Non-denoting Terms in Fuzzy Logic: An Initial Exploration

Libor Běhounek and Antonín Dvořák(✉)

Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, NSC IT4Innovations, 30. dubna 22, 701 03 Ostrava, Czech Republic
{libor.behounek,antonin.dvorak}@osu.cz
http://irafm.osu.cz

Abstract. We introduce two variants of first-order fuzzy logic that can deal with non-denoting terms, or terms that lack existing referents, e.g., Pegasus, the current king of France, the largest number, or 0/0. Logics designed for this purpose in the classical setting are known as free logics. In this paper we discuss the features of free logics and select the options best suited for fuzzification, deciding on the so-called dual-domain semantics for positive free logic with truth-value gaps and outer quantifiers. We fuzzify the latter semantics in two levels of generality, first with a crisp and subsequently with a fuzzy predicate of existence. To accommodate truth-valueless statements about nonexistent objects, we employ a recently proposed first-order partial fuzzy logic with a single undefined truth value. Combining the dual-domain semantics with partial fuzzy logic, we define several kinds of 'inner-domain' quantifiers, relativized by the predicate of existence. Finally, we make a few observations on some of the resulting rules of free fuzzy quantification that illustrate the differences between the two proposed systems of free fuzzy logic and their well known non-free or non-fuzzy variants.

Keywords: Quantifier · Free logic · Existence · Referent · Partial fuzzy logic

1 Introduction

In both formal and natural languages there are terms with no existing referents. Classical examples include 1/0, $\sum_{n=0}^{\infty}(-1)^n$, the largest natural number, Pegasus, or the current king of France. In the classical setting, dealing with such non-denoting terms falls under the domain of *free logics,* or 'logics free of existential assumptions' [4–7]. Free logics differ from classical logic mainly in the conditional validity of certain inference rules for quantifiers. These differences ensue from the modifications free logics make to the classical first-order semantics in order to accommodate terms that either have no referents at all, or have referents that fall outside the domain of existential and universal quantification. Free logics find numerous applications in the logical analysis of natural language, esp. the theory of definite descriptions, temporal and fictional

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discourse, modal logics with non-constant domains (where possible worlds can differ in existent objects), computer science (for dealing with null objects and unassigned variables), or some areas of mathematics (algebra, foundations) and philosophy [5–7].

Non-denoting terms or terms denoting nonexistent objects can, obviously, be encountered in fuzzy contexts just like in crisp contexts, e.g., when a fuzzy property is predicated of a nonexistent object or in fuzzy definite or indefinite descriptions. However, like classical logic, known systems of predicate fuzzy logic all assume that each term in the language is evaluated within the domain of quantification, and so has an existent referent. To our knowledge, no attempt at developing free fuzzy logic has yet been undertaken.

This paper aims neither at providing a definite solution to the problem of handling non-denoting terms or nonexistent objects in fuzzy contexts, nor at deriving deep mathematical results on free quantification in fuzzy logic. Rather we make the first exploration into the landscape of viable variants of free fuzzy logic, pointing out some possible desiderata and design choices, and hint at a few features in which free fuzzy logic may differ from its non-free or non-fuzzy variants.

Possible applications of free quantification in fuzzy logic are envisaged wherever non-denoting terms might be encountered in fuzzy contexts, which includes fuzzy descriptions, fuzzy temporal, fictional, or modal discourse, as well as various fuzzy methods of computer science and engineering where variables may happen to lack referents. Naturally, these applications can only be developed after the sketched systems of free fuzzy logic are elaborated in more detail. Such an elaboration is a topic for future work.

2 Non-denoting Terms in the Classical Setting

As mentioned in Sect. 1, the treatment of non-denoting terms and nonexistent objects in the crisp setting is the domain of free logics. There are several variants of free logics known from the literature, which differ in various design choices for their semantics [5–7]. In this section, we review the main available options for the semantics of crisp free logics and justify the choice of one of them as our starting point for generalization to the fuzzy setting.

In free logics, singular terms may lack referents in the domain of quantification. Most variants of free logic contain the (primitive or defined) unary *existence predicate,* traditionally denoted by $E!$, where the atomic formula $E!t$ expresses the fact that the singular term t has a referent in the domain of quantification. Besides other things, $E!$ enables an explicit expression of existential presuppositions in inferences.

There are three main families of free logics, which differ in the way they assign truth values to empty-termed atomic formulae (i.e., atomic formulae containing terms that lack referents in the domain of quantification):

– *Negative:* All empty-termed atomic formulae are considered false.

– *Positive:* Some empty-termed atomic formulae can be true.
– *Neutral:* All empty-termed atomic formulae not of the form $E!t$ are considered truth-valueless.

A further distinction regards how non-denoting terms themselves are handled in the semantics. One option is to use a single domain D of referents; the Tarski conditions then need be modified to allow singular terms to have no value in D. Another option is the so-called *dual-domain semantics.* Here, models have two domains: the *outer domain* $D_0 \neq \emptyset$ and the *inner domain* $D_1 \subseteq D_0$. In D_1, which is the range of quantification, existent objects are collected. Singular terms with non-existing referents are assigned the elements of $D_0 \setminus D_1$. In the positive dual-domain semantics, the extensions of predicates can include objects from $D_0 \setminus D_1$; this makes it possible to assign truth values to claims about nonexistent objects (e.g., that Zeus $\neq$ Pegasus or that unicorns are animals). The appeal of the dual-domain semantics lies in its closeness to the classical semantics: since every singular term has a referent in D_0, there is no need to use some non-standard way of evaluation of empty-termed atomic formulae. The dual-domain semantics is also convenient for accommodating objects that exist in possible worlds other than the actual world: in modal logics with non-constant domains, each world w comes with its own inner domain (of objects existing in w), which is a subset of a common outer domain.

An attractive option in the positive dual-domain framework is to take the so-called *outer quantifiers,* which range over the outer domain D_0, as primitive. Since all terms have referents in D_0, these quantifiers behave as the standard quantifiers of classical first-order logic. The *inner quantifiers* (ranging over D_1) are then simply restrictions of the outer quantifiers to the inner domain (delimited by the existence predicate). In particular, if we denote the outer quantifiers by $\exists^0, \forall^0$, then the inner quantifiers $\exists^1, \forall^1$ are defined as

$$(\exists^1 x)\varphi \equiv_{\mathrm{df}} (\exists^0 x)(E!x \mathbin{\&} \varphi) \tag{1}$$

$$(\forall^1 x)\varphi \equiv_{\mathrm{df}} (\forall^0 x)(E!x \rightarrow \varphi). \tag{2}$$

The ordinary meaning of the expressions “some” and “all” corresponds to *inner* quantification (over existing objects). The outer quantifiers, apart from their technical role in the semantics, are nevertheless useful in certain specific contexts: for instance, the statement “Some things do not exist”, which is not straightforwardly formalizable by means of classical or inner quantification, can be expressed by the formula $(\exists^0 x)\neg E!x$. Since $D_1, \exists^1, \forall^1$ are definable from the (classically behaving) $D_0, \exists^0, \forall^0$ and $E!$, free logic with outer quantifiers is essentially the classical logic of restricted quantification.

More details on free logics can be found in [4–7]. It remains to decide which variant(s) from the rich landscape of free logics are best suited for generalization to fuzzy contexts.

As has been observed in the literature [6,7], each of the main variants (positive, negative, and neutral) comes with some problems. Neutral free logics tend to be rather weak; also, intuitively it seems strange for statements like "Zeus = Zeus" to lack a truth value (or be false, as in negative free logics). In negative free logics, the truth values of empty-termed formulae depend on the choice of primitive predicates. In bivalent positive free logics, we are often forced to assign truth values to empty-termed formulae without any clear reason.

Therefore, for our enterprise we favor a *non-bivalent* variant of *positive* free logic, which has also been studied in the literature (see [7]) and seems most flexible compared to alternatives. In non-bivalent positive free logics, some empty-termed propositions (such as "1/0 is prime") may lack truth values, while others (such as "Zeus = Zeus") can be true and yet others (such as "Zeus = Pegasus") false. The truth-value gaps, needed in non-bivalent positive semantics, can conveniently be handled within the framework of partial fuzzy logic, recently proposed in [2,3]. Since single-domain semantics require a non-standard evaluation of singular terms (and can anyway be emulated by a dual-domain semantics with a single element in $D_0 \setminus D_1$), our choice for fuzzification is that of *dual-domain non-bivalent positive* free logic.

3 Partial Fuzzy Logic

Partial fuzzy logic, suitable for dealing with truth-valueless propositions occurring in positive free fuzzy logic, has been proposed in a propositional form in [3] and extended to a first-order variant in [2]. It represents truth value gaps by an additional truth value $*$, added to the real unit interval $[0, 1]$ or another algebra $\boldsymbol{L}$ of truth degrees of an underlying fuzzy logic L. The underlying fuzzy logic L can be any implicative expansion of the logic $\mathrm{MTL}_\triangle$ (i.e., an expansion of $\mathrm{MTL}_\triangle$ where every connective is congruent w.r.t. fully true bi-implication), for instance, $Ł_\triangle$, $\mathrm{BL}_\triangle$, $Ł\Pi$, etc. For more information on these logics see, e.g., [1]; we assume the reader's familiarity with at least one such fuzzy logic, both propositional and first-order.

The semantics of the propositional partial fuzzy logic L^* based on the fuzzy logic L is defined as follows (for additional details see [3]):

- The primitive *propositional language* of L^* contains:
  - For each propositional connective c of L, the ('Bochvar-style') connective c_B of the same arity
  - The truth constant $*$ (representing an undefined truth degree)
  - The unary connective ! (for the crisp indicator of definedness)
  - The binary connective $\wedge_\mathrm{K}$ (for 'Kleene-style' min-conjunction).
- The *intended algebras* of truth values for L^* are defined as expansions of the algebras for L by a dummy element $*$ (to be assigned to propositions with undefined truth). In the intended L^*-algebra $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$, for $\boldsymbol{L}$ an L-algebra, the connectives of L^* are interpreted by the following truth tables for each unary connective u_B, binary connective c_B (and similarly for higher arities), $\alpha, \beta \in \boldsymbol{L}$ and $\gamma, \delta \in \boldsymbol{L} \setminus \{0\}$:

	$!$
α	1
$*$	0

	u_{B}
α	$u\alpha$
$*$	$*$

c_{B}	β	$*$
α	$\alpha\, c\, \beta$	$*$
$*$	$*$	$*$

$\wedge_{\mathrm{K}}$	0	δ	$*$
0	0	0	0
γ	0	$\gamma \wedge \delta$	$*$
$*$	0	$*$	$*$

(3)

– The *tautologies* of L^* are defined as those L^*-formulae that are evaluated to 1 under all evaluations in all intended L^*-algebras. *Entailment* in L^* is defined as the transmission of the value 1 under all evaluations in all intended L^*-algebras. As usual, we write $\models \varphi$ if φ is a tautology of L^*, and $\Gamma \models \varphi$ if the set Γ of L^*-formulae entails the L^*-formula φ in L^*.

The primitive connectives of L^* make a broad class of derived connectives available in L^*. Besides the primitive *Bochvar-style* connectives c_{B}, which treat $*$ as the absorbing element, the following two important families of connectives are definable in L^*:

– The *Sobociński-style* connectives $c_{\mathrm{S}} \in \{\wedge_{\mathrm{S}}, \vee_{\mathrm{S}}, \&_{\mathrm{S}}\}$, which treat $*$ as the neutral element; and the Sobociński-style implication $\rightarrow_{\mathrm{S}}$, associated with $\&_{\mathrm{S}}$ via the residuation axiom $x \rightarrow_{\mathrm{S}} (y \rightarrow_{\mathrm{S}} z) = (x \mathbin{\&_{\mathrm{S}}} y) \rightarrow_{\mathrm{S}} z$:

c_{S}	β	$*$
α	$\alpha\, c\, \beta$	α
$*$	β	$*$

$\rightarrow_{\mathrm{S}}$	β	$*$
α	$\alpha \rightarrow \beta$	$\neg\alpha$
$*$	β	$*$

– The *Kleene-style* connectives $c_{\mathrm{K}} \in \{\wedge_{\mathrm{K}}, \vee_{\mathrm{K}}, \&_{\mathrm{K}}, \rightarrow_{\mathrm{K}}\}$, which preserve the neutral and absorbing elements of the corresponding connectives of L, and otherwise are evaluated Bochvar-style. For the primitive connective $\wedge_{\mathrm{K}}$ see (3) above; the others are defined by the following truth tables:

$\&_{\mathrm{K}}$	0	β	$*$
0	0	0	0
α	0	$\alpha \mathbin{\&} \beta$	$*$
$*$	0	$*$	$*$

$\vee_{\mathrm{K}}$	δ	1	$*$
γ	$\gamma \vee \delta$	1	$*$
1	1	1	1
$*$	$*$	1	$*$

$\rightarrow_{\mathrm{K}}$	δ	1	$*$
0	1	1	1
α	$\alpha \rightarrow \delta$	1	$*$
$*$	$*$	1	$*$

(4)

Moreover, several useful auxiliary connectives are definable in L^*, including those with the following truth tables (for $\alpha \in \boldsymbol{L}$ and $\gamma \in \boldsymbol{L} \setminus \{1\}$):

	?	$\downarrow$	$\uparrow$
α	0	α	α
$*$	1	0	1

	$\circledast$
γ	0
1	$*$
$*$	0

(5)

For examples of the logical laws governing the connectives of L^* see [3]. The semantics of the first-order extension of L^* introduced in [2] is defined as follows:

Let $\mathcal{L} = (\mathrm{Pred}_{\mathcal{L}}, \mathrm{Func}_{\mathcal{L}})$ be a first-order language with a non-empty set $\mathrm{Pred}_{\mathcal{L}}$ of predicate symbols and a set $\mathrm{Func}_{\mathcal{L}}$ of function symbols, each with an arity $n \geq 0$ (where predicate symbols of arity 0 are propositional constants and function symbols of arity 0 are object constants). Let Var be a set of object variables.

A *model* for a language $\mathcal{L}$ over an intended Ł*-algebra $\boldsymbol{L}_*$ is given as $\mathbf{M} = \big(D^{\mathbf{M}}, (P^{\mathbf{M}})_{P\in\mathrm{Pred}_{\mathcal{L}}}, (F^{\mathbf{M}})_{F\in\mathrm{Func}_{\mathcal{L}}}\big)$, where:

- $D^{\mathbf{M}}$ is a crisp non-empty set.
- $P^{\mathbf{M}}\colon (D^{\mathbf{M}})^n \to \boldsymbol{L}_*$ for each n-ary $P \in \mathrm{Pred}_{\mathcal{L}}$.
- $F^{\mathbf{M}}\colon (D^{\mathbf{M}})^n \to D^{\mathbf{M}}$ for each n-ary $F \in \mathrm{Func}_{\mathcal{L}}$.

The semantic values of a formula φ and a term t in a model $\mathbf{M}$ under an evaluation $e\colon \mathrm{Var} \to D^{\mathbf{M}}$ of object variables will be denoted by $\|\varphi\|^{\mathbf{M}}_e$ and $\|t\|^{\mathbf{M}}_e$, respectively. The evaluation that assigns $a \in D^{\mathbf{M}}$ to x and coincides with e on all other object variables will be denoted by $e[x \mapsto a]$.

The Tarski conditions for terms and atomic formulae are defined as in the first-order fuzzy logic Ł, and for propositional connectives by the truth tables (3) above. The primitive quantifiers $\exists_{\mathrm{B}}, \forall_{\mathrm{B}}$ of Ł* are interpreted *Bochvar-style,* i.e., yielding the 'undefined' value $*$ whenever an instance of the quantified formula is undefined:

$$\|(\exists_{\mathrm{B}}x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for some } a \in D^{\mathbf{M}} \\ \sup_{a\in D^{\mathbf{M}}} \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} & \text{otherwise} \end{cases}$$

$$\|(\forall_{\mathrm{B}}x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for some } a \in D^{\mathbf{M}} \\ \inf_{a\in D^{\mathbf{M}}} \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} & \text{otherwise.} \end{cases}$$

Like in the case of propositional connectives, further variants of universal and existential quantifiers are definable in Ł*, including the following important ones:

- The *Sobociński-style* quantifiers $\exists_{\mathrm{S}}, \forall_{\mathrm{S}}$, which ignore the undefined instances of the quantified formula:

$$\|(\exists_{\mathrm{S}}x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for all } a \in D^{\mathbf{M}} \\ \sup\limits_{\|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \neq *} \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} & \text{otherwise} \end{cases}$$

$$\|(\forall_{\mathrm{S}}x)\varphi\|^{\mathbf{M}}_e = \begin{cases} * & \text{if } \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} = * \text{ for all } a \in D^{\mathbf{M}} \\ \inf\limits_{\|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} \neq *} \|\varphi\|^{\mathbf{M}}_{e[x\mapsto a]} & \text{otherwise.} \end{cases}$$

They can be defined from $\exists_{\mathrm{B}}, \forall_{\mathrm{B}}$ by the Ł*-connectives (3)–(5) as follows:

$$(\exists_{\mathrm{S}}x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}}x){\downarrow}\varphi \vee_{\mathrm{B}} \circledast(\forall_{\mathrm{B}}x)?\varphi \tag{6}$$

$$(\forall_{\mathrm{S}}x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}}x){\uparrow}\varphi \vee_{\mathrm{B}} \circledast(\forall_{\mathrm{B}}x)?\varphi\ . \tag{7}$$

- The *Kleene-style* quantifiers $\exists_{\mathrm{K}}, \forall_{\mathrm{K}}$, respectively analogous to $\vee_{\mathrm{K}}$ and $\wedge_{\mathrm{K}}$, can be defined as:

$$(\exists_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}} x)\varphi \vee_{\mathrm{K}} (\exists_{\mathrm{S}} x)\varphi \tag{8}$$

$$(\forall_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}} x)\varphi \wedge_{\mathrm{K}} (\forall_{\mathrm{S}} x)\varphi \,. \tag{9}$$

As usual, *validity* in a model is defined as truth to degree 1 under all evaluations of object variables in the model; *tautologicity* as validity in all models for the given language; and *entailment* as validity in all models validating all premises. We use the usual notation $\mathbf{M} \models \varphi$ for validity, $\models \varphi$ for tautologicity, and $\varGamma \models \varphi$ for entailment.

Observation 1. It can be easily verified that, e.g., the rule of generalization is sound for all the aforementioned quantifiers: $\varphi \models (\mathrm{Q}x)\varphi$ for $\mathrm{Q} \in \{\forall_{\mathrm{B}}, \forall_{\mathrm{S}}, \forall_{\mathrm{K}}, \exists_{\mathrm{B}}, \exists_{\mathrm{S}}, \exists_{\mathrm{K}}\}$. The rule of specification, on the other hand, only holds for Bochvar and Kleene universal quantifiers: $(\mathrm{Q}x)\varphi \models \varphi$ for $\mathrm{Q} \in \{\forall_{\mathrm{B}}, \forall_{\mathrm{K}}\}$. Sobociński-style universally quantified formulae may only be instantiated with terms that do not make them undefined: $(\forall_{\mathrm{S}} x)\varphi, !\varphi(t/x) \models \varphi(t/x)$.

4 Free Fuzzy Logic with a Crisp Existence Predicate

We have now collected all requisite ingredients to brew the first system of free fuzzy logic. By a design choice justified in Sect. 2, it is going to be a fuzzy variant of positive free logic with a dual-domain semantics admitting undefined truth degrees (represented by the dummy value $*$ of a partial fuzzy logic L^*). We shall start with the simpler case when the existence predicate $E!$ is *bivalent* (i.e., total and crisp). The more general case of a *fuzzy* existence predicate will be discussed later in Sect. 5.

Let L^* be a partial fuzzy logic based on a fuzzy logic L. The semantics for a free variant of L^* will only require a minor modification to the semantics of first-order L^* described in Sect. 3:

Let $\boldsymbol{L}_*$ be an intended L^*-algebra and $\mathcal{L}$ a first-order language as in Sect. 3. A *dual-domain model* for $\mathcal{L}$ over $\boldsymbol{L}_*$ is given as $\mathbf{M} = (D_0^{\mathbf{M}}, D_1^{\mathbf{M}}, (P^{\mathbf{M}})_{P \in \mathrm{Pred}_{\mathcal{L}}}, (F^{\mathbf{M}})_{F \in \mathrm{Func}_{\mathcal{L}}})$, where:

- $D_0^{\mathbf{M}}, D_1^{\mathbf{M}}$ are crisp sets such that $D_1^{\mathbf{M}} \subseteq D_0^{\mathbf{M}} \neq \emptyset$, respectively called the *outer* and *inner domain* of $\mathbf{M}$.

Predicate and function symbols are interpreted over the *outer* domain:

- $P^{\mathbf{M}} \colon (D_0^{\mathbf{M}})^n \to \boldsymbol{L}_*$ for each n-ary $P \in \mathrm{Pred}_{\mathcal{L}}$.
- $F^{\mathbf{M}} \colon (D_0^{\mathbf{M}})^n \to D_0^{\mathbf{M}}$ for each n-ary $F \in \mathrm{Func}_{\mathcal{L}}$.

The Tarski conditions for terms, atomic formulae, and propositional connectives in $\mathbf{M}$ under an evaluation $e \colon \mathrm{Var} \to D_0^{\mathbf{M}}$ are as in Sect. 3. The additional logical predicate symbols $=$ (identity) and $E!$ (existence) are interpreted in $\mathbf{M}$ as follows:

– $E!^{\mathbf{M}}$ indicates membership in the inner domain $D_1^{\mathbf{M}}$:

$$\|E!t\|_e^{\mathbf{M}} = \begin{cases} 1 & \text{if } \|t\|_e^{\mathbf{M}} \in D_1^{\mathbf{M}} \\ 0 & \text{otherwise.} \end{cases}$$

– $=^{\mathbf{M}}$ indicates the identity across the outer domain $D_0^{\mathbf{M}}$:

$$\|t = u\|_e^{\mathbf{M}} = \begin{cases} 1 & \text{if } \|t\|_e^{\mathbf{M}} = \|u\|_e^{\mathbf{M}} \\ 0 & \text{otherwise.} \end{cases}$$

Opting for free logic with outer quantifiers (see Sect. 2), we define the primitive Bochvar-style quantifiers $\forall^0_{\mathrm{B}}, \exists^0_{\mathrm{B}}$ as ranging over the *outer* domain $D_0^{\mathbf{M}}$:

$$\left\|(\exists^0_{\mathrm{B}} x)\varphi\right\|_e^{\mathbf{M}} = \begin{cases} * & \text{if } \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} = * \text{ for some } a \in D_0^{\mathbf{M}} \\ \sup_{a\in D_0^{\mathbf{M}}} \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} & \text{otherwise} \end{cases}$$

$$\left\|(\forall^0_{\mathrm{B}} x)\varphi\right\|_e^{\mathbf{M}} = \begin{cases} * & \text{if } \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} = * \text{ for some } a \in D_0^{\mathbf{M}} \\ \inf_{a\in D_0^{\mathbf{M}}} \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} & \text{otherwise.} \end{cases}$$

The outer Sobociński and Kleene quantifiers can be defined from $\exists^0_{\mathrm{B}}, \forall^0_{\mathrm{B}}$ as in (6)–(9) of Sect. 3:

$$(\exists^0_{\mathrm{S}} x)\varphi \equiv_{\mathrm{df}} (\exists^0_{\mathrm{B}} x){\downarrow}\varphi \vee_{\mathrm{B}} \circledast(\forall^0_{\mathrm{B}} x)?\varphi \qquad (\exists^0_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\exists^0_{\mathrm{B}} x)\varphi \vee_{\mathrm{K}} (\exists^0_{\mathrm{S}} x)\varphi$$
$$(\forall^0_{\mathrm{S}} x)\varphi \equiv_{\mathrm{df}} (\forall^0_{\mathrm{B}} x){\uparrow}\varphi \vee_{\mathrm{B}} \circledast(\forall^0_{\mathrm{B}} x)?\varphi \qquad (\forall^0_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\forall^0_{\mathrm{B}} x)\varphi \wedge_{\mathrm{K}} (\forall^0_{\mathrm{S}} x)\varphi\ .$$

Analogously to (1) and (2) in Sect. 2, we would like to introduce *inner* (Bochvar-style) quantifiers $\exists^1_{\mathrm{B}}, \forall^1_{\mathrm{B}}$ by restricting the outer quantifiers $\exists^0_{\mathrm{B}}, \forall^0_{\mathrm{B}}$ to the inner domain $D_1^{\mathbf{M}}$ (delimited by $E!^{\mathbf{M}}$). In the partial fuzzy setting, there arises the question as to which of the available conjunctions and implications should be used in (1) and (2) for the relativization of quantifiers. The desired behavior of the inner quantifiers is such that they are only affected by the elements of the inner domain $D_1^{\mathbf{M}}$, i.e., iff $E!x$ evaluates to 1. In (1) and (2) we thus need to use a conjunction & and an implication $\to$ such that $0 \mathbin{\&} \alpha = 0$ and $0 \to \alpha = 1$ (to screen off the elements outside $D_1^{\mathbf{M}}$), while $1 \mathbin{\&} \alpha = \alpha$ and $1 \to \alpha = \alpha$ (not to affect the values for elements in $D_1^{\mathbf{M}}$), for all $\alpha \in \boldsymbol{L}_*$. This suggests the *Kleene* connectives $\&_{\mathrm{K}}$ and $\to_{\mathrm{K}}$ (cf. their truth tables (4) in Sect. 3) as the adequate choice for relativization. Therefore we define:

$$(\exists^1_{\mathrm{B}} x)\varphi \equiv_{\mathrm{df}} (\exists^0_{\mathrm{B}} x)(E!x \mathbin{\&_{\mathrm{K}}} \varphi) \qquad (10)$$
$$(\forall^1_{\mathrm{B}} x)\varphi \equiv_{\mathrm{df}} (\forall^0_{\mathrm{B}} x)(E!x \to_{\mathrm{K}} \varphi)\ . \qquad (11)$$

The inner Sobociński and Kleene quantifiers can again be defined from $\exists^1_{\mathrm{B}}, \forall^1_{\mathrm{B}}$ just like in (6)–(9) of Sect. 3:

$$(\exists^1_{\mathrm{S}} x)\varphi \equiv_{\mathrm{df}} (\exists^1_{\mathrm{B}} x){\downarrow}\varphi \vee_{\mathrm{B}} \circledast(\forall^1_{\mathrm{B}} x)?\varphi \qquad (\exists^1_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\exists^1_{\mathrm{B}} x)\varphi \vee_{\mathrm{K}} (\exists^1_{\mathrm{S}} x)\varphi$$
$$(\forall^1_{\mathrm{S}} x)\varphi \equiv_{\mathrm{df}} (\forall^1_{\mathrm{B}} x){\uparrow}\varphi \vee_{\mathrm{B}} \circledast(\forall^1_{\mathrm{B}} x)?\varphi \qquad (\forall^1_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\forall^1_{\mathrm{B}} x)\varphi \wedge_{\mathrm{K}} (\forall^1_{\mathrm{S}} x)\varphi\ .$$

Finally, the notions of validity, tautologicity, and entailment are defined as in Sect. 3. Let us now give some observations on this version of free fuzzy logic.

Observation 2. First, it can be observed that the definition of $=^{\mathbf{M}}$ makes all self-identity statements true to degree 1, thus $\models t = t$ for all terms t.

Secondly, since all terms denote in $D_0^{\mathbf{M}}$, the outer quantifiers behave just like the non-free quantifiers of Sect. 3. Thus, for instance (cf. Sect. 3, Observation 1):

$$\begin{aligned} \varphi &\models (\mathrm{Q}x)\varphi && \text{for } \mathrm{Q} \in \{\forall^0_{\mathrm{B}}, \forall^0_{\mathrm{S}}, \forall^0_{\mathrm{K}}, \exists^0_{\mathrm{B}}, \exists^0_{\mathrm{S}}, \exists^0_{\mathrm{K}}\} \\ (\mathrm{Q}x)\varphi &\models \varphi && \text{for } \mathrm{Q} \in \{\forall^0_{\mathrm{B}}, \forall^0_{\mathrm{K}}\} \\ (\forall^0_{\mathrm{S}}x)\varphi, !\varphi(t/x) &\models \varphi(t/x)\ . \end{aligned} \tag{12}$$

However, the behavior of inner quantifiers, which only range over $D_1^{\mathbf{M}}$, differs. For example, unlike (12), in general $(\forall^1_{\mathrm{B}}x)\varphi \not\models \varphi$, since x can be evaluated outside the inner domain $D_1^{\mathbf{M}}$. The predicate $E!$ makes it possible to indicate the existence assumptions of inner quantification explicitly; for instance, the following rules are sound:

$$\begin{aligned} \varphi(t/x), E!t &\models (\mathrm{Q}x)\varphi && \text{for } \mathrm{Q} \in \{\exists^1_{\mathrm{S}}, \exists^1_{\mathrm{K}}\} \\ (\mathrm{Q}x)\varphi, E!t &\models \varphi(t/x) && \text{for } \mathrm{Q} \in \{\forall^1_{\mathrm{B}}, \forall^1_{\mathrm{K}}\}\ . \end{aligned}$$

For $\exists^1_{\mathrm{B}}, \forall^1_{\mathrm{S}}$, on the other hand, additional definedness assumptions are needed:

$$\begin{aligned} \varphi(t/x), E!t, (\forall^1_{\mathrm{B}}x)!\varphi &\models (\exists^1_{\mathrm{B}}x)\varphi \\ (\forall^1_{\mathrm{S}}x)\varphi, E!t, !\varphi(t/x) &\models \varphi(t/x)\ . \end{aligned}$$

5 Free Fuzzy Logic with a Fuzzy Existence Predicate

In this section, we outline a variant of free fuzzy logic in which the existence predicate $E!$ need not be bivalent as in Sect. 4, but can be fuzzy. In this more general setting, the existence of the referent of a singular term can be a matter of degree. This may be useful, e.g., for modeling definite or indefinite descriptions determined by a fuzzy condition: for instance, the referent of the term *the golden mountain* can be considered to exist in a possible world w to the degree to which the greatest lump of gold in w can be considered a mountain; or the degree of purity of gold in the mountain with the most content of gold in w; or a combination thereof.

The semantics described in Sect. 4 requires just a very minor adjustment in order to admit fuzzy existence. In fact, the only change required is to assume that the inner domain $D_1^{\mathbf{M}}$ is a fuzzy (rather than crisp) subset of the outer domain $D_0^{\mathbf{M}}$. As was already the case in Sect. 4, the existence predicate $E!$ is interpreted by the membership function of $D_1^{\mathbf{M}}$. Thus the only difference to the semantics of Sect. 4 consists in the following clauses:

- $D_1^{\mathbf{M}} \colon D_0^{\mathbf{M}} \to \boldsymbol{L}$.
- $E!^{\mathbf{M}} = D_1^{\mathbf{M}}$.

The Tarski condition for $E!$ thus reads: $\|E!t\|^{\mathbf{M}}_e = D^{\mathbf{M}}_1\big(\|t\|^{\mathbf{M}}_e\big)$. All the rest of the definitions of Sect. 4, including those of the inner and outer quantifiers, remain in place.

The bivalence of $E!$ (and so the setting of Sect. 4) can easily be enforced by adding the axiom $E!x \vee_{\mathrm{B}} \neg_{\mathrm{B}} E!x$, or equivalently, $(\forall^0_{\mathrm{B}} x)(E!x \vee_{\mathrm{B}} \neg_{\mathrm{B}} E!x)$. Note that using instead the axiom $(\forall^0_{\mathrm{S}} x)(E!x \vee_{\mathrm{B}} \neg_{\mathrm{B}} E!x)$ would enforce a crisp, but possibly not totally defined predicate of existence. The question whether a partial $E!$ is meaningful, i.e., whether we may want to admit referents whose existence has no truth value (i.e., is objectively undefined, rather than just unknown), is left aside here for space reasons.

Observation 3. Obviously, the fuzziness of $E!$ does not affect the behavior of the outer quantifiers, which remains the same as in Sects. 3 and 4. What differs is the behavior of the inner quantifiers, due to the relativization to a fuzzy rather than crisp inner domain in their definition; cf. (10) and (11) in Sect. 4. For example, the following rule is sound if $E!$ is crisp, but fails in general for fuzzy $E!$:

$$!\varphi, !\psi \models (\forall^1_{\mathrm{B}} x)(\varphi \to_{\mathrm{B}} \psi) \to_{\mathrm{B}} \big((\forall^1_{\mathrm{B}} x)\varphi \to_{\mathrm{B}} (\forall^1_{\mathrm{B}} x)\psi\big)\,. \tag{13}$$

In our present setting of Sect. 5, the rule (13) only holds if $E!$ is contractive, i.e., with the additional premise $E!x \to_{\mathrm{B}} (E!x \;\&_{\mathrm{B}}\; E!x)$. (So in particular, it does hold if the underlying fuzzy logic L is Gödel or if $E!$ is crisp.)

As seen in Observation 3, the main culprit of the failure of (13), as well as many other rules for inner quantifiers, is the non-contractivity of fuzzy existence claims; i.e., the fact that $E!t$ is in general weaker than $E!t \;\&_{\mathrm{B}}\; E!t$. Taking the non-contractivity of conjunction into account, we can obtain a more fine-grained analysis of the valid rules for inner quantifiers. Let us introduce the following notation:

$$\begin{aligned}\varphi^0 &= 1\\ \varphi^{n+1} &= \varphi^n \;\&_{\mathrm{B}}\; \varphi\\ \varphi^{\triangle} &= \triangle_{\mathrm{B}}\varphi\,.\end{aligned}$$

Then we can define the *inner quantifiers of grade* n, for $n \in \mathbb{N} \cup \{\triangle\}$, as follows:

$$\begin{aligned}(\exists^n_{\mathrm{B}} x)\varphi &\equiv_{\mathrm{df}} (\exists^0_{\mathrm{B}} x)\big((E!x)^n \;\&_{\mathrm{K}}\; \varphi\big)\\ (\forall^n_{\mathrm{B}} x)\varphi &\equiv_{\mathrm{df}} (\forall^0_{\mathrm{B}} x)\big((E!x)^n \to_{\mathrm{K}} \varphi\big)\,.\end{aligned}$$

For $n \leq 1$, the definition yields the usual outer and inner quantifiers, or the quantifiers respectively relativized to the outer and inner domain. The n-grade inner quantifiers can be viewed as relativized to the n-grade inner domain $D^{\mathbf{M}}_n$, defined as the fuzzy extension of the n-times iterated existence predicate:

$$D^{\mathbf{M}}_n(a) = \|(E!x)^n\|^{\mathbf{M}}_{e[x\mapsto a]}$$

for each $a \in D^{\mathbf{M}}_0$. Higher-grade inner domains are more restrictive for the existence degrees of referents: in terms of inclusion of fuzzy sets,

$$D^{\mathbf{M}}_{\triangle} \subseteq \cdots \subseteq D^{\mathbf{M}}_2 \subseteq D^{\mathbf{M}}_1 \subseteq D^{\mathbf{M}}_0\,.$$

Consequently, higher-grade existential quantifiers are stronger and higher-grade universal quantifiers weaker than lesser-grade ones. Since the strictest inner domain $D^{\mathbf{M}}_{\triangle}$ is bivalent, the $\triangle$-grade inner quantifiers $\exists^{\triangle}_{\mathrm{B}}, \forall^{\triangle}_{\mathrm{B}}$ behave like the inner quantifiers of Sect. 4.

The stratified hierarchy of inner quantifiers makes it possible to formulate sound versions of the rule (13) of Observation 3, as well as many other contraction-sensitive rules, even for a fuzzy (non-contractive) predicate of existence:

Observation 4. In the present setting, the following modifications of the rule (13) are sound for any $m, n \geq 0$:

$$!\varphi, !\psi \models (\forall^{m}_{\mathrm{B}} x)(\varphi \to_{\mathrm{B}} \psi) \to_{\mathrm{B}} \big((\forall^{n}_{\mathrm{B}} x)\varphi \to_{\mathrm{B}} (\forall^{m+n}_{\mathrm{B}} x)\psi\big)$$
$$!\varphi, !\psi \models (\forall^{\triangle}_{\mathrm{B}} x)(\varphi \to_{\mathrm{B}} \psi) \to_{\mathrm{B}} \big((\forall^{\triangle}_{\mathrm{B}} x)\varphi \to_{\mathrm{B}} (\forall^{\triangle}_{\mathrm{B}} x)\psi\big).$$

A detailed investigation of the two variants of free fuzzy logic outlined in Sects. 4 and 5, including an axiomatic treatment, is left for future work.

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