

Hypersequent Calculi for Lewis' Conditional Logics with Uniformity and Reflexivity

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Abstract. We present the first internal calculi for Lewis' conditional logics characterized by uniformity and reflexivity, including non-standard internal hypersequent calculi for a number of extensions of the logic VTU. These calculi allow for syntactic proofs of cut elimination and known connections to S5. We then introduce standard internal hypersequent calculi for all these logics, in which sequents are enriched by additional structures to encode plausibility formulas as well as diamond formulas. These calculi provide both a decision procedure for the respective logics and constructive countermodel extraction from a failed proof search attempt.

1 Introduction

Conditional logics have a long history going back, e.g., to the works of Stalnaker, Lewis, Nute, Chellas, Burgess, Pollock in the 60's–70's [3, 4, 13, 14, 18]. In his seminal works Lewis proposed a formalization of conditional logics to capture counterfactual and other hypothetical conditionals that cannot be accommodated by the material implication of classical logic [13]. Conditional logics have since found an interest in several fields of knowledge representation, from reasoning about prototypical properties and nonmonotonic reasoning [9] to modeling belief change. A successful attempt to relate conditional logic and belief update (as opposite to belief revision) was carried out by Grahne [8], who established a precise mapping between belief update operators and Lewis' logic VCU. The relation is expressed by the so-called *Ramsey's Rule*:

$A \circ B \rightarrow C$ holds if and only if $A \rightarrow (B \boxrightarrow C)$ holds

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where the operator \circ is any *update* operator satisfying Katsuno and Mendelzon's postulates. The relation means that C is entailed by “ A updated by B ” if and only if the conditional $B \square \rightarrow C$ is entailed by A . In this sense it can be said that the conditional $B \square \rightarrow C$ expresses an hypothetical update of a piece of information A .

The family of logics studied by Lewis in [13] is semantically characterized by *sphere models*, where each world x is equipped with a set of nested sets of worlds $\text{SP}(x)$. Each set in $\text{SP}(x)$ is called a *sphere*: the intuition is that concerning x , worlds in inner spheres are more plausible than worlds belonging only to outer spheres. Lewis takes as primitive the *comparative plausibility* connective \preceq , with a formula $A \preceq B$ meaning “ A is at least as plausible as B ”. The conditional $A \square \rightarrow B$ is then defined as “ A is impossible or $A \wedge \neg B$ is less plausible than A ”. Vice versa, \preceq can be defined in terms of $\square \rightarrow$.

In this paper we continue our proof-theoretic investigation of the family of Lewis' logics, concentrating on the logics characterized by two properties: (i) *Uniformity*: all worlds have the same set of accessible worlds, where the worlds accessible from a world x are those belonging to any sphere $\alpha \in \text{SP}(x)$; (ii) *Total reflexivity*: every world x belongs to some sphere $\alpha \in \text{SP}(x)$. The basic logic is VTU ; we will then consider some of its extensions, including the above mentioned VCU . It is worth mentioning that equivalent logics are those of Comparative Concept Similarity studied in the context of ontologies [17]. These logics contain a connective \Leftarrow , which allows to express, e.g.,

$$\text{PicassoPainting} \sqsubseteq \text{BraquePainting} \Leftarrow \text{GiottoPainting}$$

asserting that “Picasso's paintings are more similar to Braque's paintings than to Giotto's ones”. The semantics is provided in terms of Distance Space Models, defined as a set of worlds equipped with a distance function. It turns out that the basic logic of Comparative Concept Similarity coincides with Lewis' logic VWU and the one defined by “minspace” Distance Models coincides with VCU , so that Distance Space Models provide an alternative simple and natural semantics for conditional logics with uniformity [1, 17].

Here we investigate *internal calculi* for logics extending VTU , i.e., calculi where each configuration of a derivation corresponds to a formula of the corresponding logic, in contrast to external calculi which make use of extra-logical elements (such as labels, terms and relations on them). Ideally, we seek calculi with the following features: (i) they should be *standard*, i.e., each connective is handled by a finite set of rules with a finite and fixed set of premises; (ii) they should be *modular*, i.e., it should be possible to obtain calculi for stronger logics by adding independent rules to a base calculus; (iii) they should have good proof-theoretical properties, such as a syntactic proof of cut admissibility; finally (iv) they should provide a decision procedure for the respective logics. In our opinion requirement (i) is particularly important: a standard calculus provides a self-explanatory presentation of the logic, thus a kind of proof-theoretic semantics.

In previous work [7], we defined calculi with many of these properties for weaker logics of the Lewis' family. For the logics with *uniformity* to the best of our knowledge no internal calculi are known; the only known external calculi for these adopt a hybrid language and a relational semantics [6]. We also consider

logics with *absoluteness*, a property stronger than uniformity stating that all worlds have the same system of spheres. It is unlikely that sequents, even extended as in [7], are sufficient to capture logics with uniformity: Since modal logic S5 can be embedded into $\mathbb{V}\mathbb{T}\mathbb{U}$, a sequent calculus for the latter would most probably also yield a sequent calculus for S5. The existence of such a calculus, however, would be very surprising. We therefore adopt the framework of *hypersequents* [2], where the basic objects are multisets of sequents.

We first provide a non-standard hypersequent calculus for $\mathbb{V}\mathbb{T}\mathbb{U}$ and its extensions and syntactically prove cut-elimination and hence completeness. We then show that by translating $\Box A$ as $\perp \preceq \neg A$ the calculi - restricted to such formulas - correspond to known hypersequent calculi for S5. Further, we construct standard calculi for all the logics by enriching the hypersequents with additional structural connectives encoding plausibility and “possible” formulas respectively. The obtained standard calculi provide decision procedures for the respective logics. Finally, we give a direct semantic completeness proof for the logics without absoluteness, by considering the invertible version of the rules and constructing a countermodel from a failed attempt at proof search. Thus, the calculi can also be used for countermodel generation, a task of independent interest.

2 Preliminaries

We consider the *conditional logics* of [13]. The set of *conditional formulae* is given by $A ::= p \mid \perp \mid A \rightarrow A \mid A \preceq A$, where $p \in \mathcal{V}$ is a propositional variable. We define the boolean connectives \wedge, \vee, \top in terms of \perp and \rightarrow as usual. Intuitively, a formula $A \preceq B$ is interpreted as “ A is at least as plausible as B ”. Lewis' *counterfactual implication* $\Box \rightarrow$ is defined by $A \Box \rightarrow B \equiv (\perp \preceq A) \vee \neg((A \wedge \neg B) \preceq A)$, whereas the *outer modality* \Box is defined by $\Box A \equiv (\perp \preceq \neg A)$. The logics we consider are defined as follows:

Definition 1. A universal sphere model (or model) is a triple $\langle W, \text{SP}, [\cdot] \rangle$, consisting of a non-empty set W of elements, called worlds, a mapping $\text{SP} : W \rightarrow 2^{2^W}$, and a propositional valuation $[\cdot] : \mathcal{V} \rightarrow 2^W$. Elements of $\text{SP}(x)$ are called spheres. We assume the following conditions:

- for every $\alpha \in \text{SP}(w)$ we have $\alpha \neq \emptyset$ (non-emptiness)
- for every $\alpha, \beta \in \text{SP}(w)$ we have $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ (sphere nesting)
- for all $w \in W$ we have $\text{SP}(w) \neq \emptyset$ (normality)
- for all $w \in W$ we have $w \in \bigcup \text{SP}(w)$ (total reflexivity)
- for all $w, v \in W$ we have $\bigcup \text{SP}(w) = \bigcup \text{SP}(v)$ (uniformity)

The valuation $[\cdot]$ is extended to all formulae by: $[\perp] = \emptyset$; $[A \rightarrow B] = (W - [A]) \cup [B]$; $[A \preceq B] = \{w \in W \mid \text{for all } \alpha \in \text{SP}(w). \text{ if } [B] \cap \alpha \neq \emptyset, \text{ then } [A] \cap \alpha \neq \emptyset\}$. We also write $w \Vdash A$ instead of $w \in [A]$ as well as $\alpha \Vdash^\forall A$ for $\forall x \in \alpha. x \Vdash A$ and $\alpha \Vdash^\exists A$ for $\exists x \in \alpha. x \Vdash A$ ¹. Validity and satisfiability of formulae in a class of models are defined as usual. Conditional logic $\mathbb{V}\mathbb{T}\mathbb{U}$ is the set of formulae valid in all universal sphere models.

¹ Using this notation we thus have: $x \Vdash A \preceq B$ iff for all $\alpha \in \text{SP}(x). \alpha \Vdash^\forall \neg B$ or $\alpha \Vdash^\exists A$.

Table 1. Lewis' logics and axioms.

(CPR) $\frac{\vdash B \rightarrow A}{\vdash A \leq B}$ (TR) $(A \leq B) \wedge (B \leq C) \rightarrow (A \leq C)$ (N) $\neg(\perp \leq \top)$ (U1) $\neg(\perp \leq A) \rightarrow (\perp \leq (\perp \leq A))$ (W) $A \rightarrow (A \leq \top)$ (A1) $(A \leq B) \rightarrow (\perp \leq \neg(A \leq B))$	(CPA) $(A \leq A \vee B) \vee (B \leq A \vee B)$ (CO) $(A \leq B) \vee (B \leq A)$ (T) $(\perp \leq \neg A) \rightarrow A$ (U2) $(\perp \leq \neg A) \rightarrow (\perp \leq \neg(\perp \leq \neg A))$ (C) $(A \leq \top) \rightarrow A$ (A2) $\neg(A \leq B) \rightarrow (\perp \leq (A \leq B))$
$\mathcal{A}_{\text{VTU}} := \{(\text{CPR}), (\text{CPA}), (\text{TR}), (\text{CO}), (\text{N}), (\text{T}), (\text{U1}), (\text{U2})\}$	
$\mathcal{A}_{\text{VWU}} := \mathcal{A}_{\text{VTU}} \cup \{(\text{W})\}$ $\mathcal{A}_{\text{VCU}} := \mathcal{A}_{\text{VTU}} \cup \{(\text{W}), (\text{C})\}$ $\mathcal{A}_{\text{VTA}} := \mathcal{A}_{\text{VTU}} \cup \{(\text{A1}), (\text{A2})\}$	
$\mathcal{A}_{\text{VWA}} := \mathcal{A}_{\text{VTU}} \cup \{(\text{W}), (\text{A1}), (\text{A2})\}$ $\mathcal{A}_{\text{VCA}} := \mathcal{A}_{\text{VTU}} \cup \{(\text{W}), (\text{C}), (\text{A1}), (\text{A2})\}$	

Extensions of VTU are defined by additional conditions on the class of models, namely:

- *weak centering*: for all $\alpha \in \text{SP}(w)$ we have $w \in \alpha$;
- *centering*: for all $w \in W$ we have $\{w\} \in \text{SP}(w)$;
- *absoluteness*: for all $w, v \in W$ we have $\text{SP}(w) = \text{SP}(v)$.

Extensions of VTU are denoted by concatenating letters for these properties: W for weak centering, C for centering, and A for absoluteness. We consider the following systems²:

$$\begin{array}{ll}
\text{VTU} & \text{VTA: VTU} + \textit{absoluteness} \\
\text{VWU: VTU} + \textit{weak centering} & \text{VWA: VTA} + \textit{weak centering} \\
\text{VCU: VTU} + \textit{centering} & \text{VCA: VTA} + \textit{centering}
\end{array}$$

These logics can be characterized by axioms in a Hilbert-style system [13, Chap. 6]. The modal axioms in the language with only the comparative plausibility operator are given in Table 1 (\vee and \wedge bind stronger than \leq). Propositional axioms and rules are standard.

3 Hypersequent Calculi

In this section we introduce calculi for VTU and extensions. We call a calculus *standard* if (a) it has a finite number of rules and (b) each rule has a finite and fixed number of premisses. With respect to this definition, the calculi introduced in this section are *non-standard*, whereas the calculi we introduce in Sect. 6 are standard.

Our calculi are based on hypersequents, where as usual a *sequent* is a pair consisting of two multisets of formulae, written as $\Gamma \Rightarrow \Delta$.

² Observe that $\text{VTA} + \textit{weak centering}$ collapses to S5 , since in any model over a set of worlds W it must be for all $w \in W$, $\text{SP}(w) = \{W\}$. Furthermore, $\text{VTA} + \textit{centering}$ collapses to Classical Logic, as in any model the set of worlds must be a singleton $\{w\}$ and $\text{SP}(w) = \{\{w\}\}$, so that $A \leq B$ is equivalent to the material implication $B \rightarrow A$. See also Proposition 16 below..

$$\begin{array}{c}
\overline{\mathcal{G} \mid \Gamma, p \Rightarrow p, \Delta} \text{ init} \quad \overline{\mathcal{G} \mid \Gamma, \perp \Rightarrow \Delta} \perp_L \quad \frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \text{ ICL} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta} \text{ ICR} \\
\frac{\frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow_L \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \rightarrow B, \Delta} \rightarrow_R}{\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow [D]_1^{k-1}, [A]_1^n : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid B_k \Rightarrow [D]_1^m, [A]_1^n : k \leq n\}}{\mathcal{G} \mid \Sigma, [C \leq D]_1^m \Rightarrow [A \leq B]_1^n, \Pi} R_{m,n}} \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_k \Rightarrow [D]_1^{k-1} : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow [D]_1^m, \Theta\}}{\mathcal{G} \mid \Sigma, [C \leq D]_1^m \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \text{ trf}_m \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow [D]_1^{k-1} : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow [D]_1^m, \Pi\}}{\mathcal{G} \mid \Sigma, [C \leq D]_1^m \Rightarrow \Pi} T_m \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow [D]_1^{k-1}, [A]_1^n : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow [D]_1^m, [A]_1^n, \Pi\}}{\mathcal{G} \mid \Sigma, [C \leq D]_1^m \Rightarrow [A \leq B]_1^n, \Pi} W_{m,n} \\
\frac{\frac{\mathcal{G} \mid \Sigma, C \Rightarrow \Pi \quad \mathcal{G} \mid \Sigma \Rightarrow D, \Pi}{\mathcal{G} \mid \Sigma, C \leq D \Rightarrow \Pi} R_C \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow A, \Pi}{\mathcal{G} \mid \Sigma \Rightarrow A \leq B, \Pi} R_W \quad \frac{\mathcal{G} \mid \Sigma, \Gamma \Rightarrow \Pi, \Delta \mid \Omega \Rightarrow \Theta}{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, \Gamma \Rightarrow \Theta, \Delta} \text{ spl}}{\frac{\frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, C \leq D \Rightarrow \Theta}{\mathcal{G} \mid \Sigma, C \leq D \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \text{ abs}_L \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow A \leq B, \Theta}{\mathcal{G} \mid \Sigma \Rightarrow A \leq B, \Pi \mid \Omega \Rightarrow \Theta} \text{ abs}_R}
\end{array}$$

$$H_{\text{VTU}} : \{\text{init}, \perp_L, \text{ICL}, \text{ICR}, \rightarrow_L, \rightarrow_R\} \cup \{R_{m,n} : m \geq 0, n \geq 1\} \cup \{\text{trf}_m : m \geq 1\} \cup \{T_m : m \geq 1\}$$

$$H_{\text{VWU}} : H_{\text{VTU}} \cup \{W_{m,n} : m + n \geq 1\} \quad H_{\text{VCU}} : H_{\text{VTU}} \cup \{R_C, R_W\}$$

$$H_{\text{VTA}} : H_{\text{VTU}} \cup \{\text{abs}_L, \text{abs}_R\} \quad H_{\text{VWA}} : H_{\text{VWU}} \cup \{\text{abs}_L, \text{abs}_R\} \quad H_{\text{VCA}} : H_{\text{VCU}} \cup \{\text{abs}_L, \text{abs}_R, \text{spl}\}$$

Fig. 1. The hypersequent calculi for VTU and extensions.

Definition 2. A hypersequent is a non-empty multiset of sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$, where $n \geq 1$ is the cardinality of the multiset. The conditional formula interpretation of a hypersequent is

$$\iota_{\leq}(\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n) := \Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1) \vee \dots \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Delta_n)$$

where \Box is the outer modality defined by $\Box A \equiv (\perp \leq \neg A)$.

The rules of the calculi $H_{\mathcal{L}}$ extend the calculi from [12] to the hypersequent setting and are given in Fig. 1. These calculi are *non-standard*, meaning that the rules have an unbounded number of premisses. We abbreviate multisets of formulae A_k, \dots, A_n to $[A]_k^n$, and $C_k \leq D_k, \dots, C_n \leq D_n$ to $[C \leq D]_k^n$ with the convention that $[A]_k^n$ is empty if $k > n$. The crucial rule for uniformity is the rule trf_m . Intuitively it unpacks a number of comparative plausibility formulae behaving like boxed formulae on the left hand side of a component in the conclusion into a different component in the rightmost premiss, most clearly seen in the case of $n = 1$. The leftmost set of premisses ensures that the comparative plausibility formulae indeed behave like boxed formulae. The rule T_m is the local version of trf_m , and essentially captures total reflexivity.

Lemma 3. *For \mathcal{L} any of the considered logics, the calculus $\mathsf{H}_{\mathcal{L}}$ is sound for \mathcal{L} .*

Proof. This follows from validity of $\Box A \rightarrow A$ in all the logics and the fact that the rules preserve soundness with respect to ι . The latter is shown for each rule by constructing a countermodel for one of the premisses from a countermodel for the conclusion, using that the sphere system is universal. For all the rules apart from trf , abs_L , abs_R , spl this follows as in [12], using that $\Box A \rightarrow \Box\Box A$ is valid. For abs_L , abs_R this follows straightforwardly from absoluteness, and for spl this follows from the fact that frames for VCA are degenerate in the sense that $\text{SP}(w) = \{\{w\}\}$ for every world w (see footnote 2).

For the rule trf , let $\mathfrak{M} = \langle W, \text{SP}, [\cdot] \rangle$ be a VTU model, let $w \in W$, and suppose that

$$\mathfrak{M}, w \models \neg\iota(\mathcal{G}) \wedge \diamond(\wedge \Sigma \wedge \bigwedge_{i=1}^m (C_i \preceq D_i) \wedge \neg \bigvee \Delta) \wedge \diamond(\wedge \Omega \wedge \neg \bigvee \Theta). \quad (1)$$

Then in particular $\mathfrak{M}, w \models \neg(\iota(\mathcal{G}) \vee \Box(\wedge \Sigma \rightarrow \Pi) \vee \Box(\wedge \Omega \rightarrow \Theta))$. Furthermore, suppose that for every $k \leq m$ we have

$$\mathfrak{M}, w \models \iota(\mathcal{G}) \vee \Box(\wedge \Sigma \rightarrow \bigvee \Pi) \vee \Box(\wedge \Omega \rightarrow \bigvee \Theta) \vee \Box\left(C_k \rightarrow \bigvee_{i=1}^{k-1} D_i\right). \quad (2)$$

Then from the case $k = 1$ of (2) we obtain $\mathfrak{M}, w \models \Box\neg C_1$. From this together with (1) and the fact that for every $v \in \bigcup \text{SP}(w)$ we have $\bigcup \text{SP}(v) = \bigcup \text{SP}(w)$ we then obtain $\mathfrak{M}, w \models \Box\neg D_1$. Similarly, using the case $k = 2$ of (2) we get $\mathfrak{M}, w \models \Box\neg D_2$ and continuing like this we get $\mathfrak{M}, w \models \Box\neg D_1 \wedge \dots \wedge \Box\neg D_m$. Together with (1) this gives $\mathfrak{M}, w \models \neg\iota(\mathcal{G}) \wedge \diamond(\wedge \Sigma \wedge \neg \bigvee \Pi) \wedge \diamond(\wedge \Omega \wedge \neg (D_1 \vee \dots \vee D_m \vee \bigvee \Theta))$ and hence we have a countermodel for the remaining premiss. \square

By induction on the formula complexity we straightforwardly obtain:

Lemma 4. *For every formula A we have $\mathsf{H}_{\mathcal{L}} \vdash \mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta$.*

As usual, a rule is *admissible* in $\mathsf{H}_{\mathcal{L}}$ if whenever the premisses are derivable in $\mathsf{H}_{\mathcal{L}}$, then so is its conclusion. It is *depth-preserving admissible*, if the depth of the derivation of its conclusion is at most the maximal depth of the derivations of its premisses.

Lemma 5. *The rules IW , EW , mrg from Fig. 2 are depth-preserving admissible in $\mathsf{H}_{\mathcal{L}}$.*

Proof. By induction on the depth of the derivation in all cases. For mrg , if the last applied rule was trf_m , we might need to replace it with T_m . \square

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{IW} \quad \frac{\mathcal{G}}{\mathcal{G} \mid \Rightarrow} \text{EW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{mrg}$$

Fig. 2. The structural rules of internal and external weakening and merge.

Observe that from admissibility of mrg using the internal contraction rules we also immediately obtain admissibility of the external contraction rules, i.e., contraction on hypersequent components. We first show completeness of the systems with the *cut rule*:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid A, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ cut}$$

Cut-free completeness then will follow from cut elimination. In the following we write $\text{H}_{\mathcal{L}}\text{cut}$ for the system $\text{H}_{\mathcal{L}}$ with the cut rule.

Lemma 6 (Completeness with cut). *For \mathcal{L} one of the considered logics the calculus $\text{H}_{\mathcal{L}}\text{cut}$ is complete for \mathcal{L} , i.e., whenever $A \in \mathcal{L}$, then $\text{H}_{\mathcal{L}}\text{cut} \vdash \Rightarrow A$.*

Proof. By deriving the axioms and using cut to simulate modus ponens and the rule (CPR). The interesting cases are the axioms (U1), (U2) for uniformity and (A1), (A2) for absoluteness. The derivation for (U1) is as follows:

$$\frac{\frac{\frac{\frac{\frac{\frac{\Rightarrow \mid A \Rightarrow \perp \mid \Rightarrow \perp \mid \perp \Rightarrow \perp}{\perp_L} \quad \frac{\Rightarrow \mid A \Rightarrow A, \perp \mid \Rightarrow \perp}{\text{Lem. 4}}}{\text{trf}_1} \quad \frac{\frac{\frac{\frac{\Rightarrow \mid A \Rightarrow \perp \mid \perp \preceq A \Rightarrow \perp}{R_{0,1}}}{\Rightarrow \perp \preceq A \mid \perp \preceq A \Rightarrow \perp} \quad \frac{\frac{\frac{\Rightarrow \perp \preceq A, \perp \preceq (\perp \preceq A)}{R_{0,1}}}{\Rightarrow \neg(\perp \preceq A) \rightarrow (\perp \preceq (\perp \preceq A))}{\neg_L, \rightarrow_R}}}{\Rightarrow \mid A \Rightarrow \perp \mid \Rightarrow \perp \mid \perp \Rightarrow \perp}}}{\Rightarrow \perp \preceq A \mid \perp \preceq A \Rightarrow \perp}}}{\Rightarrow \perp \preceq A, \perp \preceq (\perp \preceq A)}}}{\Rightarrow \neg(\perp \preceq A) \rightarrow (\perp \preceq (\perp \preceq A))} \neg_L, \rightarrow_R$$

The derivations of the remaining axioms are similar, using the rules $\text{abs}_L, \text{abs}_R$ in the case of absoluteness. \square

4 Cut Elimination

To obtain cut-free completeness for all systems we now give a syntactic proof of cut elimination. For this, in the presence of absoluteness we consider slightly extended calculi containing also versions of the rules $W_{m,n}, R_C, R_W$ where absoluteness is built in:

$$\frac{\frac{\frac{\frac{\frac{\frac{\{ \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_k \Rightarrow [\mathbf{D}]_1^{k-1}, [\mathbf{A}]_1^n : k \leq m \}}{\cup \{ \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow [\mathbf{D}]_1^m, [\mathbf{A}]_1^n, \Theta \}}}{\mathcal{G} \mid \Sigma, [\mathbf{C} \preceq \mathbf{D}]_1^m \Rightarrow [\mathbf{A} \preceq \mathbf{B}]_1^n, \Pi \mid \Omega \Rightarrow \Theta} \quad W_{m,n}^{\text{abs}}}{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, C \Rightarrow \Theta \quad \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow D, \Theta} \quad R_C^{\text{abs}} \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow A, \Theta}{\mathcal{G} \mid \Sigma \Rightarrow A \preceq B, \Pi \mid \Omega \Rightarrow \Theta} \quad R_W^{\text{abs}}}{\mathcal{G} \mid \Sigma, C \preceq D \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \quad R_C^{\text{abs}} \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow A, \Theta}{\mathcal{G} \mid \Sigma \Rightarrow A \preceq B, \Pi \mid \Omega \Rightarrow \Theta} \quad R_W^{\text{abs}}}}{\mathcal{G} \mid \Sigma, C \preceq D \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \quad R_C^{\text{abs}} \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow A, \Theta}{\mathcal{G} \mid \Sigma \Rightarrow A \preceq B, \Pi \mid \Omega \Rightarrow \Theta} \quad R_W^{\text{abs}}}}$$

Since these are derivable using the original version of the rule followed by applications of $\text{abs}_L, \text{abs}_R$, cut elimination in the extended system entails cut elimination in the original system. As can be expected, due to the presence of contraction cut elimination in a hypersequent system is rather more involved than in the sequent case of [12]. Moreover, due to the form of the absoluteness rules we cannot simply apply the general results of [11], although the strategy for the cut elimination proof is the same: Intuitively, an application of the cut rule (shown

before Lemma 6) with cut formula of maximal complexity is permuted up in the derivation of the left premiss, where applications of contraction are swallowed up in a more general induction hypothesis, until an occurrence of the cut formula is principal (Lemma 10). Then essentially the fact that contractions can be permuted above logical rules is used to obtain a single occurrence of the cut formula in the left premiss of the cut, and the cut is permuted up in the right premiss. Again, contractions are swallowed up by a generalised induction hypothesis, and once the cut formula becomes principal in the last applied rule, its complexity is reduced (Lemma 9). For technical reasons we also include the rule mrg in the calculus when proving cut elimination. By Lemma 5 it is clear that all applications of this rule can then be eliminated in the cut-free system. In the following we write $\text{H}_{\mathcal{L}}^*$ for the system $\text{H}_{\mathcal{L}}$ with cut, mrg and with the rules $W_{m,n}^{\text{abs}}, R_C^{\text{abs}}, R_W^{\text{abs}}$ where applicable, and abbreviate $\underbrace{A, \dots, A}_{n\text{-times}}$ to A^n .

Definition 7. *The cut rank of a $\text{H}_{\mathcal{L}}^*$ -derivation \mathcal{D} is the maximal complexity of a cut formula occurring in \mathcal{D} , written $\rho(\mathcal{D})$. A rule is cut-rank preserving admissible in $\text{H}_{\mathcal{L}}^*$ if whenever its premiss(es) are derivable in $\text{H}_{\mathcal{L}}^*$ with cut-rank n , then so is its conclusion.*

Lemma 8. *The rules EW, IW are depth- and cut-rank preserving admissible in $\text{H}_{\mathcal{L}}^*$.*

Proof. Standard induction on the depth of the derivation. \square

Lemma 9 (Shift Right). *Suppose that for $k > 0$ and $n_1, \dots, n_k > 0$ there are $\text{H}_{\mathcal{L}}^*$ -derivations \mathcal{D}_1 and \mathcal{D}_2 of $\mathcal{G} \mid \Omega \Rightarrow \Theta, A$ and $\mathcal{H} \mid A^{n_1}, \Xi_1 \Rightarrow \Upsilon_1 \mid \dots \mid A^{n_k}, \Xi_k \Rightarrow \Upsilon_k$ with $\rho(\mathcal{D}_1) < |A| > \rho(\mathcal{D}_2)$ and such that the displayed occurrence of A is principal in the last rule application in \mathcal{D}_1 . Then there is a $\text{H}_{\mathcal{L}}^*$ -derivation \mathcal{D} with endhypersequent $\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi_1 \Rightarrow \Theta, \Upsilon_1 \mid \dots \mid \Omega, \Xi_k \Rightarrow \Theta, \Upsilon_k$ and $\rho(\mathcal{D}) < |A|$.*

Proof. By induction on the depth of \mathcal{D}_2 . If none of the displayed occurrences of A is principal in the last rule in \mathcal{D}_2 , we apply the induction hypothesis on the premiss(es) of that rule, followed by the same rule (and possibly structural rules). If at least one of the displayed occurrences is principal in the last rule in \mathcal{D}_2 , we distinguish cases according to the last applied rule in \mathcal{D}_1 , with subcases according to the last rule in \mathcal{D}_2 . For space reasons we only consider an exemplary case, the remaining cases are similar. Suppose the last rules in \mathcal{D}_1 and \mathcal{D}_2 are $R_{m,n+1}$ and trf_s respectively, that A is the formula $E \preceq F$ and that \mathcal{D}_1 ends in:

$$\frac{\begin{array}{l} \{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid C_j \Rightarrow [\mathbf{D}]_1^{j-1}, [\mathbf{A}]_1^n, E : 1 \leq j \leq m \} \\ \cup \{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid B_j \Rightarrow [\mathbf{D}]_1^m, [\mathbf{A}]_1^n, E : 1 \leq j \leq n \} \cup \{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid F \Rightarrow [\mathbf{D}]_1^m, [\mathbf{A}]_1^n, E \} \end{array}}{\mathcal{G} \mid \Omega, [\mathbf{C} \preceq \mathbf{D}]_1^m \Rightarrow [\mathbf{A} \preceq \mathbf{B}]_1^n, E \preceq F, \Theta} R_{m,n+1}$$

First we apply the induction hypothesis to the conclusion of this and the premisses of trf_s to eliminate all the occurrences of $E \preceq F$ from the context.

Hence we assume that the only occurrences of $E \preccurlyeq F$ in the conclusion of trf_s are principal and that \mathcal{D}_2 ends in:

$$\frac{\begin{array}{l} \{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\mathbf{H}]_1^{j-1} : 1 \leq j \leq r\} \cup \{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid E \Rightarrow [\mathbf{H}]_1^r\} \\ \cup \{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\mathbf{H}]_1^{j-1}, F : r < j \leq s\} \cup \{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\mathbf{H}]_1^s, F\} \end{array}}{\mathcal{H} \mid \Xi, [\mathbf{G} \preccurlyeq \mathbf{H}]_1^r, E \preccurlyeq F, [\mathbf{G} \preccurlyeq \mathbf{H}]_{r+1}^s \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi} \text{trf}_s$$

with $E \preccurlyeq F$ not occurring in $[\mathbf{G} \preccurlyeq \mathbf{H}]_1^r$. Cuts on the formulae E and F then yield:

$$\begin{array}{l} \{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\mathbf{H}]_1^{j-1} : 1 \leq j \leq r\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid C_j \Rightarrow [\mathbf{D}]_1^{j-1}, [\mathbf{A}]_1^n, [\mathbf{H}]_1^r : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid B_j \Rightarrow [\mathbf{D}]_1^m, [\mathbf{A}]_1^n, [\mathbf{H}]_1^r : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\mathbf{H}]_1^{j-1}, [\mathbf{D}]_1^m, [\mathbf{A}]_1^n, [\mathbf{H}]_1^r : r < j \leq s \right\} \end{array}$$

Admissibility of internal weakening (Lemma 8) and an application of $R_{m+s, n+t}$ then gives:

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, [\mathbf{G} \preccurlyeq \mathbf{H}]_1^r, [\mathbf{C} \preccurlyeq \mathbf{D}]_1^m, [\mathbf{G} \preccurlyeq \mathbf{H}]_{r+1}^s \Rightarrow [\mathbf{A} \preccurlyeq \mathbf{B}]_1^n, [\mathbf{I} \preccurlyeq \mathbf{J}]_1^t, \Theta \mid \Xi \Rightarrow \Upsilon$$

Iterating this process to eliminate the remaining occurrences of $E \preccurlyeq F$ from $[\mathbf{G} \preccurlyeq \mathbf{H}]_{r+1}^s$, followed by mrg and applications of contraction then yields the desired sequent. \square

Lemma 10 (Shift Left). *Suppose that for $k > 0$ and $n_1, \dots, n_k > 0$ there are $\mathbf{H}_{\mathcal{L}}^*$ -derivations \mathcal{D}_1 and \mathcal{D}_2 of the hypersequents $\mathcal{G} \mid \Omega_1 \Rightarrow \Theta_1, A^{n_1} \mid \dots \mid \Omega_k \Rightarrow \Theta_k, A^{n_k}$ and $\mathcal{H} \mid A, \Xi \Rightarrow \Upsilon$ with $\rho(\mathcal{D}_1) < |A| > \rho(\mathcal{D}_2)$. Then there is a $\mathbf{H}_{\mathcal{L}}^*$ -derivation \mathcal{D} with endsequent $\mathcal{G} \mid \mathcal{H} \mid \Omega_1, \Xi \Rightarrow \Theta_1, \Upsilon \mid \dots \mid \Omega_k, \Xi \Rightarrow \Theta_k, \Upsilon$ and $\rho(\mathcal{D}) < |A|$.*

Proof. By induction on the depth of \mathcal{D}_1 . If none of the displayed occurrences of A is principal in the last rule in \mathcal{D}_1 or the active formula of abs_R , we apply the induction hypothesis on the premiss(es) of the last rule in \mathcal{D}_1 followed by the same rule and possibly admissibility of weakening and contraction. If one of the occurrences of A is active in abs_R , we use admissibility of EW (Lemma 8) and abs_L on \mathcal{D}_2 to obtain $\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid A \Rightarrow \cdot$. Then the induction hypothesis on this and the premiss of abs_R followed by mrg and IW yields the result. If an occurrence of A is principal in the last rule in \mathcal{D}_1 , we use the induction hypothesis to remove all the occurrences of A in the context of that rule. Then, in case this rule is $R_{m,n}, W_{m,n}, W_{m,n}^{\text{abs}}$, we apply contraction in the premisses and apply the same rule, so that only one occurrence of A is principal. Now Lemma 9 yields the result. \square

Theorem 11 (Cut Elimination). *Let $\mathcal{L} \in \{\text{VTU}, \text{VWU}, \text{VCU}, \text{VTA}, \text{VWA}, \text{VCA}\}$. If a hypersequent is derivable in $\mathbf{H}_{\mathcal{L}}^*$, then it is derivable in $\mathbf{H}_{\mathcal{L}}$.*

Proof. First we eliminate all applications of cut by induction on the tuples $\langle \rho(\mathcal{D}), \sharp(\mathcal{D}) \rangle$ under the lexicographic ordering, where $\sharp(\mathcal{D})$ is the number of applications of cut in \mathcal{D} with cut formula of complexity $\rho(\mathcal{D})$. Then applications of

$W_{m,n}^{\text{abs}}, R_C^{\text{abs}}, R_W^{\text{abs}}$ are replaced with the $W_{m,n}, R_C, R_W$ and $\text{abs}_L, \text{abs}_R$, and mrg is eliminated using Lemma 5. It is straightforward to check that applications of $W_{m,n}^{\text{abs}}, R_C^{\text{abs}}, R_W^{\text{abs}}$ are only introduced in systems including the absoluteness rules. \square

Corollary 12 (Cut-free completeness). *If $A \in \mathcal{L}$, then $\text{H}_{\mathcal{L}} \vdash \Rightarrow A$.*

5 Connections to Modal Logic

The constructed hypersequent calculi provide purely syntactical proofs of results from [13] connecting the conditional logics to, e.g., modal logic S5. We write \mathcal{L}^{\square} for the *modal fragment* of a conditional logic \mathcal{L} , i.e., the fragment where comparative plausibility formulae are restricted to the shape $(\perp \preccurlyeq \neg A)$, and we write A^{\square} for the result of replacing every subformula $\perp \preccurlyeq \neg B$ of A with $\square B$. The proofs use the fact that the hypersequent calculus H_{S5} with the propositional rules of Fig. 1, the structural rules and the rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \square A, \Delta \mid \Rightarrow A}{\mathcal{G} \mid \Gamma \Rightarrow \square A, \Delta} \square_R \quad \frac{\mathcal{G} \mid \Gamma, \square A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \square A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \square_L \quad \frac{\mathcal{G} \mid \Gamma, \square A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \square A \Rightarrow \Delta} \top$$

is cut-free complete for S5 [16], see also [11].

Lemma 13. *If $A^{\square} \in \text{S5}$, then $A \in \mathcal{L}^{\square}$ for each of the logics \mathcal{L} considered here.*

Proof. By translating H_{S5} -derivations into $\text{H}_{\mathcal{L}}$ -derivations. E.g., \square_L is translated into:

$$\frac{\frac{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid \neg \Rightarrow}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \perp_L \quad \frac{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \neg A, \Pi} \neg_L}{\frac{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{ICL}} \text{trf}_1$$

The translations of \square_R, \top are similar, using $R_{0,1}$ and \top_1 respectively. \square

The backwards direction is similar, but translates into the calculus H_{S5} above with a form of Avron's *modal splitting rule* from [2]:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, \square \Omega \Rightarrow \square \Theta, \Pi}{\mathcal{G} \mid \Gamma, \square \Omega \Rightarrow \square \Theta, \Delta \mid \Sigma \Rightarrow \Pi} \text{MS}$$

It is straightforward to check that the resulting calculus is sound for S5.

Lemma 14. *If $\mathcal{L} \neq \text{VCA}$ and $A \in \mathcal{L}^{\square}$, then $A^{\square} \in \text{S5}$.*

Proof. By translating derivations in $\text{H}_{\mathcal{L}}$ into derivations in H_{S5cut} and applying cut elimination. In particular, an application of the rule $R_{m,n}$

$$\frac{\begin{array}{l} \{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \perp \Rightarrow \neg D_1, \dots, \neg D_{j-1}, \perp^n : 1 \leq j \leq m\} \\ \cup \quad \{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \neg B_j \Rightarrow \neg D_1, \dots, \neg D_m, \perp^n : 1 \leq j \leq n\} \end{array}}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg D_1, \dots, \perp \preccurlyeq \neg D_m \Rightarrow \perp \preccurlyeq \neg B_1, \dots, \perp \preccurlyeq \neg B_n, \Delta} R_{m,n}$$

is translated into

$$\frac{\frac{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \neg B_1 \Rightarrow \neg D_1, \dots, \neg D_m, \perp^n}{\mathcal{G} \mid \Gamma, \Box D_1, \dots, \Box D_m \Rightarrow \Box B_1, \Delta \mid \neg B_1 \Rightarrow \neg D_1, \dots, \neg D_m, \perp^n} \text{IW}}{\mathcal{G} \mid \Gamma, \Box D_1, \dots, \Box D_m \Rightarrow \Box B_1, \Delta \mid D_1, \dots, D_m \Rightarrow B_1} \text{prop}}{\frac{\mathcal{G} \mid \Gamma, \Box D_1, \dots, \Box D_m \Rightarrow \Box B_1, \Delta}{\mathcal{G} \mid \Gamma, \Box D_1, \dots, \Box D_m \Rightarrow \Box B_1, \dots, \Box B_n, \Delta} \text{IW}} \Box_L, \Box_R$$

Here *prop* uses derivability of the inversions of the propositional rules using cut. Similarly, applications of T_m and trf_m are translated using m applications of \Box_L and T respectively. Applications of $W_{m,n}$ and R_C are translated by T , and R_W is replaced with weakening, using that whenever $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \perp$ is derivable in the system for $\mathsf{S5}$, then so is $\mathcal{G} \mid \Gamma \Rightarrow \Delta$. Finally, $\mathsf{abs}_L, \mathsf{abs}_R$ are replaced with the modalised splitting rule MS . \square

Theorem 15 [13, Sect. 6.3]. *Let $\mathcal{L} \neq \mathsf{VCA}$. Then $A \in \mathcal{L}^\square$ iff $A^\square \in \mathsf{S5}$.*

The proof of the previous theorem is immediate from the preceding lemmas. It is then also straightforward to derive the known collapses of the counterfactual implication $\Box \rightarrow$ in VWA and VCA . Recall that $A \Box \rightarrow B \equiv (\perp \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq A)$.

- Proposition 16.**
1. $\mathsf{H}_{\mathsf{VWA}} \vdash \Rightarrow (A \Box \rightarrow B) \leftrightarrow \Box(A \rightarrow B)$
 2. $\mathsf{H}_{\mathsf{VCA}} \vdash \Rightarrow A \leftrightarrow \Box A$
 3. $\mathsf{H}_{\mathsf{VCA}} \vdash \Rightarrow (A \Box \rightarrow B) \leftrightarrow (A \rightarrow B)$ and $\mathsf{H}_{\mathsf{VCA}} \vdash \Rightarrow (A \preccurlyeq B) \leftrightarrow (B \rightarrow A)$.

6 Standard Calculi

To convert the non-standard calculi $\mathsf{H}_{\mathcal{L}}$ into standard calculi, we consider an extended notion of sequents, where the succedent contains additional structural connectives. These sequents extend those of [7, 15] with a connective $\langle \cdot \rangle$ interpreting possible formulae.

Definition 17. *A conditional block is a tuple $[\Sigma \triangleleft C]$ containing a multiset Σ of formulae and a single formula C . A transfer block is a multiset of formulae, written $\langle \Theta \rangle$. An extended sequent is a tuple $\Gamma \Rightarrow \Delta$ consisting of a multiset Γ of formulae and a multiset Δ containing formulae, conditional blocks, and transfer blocks. An extended hypersequent is a multiset containing extended sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$.*

The *formula interpretation* of an extended sequent is (all blocks shown explicitly):

$$\begin{aligned} \iota_e(\Gamma \Rightarrow \Delta, [\Sigma_1 \triangleleft C_1], \dots, [\Sigma_n \triangleleft C_n], \langle \Theta_1 \rangle, \dots, \langle \Theta_m \rangle) \\ := \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigvee_{i=1}^n \bigvee_{B \in \Sigma_i} (B \preccurlyeq C_i) \vee \bigvee_{j=1}^m \diamond (\bigvee \Theta_j) \end{aligned}$$

The *formula interpretation* of an extended hypersequent is given by

$$\iota_e(\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n) := \Box \iota_e(\Gamma_1 \Rightarrow \Delta_1) \vee \dots \vee \Box \iota_e(\Gamma_n \Rightarrow \Delta_n)$$

The rules of the non-invertible calculi for VTU and extensions are given in Fig. 3.

$$\begin{array}{c}
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \leq B} \leq_R \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [B, \Sigma \triangleleft C] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft C]}{\mathcal{G} \mid \Gamma, A \leq B \Rightarrow \Delta, [\Sigma \triangleleft C]} \leq_L \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \triangleleft A] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \triangleleft A], [\Sigma_2 \triangleleft B]} \text{com} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{jump} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Theta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta, B \rangle}{\mathcal{G} \mid \Gamma, A \leq B \Rightarrow \Delta, \langle \Theta \rangle} \top \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \perp \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{in}_{\text{trf}} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Theta, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi} \text{jump}_U \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{jump}_T \\
\frac{}{\mathcal{G} \mid \Gamma, \perp \Rightarrow \Delta} \perp_L \quad \frac{}{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta, p} \text{init} \quad \frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \text{ICL} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \text{ICR} \\
\frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow_L \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow_R \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], [\Sigma \triangleleft A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{Con}_S \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A, A \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A \triangleleft B]} \text{Con}_B \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{W} \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma, A \leq B \Rightarrow \Delta} \text{C} \quad \frac{\mathcal{G} \mid \Sigma, \Gamma \Rightarrow \Pi, \Delta \mid \Omega \Rightarrow \Theta}{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, \Gamma \Rightarrow \Theta, \Delta} \text{spl} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, A \leq B \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, A \leq B \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_L \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow A \leq B, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow A \leq B, \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_R \\
\text{SH}_{\text{VTU}} = \{\perp_L, \text{init}, \rightarrow_L, \rightarrow_R, \text{ICL}, \text{ICR}, \text{Con}_S, \text{Con}_B\} \cup \{\leq_R, \leq_L, \text{com}, \text{jump}, \top, \text{in}_{\text{trf}}, \text{jump}_U, \text{jump}_T\} \\
\text{SH}_{\text{VWU}} = \text{SH}_{\text{VTU}} \cup \{\text{W}\} \quad \text{SH}_{\text{VCU}} = \text{SH}_{\text{VWU}} \cup \{\text{C}\} \quad \text{SH}_{\text{VTA}} = \text{SH}_{\text{VTU}} \cup \{\text{abs}_L, \text{abs}_R\} \\
\text{SH}_{\text{VWA}} = \text{SH}_{\text{VWU}} \cup \{\text{abs}_L, \text{abs}_R\} \quad \text{SH}_{\text{VCA}} = \text{SH}_{\text{VCU}} \cup \{\text{abs}_L, \text{abs}_R, \text{spl}\}
\end{array}$$

Fig. 3. The non-invertible standard calculi for extensions of VTU

Theorem 18 (Soundness). *If $\text{SH}_{\mathcal{L}} \vdash \mathcal{G}$, then $\vdash_{\mathcal{L}} \iota_e(\mathcal{G})$, and if $\text{SH}_{\mathcal{L}} \vdash \Rightarrow A$, then $A \in \mathcal{L}$.*

Proof. As for Lemma 3, by showing that the rules preserve validity under ι_e and using validity of $\Box A \rightarrow A$. For the rules $\leq_L, \leq_R, \text{com}, \text{jump}, \text{W}, \text{C}$ this is similar as in [7]. For rule \top , if the interpretation of the conclusion is falsified in \mathfrak{M}, w , then there is a world $v \in \text{SP}(w)$ with $\mathfrak{M}, v \Vdash \bigwedge \Gamma \wedge (A \leq B) \wedge \neg \bigvee \Delta \wedge \Box \neg \bigvee \Theta$. If $\llbracket B \rrbracket = \emptyset$, then in particular $\mathfrak{M}, v \Vdash \Box \neg (\bigvee \Theta \vee B)$, and the formula interpretation of the second premiss is falsified in \mathfrak{M}, w . Otherwise, from $\mathfrak{M}, v \Vdash A \leq B$ we obtain a world $x \in \bigcup \text{SP}(v) = \bigcup \text{SP}(w)$ with $\mathfrak{M}, x \Vdash A$, and from $\mathfrak{M}, v \Vdash \Box \neg \bigvee \Theta$ we also get that $\mathfrak{M}, x \Vdash \neg \bigvee \Theta$. Hence the formula interpretation of the first premiss is falsified at \mathfrak{M}, w . The remaining cases are similar. \square

Theorem 19 (Completeness). *If $A \in \mathcal{L}$ then $\text{SH}_{\mathcal{L}} \vdash \Rightarrow A$.*

Proof. By simulating derivations in $\text{H}_{\mathcal{L}}$. Most of the rules are simulated as in [7], except for the rules trf_m, \top_m . For \top_m the derivation is given in Fig. 4. The derivation of trf_m only replaces jump_T with jump_U . \square

$$\begin{array}{c}
\frac{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C_m \Rightarrow [D]_1^{m-1}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C_m \Rightarrow [D]_1^{m-1}, \perp} \text{IW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow [D]_1^m, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow [D]_1^m, \perp, \Delta} \text{IW}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle [D]_1^m, \perp \rangle} \text{jump}_T}{\mathcal{G} \mid \Gamma, C_m \leq D_m \Rightarrow \Delta, \langle [D]_1^{m-1}, \perp \rangle} \text{T} \\
\frac{\frac{\mathcal{G} \mid \Gamma, [C \leq D]_2^m \Rightarrow \Delta \mid C_1 \Rightarrow}{\mathcal{G} \mid \Gamma, [C \leq D]_2^m \Rightarrow \Delta \mid C_1 \Rightarrow \perp} \text{IW} \quad \frac{\vdots}{\mathcal{G} \mid \Gamma, [C \leq D]_2^m \Rightarrow \Delta, \langle D_1, \perp \rangle} \text{T}}{\frac{\mathcal{G} \mid \Gamma, [C \leq D]_1^m \Rightarrow \Delta, \langle \perp \rangle}{\mathcal{G} \mid \Gamma, [C \leq D]_1^m \Rightarrow \Delta} \text{in}_{\text{trf}}} \text{T}
\end{array}$$

Fig. 4. The derivation of T_m in SH_{VTU} .

7 Semantic Completeness via Invertible Calculi

An alternative completeness proof for the logics without absoluteness is given semantically by constructing a countermodel from a failed proof search. For this we consider the invertible versions $\text{SH}_{\mathcal{L}}^i$ of the calculi from Sect. 6, given in Fig. 5. Equivalence with the non-invertible calculi follows from admissibility of the structural rules, including the ones below, the proofs of which are standard by induction on the depth of the derivation:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft C]} \text{CW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft C]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A \triangleleft C]} \text{CIW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{TW}$$

Lemma 20. *The rules IW, EW, CW, CIW, TW are admissible in $\text{SH}_{\mathcal{L}}$.*

Lemma 21. *The rules ICL, ICR, Con_B , Con_S , mrg are admissible in $\text{SH}_{\mathcal{L}}^i$.*

From Lemmas 20 and 21 it immediately follows that:

Proposition 22. *The invertible and non-invertible calculi are equivalent.*

Definition 23. *An extended hypersequent \mathcal{G} is VTU -saturated if it satisfies all of the following conditions:*

1. (\leq_R) if $\Gamma \Rightarrow \Delta, A \leq B \in \mathcal{G}$, then $[\Sigma, A \triangleleft B] \in \Delta$ for some Σ ;
2. (\leq_L) if $\Gamma, C \leq D \Rightarrow \Delta, [\Sigma \triangleleft A] \in \mathcal{G}$, then $D \in \Sigma$ or $[\Sigma \triangleleft C] \in \Delta$;
3. (com) if $\Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], [\Pi \triangleleft B] \in \mathcal{G}$, then $\Sigma \subseteq \Pi$ or $\Pi \subseteq \Sigma$;
4. (jump) if $\Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \in \mathcal{G}$, then $A, \Theta \Rightarrow \Sigma, \Pi \in \mathcal{G}$ for some Θ, Π ;
5. (T) if $\Gamma, C \leq D \Rightarrow \Delta, \langle \Theta \rangle \in \mathcal{G}$, then $D \in \Theta$ or $C, \Sigma \Rightarrow \Theta, \Pi \in \mathcal{G}$ for some Σ, Π ;
6. (in_{trf}) if $\Gamma \Rightarrow \Delta \in \mathcal{G}$, then $\langle \Theta \rangle \in \Delta$ for some Θ ;
7. ($\text{jump}_U, \text{jump}_T$) if $\Gamma \Rightarrow \Delta, \langle \Theta \rangle \in \mathcal{G}$ and $\Sigma \Rightarrow \Pi \in \mathcal{G}$, then $\Theta \subseteq \Pi$;
8. (\rightarrow_L) if $\Gamma, A \rightarrow B \Rightarrow \Delta \in \mathcal{G}$, then $B \in \Gamma$ or $A \in \Delta$;
9. (\rightarrow_R) if $\Gamma \Rightarrow A \rightarrow B, \Delta \in \mathcal{G}$, then $A \in \Gamma$ and $B \in \Delta$;

$$\begin{array}{c}
\frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi, A \leq B, [A \triangleleft B]}{\mathcal{G} \mid \Sigma \Rightarrow \Pi, A \leq B} \leq_R^i \quad \frac{\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma \triangleleft A] \mid A \Rightarrow \Sigma}{\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma \triangleleft A]} \text{jump}^i \\
\frac{\mathcal{G} \mid \Omega, C \leq D \Rightarrow \Theta, [D, \Sigma \triangleleft A]}{\mathcal{G} \mid \Omega, C \leq D \Rightarrow \Theta, [\Sigma \triangleleft A]} \quad \frac{\mathcal{G} \mid \Omega, C \leq D \Rightarrow \Theta, [\Sigma \triangleleft A], [\Sigma \triangleleft C]}{\mathcal{G} \mid \Omega, C \leq D \Rightarrow \Theta, [\Sigma \triangleleft A]} \leq_L^i \\
\frac{\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma_1, \Sigma_2 \triangleleft A], [\Sigma_2 \triangleleft B]}{\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma_1 \triangleleft A], [\Sigma_1, \Sigma_2 \triangleleft B]} \text{com}^i \\
\frac{\mathcal{G} \mid \Sigma, A \leq B \Rightarrow \Pi, \langle \Theta \rangle \mid A \Rightarrow \Theta \quad \mathcal{G} \mid \Sigma, A \leq B \Rightarrow \Pi, \langle \Theta, B \rangle}{\mathcal{G} \mid \Sigma, A \leq B \Rightarrow \Pi, \langle \Theta \rangle} \top^i \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \perp \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{in}_{\text{lf}}^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Theta, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi} \text{jump}_U^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{jump}_T^i \\
\frac{\mathcal{G} \mid \Omega, \perp \Rightarrow \Theta}{\mathcal{G} \mid \Omega, \perp \Rightarrow \Theta} \perp_L \quad \frac{\mathcal{G} \mid \Omega, p \Rightarrow \Theta, p}{\mathcal{G} \mid \Omega, p \Rightarrow \Theta, p} \text{init} \\
\frac{\mathcal{G} \mid \Omega, A \rightarrow B, B \Rightarrow \Theta \quad \mathcal{G} \mid \Omega, A \rightarrow B \Rightarrow \Theta, A}{\mathcal{G} \mid \Omega, A \rightarrow B \Rightarrow \Theta} \rightarrow_L^i \quad \frac{\mathcal{G} \mid \Omega, A \Rightarrow \Theta, A \rightarrow B, B}{\mathcal{G} \mid \Omega \Rightarrow \Theta, A \rightarrow B} \rightarrow_R^i \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} W^i \quad \frac{\mathcal{G} \mid \Gamma, C \leq D, C \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, C \leq D \Rightarrow D, \Delta}{\mathcal{G} \mid \Gamma, C \leq D \Rightarrow \Delta} C^i \\
\frac{\mathcal{G} \mid \Gamma, A \leq B \Rightarrow \Delta \mid \Sigma, A \leq B \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, A \leq B \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_L^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow A \leq B, \Delta \mid \Sigma \Rightarrow A \leq B, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow A \leq B, \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_R^i \\
\text{SH}_{\text{VTU}}^i = \{\perp_L^i, \text{init}^i, \rightarrow_L^i, \rightarrow_R^i\} \cup \{\leq_R^i, \leq_L^i, \text{com}^i, \text{jump}^i, \top^i, \text{in}_{\text{lf}}^i, \text{jump}_U^i, \text{jump}_T^i\} \\
\text{SH}_{\text{VWU}}^i = \text{SH}_{\text{VTU}}^i \cup \{W^i\} \quad \text{SH}_{\text{VCU}}^i = \text{SH}_{\text{VWU}}^i \cup \{C^i\} \quad \text{SH}_{\text{VTA}}^i = \text{SH}_{\text{VTU}}^i \cup \{\text{abs}_L^i, \text{abs}_R^i\} \\
\text{SH}_{\text{VWA}}^i = \text{SH}_{\text{VWU}}^i \cup \{\text{abs}_L^i, \text{abs}_R^i\} \quad \text{SH}_{\text{VCA}}^i = \text{SH}_{\text{VCU}}^i \cup \{\text{abs}_L^i, \text{abs}_R^i\}
\end{array}$$

Fig. 5. The invertible standard calculi for extensions of VTU

It is VWU-saturated (resp. VCU-saturated) if it also satisfies (W) (resp. (C)) below:

1. (W) if $\Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \in \mathcal{G}$, then $\Sigma \subseteq \Delta$;
2. (C) if $\Gamma, C \leq D \Rightarrow \Delta \in \mathcal{G}$, then $C \in \Gamma$ or $D \in \Delta$;

A VTU-saturated extended hypersequent \mathcal{G} is called *unprovable* if it is not an instance of (init) or (\perp_L). We construct a countermodel from an unprovable VTU-saturated extended hypersequent $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ as follows:

- $W := \{1, \dots, n\}$
- $\llbracket p \rrbracket := \{i \leq n : p \in \Gamma_i\}$

The sphere systems $\text{SP}(i)$ for $i \leq n$ are then defined as follows: Assume that $\Gamma_i \Rightarrow \Delta_i$ is

$$\Gamma_i \Rightarrow \Delta'_i, [\Sigma_1 \triangleleft A_1], \dots, [\Sigma_k \triangleleft A_k]$$

where Δ'_i contains no conditional blocks. First observe that due to saturation condition 3 we may assume w.l.o.g. that $\Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_k$. Moreover, by condition 4 for every $j \leq k$ there is a component $\Gamma_{m_j} \Rightarrow \Delta_{m_j} \in \mathcal{G}$ with $A_j \in \Gamma_{m_j}$ and $\Sigma_j \subseteq \Delta_{m_j}$. Hence we set

$$\text{SP}(i) := \{\{m_k\}, \{m_k, m_{k-1}\}, \dots, \{m_k, \dots, m_1\}, W\}$$

Call the resulting structure $\mathfrak{M}_{\mathcal{G}}$.

Lemma 24. For a VTU -saturated hypersequent \mathcal{G} the structure $\mathfrak{M}_{\mathcal{G}}$ is a VTU -model.

Proof. Nesting of spheres is obvious from the fact that $\{m_k\} \subseteq \{m_k, m_{k-1}\} \subseteq \dots \subseteq \{m_k, \dots, m_1\} \subseteq W$; reflexivity and uniformity follow from the fact that $W \in \text{SP}(i)$. \square

Lemma 25. Let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be a VTU -saturated hypersequent and let $\mathfrak{M}_{\mathcal{G}}$ be define as above with world i associated to component $\Gamma_i \Rightarrow \Delta_i$. Then:

1. given a formula A , if $A \in \Gamma_i$ then $\mathfrak{M}_{\mathcal{G}}, i \Vdash A$
2. given a formula A , if $A \in \Delta_i$ then $\mathfrak{M}_{\mathcal{G}}, i \nVdash A$
3. given a block $[\Sigma \triangleleft C]$, if $[\Sigma \triangleleft C] \in \Delta_i$, then $\mathfrak{M}_{\mathcal{G}}, i \nVdash \bigvee_{B \in \Sigma} (B \preceq C)$
4. given a formula B , if $\langle \Theta, B \rangle \in \Delta_i$ for some Θ , then $\mathfrak{M}_{\mathcal{G}}, i \nVdash \Diamond B$

Proof. We prove statements 1 and 2 by mutual induction on the complexity of A . The base case and the propositional case are straightforward, hence we consider $A = E \preceq F$. Let $i \in W$ be associated to $\Gamma_i \Rightarrow \Delta_i$ with $\Delta_i = \Delta'_i, [\Sigma_1 \triangleleft D_1], \dots, [\Sigma_k \triangleleft D_k], \langle \Theta \rangle$, where Δ'_i contains no conditional block and $\Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_k$.

- Suppose $E \preceq F \in \Gamma_i$. For $\alpha \in \text{SP}(i)$, we have to show that $\alpha \Vdash^{\forall} \neg F$ or $\alpha \Vdash^{\exists} E$.

In case $\alpha \neq W$ we have $\alpha = \{m_k, \dots, m_j\}$ for some $j \leq k$ and each $m_\ell \in \alpha$ comes from a block $[\Sigma_\ell \triangleleft D_\ell]$ and is associated to a component $D_\ell, \Lambda_\ell \Rightarrow \Pi_\ell, \Sigma_\ell$ of \mathcal{G} . By saturation condition (\preceq_L), either $F \in \Sigma_j$ or $E = D_j$. In the former case with $\Sigma_j \subseteq \Sigma_{j+1} \subseteq \dots \subseteq \Sigma_k$ and the induction hypothesis we have $\mathfrak{M}_{\mathcal{G}}, m_\ell \nVdash F$, for $\ell = j, \dots, k$, showing that $\alpha \Vdash^{\forall} \neg F$. If $E = D_j$, by induction hypothesis on the component $E, \Lambda_j \Rightarrow \Pi_j, \Sigma_j$, we get $\mathfrak{M}_{\mathcal{G}}, m_j \Vdash E$, showing $\alpha \Vdash^{\exists} E$.

In case $\alpha = W$, by saturation condition (T) either $F \in \langle \Theta \rangle$, or $E, \Lambda \Rightarrow \Pi, \Theta \in \mathcal{G}$ for some Λ, Π . In the latter case for the world j associated to the component $E, \Lambda \Rightarrow \Pi, \Theta$ by induction hypothesis on E we get $\mathfrak{M}_{\mathcal{G}}, j \Vdash E$, whence $W \Vdash^{\exists} E$. In the former case we have $F \in \langle \Theta \rangle$. Any $k \in W$ (including $k=i$) is associated to a component $\Gamma_k \Rightarrow \Delta_k$, but by saturation condition ($\text{jump}_T, \text{jump}_U$) we have $\Theta \subseteq \Delta_k$, whence $F \in \Delta_k$; by induction hypothesis on F we have $\mathfrak{M}_{\mathcal{G}}, k \nVdash F$, showing $W \Vdash^{\forall} \neg F$.

- Suppose $E \preceq F \in \Delta_i$. Recall that $\text{SP}(i) = \{\{m_k\}, \{m_k, m_{k-1}\}, \dots, \{m_k, \dots, m_1\}, W\}$ with each m_ℓ associated to a sequent $D_\ell, \Lambda_\ell \Rightarrow \Pi_\ell, \Sigma_\ell \in \mathcal{G}$ coming from a block $[\Sigma_\ell \triangleleft D_\ell] \in \Delta_i$, for $\ell = j, \dots, k$. By saturation, there is $j \leq k$ with $D_j = F$ and $E \in \Sigma_j$. Consider m_j associated to the component $F, \Lambda_j \Rightarrow \Sigma_j, \Pi$. By induction hypothesis we get $\mathfrak{M}_{\mathcal{G}}, m_j \Vdash F$. Since $\Sigma_j \subseteq \Sigma_{j+1} \subseteq \dots \subseteq \Sigma_k$, we also get $\mathfrak{M}_{\mathcal{G}}, m_\ell \nVdash E$, for $\ell = j, \dots, k$. Thus for $\alpha = \{m_k, \dots, m_j\} \in \text{SP}(i)$ we get $\alpha \nVdash^{\forall} \neg F$ and $\alpha \nVdash^{\exists} E$, showing $\mathfrak{M}_{\mathcal{G}}, i \nVdash E \preceq F$.

The proof of 3 uses 2, recalling that a block is a disjunction of \preceq -formulas. The proof of 4 uses 2 with an argument as in the proof of 1 for the case of $\alpha = W$ with $B \in (\Theta)$. \square

The countermodel construction described above can be extended to $\mathbb{V}\mathbb{W}\mathbb{U}$ and $\mathbb{V}\mathbb{C}\mathbb{U}$ by modifying the definition of the model as follows. For $\mathbb{V}\mathbb{W}\mathbb{U}$, let $\text{SP}(i) := \{\{m_k, i\}, \{m_k, m_{k-1}, i\}, \dots, \{m_k, \dots, m_1, i\}, W\}$. For $\mathbb{V}\mathbb{C}\mathbb{U}$, we add $\{i\}$ to $\text{SP}(i)$ for any i . The proof of Lemma 25 can be easily extended to both cases (statements 1 and 2), using the specific saturation conditions for these systems. We leave the details to the reader; the case of Absoluteness will be handled in future work. From Lemma 25 we obtain:

Lemma 26. *For $\mathcal{L} \in \{\mathbb{V}\mathbb{T}\mathbb{U}, \mathbb{V}\mathbb{W}\mathbb{U}, \mathbb{V}\mathbb{C}\mathbb{U}\}$ let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be a \mathcal{L} -saturated hypersequent and let $\mathfrak{M}_{\mathcal{G}}$ be defined as above, then*

- for any $i \in W$ associated to sequent $\Gamma_i \Rightarrow \Delta_i$ we have $\mathfrak{M}_{\mathcal{G}, i} \not\models \iota_e(\Gamma_i \Rightarrow \Delta_i)$
- for any $i \in W$ we have $\mathfrak{M}_{\mathcal{G}, i} \not\models \iota_e(\mathcal{G})$

To use these results in a decision procedure, we consider *local loop checking*: rules are not applied if there is a premiss from which the conclusion is derivable using structural rules. Since these are all admissible in $\text{SH}_{\mathcal{L}}^i$, this does not jeopardise completeness.

Proposition 27. *Backwards proof search with local loop checking terminates and every leaf of the resulting derivation is an axiom or a saturated sequent.*

Proof. By Lemmas 20 and 21, we may assume that the proof search only considers *duplication-free* sequents, i.e., sequents containing duplicates neither of formulae nor of blocks. By the subformula property, the number of duplication-free sequents possibly relevant to a derivation of a sequent is bounded in the number of subformulae of that sequent, and hence backwards proof search for \mathcal{G} terminates. Furthermore, every leaf is either an axiom or a saturated sequent, since otherwise another rule could be applied. \square

Theorem 28 (Completeness). *If $\iota_e(\mathcal{G}) \in \mathcal{L}$, then $\text{SH}_{\mathcal{L}} \vdash \mathcal{G}$ for $\mathcal{L} \in \{\mathbb{V}\mathbb{T}\mathbb{U}, \mathbb{V}\mathbb{W}\mathbb{U}, \mathbb{V}\mathbb{C}\mathbb{U}\}$.*

Proof. By Proposition 27 backwards proof search with root \mathcal{G} terminates and every leaf of it is an axiom or a saturated sequent. By invertibility of the rules each sequent \mathcal{G}' occurring as a leaf is valid. But then \mathcal{G}' must be an axiom, since otherwise, by Lemma 26 we can build a countermodel $\mathfrak{M}_{\mathcal{G}'}$ falsifying $\iota_e(\mathcal{G}')$ and hence by monotonicity also $\iota_e(\mathcal{G})$. \square

We note that Proposition 27 gives rise to a (non-optimal) co-NEXPTIME -decision procedure for validity: Since applying backwards proof search with local loop checking to an input sequent $\Rightarrow G$ terminates and every leaf of the resulting derivation is an instance of init or \perp_L or a saturated sequent, in order to check whether $\Rightarrow G$ is derivable it suffices to non-deterministically choose a

duplication-free \mathcal{L} -saturated extended hypersequent containing only subformulas of G and containing a component $\Gamma \Rightarrow \Delta, G$. If this is not possible, then backwards proof search will produce a proof of $\Rightarrow G$. But if it is possible, then by Lemma 26 this hypersequent gives rise to a countermodel for G . Since the size of duplication-free extended hypersequents consisting of subformulae of G is bounded exponentially in the number of subformulae of G , this gives the CO-NEXPTIME complexity bound. Of course it is known that the logics of this section are EXPTIME-complete [5].

8 Conclusion

In this work we have introduced to our knowledge the first internal hypersequent calculi for Lewis' conditional logics with uniformity and reflexivity, both in non-standard and in standard form. While the former lend themselves to syntactic cut elimination, the latter are amenable to a semantic completeness proof via countermodel construction from a failed proof search and give rise to decision procedures for the considered logics.

While the treatment of these logics is an important step towards a comprehensive proof-theoretic treatment of the whole family of Lewis' logics, many interesting questions are still open. In particular, we plan to extend the semantic completeness proof also to the logics with absoluteness. Further, by moving to the framework of *grafted hypersequents* [10] we expect to be able to extend our results to the logics $\forall\mathbb{U}$ and $\forall\mathbb{N}\mathbb{U}$. Concerning Lewis' conditional logics, this would leave only the logics satisfying *Stalnaker's assumption* [13] lacking a satisfactory internal proof system. Their proof-theoretic investigation will be subject of future research. Finally, we aim at providing complexity-optimal proof methods for the logics under consideration. In particular, for logics with absoluteness, one could make the blocks "global" to the whole hypersequent. We conjecture that such calculi could yield complexity-optimal decision procedures.

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