

Equilibrium Fluctuations for the Slow Boundary Exclusion Process

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Abstract We prove that the equilibrium density fluctuations of the symmetric simple exclusion process in contact with slow boundaries is given by an Ornstein–Uhlenbeck process with Dirichlet, Robin or Neumann boundary conditions depending on the range of the parameter that rules the slowness of the boundaries.

Keywords Equilibrium density fluctuations · Exclusion process
Slow boundaries · Ornstein–Uhlenbeck process

1 Introduction

The study of nonequilibrium behavior of interacting particle systems is one of the most challenging problems in the field and it has only been completely solved in very particular cases. The toy model for the study of a system in a nonequilibrium scenario is the symmetric simple exclusion process (SSEP) whose dynamics is rather simple to explain and it already captures many features of more complicated systems.

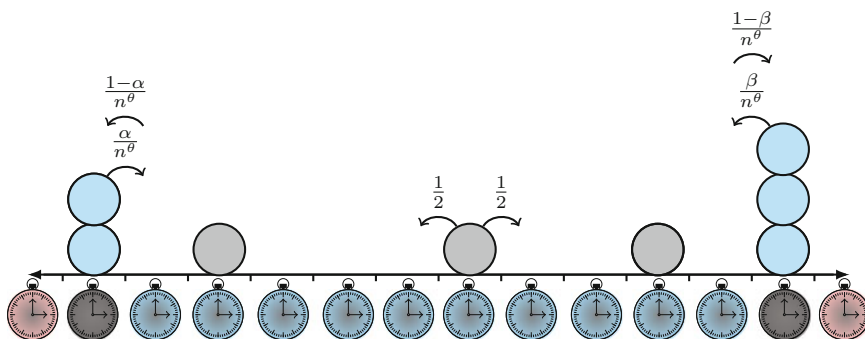
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The dynamics of this model can be described as follows. We fix a scaling parameter n and we consider the SSEP evolving on the discrete space $\Sigma_n = \{1, \dots, n - 1\}$ to which we call the bulk. To each pair of bonds $\{x, x + 1\}$ with $x = 1, \dots, n - 2$ we associate a Poisson process $N_{x,x+1}(t)$ of rate 1. Now we artificially add two end points at the bulk, namely, we add the sites $x = 0$ and $x = n$ and we superpose the exclusion dynamics with a Glauber dynamics which has only effect at the boundary points of the bulk, namely at the sites $x = 1$ and $x = n - 1$. For that purpose, we add extra Poisson processes at the bonds $\{0, 1\}$ and $\{n - 1, n\}$. In each one of these bonds there are two Poisson processes: $N_{0,1}(t)$ with parameter $\alpha n^{-\theta}$, $N_{1,0}(t)$ with parameter $(1 - \alpha)n^{-\theta}$, $N_{n-1,n}(t)$ with parameter $\beta n^{-\theta}$ and $N_{n,n-1}(t)$ with parameter $(1 - \beta)n^{-\theta}$. All the Poisson processes are independent. Above $\alpha, \beta \in (0, 1)$ and $\theta \geq 0$ is a parameter that rules the slowness of the boundary dynamics. Below in the figure we colored the Poisson clocks associated to the bonds in the bulk in the blue color, while the Poisson clocks associated to the bonds at the boundary are colored in the gray and pink colors, to emphasize that they have different rates.

Now that the clocks are fixed we can explain the dynamics. For that purpose, initially we place particles in the bulk according to some probability measure and we denote this configuration of particles and holes by $\eta = (\eta(1), \dots, \eta(n - 1))$, so that for $x \in \Sigma_n$, $\eta(x) = 1$ if there is a particle at the site x and $\eta(x) = 0$ if the site x is empty. Now, if a clock rings for a bond $\{x, x + 1\}$ in the bulk, then we exchange the coordinates x and $x + 1$ of η , that is we exchange $\eta(x)$ with $\eta(x + 1)$ at rate 1. If the clock rings for the bond at the boundary as, for example, from the Poisson process $N_{0,1}(t)$ then a particle gets into the bulk through the site 1 at rate $\alpha n^{-\theta}$ if and only if there is no particle at the site 1, otherwise nothing happens. If the clock rings from the Poisson process $N_{1,0}(t)$ and there is a particle at the site 1, then it exits the bulk from the site 1 at rate $(1 - \alpha)n^{-\theta}$. Note that the higher the value of θ the slower is the dynamics at the boundaries. For a display of the description above, see the figure below.



The dynamics just described is Markovian and can be completely characterized in terms of its infinitesimal generator given below in (1). We note that the space state of this Markov process is $\Omega_n := \{0, 1\}^{\Sigma_n}$. Observe that the bulk dynamics preserves the number of particles and our interest is to describe the space-time evolution of

this conserved quantity as a solution of some partial differential equation called the *hydrodynamic equation*.

Note that for the choice $\alpha = \beta = \rho$ a simple computation shows that the Bernoulli product measure of parameter ρ given by: $\nu_\rho^n(\eta \in \Omega_n : \eta(x) = 1) = \rho$ is invariant under the dynamics. For this choice of the parameters the boundary reservoirs have the same intensity and we do not see any induced current on the system. Nevertheless, in the case $\alpha \neq \beta$, let us say for example $\alpha < \beta$, there is a tendency to have more particles entering into the bulk from the right reservoir and leaving the system from the left reservoir. This is a current which is induced by the difference of the density at the boundary reservoirs. Note that in the bulk the dynamics is symmetric. In the case $\alpha \neq \beta$, since we have a finite state Markov process, there is only one stationary measure that we denote by μ_n^{ss} which is no longer a product measure as in the case $\alpha = \beta$. By using the matrix ansatz method developed by [3, 10, 11] and references therein, it is possible to compute the correlation function in the stationary state and an important problem is to analyze the behavior of the system starting from this non-equilibrium stationary state.

We note that the hydrodynamic limit of this model was studied in [1] and the hydrodynamic equations consist in the heat equation with different types of boundary conditions depending on the range of the parameter θ . More precisely, for $0 \leq \theta < 1$ the heat equation has Dirichlet boundary conditions which fix the value of the density profile at the points 0 and 1 to be α and β , respectively. In this case we do not see any difference at the macroscopic level with respect to the case $\theta = 0$. Nevertheless, for $\theta = 1$ the boundary dynamics is slowed enough in such a way that macroscopically the Dirichlet boundary conditions are replaced by a type of Robin boundary conditions. These Robin boundary conditions state that the rate at which particles are injected into the system through the boundary points, is given by the difference of the density at the bulk and the boundary. Finally for $\theta > 1$, the boundaries are sufficiently slowed so that the Robin boundary conditions are replaced by Neumann boundary conditions stating that macroscopically there is no flux of particles from the boundary reservoirs.

We emphasize that there are many similar models to the one studied in these notes which we summarize as follows. In [7–9], the authors consider a model where removal of particles can only occur at an interval around the left boundary and the entrance of particles is allowed only at an interval around the right boundary. Their model presents a current exchange between the two reservoirs and shows some similarities with our model for the choice $\theta = 1$. Another case already studied in the literature (see [12, 19]) is when the boundary is not slowed, that corresponds to our model for the choice $\theta = 0$. As mentioned above, the hydrodynamic equation of this model has Dirichlet boundary conditions, see [12] or the Eq.(7). A similar model, whose hydrodynamic equation has both Dirichlet boundary conditions and Neumann boundary conditions, was studied in [6]. The main difference, at the macroscopic level, is that the end points of the boundary conditions vary with time. The microscopic dynamics there is given by the SSEP evolving on \mathbb{Z} with additional births and deaths restricted to a subset of configurations where there is a leftmost hole and a rightmost particle. In this situation, at a fixed rate j birth of particles

occur at the position of the leftmost hole and at the same rate, independently, the rightmost particle dies. Another model which has a current is considered in [4]. The dynamics evolves on the discrete torus $\mathbb{Z}/n\mathbb{Z}$ without reservoirs, but has a surprising phenomenon: a “battery effect”. This effect produces a current of particles through the system and is due to a single abnormal bond, where the rates to cross from left to right and from right to left are different. Finally, another model which has similarities with the model we consider in these notes is the SSEP with a slow bond, which was studied in [13–15]. The dynamics evolves on the discrete torus $\mathbb{Z}/n\mathbb{Z}$, and particles exchange positions between nearest neighbor bonds at rate 1, except at one particular bond, where the exchange occurs at rate $n^{-\beta}$. In this case $\beta > 0$ is a parameter that rules the slowness of the bond and for that reason the bond is called the *slow bond*. The similarity between the slow bond model and the slow boundary model considered in these notes is that if we “open” the discrete torus exactly at the position of the slow bond, then the slow bond gives rise to a slow boundary. In [13, 15] different hydrodynamic behaviors were obtained, depending on the range of the parameter β , more precisely, the hydrodynamic equation is, in all cases, the heat equation but the boundary conditions vary with the value of β , exhibiting three different regimes as for the slow boundary model, see [1].

Our interest in these notes is to go beyond the hydrodynamical behavior, analyzing the fluctuations around the hydrodynamical profile. To accomplish this, we restrict ourselves to the case $\alpha = \beta = \rho$ and starting from the stationary measure ν_ρ^n defined above. Our result states that the fluctuations starting from ν_ρ^n are given by an Ornstein–Uhlenbeck process solution of

$$d\mathcal{Y}_t = \Delta_\theta \mathcal{Y}_t dt + \sqrt{2\chi(\rho)t} \nabla_\theta d\mathcal{W}_t,$$

where $\chi(\rho)$ is the variance of $\eta(x)$ with respect to ν_ρ^n , \mathcal{W}_t is a space-time white noise of unit variance and Δ_θ and ∇_θ are, respectively, the Laplacian and derivative operators defined on a space of test functions with different types of boundary conditions depending on the value of θ . We note that the case $\theta = 0$ was studied in [19] and the case $\theta = 1$ was studied in [16]. In those articles, the nonequilibrium fluctuations were obtained starting from general initial measures, which include the equilibrium case ν_ρ^n treated here. We note however, that the case $\theta \neq 1$ is quite difficult to attack at the nonequilibrium scenario since we need to establish a local replacement (see Lemma 3) in order to close the martingale problem, which we can only prove starting the system from the equilibrium state. In a future work, we will dedicate to extending this result to the nonequilibrium situation as, for example, starting the system from the steady state when $\alpha \neq \beta$.

Here follows an outline of these notes. In Sect. 2 we give the definition of the model, we recall from [1] the hydrodynamic limit and in Sect. 3 we state our main result, namely, Theorem 3. In Sect. 4 we characterize the limit process by means of a martingale problem. Tightness is proved in Sect. 5 and in Sect. 6 we prove the Replacement Lemma which is the most technical part of these notes.

2 Statement of Results

2.1 The Model

For $n \geq 1$, we denote by Σ_n the set $\{1, \dots, n-1\}$, which will be referred by the expression *bulk*. The symmetric simple exclusion process with slow boundaries is a Markov process $\{\eta_t : t \geq 0\}$ with state space $\Omega_n := \{0, 1\}^{\Sigma_n}$. The slowness of the boundaries is ruled by a parameter that we denote by $\theta \geq 0$. If η is a configuration of the state space Ω , then for $x \in \Sigma_n$, the random variable $\eta(x)$ can take only two values, namely 0 or 1. If $\eta(x) = 0$, it means that the site x is vacant, while $\eta(x) = 1$ means that the site x is occupied. The dynamics of this model can be described as follows. In the bulk particles move according to continuous time random walks, but whenever a particle wants to jump to an occupied site, the jump is suppressed. At the left boundary, particles can be created (resp. removed) at rate $\alpha n^{-\theta}$ (resp. $(1-\alpha)n^{-\theta}$). At the right boundary, particles can be created (resp. removed) at rate $\beta n^{-\theta}$ (resp. $(1-\beta)n^{-\theta}$).

Fix now a finite time horizon T . The Markov process $\{\eta_t(x) : x \in \Sigma_n; t \in [0, T]\}$ can be characterized in terms of its infinitesimal generator that we denote by \mathcal{L}_n^θ and is defined as follows. For a function $f : \Omega_n \rightarrow \mathbb{R}$, we have that

$$\begin{aligned} (\mathcal{L}_n^\theta f)(\eta) &= \left[\frac{\alpha}{n^\theta} (1 - \eta(1)) + \frac{(1-\alpha)}{n^\theta} \eta(1) \right] (f(\eta^1) - f(\eta)) \\ &+ \left[\frac{\beta}{n^\theta} (1 - \eta(n-1)) + \frac{(1-\beta)}{n^\theta} \eta(n-1) \right] (f(\eta^{n-1}) - f(\eta)) \quad (1) \\ &+ \sum_{x=1}^{n-2} (f(\sigma^{x,x+1}\eta) - f(\eta)) , \end{aligned}$$

where $\sigma^{x,x+1}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$, that is,

$$(\sigma^{x,x+1}\eta)(y) = \begin{cases} \eta(x+1), & \text{if } y = x, \\ \eta(x), & \text{if } y = x+1, \\ \eta(y), & \text{otherwise.} \end{cases} \quad (2)$$

and for $x = 1, n-1$ η^x is the configuration obtained from η by flipping the occupation variable $\eta(x)$:

$$(\eta^x)(y) = \begin{cases} 1 - \eta(y), & \text{if } y = x, \\ \eta(y), & \text{otherwise.} \end{cases} \quad (3)$$

Let $\mathcal{D}([0, T], \Omega_n)$ be the space of trajectories which are right continuous and with left limits, taking values in Ω_n . Denote by $\mathbb{P}_{\mu_n}^{\theta,n}$ the probability on $\mathcal{D}([0, T], \Omega_n)$

induced by the Markov process with generator $n^2 \mathcal{L}_n^\theta$ and the initial measure μ_n and denote by $\mathbb{E}_{\mu_n}^{\theta, n}$ the expectation with respect to $\mathbb{P}_{\mu_n}^{\theta, n}$.

2.2 Stationary Measures

The stationary measure μ_n^{ss} for this model when $\alpha = \beta = \rho \in (0, 1)$ is the Bernoulli product measure given by

$$\nu_\rho^n \left(\eta \in \Omega_n : \eta(x) = 1 \right) = \rho.$$

But in the general case, where $\alpha \neq \beta$, the stationary measure μ_n^{ss} does not have independent marginals, see [10]. What we can say about the stationary behavior of this model is that the density of particles has a behavior very close to a linear profile, which depends on the range of θ in the sense of the following definition:

Definition 1 Let $\gamma : [0, 1] \rightarrow [0, 1]$ be a measurable profile. A sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is said to be *associated* to γ if, for any $\delta > 0$ and any continuous function $f : [0, 1] \rightarrow \mathbb{R}$ the following limit holds:

$$\lim_{n \rightarrow \infty} \mu_n \left(\eta : \left| \frac{1}{n} \sum_{x=1}^{n-1} f\left(\frac{x}{n}\right) \eta(x) - \int f(u) \gamma(u) du \right| > \delta \right) = 0.$$

For μ_n equal to the stationary measure μ_n^{ss} , the limit above is called the *hydrostatic limit*.

Theorem 1 (Hydrostatic Limit, [1]) *Let μ_n^{ss} be the stationary probability measure in Ω_n wrt the Markov process with infinitesimal generator $n^2 \mathcal{L}_n^\theta$, defined in (1). The sequence $\{\mu_n^{ss}\}_{n \in \mathbb{N}}$ is associated (in the sense of Definition 1) to the profile $\bar{\rho} : [0, 1] \rightarrow \mathbb{R}$ given by*

$$\bar{\rho}(u) = \begin{cases} (\beta - \alpha)u + \alpha, & \text{if } \theta \in [0, 1), \\ \frac{\beta - \alpha}{3}u + \alpha + \frac{\beta - \alpha}{3}, & \text{if } \theta = 1, \\ \frac{\beta + \alpha}{2}, & \text{if } \theta \in (1, \infty), \end{cases} \tag{4}$$

for all $u \in [0, 1]$.

Another feature that we can say about the stationary state of the model studied in this paper is that the profiles in (4) are very close to the mean of $\eta(x)$ taken with respect to the stationary measure μ_n^{ss} . To state this result properly, we start by defining for an initial measure μ_n in Ω_n , for $x \in \Sigma_n$ and for $t \geq 0$ the empirical mean given by

$$\rho_t^n(x) := \mathbb{E}_{\mu_n}^{\theta, n}[\eta_t(x)]. \tag{5}$$

If in the expression above $\mu_n = \mu_n^{ss}$, then $\rho_t^n(x)$ does not depend on t , so that $\rho_t^n(x) = \rho^n(x)$. From [1], we have that $\rho^n(x)$ satisfies the following recurrence relations:

$$\begin{cases} 0 = [\rho^n(x + 1) - \rho^n(x)] + [\rho^n(x - 1) - \rho^n(x)], & \text{if } x \in \{2, \dots, n-2\}, \\ 0 = [\rho^n(2) - \rho^n(1)] + n^{-\theta}[\alpha - \rho^n(1)], \\ 0 = n^{-\theta}[\beta - \rho^n(n-1)] + [\rho^n(n-2) - \rho^n(n-1)]. \end{cases}$$

A simple computation shows that $\rho^n(x)$ is given by $\rho^n(x) = a_n x + b_n$, for all $x \in \Sigma_n$, where $a_n = \frac{\beta - \alpha}{2n^\theta + n - 2}$ and $b_n = \alpha + a_n(n^\theta - 1)$. Moreover, we conclude that

$$\lim_{n \rightarrow \infty} \left(\max_{x \in \Sigma_n} \left| \rho^n(x) - \bar{\rho}\left(\frac{x}{n}\right) \right| \right) = 0.$$

2.3 Hydrodynamic Limit

In [1] it was established the hydrodynamic limit of the model for any $\theta \geq 0$. For completeness we recall that result now. Fix a measurable density profile $\rho_0 : [0, 1] \rightarrow [0, 1]$ and for each $n \in \mathbb{N}$, let μ_n be a probability measure on Ω_n .

Theorem 2 (Hydrodynamic Limit, [1]) *Suppose that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is associated to a profile $\rho_0(\cdot)$ in the sense of Definition 1. Then, for each $t \in [0, T]$, for any $\delta > 0$ and any continuous function $f : [0, 1] \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow +\infty} \mathbb{P}_{\mu_n}^{\theta, n} \left[\eta : \left| \frac{1}{n} \sum_{x=1}^{n-1} f\left(\frac{x}{n}\right) \eta_{tn^2}(x) - \int f(u) \rho(t, u) du \right| > \delta \right] = 0,$$

where $\rho(t, \cdot)$ is the unique weak solution of the heat equation

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & \text{for } t > 0, u \in (0, 1), \\ \rho(0, u) = \rho_0(u), & u \in [0, 1]. \end{cases} \tag{6}$$

with boundary conditions that depend on the range of θ , which are given by:

$$\text{For } \theta < 1, \partial_u \rho(t, 0) = \alpha \text{ and } \partial_u \rho(t, 1) = \beta, \text{ for } t > 0. \tag{7}$$

$$\text{For } \theta = 1, \partial_u \rho(t, 0) = \rho(t, 0) - \alpha \text{ and } \partial_u \rho(t, 1) = \beta - \rho(t, 1), \text{ for } t > 0. \tag{8}$$

$$\text{For } \theta > 1, \partial_u \rho(t, 0) = \partial_u \rho(t, 1) = 0, \text{ for } t > 0. \tag{9}$$

Remark 1 We note that the profiles in (4) are stationary solutions of the heat equation with the corresponding boundary conditions given above.

3 Density Fluctuations

3.1 The Space of Test Functions

The space $C^\infty([0, 1])$ is the space of functions $f : [0, 1] \rightarrow \mathbb{R}$ such that f is continuous in $[0, 1]$ as well as all its derivatives.

Definition 2 Let \mathcal{S}_θ denote the set of functions $f \in C^\infty([0, 1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that

- (1) for $\theta < 1$: $\partial_u^{2k} f(0) = \partial_u^{2k} f(1) = 0$.
- (2) for $\theta = 1$: $\partial_u^{2k+1} f(0) = \partial_u^{2k} f(0)$ and $\partial_u^{2k+1} f(1) = -\partial_u^{2k} f(1)$.
- (3) for $\theta > 1$: $\partial_u^{2k+1} f(0) = \partial_u^{2k+1} f(1) = 0$.

Definition 3 For $\theta \geq 0$, let $-\Delta_\theta$ be the positive operator, self-adjoint on $L^2[0, 1]$, defined on $f \in \mathcal{S}_\theta$ by

$$\Delta_\theta f(u) = \begin{cases} \partial_u^2 f(u), & \text{if } u \in (0, 1), \\ \partial_u^2 f(0^+), & \text{if } u = 0, \\ \partial_u^2 f(1^-), & \text{if } u = 1. \end{cases} \tag{10}$$

Above, $\partial_u^2 f(a^\pm)$ denotes the side limits at the point a . Analogously, let $\nabla_\theta : \mathcal{S}_\theta \rightarrow C^\infty([0, 1])$ be the operator given by

$$\nabla_\theta f(u) = \begin{cases} \partial_u f(u), & \text{if } u \in (0, 1), \\ \partial_u f(0^+), & \text{if } u = 0, \\ \partial_u f(1^-), & \text{if } u = 1. \end{cases} \tag{11}$$

Definition 4 Let $T_t^\theta : \mathcal{S}_\theta \rightarrow \mathcal{S}_\theta$ be the semigroup associated to (6) with the corresponding boundary conditions for the case $\alpha = \beta = 0$. That is, given $f \in \mathcal{S}_\theta$, by $T_t^\theta f$ we mean the solution of the homogeneous version of (6) with initial condition f .

Definition 5 Let \mathcal{S}'_θ be the topological dual of \mathcal{S}_θ with respect to the topology generated by the seminorms

$$\|f\|_k = \sup_{u \in [0, 1]} |\partial_u^k f(u)|, \tag{12}$$

where $k \in \mathbb{N} \cup \{0\}$. In other words, \mathcal{S}'_θ consists of all linear functionals $f : \mathcal{S}_\theta \rightarrow \mathbb{R}$ which are continuous with respect to all the seminorms $\|\cdot\|_k$.

Let $\mathcal{D}([0, T], \mathcal{S}'_\theta)$ (resp. $\mathcal{C}([0, T], \mathcal{S}'_\theta)$) be the space of trajectories which are right continuous and with left limits (resp. continuous), taking values in \mathcal{S}'_θ .

The expression for T_t^θ , $\theta \geq 0$, is presented in the next proposition:

Proposition 1 *Let $\theta \geq 0$. Suppose that $\rho_0 \in L^2[0, 1]$. Then*

$$(T_t^\theta \rho_0)(u) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \Psi_n(u), \tag{13}$$

where $\{\Psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2[0, 1]$ constituted by eigenfunctions of the associated Regular Sturm-Liouville Problem (concerning the operator Δ_θ) and a_n are the Fourier coefficients of ρ_0 in the basis $\{\Psi_n\}_{n \in \mathbb{N}}$.

- For $\theta < 1$, the corresponding orthonormal basis of $L^2[0, 1]$ is

$$\begin{cases} \Psi_n(u) = \sqrt{2} \sin(n\pi u), & \text{for } n \in \mathbb{N}, \\ \Psi_0(u) \equiv 1. \end{cases}$$

The eigenvalues of the associated Regular Sturm-Liouville Problem (concerning the operator Δ_θ) are given by $\lambda_n = n^2\pi^2$.

- For $\theta = 1$, the corresponding orthonormal basis of $L^2[0, 1]$ is a linear combination of sines and cosines, namely,

$$\Psi_n(u) = A_n \sin(\sqrt{\lambda_n}u) + A_n \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}u), \text{ for } n \in \mathbb{N} \cup \{0\},$$

where A_n is a normalizing constant. The eigenvalues λ_n do not have an explicit formula, but it can be verified that $\lambda_n \sim n^2\pi^2$.

- For $\theta > 1$, the corresponding orthonormal basis of $L^2[0, 1]$ is

$$\begin{cases} \Psi_n(u) = \sqrt{2} \cos(n\pi u), & \text{for } n \in \mathbb{N}, \\ \Psi_0(u) \equiv 1. \end{cases}$$

The eigenvalues of the associated Regular Sturm-Liouville Problem (concerning the operator Δ_θ) are given by $\lambda_n = n^2\pi^2$.

Proof For $\theta = 1$ the expression for T_t^θ has been obtained in [16]. For the case $\theta \neq 1$, as in [16], we state the associated Regular Sturm-Liouville Problem (for details on this subject we refer to [2], for instance):

$$\text{For } \theta < 1 : \begin{cases} \Psi''(u) + \lambda\Psi(u) = 0, & u \in (0, 1), \\ \Psi(0) = 0, \quad \Psi(1) = 0; \end{cases}$$

$$\text{For } \theta \geq 1 : \begin{cases} \Psi''(u) + \lambda\Psi(u) = 0, & u \in (0, 1), \\ \Psi'(0) = 0, \quad \Psi'(1) = 0. \end{cases}$$

The solution of each one of the problems above (the eigenvalues λ_n and the eigenfunctions Ψ_n) can be found in Chap. 10 of [5].

As a consequence, the series (13) converges exponentially fast, implying that $(T_t^\theta \rho_0)(u)$ is smooth in space and time for any $t > 0$. This observation implies a property of $T_t^\theta : \mathcal{S}_\theta \rightarrow \mathcal{S}_\theta$ stated in the next corollary.

Corollary 1 *If $f \in \mathcal{S}_\theta$, then for any $t > 0$, $T_t^\theta f \in \mathcal{S}_\theta$ and $\Delta_\theta T_t^\theta f \in \mathcal{S}_\theta$.*

We observe that the previous result is needed in the proof of uniqueness of the corresponding Ornstein–Uhlenbeck process (which is defined in the next section). Its proof is a consequence of the formula (13), see [16] for more details.

3.2 Ornstein–Uhlenbeck Process

Fix $\rho \in (0, 1)$. Based on [17, 18], we give here a characterization of the generalized Ornstein–Uhlenbeck process which is a solution of

$$d\mathcal{Y}_t = \Delta_\theta \mathcal{Y}_t dt + \sqrt{2\chi(\rho)t} \nabla_\theta d\mathcal{W}_t, \tag{14}$$

where \mathcal{W}_t is a space-time white noise of unit variance and $\chi(\rho) = \int (\eta(x) - \rho)^2 d\nu_\rho^n = \rho(1 - \rho)$, in terms of a martingale problem. We will see below that this process governs the equilibrium fluctuations of the density of particles of our model. In spite of having a dependence of \mathcal{Y}_t on θ , we do not index on it to not overload notation. Denote by \mathbf{Q}_ρ^θ the distribution of \mathcal{Y} . and $\mathbf{E}_{\mathbf{Q}_\rho^\theta}$ the expectation with respect to \mathbf{Q}_ρ^θ .

Define the inner product between the functions $f, g : [0, 1] \rightarrow \mathbb{R}$ by

$$\langle f, g \rangle_{L_\rho^{2,\theta}} = 2\chi(\rho) \left[\int_0^1 f(u) g(u) du + \left(f(0)g(0) + f(1)g(1) \right) \mathbf{1}_{\theta=1} \right],$$

where $\mathbf{1}$. is the indicator function. Then, $L_\rho^{2,\theta}([0, 1])$ is the space of functions $f : [0, 1] \rightarrow \mathbb{R}$ with $\|f\|_{L_\rho^{2,\theta}} < \infty$, where

$$\|f\|_{L_\rho^{2,\theta}}^2 = \langle f, f \rangle_{L_\rho^{2,\theta}}. \tag{15}$$

Proposition 2 *There exists an unique random element \mathcal{Y} taking values in the space $\mathcal{C}([0, T], \mathcal{S}'_\theta)$ such that:*

- (i) *For every function $f \in \mathcal{S}_\theta$, $\mathcal{M}_t(f)$ and $\mathcal{N}_t(f)$ given by*

$$\begin{aligned} \mathcal{M}_t(f) &= \mathcal{Y}_t(f) - \mathcal{Y}_0(f) - \int_0^t \mathcal{Y}_s(\Delta_\theta f) ds, \\ \mathcal{N}_t(f) &= (\mathcal{M}_t(f))^2 - 2\chi(\rho) t \|\nabla_\theta f\|_{L_\rho^{2,\theta}}^2 \end{aligned} \tag{16}$$

are \mathcal{F}_t -martingales, where for each $t \in [0, T]$, $\mathcal{F}_t := \sigma(\mathcal{Y}_s(f); s \leq t, f \in \mathcal{S}_\theta)$.
(ii) \mathcal{Y}_0 is a Gaussian field of mean zero and covariance given on $f, g \in \mathcal{S}_\theta$ by

$$\mathbf{E}_{\mathbf{Q}_\rho^\theta}[\mathcal{Y}_0(f)\mathcal{Y}_0(g)] = \langle f, g \rangle_{L_\rho^{2,\theta}} \tag{17}$$

Moreover, for each $f \in \mathcal{S}_\theta$, the stochastic process $\{\mathcal{Y}_t(f); t \geq 0\}$ is Gaussian, being the distribution of $\mathcal{Y}_t(f)$ conditionally to \mathcal{F}_s , for $s < t$, normal of mean $\mathcal{Y}_s(T_{t-s}^\theta f)$ and variance $\int_0^{t-s} \|\nabla_\theta T_r^\theta f\|_{L_\rho^{2,\theta}}^2 dr$, where T_t^θ was given in Definition 4.

The random element \mathcal{Y} is called the generalized Ornstein–Uhlenbeck process of characteristics Δ_θ and ∇_θ . From the second equation in (16) and Lévy’s Theorem on the martingale characterization of Brownian motion, for each $f \in \mathcal{S}_\theta$, the process

$$\mathcal{M}_t(f) \left(2\chi(\rho)t \|\nabla_\theta f\|_{L_\rho^{2,\theta}}^2 \right)^{-1/2} \tag{18}$$

is a standard Brownian motion. Therefore, in view of Proposition 2, it makes sense to say that \mathcal{Y} is the formal solution of (14).

3.3 The Density Fluctuation Field

We define the density fluctuation field \mathcal{Y}^n as time-trajectory of the linear functional acting on functions $f \in \mathcal{S}_\theta$ as

$$\mathcal{Y}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} f\left(\frac{x}{n}\right) \left(\eta_{tm^2}(x) - \rho_t^n(x) \right), \quad \text{for all } t \geq 0, \tag{19}$$

where ρ_t^n was defined in (5). Our results are given for the case $\alpha = \beta = \rho$ and for μ_n being equal to ν_ρ^n , that is, the Bernoulli product measure with parameter $\rho \in (0, 1)$, so that $\rho_t^n(x) = \rho$, for all $x \in \Sigma_n$ and $t \geq 0$. Let $\mathbb{Q}_\rho^{\theta,n}$ be the probability measure on $\mathcal{D}([0, T], \mathcal{S}_\theta)$ induced by the density fluctuation field \mathcal{Y}^n and ν_ρ^n . We note that since we will consider only the initial measure μ_n as ν_ρ^n , we will simplify the notations $\mathbb{P}_{\nu_\rho^n}^{\theta,n}$ and $\mathbb{E}_{\nu_\rho^n}^{\theta,n}$ as $\mathbb{P}_\rho^{\theta,n}$ and $\mathbb{E}_\rho^{\theta,n}$, respectively. Our main result is the following theorem.

Theorem 3 (Ornstein–Uhlenbeck limit) *For $\alpha = \beta = \rho \in (0, 1)$, if we take the initial measure to be ν_ρ^n , namely, the Bernoulli product measure with parameter ρ , then, the sequence $\{\mathbb{Q}_\rho^{\theta,n}\}_{n \in \mathbb{N}}$ converges, as $n \rightarrow \infty$, to a generalized Ornstein–Uhlenbeck*

(O.U.) process, which is the formal solution of equation (14). As a consequence, the variance of the limit field \mathcal{Y}_t is given on $f \in \mathcal{S}_\theta$ by

$$\mathbf{E}_{\mathbf{Q}_\rho^\theta} [\mathcal{Y}_t(f)\mathcal{Y}_s(f)] = \chi(\rho) \int_0^1 (f(u))^2 du + \int_0^s \|\mathbf{T}_{t-r}^\theta f\|_{L_\rho^{2,\theta}}^2 dr, \quad (20)$$

where $\|\cdot\|_{L_\rho^{2,\theta}}$ was defined in (15).

4 Proof of Theorem 3

4.1 Characterization of Limit Points

Fix a test function f . By Dynkin’s formula, we have that

$$M_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0(f) - \int_0^t (\partial_s + n^2 \mathcal{L}_n^\theta) \mathcal{Y}_s^n(f) ds, \quad (21)$$

$$N_t^n(f) = (M_t^n(f))^2 - \int_0^t n^2 \mathcal{L}_n^\theta \mathcal{Y}_s^n(f)^2 - 2\mathcal{Y}_s^n(f) n^2 \mathcal{L}_n^\theta \mathcal{Y}_s^n(f) ds \quad (22)$$

are martingales with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : s \leq t)$. To simplify notation we denote $\Gamma_s^n(f) := (\partial_s + n^2 \mathcal{L}_n^\theta) \mathcal{Y}_s^n(f)$. A long but elementary computation shows that

$$\begin{aligned} \Gamma_s^n(f) &= \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} \Delta_n f\left(\frac{x}{n}\right) (\eta_s(x) - \rho) \\ &\quad + \sqrt{n} \nabla_n^+ f(0) (\eta_s(1) - \rho) - \sqrt{n} \nabla_n^- f(n) (\eta_s(n-1) - \rho) \\ &\quad - \frac{n^{3/2}}{n^\theta} f\left(\frac{1}{n}\right) (\eta_s(1) - \rho) - \frac{n^{3/2}}{n^\theta} f\left(\frac{n-1}{n}\right) (\eta_s(n-1) - \rho). \end{aligned} \quad (23)$$

Above

$$\Delta_n f(x) := n^2 \left[f\left(\frac{x+1}{n}\right) + f\left(\frac{x-1}{n}\right) - 2f\left(\frac{x}{n}\right) \right],$$

$$\nabla_n^+ f(x) := n \left[f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right) \right]$$

and

$$\nabla_n^- f(x) := n \left[f\left(\frac{x}{n}\right) - f\left(\frac{x-1}{n}\right) \right].$$

We note that for the choice $\theta = 0$, using the fact that $f(0) = 0 = f(1)$, the expression (23) reduces to

$$\Gamma_s^n(f) = \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} \Delta_n f\left(\frac{x}{n}\right) (\eta_s(x) - \rho), \tag{24}$$

which is $\mathcal{Y}_s^n(\Delta_n f)$.

Now, we close the Eq. (23) for each regime of θ . The goal is to show that we can rewrite (23) as (24) plus a term which vanishes as $n \rightarrow \infty$.

- The case $\theta < 1$: we note that since $f \in \mathcal{S}_\theta$ we can write $\Gamma_s^n(f)$ as

$$\mathcal{Y}_s^n(\Delta_n f) + \sqrt{n}(1 - n^{-\theta}) \left\{ \nabla_n^+ f(0)(\eta_s(1) - \rho) - \nabla_n^- f(n)(\eta_s(n - 1) - \rho) \right\}.$$

In order to close the equation for the martingale we need to show that:

$$\lim_{n \rightarrow \infty} \mathbb{E}_\rho^{\theta, n} \left[\left(\int_0^t \sqrt{n}(\eta_s(x) - \rho) ds \right)^2 \right] = 0, \quad \text{for } x = 1, n - 1, \tag{25}$$

which is a consequence of Lemma 3, see Remark 2.

- The case $\theta = 1$: we can write $\Gamma_s^n(f)$ as

$$\begin{aligned} &\mathcal{Y}_s^n(\Delta_n f) + \sqrt{n}(\partial_u f(0) - f(0))(\eta_s(1) - \rho) \\ &+ \sqrt{n}(\partial_u f(1) + f(1))(\eta_s(n - 1) - \rho) + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

Since $f \in \mathcal{S}_\theta$ the last expression equals to $\mathcal{Y}_s^n(\Delta_n f)$.

- The case $\theta > 1$: we can repeat the computations above and since $f \in \mathcal{S}_\theta$, $\Gamma_s^n(f)$ can be rewritten as

$$\mathcal{Y}_s^n(\Delta_n f) - \frac{n^{3/2}}{n^\theta} f\left(\frac{1}{n}\right) (\eta_s(1) - \rho) - \frac{n^{3/2}}{n^\theta} f\left(\frac{n-1}{n}\right) (\eta_s(n - 1) - \rho) + O\left(\frac{1}{\sqrt{n}}\right).$$

Then, in order to close the equation for the martingale term we need to show that

$$\lim_{n \rightarrow \infty} \mathbb{E}_\rho^{\theta, n} \left[\left(\int_0^t \frac{n^{3/2}}{n^\theta} (\eta_s(x) - \rho) ds \right)^2 \right] = 0, \quad \text{for } x = 1, n - 1, \tag{26}$$

which is a consequence of Lemma 3, see Remark 2.

From the previous observations, for each regime of θ we can rewrite (23) as (24) plus a negligible term.

Lemma 1 For all $\theta \geq 0$, $t > 0$ and $f \in \mathcal{S}_\theta$ it holds that

$$\lim_{n \rightarrow \infty} \mathbb{E}_\rho^{\theta, n} [|M_t^n(f)|^2] = t \|\nabla_\theta f\|_{L_\rho^{2, \theta}},$$

where the norm above was defined in (15).

Proof A simple computation shows that the integral part of the martingale $N_t^n(f)$ can be written as

$$\begin{aligned} n^2 \mathcal{L}_n^\theta \mathcal{Y}_s^n(f)^2 - 2 \mathcal{Y}_s^n(f) n^2 \mathcal{L}_n^\theta \mathcal{Y}_s^n(f) &= \frac{1}{n} \sum_{x=1}^{n-2} \left(\nabla_n^+ f\left(\frac{x}{n}\right) \right)^2 \left(\eta_s(x) - \eta_s(x+1) \right)^2 \\ &\quad + \frac{n}{n^\theta} \left(f\left(\frac{1}{n}\right) \right)^2 \left(\rho - 2\rho\eta_s(1) + \eta_s(1) \right) \\ &\quad + \frac{n}{n^\theta} \left(f\left(\frac{n-1}{n}\right) \right)^2 \left(\rho - 2\rho\eta_s(n-1) + \eta_s(n-1) \right), \end{aligned}$$

from where we get that

$$\begin{aligned} &\mathbb{E}_\rho^{\theta, n} [|M_t^n(f)|^2] \\ &= 2\chi(\rho) t \left\{ \frac{1}{n} \sum_{x=1}^{n-2} \left(\nabla_n^+ f\left(\frac{x}{n}\right) \right)^2 + \frac{n}{n^\theta} \left(\left(f\left(\frac{1}{n}\right) \right)^2 + \left(f\left(\frac{n-1}{n}\right) \right)^2 \right) \right\}. \end{aligned} \tag{27}$$

Let $f \in \mathcal{S}_\theta$. The first term at the right hand side of the previous expression converges to $2\chi(\rho) \int_0^1 \left(\nabla_\theta f(u) \right)^2 du$, for all $\theta \geq 0$. The second term at the right hand side of last expression has to be analyzed for each case of θ separately:

- The case $\theta < 1$: since $f(0) = 0 = f(1)$, the second term at the right hand side of (27) can be rewritten as $2\chi(\rho) t$ times

$$\frac{n}{n^\theta} \left(\left(f\left(\frac{1}{n}\right) \right)^2 + \left(f\left(\frac{n-1}{n}\right) \right)^2 \right) = \frac{1}{n^{1+\theta}} \left(\left(\nabla_n^+ f(0) \right)^2 + \left(\nabla_n^- f(n) \right)^2 \right),$$

which goes to zero as $n \rightarrow \infty$.

- The case $\theta = 1$: the second term at the right hand side of (27) converges, as $n \rightarrow \infty$, to

$$2\chi(\rho) t \left(f^2(0) + f^2(1) \right).$$

Recalling that $f(0) = \partial_u f(0)$ and $f(1) = -\partial_u f(1)$, the proof ends.

- The case $\theta > 1$: since $f \in \mathcal{S}_\theta$ and $\frac{n}{n^\theta} \rightarrow 0$, as $n \rightarrow \infty$, the second term at the right hand side of (27) converges to zero when $n \rightarrow \infty$.

We have just proved that the quadratic variation of the martingale converges in mean. In the next Lemma we state the stronger convergence of the martingales to a Brownian motion.

Lemma 2 For $f \in \mathcal{S}_\theta$, the sequence of martingales $\{M_t^n(f); t \in [0, T]\}_{n \in \mathbb{N}}$ converges in the topology of $\mathcal{D}([0, T], \mathbb{R})$, as $n \rightarrow \infty$, towards a Brownian motion $\mathcal{W}_t(f)$ of quadratic variation given by $t \|\nabla_\theta f\|_{L_\rho^{2,\theta}}$ where $\|\cdot\|_{L_\rho^{2,\theta}}$ was defined in (15).

Proof We can repeat here the same proof of [14, p. 4170], which is based on Lemma 1 and the fact that a limit in distribution of a uniformly integrable sequence of martingales is a martingale. We leave the details to the interested reader.

4.2 Convergence at Initial Time

Proposition 3 The sequence $\{\mathcal{Y}_0^n\}_{n \in \mathbb{N}}$ converges in distribution to \mathcal{Y}_0 , where \mathcal{Y}_0 is a Gaussian field with mean zero and covariance given by (17).

Proof We first claim that, for every $f \in \mathcal{S}_\theta$ and every $t > 0$,

$$\lim_{n \rightarrow +\infty} \log \mathbb{E}_\rho^{\theta,n} \left[\exp\{i \lambda \mathcal{Y}_0^n(f)\} \right] = -\frac{\lambda^2}{2} \chi(\rho) \int_0^1 f^2(u) du .$$

Since ν_ρ^n is a Bernoulli product measure,

$$\begin{aligned} \log \mathbb{E}_\rho^{\theta,n} [\exp\{i \lambda \mathcal{Y}_0^n(f)\}] &= \log \int \left[\exp \left\{ \frac{i \lambda}{\sqrt{n}} \sum_{x \in \Sigma_n} (\eta_0(x) - \rho) f \left(\frac{x}{n} \right) \right\} \right] d\nu_\rho^n \\ &= \sum_{x \in \Sigma_n} \log \int \left[\exp \left\{ \frac{i \lambda}{\sqrt{n}} (\eta_0(x) - \rho) f \left(\frac{x}{n} \right) \right\} \right] d\nu_\rho^n . \end{aligned}$$

Since f is smooth and using Taylor’s expansion, the right hand side of last expression is equal to

$$-\frac{\lambda^2}{2n} \sum_{x \in \Sigma_n} f^2 \left(\frac{x}{n} \right) \chi(\rho) + O \left(\frac{1}{\sqrt{n}} \right) .$$

Taking the limit as $n \rightarrow +\infty$ and using the continuity of f , the proof of the claim ends. Replacing f by a linear combination of functions and recalling the Cramér-Wold device, the proof finishes.

5 Tightness

Now we prove that the sequence of processes $\{\mathcal{Y}_t^n; t \in [0, T]\}_{n \in \mathbb{N}}$ is tight. Recall that we have defined the density fluctuation field on test functions $f \in \mathcal{S}_\theta$. Since we want to use Mitoma's criterium [20] for tightness, we need the following property from the space \mathcal{S}_θ .

Proposition 4 *The space \mathcal{S}_θ endowed with the semi-norms given in (12) is a Fréchet space.*

Proof The definition of a Fréchet space can be found, for instance, in [21]. Since $C^\infty([0, 1])$ endowed with the semi-norms (12) is a Fréchet space, and a closed subspace of a Fréchet space is also a Fréchet space, it is enough to show that \mathcal{S}_θ is a closed subspace of $C^\infty([0, 1])$, which is a consequence of the fact that uniform convergence implies point-wise convergence.

As a consequence of Mitoma's criterium [20] and Proposition 4, the proof of tightness of the \mathcal{S}'_θ valued processes $\{\mathcal{Y}_t^n; t \in [0, T]\}_{n \in \mathbb{N}}$ follows from tightness of the sequence of real-valued processes $\{\mathcal{Y}_t^n(f); t \in [0, T]\}_{n \in \mathbb{N}}$, for $f \in \mathcal{S}_\theta$.

Proposition 5 (Mitoma's criterium, [20]) *A sequence of processes $\{x_t; t \in [0, T]\}_{n \in \mathbb{N}}$ in $\mathcal{D}([0, T], \mathcal{S}'_\theta)$ is tight with respect to the Skorohod topology if, and only if, the sequence $\{x_t(f); t \in [0, T]\}_{n \in \mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$, for any $f \in \mathcal{S}_\theta$.*

Now, to show tightness of the real-valued process we use the Aldous' criterium:

Proposition 6 *A sequence $\{x_t; t \in [0, T]\}_{n \in \mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$ if:*

- (i) $\lim_{A \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{P}_{\mu_n} \left(\sup_{0 \leq t \leq T} |x_t| > A \right) = 0,$
- (ii) for any $\varepsilon > 0$, $\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} \sup_{\lambda \leq \delta} \sup_{\tau \in \mathcal{T}_T} \mathbb{P}_{\mu_n} (|x_{\tau+\lambda} - x_\tau| > \varepsilon) = 0,$

where \mathcal{T}_T is the set of stopping times bounded by T .

Fix $f \in \mathcal{S}_\theta$. By (21), it is enough to prove tightness of $\{\mathcal{Y}_0^n(f)\}_{n \in \mathbb{N}}$, $\{\int_0^t \Gamma_s^n(f) ds; t \in [0, T]\}_{n \in \mathbb{N}}$, and $\{\mathcal{M}_t^n(f); t \in [0, T]\}_{n \in \mathbb{N}}$.

5.1 Tightness at the Initial Time

This follows from Proposition 3.

5.2 Tightness of the Martingales

By Lemma 2, the sequence of martingales converges. In particular, it is tight.

5.3 Tightness of the Integral Terms

The first claim of Aldous' criterium can be easily checked for the integral term $\int_0^t \Gamma_s^n(f) ds$, where the expression for $\Gamma_s^n(f)$ can be found in (23). Let $f \in \mathcal{S}_\theta$.

- The case $\theta < 1$: by Young's inequality and Cauchy-Schwarz's inequality we have that

$$\begin{aligned} & \mathbb{E}_\rho^{\theta,n} \left[\sup_{t \leq T} \left(\int_0^t \Gamma_s^n(f) ds \right)^2 \right] \\ & \leq CT \int_0^T \mathbb{E}_\rho^{\theta,n} \left[\left(\frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} \Delta_n f\left(\frac{x}{n}\right) (\eta_{sn^2}(x) - \rho) \right)^2 \right] ds \\ & + C (\nabla_n^+ f(0))^2 T \int_0^T \mathbb{E}_\rho^{\theta,n} \left[\left(\sqrt{n} (\eta_{sn^2}(1) - \rho) \right)^2 \right] ds \\ & + C (\nabla_n^- f(1))^2 T \int_0^T \mathbb{E}_\rho^{\theta,n} \left[\left(\sqrt{n} (\eta_{sn^2}(n-1) - \rho) \right)^2 \right] ds. \end{aligned}$$

Since $f \in \mathcal{S}_\theta$ and by (25), the second and third terms at the right hand side of the previous expression go to zero, as $n \rightarrow \infty$. Then there exists $C > 0$ such that these two terms are bounded from above by CT . The first term at the right hand side of last expression is bounded from above by T^2 times

$$\frac{1}{n} \sum_{x=1}^{n-1} \left(\Delta_n f\left(\frac{x}{n}\right) \right)^2 \chi(\rho). \tag{28}$$

Now, since $f \in \mathcal{S}_\theta$ last expression is bounded from above by a constant. Now we need to check the second claim of Aldous' criterium. For that purpose, fix a stopping time $\tau \in \mathcal{T}_T$. By Chebyshev's inequality together with (28), we get that

$$\mathbb{P}_\rho^{\theta,n} \left(\left| \int_\tau^{\tau+\lambda} \Gamma_s^n(f) ds \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E}_\rho^{\theta,n} \left[\left(\int_\tau^{\tau+\lambda} \Gamma_s^n(f) ds \right)^2 \right] \leq \frac{\delta C}{\varepsilon^2},$$

which vanishes as $\delta \rightarrow 0$.

- The case $\theta = 1$: we note that it was treated in [16].
- The case $\theta > 1$: as in the case $\theta < 1$, we have that

$$\begin{aligned} & \mathbb{E}_\rho^{\theta,n} \left[\sup_{t \leq T} \left(\int_0^t \Gamma_s^n(f) ds \right)^2 \right] \\ & \leq CT \int_0^T \mathbb{E}_\rho^{\theta,n} \left[\left(\frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} \Delta_n f\left(\frac{x}{n}\right) (\eta_{sn^2}(x) - \rho) \right)^2 \right] ds \end{aligned}$$

$$\begin{aligned}
 &+ C f^2\left(\frac{1}{n}\right) T \int_0^T \mathbb{E}_\rho^{\theta,n} \left[\left(\frac{n^{3/2}}{n^\theta} (\eta_{sn^2}(1) - \rho) \right)^2 \right] ds \\
 &+ C f^2\left(\frac{n-1}{n}\right) T \int_0^T \mathbb{E}_\rho^{\theta,n} \left[\left(\frac{n^{3/2}}{n^\theta} (\eta_{sn^2}(n-1) - \rho) \right)^2 \right] ds,
 \end{aligned}$$

plus a term of order $\frac{1}{\sqrt{n}}$. To bound the first term at the right hand side of the previous inequality we repeat the same computations as in the case $\theta < 1$. In order to bound the second and the third terms at the right hand side of the previous inequality, we use (26) and the proof follows as in the case $\theta < 1$.

6 Replacement Lemma

This section is devoted to estimate the expectations (25) and (26). In order to do this we start introducing some notations. Let μ be an initial measure. For $x = 0, 1, \dots, n - 1$, define

$$I_{x,x+1}(f, \mu) := \int r_{x,x+1}(\eta) (f(\sigma^{x,x+1}\eta) - f(\eta))^2 d\mu,$$

where $\sigma^{x,x+1}\eta$ was defined in (2), for $x = 1, \dots, n - 2$, $\sigma^{0,1}\eta := \eta^1$, $\sigma^{n-1,n}\eta := \eta^{n-1}$ (the configurations η^1 and η^{n-1} were defined in (3)), and the rates are given by

$$\begin{aligned}
 r_{0,1}(\eta) &:= r_\alpha(\eta) := \frac{\alpha}{n^\theta} (1 - \eta(1)) + \frac{1 - \alpha}{n^\theta} \eta(1), \\
 r_{n-1,n}(\eta) &:= r_\beta(\eta) := \frac{\beta}{n^\theta} (1 - \eta(n-1)) + \frac{1 - \beta}{n^\theta} \eta(n-1), \\
 r_{x,x+1}(\eta) &:= 1, \quad \text{if } x = 1, \dots, n - 2.
 \end{aligned}$$

Define the quantity:

$$\mathcal{D}_n(f, \mu) := \sum_{x=0}^{n-1} I_{x,x+1}(f, \mu) = \sum_{x=0}^{n-1} \int r_{x,x+1}(\eta) (f(\sigma^{x,x+1}\eta) - f(\eta))^2 d\mu. \tag{29}$$

The Dirichlet form is defined by $\langle -\mathcal{L}_n^\theta f, f \rangle_\mu$, where we can rewrite for short the infinitesimal generator as

$$\mathcal{L}_n^\theta f(\eta) := \sum_{x=0}^{n-1} L_{x,x+1} f(\eta) := \sum_{x=0}^{n-1} r_{x,x+1}(\eta) (f(\sigma^{x,x+1}\eta) - f(\eta)).$$

Now, we recall that we consider the case $\alpha = \beta = \rho \in (0, 1)$, so that the measure ν_ρ^n (the Bernoulli product measure) is invariant for this process and it satisfies

$$r_{x,x+1}(\eta) \nu_\rho^n(\eta) = r_{x,x+1}(\sigma^{x,x+1}\eta) \nu_\rho^n(\sigma^{x,x+1}\eta), \tag{30}$$

for all $x \in \{0, 1, \dots, n - 1\}$. Let us check this equality in the case $x = 0$, the case $x = n - 1$ is similar and the others are also very simple to check. Note that

$$r_{0,1}(\sigma^{0,1}\eta) \frac{\nu_\rho^n(\sigma^{0,1}\eta)}{\nu_\rho^n(\eta)} = \left[\frac{\rho}{n^\theta} (1 - \eta^1(1)) + \frac{1 - \rho}{n^\theta} \eta^1(1) \right] \frac{\nu_\rho^n(\eta^1)}{\nu_\rho^n(\eta)}. \tag{31}$$

Since

$$\frac{\nu_\rho^n(\eta^1)}{\nu_\rho^n(\eta)} = \mathbf{1}_{\eta(1)=1} \frac{1 - \rho}{\rho} + \mathbf{1}_{\eta(1)=0} \frac{\rho}{1 - \rho}, \tag{32}$$

then (31) becomes

$$\mathbf{1}_{\eta(1)=1} \left[\frac{\rho}{n^\theta} \right] \frac{1 - \rho}{\rho} + \mathbf{1}_{\eta(1)=0} \left[\frac{1 - \rho}{n^\theta} \right] \frac{\rho}{1 - \rho} = r_{0,1}(\eta).$$

Thus, using (30), we get

$$\langle -\mathcal{L}_n^\theta f, f \rangle_{\nu_\rho^n} = \frac{1}{2} \mathcal{D}_n(f, \nu_\rho^n). \tag{33}$$

Lemma 3 (Replacement Lemma) *Let $x = 1, n - 1$ and $t > 0$ fixed. Then*

$$\mathbb{E}_\rho^{\theta,n} \left[\left(\int_0^t c_n(\eta_s(x) - \rho) ds \right)^2 \right] \leq C \frac{c_n^2 n^\theta}{n^2}.$$

Remark 2 Recall that for $\theta < 1$ we have in (25) $c_n = \sqrt{n}$, so that the error above becomes n^θ/n , which vanishes as $n \rightarrow \infty$. Recall that for $\theta > 1$ we have in (26) $c_n = n^{3/2}/n^\theta$, so that the error above becomes n/n^θ , which vanishes as $n \rightarrow \infty$.

Proof The proof follows by a classical argument combining both the Kipnis–Varadhan’s inequality (see [18, p. 333, Lemma 6.1]) with Young’s inequality. For that purpose let $x = 1$ (the other case is completely analogous) and note that the expectation in the statement of the lemma can be bounded from above by

$$\sup_{f \in L_{\nu_\rho^n}^2} \left\{ \int c_n(\eta(1) - \rho) f(\eta) d\nu_\rho^n + n^2 \langle \mathcal{L}_n^\theta f, f \rangle_{\nu_\rho^n} \right\}, \tag{34}$$

where $L_{\nu_\rho^n}^2$ is the space of functions f such that $\int f^2(\eta) d\nu_\rho^n < +\infty$. We start by writing the integral $\int c_n(\eta(1) - \rho) f(\eta) d\nu_\rho^n$ as twice its half and in one of the terms

we make the exchange $\eta \rightarrow \eta^1$ to have

$$\frac{1}{2} \int c_n(\eta(1) - \rho) f(\eta) d\nu_\rho^n + \frac{1}{2} \int c_n(1 - \eta(1) - \rho) f(\eta^1) \frac{\nu_\rho^n(\eta^1)}{\nu_\rho^n(\eta)} d\nu_\rho^n,$$

see (32) to get the expression of $\frac{\nu_\rho^n(\eta^1)}{\nu_\rho^n(\eta)}$. A simple computation shows that the integral at the right hand side of last expression is equal to

$$-\frac{1}{2} \int c_n(\eta(1) - \rho) f(\eta^1) d\nu_\rho^n,$$

so that the display above is equal to

$$\frac{1}{2} \int c_n(\eta(1) - \rho) (f(\eta) - f(\eta^1)) d\nu_\rho^n.$$

By Young's inequality we can bound the previous expression by

$$B \int c_n^2(\eta(1) - \rho)^2 d\nu_\rho^n + \frac{1}{4B} \int (f(\eta) - f(\eta^1))^2 d\nu_\rho^n.$$

Now, remember the notation $\eta^1 = \sigma^{0,1}\eta$ and multiply and divide by $r_{0,1}(\eta)$ the integrand function inside the second integral above. We can do it, because there exists \tilde{C}_ρ such that $\frac{\tilde{C}_\rho}{n^\theta} \leq r_{0,1}(\eta) \leq \frac{C_\rho}{n^\theta}$. Then we can bound the previous expression from above by

$$B \int c_n^2(\eta(1) - \rho)^2 d\nu_\rho^n + \frac{n^\theta}{4B\tilde{C}_\rho} \int r_{0,1}(\eta) (f(\sigma^{0,1}\eta) - f(\eta))^2 d\nu_\rho^n.$$

Using (29) the second integral in the last expression is bounded from above by $\mathcal{D}_n(f, \nu_\rho^n)$. Recalling (33), we get

$$\int c_n(\eta(1) - \rho) f(\eta) d\nu_\rho^n \leq B c_n^2 \int (\eta(1) - \rho)^2 d\nu_\rho^n + \frac{n^\theta}{2B\tilde{C}_\rho} \langle -\mathcal{L}_n^\theta f, f \rangle_{\nu_\rho^n}.$$

Putting this inequality in (34) and choosing $B = n^{\theta-2}/2\tilde{C}_\rho$, the term at the right hand side of the last expression cancels with $n^2 \langle \mathcal{L}_n^\theta f, f \rangle_{\nu_\rho^n}$. Therefore, the expectation appearing in the statement of the lemma is bounded from above by

$$\frac{c_n^2 n^\theta}{2\tilde{C}_\rho n^2} \int (\eta(1) - \rho)^2 d\nu_\rho^n.$$

Since η is bounded the proof ends.

Acknowledgements A. N. was supported through a grant “L’ORÉAL - ABC - UNESCO Para Mulheres na Ciência”. P. G. thanks FCT/Portugal for support through the project UID/MAT/04459/2013. T. F. was supported by FAPESB through the project Jovem Cientista-9922/2015. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovative programme (grant agreement No 715734).

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