drift of a heavy gyroscope in the gimbal suspension [14], and many others. In [5,6,16], the inverse problems of dynamics are studied under the additional assumptions of presence of random perturbations from the class of Wiener processes. In particular, the following problems are solved by the method of quasi-inversion:

(i) the main inverse problem of dynamics, i.e., the construction of the set of Itô-type second-order stochastic differential equations with given integral manifold;

(ii) the problem of reconstruction of the equations of motion, i.e., the construction of the set of control parameters contained in a given system of Itô-type second-order stochastic differential equations according to a given integral manifold, and

(iii) the problem of closure of the equations of motion, i.e., the construction of the set of closing Itô-type second-order stochastic differential equations for a given system of equations and a given integral manifold.

The problems of construction the Lagrange, Hamilton, and Birkhoff equations on the given properties of the motion (1) were considered in the class of ordinary differential equations in [2–4], and in the class of Ito stochastic differential equations in [5,6]. In [6], Lagrange, Hamilton and Birkhoff equations are constructed for the given properties of motion in the class of stochastic differential equations of Ito-type under the assumption that random perturbations are from a class of Wiener processes.

In this paper, in contrast to [6], the force function is constructed under the assumption that random perturbations are from a more general class than Wiener processes, namely, from the class of processes with independent increments.

Previously at the first stage, the Ito equation

$$\ddot{x} = f(x, \dot{x}, t) + \sigma(x, \dot{x}, t)\dot{\xi} \tag{3}$$

is constructed so that the set (1) is an integral manifold of the constructed equation (3). For that, we use the quasi-inversion method [7,8] in combination with the Erugin method [9,10] and the stochastic differentiation of a composite function [1].

We say that a function $g(y,t)$ belongs to the class K, $g \in K$, if g is continuous in t, $t \in [0,\infty]$, satisfies the Lipschitz condition with respect to y in the entire space $y = (x^T, \dot{x}^T)^T \in R^{2n}$, i.e.,

$$\|g(y,t) - g(\widetilde{y}, t\| \le M\|y - \widetilde{y}\|$$

and satisfies the condition of linear growth

$$\|g(y,t)\| \le M(1 + \|y\|)$$

with a certain constant M.

It is assumed that the vector functions f and also the matrix σ belong to the class K that guarantees the existence and uniqueness (to within the stochastic equivalence) of the solution $y(t) = \left(x(t)^T, \dot{x}(t)^T\right)^T$ of the equation (3) with the initial condition $y(t_0) = \left(x(t_0)^T, \dot{x}(t_0)^T\right)^T = = (x_0^T, \dot{x}_0^T)^T$. This solution is a strictly Markov process continuous with probability 1 [[5], p.107].

Then, in the second stage, on the basis of the obtained Ito equation of the second order, we construct equivalent stochastic equations of the Lagrange type. And in the third stage, under the assumption that the generalized Lagrangian has the form

$$L = T\left(x, \dot{x}, t\right) + U\left(x, \dot{x}, t\right), \quad \text{where} \quad T = a_{ij}\dot{x}_i\dot{x}_j, \quad (i, j = \overline{1, n}), \tag{4}$$

we define the desired force function in the form

$$U\left(x, \dot{x}, t\right) = L\left(x, \dot{x}, t\right) - a_{ij}\dot{x}_i\dot{x}_j.$$

2 Construction of the Force Function on the Basis of Given Properties of Motion

Previously, using the rule of Ito stochastic differentiation, we compose the equation of perturbed motion

$$\ddot{\lambda} = \frac{\partial \lambda}{\partial x}\left(f + \sigma\dot{\xi}\right) + \dot{x}^{\mathrm{T}}\frac{\partial^2 \lambda}{\partial x \partial x}\dot{x} + 2\frac{\partial^2 \lambda}{\partial x \partial t} + \frac{\partial^2 \lambda}{\partial t^2}.$$

Then, following the Erugin method [9], we introduce a vector function A and a matrix function B such that $A(0, 0, x, \dot{x}, t) \equiv 0$, $B(0, 0, x, \dot{x}, t) \equiv 0$, and

$$\ddot{\lambda} = A(\lambda, \dot{\lambda}, x, \dot{x}, t) + B(\lambda, \dot{\lambda}, x, \dot{x}, t)\dot{\xi}. \tag{5}$$

Comparing Eqs. (5) and (7), we obtain

$$\begin{cases} \dfrac{\partial \lambda}{\partial x} f = A - \dot{x}^{\mathrm{T}}\dfrac{\partial^2 \lambda}{\partial x \partial x}\dot{x} - 2\dfrac{\partial^2 \lambda}{\partial x \partial t} - \dfrac{\partial^2 \lambda}{\partial t^2}, \\ \dfrac{\partial \lambda}{\partial x}\sigma = B. \end{cases} \tag{6}$$

Using these relations, we determine the vector-function f and the matrix σ by the method of quasi-inversion [[6], p. 12]:

$$f = k\left[\frac{\partial \lambda}{\partial x}C\right] + \left(\frac{\partial \lambda}{\partial x}\right)^{+}\left(A - \dot{x}^{\mathrm{T}}\frac{\partial^2 \lambda}{\partial x \partial x}\dot{x} - 2\frac{\partial^2 \lambda}{\partial x \partial t} - \frac{\partial^2 \lambda}{\partial t^2}\right),$$

$$\sigma_i = s_i\left[\frac{\partial \lambda}{\partial x}C\right] + \left(\frac{\partial \lambda}{\partial x}\right)^{+} B_i, \quad i = \overline{1, r}, \tag{7}$$

where $\sigma_i = (\sigma_{1i}, \sigma_{2i}, \ldots, \sigma_{ni})^T$ is the ith column of the matrix $\sigma = (\sigma_{\nu j})$, $\left(\nu = \overline{1, n},\ \ j = \overline{1, r}\right)$; $B_i = (B_{1i}, B_{2i}, \ldots, B_{mi})^T$ is the ith column of the matrix $B = (B_{\mu j})$, $\left(\mu = \overline{1, m},\ \ j = \overline{1, r}\right)$; s_i and k are arbitrary scalar values.

Thus, it follows from (9) and (10) that the set of Itô differential equations of the second order with given integral manifold (1) has the form

$$\ddot{x} = k\left[\frac{\partial \lambda}{\partial x}C\right] + \left(\frac{\partial \lambda}{\partial x}\right)^{+}\left(A - \dot{x}^{\mathrm{T}}\frac{\partial^2 \lambda}{\partial x \partial x}\dot{x} - 2\frac{\partial^2 \lambda}{\partial x \partial t} - \frac{\partial^2 \lambda}{\partial t^2}\right) +$$
$$+ (s_1\left[\frac{\partial \lambda}{\partial x}C\right] + \left(\frac{\partial \lambda}{\partial x}\right)^{+} B_1, \ldots, s_r\left[\frac{\partial \lambda}{\partial x}C\right] + \left(\frac{\partial \lambda}{\partial x}\right)^{+} B_r)\dot{\xi}.$$

Using the rule of Ito stochastic differentiation, we write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right) = \frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}\ddot{x}_k + S_{1\nu} + S_{2\nu} + S_{3\nu}, \tag{8}$$

where $S_{1\nu} = \dfrac{1}{2}\dfrac{\partial^3 L}{\partial \dot{x}_\nu \partial \dot{x}_i \partial \dot{x}_k}\sigma_{ij}\sigma_{kj}$, $S_{2\nu} = \int\{\dfrac{\partial L(x, \dot{x} + \sigma c(y), t)}{\partial \dot{x}_\nu} - \dfrac{\partial L(x, \dot{x}, t)}{\partial \dot{x}_\nu}\}dy$, $S_{3\nu} = \int[\dfrac{\partial L(x, \dot{x} + \sigma c(y), t)}{\partial \dot{x}_\nu} - \dfrac{\partial L(x, \dot{x}, t)}{\partial \dot{x}_\nu}]P^0(t, dy)$.

With regard for (11), Eq. (2) takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\nu}\right) - \frac{\partial L}{\partial x_\nu} - \sigma^{'}_{\nu j}(x,\dot{x},t)\dot{\xi}^j = \frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}\ddot{x}_k +$$

$$+ S_{1\nu} + S_{2\nu} + S_{3\nu} - \frac{\partial L}{\partial x_\nu} - \sigma^{'}_{\nu j}(x,\dot{x},t)\dot{\xi}, \tag{9}$$

or, with regard for (12) and Eq. (9),

$$\frac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \frac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}\ddot{x}_k + S_{1\nu} + S_{2\nu} + S_{3\nu} -$$

$$- \frac{\partial L}{\partial x_\nu} - \sigma^{'}_{\nu j}(x,\dot{x},t)\dot{\xi}^j \equiv \ddot{x}_\nu - f_\nu(x,\dot{x},t) - \sigma_{\nu j}(x,\dot{x},t)\dot{\xi}, \tag{10}$$

where f_ν is the νth component of the vector function f (9), and the jth column of the matrix $\sigma = (\sigma_{\nu j})$ has the form (10).

Using relation (13), we obtain

$$\begin{cases} \dfrac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \dfrac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + S_{1\nu} + S_{2\nu} + S_{3\nu} - \dfrac{\partial L}{\partial x_\nu} = -f_\nu, \\ \dfrac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k} = \delta^k_\nu, \quad \sigma^{'}_{\nu j}(x,\dot{x},t) = \sigma_{\nu j}. \end{cases} \tag{11}$$

Conditions (14) ensure the solving of the problem of direct representation of the equivalent stochastic equation of a Lagrangian structure of the form (2) by the constructed Ito equation (5).

Now let us consider the problem of indirect representation of a Lagrange-type equation. To this end, we introduce a matrix $H = (h^k_\nu)$ and consider the relation

$$h^k_\nu\left(\ddot{x}_k - f_k - \sigma_{kj}\dot{\xi}^j\right) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial x_\nu}\right) - \frac{\partial L}{\partial x_\nu} - \sigma^{'}_{\nu j}\dot{\xi}^j. \tag{12}$$

By analogy with the analysis of the direct representation of the Lagrange-type equation (10) for identity (12) to be true following conditions must be satisfied:

$$\begin{cases} h^k_\nu = \dfrac{\partial^2 L}{\partial \dot{x}_\nu \partial \dot{x}_k}, \quad h^k_\nu \sigma_{kj} = \sigma^{'}_{\nu j}, \\ -h^k_\nu f_k = \dfrac{\partial^2 L}{\partial \dot{x}_\nu \partial t} + \dfrac{\partial^2 L}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + S_{1\nu} + S_{2\nu} + S_{3\nu} - \dfrac{\partial L}{\partial x_v}. \end{cases} \tag{13}$$

For the Lagrange function of the form (4) and, correspondingly, the force function of the form (5), conditions are equivalent to the following system of partial differential equations

$$\frac{\partial^2 U}{\partial \dot{x}_\nu \partial \dot{x}_k} = h^k_\nu - a_{\nu k}; \frac{\partial U}{\partial x_\nu} = \frac{\partial^2 U}{\partial \dot{x}_\nu \partial t} + \frac{\partial^2 U}{\partial \dot{x}_\nu \partial x_k}\dot{x}_k + \tilde{S}_{1\nu} + \tilde{S}_{2\nu} + \tilde{S}_{3\nu} + h^k_v f_k, \tag{14}$$

$$\sigma'_{\nu j}(x, \dot{x}, t) = h^k_\nu \sigma_{kj},$$

where

$$\tilde{S}_{1\nu} = \frac{1}{2}\frac{\partial^3 U}{\partial \dot{x}_\nu \partial \dot{x}_i \partial \dot{x}_k}\sigma_{ij}\sigma_{kj}, \tilde{S}_{2\nu} = \int \{\frac{\partial U(x, \dot{x} + \sigma c(y), t)}{\partial \dot{x}_\nu} - \frac{\partial U(x, \dot{x}, t)}{\partial \dot{x}_\nu}\}dy,$$

$$S_{3\nu} = \int [\frac{\partial U(x, \dot{x} + \sigma c(y), t)}{\partial \dot{x}_\nu} - \frac{\partial U(x, \dot{x}, t)}{\partial \dot{x}_\nu}]\dot{P}^0(t, dy).$$

Thus, it is proved [11,15,17–21].

Theorem 1. *For the construction of the set of stochastic equations of the Lagrangian structure (2) on the basis of the given set (1) in the indirect representation such that set (1) is the integral manifold of Eq. (2), it is necessary and sufficient that*
1) the generalized force function $U = U(x, \dot{x}, t)$ *satisfies the conditions,*
2) the vector-function f *and the matrix* σ *satisfy respectively conditions (9), (10).*

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Abramov, N.V., Mukharlyamov, R.G., Kirgizbaev, Z.H.K.: Control the dynamics of systems with program communications. Publishing House of Nizhnevartovsk. State. Univ., Nizhnevartovsk (2013) (in Russian)
2. Erugin, N.P.: Construction of the entire set of system of differential equations with integral curve. Prikl. Mat. Mech. **10**(6), 659–670 (1952)
3. Galiullin, A.S.: Methods for the Solution of Inverse Problems of Dynamics. Nauka, Moscow (1986) (in Russian)
4. Gikhman, I.I., Skorokhod, A.V.: Stochastic Differential Equations. Naukova Dumka, Kiev (1968) (in Russian)
5. Ibraeva, G.T., Tleubergenov, M.I.: Main inverse problem for differential systems with degenerate diffusion. Ukrainian Math. J. **65**(5), 787–792 (2013)
6. Ibraeva, G.T., Tleubergenov, M.I.: On the problem of differential systems' closure with degenerate diffusion. Vest. Ross. Univers. Druzh. Narodov. Ser. Mat., Inform., Fiz. Moscow **1**, 12–19 (2008)
7. Khas'minskiĭ, R.Z.: Stochastic Stability of Differential Equations. Sijthoff & Noordhoff (1980)
8. Llibre J., Ramirez R.: Inverse problems in ordinary differential equations and applications. In: Progress in Mathematics, vol. 313. Springer International Publishing, Switzerland (2016)
9. Mukhamedzianov, I.A., Mukharliamov, R.G.: Equations of Program Motions. Moscow (1986) (in Russian)
10. Mukharlyamov, R.G.: Differential-algebraic equations of programmed motions of lagrangian dynamical systems. Mech. Solids **46**(4), 534–543 (2011)

11. Mukharlyamov, R.G.: Simulation of control processes, stability and stabilization of systems with program contraints. J. Comput. Syst. Sci. Int. **54**(1), 13–26 (2011)
12. Pugachev, V.S., Sinitsyn, I.N.: Stochastic Differential Systems. Analysis and Filtration, Nauka, Moscow (1990) (in Russian)
13. Sagirov, P.: Stochastic methods in the dynamics of satellites. Mekhanika. Sbornik Perevodov. **5**, 28–47, **6**, 3–38 (1974)
14. Samoilenko, A.M., Stanzhytskyi, O.: Qualitative and Asymptotic Analysis of Differential Equations with Random Perturbations. World Scientific, Singapore (2011)
15. Sinitsyn, I.N.: On the fluctuations of a gyroscope in gimbal suspension. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela. **3**, 23–31 (1976)
16. Tleubergenov, M.I.: On the inverse stochastic reconstruction problem. Differ. Equ. **50**(2), 274–278 (2014)
17. Tleubergenov, M.I.: Inverse problems of stochastic differential systems. In: Abstract of the Doctoral-Degree Thesis (Physics and Mathematics), Alma-Ata (1999)
18. Tleubergenov, M.I., Azhymbaev, D.T.: On the construction of a set of stochastic differential equations on the basis of a given integral manifold independent of velocities. Ukrainian Math. J. **68**(1), 1163–1173 (2010)
19. Tuladhar, B.M.: On the construction of the programmed motion. Hadron. J. **5**(5), 96–101 (1983)
20. Tuladhar, B.M.: On the construction of the canonical equations of motion. Mécanique Appliquée **28**(3), 53–58 (1983)
21. Tuladhar, B.M.: Construction of equations of Lagrange, Hamilton, and Birkhoff types on the basis of given properties of motion. In: Abstract of the Doctoral-Degree Thesis (Physics and Mathematics), Moscow (1983)

An Inverse Coefficient Problem for a Quasilinear Parabolic Equation of High Order

Asal V. Yuldasheva(✉)

National University of Uzbekistan, 4 Universitetskaya st.,Tashkent100174, Uzbekistan
yuasv86@mail.ru

Abstract. In this paper an inverse problem of finding the time-dependent coefficient of heat capacity together with solution of high-order heat equation with nonlocal boundary and integral overdetermination conditions is considered. The existence and uniqueness of a solution of the inverse problem are proved by using the Fourier method and the iteration method. Continuous dependence upon the data of the inverse problem is shown.

Keywords: Inverse problem · Quasilinear parabolic equation · Overdetermination data · Classical solution · Fourier method · Iteration method

1 Introduction

Denote the domain D by

$$D := \{0 < x < \pi,\ \ 0 < t < T\}.$$

We consider quasilinear high-order parabolic equation

$$\frac{\partial u}{\partial t} = (-1)^k a^2 \frac{\partial^{2k} u}{\partial x^{2k}} - p(t)u + f(t, x, u),\ k \in N, \tag{1}$$

with initial condition

$$u(x, 0) = \varphi(x),\ 0 \le x \le \pi, \tag{2}$$

the nonlocal boundary conditions

$$\frac{\partial^i u}{\partial x^i}\mid_{x=0} = \frac{\partial^i u}{\partial x^i}\mid_{x=\pi},\ \ i = 0, 1, ..., 2k-1,\ 0 \le t \le T, \tag{3}$$

and the integral overdetermination data

$$\int_0^\pi u(x, t)dx = E(t),\ 0 \le t \le T. \tag{4}$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_40

The functions $E(t)$, $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,T]$, $[0,\pi]$ and $\overline{D}\times(-\infty,\infty)$, respectively.

The problem of finding the pair $\{p(t),u(x,t)\}$ in (1)-(4) are called inverse problem.

Definition 1.
The pair $\{p(t),u(x,t)\}$ from the class $C[0,T]\times(C^{2k,1}(D)\bigcap C^{2k-1,0}(\overline{D}))$, for which conditions (1)-(4) are satisfied and $p(t)\geq 0$ on the interval [0,T], is called the classical solution of the inverse problem (1)-(4).

The problem of indentifying a coefficient in a nonlinear parabolic equation is an interesting problem for many scientists [1–3].

Inverse problem for parabolic equations with nonlocal conditions are investigated in [4,5]. This kind of conditions arise from many important applications in heat transfer, life sciences, etc.

Using the Fourier method and the iteration method we prove the existence, uniqueness and continuous dependence upon the data of the solution under some natural regularity and consistency conditions on the input data.

2 Existence and Uniqueness of the Solution of the Inverse Problem

The main result of the existence and uniqueness of the solution of the inverse problem (1)-(4) is presented as follows. We have the following assumptions on the data of the problem (1)-(4):

$(C_1) E(t)\in C^1[0,T], E(t)>0, E'(t)\leq 0,$
$(C_2) \varphi(x)\in C^3[0,\pi], \varphi^{(i)}(0)=\varphi^{(i)}(\pi), i=0,1,...,k+1,$
$(C_3) f(x,t,u)\in C(\overline{D}\times(-\infty,\infty)), |\frac{\partial^n f(x,t,u)}{\partial x^n}-\frac{\partial^n f(x,t,\widetilde{u})}{\partial x^n}|\leq b(x,t)|u-\widetilde{u}|,$ $n=0,1,...,2k$
where $b(x,t)\in L_2(D)$, $b(x,t)\geq 0$, $f^{(i)}(x,t,u)|_{x=0}=f^{(i)}(x,t,u)|_{x=\pi}$, $i=\overline{0,2k}$, $f_0(t)\geq 0$, $t\in[0,T]$, where

$$f_0(t)=\int_0^\pi \frac{1}{2}f(x,t,u)dx.$$

A solution of (1)-(3) for arbitrary $p(t)\in C[0,T]$ we find in following form:

$$u(x,t)=\frac{u_0(t)}{2}+\sum_{n=1}^{\infty}(u_{cn}(t)cos2nx+u_{sn}(t)sin2nx), \tag{5}$$

It is clear that (5) satisfies conditions (3). In order to determine unknown functions $u_0(t),u_{cn}(t),u_{sn}(t)$ using equation (1) and condition (2) with

$$\varphi(x)=\frac{\varphi_0(t)}{2}+\sum_{n=1}^{\infty}(\varphi_{cn}cos2nx+\varphi_{sn}sin2nx),$$

we get the infinite system of integral equations:

$$u_0(t)=\varphi_0 e^{-\int_0^t p(s)ds}+\frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u)e^{\int_0^\tau p(s)ds}d\xi d\tau,$$

$$u_{cn}(t)=\varphi_{cn}e^{-a^2(2n)^{2k}t-\int_0^t p(s)ds}+\frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u)e^{-a^2(2n)^{2k}(t-\tau)-\int_\tau^t p(s)ds}cos2n\xi d\xi d\tau$$

$$u_{sn}(t)=\varphi_{sn}e^{-a^2(2n)^{2k}t-\int_0^t p(s)ds}+\frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u)e^{-a^2(2n)^{2k}(t-\tau)-\int_\tau^t p(s)ds}sin2n\xi d\xi d\tau.$$

Under conditions $(C_1)--(C_3)$, the series (5) and its derivatives converge uniformly in $\overline{D}$ since their majorizing sums are absolutely convergent.

Differentiating (4) under condition (C_1), we obtain

$$\int_0^\pi u_t(x,t)dx=E'(t),\ 0\le t\le T. \tag{6}$$

Equations (5) and (6) yield

$$p(t)=\frac{1}{E(t)}[-E'(t)+\frac{1}{2}f_0(t)]. \tag{7}$$

Definition 2. Denote the set

$$\{u(t)\}=\{u_0(t),u_{c1}(t),u_{s1}(t),...,u_{cn}(t),u_{sn}(t),...\},$$

of continuous on $[0,T]$ functions satisfying the condition

$$\max_{t\in[0,T]}|u_0(t)|+\sum_{n=1}^{\infty}[\max_{t\in[0,T]}|u_{cn}(t)|+\max_{t\in[0,T]}|u_{sn}(t)|]<\infty$$

by B_1. Let

$$\|u(t)\|=\max_{t\in[0,T]}|u_0(t)|+\sum_{n=1}^{\infty}[\max_{t\in[0,T]}|u_{cn}(t)|+\max_{t\in[0,T]}|u_{sn}(t)|]$$

be norm in B_1.

Let us denote

$$B_2=\{p(t)\in C[0,T]:p(t)\ge 0\},$$

$\|p(t)\|=\max_{t\in[0,T]}|p(t)|$ to be the norm in B_2.

It can be shown that B_1 and B_2 are Banach spaces.

Theorem 1. *Let assumptions $(C_1)--(C_3)$ be satisfied. The inverse problem (1)-(4) has a unique solution.*

Proof. An iteration for (5) is defined as follows:

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u^{(N)})e^{\int_0^\tau p^{(N)}(s)ds}d\xi d\tau,$$

$$u_{cn}^{(N+1)}(t) = u_{cn}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u^{(N)})e^{-a^2(2n)^{2k}(t-\tau)-\int_\tau^t p^{(N)}(s)ds}cos2n\xi d\xi d\tau,$$

$$u_{sn}^{(N+1)}(t) = u_{sn}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u^{(N)})e^{-a^2(2n)^{2k}(t-\tau)-\int_\tau^t p^{(N)}(s)ds}sin2n\xi d\xi d\tau, \tag{8}$$

where $N = 0, 1, 2, ..$ and

$$u_0^{(0)}(t) = \varphi_0 e^{-\int_0^t p(s)ds},$$

$$u_{cn}^{(0)}(t) = \varphi_{cn} e^{-a^2(2n)^{2k}t-\int_0^t p(s)ds},$$

$$u_{sn}^{(0)}(t) = \varphi_{sn} e^{-a^2(2n)^{2k}t-\int_0^t p(s)ds}.$$

From the conditions of the theorem, we have $u^0(t) \in B_1$ and $p^0 \in B_2$. Let us write $N = 0$ in (8)

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,u^{(0)})d\xi d\tau$$

Adding and subtracting $\frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,0)d\xi d\tau$ to and from both sides of the last equation, we obtain

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi [f(\xi,\tau,u^{(0)}) - f(\xi,\tau,0)]d\xi d\tau + \frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,0)d\xi d\tau.$$

Applying the Cauchy inequality and the Lipschitz condition to the last equation and taking the maximum of both sides of the last inequality yield the following:

$$\max_{t\in[0,T]} |u_0^{(1)}(t)| \le |\varphi_0| + \sqrt{\frac{T}{\pi}}\|b\|_{L_2(D)}\|u^{(0)}(t)\|_{B_1} + \sqrt{\frac{T}{\pi}}\|f(x,t,0)\|_{L_2(D)}.$$

Analogously, we get

$$|u_{cn}^{(1)}(t)| = |\varphi_{cn}| + \frac{\sqrt{\pi}}{\sqrt{2}a(2n)^k}(\int_0^T \frac{2}{\pi}\int_0^t(\int_0^\pi [f(\xi,\tau,u^0) - f(x,t,0)]cos2n\xi d\xi)^2 d\tau)^{1/2} +$$
$$+\frac{\sqrt{\pi}}{\sqrt{2}a(2n)^k}(\int_0^T \frac{2}{\pi}\int_0^t(\int_0^\pi f(x,t,0)cos2n\xi d\xi)^2 d\tau)^{1/2} \tag{9}$$

Applying the Cauchy inequality, the Hölder inequality, the Bessel inequality, the Lipschitz condition and taking maximum of both sides of the last inequality yield the following

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} ||u_{cn}^{(1)}(t)| \leq \sum_{n=1}^{\infty} |\varphi_{cn}| + M[\|b(t,x)\|_{L_2(D)}\|u^{(0)}(t)\|_{B_1} + \|f(t,x,0)\|_{L_2(D)}],$$

where $M = \sqrt{\frac{\pi}{2}}(\sum_{n=1}^{\infty} \frac{1}{a^2(2n)^{2k}})^{1/2}$.

Applying the same estimations, we obtain

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} ||u_{sn}^{(1)}(t)| \leq \sum_{n=1}^{\infty} |\varphi_{sn}| + M[\|b(t,x)\|_{L_2(D)}\|u^{(0)}(t)\|_{B_1} + \|f(t,x,0)\|_{L_2(D)}],$$

Finally, we have the following inequality:

$$\|u^{(1)}(t)\|_{B_1} = \max_{t\in[0,T]} |u_0^{(1)}(t)| + \sum_{n=1}^{\infty}[\max_{t\in[0,T]} ||u_{cn}^{(1)}(t)| + \max_{t\in[0,T]} ||u_{sn}^{(1)}(t)|] \leq$$

$$|\varphi_0| + \sum_{n=1}^{\infty}(|\varphi_{cn}| + |\varphi_{sn}|) + 2M[\|b(t,x)\|_{L_2(D)}\|u^{(0)}(t)\|_{B_1} + \|f(t,x,0)\|_{L_2(D)}]$$

Hence $u^1(t) \in B_1$. In the same way, for a general value of N, we have

$$\|u^{(N)}(t)\|_{B_1} = \max_{t\in[0,T]} |u_0^{(N)}(t)| + \sum_{n=1}^{\infty}[\max_{t\in[0,T]} ||u_{cn}^{(N)}(t)| + \max_{t\in[0,T]} ||u_{sn}^{(N)}(t)|] \leq$$

$$|\varphi_0| + \sum_{n=1}^{\infty}(|\varphi_{cn}| + |\varphi_{sn}|) + 2M[\|b(t,x)\|_{L_2(D)}\|u^{(N-1)}(t)\|_{B_1} + \|f(t,x,0)\|_{L_2(D)}].$$

Since $u^{N-1}(t) \in B_1$, we have $u^N(t) \in B_1$

$$\{u(t)\} = \{u_0(t), u_{c1}(t), u_{s1}(t), ..., u_{cn}(t), u_{sn}(t), ...\} \in B_1.$$

An iteration for (7) is defined as follows:

$$p^{(N)}(t) = \frac{1}{E(t)}[-E'(t) + \frac{1}{2}\int_0^t f(\xi,\tau,u^{(N)})d\xi],$$

where $N = 0, 1, 2...$

$$p^{(1)}(t) = \frac{1}{E(t)}[-E'(t) + \frac{1}{2}\int_0^t f(\xi,\tau,u^{(1)})d\xi].$$

Applying the Cauchy inequality, we get

$$\|p^{(1)}(t)\|_{B_2} = |\frac{-E'(t)}{E(t)}| + \frac{1}{E(t)}[\|b(t,x)\|_{L_2(D)}\|u^{(1)}(t)\|_{B_1} + \|f(t,x,0)\|_{L_2(D)}].$$

Hence $p^{(1)}(t) \in B_2$. In the same way, for a general value of N, we have

$$\|p^{(N)}(t)\|_{B_2} = |\frac{-E'(t)}{E(t)}| + \frac{1}{E(t)}[\|b(t,x)\|_{L_2(D)}\|u^{(N)}(t)\|_{B_1} + \|f(t,x,0)\|_{L_2(D)}],$$

we deduce that $p^{(N)}(t) \in B_2$.

Now we prove the iterations $u^{N+1}(t)$ and $p^{(N+1)}(t)$ converge as $N \to \infty$ in B_1 and B_2, respectively. Analogously as above for arbitrary N, we have

$$\|p^{(N+1)}(t) - p^{(N)}(t)\|_{B_2} \leq \frac{1}{E(t)}[\|b(t,x)\|_{L_2(D)}\|u^{(N+1)}(t) - u^{(N)}(t)\|_{B_1}$$

$$\|u^{(N+1)}(t) - u^{(N)}(t)\|_{B_1} \leq (M(1+\frac{TM}{E})^N \frac{\pi^{N/2}}{\sqrt{N!}}\|b(t,x)\|_{L_2^N(D)}. \quad (10)$$

Using $(C_1) - -(C_3)$ and the comparison test, we deduce from (10) that the series $\sum_{N=1}^{\infty}[u^{(N+1)}(t) - u^{(N)}(t)]$ is uniformly convergent to an element of B_1. However, the general term of the sequence $\{u^{(N+1)}(t)\}$ may be written as

$$u^{(N+1)}(t) = u^{(0)}(t) + \sum_{N=1}^{\infty}[u^{(N+1)}(t) - u^{(N)}(t)].$$

So the sequence $\{u^{(N+1)}(t)\}$ is uniformly convergent to an element of B_1 because the sum on the right-hand side is the Nth partial sum of the aforementioned uniformly convergent series. Therefore $u^{N+1}(t)$ and $p^{(N+1)}(t)$ converge in B_1 and B_2, respectively. Furthermore these limits satisfy (8).

For the uniqueness, we assume that problem (1)-(4) has two solution pairs $(p,u),(q,v)$. Applying the Cauchy inequality, the Hölder inequality, the Bessel inequality, the Lipschitz condition to $|u(t) - v(t)|$ and $|p(t) - q(t)|$, we obtain

$$\|p(t) - q(t)\|_{B_2} \leq \frac{1}{E(t)}(\int_0^t\int_0^\pi b^2(\xi,\tau)|u(\tau) - v(\tau)|^2 d\xi d\tau)^{1/2}$$

$$\|u^{(N+1)}(t) - u^{(N)}(t)\|_{B_1} \leq (\|\varphi\| + M)\frac{T}{E} + \frac{1}{2}M)$$

$$(\int_0^t\int_0^\pi b^2(\xi,\tau)|u(\tau) - v(\tau)|^2 d\xi d\tau)^{1/2}. \quad (11)$$

Applying the Gronwall inequality to (11), we have $u(t) = v(t)$. Hence $p(t) = q(t)$. The theorem is proved. □

3 Continuous Dependence upon the Data

Theorem 2. *Under assumptions $(C_1) - (C_2)$, the solution (p,u) of problem (1)-(4) depends continuously upon the data φ, E.*

Proof. Let $\Phi = \{\varphi, E, f\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{E}, f\}$ be two sets of the data, which satisfy the assumptions $(C_1) - (C_2)$. Suppose that there exist positive constants $M_i, i = 0, 1, 2$ such that

$$0 < M_0 \leq |E|,\ 0 < M_0 \leq |\overline{E}|,$$

$$\|E\|_{C^1[0,T]} \leq M_1,\ \|E\|_{C^1[0,T]} \leq M_1,$$

$$\|\varphi\|_{C^{k+1}[0,\pi]} \leq M_2,\ \|\overline{\varphi}\|_{C^{k+1}[0,\pi]} \leq M_2.$$

Let us denote $\|\Phi\| = (\|E\|_{C^1[0,T]} + \|\varphi\|_{C^{k+1}[0,\pi]} + \|f\|_{C^{2k+1,0}(\overline{D})})$. Let (p, u) and $(\overline{p}, \overline{u})$ be solution of inverse problem (1)-(4) corresponding to the data $\Phi = \{\varphi, E, f\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{E}, f\}$, respectively.

Using results of section 2, we get

$$\|p(t) - \overline{p(t)}\| \leq M_3\|E - \overline{E}\|_{C^1[0,T]} + M_4\left(\int_0^t \int_0^\pi b^2(\xi, \tau)|u(\tau) - \overline{u}(\tau)|^2 d\xi d\tau\right)^{1/2},$$

$$\|u(t) - \overline{u}(t)\| \leq 2M_5\|\Phi - \overline{\Phi}\|^2 exp 2M_6^2\left(\int_0^t \int_0^\pi b^2(\xi, \tau) d\xi d\tau\right),$$

here $M_i, i = 3, 4, 5$ are constants that are determined by M_0, M_1 and M_2. If $\Phi \to \overline{\Phi}$, then $u \to \overline{u}$. Hence $p \to \overline{p}$. Theorem is proved. □

References

1. Cannon, J.: The solution of the heat equation subject to specification of energy. Q. Appl. Math. **21**, 155–160 (1963)
2. Cannon, J., Lin, Y.: Determination of parameter p(t) in Holder classes for some semilinear parabolic equations. Inverse Probl. **4**, 595–606 (1988)
3. Gatti, S.: An existence result for an inverse problem for a quasilinear parabolic equation. Inverse Probl. **14**, 53–65 (1998)
4. Ismailov, M., Kanca, F.: An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions. Math. Methods Appl. Sci. **34**, 692–702 (2011)
5. Kanka, F., Baglan, I.: An inverse problem for a quasilinear parabolic equation with nonlocal boundary and overdetermination conditions. J. Ineq. Appl. **76**, 1–16 (2014)

Fundamental Solutions of Biot Equations for Moving Loads

Gulmira K. Zakiryanova(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin st., Almaty 050010, Kazakhstan
gulmzak@mail.ru

Abstract. Here the two-component medium of M. Biot consisting of solid and fluid components is considered under action of moving loads. The fundamental and generalized solutions of Biot equations have been constructed for subsonic and supersonic velocities of loads.

Keywords: Continuum mechanics · Medium of M. Biot · Wave propagation · Hyperbolic equations · Fundamental solutions · Fourier transform · Moving loads · Subsonic velocities

1 Introduction

A special class of dynamic problems of continuum mechanics is transportation problem in which the acting external forces and loads move with certain some velocities and their shape does not change over time. This class of problems is a model in the study of impact various transport load in mediums. Real mediums are usually multi-component, so the study of the transport problems in such mediums is actual. Type of equations system motion of the medium depends on the ratio of the motion velocity of the moving load c to the sound velocities of disturbances propagation in the medium c_j, which may be several.

For transonic and supersonic velocities there exists the load in the medium shock waves. This class of problems in elastic media were studied by M.V. Eisenberg-Stepanenko, Sh.M. Aytaliev, L.A. Alexeyeva, V. Katsumi, V.N. Ukrainec etc. Medium of Biot is more difficult than isotropic elastic medium. In mainly by the Fourier method a class of solutions was studied in the works Kh.L. Rakhmatullin, Ya.U. Saatov, I.G. Philippov, T.U. Artykov, I.A. Kiyko, V.V. Shershnev [1–3] etc.

Here two-component medium of M. Biot consisting of solid and fluid components is considered under action of moving loads. This medium has three sound velocities [4] which describe the propagation velocities of longitudinal waves in solid and fluid components of medium, and the propagation velocity of shear wave. The fundamental solutions of Biot equations have been constructed for subsonic and supersonic velocities of loads.

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_41

2 Motion Equations for Medium of M. Biot

The motion of a homogeneous isotropic two-component medium of M. Biot in the absence of fluid viscosity are described by the following system of second order hyperbolic equations [1,4]:

$$(\lambda+\mu)u^s_{j},_{ji}+\mu u^s_{i},_{jj}+Qu^f_{j},_{ji}+G^s_i=\rho_{11}\ddot{u}^s_i+\rho_{12}\ddot{u}^f_i \tag{1}$$

$$Qu^s_{j},_{ji}+Ru^f_{j},_{ji}+G^f_i=\rho_{12}\ddot{u}^s_i+\rho_{22}\ddot{u}^f_i,(x,t)\in R^3\times[0,\infty), \tag{2}$$

where u^s_i and u^f_i are the elastic components and the fluid components of the displacement vector, G^s_i, G^f_i are the body forces acting respectively on the solid and fluid components. The constants λ,μ,Q,R have the dimension of stress, ρ_{11}, ρ_{12}, ρ_{22} are related to the particle mass density of the elastic component ρ_s and fluid component ρ_f by relations: $\rho_{11}=(1-m)\rho_s-\rho_{12}$, $\rho_{22}=m\rho_f-\rho_{12}$, $\rho_{22}=m\rho_f-\rho_{12}$, m is the porosity of the medium. The elastic stress tensor components and the fluid pressure are

$$\sigma_{ij}=\mu(u^s_{i},_{j}+u^f_{i},_{j})+(\lambda u^s_{k},_{k}+Qu^f_{k},_{k})\delta_{ij},$$

$$p=-\left(Qu^s_{k},_{k}+Ru^f_{k},_{k}\right)/m,$$

here δ_{ij} is Kronecker delta, $u_{i,j}=\partial u_i/\partial x_j$. Everywhere summation is carried over like indexes from 1 to 3.

In this medium there are three sound velocities of wave propagation:

$$c^2_{1,2}=\frac{(\lambda+2\mu)\rho_{22}+R\rho_{11}-2Q\rho_{12}}{2(\rho_{11}\rho_{22}-\rho_{12}^2)}\pm$$

$$\pm\frac{\sqrt{((\lambda+2\mu)\rho_{22}-R\rho_{11})^2+4((\lambda+2\mu)\rho_{12}-Q\rho_{11})(R\rho_{12}-2Q\rho_{22})}}{2(\rho_{11}\rho_{22}-\rho_{12}^2)}$$

$$c_3=\sqrt{\frac{\mu\rho_{22}}{(\rho_{11}\rho_{22}-\rho_{12}^2)}},$$

where c_1, c_2 describe the propagation velocity of longitudinal waves and the third c_3 is the propagation velocity of shear wave $(c_2<c_3<c_1)$. For these velocities take place the following relations:

$$c_1^2+c_2^2=\frac{(\lambda+2\mu)\,\rho_{22}+R\rho_{11}-2Q\rho_{12}}{\rho_{22}\rho_{11}-\rho_{12}^2},\quad c_1^2c_2^2=\frac{(\lambda+2\mu)\,R-Q^2}{\rho_{22}\rho_{11}-\rho_{12}^2}$$

We introduce Mach numbers $M_j=c/c_j$, $j=1,2,3$ and we have $c<\min\{c_1,c_2,c_3\}\Rightarrow M_j<1\quad\forall j$ for subsonic load and $c>\max\{c_1,c_2,c_3\}\Rightarrow M_j>1$ $\forall j$ for supersonic load. For transonic loads there is $c\in(c_2,\,c_1)$, $c\neq c_3$. In the case of sound there is $c=c_j\Rightarrow M_j=1$.

When communication fails between the fluid and the elastic solid

$$Q \to 0, \ \ \rho_{12} \to 0,$$

we have $c_3 \to \sqrt{\mu/\rho_{11}}$, $c_1 \to c_s$, $c_2 \to c_f$, where $c_s = \sqrt{(\lambda + 2\mu)/\rho_{11}}$ is the longitudinal wave velocity in a dry porous skeleton, $c_f = \sqrt{R/\rho_{22}}$ is the velocity of longitudinal waves in the liquid, and

$$\zeta_1 = \frac{\lambda + 2\mu - c_1^2\rho_{11}}{c_1^2\rho_{12} - Q} \equiv \frac{Q - c_1^2\rho_{12}}{c_1^2\rho_{22} - R} \ \to 0,$$

$$\zeta_2 = \frac{R - c_2^2\rho_{22}}{c_2^2\rho_{12} - Q} \equiv \frac{Q - c_2^2\rho_{12}}{c_2^2\rho_{11} - \lambda - 2\mu} \ \to 0,$$

$$\zeta_3 = -\frac{\rho_{12}}{\rho_{22}} \ \to 0.$$

3 Motion Equations for Moving Loads

It is assumed that the mass forces in the medium of M. Biot move with constant velocity c along the axis x_3 and are represented as

$$G_i = G_i(x_1, x_2, x_3 + ct).$$

Solutions for u_i have the same structure:

$$u_i = u_i(x_1, x_2, x_3 + ct).$$

In the coordinate system $(x_1', x_2', x_3') = (x_1, x_2, x_3 + ct)$ motion equations of Biot equations (1), (2) are

$$(\lambda + 2\mu)u_j^s,_{ji} + Qu_j^f,_{ji} + \mu u_i^s,_{jj} + G_i^s = c^2(\rho_{11}u_i^s,_{33} + \rho_{12}u_i^f,_{33}), \tag{3}$$

$$Qu_i^s,_{ji} + Ru_j^f,_{ji} + G_i^f = c^2(\rho_{12}u_i^s,_{33} + \rho_{22}u_i^f,_{33}), \tag{4}$$

here $u_{i,j} = \partial u_i/\partial x_j'$

For subsonic load we have elliptic equations. Equations (3), (4) are strong hyperbolic for supersonic load. In the case of transonic loads and sound equations are of mixed type depending on j.

Thereafter, for notational convenience, we introduce the vector $u = \{u^s, u^f\} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ of dimension 6, assuming that u_i are the displacement components of the solid phase for $i = 1, 2, 3$ and fluid for $i = 4, 5, 6$. Similarly, we introduce the vector of mass forces $G = \{G^s, G^f\} = \{G_1, G_2, G_3, G_4, G_5, G_6\}$.

4 Fourier Transform of Fundamental Solutions

Let mass forces in medium of Biot be concentrated impulse functions:

$$G_i(x) = \delta_{ij}\delta(x_1)\delta(x_2)\delta(x_3 + ct).$$

In this case, the system of equations (3), (4) for the fundamental solutions U_{ij} (of dimension 6×6) can be rewritten as

$$(\lambda + 2\mu)U_{ik},_{kj} + QU_{i(k+3)},_{kj} + \mu U_{ij},_{kk} -$$

$$c^2\rho_{11}U_{ij},_{33} - c^2\rho_{12}U_{i(j+3)},_{33} + \delta_{ij}\delta(x') = 0,$$

$$QU_{ik},_{kj} + RU_{i(k+3)},_{kj} - c^2\rho_{12}U_{ij},_{33} - c^2\rho_{22}U_{i(j+3)},_{33} + \delta_{i(j+3)}\delta(x') = 0.$$

Green's tensor components have the following physical meaning: at $1 \le j \le 3$ it is j-component of solid phase displacement, at $4 \le j \le 6$ we have $(j-3)$-th components of displacements of the liquid from the action of a concentrated force along the i-th coordinate axis on a solid phase (at $1 \le i \le 3$) or from the action of a concentrated force along the $(i-3)$-th axis of coordinates on the liquid (at $4 \le i \le 6$).

In the construction of the Green's tensor it is commonly used apparatus of integral Fourier transforms allowing you to transfer from the differential equations for the tensor to linear algebraic equations for his image. Allowing the latter to determine the transform of the tensor in the form of fractional - rational function of the variables of integral Fourier transforms and then to restore the original tensor, using the inverse transformation.

It is constructed its Fourier transform, which is as follows:

$$\overline{U}_{kj} = \frac{c_3^2}{\mu}\left(\frac{b_{k3}\delta_{kj}}{c_3^2\left(\xi^2 - M_3^2\xi_3^2\right)} - \frac{\xi_k\xi_j}{c^2\xi_3^2}\sum_{l=1}^{3}\frac{b_{ki}}{\xi^2 - M_3^2\xi_3^2}\right) \quad for\ k = \overline{1,3},\ j = \overline{1,6};$$

$$\overline{U}_{kj} = \frac{\delta_{kj}}{\rho_{22}c^2\xi_3^2} + \frac{c_3^2}{\mu}\left(\frac{d_3\delta_{kj}}{c_3^2\left(\xi^2 - M_3^2\xi_3^2\right)} - \frac{\xi_{k-3}\xi_{j-3}}{c^2\xi_3^2}\sum_{l=1}^{3}\frac{d_l}{\xi^2 - M_l^2\xi_3^2}\right) \tag{5}$$

$$for\ k = \overline{4,6},\ j = \overline{4,6};$$

$$\overline{U}_{kj} = \overline{U}_{jk} \quad for\ k = \overline{4,6},\ j = \overline{1,3}; \quad \xi^2 = \sum_{j=1}^{3}\xi_j\xi_j,$$

where

$$b_{k1} = \frac{c_1^2 - c_f^2}{c_1^2 - c_2^2},\ \ b_{k2} = \frac{c_2^2 - c_f^2}{c_1^2 - c_2^2},\ \ b_{k3} = -1,\ \ k = \overline{1,3},$$

$$b_{k1} = \zeta_1 \frac{c_1^2 - c_f^2}{c_2^2 - c_f^2}, \quad b_{k2} = -\zeta_2 \frac{c_2^2 \rho_{11} - c_s^2}{c_1^2 \rho_{22} - c_2^2}, \quad b_{k3} = -\zeta_3, \quad k = \overline{4,6},$$

$$d_1 = \frac{c_1^2 \rho_{11} - c_s^2}{c_1^2 \rho_{22} - c_2^2}, \quad d_2 = \frac{c_2^2 \rho_{11} - c_s^2}{c_1^2 \rho_{22} - c_2^2}, \quad d_3 = -\zeta_3, \quad k = \overline{4,6}.$$

In (5) $M_l = c/c_l$ $(l = \overline{1,3})$ are Mach numbers, $m_l = \sqrt{1 - M_l^2}$. For subsonic load we have $M_l < 1$ and for supersonic load there is $M_l > 1$.

5 Fundamental Solutions of the Motion Equations of Biot Medium

Using the properties of the inverse Fourier transform of generalized functions and the original of functions

$$\overline{f}_{0j}(\xi) = \frac{1}{\left(\xi^2 - M_j^2 \xi_3^2\right)}, \quad \overline{f}_{2j}(\xi) = \frac{\overline{f}_{0j}}{(-i\xi_3)^2} = \frac{1}{\left(\xi^2 - M_j^2 \xi_3^2\right) \xi_3^2},$$

which are constructed in [5] for various M_j, it is obtained the original of fundamental solutions $U_{kj}(x)$.

For $M_j < 1$ we have

$$\overline{f}_{0j}(\xi) \quad \leftrightarrow \quad f_{0j}(r, x_3, m_j) = \frac{1}{4\pi\sqrt{m_j^2 r^2 + x_3^2}}$$

$$r = \sqrt{x_k x_k}, \quad k = 1, 2,$$

$$\overline{f}_{2j}(\xi) \leftrightarrow$$

$$f_{2j}(r, x_3, m_j) = \left(|x_3| \ln \frac{\left(|x_3| + \sqrt{m_j^2 r^2 + x_3^2}\right)}{m_j r} - \sqrt{m_j^2 r^2 + x_3^2} \right) /4\pi.$$

For $M_l > 1$ we have

$$\overline{f}_{0j}(\xi) \quad \leftrightarrow \quad f_{0l}(r, x_3, m_j) = \frac{H(-x_3 - m_j r)}{2\pi\sqrt{x_3^2 - m_j^2 r^2}},$$

$$\overline{f}_{2j}(\xi) \quad \leftrightarrow \quad f_{2l}(r, x_3, m_j) = -\frac{H(-x_3 - m_j r)}{2\pi} \times$$

$$\times \left(|x_3| \ln \frac{|x_3| + \sqrt{x_3^2 - m_j^2 r^2}}{m_j r} - \sqrt{x_3^2 - m_j^2 r^2} \right).$$

For $M_l = 1$ we have

$$\overline{f}_{0j}(\xi) \quad \leftrightarrow \quad f_{0j}(r, x_3) = \frac{1}{2\pi}\delta(x_3)\ln\frac{1}{r},$$

$$\overline{f}_{2j}(\xi) \quad \leftrightarrow \quad f_{2l}(r, x_3) = -\frac{1}{2\pi}H(-x_3)\,|x_3|\ln r.$$

So, fundamental solutions of the motion equations of Biot medium at subsonic velocities are

$$U_{kj} = \frac{b_{k3}\delta_{kj}}{4\pi\mu\sqrt{m_3^2 r^2 + x_3^2}} -$$
$$-\frac{c_3^2}{4\pi\mu c^2}\sum_{l=1}^{3} b_{kl}\frac{\left(x_3^2 x_k x_j - x_3(\delta_{k3}x_j + \delta_{j3}x_k)r^2 + \delta_{k3}\delta_{j3}r^4\right)}{r^4\sqrt{m_l^2 r^2 + x_3^2}} -$$
$$-\frac{c_3^2}{4\pi\mu c^2}\sum_{l=1}^{3} b_{kl}\frac{\left(\sqrt{m_l^2 r^2 + x_3^2} - m_l r\right)\left(\delta_{kj}r^2 - x_k x_j\right)}{r^4}, \tag{6}$$

$$k = 1, 2, 3, \;\; j = 1,\overline{6}$$

$$U_{jk} = U_{kj}, \;\; k = 4, 5, 6, \;\; j = 1, 2, 3$$

$$U_{kj} = -\frac{\delta_{kj}\,|x_3|}{2\rho_{22}c^2} + \frac{d_3\delta_{kj}}{4\pi\mu\sqrt{m_3^2 r^2 + x_3^2}} -$$
$$-\frac{c_3^2}{4\pi\mu c^2}\sum_{l=1}^{3} d_l\frac{\left(x_3^2 x_k x_j - x_3(\delta_{k3}x_j + \delta_{j3}x_k)r^2 + \delta_{k3}\delta_{j3}r^4\right)}{r^4\sqrt{m_l^2 r^2 + x_3^2}} -$$
$$-\frac{c_3^2}{4\pi\mu c^2}\sum_{l=1}^{3} d_l\frac{\left(\sqrt{m_l^2 r^2 + x_3^2} - m_l r\right)\left(\delta_{kj}r^2 - x_k x_j\right)}{r^4},$$
$$k = 4, 5, 6, \;\; j = 4, 5, 6.$$

Fundamental solutions (6) for $\|x\| \to \infty$ have the following asymptotics

$$U_{kj} = O\left(|x|^{-1}\right).$$

6 Generalized Solutions of M.Biot Equations

A generalized solution of equations (1), (2) for arbitrary mass forces can be represented as a convolution

$$u_j = U_{jk} * G_k,$$

which for regular functions $G(x)$ have the following integral form:

$$\hat{u}_i = \int_{R^3} U_{ik}(x-y) G_k(y) dy_1 dy_2 dy_3, \;\; i,k = 1,...,6,$$

where $u = \left\{u^s, u^f\right\} = \{u_1, ..., u_6\}, \;\; G = \left\{G^s, G^f\right\} = \{G_1, ..., G_6\}.$

For concentrated source such singular generalized functions with point support (dipole, multipole etc.) it is necessary the convolution to take by rules of the theory of generalized functions.

Acknowledgements. This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

Appendix

The constructed fundamental solutions can be used for solving boundary-value problems in a Biot medium with cylindrical boundaries on the basis of methods of boundary equations and boundary-element. A similar problem for an elastic medium is considered in [6].

The solutions obtained here can also be used for investigating the dynamics of the massif in the neighborhood of underground constructions such as tunnels, transport pipelines depending on the properties of water saturation of the medium, the velocity and type of existing transport loads.

References

1. Rahmatullin, Kh.L., Saatov, Ya.U., Philippov, I.G., Artykov T.U.: Waves in two-component media. Science, Tashkent (1974)
2. Alexeyeva, L.A.: Fundamental solutions in elastic space in the case of running loads. Appl. Math. Mech. **55**(5), 854–862 (1991)
3. Alexeyeva, L.A.: Singular border integral equations of the BVP of elastodynamics in the case of subsonic running loads. Differ. Equ. **46**(4), 512–519 (2010)
4. Alexeyeva, L.A., Shershnev, V.V.: Fundamental solutions of the motion equations of Bio medium. Rep NAS RK **1**, 3–8 (1994)
5. Biot, M.A.: Mechanics of deformation and acoustic propagation in porous media. J. Appl. Phys. **33**, 1482–1498 (1962)
6. Kiyko, I.A.: Harmonic waves propagation in a two-lane two-layer. Bull. Mosc. Univ. Ser. 1. Math. Mech. **5**, 321–354 (1976)

On Instability of a Program Manifold of Basic Control Systems

Sailaubay S. Zhumatov(✉)

Institute of Mathematics and Mathematical Modeling, 125 Pushkin St., Almaty 050010, Kazakhstan
sailau.math@mail.ru

Abstract. The methodology of stability analysis is expounded to the systems automatic control feedback at presence of non-linearity. The conditions of asymptotically instability of the basic control systems are considered in the neighborhood of a program manifold. Nonlinearity satisfies to generalized conditions of local quadratic relations. The sufficient conditions of instability of the program manifold have been obtained relatively to a given vector-function by means of construction of Lyapunov function, in the form "quadratic form plus an integral from nonlinearity". It is solved more general inverse problem of dynamics: not only builds the corresponding system of differential equations, but also investigates the instability, which is very important for a variety of mathematical models mechanics.

Keywords: Instability · Program manifold · Baswic control systems · Lyapunov function

1 Introduction

The problem of the construction of the complete set of systems of differential equations with given integral curve was posed in [1],where a method for its solving was also presented. This problem further developed to the construction of systems of differential equations on the basis of a given integral manifold, to solving various inverse problem of dynamics, and constructing systems of program motion. It should be noted that, in the course of investigation of these problems, the construction of stable systems (which is one of the main problems in the theory of stability) was developed into an independent theory. A detailed survey of these works can be found in [2]. The works [6–8] are devoted to the construction of automatic control systems on the basis of a given manifold. In these works, control systems were constructed for a scalar nonlinear function $\varphi(\sigma)$, and sufficient conditions for absolute stability were established. The problem of the construction of automatic control systems for a vector nonlinear function with locally quadratic relations was solved in [5,10].In this paper it is studied the

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_42

inverse dynamics problem: for a given manifold to restore a force field, which lies in the tangent subspace to a manifold. It is solved more general inverse problem of dynamics: not only it is built the corresponding system of differential equations, but also it is investigated the instability, which is very important for a variety of mathematical models mechanics.

The frequency conditions of instability are received in [3,9] for nonlinear control systems with respect to zero position of equilibrium. In this paper the conditions of instability of the basic control systems are investigated in the neighborhood of a program manifold.

Consider the problem of the construction of a stable control of the form

$$\dot{x} = f(t,x) - B\xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T\omega, \quad t \in I = [0, \infty), \tag{1}$$

on the basis of a given $(n-s)$-dimensional smooth integral manifold $\Omega(t)$ defined by the vector equation

$$\omega(t,x) = 0, \tag{2}$$

where $x \in R^n$ is a state vector of the object, $f \in R^n$ is a vector-function, satisfying conditions of existence and uniqueness of a solution $x(t) = 0$, $B \in R^{n\times r}$, $P \in R^{s\times r}$, are constant matrices, $\omega \in R^s (s \leq n)$ is a vector, $\xi \in R^r$ is a vector-function of control on deviation from the given program manifold, satisfying conditions of local quadratic relations

$$\begin{gathered}\varphi(0) = 0 \ \wedge \ \varphi^T\theta(\sigma - K^{-1}\varphi) > 0, \quad \forall\, \sigma \neq 0, \\ \theta = diag\,\|\,\theta_1, \ldots, \theta_r\|\,, \quad K = K^T > 0\end{gathered} \tag{3}$$

differentiable in σ and $\dfrac{\partial\varphi}{\partial\sigma}$ satisfies the conditions

$$K_1 \leq \frac{\partial\varphi}{\partial\sigma} \leq K_2, \quad K_i = diag\,\|n_1, \ldots, n_r\| \quad (i = 1, 2)\,, \quad K_2 \gg 0. \tag{4}$$

The given program $\Omega(t)$ is exactly realized only if the initial values of the state vector satisfy the condition $\omega(t,x) = 0$. However, this condition cannot always be exactly satisfied. Therefore, in the construction of systems of program motion6 the requirement of the stability of the program manifold $\Omega(t)$ with respect to the vector function ω should also be taken into account.

Actuality of studying these problems is caused by existence of number of the inverse dynamics problems.

To the construction of the systems of equations on the given program manifold, possessing properties of stability, optimality and establishment of estimations of indexes' quality of transient in the neighborhood of a program manifold and to solving of various inverse problems of dynamics there was devoted a great number of works, for example, see [1,2,4–8,10,11]. The detailed reviews of these works were shown in [2,4,11].

In this paper we use Lyapunov function in the form "quadratic form plus integral from nonlinearity" and estimates of positive defined quadratic form.

In the space R^n we select the domain $G(R)$:

$$G(R) = \{(t, x) : t \in I \wedge \|\omega(t, x)\| \leq \rho < \infty\}. \tag{5}$$

Taking into account that $\Omega(t)$ is the integral manifold for the system (1)-(3), we have

$$\dot{\omega} = \frac{\partial \omega}{\partial t} + Hf(t, x) = F(t, x, \omega), \tag{6}$$

where $H = \frac{\partial \omega}{\partial x}$ is the Jacobi matrix and $F(t, x, 0) \equiv 0$ is a certain s-dimensional Erugin vector function [1].

Assuming that the Erugin function $F(t, x, \omega) = -A\omega, -A \in R^{s\times s}$ is Hurwitz matrix and differentiating the manifold $\Omega(t)$ with respect to time t along the solutions of system (1)-(3), we get

$$\dot{\omega} = -A\omega - HB\xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T\omega. \tag{7}$$

Statement of the problem To get the condition of instability of a program manifold $\Omega(t)$ of the basic control systems in relation to the given vector-function ω.

The system (7) has only a position of equilibrium $x = \sigma = 0$ if and only if, when

$$\det \left\| A + HBhP^T \right\| \neq 0, \quad \forall\, h \in (0, K].$$

Definition 1. A program manifold $\Omega(t)$ is called instable on the whole in relation to vector-function ω, if in phase space there is an unlimited open domain $G(R)$, including a neighborhood of the given program manifold and possessing such property, that all solutions in relation to a vector-function ω beginning in this domain, are unlimited at $t \to \infty$.

Definition 2. A program manifold $\Omega(t)$ is called absolutely instable in relation to a vector-function ω, if it is instable on the whole at all functions $\varphi(\sigma)$ satisfying the conditions (3).

Definition 3. The continuous function $V(\omega)$ will be called positive in the domain (5) if $V(0) = 0$ and $\lim\limits_{\omega_k \to \infty} V(\omega) = \infty$ at least for one value $k \leq s$.

Definition 4. The function $V(t, \omega)$ will be called positive and admitting a positive upper limit in the domain $G(R)$ if there exist two continues functions $V_1(\omega), V_2(\omega)$ such that for $\forall\, \omega$ are valid the following inequalities

$$V_1(\omega) \leq V(t, \omega) \leq V_2(\omega) \tag{8}$$

2 The Main Theorem on Instability

Theorem 1. *If for the system (7) there is found a positive function $V(t, \omega)$ admitting a positive upper limit in the domain $G(R)$ derivative which is*

$$\dot{V}(t, \omega) \geq \gamma > 0 \quad \forall\, \omega \in G(R) \wedge t \in I, \tag{9}$$

then the program manifold $\Omega(t)$ is instable as a whole with respect to vector-function ω.

Proof. Assume that $\omega = \omega(t; t_0, x_0)$ is non-trivial solution of the system (7) defined by the initial conditions

$$\omega(t; t_0, x_0) = \omega_0 \neq 0. \tag{10}$$

Since the function $V(t, \omega(t))$ is positive in the domain $G(R)$ the next inequality holds

$$V(t_0, \omega_0) = \alpha > 0,$$

where α is a certain number.

At $V(t, \omega(t)) > 0$ owing to conditions (9), we have

$$\dot{V}(t, \omega) > 0 \quad \forall\, \omega \in \Xi \wedge t \in I.$$

Hence at $t \geq t_0$ we get

$$V(t, \omega(t)) > V(t_0, \omega_0) = \alpha.$$

If the solution $\omega(t)$ leaves the domain $G(R)$, then for some $t_1 > t_0$ we will get $V(t_1, \omega(t_1)) = 0$. Moreover

$$V(t, \omega(t)) \geq \alpha > 0 \quad for \quad t_0 \leq t \leq t_1. \tag{11}$$

Going to the limit in (11) at $t \to t_1 - 0$, we will have $V(t_1, \omega(t_1)) \geq \alpha > 0$, what is impossible. Therefore the solution $\omega(t)$ lies entirely in the domain $G(R)$.

Taking into account condition (9), we have an inequality

$$\dot{V}(t, \omega) \geq \gamma > 0 \quad \forall\, \omega \in \Xi \wedge t \in I \tag{12}$$

Integrating the inequality (12) term by term, we will have at $t \geq t_0$

$$V(t, \omega(t)) > V(t_0, \omega(t_0)) + \gamma(t - t_0). \tag{13}$$

From the inequality (13) follows

$$\lim_{t \to \infty} V(t, \omega(t)) = \infty.$$

Then by definition

$$\lim_{\omega_k \to \infty} V_1(\omega_1, \ldots, \omega_s) = \infty$$

and (13) we have

$$\lim_{t \to \infty} \|\omega(t; t_0, \omega_0))\| = \infty.$$

Therefore program manyfold $\Omega(t)$ is instable as a whole with respect to vector-function ω.

3 Asymptotical Instability of the Program Manifold

Consider the case when Erugin's function $F(t, x, \omega) = F(t, \omega)$ is non-autonomous, then we get system

$$\dot{\omega} = F(t, \omega), \tag{14}$$

where $F(t, 0) = 0$ and $F(t, \omega)$ satisfy the conditions of existence and uniqueness of the trivial solution of the system (14).

Assume that there exists a nonnegative local quadratic relation:

$$S = S(t, \omega) \geq 0 \wedge S(0, \omega) = 0. \tag{15}$$

Theorem 2. *Suppose that there exist a positive-definite function*

$$V = V(t, \omega) > 0 \tag{16}$$

and a nonnegative number α *such that*

$$M[t,\ \omega(t)] = V(t, ; \omega) + \int_{t_0}^{t} S(\tau, \omega(\tau)) d\tau > 0, \tag{17}$$

where $\omega(t)$ *is an arbitrary solution satisfying condition (15) and*

$$\dot{M}\Big|_{(14)} = W[\omega(t)] > 0, \tag{18}$$

then the program manifold $\Omega(t)$ *is asymptotically instable at conditions (15) with respect to vector-function* ω*.*

Proof. By virtue of (18) we have

$$\frac{dM}{dt} \geq \alpha_0 > 0, \tag{19}$$

where α_0 is a certain number.

Then, for any $t > t_0$, with regard for inequality (15), we get

$$M[\omega(t)] = M[\omega(t_0)] + \int_{t_0}^{t} S(\tau, \omega(\tau)) d\tau \leq M[\omega(t_0)] + \alpha_0 (t - t_0). \tag{20}$$

For sufficiently large t, it follows from (15) that $M[\omega(t)]$ becomes more large. Therefore

$$\lim_{t \to \infty} M[\omega(t)] = \infty. \tag{21}$$

By virtue of supposition (13) relation (21) holds if and only if

$$\lim_{t \to \infty} \|\omega(t; t_0, x_0))\| = \infty.$$

Therefore program manyfold $\Omega(t)$ is asymptotically instable with respect to vector-function ω.

The following theorem can be proved similarly.

Theorem 3. *If there exist a positive-definite function* $V = V(\omega) > 0$ *that possesses the property*

$$V = V(\omega) \to \infty \quad as \quad \|\omega(t; t_0, x_0))\| \to \infty \tag{22}$$

and a nonnegative number α *such that relations (16) and (18) are true, then program manyfold* $\Omega(t)$ *is asymptotically instable as a whole at conditions (15) with respect to vector-function* ω.

Now consider system (7) in the case, where the control is direct and scalar:

$$\dot{\omega} = -A\omega - b\xi, \quad \xi = \varphi(\sigma), \quad \sigma = c^T\omega. \tag{23}$$

Using Theorem 2, one can obtain sufficient conditions for the program manyfold $\Omega(t)$ to be asymptotically instable as a whole if

$$\varphi(0) = 0 \ \wedge \ k_1\sigma^2 < \sigma\varphi(\sigma) < k_1\sigma^2, \quad \forall\ \sigma \neq 0, \tag{24}$$

where k_1 and k_2 are certain constants, inequality (16)-(18) are true, and condition (24) is satisfied for

$$V(\omega,\ \xi) = \omega^T L\omega + \beta\int_0^\sigma \varphi(\sigma)d\sigma > 0, \tag{25}$$

where β is a nonnegative number.

In other words, the following theorem is true:

Theorem 4. *If there exist a real matrix* $L = L^T > 0$ *and nonnegative numbers* α *and* β *such that one of the conditions*

$$Q = \begin{Vmatrix} 2G & g \\ g^T & \rho \end{Vmatrix} \tag{26}$$

$$2\rho G - gg^T < 0, \tag{27}$$

$$2G > 0 \ \wedge \ \rho - 2^{-1}g^T G^{-1} g < 0 \tag{28}$$

is satisfied, then program manyfold $\Omega(t)$ *is asymptotically instable as a whole with respect to vector-function* ω *at the conditions*

$$S(\omega) = (\sigma - k_2^{-1}\varphi(\sigma))(\varphi(\sigma) - k_1\sigma) \geq 0, \tag{29}$$

where $k \geq 0$ *and* $k_2 \leq \infty$, *and*

$$A^T L + LA + k_1\alpha cc^T = 2G \tag{30}$$

$$g = Lb + 2^{-1}\left[\beta A^T c - \alpha\left(1 + k_1 k_2^{-1}\right)\right] \tag{31}$$

$$\rho = \alpha k_2^{-1} + \beta c^T b. \tag{32}$$

Acknowledgments. This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 3357/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

1. Erugin, N.P.: Construction of the entire set of systems of differential equations that have a given integral curve. Prikl. mat. Mech. **10**, 659–670 (1952) (in Russian)
2. Galiullin, A.S., Mukhametzyanov, I.A., Mukharlyamov,R.G.: Review of researches on the analytical construction of the systems programmatic motions. Vestnik RUDN. (1), 5–21 (1994) (in Russian)
3. Gelig, AKh, Leonov, G.A., Yakubovich, V.A.: Stability of Nonlinear systems with nonunique equilibrium state. Nauka, Moskow (1981) (in Russian)
4. Llibre, J., Ramirez, R.: Inverse Problems in Ordinary Differential Equations and Applications. Springer International Publishing, Switzerland, Geneva (2016)
5. Maygarin, B.G.: Stability and Quality of Process of Nonlinear Automatic Control System. Nauka, Alma-Ata (1981) (in Russian)
6. Mukhametzyanov, I.A.: On stability of a program manifold.I. Differen. uravneniya. **30**, 846–856 (1973) (in Russian)
7. Mukhametzyanov, I.A.: On stability of a program manifold.II. Differen. uravneniya. **30**, 1037–1048 (1973) (in Russian)
8. Mukharlyamov, R.G.: Differential-algebraic equations of programmed motions of lagrangian dynamical systems. Mech. Solids **46**, 534–543 (2011)
9. Yakubovich, V.A.: Absolute instability of nonlinear control systems. I. General frequency criteria. Avtomatika and Telemekhanika. (12), 5–12 (1970) (in Russian)
10. Zhumatov, S.S., Krementulo, B.B., Maygarin, B.G.: Lyapunov's second method in the problems of stability and control by motion. Gylym, Almaty (1999)
11. Zhumatov, S.S.: Frequently conditions of convergence of control systems in the neighborhoods of program manifold. Nelineinye Koleb. **28**, 367–375 (2016)

Geometric Approach to Domain Wall Solution

Zhanat Kh. Zhunussova(✉)

Al-Farabi Kazakh National University, 71 Al-Farabi Avenue, Almaty 050040, Kazakhstan
zhanat.zhunusova@kaznu.kz

Abstract. Some generalizations of the Landau-Lifschitz equation are integrable, admit physically interesting exact solutions and these integrable equations are solvable by the inverse scattering method. Investigations of the integrable spin equations in (1+1)-, (2+1)-dimensions are topical both from the mathematical and physical points of view. Integrable equations admit different kinds of physically interesting equations as domain wall solutions. We consider an integrable spin equation. There is a corresponding Lax representation. Moreover the equation allows an infinite number of integrals of motion. We construct a surface corresponding to domain wall solution of the equation. Further, we investigate some geometrical features of the surface.

Keywords: Inegrable equation · Lax reprezentation · Integrals of motion · Exact solution · Domain wall solutuin · Surface

1 Introduction

We use the geometric approach to one of the generalized Landau-Lipshitz equation [1–4]

$$\mathbf{S}_t = (\mathbf{S} \times \mathbf{S}_y + u\mathbf{S})_x, \tag{1a}$$

$$u_x = -(\mathbf{S}, (\mathbf{S}_x \times \mathbf{S}_y)), \tag{1b}$$

where $\mathbf{S}$ is a spin vector, $S_1^2 + S_2^2 + S_3^2 = 1$, $\times$ is a vector product, u is a scalar function. The equation allows an infinite number of motion integrals and has several exact solutions. One of them is the domain wall solution [3]. We identify the spin vector $\mathbf{S}$ and vector $\mathbf{r}_x$ according to the geometric approach [4]

$$\mathbf{S} \equiv \mathbf{r}_x. \tag{2}$$

Then (1a), (1b) take the form

$$\mathbf{r}_{xt} = (\mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x)_x \tag{3a}$$

$$u_x = -(\mathbf{r}_x, (\mathbf{r}_{xx} \times \mathbf{r}_{xy})). \tag{3b}$$

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9_43

If we integrate (3a) by x, then it takes the form

$$\mathbf{r}_t = \mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x.$$

Taking into account Gauss-Weingarten equation and $E = \mathbf{r}_x^2 = 1$, the system is defined as

$$\mathbf{r}_t = (u + \frac{MF}{\sqrt{\Lambda}})\mathbf{r}_x - \frac{M}{\sqrt{\Lambda}}\mathbf{r}_y + \Gamma_{12}^2\sqrt{\Lambda}\mathbf{n},$$
$$u_x = \sqrt{\Lambda}(L\Gamma_{12}^2 - M\Gamma_{11}^2),$$

where

$$\Gamma_{11}^2 = \frac{2EF_x - EE_t - FE_x}{2\Lambda},$$
$$\Gamma_{12}^2 = \frac{EG_x - FE_t}{2\Lambda},$$

$\Lambda = EG - F^2$. Equations (1a), (1b) is integrable equation and has solutions.

2 Construction of Surface Corresponding to Domain Wall Solution

Here we present the domain wall solution of the equation (1a), (1b) [4],

$$S^+(x, y, t) = \frac{\exp(iby)}{\cosh[a(x - bt - x_0)]}, \tag{4a}$$
$$S_3(x, y, t) = -\tanh[a(x - bt - x_0)], \tag{4b}$$

where a, b are real constants.

Theorem. *Domain wall solution* (4a)–(4b) *of the spin system* (1a), (1b) *can be represented as components of the vector* $\mathbf{r}_x$, *where*

$$r_1 = \frac{1}{a}\cos(by)\arctan(\sinh[a(x - bt - x_0)]) + c_1, \tag{5a}$$
$$r_2 = \frac{1}{a}\sin(by)\arctan(\sinh[a(x - bt - x_0)]) + c_2, \tag{5b}$$
$$r_3 = -\frac{1}{a}\ln|\cosh[a(x - bt - x_0)]| + c_3, \tag{5c}$$

where c_1, c_2, c_3 *are constants. Solution of the form* (5a)–(5c) *corresponds to the surface with the following coefficients of the first and second fundamental forms*

$$E = \frac{2 + \sinh^2[a(x - bt - x_0)]}{(1 + \sinh^2[a(x - bt - x_0)])^2}, \quad F = 0, \tag{6a}$$
$$G = \frac{b^2}{a^2}\arctan^2(\sinh[a(x - bt - x_0)]), \quad L = 0, \tag{6b}$$
$$M = 0, \quad N = -\frac{b^3\arctan^2(\sinh[a(x - bt - x_0)])}{\sqrt{\Lambda}a^2\cosh[a(x - bt - x_0)]}. \tag{6c}$$

Proof. From (2) we have

$$(S_1, S_2, S_3) = (r_{1x}, r_{2x}, r_{3x}), \tag{7}$$

i.e.

$$r_{1x} = S_1, \quad r_{2x} = S_2, \quad r_{3x} = S_3. \tag{8}$$

Hence

$$r_1 = \int S_1 dx + c_1, \tag{9a}$$

$$r_2 = \int S_2 dx + c_2, \tag{9b}$$

$$r_3 = \int S_3 dx + c_3, \tag{9c}$$

where c_1, c_2, c_3 are constants of integration. Note

$$S^+ = S_1 + iS_2 = r_x^+,$$

then

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+, \tag{10}$$

where c^+ is constant of integration. Substituting (4b) to the equation (9c), we have

$$r_3 = \int S_3 dx + c_3 = -\int [\tanh[a(x - bt - x_0)] dx + c_3 =$$
$$= -\frac{1}{a} \ln|\cosh[a(x - bt - x_0)]| + c_3, \tag{11}$$

where c_3 is constant. Thus

$$r_3 = -\frac{1}{a} \ln|\cosh[a(x - bt - x_0)]| + c_3. \tag{12}$$

Substituting (4a) to (10), we have

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+ =$$
$$= \int \frac{\exp(iby)}{\cosh[a(x - bt - x_0)]} dx + c^+,$$

then

$$r^+ = \frac{1}{a} \cos(by) \arctan(\sinh[a(x - bt - x_0)]) + c_1 +$$
$$+ i(\frac{1}{a} \sin(by) \arctan(\sinh[a(x - bt - x_0)]) + c_2),$$

i.e. we have obtained

$$r_1 = \frac{1}{a}\cos(by)\arctan(\sinh[a(x-bt-x_0)]) + c_1,$$
$$r_2 = \frac{1}{a}\sin(by)\arctan(\sinh[a(x-bt-x_0)]) + c_2. \tag{13}$$

Thus, (12), (13) give us (5a)–(5c).

We proceed to prove the second part of the theorem. From (12) and (13) we have

$$r_{1x} = \frac{\cos(by)}{1+\sinh^2[a(x-bt-x_0)]}, \quad r_{2x} = \frac{\sin(by)}{1+\sinh^2[a(x-bt-x_0)]}, \tag{14a}$$

$$r_{3x} = -\frac{1}{\cosh^2[a(x-bt-x_0)]}, \quad r_{1y} = -\frac{b}{a}\sin(by)\arctan(\sinh[a(x-bt-x_0)]), \tag{14b}$$

$$r_{2y} = \frac{b}{a}\cos(by)\arctan(\sinh[a(x-bt-x_0)]), \quad r_{3y} = 0. \tag{14c}$$

Then we can calculate

$$\begin{aligned} E = \mathbf{r}_x^2 = r_{1x}^2 + r_{2x}^2 + r_{3x}^2 = \\ = \frac{\cos^2(by)}{(1+\sinh^2[a(x-bt-x_0)])^2} + \\ + \frac{\sin^2(by)}{(1+\sinh^2[a(x-bt-x_0)])^2} + \frac{1}{\cosh^2[a(x-bt-x_0)]} = \frac{2+\sinh^2[a(x-bt-x_0)]}{(1+\sinh^2[a(x-bt-x_0)])^2}. \end{aligned} \tag{15}$$

Similarly, using (13) and (14c), we obtain

$$G = \mathbf{r}_y^2 = r_{1y}^2 + r_{2y}^2 + r_{3y}^2 = \frac{b^2}{a^2}\arctan^2(\sinh[a(x-bt-x_0)]), \tag{16}$$

$$F = (\mathbf{r}_x, \mathbf{r}_y) = r_{1x}r_{1y} + r_{2x}r_{2y} + r_{3x}r_{3y} = 0. \tag{17}$$

Formulas (15)–(17) give us the first three equations (6a)–(6c). Using (15)–(17), we compute

$$\Lambda = EG - F^2 = \frac{b^2(2+\sinh^2[a(x-bt-x_0)])}{a^2(1+\sinh^2[a(x-bt-x_0)])^2}\arctan^2(\sinh[a(x-bt-x_0)]).$$

We calculate the components of the vector $\mathbf{n}$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\sqrt{\Lambda}} = \frac{1}{\sqrt{\Lambda}}(n_1, n_2, n_3),$$
$$n_1 = \frac{1}{\sqrt{\Lambda}}(r_{2x}r_{3y} - r_{3x}r_{2y}) = \frac{b\cos(by)\arctan(\sinh[a(x-bt-x_0)])}{\sqrt{\Lambda}ach[a(x-bt-x_0)]}. \tag{18}$$

Similarly, for the components

$$n_2 = \frac{1}{\sqrt{\Lambda}}(r_{3x}r_{1y} - r_{1x}r_{3y}) = \frac{b\sin(by)\arctan(\sinh[a(x-bt-x_0)])}{\sqrt{\Lambda}ach[a(x-bt-x_0)]}, \tag{19a}$$

$$n_3 = \frac{1}{\sqrt{\Lambda}}(r_{1x}r_{2y} - r_{2x}r_{1y}) = \frac{b\arctan(\sinh[a(x-bt-x_0)])}{\sqrt{\Lambda}a(1+\sinh^2[a(x-bt-x_0)])}. \tag{19b}$$

Now, from (14a), (14b) we have

$$r_{1xx} = -\frac{2\arccos(by)\sinh[a(x-bt-x_0)]\cosh[a(x-bt-x_0)]}{(1+\sinh^2[a(x-bt-x_0)])^2}, \tag{20a}$$

$$r_{2xx} = -\frac{2\arcsin(by)\sinh[a(x-bt-x_0)]\cosh[a(x-bt-x_0)]}{(1+\sinh^2[a(x-bt-x_0)])^2}, \tag{20b}$$

$$r_{3xx} = \frac{ash[a(x-bt-x_0)]}{\cosh^2[a(x-bt-x_0)]}. \tag{20c}$$

Thus, using (18), (19a), (19b), (20a)–(20c), we can compute

$$L = (\mathbf{n}, \mathbf{r}_{xx}) = n_1 r_{1xx} + n_2 r_{2xx} + n_3 r_{3xx}.$$

It follows that

$$L = 0. \tag{21}$$

Similarly, we calculate other coefficients of the second fundamental form

$$M = 0, \tag{22}$$

$$N = -\frac{b^3\arctan^2(\sinh[a(x-bt-x_0)])}{\sqrt{\Lambda}a^2\cosh[a(x-bt-x_0)]}. \tag{23}$$

The formulas (21)–(23) give us the last three equations (6a)–(6c). Finally, Theorem is proved.

Finally, we use possibilities of the editor Maple and construct the surface at some values of the parameters. The components of the vector $\mathbf{r}_x$ (5a)–(5c) can be represented as a function $r_3 = f(r_1, r_2)$, i.e.

$$r_3 = -\frac{1}{a}\ln\left|\sqrt{\tanh(a((r_1)^2) + (r_2)^2)) + 1}\right|, \tag{24}$$

under $c_1 = c_2 = c_3 = 0$. By varying the parameter a and by choosing the segments for the values r_1, r_2, we obtain the figures presented below.

3 Conclusion

Based on the results of work [4], where Gauss-Codazzi-Mainardi equation is considered in multidimensional space, we have studied generalized Landau-Lipshitz equation and built the surface corresponding to domain wall solution. Thus, this work fully reveals the meaning of the geometric approach in (2+1)-dimensions (Figs. 1 and 2).

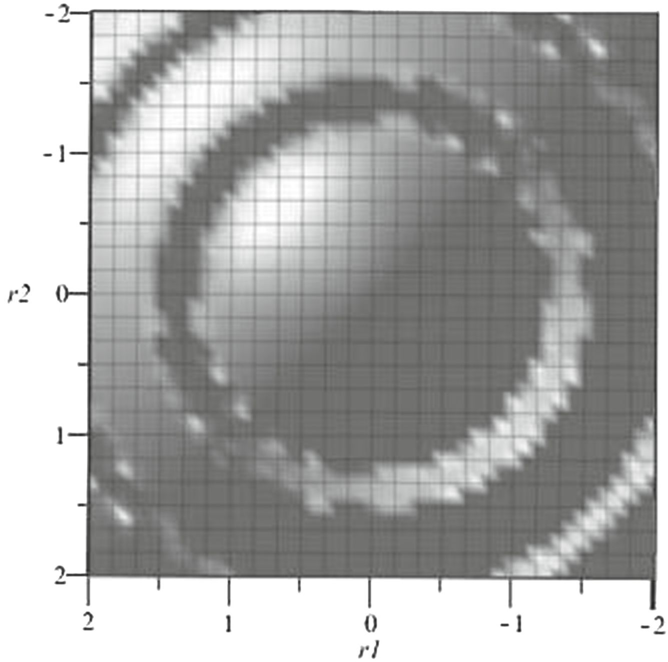


Fig. 1. The surface is illustrated at $a := 1, r_1 : [-2, 2], r_2 : [-2, 2]$

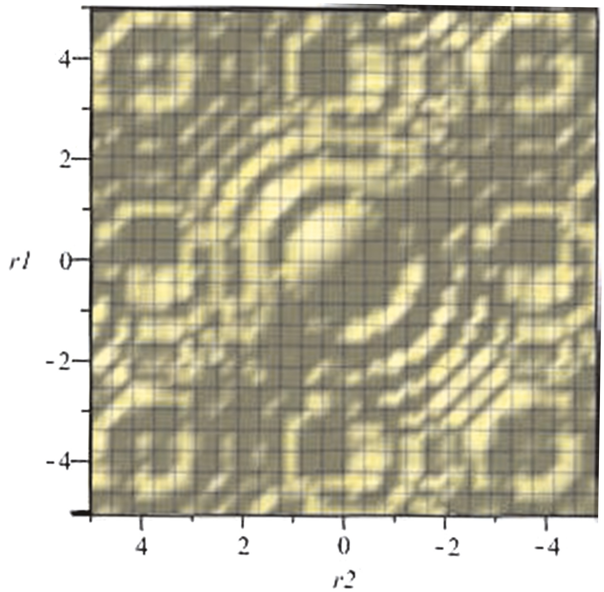


Fig. 2. The surface is illustrated at $a := 1, r_1 : [-5, 5], r_2 : [-5, 5]$

References

1. Ablowitz, M.J., Clarkson, P.A.: Solitons, Non-linear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1992)
2. Bliev, N.K., Myrzakulov, R., Zhunussova, Zh.Kh.: Some exact solutions of the non-linear sigma model. Doclady AN RK **5**, 3–10 (1999)
3. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. **19**, 1095–1097 (1967)
4. Lakshmanan, M., Myrzakulov, R., et al.: Motion of curves and surfaces and non-linear evolution equations in 2+1 -dimensions. J. Math. Phys. **39**(7), 3765–3771 (1998)

Index

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9

Author Index

T.Sh. Kalmenov et al. (eds.), *Functional Analysis in Interdisciplinary Applications*,
Springer Proceedings in Mathematics & Statistics 216,
https://doi.org/10.1007/978-3-319-67053-9