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Advances in Fuzzy Logic and Technology 2017

Proceedings of: EUSFLAT-2017 – The 10th Conference of the European Society for Fuzzy Logic and Technology, September 11–15, 2017, Warsaw, Poland IWIFSGN'2017 – The Sixteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, September 13–15, 2017, Warsaw, Poland, Volume 1

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Foreword

This volume constitutes the proceedings of the two collocated international conferences. The main part includes the papers accepted, after a strict peer review process, for the presentation at, and for the inclusion in the proceedings of the 10th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2017) held in Warsaw, Poland, on September 11–15, 2017. It is combined with the papers accepted, also after a strict peer review process, for the presentation at, and for the inclusion in the proceedings of the Sixteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN'2017) held in Warsaw, Poland, on September 13–15, 2017.

The EUSFLAT-2017 Conference was organized by the Systems Research Institute, Polish Academy of Science, Department IV of Engineering Sciences, Polish Academy of Sciences, and the Polish Operational and Systems Research Society. It is the 10th jubilee edition of the flagship conference of the European Society for Fuzzy Logic and Technology (EUSFLAT). The aim of the conference, in line with the mission of the EUSFLAT Society, is to bring together theoreticians and practitioners working on fuzzy logic, fuzzy systems, soft computing, and related areas and to provide for them a platform for the exchange of ideas, discussing newest trends and networking.

The papers included in the proceedings volume have been subject to a thorough review process by highly qualified peer reviewers. Comments and suggestion from them have considerably helped improve the quality of the papers but also the assignment of the papers to best suited sessions in the conference program. In the proceedings volume, the papers have been ordered alphabetically with respect to the name of the first author, and a convenient author's index is included at the end of the volume.

Thanks are due to many people and parties involved. First, in the early stage of the preparation of the conference general perspective, scope, topics, and coverage, we have received an invaluable help from the members of the International Committees of both conferences, notably the chairs responsible for various aspects of the conferences, as well as many people from the European Society for Fuzzy Logic and Technology (EUSFLAT). That help during the initial planning stage had resulted in a very attractive and up-to-date proposal of the scope and coverage that had clearly implied a considerable interest of the international research communities active in the areas covered who submitted a large number of very interesting and high-level papers. An extremely relevant role of the organizers of special sessions, competition, and other events should also be greatly appreciated. Thanks to their vision and hard work, we had been able to collect many papers on focused topics which had then resulted, during the conferences, in very interesting presentations and stimulating discussions at the sessions.

Though EUSFLAT-2017 is a subsequent edition of the main European conference on the broadly perceived fuzzy logic and technology, and an overwhelming majority of participants come from Europe, many people from other continents have also decided to submit their contributions. This has clearly resulted in a "globalization" of the EUSFLAT conferences which we have been able to increasingly notice since its founding. Of a particular importance in this respect is that among the plenary and keynote speakers, there are top researchers and scholars, as well as practitioners, not only from Europe but also from other continents.

The members of the Program Committee, together with the session organizers, and a group of other anonymous peer reviewers have undertaken a very difficult task of selecting the best papers, and they have done it excellently. They deserve many thanks for their great job for the entire community who is always concerned with quality and integrity. We also wish to thank the members of the EUSFLAT Board for their support throughout the organization process.

At the stage of the running of the conference, many thanks are due to the members of the Organizing Committee, chaired by Ms. Krystyna Warzywoda and Ms. Agnieszka Jóźwiak, and supported by their numerous collaborators.

And last but not least, we wish to thank Dr. Tom Ditzinger, Dr. Leontina di Cecco, and Mr. Holger Schaepe for their dedication and help to implement and finish this large publication project on time maintaining the highest publication standards.

June 2017 The Editors

Contents

Optimal Control of a Ball and Beam Nonlinear Model Based on Takagi-Sugeno Fuzzy Model

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Abstract. In this work, an improved approach for Takagi-Sugeno system identification is used. Linear Quadratic Regulator is applied for an optimal state feedback. Duality theorem and Linear Quadratic Regulator is applied for an optimal state estimation. Simulation results over the ball and beam nonlinear model show a stable closed loop in the full range and good transient response.

Keywords: Ball and beam · Takagi-Sugeno model · State model · Linear Quadratic Regulator · Duality theorem

1 Introduction

The ball and beam system [1] is a classical mechanical system with two degrees of freedom. The beam rotates, driven by a torque at the center of rotation. The ball rolls freely along the beam and in contact with the beam. Despite its mechanical simplicity, the ball and beam system presents significant challenges from the point of view of automation; the system is nonlinear and unstable.

The ball and beam is a common didactical plant in many control laboratories around the world [2], as it is very nonlinear, unstable, which means that it is difficult to control, and can be a benchmark for testing several advanced control techniques [3].

Takagi-Sugeno (T-S) fuzzy model [4] has been an important tool for the modelling and control of nonlinear systems, since it builds the full nonlinear model by a linear model at each fuzzy rule and the fuzzy interpolation among them. Moreover, the T-S fuzzy identification allows the identification of all the fuzzy parameters of the full nonlinear system minimizing a global error index.

Optimal control has been a significant method for the controllers design. Linear Quadratic Regulator [5], is an optimal control design method for state space linear models which allows the minimization of a cost function in which state dynamics and control action are weighted.

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One of the most important problems in state space feedback is that usually the states are not directly accessible, since not all the state variables are measurable. For this propose, state observers [6], can create a surrogate state vector, which tends asymptotically to the real state vector and can be used for state feedback.

The duality theorem [6] allows the design of an state observer matrix with the same techniques for an state feedback controller, including the LQR method [5]. For that propose a dual system can be built from the state space model, and the duality theorem says that a controller designed in the dual system is equivalent to an observer designed for the state space model.

The rest of the work is organized as follows. Section 2 describes ball and beam nonlinear model. The fuzzy T-S model and the fuzzy identification method are described in Sect. 3. Optimal state controller and optimal state observer designs are described in Sect. 4. In Sect. 5, the proposed fuzzy optimal controller is applied to the ball and beam nonlinear model and the results are obtained and discussed.

2 Ball and Beam Nonlinear Model

In this work, we use the AMIRA BW500 ball and beam model (Fig. 1) [1]. The ball position p, considered as system output, is supposed to be measured by a camera, therefore the discrete sample time is supposed to be large. The beam angle α, considered as measurable internal variable, is supposed to be measured by an incremental encoder. The system input F is supposed to be a force produced by a DC motor, which causes the beam to rotate around its center.

The nonlinear differential equations of the ball and beam model [1], used for the simulation model, are:

$$\left(m + \frac{I_b}{r^2}\right)\ddot{p} + \left(mr^2 + I_b\right)\frac{1}{r}\ddot{\alpha} - mp\dot{\alpha}^2 = mg\sin(\alpha) \tag{1}$$

$$\begin{aligned}&\left(mp^2 + I_b + I_W\right)\ddot{\alpha} + \left(2mp\dot{p} + bl^2\right)\dot{\alpha} + Kl^2\alpha + \\ &\left(mr^2 + I_b\right)\frac{1}{r}\ddot{p} - mgp\cos(\alpha) = Fl\cos(\alpha)\end{aligned} \tag{2}$$

where p is the position of the ball, α is the angle of the beam and F is the force of the drive mechanics. Table 1 summarizes the parameters of the model and its values.

In this model, some restrictions from the real AMIRA BW500 ball and beam model [1] have to be added. The ball position p has to be contained in $[-0.4, 0.4]\,m$, the beam angle α has to be contained in $[-0.69, 0.69]\,rad$ and the input force F has to be contained in $[-5, 5]\,N$.

Since we consider α just as a measurable internal variable, the system is single-input-single-output (SISO), and the relation between p as output and F as input is described by two second order differential equations, so the global system is fourth order.

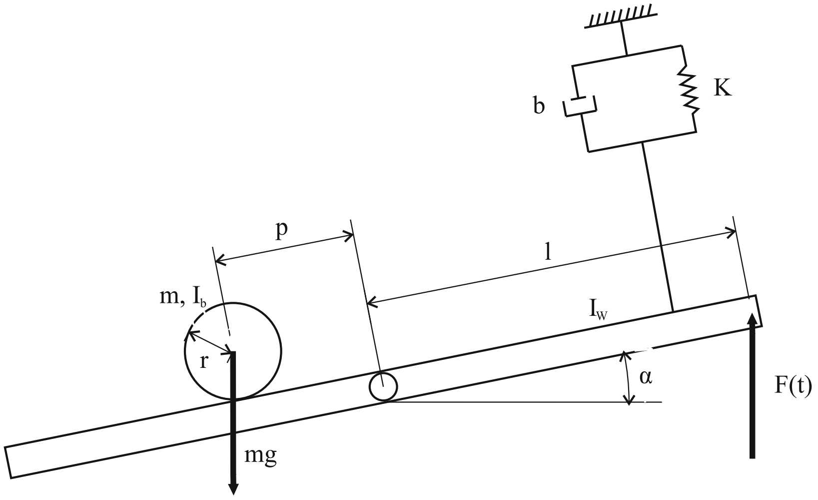

Fig. 1. Ball and beam system.

Table 1. Ball and beam parameters

Parameter	Meaning	Value
m	Mass of the ball	0.025 Kg
g	Gravity	9.81 m/s^2
r	Roll radius of the ball	0.0167 m
I_b	Inertia moment of the ball	$3.516 \cdot 10^{-6}$ Kgm^2
I_W	Inertia moment of the beam	0.09 Kgm^2
b	Friction coefficient of the drive mechanics	1.0 Ns/m
K	Stiffness of the drive mechanics	0.001 N/m
l	Radius of force application	0.49 m

3 Fuzzy Takagi-Sugeno Model and System Identification

3.1 Fuzzy T-S Model

Nonlinear systems can be modelled by T-S model, supposing known a set of measurable nonlinear variables $[z_1(k), z_2(k), \ldots, z_m(k)]$ of the system. By choosing $[r_1, r_2, \ldots, r_m]$ number of fuzzy sets for these variables, a monovariable fuzzy system can be defined as follows:

$$S^{(i_1 \ldots i_m)}\text{: If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \ldots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:}$$

$$y(k) = a_0^{(i_1 \ldots i_m)} + a_1^{(i_1 \ldots i_m)} y(k-1) + \cdots + a_n^{(i_1 \ldots i_m)} y(k-n) \qquad (3)$$
$$+ b_1^{(i_1 \ldots i_m)} u(k-1) + \cdots + b_n^{(i_1 \ldots i_m)} u(k-n)$$

In each rule, we can transform the difference Eq. (3) to state model with affine term as follows:

$$
\begin{aligned}
&S^{(i_1\dots i_m)}\text{: If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:}\\
&\quad x(k) \in \Re^n\\
&x(k+1) = \begin{bmatrix} a_0^{(i_1\dots i_m)} \cdot a_1^{(i_1\dots i_m)} \\ a_0^{(i_1\dots i_m)} \cdot a_2^{(i_1\dots i_m)} \\ \vdots \\ a_0^{(i_1\dots i_m)} \cdot a_n^{(i_1\dots i_m)} \end{bmatrix} + \\
&\qquad + \begin{bmatrix} a_1^{(i_1\dots i_m)} & 1 & \cdots & 0 \\ a_2^{(i_1\dots i_m)} & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ a_n^{(i_1\dots i_m)} & 0 & \cdots & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1^{(i_1\dots i_m)} \\ b_2^{(i_1\dots i_m)} \\ \vdots \\ b_n^{(i_1\dots i_m)} \end{bmatrix} u(k)\\
&\quad y(k) = a_0^{(i_1\dots i_m)} + \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x(k)
\end{aligned}
\tag{4}
$$

This means

$$
\begin{aligned}
&S^{(i_1\dots i_m)}\text{: If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:}\\
&x(k+1) = a_x^{(i_1\dots i_m)} + A^{(i_1\dots i_m)} x(k) + B^{(i_1\dots i_m)} u(k)\\
&\quad y(k) = a_y^{(i_1\dots i_m)} + Cx(k)
\end{aligned}
\tag{5}
$$

3.2 Estimation of T-S Model Parameters

The identification method of T-S fuzzy models [4] is based on the estimation of the fuzzy system parameters minimizing a quadratic performance index. The traditional T-S identification method [4] fails if the membership functions of the fuzzy rules are overlapped triangular in shape, since the T-S matrix is not of full rank and then it is not invertible [7]. Thus, in [7] was proposed a generalized T-S identification, using a parameters weighting method.

The fuzzy estimation of the output becomes:

$$
\begin{aligned}
\hat{y} = &\sum_{i_1=1}^{r_1} \cdots \sum_{i_m=1}^{r_m} \beta^{(i_1\dots i_m)} \left(z_{(i_1\dots i_m)}(k) \right) \Big[a_0^{(i_1\dots i_m)} + a_1^{(i_1\dots i_m)} y(k-1) \\
&+ \cdots + a_n^{(i_1\dots i_m)} y(k-n) + b_1^{(i_1\dots i_m)} u(k-1) + \cdots + b_n^{(i_1\dots i_m)} u(k-n) \Big]
\end{aligned}
\tag{6}
$$

where

$$
\beta^{(i_1\dots i_m)} \left(z_{(i_1\dots i_m)}(k) \right) = \frac{\mu_{1i_1}(z_1) \dots \mu_{mi_m}(z_m)}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_m=1}^{r_m} \left(\mu_{1i_1}(z_1) \dots \mu_{mi_m}(z_m) \right)} \tag{7}
$$

with $\mu_{ji_j}(z_j)$ being the membership function corresponding to the fuzzy set $M_j^{i_j}$.

We have supposed to have a set of input/output system samples and a first affine linear parameters estimation:

$$p^0 = \left[\, a_0^0 \; a_1^0 \ldots a_n^0 \; b_1^0 \ldots b_n^0 \,\right] \tag{8}$$

These parameters could be obtained by a classical input/output identification of the data, for example with least squares method. This first approximation can be utilized as reference parameters for all the subsystems. Then, the fuzzy model parameters can be obtained minimizing:

$$\begin{aligned} J &= \sum_{k=1}^{s} \left(y(k) - \hat{y}(k)\right)^2 + \gamma^2 \sum_{i_1=1}^{r_1} \cdots \sum_{i_m=1}^{r_m} \sum_{j=0}^{n} \left(p_j^0 - p_j^{(i_1 \ldots i_m)}\right)^2 \\ &= \|Y - XP\|^2 + \gamma^2 \|P_0 - P\|^2 = \left\| \begin{bmatrix} Y \\ \gamma P_0 \end{bmatrix} - \begin{bmatrix} X \\ \gamma I \end{bmatrix} P \right\|^2 = \|Y_a - X_a P\|^2 \end{aligned} \tag{9}$$

where Y are the output data, X are the input/output fuzzy data, P_0 are the linear estimated parameters repeated as many times as the number of fuzzy rules ($P_0 = [p_0, p_0, \ldots, p_0]$), and P are the fuzzy T-S model parameters. The γ factor represents the degree of confidence of the linear estimated parameters, and it must be tuned by try and error. It should be noted that the matrix X_a is of full rank, which solves the problem where the traditional T-S identification method fails. Thus, the vector P can be computed as:

$$P = \left(X_a^t X_a\right)^{-1} X_a^t Y_a \tag{10}$$

4 Fuzzy Controller and Observer Design

In order to calculate the coefficients of the state feedback controller, discrete LQR [5] method is chosen, which allows optimal control weighting the dynamic response and the control action.

In LQR method, the goal is to minimize the cost index J:

$$J = \sum_{k=0}^{\infty} \left[\left(x(k) - x_r\right)^t Q \left(x(k) - x_r\right) + \left(u(k) - u_r\right)^t R \left(u(k) - u_r\right) \right] \tag{11}$$

LQR method is completely optimal for linear systems, however, in the case of nonlinear systems, it is complex to propose the minimization of any objective function for the global system. In order to solve this problem, we suggest minimizing the cost of each fuzzy rule instead of the global cost. The solution will be a suboptimal one but with the great advantage of being easy to calculate. With this method, the global stability is not guaranteed, which needs to be analyzed a posteriori, although gaining in return a balance between static and dynamic behavior of the system with admissible control actions.

The state observer [6] is a parallel dynamic system with a correction term that approximates the estimated state to the real one:

$$\begin{aligned} x_e(k+1) &= a_x + A x_e(k) + B u(k) + H \left(y(k) - y_e(k)\right) \\ y_e(k) &= a_y + C x_e(k) \end{aligned} \tag{12}$$

The estimation error is:

$$\begin{aligned}\varepsilon(k+1) = x(k+1) - x_e(k+1) = (a_x + Ax(k) + Bu(k)) \\ - (a_x + Ax_e(k) + Bu(k) + H(y(k) - (a_y + Cx_e(k))))\end{aligned} \tag{13}$$

which can be rewritten as follows:

$$\varepsilon(k+1) = (A - HC)\,\varepsilon(k) \tag{14}$$

The duality theorem [6] states that the design of a state observer is equivalent to designing a state feedback controller using some transformations in the state matrices. Based on the linear discrete system described as:

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k)\end{aligned} \tag{15}$$

The corresponding dual system becomes:

$$\begin{aligned}x_d(k+1) &= A_d x_d(k) + B_d u(k) \\ y_d(k) &= C_d x_d(k)\end{aligned} \tag{16}$$

where:

$$\begin{aligned}A_d &= A^t \\ B_d &= C^t \\ C_d &= B^t\end{aligned} \tag{17}$$

Therefore, it is possible to calculate a control matrix for the dual system K_d equivalent to the observation matrix for the original system H:

$$H = K_d^t \tag{18}$$

In this way, is possible calculate the observation matrix H, obtaining the controller matrix for the dual system K_d by any state controller design method in the dual system. In this case, a discrete LQR [5] is proposed, obtaining the H observer matrix from the dual system matrices $A_d = A^t$ and $B_d = C^t$, and the weighing matrices Q_d and R_d, minimizing the following index cost:

$$J = \sum_{k=0}^{\infty} \left[x_d(k)^t Q_d x_d(k) + u(k)^t R_d u(k)\right] \tag{19}$$

5 Results

In this section, we apply the proposed fuzzy optimal controller to the ball and beam nonlinear model described in Sect. 2. The ball and beam model works in continuous time, but the controller has been developed in discrete time, so a sampler and zero order holding device have been added to the model, supposing the proposed sampling time is $T = 0.05$ s. All system variables are supposed to be ideally measured.

As first step, a linear identification of the system has been made by least squares method, obtaining a first affine linear parameters estimation:

$$\begin{aligned} p(k) &= 0 + 3.92p(k-1) - 5.75p(k-2) + 3.75p(k-3) - 0.92p(k-4) \\ &\quad - 0.0001F(k-1) + 0.0001F(k-2) + 0.0001F(k-3) - 0.0001F(k-4) \end{aligned}$$

For a T-S model identification of the system, an iterative adjustment of the membership functions of the fuzzy rules and the weighting factor γ have been made, adjusting by try and error the fuzzy membership functions defined in Fig. 2 and the weighting factor $\gamma = 3.7 \cdot 10^{-6}$, obtaining an identification error of $1.61691\dot{0}^{-11}$.

With the generalized T-S identification method and Eq. (4), a fuzzy T-S state matrices of the system has been obtained:

$S^{(1,1,1,1)}$: If $p(k)$ is M_1^1 and $\dot{p}(k)$ is M_2^1 and $\alpha(k)$ is M_3^1 and $\dot{\alpha}(k)$ is M_4^1 then:

$$a_x^{(1,1,1,1)} = 10^{-5} \begin{bmatrix} 0.3146 & -0.4620 & 0.3014 & -0.0737 \end{bmatrix}^t$$

$$A^{(1,1,1,1)} = \begin{bmatrix} 3.9181 & 1 & 0 & 0 \\ -5.7542 & 0 & 1 & 0 \\ 3.7543 & 0 & 0 & 1 \\ -0.9181 & 0 & 0 & 0 \end{bmatrix}$$

$$B^{(1,1,1,1)} = 10^{-3} \begin{bmatrix} -0.0578 & 0.1418 & 0.1160 & -0.0544 \end{bmatrix}^t$$

$$a_x^{(1,1,1,1)} = 8.0285 \cdot 10^{-7}$$

$$C^{(1,1,1,1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\vdots$$

Thus, the controller matrix K is designed in each rule by discrete LQR [5] algorithm, using the system matrices A and B, and the positive definite weighting matrices $Q = I$ and $R = 1$. Obtaining the fuzzy controller matrix:

$S^{(1,1,1,1)}$: If $p(k)$ is M_1^1 and $\dot{p}(k)$ is M_2^1 and $\alpha(k)$ is M_3^1 and $\dot{\alpha}(k)$ is M_4^1 then:

$$K^{(1,1,1,1)} = 10^3 \begin{bmatrix} 2.9586 & 2.5483 & 2.1733 & 1.8322 \end{bmatrix}$$

$$\vdots$$

The observer algorithm is designed in each rule by duality theorem [6] and discrete LQR algorithm [5], using the dual system matrices $A_d = A^t$ and $B_d = C^t$, and the positive definite weighting matrices $Q_d = I$ and $R_d = 1$. Obtaining the fuzzy observer matrix by the Eq. (18):

$S^{(1,1,1,1)}$: If $p(k)$ is M_1^1 and $\dot{p}(k)$ is M_2^1 and $\alpha(k)$ is M_3^1 and $\dot{\alpha}(k)$ is M_4^1 then:

$$H^{(1,1,1,1)} = \begin{bmatrix} 2.7955 & -4.9873 & 3.4842 & -0.8789 \end{bmatrix}^t$$

$$\vdots$$

With this controller and observer design method, the global stability cannot be proved theoretically, so it has to be analyzed a posteriori with the simulation results.

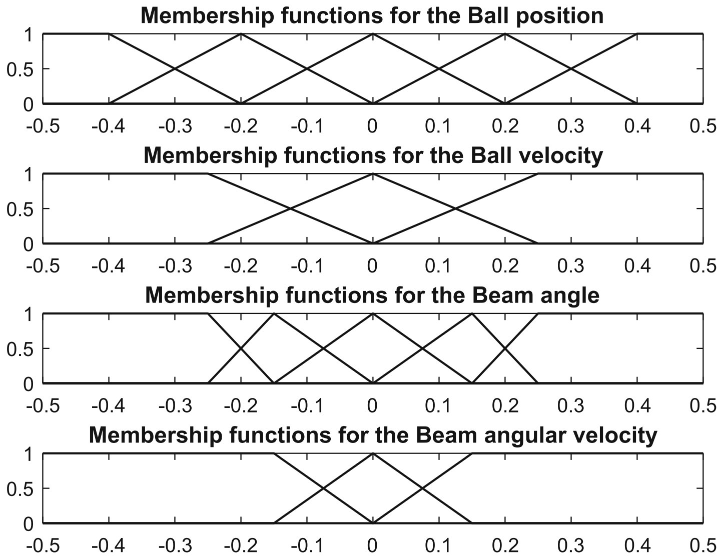

Fig. 2. Membership functions of the fuzzy sets.

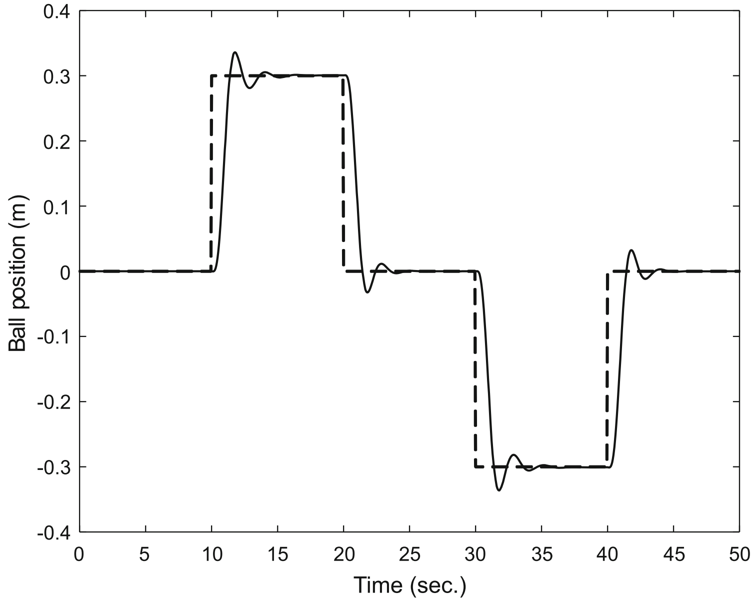

Fig. 3. Ball position.

Figures 3, 4, 5 and 6 show each one the position of the ball, the angle of the beam, the input force and the observation error of the ball position, when the controlled system is subjected to changes in the ball position reference.

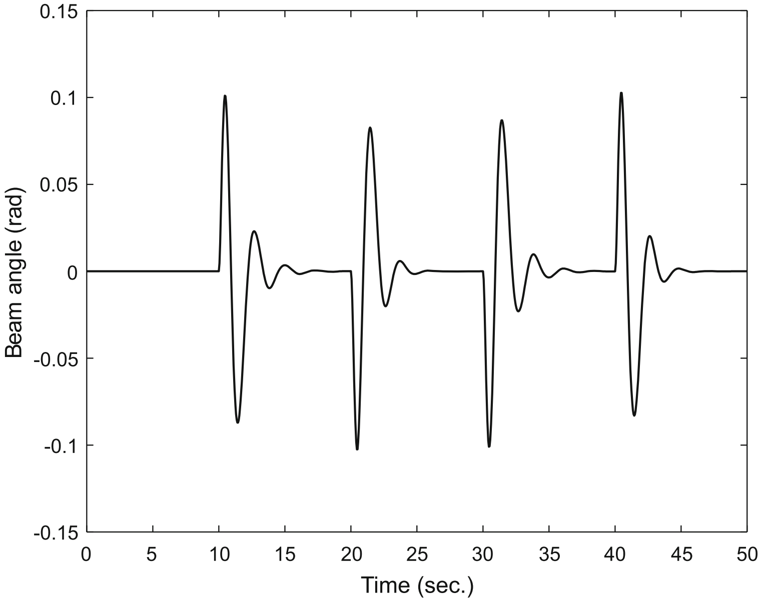

Fig. 4. Beam angle.

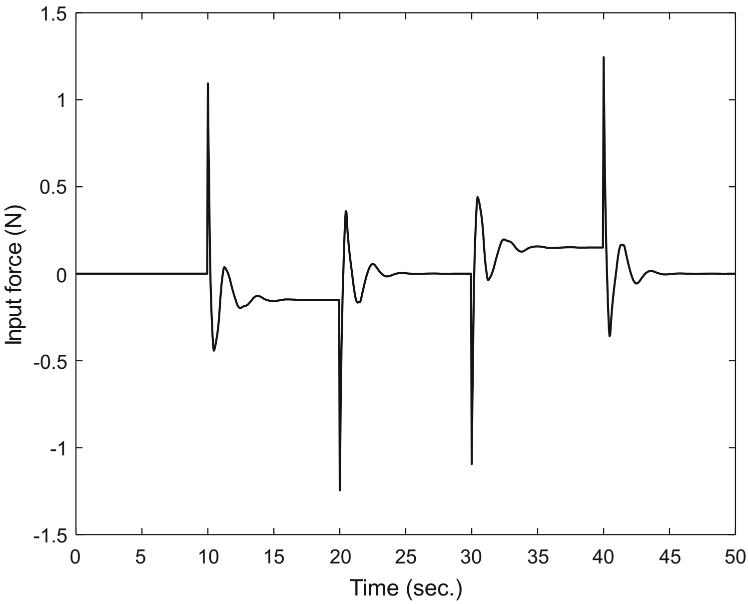

Fig. 5. Input force.

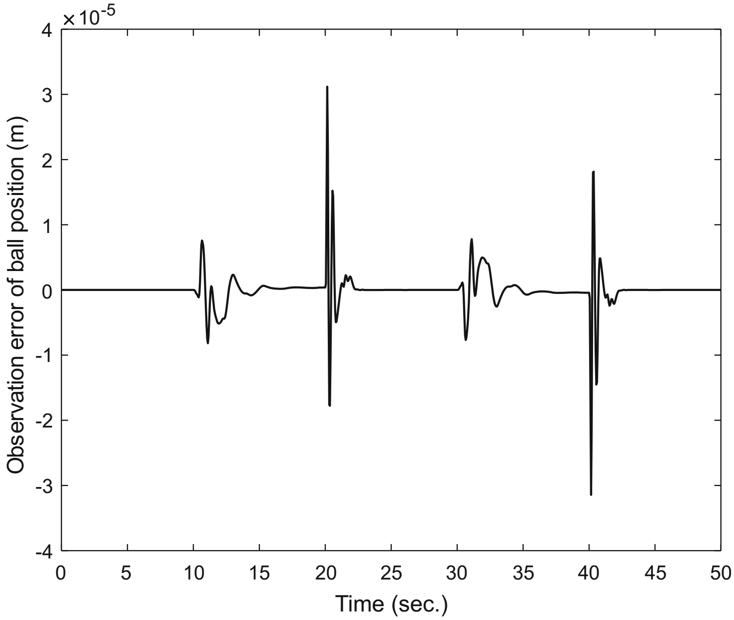

Fig. 6. Observation error of ball and beam position.

In Figs. 3, 4 and 5 it can be seen that, the system variables are in the physical range of the ball and beam, and all these variables present smooth and stable transient responses. In Fig. 6 it is shown that the observation error is small and tends to zero. Thus, the controlled ball and beam model has a stable response in the full range of the system and presents a good transient response.

6 Conclusion

In this work, we have shown the obtained results that a generalized T-S identification method and an optimal state controller and observer designed in each fuzzy rule, applied in a ball and beam nonlinear model. The fuzzy generalized T-S identification method is based on a weighting parameter of the previously estimated linear model. The optimal controller and observer has been designed in each fuzzy rule, so a suboptimal solution have been found, but easy of calculate and compute. The results show that the ball and beam controlled nonlinear model has a stable behavior and good transient response on the full range of the ball and beam system.

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The Classification of All the Subvarieties of $\mathbb{DNMG}$

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Abstract. MTL is the logic of all left-continuous t-norms and their residua. The equivalent algebraic semantics of MTL is constituted by the variety of MTL-algebras, $\mathbb{MTL}$. The variety $\mathbb{WNM}$ of *weak nilpotent minimum* algebras is a major subvariety of $\mathbb{MTL}$, containing several subvarieties of $\mathbb{MTL}$ which have been subjects of study in the literature, such as Gödel algebras, Nilpotent Minimum algebras, Drastic Product and Revised Drastic Product algebras, NMG-algebras, as well as Boolean algebras. In this paper we introduce and axiomatise $\mathbb{DNMG}$, a proper subvariety of $\mathbb{WNM}$ which contains all the aforementioned varieties. We show that $\mathbb{DNMG}$ is singly generated by a standard algebra. Further, we determine the structure of the lattice of subvarieties of $\mathbb{DNMG}$, and we provide the axiomatisation of every subvariety.

Keywords: WNM-algebras · DNMG-algebras · NM-algebras · Gödel-algebras · DP-algebras · Axiomatisations of subvarieties · Single chain completeness

1 Introduction

Nilpotent minimum t-norm $*_{\mathrm{NM}}$ [14] was one of the first examples of a left-continuous but not continuous t-norm. The logic related to $*_{\mathrm{NM}}$, NM, was introduced by Esteva and Godo in [12]. In the same paper they presented a generalisation of NM, the logic of Weak Nilpotent Minimum, WNM, and the related algebraic semantics, the variety of WNM-algebras $\mathbb{WNM}$. WNM is an extension of MTL, the logic of all left continuous t-norms and their residua [12,20]. Several extensions of WNM have been extensively studied in the literature. In particular, Gödel logic G, Drastic Product DP ([4], firstly introduced as $\mathrm{S}_3\mathrm{MTL}$ in [23]), Revised Drastic product RDP ([9,25], based on the t-norm introduced in [19]), NMG [26], NM [12], and classical Boolean logic B. During the years a number of topics concerning WNM and its algebraic semantics has been investigated: the papers [5,13,16,21,24] are only few examples. WNM has been extensively studied in [23], where the problem of axiomatising its extensions has been partially

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solved. The task of characterising and axiomatising the lattice of extensions of a given extension of WNM has been accomplished in some cases. Gispert [17], solved the case for NM. The lattice of extensions of G is well known. The extensions of EMTL, which is the largest common fragment of G and DP, have been axiomatised in [6].

In this article we introduce the variety $\mathbb{DNMG}$, the algebraic semantics of DNMG, which is an extension of WNM which is a particularly tame single-chain complete common fragment of G, NM and DP (and hence, of all the aforementioned extensions of WNM). We prove standard completeness for DNMG. Generalising Gispert's result, we describe the structure of the lattice of subvarieties of $\mathbb{DNMG}$, showing that each one of them is generated by finitely many chains. We further provide a uniform way to axiomatise each one of these subvarieties.

2 Preliminaries

We assume that the reader is acquainted with many-valued logics in Hájek's sense, and with their algebraic semantics. We refer to [11,18] for any unexplained notion. We recall that MTL is the logic, on the language $\{\&, \wedge, \vee, \rightarrow, \neg, \bot, \top\}$, of all left-continuous t-norms and their residua, and that its associated algebraic semantics in the sense of Blok and Pigozzi [7] is the variety $\mathbb{MTL}$ of *MTL-algebras*, that is, prelinear, commutative, bounded, integral, residuated lattices [11]. In an MTL-algebra $\mathcal{A} = (A, *, \Rightarrow, \sqcap, \sqcup, \sim, 0, 1)$ the connectives $\&, \rightarrow, \wedge, \vee, \neg, \bot, \top$ are interpreted, respectively, by $*, \Rightarrow, \sqcap, \sqcup, \sim, 0, 1$. Totally ordered MTL-algebras are called MTL-chains. In every chain $\sqcap = \min$ and $\sqcup = \max$. An MTL-algebra is called standard whenever its lattice reduct is $([0,1], \leq)$, with the usual order.

Given an MTL-algebra $\mathcal{A}$, with $\mathbf{V}(\mathcal{A})$ we mean the variety generated by $\mathcal{A}$, which is said to be *generic* for $\mathbf{V}(\mathcal{A})$. A logic L is the extension of MTL via a set of axioms $\{\varphi_i\}_{i\in I}$ if and only if $\mathbb{L}$ is the subvariety of MTL-algebras satisfying $\{\bar{\varphi}_i = 1\}_{i\in I}$, where $\bar{\varphi}_i$ is obtained from φ_i by replacing the connectives with the corresponding operations, and every propositional variable in φ with an individual variable. With $\mathcal{A} \models \bar{\varphi} = 1$ we mean that $\mathcal{A}$ satisfies $\bar{\varphi} = 1$.

The logic WNM [12] is axiomatised as MTL plus:

$$\neg(\varphi\&\psi) \vee ((\varphi \wedge \psi) \rightarrow (\varphi\&\psi)). \tag{wnm}$$

The logics G, DP, EMTL, RDP, NM, NMG [1,4,6,12,25,26] are axiomatised as WNM plus, respectively:

$$\varphi \rightarrow (\varphi\&\varphi). \tag{id}$$
$$\varphi \vee \neg(\varphi\&\varphi). \tag{dp}$$
$$(\varphi \rightarrow (\varphi\&\varphi)) \vee (\psi \vee \neg(\psi\&\psi)). \tag{emtl}$$
$$(\varphi \rightarrow \neg\varphi) \vee \neg\neg\varphi. \tag{rdp}$$
$$\neg\neg\varphi \rightarrow \varphi. \tag{inv}$$
$$\neg\neg\varphi \vee (\neg\neg\varphi \rightarrow \varphi). \tag{nmg}$$

NM$^-$ [17] is axiomatised as NM plus $\neg((\neg(\varphi^2))^2) \leftrightarrow (\neg((\neg\varphi)^2))^2$, where φ^2 stands for $\varphi\&\varphi$. B is classical logic, axiomatised as MTL plus $\varphi \vee \neg\varphi$.

As shown in [17], the operations $*, \Rightarrow$ of a WNM-chain $\mathcal{A}$, are:

$$x * y = \begin{cases} 0 & \text{if } x \leq {\sim}y \\ \min(x, y) & \text{Otherwise.} \end{cases} \qquad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max({\sim}x, y) & \text{Otherwise.} \end{cases} \tag{1}$$

Where $\sim$, the negation of $\mathcal{A}$, is a *(generalised) weak negation*, that is a map ${\sim}: A \to A$ such that ${\sim}1 = 0$, ${\sim}{\sim}a \geq a$, and if $a \leq b$, then ${\sim}a \geq {\sim}b$. Each weak negation is the negation of a uniquely determined WNM-chain. $\mathbb{WNM}$ is locally finite, i.e. for every WNM-algebra each one of its finitely generated subalgebras is finite. For each integer $n \geq 2$, with $\mathbb{G}_n$, $\mathbb{DP}_n, \mathbb{NM}^-_{2n}$ we will denote the variety generated, respectively, by the Gödel chain with n elements, the DP-chain with n elements, and the NM$^-$-chain with $2n$ elements.

3 DNMG-algebras

$\mathbb{DNMG}$ is the variety of WNM-algebras satisfying the following identity.

$${\sim}{\sim}x \sqcup ({\sim}{\sim}x \Rightarrow x) \sqcup ({\sim}{\sim}x \Leftrightarrow {\sim}x) = 1. \tag{DNMG}$$

Since each variety of MTL-algebras is generated by its chains, we immediately have that two subvarieties of MTL-algebras coincide iff they have the same class of chains. We are then going to analyse the structure of DNMG-chains.

Definition 1. *Let $\mathcal{A}$ be a WNM-chain. Let us define the following sets.*

- $A^+ = \{a \in A : a > {\sim}a\}$, *and* $A^- = \{a \in A : a < {\sim}a\}$.
- $S(A) = \{a \in A \mid {\sim}{\sim}a = 1, a \neq 1\}$.
- $F(A) = \{a \in A \mid {\sim}a = {\sim}{\sim}a\}$.
- $I^-(A) = \{a \in A \mid {\sim}{\sim}a = a, 0 < a < {\sim}a\}$.
- $I^+(A) = \{a \in A \mid {\sim}{\sim}a = a, 1 > a > {\sim}a\}$.
- $I(A) = I^-(A) \cup I^+(A)$.

Clearly, $I^-(A) \cap I^+(A) = \emptyset$. Further, $\sim$ is a bijection of $I(A)^+$ onto $I(A)^-$. Notice that $S(A)$ is disjunct from $I(A)$ and $F(A)$. Given $B, C \subseteq A$, we write $B \prec C$ whenever $b <_A c$, for every $b \in B$ and $c \in C$. The following is immediate.

Proposition 1. *For every WNM-chain $\mathcal{A}$ it holds that $I^-(A) \prec F(A) \prec I^+(A) \prec S(A)$. In particular, $I(A)^- \cup F(A) \cup \{0\} = A \setminus A^+$, and $I(A)^+ \cup S(A) \cup \{1\} = A^+$.*

By Proposition 1, the sets $S(A), F(A), I^-(A), I^+(A)$ are pairwise disjoint. For any subset $S \subseteq A$, let $\langle S \rangle$ be the subalgebra of $\mathcal{A}$ generated by S.

Proposition 2. *Let $\mathcal{A}$ be a WNM-chain and pick $S \subseteq A$.*

1. *If $S \subseteq I(A)$, then $\langle S\rangle = S \cup \{\sim a \mid a \in S\} \cup \{0,1\}$ is an NM$^-$ algebra.*
2. *If $S \subseteq F(A)$, then $\langle S\rangle = S \cup \{\sim a \mid a \in S\} \cup \{0,1\}$ is a DP-algebra.*
3. *If $S \subseteq S(A)$ then $\langle S\rangle = S \cup \{0,1\}$ is a Gödel algebra.*

Proof. Using (1) an easy check shows that the sets $(S \cap I(A)) \cup \{\sim a \mid a \in S \cap (I(A))\} \cup \{0,1\}$, $(S \cap F(A)) \cup \{\sim a \mid a \in S \cap (F(A))\} \cup \{0,1\}$ and $(S \cap S(A)) \cup \{0,1\}$ are subuniverses of A. The rest follows by Definition 1. □

Theorem 1. *A WNM-chain $\mathcal{A}$ is a DNMG-chain iff $A = S(A) \cup F(A) \cup I(A) \cup \{0,1\}$.*

Proof. Let $\mathcal{A}$ be a WNM-chain such that $A = S(A) \cup F(A) \cup I(A) \cup \{0,1\}$. Then, each element in $a \in S(A)$ satisfies $\sim\sim a = 1$, each element $a \in F(A)$ satisfies $\sim\sim a = \sim a$, and each element $a \in I(A)$ satifies $\sim\sim a = a$. The elements 0 and 1 both satisfy $\sim\sim a = a$. Whence $\mathcal{A}$ satisfies the identity (DNMG). Conversely, let $\mathcal{A}$ be a WNM-chain such that there is $a \in A \setminus (S(A) \cup F(A) \cup I(A) \cup \{0,1\})$. Then $\sim\sim a \neq 1$, $\sim\sim a \neq \sim a$ and $\sim\sim a \neq a$. Whence $\mathcal{A}$ is not a DNMG-chain. □

Lemma 1. *Let $\mathbb{V} \in \{\mathbb{DP}, \mathbb{NM}^-, \mathbb{G}\}$. For each subvariety $\mathbb{W}$ of $\mathbb{V}$, a chain $\mathcal{A}$ is generic for $\mathbb{W}$ iff each finite chain in $\mathbb{W}$ embeds into $\mathcal{A}$. Moreover, $\mathbf{V}(\mathcal{A}) = \mathbb{V}$ iff $\mathcal{A}$ is infinite.*

Proof. Let $\mathbb{V} \in \{\mathbb{DP}, \mathbb{NM}^-, \mathbb{G}\}$, and pick two chains $\mathcal{A}, \mathcal{B} \in \mathbb{V}$. By the results of [6,17] we have two consequences. First, $\mathbf{V}(\mathcal{A}) = \mathbb{V}$ iff $\mathcal{A}$ is infinite. Next, if $|\mathcal{A}| < |\mathcal{B}|$ then $\mathcal{A}$ embeds into $\mathcal{B}$. The claim follows immediately. □

Definition 2. *Let $C \subseteq \{F, I, S\}$, and let $\mathcal{A}$ be a WNM-chain. We say that $\mathcal{A}$ is C*-semigeneric *iff for any $X \in C$ and for any finite chain $\mathcal{B} \in \mathbb{WNM}$, it holds that $\langle X(B)\rangle$ embeds into $\langle X(A)\rangle$.*

In the following, for any $X \in \{F, I, S\}$, by $L(X)$-chain we mean: DP-chain if $X = F$, NM$^-$-chain if $X = I$, and G-chain if $X = S$.

Lemma 2. *For each $X \in \{F, I, S\}$, a WNM-chain $\mathcal{A}$ is X-semigeneric iff $\langle X(A)\rangle$ is generic for the variety generated by $L(X)$-chains.*

Proof. By Lemma 1 and Proposition 2. □

Definition 3. *For each $n > 0$, we let $e_n(F), e_n(I)$ and $e_n(S)$ denote the following terms*

$$e_n(F) = \bigsqcup_{i=1}^{n} ((\sim\sim x_i) \sqcup ((\sim\sim x_i \Rightarrow x_i) \sqcap (\sim((\sim(x_i^2))^2) \Leftrightarrow (\sim((\sim x_i)^2)))^2),$$

$$e_n(I) = \bigsqcup_{i=1}^{n} ((\sim\sim x_i) \sqcup (\sim x_i) \sqcup (\sim\sim x_i \Leftrightarrow \sim x_i)),$$

$$e_n(S) = \bigsqcup_{i=1}^{n} ((\sim\sim x_i \Leftrightarrow \sim x_i) \sqcup (\sim\sim x_i \Rightarrow x_i)).$$

Lemma 3. *Let $\mathcal{A}$ be a DNMG-chain, $X \in \{F, I, S\}$ and, for each $n > 0$, let $(a_1, a_2, \ldots, a_n) \in A^n$. Then, $(e_n(X))(a_1, \ldots, a_n) < 1$ iff $(a_1, \ldots, a_n) \in X(A)$.*

Proof. By Definition 1, it is sufficient to note that the ith disjunct of $e_n(X)$ evaluates to 1 iff $a_i \in A \setminus X(A)$. □

Lemma 4. *Let $\mathcal{A}$ be a non-trivial DNMG-chain, and $t(x_1, \ldots, x_n) = 1$ be an equation. Then, for every $X \in \{F, I, S\}$, the equation $t = 1$ holds in $\langle X(A)\rangle$ iff $t \sqcup e_n(X) = 1$ holds in $\mathcal{A}$.*

Proof. If $X(A) = \emptyset$, then $\langle X(A)\rangle$ is isomorphic with the two-element Boolean algebra $\{0, 1\}$, and by Lemma 3 $\mathcal{A} \models e_n(X) = 1$. The claim follows immediately. We then assume $X(A) \neq \emptyset$: by Lemma 3 we have that $\langle X(A)\rangle \not\models e_n(X) = 1$. Assume first that $\mathcal{A} \models t \sqcup e_n(X) = 1$: we must have $\langle X(A)\rangle \models t = 1$. Conversely, suppose $\mathcal{A} \not\models t \sqcup e_n(X) = 1$: by Lemma 3 we have that for some $a_1, \ldots, a_n \in \langle X(A)\rangle$, $e_n(X)(a_1, \ldots, a_n) < 1$ and $t(a_1, \ldots, a_n) < 1$. We conclude that $\mathcal{A} \not\models t = 1$. □

Lemma 5. *A DNMG-chain $\mathcal{A}$ embeds into a DNMG-chain $\mathcal{B}$ iff $\langle X(A)\rangle$ embeds into $\langle X(B)\rangle$ for each $X \in \{F, I, S\}$.*

Proof. One direction is trivial. Assume then that there is $X \in \{F, I, S\}$ such that $\langle X(A)\rangle$ does not embed into $\langle X(B)\rangle$, and assume further, by contradiction, that $f\colon A \to B$ is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Then there is $a \in \langle X(A)\rangle$ such that $f(a) \notin \langle X(B)\rangle$, for otherwise f would embed $\langle X(A)\rangle$ into $\langle X(B)\rangle$. Assume $X = F$ and $f(a) \in B \setminus \langle F(B)\rangle$. Then $\sim\sim a = \sim a$, while $\sim\sim f(a) \neq \sim f(a)$, contradicting the fact that f is an homomorphism. The other cases $X = I$ or $X = S$, are dealt with analogously. □

Lemma 6. *Let $\mathbb{V}$ be a subvariety of $\mathbb{DNMG}$. Then a DNMG-chain $\mathcal{A} \in \mathbb{V}$ is generic for $\mathbb{V}$ iff each finite chain $\mathcal{B}$ in $\mathbb{V}$ embeds into $\mathcal{A}$.*

Proof. Assume that each finite chain $\mathcal{B}$ in $\mathbb{V}$ embeds into $\mathcal{A}$. Since $\mathbb{V}$ has the finite model property, being locally finite, then it is generated by the class of its finite chains. This implies that $\mathbf{V}(\mathcal{A}) = \mathbb{V}$. On the other hand, assume that $\mathcal{B}$ does not embed into $\mathcal{A}$. Then, by Lemma 5, there is $X \in \{F, I, S\}$ such that $\langle X(B)\rangle$ does not embed into $\langle X(A)\rangle$. By Lemma 1, $\langle X(B)\rangle$ is an $L(X)$-chain which does not belong to the variety generated by the $L(X)$-chain $\langle X(A)\rangle$. Whence there is an equation $t(x_1, \ldots, x_n) = 1$ holding in $\langle X(A)\rangle$ and failing in $\langle X(B)\rangle$. Then, by Lemma 4, the equation $t \sqcup e_n(X) = 1$ holds in $\mathcal{A}$ and fails in $\mathcal{B}$, proving that $\mathcal{A}$ is not generic for $\mathbb{V}$. □

Lemma 7. *Let $\mathbb{V} \subseteq \mathbb{DNMG}$ be a single chain generated variety. Then there is a chain $\mathcal{B} \in \mathbb{V}$ such that every countable chain in $\mathbb{V}$ embeds into it, and $\mathbf{V}(\mathcal{B}) = \mathbb{V}$.*

Proof. Immediate by Lemma 6, [10, Theorem 3.8] and [22, Theorems 3 and 5].

Lemma 8. *A DNMG-chain is $\{F, I, S\}$-semigeneric iff it is generic for $\mathbb{DNMG}$.*

Proof. Immediate, from Lemmas 2, 5 and 6. □

Theorem 2. *A chain $\mathcal{A}$ is generic for $\mathbb{DNMG}$ iff the sets $F(A), I(A)$ and $S(A)$ are infinite.*

Proof. By Lemmas 1 and 8. □

We are ready to prove standard completeness for the logic DNMG whose associated algebraic semantics is given by $\mathbb{DNMG}$.

Definition 4. *Let $*\colon [0,1]^2 \to [0,1]$ be defined by:*

$$x * y = \begin{cases} 0 & \textit{if } x+y \le 3/4 \textit{ or } \max\{x,y\} \le 1/2 \\ \min\{x,y\} & \textit{otherwise.} \end{cases}$$

It is immediate to check that $*$ is a left-continuous t-norm. Moreover $*$ is the monoidal operation of the standard WNM-chain $[0,1]_*$ determined by the following negation: $\sim 0 = 1$, $\sim x = 3/4 - x$ for all $x \in (0,1/4) \cup (1/2,3/4]$, $\sim x = 1/2$ for all $x \in [1/4,1/2]$, $\sim x = 0$ for all $x \in [3/4,1]$.

Lemma 9. *The WNM-chain $[0,1]_*$ determined by the t-norm in Definition 4 is a DNMG-chain.*

Proof. We just check that any element $a \in [0,1/4) \cup (1/2,3/4)$ is such that $\sim\sim a = a$, while any element $a \in [1/4,1/2]$ is such that $\sim\sim a = \sim a = 1/2$, and finally any element $a \in [3/4,1]$ is such that $\sim\sim a = 1$. □

Theorem 3. *The logic DNMG is standard complete, since the variety $\mathbb{DNMG}$ is generated by $[0,1]_*$.*

Proof. By Lemma 9, $[0,1]_* \in \mathbb{DNMG}$. Now, $F([0,1]_*) = [1/4,1/2]$, $I([0,1]_*) = (0,1/4) \cup (1/2,3/4)$, and $S([0,1]_*) = [3/4,1)$. By Lemma 8 and Theorem 2, the standard chain $[0,1]_*$ is generic for $\mathbb{DNMG}$. □

4 The Lattice of Subvarieties of $\mathbb{DNMG}$

Let ω denote the ordinal $\{0,1,2,\ldots\}$ of the natural numbers, and let $\omega+1$ be the ordinal $\omega \cup \{\omega\}$. For any integer $n > 0$ and any sequence $\kappa_1, \kappa_2, \ldots, \kappa_n$ of ordinals, the *direct product* $\kappa_1 \times \kappa_2 \times \cdots \times \kappa_n$ is the poset obtained equipping the cartesian product with the pointwise order: $(a_1,\ldots,a_n) \le (b_1,\ldots,b_n)$ iff $a_i \le b_i$ for all $i \in \{1,2,\ldots,n\}$. We write $\kappa^{(n)}$ to mean the nth *direct power* of the ordinal κ, that is, the direct product of n copies of κ.[1] A subset $S \subseteq \kappa_1 \times \cdots \times \kappa_n$ is an *antichain* if for each $a, b \in S$, neither $a \le b$ nor $b \le a$. The set of all antichains of a poset P is denoted $\mathcal{AC}(P)$.

Lemma 10. *For $n > 0$, every antichain of $P = \kappa_1 \times \kappa_2 \times \cdots \times \kappa_n$ is finite.*

[1] We use this notation to distinguish direct powers from ordinal exponentiation.

Proof. By [8, Exercise 2.3.4] a poset is a well-quasi-order (wqo) iff (1) it has no infinite strictly decreasing sequences, i.e. $x_0 > x_1 > \dots$, and (2) it has no infinite antichains. As every ordinal κ_i is well-ordered, then it satisfies (1) and (2): whence it is a wqo. By [8, Lemma 2.3.9] P is also a wqo, whence each one of its antichains is finite. □

Corollary 1. *For each integer $n > 0$, the set $\mathcal{AC}((\omega+1)^{(n)})$ has cardinality $\aleph_0$.*

Proof. By Lemma 10, every antichain of $(\omega+1)^{(n)}$ is a finite set of n-tuples of $\omega \cup \{\omega\}$. Hence it can be coded in the natural numbers, i.e. there is an injective map $\mathcal{AC}((\omega+1)^{(n)}) \to \omega$. Whence, $|\mathcal{AC}((\omega+1)^{(n)})| \leq \aleph_0$. To conclude, note that $\mathcal{AC}((\omega+1)^{(1)}) = \omega+1$, whence, $|\mathcal{AC}((\omega+1)^{(n)})| \geq |\mathcal{AC}((\omega+1)^{(1)})| = \aleph_0$. □

The set $\mathcal{AC}((\omega+1)^{(n)})$ can be equipped with a lattice structure by putting $X \leq Y$ if for each n-tuple $x \in X$ there is $y \in Y$ such that $x \leq y$, for all $X, Y \in \mathcal{AC}((\omega+1)^{(n)})$. In this section we shall prove that the lattice of subvarieties of $\mathbb{DNMG}$ is isomorphic with $\mathcal{AC}((\omega+1)^{(3)})$.

Definition 5. *Let $\mathcal{A}$ be a DNMG-chain. Then the* triplet $T(\mathcal{A})$ *associated with $\mathcal{A}$ is an element $(a, b, c) \in (\omega+1)^{(3)}$ defined as follows.*

1. *If $S(A)$ is infinite then $a = \omega$, otherwise $a = |S(A)|$.*
2. *If $I^-(A)$ is infinite then $b = \omega$, otherwise $b = |I^-(A)|$.*
3. *If $F(A)$ is infinite then $c = \omega$, otherwise $c = |F(A)|$.*

Lemma 11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two DNMG-chains with $\mathcal{A}$ of finite cardinality. Then $\mathcal{A}$ embeds into $\mathcal{B}$ iff $T(\mathcal{A}) \leq T(\mathcal{B})$ in the pointwise order.*

Proof. $\mathcal{A}$ embeds into $\mathcal{B}$ iff, by Lemma 5, $\langle S(A)\rangle$ embeds into $\langle S(B)\rangle$, $\langle I(A)\rangle$ embeds into $\langle I(B)\rangle$ and $\langle F(A)\rangle$ embeds into $\langle F(B)\rangle$, iff $T(\mathcal{A}) \leq T(\mathcal{B})$. □

Lemma 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be two DNMG-chains. Then $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$ iff $T(\mathcal{A}) \leq T(\mathcal{B})$. As a consequence, $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{B})$ iff $T(\mathcal{A}) = T(\mathcal{B})$.*

Proof. Assume first that $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$. Then each chain in $\mathbf{V}(\mathcal{A})$ belongs to $\mathbf{V}(\mathcal{B})$, too. By Lemma 6, each finite chain $\mathcal{C} \in \mathbf{V}(\mathcal{A})$ embeds into both $\mathcal{A}$ and $\mathcal{B}$. By Lemma 11 this occurs only if $T(\mathcal{C}) \leq T(\mathcal{A})$ and $T(\mathcal{C}) \leq T(\mathcal{B})$. But then $\langle X(A)\rangle$ embeds into $\langle X(B)\rangle$ for each $X \in \{F, I, S\}$, or they are both of infinite cardinality. In both cases $T(\mathcal{A}) \leq T(\mathcal{B})$.

For the other way round, assume $T(\mathcal{A}) \leq T(\mathcal{B})$. Take now any finite chain $\mathcal{C} \in \mathbf{V}(\mathcal{A})$. Then by Lemma 6, $\mathcal{C}$ embeds into $\mathcal{A}$, and by Lemma 11, $T(\mathcal{C}) \leq T(\mathcal{A})$. By our standing assumption, $T(\mathcal{C}) \leq T(\mathcal{B})$, too. Whence, again by Lemma 11, $\mathcal{C}$ embeds into $\mathcal{B}$, which implies that each finite chain in $\mathbf{V}(\mathcal{A})$ belongs to $\mathbf{V}(\mathcal{B})$, too. We conclude $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$. □

Notice that by Lemmas 2 and 8 and Theorem 2, if C is a set of DNMG-chains containing an infinite chain $\mathcal{A}$ of any cardinality, then $\mathbf{V}(C) = \mathbf{V}(C')$ for $C' = (C \cup \{\mathcal{A}'\}) \setminus \{\mathcal{A}\}$, for a suitable $\mathcal{A}'$. More precisely, $\mathcal{A}'$ is obtained from $\mathcal{A}$ by replacing, for every $X \in \{F, I, S\}$ such that $X(A)$ is infinite, the subalgebra

$\langle X(A)\rangle$ with a fresh copy of a fixed denumerable $L(X)$-chain $\mathcal{B}$, where we safely identify the extremes of $\mathcal{B}$ with those of $\mathcal{A}$. We call $\mathcal{A}'$ a *regular* chain.

Then we may assume that each subvariety of $\mathbb{DNMG}$ is generated by a set of regular chains. Two regular DNMG-chains $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $T(\mathcal{A}) = T(\mathcal{B})$. We now fix, once and for all, one *representative* regular chain $\mathcal{A}_T$, for each triple $T \in (\omega+1)^{(3)}$. Notice that given two regular representative DNMG-chains $\mathcal{A}$ and $\mathcal{B}$, we have that $\mathcal{A}$ embeds into $\mathcal{B}$ iff $T(\mathcal{A}) \leq T(\mathcal{B})$. A set C of DNMG-chains is *irredundant* if each $\mathcal{A} \in C$ is representative regular and $\mathbf{V}(C \setminus \{\mathcal{A}\}) \subsetneq \mathbf{V}(C)$ for all $\mathcal{A} \in C$. Otherwise C is *redundant*.

Lemma 13. *Every irredundant set C of DNMG-chains is finite. Moreover, the map $\mathcal{A} \mapsto T(\mathcal{A})$ is a bijection between C and $C_T = \{T(\mathcal{A}) : \mathcal{A} \in C\}$, which is the underlying set of a finite antichain of $(\omega+1)^{(3)}$.*

Proof. Let C be an irredundant set of DNMG-chains. Then, for every $\mathcal{A}, \mathcal{B} \in C$, with $\mathcal{A} \neq \mathcal{B}$ we have that $T(\mathcal{A})$ is incomparable with $T(\mathcal{B})$. Indeed, if not, by Lemma 12 either $\mathbf{V}(C \setminus \{\mathcal{A}\}) = \mathbf{V}(C)$ or $\mathbf{V}(C \setminus \{\mathcal{B}\}) = \mathbf{V}(C)$. In both cases we have a contradiction. By the previous observations we have that C_T must be the underlying set of an antichain of $(\omega+1)^{(3)}$, and by Lemma 10 C_T is finite. The proof is settled by noticing that the map $\mathcal{A} \mapsto T(\mathcal{A})$ is a bijection between C and C_T. □

Lemma 14. *The lattice $\Lambda(\mathbb{DNMG})$ of all subvarieties of $\mathbb{DNMG}$, ordered by inclusion, is isomorphic with the lattice $\Gamma(\mathbb{DNMG})$ of all irredundant sets of DNMG-chains, ordered by inclusion.*

Proof. Clearly, each irredundant set of DNMG-chains generates a subvariety of $\mathbb{DNMG}$, and each subvariety is generated by some irredundant set of DNMG-chains. We prove than no subvariety can be generated by two distinct irredundant sets C, D of DNMG-chains, with $C \neq D$: by Lemma 13 we can assume $C = \{\mathcal{A}_1, \ldots, \mathcal{A}_h\}$ and $D = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$. By contradiction, suppose $\mathbf{V}(C) = \mathbf{V}(D)$.

Then, without loss of generality, there is a chain $\mathcal{A}_r \in C \setminus D$. By [6, Theorem 7], we have that the class of chains in $\mathbf{V}(C) = \mathbf{V}(D)$ coincides with the one in $\bigcup_{i=1}^{h} \mathbf{V}(\mathcal{A}_i)$ and with the one in $\bigcup_{i=1}^{k} \mathbf{V}(\mathcal{B}_i)$. This means that there is $\mathcal{B}_s \in D$ such that $\mathbf{V}(\mathcal{A}_r) \subseteq \mathbf{V}(\mathcal{B}_s)$, and clearly $\mathcal{B}_s \notin C$ (otherwise C would be redundant). This implies $T(\mathcal{A}_r) \leq T(\mathcal{B}_s)$ and hence $\mathcal{A}_r$ embeds into $\mathcal{B}_s$, being both chains regular representative. On the other hand, for the same reasons, this in turns implies that $\mathcal{B}_s$ embeds into some chain $\mathcal{A}_t \in C$. Note that $t \neq r$, as otherwise $\mathcal{A}_r \simeq \mathcal{B}_s$, and since both chains are regular representative we would conclude $\mathcal{A}_r = \mathcal{B}_s$, in contrast with the fact that $\mathcal{A}_r \notin D$. But then also $\mathcal{A}_r$ embeds into $\mathcal{A}_t$, contradicting the fact that C is irredundant. It is obvious that both the bijective correspondence $C \mapsto \mathbf{V}(C)$ and its inverse are order-preserving. □

Theorem 4. *The lattice $\Lambda(\mathbb{DNMG})$ is isomorphic with $\mathcal{AC}((\omega+1)^{(3)})$.*

Proof. By Lemma 14, we prove that $\Gamma(\mathbb{DNMG})$ is isomorphic with $\mathcal{AC}((\omega+1)^{(3)})$. Consider a set C of pairwise non-isomorphic representative chains in $\mathbb{DNMG}$

such that the set $C_T = \{T(\mathcal{A}) \mid \mathcal{A} \in C\}$ is not an antichain. Then there are chains $\mathcal{A}$ and $\mathcal{B} \in C$ such that $T(\mathcal{A}) \leq T(\mathcal{B})$. By Lemma 12, $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$. Whence, $\mathbf{V}(C) = \mathbf{V}(C \setminus \{\mathcal{A}\})$. That is C is redundant. Whence, each element of $\Gamma(\mathbb{DNMG})$ is a set C of chains such that C_T is an antichain, that is T maps $\Gamma(\mathbb{DNMG})$ to $\mathcal{AC}((\omega+1)^{(3)})$. T is injective by construction. For surjectivity, let $A \in \mathcal{AC}((\omega+1)^{(3)})$. With each triple $(a,b,c) \in A$ we associate the representative regular DNMG-chain $\mathcal{A}_{(a,b,c)}$ such that $T(\mathcal{A}_{(a,b,c)}) = (a,b,c)$. Call $C(A)$ the set of chains so obtained. Clearly, $T(C(A)) = A$ and all chains in $C(A)$ are representative and regular. Finally, for any triple $(a,b,c) \in A$ we have that $\mathbf{V}(C(A) \setminus \{\mathcal{A}_{(a,b,c)}\}) \subsetneq \mathbf{V}(C(A))$, for otherwise there is a triple $(d,e,f) \in A$ such that $(a,b,c) \leq (d,e,f)$, contradicting the fact that A is an antichain. Whence $C(A)$ is irredundant, that is, it belongs to $\Gamma(\mathbb{DNMG})$. Surjectivity is proved. □

Corollary 2. *There are countably many subvarieties of* $\mathbb{DNMG}$*. Every subvariety of* $\mathbb{DNMG}$ *is generated by a finite number of DNMG-chains.*

Proof. By Theorem 4, Lemma 10 and Corollary 1. □

5 Uniform Axiomatisations

In this section we axiomatise all the subvarieties of $\mathbb{DNMG}$ in a uniform way.

Definition 6. *For each integer $n > 0$ let Q_n be the term:*

$$Q_n = \bigsqcup_{1 \leq i \neq j \leq n+1} (x_i \Leftrightarrow x_j)\,.$$

Furthermore let $Q_0 = 0$ and $Q_\omega = 1$, and $e_{\omega+1}(X) = 1$ for each $X \in \{F, I, S\}$.

Given $(a,b,c) \in (\omega+1)^{(3)}$, we write $\mathbf{V}(a,b,c)$ to mean the variety generated by a DNMG-chain $\mathcal{A}$ such that $T(\mathcal{A}) = (a,b,c)$.

Theorem 5. *For each $(a,b,c) \in (\omega+1)^{(3)}$, the variety $\mathbf{V}(a,b,c)$ is the subvariety of* $\mathbb{DNMG}$ *satisfying the following equation.*

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_b \sqcup e_{b+1}(I)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

Proof. Let $\mathcal{D}$ be a DNMG-chain such that $T(\mathcal{D}) = (a,b,c)$. We show that $\mathcal{D} \models Q_a \sqcup e_{a+1}(S) = 1$. As a matter of fact, if $a = 0$ then by Lemma 3 $1 = e_{a+1}(d_1, \ldots, d_{a+1}) = (Q_a \sqcup e_{a+1}(S))(d_1, \ldots, d_{a+1})$, since $d_i \in D \setminus S(D)$ for all $i \in \{1, 2, \ldots, a+1\}$. If $a = \omega$ then $(Q_a \sqcup e_{a+1}(S))(d_1, \ldots, d_{m+1}) = 1$ for $Q_a = 1$. If a is a positive integer, then $(Q_a \sqcup e_{a+1}(S))(d_1, \ldots, d_{a+1}) = 1$. Indeed, if $d_i \in S(D)$ for all $i \in \{1, 2, \ldots, a+1\}$ then there are distinct indices i, j such that $d_i \Leftrightarrow d_j = 1$, for $|S(A)| = a$, whence $(Q_a \sqcup e_{a+1}(S))(d_1, \ldots, d_{a+1}) = Q_a(d_1, \ldots, d_{a+1}) = 1$. If on the other hand there is some $d_i \in D \setminus S(D)$, then, by Lemma 3, $e_{a+1}(S)(d_1, \ldots, d_{a+1}) = 1 = (Q_a \sqcup e_{a+1}(S))(d_1, \ldots, d_{a+1})$. We proceed similarly, to show that $\mathcal{D} \models Q_b \sqcup e_{b+1}(I) = 1$ and $\mathcal{D} \models Q_c \sqcup e_{c+1}(F) = 1$, *mutatis mutandis.* Let now $\mathcal{D}$ be a DNMG-chain not in $\mathbf{V}(a,b,c)$, whence $T(\mathcal{D}) = (a', b', c')$

with $(a', b', c') \not\leq (a, b, c)$. If $a' > a$ then picking pairwise distinct elements $a_1, \ldots, a_{a+1} \in S(D)$ we have that $Q_a(d_1, \ldots, d_{a+1}) < 1$. Further, by Lemma 3, $e_{a+1}(S)(d_1, \ldots, d_{a+1}) < 1$, too. We conclude that $\mathcal{D} \not\models Q_a \sqcup e_{a+1}(S) = 1$. The cases $b' > b$ and $c' > c$ are dealt with in the same manner, *mutatis mutandis.* □

Clearly, if some index a, b, c is zero, the associated conjunct in the axiomatising equation can be simplified: $Q_a \sqcup e_{a+1}(S)$ to $e_{a+1}(S)$ and so forth. Analogously, if some index a, b, c is ω, the associated conjunct can be totally disregarded. For instance $\mathbf{V}(\omega, \omega, \omega) = \mathbb{DNMG}$, and as a matter of fact the axiomatising equation in this case is identically $1 = 1$. As $\mathbb{DNMG}$ contains all major subvarieties of $\mathbb{WNM}$, several varieties of the form $\mathbf{V}(a, b, c)$ have already been studied and axiomatised in the literature. The following theorems report on this aspect.

Theorem 6. $\mathbf{V}(0, 0, 0) = \mathbb{B}$. *For each integer* $n > 0$, $\mathbf{V}(n, 0, 0) = \mathbb{G}_{n+2}$, $\mathbf{V}(0, n, 0) = \mathbb{NM}_{2n+2}$, $\mathbf{V}(0, 0, n) = \mathbb{DP}_{n+2}$. *Also,* $\mathbf{V}(\omega, 0, 0) = \mathbb{G}$, $\mathbf{V}(0, \omega, 0) = \mathbb{NM}^-$ *and* $\mathbf{V}(0, 0, \omega) = \mathbb{DP}$.

Proof. Immediate by [6, 17], and Definition 5. □

We now show how the axiomatisation provided in Theorem 5 can be simplified, when exactly one element in the triplet (a, b, c) is zero. In this case $\mathbf{V}(a, b, c)$ is either a subvariety of $\mathbb{RDP} = \mathbf{V}(\omega, 0, \omega)$, or of one of the following two subvarieties. The variety $\mathbb{DNM} = \mathbf{V}(0, \omega, \omega)$, axiomatised as $\mathbb{WNM}$ plus:

$$(\sim\sim x \Rightarrow x) \sqcup (\sim\sim x \Leftrightarrow \sim x) = 1. \tag{IF}$$

The variety $\mathbb{NMG}^- = \mathbf{V}(\omega, \omega, 0)$, axiomatised as $\mathbb{NMG}$ plus:

$$\sim((\sim(x^2))^2) \Leftrightarrow (\sim((\sim x)^2))^2 = 1. \tag{NF}$$

Theorem 7. *1. For all* $b, c \in (\omega + 1)^{(2)}$ *with* $b \neq 0 \neq c$, *the variety* $\mathbf{V}(0, b, c)$ *is axiomatised as* $\mathbb{DNM}$ *plus*

$$(Q_b \sqcup e_{b+1}(I)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

2. For all $a, c \in (\omega + 1)^{(2)}$ *with* $a \neq 0 \neq c$, *the variety* $\mathbf{V}(a, 0, c)$ *is axiomatised as* $\mathbb{RDP}$ *plus*

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

3. For all $a, b \in (\omega + 1)^{(2)}$ *with* $b \neq 0 \neq a$, *the variety* $\mathbf{V}(a, b, 0)$ *is axiomatised as* $\mathbb{NMG}^-$ *plus*

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_b \sqcup e_{b+1}(I)) = 1\,.$$

Proof. Immediate by Theorem 5.

We now provide the general criterion to axiomatise the subvarieties of $\mathbb{DNMG}$.

Theorem 8. *Let $C = \{\mathcal{A}_i\}_{i \in I}$ be an irredundant set of DNMG-chains. Let further $t_i(x_1, \dots, x_{n_i}) = 1$ be the equation axiomatising $\mathbf{V}(\mathcal{A}_i)$ for each $i \in I$, as given by Theorem 5. Then $\mathbf{V}(C)$ contains exactly the DNMG-algebras satisfying the equation*

$$\bigsqcup_{i \in I} t_i(y_{i,1}, \dots, y_{i,n_i}) = 1\,,$$

where all the variables $y_{i,j}$, for $i \in I$, and $j \in \{1, \dots, n_i\}$, are pairwise distinct.

Proof. First notice that by Corollary 2, $\bigsqcup_{i \in I} t_i = 1$ is indeed an equation, as I is a finite index set. The proof is settled by noting that $\mathbf{V}(C) = \bigsqcup_{i \in I} \mathbf{V}(\mathcal{A}_i)$, and by using [15, Lemma 5.25]. □

Corollary 3. *Every element of $\Lambda(\mathbb{DNMG})$ is the join of a finite set of join irreducible elements.*

Proof. By [2, Theorem 5.1] a variety of MTL-algebras is join irreducible, in the lattice of the subvarieties of $\mathbb{MTL}$, if and only if it is generated by a single chain. The claim follows by Theorem 8. □

Theorem 9. $\mathbb{DNMG}$ *is the smallest subvariety in $\Lambda(\mathbb{DNMG})$ which contains* $\mathbb{DP}$, $\mathbb{NM}^-$, $\mathbb{G}$ *and it is generated by a single chain.*

Proof. Immediate by Theorem 6 and Lemma 12, since $\mathbb{DNMG} = \mathbf{V}(\omega, \omega, \omega)$. □

Remark 1. Notice that $\mathbb{NM} = \mathbf{V}(0, \omega, 1)$, and its lattice of subvarieties is given by all antichains $C \in \mathcal{AC}((\omega + 1)^{(3)})$ such that all $T \in C$ have either the form $T = (0, b, 1)$ or $T = (0, b, 0)$ for some integer $b \geq 0$, or $T = (0, \omega, 0)$, whose corresponding variety is $\mathbb{NM}^-$.

The almost minimal subvarieties of $\mathbb{DNMG}$ are exactly $\mathbb{G}_3 = \mathbf{V}(1, 0, 0)$, $\mathbb{NM}_4 = \mathbf{V}(0, 1, 0)$, and $\mathbb{NM}_3 = \mathbb{DP}_3 = \mathbf{V}(0, 0, 1)$. Whence they coincide with the almost minimal subvarieties of $\mathbb{WNM}$ (see [3]).

By Lemma 7 and [10, Theorem 3.5], every variety of DNMG-algebras of the form $\mathbf{V}(a, b, c)$ is such that the corresponding logic has the strong single chain completeness (see [2, 22]).

The subvarieties of $\mathbb{DNMG}$ generated by a standard algebra are exactly $\mathbb{G} = \mathbf{V}(\omega, 0, 0)$, $\mathbb{NM} = \mathbf{V}(0, \omega, 1)$, $\mathbb{NMG} = \mathbf{V}(\omega, \omega, 1)$, $\mathbb{RDP} = \mathbf{V}(\omega, 0, \omega)$, $\mathbb{DNM} = \mathbf{V}(0, \omega, \omega)$, and, clearly, $\mathbb{DNMG} = \mathbf{V}(\omega, \omega, \omega)$.

Finally, $\mathbb{EMTL} = \mathbf{V}(\{(\omega, 0, 0), (0, 0, \omega)\})$ is an example of a subvariety of $\mathbb{DNMG}$ which cannot be generated by a single chain [6].

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Fuzzy Heyting Algebra

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Abstract. In this paper, we introduce the concept of fuzzy Heyting algebra (FHA) as an extension of Heyting algebra. We also characterize fuzzy Heyting algebra using the properties of Heyting algebra(HA) and distributive fuzzy lattices. We, finally, state and prove some results on fuzzy Heyting algebra.

Keywords: Heyting algebra · Fuzzy Heyting algebra · Fuzzy relation · Fuzzy poset · Distributive fuzzy lattices

MSC Code: 06D20 · 06D72 · 06D75

1 Introduction and Preliminaries

The concept of fuzzy set was first itroduced by Zadeh [8] and this concept was adapted by Goguen [11] to define and study fuzzy relations. G. Birkhoff [10] introduced the concept of Brouwerian lattice as a distributive lattice or Heyting algebra as a bounded distributive lattice in which for any two elements a, b there exists a largest element $a \to b$ such that $a \wedge (a \to b) \leq b$. Heyting Algebra is a relatively pseudo complemented distributive lattice. It arises from non classical logic and was first investigated by Skolem T [2]. It is named as Heyting Algebra after the Dutch Mathematician Arend Heyting [1]. In this paper we introduced the concept of Fuzzy Heyting Algebra (FHA) and studied some important properties of Fuzzy Heyting Algebra using fuzzy relation and fuzzy poset defined by Chon [4]. We also characterized fuzzy Heyting algebra using the directed above fuzzy poset and proved that any distributive fuzzy lattice is fuzzy Heyting algebra iff there exists a largest element c of H(Heyting Algebra) such that $A(a \wedge c, b) > 0$, for all a, b $\in$ H.

Definition 1.1. An algebra $(H, \vee, \wedge \to, 0, 1)$ is called a Heyting algebra if it satisfies the following

(1) $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice
(2) $a \to a = 1$
(3) $b \leq a \to b$
(4) $a \wedge (a \to b) = a \wedge b$

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(5) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$
(6) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$, for all a, b, c $\in$ H

Definition 1.2. A bounded distributive lattice$(H, \vee, \wedge, 0, 1)$ is said to be a Heyting Algebra if there exist a binary operation $\to$ on H such that, for any x, y, z $\in$ H, $x \wedge z \leq y \Leftrightarrow z \leq x \to y$

Theorem 1.3. *Let H be a Heyting algebra, then for any a, b, c $\in$ H, the following hold:*

(i) $a \wedge c \leq b \Leftrightarrow c \leq a \to b$
(ii) $a \leq b \Leftrightarrow a \to b = 1$

Lemma 1.4. *In any Heyting algebra H, the following hold:*

(a) $a \to (b \wedge a) = a \to b$
(b) $a \leq b \Rightarrow x \to a \leq x \to b$
(c) $a \leq b \Rightarrow b \to x \leq a \to x$*, for all* $a, b, c, x \in H$

Theorem 1.5. *If* $(H, \vee, \wedge, \to, 0, 1)$ *is a Heyting Algebra and a, b $\in$ H, then* $a \to b$ *is the largest element c of H such that* $a \wedge c \leq b$

Theorem 1.6. *The following are equivalent:*

(1) H is Heyting algebra
(2) For any a, b, c $\in$ H, $a \wedge c \leq b \Leftrightarrow c \leq a \to b$
(3) $b \leq a \to b, forall\ a, b \in H$

Definition 1.7. Let X be a set. A function A:$X \times X \to [0, 1]$ is called a fuzzy relation in X. The fuzzy relation A in X is reflexive iff $A(x, x) = 1$, for all $x \in X$. The fuzzy relation A in X is anti symmetric iff $A(x, y) > 0$ and $A(y, x) > 0 \Rightarrow x = y$. The fuzzy relation A in X is transitive iff $A(x, z) \geq Sup_{y \in X}(min(A(x, y), A(y, z)))$. A fuzzy relation A is fuzzy partial order relation if A is reflexive, symmetric and transitive. A fuzzy partial order relation A is fuzzy total order relation iff $A(x, y) > 0$ or $A(y, x) > 0$, for all $x, y \in$ H. If A is a fuzzy partial order relation on a set X, then (X,A) is called a fuzzy partially ordered set or a fuzzy poset. If A is a fuzzy total order relation in a set X, then (X,A) is called a fuzzy totally ordered set or a fuzzy chain.

Definition 1.8. Let (X,A) be a fuzzy poset and B $\subseteq$ X. An element $u \in$ X is said to be an upper bound for a subset B iff $A(b, u) > 0, \forall b \in$ B. An upper bound u_0 for a subset B is least upper bound of B iff $A(u_0, u) > 0$ for every upper bound u for B. An element $v \in$ X is said to be an lower bound for a subset B iff $A(v, b) > 0, \forall b \in$ B. A lower bound v_0 for a subset B is the greatest lower bound of B iff $A(v, v_0) > 0$ for every lower bound v for B. We denote the lub of the set $\{x, y\} = x \vee y$ and glb of the set $\{x, y\} = x \wedge y$

Definition 1.9. Let (X,A) be a fuzzy poset. (X,A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exists for all x, y $\in$ X.

Proposition 1.10. *Let (X,A) be a fuzzy lattice and x, y, z* $\in X$*. Then*

(i) $A(x, x \vee y) > 0, A(y, x \vee y) > 0, A(x \wedge y, x) > 0, A(x \wedge y, y) > 0$
(ii) $A(x, z) > 0$ *and* $A(y, z) > 0 \Rightarrow A(x \vee y, z) > 0$
(iii) $A(z, x) > 0$ *and* $A(z, y) > 0 \Rightarrow A(z, x \wedge y) > 0$
(iv) $A(x, y) > 0$ *iff* $x \vee y = y$
(v) $A(x, y) > 0$ *iff* $x \wedge y = x$
(vi) If $A(y, z) > 0$*, then* $A(x \wedge y, x \wedge z) > 0$ *and* $A(x \vee y, x \vee z) > 0$

Proposition 1.11. *Let (X,A) be a fuzzy lattice and x, y, z* $\in X$*. Then*

(1) $x \vee x = x, x \wedge x = x$
(2) $x \vee y = y \vee x, y \wedge x = x \wedge y$
(3) $(x \vee y) \vee z = x \vee (y \vee z), (x \wedge y) \wedge z = x \wedge (y \wedge z)$
(4) $(x \vee y) \wedge x = x, (x \wedge y) \vee x = x$

Definition 1.12. Let (H,A) be a fuzzy lattice. (H,A) is distributive iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$, for all x, y, z $\in$ H.

From distributive inequalities (H,A) is distributive iff $A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) > 0$ and $A((x \vee y) \wedge (x \vee z), x \vee (y \vee z)) > 0$ From now onwards by H, we mean Heyting algebra unless otherwise specified.

2 Fuzzy Heyting Algebra

In this section, we introduced the concept of fuzzy heyting algebra (FHA) and studied some important properties.

Definition 2.1. Let (H,A) be a fuzzy lattice. Then (H,A) is said to be a bounded fuzzy lattice iff

(1) $A(x \wedge 0, 0) = A(0, x \wedge 0) = 1$
(2) $A(x \vee 1, 1) = A(1, x \vee 1) = 1$, for all $x \in H$

Definition 2.2. A bounded distributive fuzzy lattice(H,A) is said to be a Fuzzy Heyting Algebra if there exists a binary operation $\rightarrow$ such that, for any x, y, z $\in H, A(x \wedge z, y) > 0 \Leftrightarrow A(z, x \rightarrow y) > 0$

Theorem 2.3. *Let (H,A) be a bounded distributive fuzzy lattice, then (H,A) is called a fuzzy Heyting algebra if it satisfies the following axioms:*

(1) $A(a \rightarrow a, 1) = A(1, a \rightarrow a) = 1$
(2) $A(b, a \rightarrow b) > 0$
(3) $A(a \wedge (a \rightarrow b), a \wedge b) = A(a \wedge b, a \wedge (a \rightarrow b)) = 1$
(4) $A(a \rightarrow (b \wedge c), (a \rightarrow b) \wedge (a \rightarrow c)) = A((a \rightarrow b) \wedge (a \rightarrow c), a \rightarrow (b \wedge c)) = 1$
(5) $A((a \vee b) \rightarrow c), (a \rightarrow c) \wedge (b \rightarrow c)) = A((a \rightarrow c) \wedge (b \rightarrow c), (a \vee b) \rightarrow c)) = 1$
for all a, b, c $\in H$

Example 2.4. Let $(B, \vee, \wedge, ', 0, 1)$ be a Boolean algebra and a, b $\in$ B and A:$B \times B \to$ [0,1] is a fuzzy relation. Define $a \to b = a^{'} \vee b$. Then (B,A) is a Fuzzy Heyting algebra

Proof: Clearly, $(B, \vee, \wedge, \to, 0, 1)$ is a Heyting algebra and (B,A) is a bounded distributive fuzzy lattice.

(1) $A(a \to a, a^{'} \vee a) = A(a^{'} \vee a, a \to a) = 1$
(2) $A((a \to b) \wedge b, (a^{'} \vee b) \wedge b) = A((a^{'} \vee b) \wedge b, (a \to b) \wedge b) = 1$
(3) $A(a \wedge (a \to b), a \wedge b) = A(a \wedge b, a \wedge (a \to b)) = 1$
(4) $A(a \to (b \wedge c), a^{'} \vee (b \wedge c)) = A(a \to b) \wedge (a \to c), a \to (b \wedge c)) = 1$
(5) $A((a \vee b) \to c, (a \to c) \wedge (b \to c)) = A((a \to c) \wedge (b \to c, (a \vee b) \to c)$, for all a, b, c $\in$ B. Thus,(B,A) is a fuzzy Heyting algebra

Lemma 2.5. *Let (H,A) be a bounded distributive fuzzy lattice. Then* $(H, \vee, \wedge, \to, 0, 1)$ *is a Heyting Algebra iff (H,A) is a fuzzy heyting algebra.*

From the definition of Heyting Algebra and fuzzy lattice property, we have the following lemma.

Lemma 2.6. $A(b, a \to b) > 0$ *iff* $b \wedge (a \to b) = b$ *or equivalently* $b \vee (a \to b) = a \to b$, $\forall a, b \in H$

Lemma 2.7. *In any Fuzzy Heyting Algebra the following holds:*

(i) $A(a \to (b \wedge a), a \to b) = 1$
(ii) $A(a, b) > 0 \Rightarrow A(x \to a, x \to b) > 0$
(iii) $A(a, b) > 0 \Rightarrow A(b \to x, a \to x) > 0$

Proof:

(i) $A(a \to (b \wedge a), a \to b)$= $A((a \to b) \wedge (a \to a), a \to b)$=$A((a \to b) \wedge 1, a \to b)$=$A(a \to b, a \to b)$=1
(ii) $A(a, b) > 0$ iff $a \wedge b = a$ or $a \vee b = b$ [By proposition 1.10] $A(x \to a, x \to b) = A(x \to (a \wedge b), x \to b)$[since a $\wedge$ b = a] =$A((x \to a) \wedge (x \to b), x \to b) > 0$ [since (x$\to a) \wedge (x \to b) \leq x \to b$]
(iii) $A(a, b) > 0$ iff $a \vee b = b$ [Proposition 1.10].
Now, $A(b \to x, a \to x) = A((a \vee b) \to x, a \to x) = A((a \to x) \wedge (b \to x), a \to x) > 0$

Theorem 2.8. *If (H,A) is a FHA and a, b* $\in$*H, then* $a \to b$ *is the largest element of the set S=*$\{c \in H : A(a \wedge c, b) > 0\}$

Proof: Let H be a FHA

We shall show that $a \to b \in$ S. Let a, b $\in$ H. Then $A(a \wedge (a \to b), a \wedge b) > 0$ clearly, $A(a \wedge b, b) > 0$. This implies, $A(a \wedge (a \to b), b) \geq sup_{a \wedge b}(minA(a \wedge (a \to b), a \wedge b), A(a \wedge b, b)) > 0$.
$\Rightarrow A(a \wedge (a \to b), b) > 0$.
$\Rightarrow a \to b \in$ S. Let d be such that $a \wedge d \leq b$. Then $A(a \wedge d, b) > 0$

$\Rightarrow A(a \to (a \wedge b), a \to b) > 0$
$\Rightarrow A(a \to d, a \to b) > 0.$
$\Rightarrow (d, a \to b) > 0$
$\Rightarrow A(d, a \to b) \geq Sup_{a \to d \in H}(min(A(d, a \to d), A(a \to d, a \to b))) > 0$
$\Rightarrow A(d, a \to b) > 0$
$\Rightarrow a \to b$ is an upper bound of d.
Thus, $a \to b$ is the largest element of S.

Lemma 2.9. *Let (H,A) be a fuzzy Heyting algebra, then for any a, b, c $\in$ H, we have* $A(a, b) > 0 \Leftrightarrow A(a \to b, 1) = 1 = A(1, a \to b)$

Proof. Suppose $A(a, b) > 0$. Then $A(a \to a, a \to b) > 0$.
$\Rightarrow A(1, a \to b) > 0$. But a$\to b \leq 1$, as 1 is the largest element.
$\Rightarrow (a \to b, 1) > 0 \Rightarrow a \to b = 1$. Hence the result. Conversely, assume $A(a \to b, 1) = A(1, a \to b)$. Then, $a \wedge (a \to b) = a \wedge 1$
$\Rightarrow a \wedge (a \to b) = a$
$\Rightarrow a \wedge b = a$
$\Rightarrow a \leq b$. Hence, $A(a, b) > 0$.

Theorem 2.10. *let (H,A) be a fuzzy Heyting algebra, then the following are equivalent.*

(1) $A(a \wedge c, b) > 0$
(2) $A(a \to c, a \to b) > 0$
(3) $A(c, a \to b) > 0$, *for* $a, b, c \in H$

Proof: straightforward

Theorem 2.11. *Every Fuzzy Heyting algebra is a distributive Fuzzy lattice*

Proof: Since $A(x, x \vee y) > 0$, we have $A(y \wedge x, (y \wedge x) \vee (z \wedge x)) > 0$. Hence $A(y, x \to (y \wedge x) \vee (z \vee x)) > 0$. Similarly, $A(z, x \to (y \wedge x) \vee (z \wedge x)) > 0$. This implies $A(y \vee z, x \to (y \wedge x) \vee (z \wedge x)) > 0$.
$\Rightarrow A(x \wedge (y \vee z), x \wedge (x \to (y \wedge x) \vee (z \wedge x))) > 0.$
$\Rightarrow A(x \wedge (y \vee z), x \wedge (y \wedge x) \vee (z \wedge x)) > 0.$
$\Rightarrow A(x \wedge (y \vee z), (y \wedge x) \vee (z \wedge x)) > 0.$ *
From $A(y, y \vee z) > 0$ and $A(y \wedge x, y) > 0$ and $A(y \wedge x, x) > 0$, We have $A(y \wedge x, (y \vee z) \wedge x) > 0$. Similarly, $A(z \wedge x, (y \vee z) \wedge x) > 0$ Thus, $A((y \wedge x) \vee (z \wedge x), (y \vee z) \wedge x) > 0$. **
From * and ** we have the result. Hence the theorem follows.

Definition 2.12. The fuzzy poset (H,A) is said to be directed above if $\forall a, b, c \in H, A(a, c) > 0$ and $A(b, c) > 0$, then $\exists x \in$ H such that $A(x, c) > 0$.

Theorem 2.13. *The following are equivalent:*

(1) (H,A) is a fuzzy Heyting algebra
(2) The fuzzy poset (H,A) is directed above

(3) (H,A) is a distributive fuzzy lattice

Proof: (1) $\Rightarrow$ (2) Let a, b $\in$ H. Then $\exists c \in H$ such that $A(a,c) > 0$ and $A(b,c) > 0$ $\Rightarrow A(a \vee b, c) > 0$. Take $x = a \vee b \in H$. Hence $A(x,c) > 0$ (2) $\Rightarrow$ (3) Suppose (2) holds.
Then, $A(a \vee b, c) > 0$.
$\Rightarrow A(c \wedge (a \vee b), c \wedge c) > 0$
Claim: $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) > 0$ and $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) > 0$.
We know $A(a \vee b, c) > 0, A(a,c) > 0, A(b,c) > 0, A(c,c) > 0$.
$\Rightarrow A(b \vee c, c) > 0$.
$\Rightarrow A(a \wedge (b \vee c), a \wedge c) > 0$. It is clear that $A((a \vee b) \wedge (a \vee c), a \vee c) > 0$ and $A((a \wedge b), (a \wedge b) \vee (a \wedge c)) > 0$. Also $A((a \wedge c), (a \wedge b) \vee (a \wedge c)) > 0$ Thus, we have $A(a \wedge (b \vee c), (a \wedge b) \vee (a \vee c)) \geq Sup_{a \wedge c \in H}(min(A(a \wedge (b \vee c), a \wedge\ c),$ $A((a \wedge c), (a \wedge c) \vee (a \wedge b))) > 0$. Thus, $A(a \wedge (b \vee c), (a \wedge b) \vee (a \vee c)) > 0$. similarly $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) > 0$. Therefore, (H,A) is a distributive fuzzy lattice (3) $\Rightarrow$ (1) Suppose (H,A) is a distributive fuzzy lattice such that $A(a \wedge c, b) > 0$
Claim: $A(c, a \rightarrow b) > 0$. Clearly, $A(a \wedge c, a \wedge b) > 0$
$\Rightarrow A(a \rightarrow (a \wedge c), a \rightarrow (a \wedge b)) > 0$
$\Rightarrow A(a \rightarrow c, a \rightarrow b) > 0$,but $A(c, a \rightarrow c) > 0$
$\Rightarrow A(c, a \rightarrow b) \geq sup_{a \rightarrow c \in H}(min(A(c, a \rightarrow c), A(a \rightarrow c, a \rightarrow b))) > 0$ Hence, (H,A)is a fuzzy Heyting Algebra

Lemma 2.14. *If $A(a,c) > 0$ and $A(b,c) > 0$, then we have the following.*

(1) $A(a \wedge b, b \wedge a) > 0$
(2) $A(a \vee b, b \vee a) > 0$
(3) $A((a \rightarrow c) \wedge (b \rightarrow c, 1) > 0$
(4) $A((c \rightarrow a) \wedge (c \rightarrow b), 1) > 0$
(5) $A(((a \rightarrow c) \wedge (b \rightarrow c)) \vee ((c \rightarrow a) \wedge (c \rightarrow b)), 1) > 0$

Theorem 2.15. *Let (H,A) be a distributive fuzzy lattice. Then (H,A) is a fuzzy Heyting algebra iff for any $a, b \in H$, there exists a largest element $c \in H$ such that $A(a \wedge c, b) > 0$.*

Proof: $(\Rightarrow).Clearly, A(a \wedge (a \rightarrow b), a \wedge b) > 0$ and $A(a \wedge b, b) > 0$. This implies $A(a \wedge c, b) > 0$ Let $d \in$ H such that $A(a \wedge d, b) > 0$. We shall prove that $A(d,c) > 0$. $A(a \wedge d, b) > 0$.
$\Rightarrow A(a \rightarrow d, a \rightarrow b) > 0$, but $A(d, a \rightarrow d) > 0$
$\Rightarrow A(d, a \rightarrow b) > 0$. Taking $c = a \rightarrow b$, we have $A(d,c) > 0$. Therefore, there is a largest element $c \in$ H such that $A(a \wedge c, b) > 0$.
Conversely, suppose the given condition holds. Define a binary operation $\rightarrow$ on H such that $a \rightarrow b$ is the largest element of the set$\{c \in H : A(a \wedge c, b) > 0\}$.
we prove that (H,A) is a fuzzy Heyting algebra

(1) Clearly $A(b, a \rightarrow b) > 0$. For $b \in$ H, since $A(a \wedge b, a) > 0$, we have
$A(a \rightarrow (a \wedge b), a \rightarrow a) > 0$
$\Rightarrow A(b, a \rightarrow a) > 0$
$\Rightarrow a \rightarrow a$ is an upper bound of b.

But $a \to$ a=1
$\Rightarrow A(a \to a, 1) = A(1, a \to a) = 1$

(2) since $A(a \wedge b, b) > 0$. Then $A(a \to b, a \to b) > 0$
$\Rightarrow A(b, a \to b) > 0$

(3) Since $A(a \wedge (a \to b), b) > 0$, we have $A(a \wedge (a \wedge (a \to b)), a \wedge b) > 0$
$\Rightarrow A(a \wedge (a \to b), a \wedge b) > 0$, on the other hand, $A(a \wedge b, b) > 0$.
$\Rightarrow A(b, a \to b) > 0$
$\Rightarrow A(a \wedge b, a \wedge (a \to b)) > 0$ from anti symmetry we have $a \wedge (a \to b) = a \wedge b$
Thus, we have $A(a \wedge (a \to b), a \wedge b) = A(a \wedge b, a \wedge (a \to b) = 1$

(4) From Heyting algebra, we have $a \wedge (a \to (b \wedge c)) = a \wedge a \wedge (b \wedge c) \leq b$, we have $A(a \wedge (a \to (b \wedge c), b) > 0$
$\Rightarrow A(a \wedge (b \wedge c), b) > 0$
$\Rightarrow A(a \to (b \wedge c), a \to b) > 0$. Similarly, $A(a \to (b \wedge c), a \to c) > 0$
$\Rightarrow a \to (b \wedge c)$ is a lower bound of $\{a \to b, a \to c\}$
$\Rightarrow A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) > 0$.On the other hand $A(a \wedge (a \to b) \wedge (a \to c), a \wedge b \wedge (a \to c)) > 0$
$\Rightarrow A(a \wedge (a \to b) \wedge (a \to c), b \wedge a \wedge (a \to c)) > 0$
$\Rightarrow A(a \wedge b \wedge c, b \wedge a \wedge c) > 0$
$\Rightarrow A((a \to b) \wedge (a \to c), (a \to b) \wedge (a \to c)) > 0 \Rightarrow A((a \to b) \wedge (a \to c), a \to (b \wedge c)) > 0$
Therefore, $A((a \to b) \wedge (a \to c), a \to (b \wedge c)) = A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) = 1$

(5) Consider $A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \wedge (a \to c) \wedge (b \to c)) \vee (b \wedge (a \to c) \wedge (b \to c)) > 0$
$\Rightarrow A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \wedge c) \vee (b \wedge c)) > 0$
$\Rightarrow A((a \vee b) \wedge (a \to c) \wedge (b \to c), (a \vee b) \wedge c) > 0$ but $A((a \vee b) \wedge c, c) > 0$
$\Rightarrow A((a \vee b) \wedge (a \to c) \wedge (b \to c), c) > 0$ By
Theorem 1.3 we have $A((a \to c) \wedge (b \to c), (a \vee b) \to c) > 0$. On the other hand $A(a, a \vee b) > 0 \Rightarrow A((a \vee b) \to c, a \to c) > 0$. Similarly, $A((a \vee b) \to c, b \to c) > 0$
$\Rightarrow A((a \vee b) \to c, (a \to c) \wedge (b \to c)) > 0$
Hence, $A((a \vee b) \to c, (a \to c) \wedge (b \to c)) = A((a \to c) \wedge (b \to c), (a \vee b) \to c)$
Therefore (H,A) is a fuzzy Heyting Algebra

Definition 2.16. Let (H,A) be a distributive fuzzy lattice. Then the fuzzy Heyting algebra (H,A) satisfies the infinite meet distributive fuzzy law if $A(a \wedge (\vee_{i \in I} s_i), \vee_{i \in I}(a \wedge s_i))) = A(\vee_{i \in I}(a \wedge s_i), a \wedge (\vee_{i \in I} s_i)) = 1$
where

$$\{s_i : i \in I\} \subseteq H$$

Theorem 2.17. *Let (H,A) be a distributive fuzzy lattice. Then (H,A)is a fuzzy heyting algebra iff it satisfies the infinite meet distributive fuzzy law. That is for any family*

$$\{s_i : i \in I\} \subseteq H$$

if $\vee_{i \in I} s_i$ *exists, then* $\vee_{i \in I}(a \wedge s_i)$ *exists for any* $a \in H$ *and it is equal to*

$$a \wedge (\vee_{i \in I} s_i).$$

Proof: Let (H,A)be a distributive fuzzy lattice and a, b $\in$ H. Define

$$a \rightarrow b = \vee_{s \in S_{ab}} s, \text{where } S_{ab} = \{s \in H : A(a \wedge s, b) > 0\}$$

Now, let a, b, c $\in$ H, Then

(1) $S_{aa} = \{s \in H : A(a \wedge s, a) > 0\} = (H, A)$
$\Rightarrow a \rightarrow a = \vee H = 1$
Thus, $A(a \rightarrow a, 1) = A(1, a \rightarrow a) = 1$

(2) since $A(a \wedge b, b) > 0$, we have $b \in S_{ab}$. This implies $A(b, a \rightarrow b) > 0$. Thus, $A((a \rightarrow b) \wedge b, b) = A(b, (a \rightarrow b) \wedge b) = 1$

(3) $A(a \wedge (a \rightarrow b), a) > 0$ and $A(a \wedge (a \rightarrow b), a \wedge (\vee_{s \in S_{ab}} s)) > 0$
$\Rightarrow A(\vee_{s \in S_{ab}} (a \wedge s), b) > 0$
$\Rightarrow A(a \wedge (a \rightarrow b), \vee_{s \in S_{ab}} (a \wedge s)) > 0$
$\Rightarrow A(a \wedge (a \rightarrow b), b) > 0.$
Hence $a \wedge (a \rightarrow b)$ is a lower bound of $\{a, b\}$
$\Rightarrow A(a \wedge (a \rightarrow b), a \wedge b) > 0$. On the other hand, we have $A(a \wedge (a \wedge b), b) > 0.$
$\Rightarrow a \wedge b \in S_{ab}$
$\Rightarrow A(a \wedge b, a \rightarrow b) > 0$
$\Rightarrow A(a \wedge (a \wedge b), a \wedge (a \rightarrow b)) > 0$
$\Rightarrow A(a \wedge b, a \wedge (a \rightarrow b)) > 0$ Thus, $A(a \wedge b, a \wedge (a \rightarrow b)) = A(a \wedge (a \rightarrow b), a \wedge b) = 1$

(4) Since $A(a \wedge (a \rightarrow (b \wedge c)), b) > 0.$
$\Rightarrow A(a \rightarrow (b \wedge c), a \rightarrow b) > 0$
Similarly, $A(a \rightarrow (b \wedge c), a \rightarrow c) > 0.$
$\Rightarrow a \rightarrow (b \wedge c)$ is a lower bound of $\{a \rightarrow b, a \rightarrow c\}$
$\Rightarrow A(a \rightarrow (b \wedge c), (a \rightarrow b) \wedge (a \rightarrow c)) > 0$. On the other hand, $A(a \wedge (a \rightarrow b) \wedge (a \rightarrow c), a \wedge b \wedge (a \rightarrow c)) > 0$
$\Rightarrow A(a \wedge (a \rightarrow b) \wedge (a \rightarrow c), a \wedge b \wedge c) > 0$
$\Rightarrow A(a \wedge (a \rightarrow b) \wedge (a \rightarrow c), b \wedge c) > 0$
$\Rightarrow A((a \rightarrow b) \wedge (a \rightarrow c), a \rightarrow (b \wedge c)) > 0$. Hence, $A((a \rightarrow b) \wedge (a \rightarrow c), a \rightarrow (b \wedge c)) = A(a \rightarrow (b \wedge c), (a \rightarrow b) \wedge (a \rightarrow c)) = 1$

(5) Consider, $A((a \vee b) \wedge (a \rightarrow c) \wedge (b \rightarrow c), (a \wedge (a \rightarrow c) \wedge (b \wedge c)) \vee (b \wedge (a \rightarrow c) \wedge)(b \wedge c)) > 0$
$\Rightarrow A((a \rightarrow) \wedge (b \rightarrow c), (a \vee b) \rightarrow c) > 0$. Since $A(a, a \vee b) > 0, A(b, a \vee b) > 0$. This implies $A((a \vee b) \rightarrow c, a \rightarrow c) > 0$ and $A((a \vee b) \rightarrow c, b \rightarrow c) > 0$
$(a \vee b) \rightarrow c$ is a lower bound of $\{a \rightarrow c, b \rightarrow c\}$.
$\Rightarrow A((a \vee b) \rightarrow c, (a \rightarrow c) \wedge (b \rightarrow c)) > 0$. Hence, $A((a \vee b) \rightarrow c, (a \rightarrow c) \wedge (b \rightarrow c)) = A((a \rightarrow) \wedge (b \rightarrow c), (a \vee b) \rightarrow c) = 1$. Therefore,(H,A) is Fuzzy Heyting algebra. Conversely,Suppose (H,A) be a fuzzy Heyting algebra. Let a$\in$ H, $\{s_i : i \in I\} \subseteq$H. Then $a \wedge s_i \in$ H. Since $A(s_i, \vee s_i) > 0$ we have $A(\vee_{i \in I}(a \wedge s_i), a \wedge (\vee_{i \in I} s_i)) > 0$.On the other hand $A(a \wedge s_i, \vee_{i \in I}(a \wedge s_i)) > 0$.
$\Rightarrow A(s_i, a \rightarrow \vee_{i \in I}(a \wedge s_i)) > 0, \forall i \in I$
$\Rightarrow A(\vee_{i \in I} s_i, a \rightarrow \vee_{i \in I}(a \wedge s_i)) > 0$
$\Rightarrow A(a \wedge (\vee_{i \in I} s_i), a \wedge (a \rightarrow \vee_{i \in I}(a \wedge s_i))) > 0$
but, $A(a \wedge \vee_{i \in I}(a \wedge s_i), \vee_{i \in I}(a \wedge s_i)) > 0$

$\Rightarrow A(a \wedge (\vee_{i\in I} s_i), \vee_{i\in I}(a \wedge s_i))) > 0$
$A(a \wedge (\vee_{i\in I} s_i), \vee_{i\in I}(a \wedge s_i))) = A(\vee_{i\in I}(a \wedge s_i), a \wedge (\vee_{i\in I} s_i)) = 1$

Future Work and Plan:

From now on wards, we will try to introduce fuzzy Heyting ideals and filters using the concept of fuzzy ideals and filters of fuzzy lattices.

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A New Distance on Generalized Fuzzy Numbers and a Glimpse on Their Properties

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Abstract. Normalization is the dominant but inexact method to handle any nonnormal fuzzy sets data. This stems from the fact that normalization ignores some parts of such data in order to prepare them for being used in computational operations. A subset of such data which satisfies the property of convexity is called Generalized Fuzzy Numbers (GFN). In this paper, a new distance is presented on the set of GFNs. In the special case, when GFNs are normal (i.e. Fuzzy Numbers), the proposed distance is converted to a well-known distance which in the fuzzy literature has already been proved to be a metric. Also, some of the features of the proposed distance are studied through several examples.

1 Introduction

Till now, a plenty of fuzzy logic distances have been proposed for fuzzy normal and convex fuzzy sets named fuzzy numbers. For instance: Amirfakhrian and Abbasbandy [1,2] introduced source distance between fuzzy numbers, Gerzgorzewski [10,11] proposed P- distance and Allahviranloo, et al. [3] presented TRD distance. However, in reality, a large portion of fuzzy data is non-normal which meet the condition of convexity. A special case of such data called Generalized Fuzzy Numbers (GFNs) has been studied by Chen in 1985 [5]. The principal trend till now for treating any non-normal fuzzy sets is converting them into fuzzy numbers via normalization. However, such an approach notoriously suffers from data loss. To mitigate this, in this paper we present a distance independent of height on GFNs. We also investigate some of the important properties of this distance.

2 Basic Concepts

In this section, the essential basic concepts used throughout the paper are given. Let $\mathcal{F}(\mathbb{R})$ be the set of all fuzzy numbers (the set of all normal and convex fuzzy sets [14,16]) on the real line.

Definition 1. *A generalized LR fuzzy number* $\tilde{u}$ *with the membership function* $\mu_{\tilde{u}}(x)$, $x \in \mathbb{R}$ *can be defined as [1, 7, 8]:*

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$$\mu_{\tilde{u}}(x) = \begin{cases} L_{\tilde{u}}(x), & a \le x \le b, \\ 1, & b \le x \le c, \\ R_{\tilde{u}}(x), & c \le x \le d, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $L_{\tilde{u}}$ is the left membership function and $R_{\tilde{u}}$ is the right membership function. It is assumed that $L_{\tilde{u}}$ is increasing in $[a, b]$ and $R_{\tilde{u}}$ is decreasing in $[a, b]$, and that $L_{\tilde{u}}(a) = R_{\tilde{u}}(d) = 0$ and $L_{\tilde{u}}(b) = R_{\tilde{u}}(c) = 1$. In addition, if $L_{\tilde{u}}$ and $R_{\tilde{u}}$ are linear, then $\tilde{u}$ is a trapezoidal fuzzy number, which is denoted by $\tilde{u} = (a, b, c, d)$. If $b = c$, we denoted it by $\tilde{u} = (a, c, d)$, which is a triangular fuzzy number.

Definition 2. *A fuzzy set $\tilde{A}$ is convex if $\forall x, y \in \tilde{A}$ and $\forall \lambda \in [0, 1]$ we have [16]:*

$$\mu_{\tilde{A}}(\lambda x + (1 - \lambda)y) \ge \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\} \tag{2}$$

Accordingly, if all α-cuts of $\tilde{A}$ are convex, then $\tilde{A}$ is a convex fuzzy set.

Definition 3. *The least upper bound of $\mu_{\tilde{u}}(x)$ is height of fuzzy set $\tilde{A}$ [6,15], i.e.*

$$hgt(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x) \tag{3}$$

$\tilde{A}$ is normal if and only if $\exists x \in X, \mu_{\tilde{A}}(x) = 1$.

The parametric form of a fuzzy number is given by $\tilde{u} = (\underline{u}, \overline{u})$, where $\underline{u}$ and $\overline{u}$ are functions defined over $[0, 1]$ and satisfy the following requirements [12,13]:

(1) $\underline{u}$ is a monotonically increasing left continuous function.
(2) $\overline{u}$ is a monotonically decreasing left continuous function.
(3) $\underline{u} \le \overline{u}$, in $[0, 1]$.

We name $\underline{u}$ and $\overline{u}$, left and right spread functions, respectively. If a is a crisp number, then $\underline{u}(r) = \overline{u}(r) = a$, for $\forall r \in [0, 1]$.

Definition 4. *We say that a fuzzy number $\tilde{v}$ has an $m-$degree polynomial form, if there exist two polynomials p and q of degree at most m such that $\tilde{v} = (p, q)$ [4].*

Let $\tilde{v} \in \mathcal{F}_m(\mathbb{R})$ be the set of all $m-$degree polynomial form fuzzy numbers. For $0 < \alpha \le 1$, α-cut of a fuzzy number $\tilde{u}$ is defined by [7] as follows:

$$[\tilde{u}]^{\alpha} = \{t \in \mathbb{R} \mid \mu_{\tilde{u}}(t) \ge \alpha\}. \tag{4}$$

The core of a fuzzy number is defined by [8] as follows:

$$core(\tilde{u}) = \{t \in \mathbb{R} \mid \mu_{\tilde{u}}(t) = 1\}. \tag{5}$$

Let $\mathcal{F}_c(\mathbb{R})$ be the set of all fuzzy numbers with continuous left and right spread functions and let $\mathcal{F}_m(\mathbb{R})$ be the set of all $m-$degree polynomial form fuzzy numbers [4]. We also consider Π_m as the set of all polynomials of degree at most m.
We can write a fuzzy number $\tilde{u} \in \mathcal{F}_m(\mathbb{R})$ as follows:

$$\tilde{u} = (\underline{u}, \overline{u}), \tag{6}$$

where $\underline{u}$, $\overline{u} \in \Pi_m$.

Definition 5. *A fuzzy set whose its membership function μ is defined with the following conditions is called a Generalized Fuzzy Number, GFN,[5]:*

(i) μ *is a continuous mapping from* $\mathbb{R}$ *to the closed interval* $[0, h]$, $0 < h \leq 1$*;*
(ii) $\mu(x) = 0$ *for* $x \in [\infty, a]$*;*
(iii) μ *is strictly increasing on* $[a, b]$*;*
(iv) $\mu(x) = h$ *for* $x \in [b, c]$*;*
(v) μ *is strictly decreasing on* $[b, c]$*;*
(vi) $\mu(x) = 0$ *for* $x \in [\infty, a]$*.*

On the other word, GFN is a fuzzy set with following membership function:

$$\mu_{\tilde{u}}(x) = \begin{cases} L_{\tilde{u}}(x), & a \leq x \leq b, \\ h, & b \leq x \leq c, \\ R_{\tilde{u}}(x), & c \leq x \leq d, \\ 0, & otherwise, \end{cases} \tag{7}$$

where $0 < h \leq 1$, $L_{\tilde{u}}$ is strictly increasing function on $[a, b]$ and $R_{\tilde{u}}$ is strictly decreasing function on $[b, c]$. If $L_{\tilde{u}}$ and $R_{\tilde{u}}$ corresponds to straight lines we have a Trapezoidal Generalized Fuzzy Number which shows by $(a, b, c, d; h)$, and when $b = c$ we have Triangular Generalized Fuzzy Number that shows by $(a, b, d; h)$. Moreover, when $a = b = c = d =$ we have crisp-value and denote that with $\{a; h\}$.

3 A Parametric Distance

To quantify the separation between two fuzzy numbers, we start with presenting a new distance.

Definition 6. *For $\tilde{u}, \tilde{v} \in GFN$, with heights h_u and h_v, respectively, let* $h_{\max} = \max\{h_u, h_v\}$ *and* $h_{\min} = \min\{h_u, h_v\}$*. We define a new distance D as follows:*

$$D_{p,q}(\tilde{u},\tilde{v}) = \{\frac{1}{h_{\min}}\int_0^{h_{\min}} q\,|\underline{u}(r)-\underline{v}(r)|^p + (1-q)\,|\overline{u}(r)-\overline{v}(r)|^p\ dr\}^{\frac{1}{p}} + \left|\int_{h_{\min}}^{h_{\max}} r(R(r)-L(r))\,dr\right|.$$

where $R(r)$ and $L(r)$ are defined as follows:

$$R(r) = \begin{cases} \overline{u}, & h_{\max} = h_u, \\ \overline{v}, & h_{\max} = h_v, \end{cases}$$

$$L(r) = \begin{cases} \underline{u}, & h_{\max} = h_u, \\ \underline{v}, & h_{\max} = h_v, \end{cases}$$

and $p > 0$ is a real positive number and $q \in [0,1]$.

Theorem 1. *For two pair crisp-values $\tilde{u}$ and $\tilde{v}$ with arbitrary heights with following membership functions:*

$$\mu_{\tilde{u}}(x) = \begin{cases} h\ , & x = a, \\ 0\ , & otherwise, \end{cases}$$

$$\mu_{\tilde{v}}(x) = \begin{cases} h\ , & x = b, \\ 0\ , & otherwise. \end{cases}$$

we have:

$$D_{p,q}(\tilde{u},\tilde{v}) = |a-b|,$$

where $p \in \mathbb{R}^+$ and $q \in [0,1]$.

Theorem 2. *For $\tilde{u},\tilde{v},\tilde{w} \in GFN(\mathbb{R})$ the distance, $D_{p,q}$, satisfies the following properties:*

(1) $D_{p,q}(\tilde{u},\tilde{u}) = 0$,
(2) $D_{p,q}(\tilde{u},\tilde{v}) = D_{p,q}(\tilde{v},\tilde{u})$,
(3) $D_{p,q}(\tilde{u},\tilde{w}) \le D_{p,q}(\tilde{u},\tilde{v}) + D_{p,q}(\tilde{v},\tilde{w})$,

where $p \in \mathbb{R}^+$ and $q \in [0,1]$.

For the set of all fuzzy semi-numbers with the same height, we prove the following theorem.

Theorem 3. *For $\tilde{u},\tilde{v},\tilde{u}',\tilde{v}'$ in Trapezoidal Generalized Fuzzy Number and non-negative real number k, the distance $D_{p,q}$ satisfies the following properties:*

(1) $D_{p,q}(k\tilde{u},k\tilde{v}) = kD_{p,q}(\tilde{u},\tilde{v})$,
(2) $D_{p,q}(\tilde{u}+\tilde{v},\tilde{u}'+\tilde{v}') \le D_{p,q}(\tilde{u},\tilde{u}') + D_{p,q}(\tilde{v},\tilde{v}')$,

where $p \in \mathbb{R}^+$ and $q \in [0,1]$.

If $h = 1$, $\tilde{u}$ and $\tilde{v}$ are fuzzy numbers and the distance we defined became as follows:

$$D_{p,q}(\tilde{u},\tilde{v}) = \Bigg(\int_0^1 q|\underline{u}(r) - \underline{v}(r)|^p \mathrm{d}r + \int_0^1 (1-q)|\overline{u}(r) - \overline{v}(r)|^p \mathrm{d}r\Bigg)^{\frac{1}{p}}, \tag{8}$$

that presented by Grzegorzewski [9].

Theorem 4. *$D_{p,q}$ is a metric on the set of fuzzy numbers [9].*

Proof. It can be found in [9]. □

Theorem 5. *$D_{p,q}$ is a metric on the set of all equiheigh generalized fuzzy numbers.*

Proof. For the set of all equiheigh generalized fuzzy numbers we have $h_{\min} = h_{\max}$, therefore the proof is completed. □

we follow our paper with investigating our distance in some examples.

4 Numerical Examples

In this section we present some examples which have been solved by Mathematica software using 11 decimal digits.

Example 1. *Distance between two GFN's $\tilde{u} = (0,1,2,4;\frac{1}{2})$ and $\tilde{v} = (1,3,4,5;\frac{1}{3})$ which shows in Fig. 1, with $p = 2$ and $q = \frac{1}{2}$ is:*

$$D_{2,0.5}(\tilde{u},\tilde{v}) = 1.5675$$

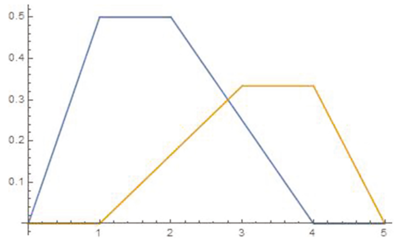

Fig. 1. Two Trapezoidal Generalized Fuzzy Numbers $\tilde{u}$ and $\tilde{v}$.

Moreover, Distances between $\tilde{u}$ and $\tilde{v}$ when $p = \{1, 1.1, 1.2, ..., 2\}$ and $q = \{0, 0.2, 0.4, ..., 1\}$ were calculated and showed in the Table 1.

Table 1. $D_{2,0.5}(\tilde{u}, \tilde{v})$ with different p and q

p	q = 0	q = 0.2	q = 0.4	q = 0.6	q = 0.8	q = 1
1	1.26852	1.36852	1.46852	1.56852	1.66852	1.76852
1.1	1.26892	1.37124	1.47275	1.57351	1.67358	1.77303
1.2	1.26931	1.37403	1.47704	1.57854	1.67865	1.77751
1.3	1.26971	1.37687	1.48139	1.58359	1.68371	1.78197
1.4	1.27011	1.37978	1.48581	1.58868	1.68877	1.7864
1.5	1.2705	1.38275	1.49028	1.59379	1.69382	1.7908
1.6	1.2709	1.38579	1.4948	1.59892	1.69886	1.79518
1.7	1.2713	1.38888	1.49938	1.60408	1.70389	1.79953
1.8	1.27169	1.39205	1.50402	1.60925	1.70891	1.80385
1.9	1.27209	1.39528	1.50871	1.61444	1.71391	1.80813
2	1.27248	1.39857	1.51344	1.61965	1.71889	1.81239

As it seems, increasing both values of p and q will increase the value of distance between these two generalized fuzzy numbers. The minimum and maximum values for distance between these two generalized fuzzy numbers for various values of $p = 1 : 0.1 : 2$ and $q = 0 : 0.1 : 1$ are $D_{1,0}(\tilde{u}, \tilde{v}) = 1.26852$ and $D_{2,1}(\tilde{u}, \tilde{v}) = 1.81239$, respectively.

Example 2. *Let $\tilde{u} = ((x - 1.3)^2, -(x - 2.8)^3)$ and $\tilde{v} = (\sqrt{x-2}, \sqrt{3-x})$ be the parametric forms with left and right spreads of two Generalized Fuzzy Numbers (Fig. 2).*

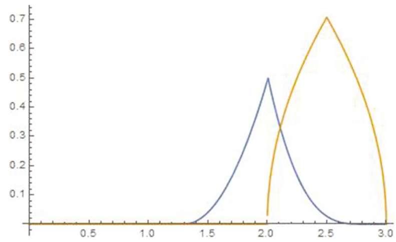

Fig. 2. Two Generalized Fuzzy Numbers $\tilde{u}$ and $\tilde{v}$.

As seen in Fig. 2, we have $hgt(\tilde{u}) = 0.4993$ and $hgt(\tilde{u}) = 0.7071$. The distance between these two GFNs is as follows:

$$D_{2,0.5}(\tilde{u}, \tilde{v}) = 0.933807$$

Example 3. *For two Generalized Fuzzy Numbers* $\tilde{u} = (0, 2, 3, 4; \frac{1}{2})$ *and* $\tilde{v} = (1, 2, 4, 5; h)$ *the distance with* $p = 2$ *and* $q = \frac{1}{2}$ *is plotted when* $h \in [0, 1]$ *(Fig. 3):*

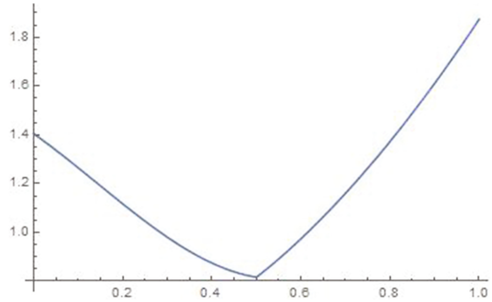

Fig. 3. $D_{2,0.5}(\tilde{u}, \tilde{v})$ when $h \in [0, 1]$.

The Figure shows that the minimum distance occurs when $h = \frac{1}{2}$.

Example 4. *Suppose* $\tilde{u} = (0, 2, 3, 4; \frac{1}{2})$ *and* $\tilde{v} = (1, 2, 4, 5; h)$ *are Generalized Fuzzy Numbers, for minimum distance with* $p = 2$ *between* $\tilde{u}$ *and* $\tilde{v}$ *the height* h *is approximated when* $q \in \{0, \frac{1}{20}, \frac{1}{10}, ..., \frac{19}{20}, 1\}$ *(as shown in Table 2).*

Table 2. Approximating of height while $D_{2,q}$ is minimum.

q	h	q	h	q	h
0	0.001	0.35	0.5	0.7	0.5
0.05	0.001	0.4	0.5	0.75	0.5
0.1	0.001	0.45	0.5	0.8	0.516337
0.15	0.35817	0.5	0.5	0.85	0.532521
0.2	0.444457	0.55	0.5	0.9	0.548632
0.25	0.5	0.6	0.5	0.95	0.564758
0.3	0.5	0.65	0.5	1	0.580994

Moreover, distance with $p = 2$ *between* $\tilde{u}$ *and* $\tilde{v}$ *is calculated while the height* $h = 0.0001, 0.1, 0.2, ..., 1$ *and* $q = 0 : 0.2 : 1$ *(as shown in Table 3).*

As it seems, for $q = 0$*, the values of distances are increasing when* h *increases, whereas for other fixed values of* q*, the values of* h *decrease and after reaching the minimum value, again they increase. From Table 2, for* $q = 0$*, the nearest Generalized Fuzzy Number has the height* $h = 0.0001$ *(the minimum distance which occurs at the first row of the first column is* 0.82741*). For* $q = 0.2$*, the*

Table 3. $D_{2,q}(\tilde{u}, \tilde{v})$ with different q and h.

h	q = 0	q = 0.2	q = 0.4	q = 0.6	q = 0.8	q = 1
0.0001	0.82741	1.1063	1.31447	1.48813	1.64026	1.77731
0.1	0.87491	1.05338	1.19947	1.32623	1.43975	1.54349
0.2	0.90711	0.99758	1.07894	1.15347	1.22266	1.28751
0.3	0.93229	0.95185	0.97096	0.98964	1.00793	1.02585
0.4	0.95985	0.92672	0.89227	0.85633	0.81870	0.77911
0.5	1	0.93095	0.85635	0.77460	0.68313	0.57735
0.6	1.20329	1.11796	1.02463	0.92051	0.80066	0.65465
0.7	1.41821	1.32281	1.21783	1.09964	0.96146	0.78746
0.8	1.64992	1.5477	1.43485	1.30713	1.15646	0.96268
0.9	1.90153	1.79439	1.67588	1.54135	1.38173	1.17362
1	2.17497	2.06413	1.94136	1.80173	1.63546	1.41667

nearest Generalized Fuzzy Number has the height $h = 0.4$ *(the minimum distance which occurs at the fifth row of the second column is* 0.92672*). For all* $q = 0.4 : 0.2 : 1$*, the nearest Generalized Fuzzy Numbers have the height* $h = 0.5$ *(the minimum values occur at the sixth row of the respective column).*

This example shows that the optimum value of h *(with respect to the minimum value of distance) is related to* q*!*

Therewith, the figure is plotted the distance with $p = 2$ *between* $\tilde{u}$ *and* $\tilde{v}$ *when* $h \in [0, 1]$ *and* $q \in [0, 1]$ *(Fig. 4):*

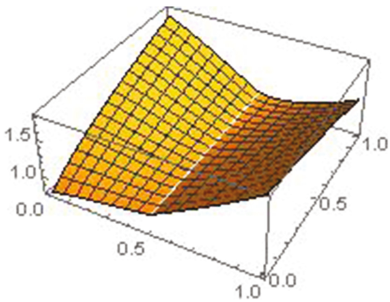

Fig. 4. $D_{2,q}(\tilde{u}, \tilde{v})$ when $h \in [0, 1]$ and $q \in [0, 1]$.

5 Conclusion

In this paper, a new distance was presented on the set of Generalized Fuzzy Numbers, with continuous left and right spread functions. Also, for fuzzy numbers, we showed this distance is the same as the one proposed by Gerzgorzewski [9] which proved that this distance is metric.

In the following, through some examples, the paper explored some of the capabilities of the proposed distance. One of these examples approximated the height of a given Generalized Fuzzy Number (GFN) while it had the minimum distance to any arbitrary GFN. Till now, distances presented on GFNs are applied only on trapezoidal forms but the second example showed the way the distance proposed in this paper is utilized on GFNs generally. Also, with some examples we show that this distance related with its parameter.

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Fuzzy Logic Load Balancing for Cloud Architecture Network - A Simulation Test

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Abstract. This article presents the algorithm of resource usage optimization for highly complex computer system architectures, such as Cloud Computing solutions. The main problem of such solutions is predicting the resources usage for allocating and dismissal. The proposed algorithm, based on OFN, allows to recognize the trend in the processed requests by the servers. In effect, the CC solutions allow to add resources dynamically, according to the amount of connections, and manage them in real time. This article proposes a fuzzy logic load balancing method for highly complex system architecture, which makes possible to use the resources in more efficient way. Description of the proposed algorithm is followed by simulation test results.

Keywords: Cloud Computing · Complex architecture · Fuzzy logic

1 Introduction

Cloud Computing (CC) is currently a widely applied technology, but still represents a fresh approach towards utilizing resources. From the market's point of view CC offers a significant operating cost reduction. The users who take advantage of this data storage capability and access to applications on demand, may use the resources easily, on various devices, for a long time [1,21]. However, such solutions as CC require considering the issues related to highly complex computer system architectures. The main problems include the security of the data and managing the system resources in most optimal way. Usually big companies use CC for providing big amount of data and process a large number of server requests. It is impossible to meet such demands basing on single server solutions, thusa method of load balancing, resources virtualization and a special way of their management was developed. This results in creating a highly complex system the architecture of which provides the means to handle the users' demands. It is not an uncommon case that both private and commercial users seek abundant resources under as low price as possible. CC solutions allow to add resources dynamically, depending on the amount of connections, and manage them in real time. For example, if one server is not able to handle the connections, another server instance is launched. Inability of the second

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server to handle the connections leads to starting a third instance of it, which provides more resources. However, such method of allocation of resources is not optimal. This article proposes a fuzzy logic load balancing method for highly complex system architecture, which allows to improve the efficiency of resources usage. This article is organized as follows: In Sect. 2 a general idea of the system architecture for cloud computing is provided. Section 3 proposes the method for optimal usage of the resources. The quoted definitions were originally introduced by Czerniak et al. for network security purposes. Section 4 presents the results of simulations. The final Sect. 5 covers the conclusions.

2 Cloud Computing Solution Architecture

As it was already mentioned, the main problem of CC is security and the optimal allocation of the resources. The issue of assuring the safety of data which is stored in the cloud, in different sites, is a significant challenge. The lack of security standards until recently and a difficulty with their determination and application, along with a limited confidence in this technology, are the main obstacles to broader implementation of CC [25,30,31]. Thereby, in the Authors' opinion, the elaboration of standards and mechanisms for assuring the stored data's security and integrity is an indispensable condition for future popularization of CC. It has been already pointed out that the cloud is a perfect solution for business, since it offers cost reduction, as after a little investment in the infrastructure, it provides a quick access to multiple services. However, the strategy of ensuring the security and trust should be considered at the initial level of designing the system. It is a widely acknowledged fact that it is difficult - if not impossible - to develop a single common solution dedicated to every possible system architecture [29]. The emphasis on security issues escalates continuously. The security should be treated globally, on all layers of communications, starting from hardware protection, through the end point restrictions, to the application level.

The second problem brought out at the beginning of this section - the optimal allocation of the resources - seems also a complicated matter. The users tend to choose the solutions which offer more at lower cost. Nowadays the model of providing the resources and software employs the time of usage as a main evaluator. As the resources' requirement grow, subsequent cloud resources are allocated and dedicated to processing the user's requests. Figure 1 presents the possible architecture of the cloud.

The presented architecture consists of four layers:

- servers layer,
- router connections layer,
- switching layer,
- load balancing layer.

The servers layer is responsible for providing the servers - or, to be exact - the CPU and data storage for the users. The servers resources are the main source of the cost borne by the users - since they generate the main cost for the cloud

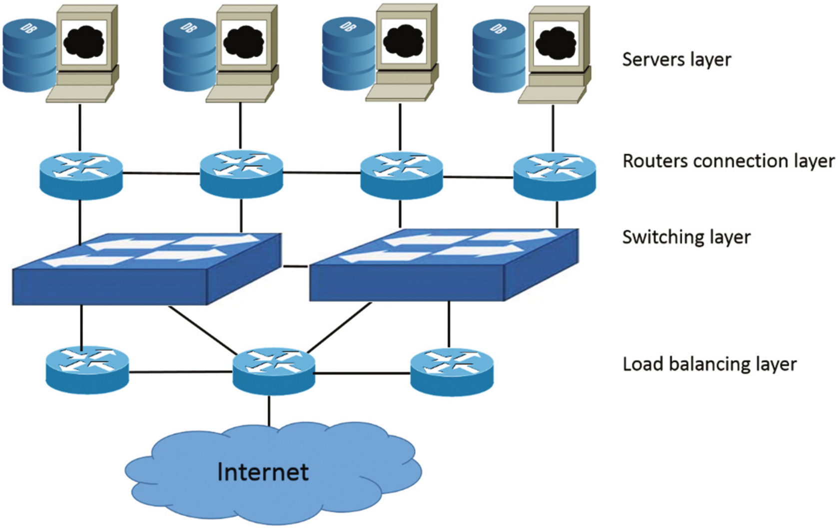

Fig. 1. Possible cloud architecture

provider, who has to pay for the material resources, while he is selling the virtual ones, for the specified time of usage.

The router connections layer is accountable for connecting the servers in a proper way. It provides the routing protocols and transfers the data to an appropriate server.

The switching layer is responsible for fast switching the packets between the load balancing layer and the router.

The load balancing layer is the first point of contact between the cloud and the outside network, which usually is the Internet network. This layer is in charge of providing one representative point of contact, which during the operation is multiplied. This allows to provide more resources than a single server is capable to offer. The main algorithm used for this process is described in the following steps:

– step 1: oobtaining a connection request from the user,
– step 2: allocating the server resources,
– step 3: transfer connection from the user to the server,
– step 4: in the case when the connections to the server exceed the servers limit, another server is allocated.

3 Fuzzy Logic Method for Load Balancing

The algorithm for load balancing mentioned in the previous section is not an optimal method for the resources' utilization. The number of user connections is not a constant value. In normal conditions the users shut down their connections after finishing their tasks, which results in a decrease in the total number of connections. This could allow to release some server resources, but in most cases it does not happen. It is because the connections are not divided between the servers instances in most optimal way. For this purpose, some predictions concerning the connections would be required and in this matter the solutions based on fuzzy logic can be applied, especially the Ordered Fuzzy Numbers [10,22,27]. The implementation of OFN may accelerate the decision to release the server instances because they allow to predict an upcoming decrease in the number of connections.

In this Section the algorithm which uses OFN for detecting a decrease in the total number of connections is described. The algorithm measures the active connections in all servers in use during the data processing. This measurement is performed continuously in a given period of time referred to as a timeslot. Four subsequent timeslots are described as:

$$t_i, t_{(i-1)}, t_{(i-2)}, t_{(i-3)} \tag{1}$$

where t_i is represent the current timeslot.

These four results together provide a fuzzy number in OFN notation, presented in Fig. 2, where

- $f_A(0)$ respond to $t_{(i-3)}$,
- $f_A(1)$ respond to $t_{(i-2)}$,
- $g_A(1)$ respond to $t_{(i-1)}$,
- $g_A(0)$ respond to t_i.

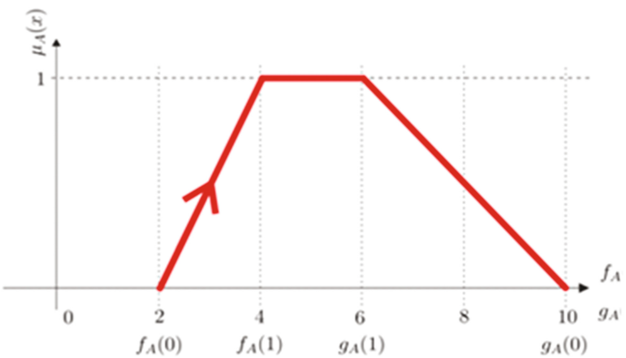

Fig. 2. Fuzzy number in OFN notation

As it couldcan be noticed, oin Fig. 2, the OFN has gotpresent a trapezoid form, their shape deriving from the and the value of function f_A and g_A.

The definition of fuzzy observance of the connection used is as follows:

Definition 1. Fuzzy observance of C connection in time t_i is a set

$$C/t_i = \{f_C(0)/t_{i-3}, f_C(1)/t_{i-2}, g_C(1)/t_{i-1}, g_C(0))/t_i\} \tag{2}$$

where

$$t_i > t_{i-1} > t_{i-2} > t_{3-1} \ |t_i - t_{i-1}| = |t_{i-1} - t_{i-2}| = |t_{i-2} - t_{i-3}| = t_n,$$
timeslot of the measurement
$$f_C(0) \leq f_C(1) \leq g_C(1) \leq g_C(0)$$

This provide the Lemma 1.

Lemma 1.

$$C_{positive} = \begin{cases} f_C(0) < f_C(1) < g_C(1) \\ or \\ f_C(1) < g_C(1) < g_C(0) \end{cases} \tag{3}$$

In other situation $C_{negative}$.

According to this definition, the counters of active connections to the servers should give:

– positive order of OFN when the number of connections increases,
– negative order of OFN when the number of connections decreases.

The four measurements performed during the cloud operations allow to prepare fuzzy numbers in OFN notation. This gives a fuzzy observance of the whole cloud CC_m, defined as follows:

Definition 2. Fuzzy observance of the cloud connections is described by following formula:

$$CC_m = \sum_{i=1}^{n} \left\{ \begin{matrix} C_{positive} | C_{negative} \\ C_i * w_i | - C_i * w_i \end{matrix} \right\}. \tag{4}$$

Where $w_i \in \{w_i, ..., w_n\}$ describes the impact of the server on whole cloud architecture. This value should be considered by the network administrator, jointly with the server resources and the impact on whole network. CC_m will provide the information for the network administrator that according to the network trend some servers can be released or that they should still process the user's request.

As it is known, the CC solutions base on a highly complicated architecture, so the number of servers they use may be very high. The result can be visualized by means of pipe of samples [9], presented in Fig. 3. Each time a connection is established, a number of measurements take place, the amount of which corresponds to the number of servers in use. The results are collected into a pipe, where a single slice of it contains the measurements from a single timeslot. Over

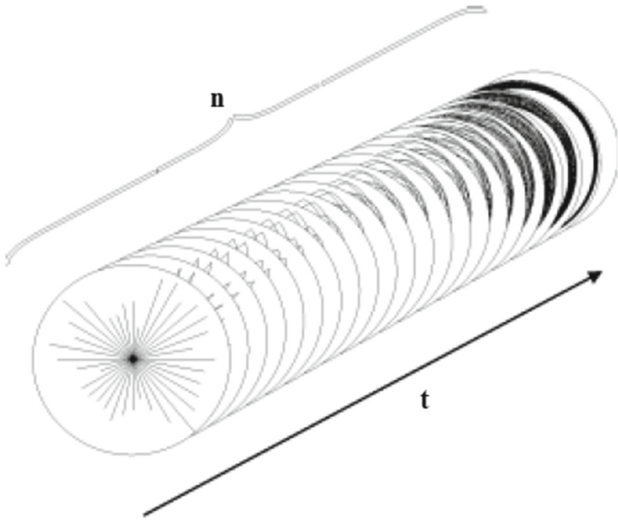

Fig. 3. The pipe of samples for data storing

the time a pipe is formed, being an accumulation of n slices. The original solutions [10] where prepared for 360 servers. Figure 4 presents the results with only server 1 taken into account. As it was mentioned, four measurements give an OFN number. Two of possible notations of such numbers' order are presented in Fig. 4. According to definition 2, it is possible to make a decision on:

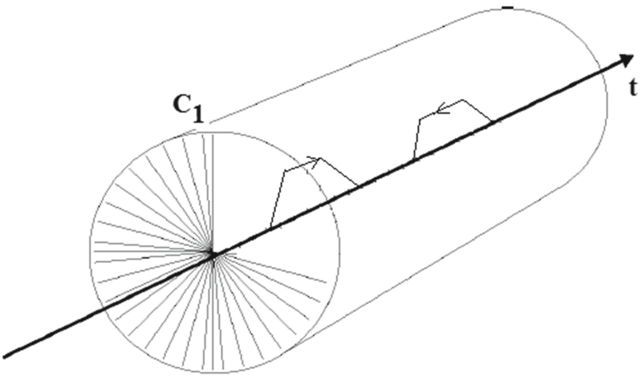

Fig. 4. The pipe of samples for data storing

- not allocating another server when a new request appears,
- passing the connections to a different server and releasing one of the instances.

The first mentioned situation will occur when the load balance layer registers that the incoming connections are not increasing the total amount of connections on the servers. This is achieved by recognizing the trend showing that the servers are processing less requests in general. In consequence, the solution allows to avoid creating new, unnecessary instances of the server.

The second mentioned situation will take place when a global trend of decreasing number of processed connections is recognized. This leads to moving the processed requests to other servers and releasing some of the instances. The above-described mechanism allows to use the *CC* resources in most optimal way.

4 Simulation

The simulation was performed for the proposed CC architecture and the measurement was simulated on the servers as presented in Fig. 5. There were 4 servers (C1, C2, C3, C4) responsible for processing the users' requests. Each server was processing 1000 user requests as a maximum value. When the number of requests exceeded 1000, another server was switched in. The process of arriving of the users' requests and the connections is presented in Table 1. The column "time" presents the current timeslot. The columns "incoming connections" and "released connections" present, respectively, the number of connections established basing on incoming requests and the ones released during the timeslots. The columns C1-C4 present the total number of requests processed by the servers. As it can be noticed, the server C2 was launched in the timeslot 2, while the server C3 in timeslot 8 and the server C4 in timeslot 12. It can also be observed that the server C1 was released in timeslot 20, as it had finished processing the requests. The same situation was with server C2, but it took place in timeslot 25. Using the definition 2, the CC_m for timeslot 16 can be calculated. According to the Table 1, the Cs for each server are as follows:

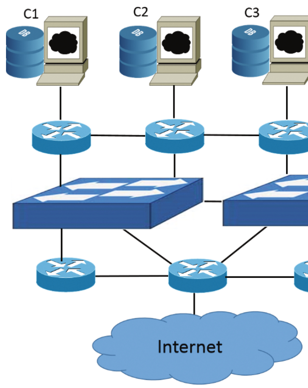

Fig. 5. Simulation test architecture

Table 1. User request processing during simulation

Time	Incomming connections	Relased connections	Sum of connections	Servers			
			Sum	C1	C2	C3	C4
0	500	0	500	500			
1	300	0	800	800			
2	300	0	1100	1000	100		
3	200	0	1300	1000	300		
4	100	0	1400	1000	400		
5	200	0	1600	1000	600		
6	300	0	1900	1000	900		
7	100	0	2000	1000	1000		
8	200	0	2200	1000	1000	200	
9	100	0	2300	1000	1000	300	
10	300	0	2600	1000	1000	600	
11	250	0	2850	1000	1000	850	
12	450	0	3300	1000	1000	1000	300
13	50	50	3300	940	990	1000	350
14	50	100	3250	900	950	1000	400
15	50	300	3000	720	830	1000	450
16	50	300	2750	680	820	1000	500
17	50	200	2600	380	720	950	550
18	50	300	2350	190	700	860	600
19	50	200	2200	20	680	850	650
20	50	100	2150	0	610	840	700
21	100	100	2150	0	560	790	800
22	100	200	2050	0	480	670	900
23	100	300	1850	0	280	570	1000
24	100	200	1750	0	180	570	1000
25	100	300	1550	0	0	550	1000

$$C_1 = [940, 900, 720, 680], \text{ with negative order,}$$
$$C_2 = [990, 950, 830, 820], \text{ with negative order,}$$
$$C_3 = [1000, 1000, 1000, 1000], \text{ with positive order,}$$
$$C_4 = [350, 400, 450, 500], \text{ with positive order,}$$

Calculatinge the CC_m according to Definition 2 is presented below:

$$CC_m = -C_1 - C_2 + C_3 + C_4$$

So finally fuzzy observance of the cloud servers give as a OFN number in timeslots 16 as

$$CC_m = [-580, -450, -100, 0],$$

with a negative order is obtained. It means that a prediction for the servers' usage is as follows: the number of the process requests is decreasing and the server C1 can be released in timeslot 17. It also means that it can be released 3 timeslots earlier than in the original algorithm. When the same fuzzy observance is performed for the timeslot 22, the Cs for each server are as follows:

$$C_1 = [20, 0, 0, 0], \text{ with negative order,}$$
$$C_2 = [680, 610, 560, 480], \text{ with negative order,}$$
$$C_3 = [850, 840, 790, 670], \text{ with negative order,}$$
$$C_4 = [650, 700, 800, 900], \text{ with positive order.}$$

The calculation of the CC_m according to Definition 2 gives:

$$CC_m = -C_1 - C_2 - C_3 + C_4$$

So finally:

$$CC_m = [-900, -750, -550, -250],$$

with a negative order is obtained. It means that a prediction for the servers' usage is as follows: the number of the process request is decreasing and the server C2 can be released in timeslot 22. This shows that it can be released 3 timeslots earlier than in the original algorithm.

The servers' usage during each timeslot is presented in Fig. 6. It is noteworthy that during the simulation the proposed algorithm was using fewer servers than the original one.

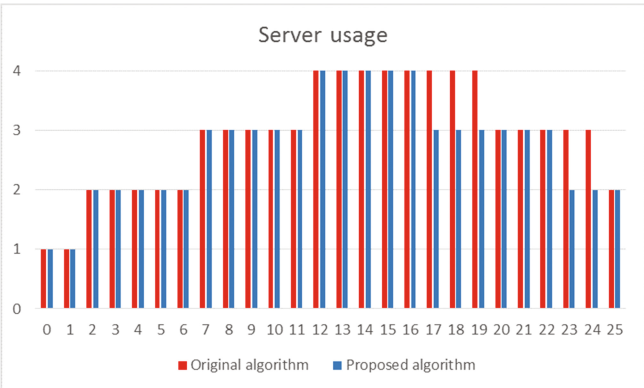

Fig. 6. Server usage simulation

5 Summary

This article presented the algorithm for the optimization of the resources' usage for highly complex computer system architectures, such as Cloud Computing solutions. Their main problem is predicting the resources' usage for the purpose od allocation and dismissal. The proposed algorithm based on OFN usage allows to recognize the trend in the processed handled by the servers and trace the signs of a decrease in the number of requests. It also allows to avoid creating new, redundant instances of the servers, The presented solution helps to move the processed requests to other servers and release some of the instances faster, which contributes to more optimal utilization of the CC resources. By means of the presented algorithm, the CC solutions may add the resources dynamically, according to the amount of connections, and manage them in real time. This article proposes a fuzzy logic load balancing method for highly complex system architectures, which allows to use the resources in a more efficient way and load balancing, aside from the security issues [23,24,28] remain one of the main problems the CC solutions have to deal with [7].

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Dynamical Control of Computations Using the Iterative Methods to Solve Fully Fuzzy Linear Systems

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Abstract. A linear system with fuzzy coefficients matrix, unknown and right hand side fuzzy vectors is called a fully fuzzy linear system (FFLS). Solving these kinds of systems via iterative methods to find the optimal number of iterations and optimal solution is important computationally. In this study, a FFLS is solved in the stochastic arithmetic to find this optimal solution. To this end, the CESTAC (Controle et Estimation Stochastique des Arrondis de Calculs) method and the CADNA (Control of Accuracy and Debugging for Numerical Application) library are considered to evaluate the round-off error effect on computed results. The Gauss-Seidel, Jacobi, Richardson and SOR iterative methods are used to solve FFLS. Also, an efficient algorithm is presented based on the proposed approach to compute the optimal results. Finally, two numerical examples are solved to validate the results and show the importance of using the stochastic arithmetic in comparison with the common floating-point arithmetic.

Keywords: Iterative methods · Fully fuzzy linear systems (FFLS) · Stochastic arithmetic · CESTAC method · CADNA library · Hausdorff distance

1 Introduction

The fuzzy system of linear equations as $A\tilde{x} = \tilde{b}$ studied by Friedman et al. [16] where A is a crisp matrix and $\tilde{x}, \tilde{b}$ are fuzzy vectors. They replaced a $2n \times 2n$ crisp linear system to solve the original $n \times n$ fuzzy linear system. They derived conditions for the existence and uniqueness of the solution and presented a numerical algorithm for estimation the solution. Recently, some authors have proposed some works to solve the FFLS. In [6,12,18] a linear programming method proposed to calculate the solution of FFLS. A FFLS can be solved by interval systems, however, the classical solution often fails to exist because the solution of linear interval equations is not necessarily intervals. The direct method, LU decomposition method, iterative method and Adomian decomposition method are applied to find the solutions of fully fuzzy linear systems [12].

When an iterative scheme is chosen to solve a FFLS, the solution is calculated in the floating-point arithmetic via a package like Matlab or Mathematica

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that the termination criterion depends on a positive number like ϵ. Also, the exact solution of the system must be accessible. In this case, the number of iteration in results may be increased without increasing the accuracy or results may not be accurate. Because of the round-off error propagation, the computer may not able to improve the accuracy of the results. Therefore, to validate the results and to improve the accuracy, we apply the iterative methods in the stochastic arithmetic. In recent years, the CESTAC method, which is based on the discrete stochastic arithmetic, is used to validate many problems in mathematics and physics such as interpolation polynomials [2], ill-condition functions [3], numerical integration [1,4,17] and others [8,9,24].

In [13], the authors solved fuzzy linear system with crisp coefficient matrix, based on the Gauss-Seidel and Jacobi iterative methods in the stochastic arithmetic. This paper is a development of this work on the FFLS.

This paper is organized as follows. We give the definition of fuzzy number, and fully fuzzy linear systems in Sect. 2. Iterative methods are presented in Sect. 3. In Sect. 4, we recall the stochastic round-off analysis, the fuzzy CESTAC method in the stochastic arithmetic. Section 5 describes datasets and experiments by using the proposed algorithm which is performed by the CADNA library based on the discrete stochastic arithmetic and the CESTAC method.

2 Preliminaries

2.1 Fuzzy Numbers and Fully Fuzzy Linear Systems

Here, we give some basic definitions on fuzzy numbers and brief description of FFLS.

Definition 2.1.1. A fuzzy number $\tilde{A}$ is called to be an LR fuzzy number, if there exist shape function L (for left), R (for right) and scalars α, β with

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}) & x \leq a, \quad \alpha > 0 \\ 1 & x = a \\ R(\frac{x-a}{\beta}) & x \geq a, \quad \beta > 0 \end{cases}$$

where a is the mean value of $\tilde{A}$, $\mu_{\tilde{A}}$ its membership function, α and β are the left and right spreads, respectively. An LR fuzzy number $\tilde{A}$ is shown as $\tilde{A} = (a, \alpha, \beta)_{LR}$. If $a - \alpha > 0$ then $\tilde{A}$ is said positive [11].

Definition 2.1.2 [11]. An arbitrary fuzzy number is presented by an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$, $0 \leqslant r \leqslant 1$ which satisfy the following conditions:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1].
2. $\bar{u}(r)$ is a bounded left continuous non-increasing function over [0,1].
3. $\underline{u}(r) \leqslant \bar{u}(r), 0 \leqslant r \leqslant 1$.

   The set of the fuzzy numbers is denoted by E^1.

Definition 2.1.3 [**13**]. Let $\tilde{u} = (\underline{u}(r), \bar{u}(r))$ and $\tilde{v} = (\underline{v}(r), \bar{v}(r))$ are two arbitrary fuzzy numbers, the Hausdorff distance between them is defined by:
$H.D(\tilde{u}, \tilde{v}) = sup_{0 \leqslant r \leqslant 1}\ max \{| \underline{u}(r) - \underline{v}(r) |, | \bar{u}(r) - \bar{v}(r) |\}$.

Definition 2.1.4. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix if each element in A is a fuzzy number [11].
In the fuzzy matrix $\tilde{A} = (A, M, N)$, the crisp matrix $A = (a_{ij})$ called the main matrix and $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are the right and left spread matrices, respectively.

Consider the $n \times n$ linear system of equations:

$$\begin{cases} (\tilde{a_{11}} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{cases} \tag{1}$$

The matrix form of this system is

$$\tilde{A} \otimes \tilde{X} = \tilde{b}. \tag{2}$$

Here, the matrix $\tilde{A}_{n\times n} = (\tilde{a}_{ij}), 1 \leqslant i, j \leqslant n$ is a fuzzy matrix and $\tilde{X}_i, \tilde{b}_i (1 \leqslant i \leqslant n)$ are two fuzzy vectors in E^1. The system (2) is called a fully fuzzy linear system (FFLS). In Eq. (2), if each element of matrix coefficients $\tilde{A}$ and constant vector $\tilde{b}$ is a positive LR fuzzy number, then fuzzy system is called a positive FFLS.

Definition 2.2.1 [**10**]. A fuzzy matrix $\tilde{A}_{n\times n} = (\tilde{a}_{ij})$ is called an lower (upper) triangular fuzzy matrix, if $\tilde{a}_{ij} = \tilde{0} = (0, 0, 0), \quad \forall i < j\ (i > j)$.

Definition 2.2.2 [**10**]. Let $\tilde{A}_{m\times n} = (\tilde{a}_{ij})$ and $\tilde{B}_{n\times p} = (\tilde{b}_{ij})$ be two fuzzy matrices. Operation multiplication is defined as $\tilde{A} \otimes \tilde{B} = \tilde{C}$, which $\tilde{C}$ is the $m \times p$ fuzzy matrix where $\tilde{c}_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik} \otimes \tilde{b}_{kj}$.

3 Iterative methods

In this section, we introduce the iterative methods which are applied for solving the FFLS.

Definition 3.1. Suppose Eq. (2) be the positive FFLS, after operating the multiplication and summarizing, $\tilde{X}$ is a solution of Eq. (2), if and only if

$$\begin{cases} AX = b, \\ MX + AY = g, \\ NX + AZ = h. \end{cases} \tag{3}$$

With conditions $Y \geqslant 0, Z \geqslant 0$ and $X - Y \geqslant 0$, the solution $\tilde{X} = (X, Y, Z)$ is called a consistent solution of positive FFLS or for abbreviation consistent solution [10]. Otherwise, it will be called dummy solution. In this study, to determine the consistent solution of the FFLS is the final target. But unfortunately, the iterative methods may converge to the dummy solutions.

Theorem 3.1. Suppose $\tilde{A} = (A, M, N)$ and $\tilde{b} = (b, g, h)$ be a non-negative fuzzy matrix and a non-negative fuzzy vector, and let A be the product of a permutation matrix by a diagonal matrix with positive diagonal entries. The system $\tilde{A}\tilde{x} = \tilde{b}$ has a positive fuzzy solution, if $h \geqslant MA^{-1}b, g \geqslant NA^{-1}b$ and $(MA^{-1} + I)b \geqslant h$ [10].

For matrix A, the following decomposition is used.

$$A = D + L + U, \tag{4}$$

where D is the diagonal, L its strict lower part and U its strict upper part of matrix A. We can assume that the diagonal entries of the matrix A are all non-zero.

The Jacobi iterative method to find the solution of the Eq. (2) is as follows:

$$\begin{cases} X^{(k+1)} = -D^{-1}(L+U)X^{(k)} + D^{-1}b, \\ Y^{(k+1)} = -D^{-1}(L+U)Y^{(k)} - D^{-1}MX^{k} + D^{-1}g, \\ Z^{(k+1)} = -D^{-1}(L+U)Z^{(k)} - D^{-1}NX^{k} + D^{-1}h. \end{cases} \tag{5}$$

Also, an other iterative method for solving Eq. (2) is the Gauss-Seidel method, which is defined as follows:

$$\begin{cases} X^{(k+1)} = -(D+L)^{-1}UX^{(k)} + (D+L)^{-1}b, \\ Y^{(k+1)} = -(D+L)^{-1}UY^{(k)} - (D+L)^{-1}MX^{k} + (D+L)^{-1}g, \\ Z^{(k+1)} = -(D+L)^{-1}UZ^{(k)} - (D+L)^{-1}NX^{k} + (D+L)^{-1}h. \end{cases} \tag{6}$$

The Richardson extrapolation method is the other iterative method to solve Eq. (2), which is as follows:

$$\begin{cases} X^{(k+1)} = (I - \gamma A)X^{(k)} + \gamma b, \\ Y^{(k+1)} = (I - \gamma A)Y^{(k)} - \gamma MX^{k} + \gamma g, \\ Z^{(k+1)} = (I - \gamma A)Z^{(k)} - \gamma NX^{k} + \gamma h, \end{cases} \tag{7}$$

where γ is called the extrapolation parameter. If the matrix A is symmetric positive definite, then the optimum parameter would be $\gamma_{opt} = \dfrac{2}{m(A) + M(A)}$.

The following SOR method is the fourth iterative method, that we apply it for solving FFLS in the stochastic arithmetic:

$$\begin{cases} X^{(k+1)} = -(D+\omega L)^{-1}[(\omega-1)D + \omega U]X^{(k)} + \omega(D+\omega L)^{-1}b, \\ Y^{(k+1)} = -(D+\omega L)^{-1}[(\omega-1)D + \omega U]Y^{(k)} - \omega(D+\omega L)^{-1}MX^{k} + (D+\omega L)^{-1}g, \\ Z^{(k+1)} = -(D+\omega L)^{-1}[(\omega-1)D + \omega U]Z^{(k)} - \omega(D+\omega L)^{-1}NX^{k} + (D+\omega L)^{-1}h. \end{cases} \tag{8}$$

Theorem 3.2. The iterative method for solving the FFLS $\tilde{A}\tilde{x} = \tilde{b}$, converges if and only the classical iterative method converges for solving the crisp linear system $Ax = b$ [10].

Theorem 3.3. If the SOR method be convergent then $0 < \omega < 2$ [20].

Theorem 3.4. Let $A \in L(R^n)$ be symmetric positive definite and assume that $0 < \omega < 2$. Then the SOR method converges for any choice of initial vector X^0 [20].

4 Fuzzy CESTAC Method

In this section, a theorem is proved to show the accuracy of the iterative methods to solve a FFLS based on the concept of the common significant digits between two vectors. The results of this theorem is applied to designate the termination criterion in the proposed algorithm in next section. At first, we introduce the fuzzy CESTAC method.

Let $\tilde{x} = (x, y, z)$ be a fuzzy number in E^1. Then, $\tilde{x}$ in the computer is represented by $\tilde{X} = (X, Y, Z)$ as follows:

$$X = x - \epsilon_1 2^{E_1 - P} \alpha_1, \tag{9}$$

$$Y = y - \epsilon_2 2^{E_2 - P} \alpha_2, \tag{10}$$

$$Z = z - \epsilon_3 2^{E_3 - P} \alpha_3, \tag{11}$$

where, ϵ_1, ϵ_2 and ϵ_3 are the signs of x, y and z respectively and $2^{-P}\alpha_1, 2^{-P}\alpha_2$ and $2^{-P}\alpha_3$ are the lost part of the mantissa due to round-off error and E_1, E_2 and E_3 are the binary exponents of the results. In the floating-point arithmetic and double precision case, $P = 53$ and usually $-1 \leqslant \alpha_1, \alpha_2, \alpha_3 \leqslant 1$. The CESTAC method is applied in order to implement an algorithm, any result is a random variable with mean μ and variance σ^2. Hence, a stochastic arithmetic should be used. The last mantissa bit is perturbed N times, to estimate μ and σ^2. Then, the mean of samples is considered as estimation of the result and the variance of them is used to determine the accuracy of the result. The main idea of the CESTAC method is to evaluate the number of significant digits of the result effectively [24]. The algorithm of the fuzzy CESTAC method is as follows:

Algorithm 1.

1. Let $X_1, X_2, \ldots, X_N$, $Y_1, Y_2, \ldots, Y_N$, $Z_1, Z_2, \ldots, Z_N$ be N samples for X, Y and Z by perturbation the last mantissa bit.
2. Calculate the mean of samples as

$$X_{ave} = \frac{\sum_{i=1}^{N} X_i}{N}, Y_{ave} = \frac{\sum_{i=1}^{N} Y_i}{N}, Z_{ave} = \frac{\sum_{i=1}^{N} Z_i}{N}.$$

3. Calculate the variance of samples as

$$S_X^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - X_{ave})^2, \; S_Y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i - Y_{ave})^2, \; S_Z^2 = \frac{1}{N-1} \sum_{i=1}^{N} (Z_i - Z_{ave})^2.$$

4. Calculate

$$C_{X_{ave},X} = log_{10}\frac{\sqrt{N}|X_{ave}|}{\tau_\beta S_X}, \quad C_{Y_{ave},Y} = log_{10}\frac{\sqrt{N}|Y_{ave}|}{\tau_\beta S_Y}, \quad C_{Z_{ave},Z} = log_{10}\frac{\sqrt{N}|Z_{ave}|}{\tau_\beta S_Z},$$

as the common significant digits of each corresponding components of the exact solution and the approximate value, where τ_β is the value of T distribution with $N-1$ degree of freedom and confidence interval $1-\beta$ (If $N=3$ and $\beta = 0.05$ then $\tau_\beta = 4.303$).
5. if $C_{X_{ave},X} \leqslant 0 \;\; or X_{ave} = 0, C_{Y_{ave},Y} \leqslant 0 \;\; or Y_{ave} = 0, C_{Z_{ave},Z} \leqslant 0 \;\; or Z_{ave} = 0.$ then write $\tilde{X} = @.0$

Definition 4.1. In the CESTAC method $X = @.0$ is an "informatical zero", if and only if, $X_{ave} = 0$ or $C_{X_{ave},X} \leq 0$ [15].

For solving the crisp linear system $AX = b$ by the iterative methods, this system converts into an equivalent system of the form $X = QX + C$ for some fixed matrix Q and vector C. With the initial vector $X^{(0)}$, the iterative equation to approximate the solution vectors is as follows:

$$X^{(k+1)} = QX^{(k)} + C, \qquad k = 1, 2, 3, \ldots \tag{12}$$

Corollary 4.1. For any $X^{(0)} \in \mathbb{R}^n$, the iterative method $X^{(k+1)} = QX^{(k)} + C$, for each $k \geq 1$, converges to the unique solution of $X = QX + C$ if and only if $\rho(Q) < 1$ [20].

Corollary 4.2. If $\| \, Q \, \| < 1$ for any natural matrix norm, and C is a given vector, then the iterative method $X^{(k+1)} = QX^{(k)} + C$ converges, for any $X^{(0)} \in \mathbb{R}^n$ [20].

Definition 4.2 [13,19]. For two distinct vectors $X, Y \in \mathbb{R}^n$, the number of common significant digits is defined as:

$$C_{X,Y} = \log_{10} \frac{||X+Y||_2}{2\sqrt{n}||X-Y||_2}. \tag{13}$$

for $X = Y : C_{X,Y} = +\infty$.

Now, we can prove the following theorem for computing of the common significant digits of each corresponding components of the approximated solution and exact solution for a linear system using an iterative method.

Theorem 4.1. Let $X^{(k+1)} = P_1X^{(k)} + Q_1, Y^{(k+1)} = P_2Y^{(k)} + Q_2$ and $Z^{(k+1)} = P_3Z^{(k)} + Q_3, k \geqslant 0$ be convergent iterexact solution (X, Y, Z) of the system (3) with $Q_1, Q_2, Q_3 \neq 0$. Then, for sufficiently large value of k,

$$log_{10}(1- \| P_1 \|_2) \leqslant C_{X^{(k)},X} - C_{X^{(k)},X^{(k+1)}} \leqslant log_{10}(1+ \| P_1 \|_2), \tag{14}$$
$$log_{10}(1- \| P_2 \|_2) \leqslant C_{Y^{(k)},Y} - C_{Y^{(k)},Y^{(k+1)}} \leqslant log_{10}(1+ \| P_2 \|_2), \tag{15}$$
$$log_{10}(1- \| P_3 \|_2) \leqslant C_{Z^{(k)},Z} - C_{Z^{(k)},Z^{(k+1)}} \leqslant log_{10}(1+ \| P_3 \|_2). \tag{16}$$

Proof. First we prove (14), then (15) and (16) can be proved similarly. According to Eq. (13)

$$C_{X^{(k)},X} - C_{X^{(k)},X^{(k+1)}} = \log \frac{\|X^{(k)} + X\|_2}{2\sqrt{n}\|X^{(k)} - X\|_2} - \log \frac{\|X^{(k)} + X^{(k+1)}\|_2}{2\sqrt{n}\|X^{(k)} - X^{(k+1)}\|_2}$$

$$= \log \frac{\|X^{(k)} + X\|_2}{\|X^{(k)} + X^{(k+1)}\|_2} + \log \frac{\|X^{(k)} - X^{(k+1)}\|_2}{\|X^{(k)} - X\|_2}. \quad (17)$$

Since by increasing k, $X^{(k)} \simeq X$ and $X^{(k+1)} \simeq X$, then first term of this equation is almost equal to zero. Let $X^{(k)} - X = E^{(k)}$, then one can see

$$X^{(k)} - X = P_1^{(k)}(X_0 - X) = P_1^{(k)} E^{(0)}. \quad (18)$$

For the second term of Eq. (17) we can write

$$\frac{\|X^{(k)} - X^{(k+1)}\|_2}{\|X^{(k)} - X\|_2} = \frac{\|X^{(k)} - X + X - X^{(k+1)}\|_2}{\|X^{(k)} - X\|_2} \le \frac{\|X^{(k)} - X\|_2 + \|X^{(k+1)} - X\|_2}{\|X^{(k)} - X\|_2}$$
$$= 1 + \frac{\|X^{(k+1)} - X\|_2}{\|X^{(k)} - X\|_2} = 1 + \frac{\|P_1^{(k+1)} E^{(0)}\|_2}{\|P_1^{(k)} E^{(0)}\|_2} = 1 + \|P_1\|_2. \quad (19)$$

According to Eq. (18),

$$X^{(k+1)} - X^{(k)} = P_1^k (I - P_1) E^{(0)}.$$

Therefore,

$$\frac{\|X^{(k)} - X^{(k+1)}\|_2}{\|X^{(k)} - X\|_2} = \frac{\|P_1^k (I - P_1) E^{(0)}\|_2}{\|P_1^k E^{(0)}\|_2} \ge \frac{\|P_1^k E^{(0)}\|_2 - \|P_1\|_2 \|P_1^k E^{(0)}\|}{\|P_1^k E^{(0)}\|_2} = 1 - \|P_1\|_2. \quad (20)$$

According to (19) and (20),

$$1 - \|P_1\|_2 \le \frac{\|X^{(k)} - X^{(k+1)}\|_2}{\|X^{(k)} - X\|_2} \le 1 + \|P_1\|_2.$$

And finally,

$$\log(1 - \|P_1\|_2) \le \log \frac{\|X^{(k)} - X^{(k+1)}\|_2}{\|X^{(k)} - X\|_2} \le \log(1 + \|P_1\|_2). \quad (21)$$

When $\| P_1 \|_2 \ll 1$, then terms of Eq. (21) are almost zero and according to Eq. (14), we prove that the common significant digits of each corresponding components of the solution in two successive iterations and the common significant digits of the computed solution and the exact solution are almost equal.

$$C_{X^{(k)},X} \simeq C_{X^{(k)},X^{(k+1)}}.$$

5 Numerical Examples

For solving full fuzzy linear systems in the stochastic arithmetic, we present two sample examples by using the following algorithm and mentioned iterative methods with the initial vector $(X^{(0)}, Y^{(0)}, Z^{(0)})$. The programs have been written by C++ code and implemented on a PC in double precision and executed by using CADNA library on a Linux machine. CADNA is a library which is able to perform the CESTAC method on a code written in C++ or Fortran [8,24]. For the termination criterion, we consider the Hausdorff distance to be an informatical zero (@.0). The successive values $X^{(k)}, Y^{(k)}$ and $Z^{(k)}$ are computed and at each iteration. The computations in the stochastic arithmetic of the sequence $X^{(k)}, Y^{(k)}, Z^{(k)}$ are stopped when for an index like k_{opt}, the number of common significant digits in corresponding components in the norm of difference between $X^{(k_{opt})}, Y^{(k_{opt})}, Z^{(k_{opt})}$ and $X^{(k)}, Y^{(k)}, Z^{(k)}$ respectively become zero.

Algorithm 2.

1. Type (*double_st*) The list of the real variables.
2. Call cadna-init(-1)
3. $k = 0$
4. $cin >> X_0, Y_0, Z_0$
5. Do {
6. $X^{(k+1)} = P_1 X^{(k)} + Q_1, Y^{(k+1)} = P_2 X^{(k)} + Q_2, X^{(k+1)} = P_3 X^{(k)} + Q_3$, $k = 0, 1, 2, \ldots$
7. $cout <<$ "$X =$", $strp(X^{k+1})$, "$Y =$", $strp(Y^{k+1})$, "$Z =$", $strp(Z^{k+1})$
8. $k = k + 1$ }
9. While $Max\{Max\|(X^{(k+1)} - X^{(k)})\|_2, Max\|(Y^{(k+1)} - Y^{(k)})\|_2, Max\|(Z^{(k+1)} - Z^{(k)})\|_2\} \neq @.0$.
10. cadna-end().

The function "Strp" in the output instruction shows only the significant digits of the value.

Example 1. Consider the following FFLS with triangular fuzzy numbers as follows:

$$\begin{cases} (3, 0.2, 0.2)\tilde{x}_1 + (0, 0.2, 0.2)\tilde{x}_2 + (1, 0.1, 0.1)\tilde{x}_3 = (2, 1, 3), \\ (1, 0.2, 0.2)\tilde{x}_1 + (2, 0.2, 0.2)\tilde{x}_2 + (0, 0.1, 0.1)\tilde{x}_3 = (1, 1, 2), \\ (0, 0.2, 0.2)\tilde{x}_1 + (2, 0.2, 0.2)\tilde{x}_2 + (5, 0.1, 0.1)\tilde{x}_3 = (3, 2, 2). \end{cases} \quad (22)$$

By applying Algorithm 2 with $X^{(0)}$ =[0,0,0]^T, $Y^{(0)}$ =[0,0,0]^T, and $Z^{(0)}$ =[0,0,0]^T, as the initial values and using the iterative methods in stochastic arithmetic, the optimal number of iterations for Jacobi method 42, for the Gauss-seidel method is 16 iterations, Richardson method with $\gamma = 0.26$, is 41 and for the SOR iterative method with $\omega = 0.96196384$ is 11. The optimal solution in the stochastic arithmetic for this system listed in Tables 1, 2, 3 and 4 for each method.

Table 1. Jacobi method

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (0.666666666666667, 0.333333333333333, 1)$	1
	$\tilde{x}_2 = (0.5, 0.5, 1)$	
	$\tilde{x}_3 = (0.6, 0.4, 0.4)$	
⋮	⋮	⋮
41	$\tilde{x}_1 = (0.5, 0.1875, 0.875)$	0.1E-014
	$\tilde{x}_2 = (0.25, 0.30625, 0.4625)$	
	$\tilde{x}_3 = (0.5, 0.2375, 0.175)$	
42	$\tilde{x}_1 = (0.5, 0.187499999999999, 0.874999999999999)$	@.0
	$\tilde{x}_2 = (0.25, 0.306249999999999, 0.462499999999999)$	
	$\tilde{x}_3 = (0.5, 0.237499999999999, 0.174999999999999)$	

Table 2. Gauss-Seidel method

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (0.666666666666667, 0.333333333333333, 1)$	1
	$\tilde{x}_2 = (0.166666666666667, 0.333333333333333, 0.5)$	
	$\tilde{x}_3 = (0.533333333333333, 0.266666666666667, 0.2)$	
⋮	⋮	⋮
15	$\tilde{x}_1 = (0.5, 0.1875, 0.875)$	0.8E-015
	$\tilde{x}_2 = (0.25, 0.306249999999999, 0.4625)$	
	$\tilde{x}_3 = (0.5, 0.2375, 0.175)$	
16	$\tilde{x}_1 = (0.5, 0.187499999999999, 0.875)$	@.0
	$\tilde{x}_2 = (0.25, 0.30625, 0.4625)$	
	$\tilde{x}_3 = (0.5, 0.237499999999999, 0.175)$	

Table 3. Richardson method with $\gamma = 0.26$

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (0.666666666666667, 0.333333333333333, 0.8)$	1
	$\tilde{x}_2 = (0.6, 0.5, 1)$	
	$\tilde{x}_3 = (0.633333333333333, 0.410000000000000, 0.350000000000000)$	
⋮	⋮	⋮
40	$\tilde{x}_1 = (0.5, 0.187499999999999, 0.875)$	0.1E-013
	$\tilde{x}_2 = (0.25, 0.306249999999999, 0.462499999999999)$	
	$\tilde{x}_3 = (0.5, 0.237499999999999, 0.174999999999999)$	
41	$\tilde{x}_1 = (0.5, 0.1875, 0.875)$	@.0
	$\tilde{x}_2 = (0.25, 0.30625, 0.4625)$	
	$\tilde{x}_3 = (0.5, 0.2375, 0.175)$	

Table 4. SOR Method with $\omega = 0.96196384$

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (0.641309226666666, 0.320654613333333, 0.961963839999999)$	0.961963840000000
	$\tilde{x}_2 = (0.172523776844151, 0.326752848422075, 0.499276625266227)$	
	$\tilde{x}_3 = (0.510793650054278, 0.259055766080384, 0.192671112134663)$	
2	$\tilde{x}_1 = (0.501913826670359, 0.181212961559680, 0.868201674123549)$	0.139441651773652
	$\tilde{x}_2 = (0.246132585953391, 0.303394142272884, 0.460509054265435)$	
	$\tilde{x}_3 = (0.501898673987306, 0.236754927645707, 0.173774364702036)$	
$\vdots$	$\vdots$	$\vdots$
10	$\tilde{x}_1 = (0.49999999999999, 0.18749999999999, 0.87500000000000)$	0.1E-014
	$\tilde{x}_2 = (0.25000000000000, 0.306249999999999, 0.46250000000000)$	
	$\tilde{x}_3 = (0.49999999999999, 0.237500000000000, 0.17500000000000)$	
11	$\tilde{x}_1 = (0.5, 0.1875, 0.875)$	@.0
	$\tilde{x}_2 = (0.25, 0.30625, 0.4625)$	
	$\tilde{x}_3 = (0.5, 0.2375, 0.175)$	

Table 5. Jacobi method

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (3.05555555555555, 1.03333333333333, 2.12500000000000)$	3.05555555555555
	$\tilde{x}_2 = (2.53571428571428, 1.28928571428571, 1.21785714285714)$	
	$\tilde{x}_3 = (2.20833333333333, 1.15000000000000, 2.12083333333333)$	
	$\tilde{x}_4 = (2.450000000000000, 1.002500000000000, 1.84750000000000)$	
2	$\tilde{x}_1 = (0.948809523809523, -0.247464726631393, 0.183831569664902)$	2.15687830687830
	$\tilde{x}_2 = (0.681746031746031, 0.148316326530612, -0.574909297052154)$	
	$\tilde{x}_3 = (0.051455026455026, 0.294523809523809, 0.294259259259259)$	
	$\tilde{x}_4 = (0.382936507936508, 0.404888888888888, -0.308015873015870)$	
$\vdots$	$\vdots$	$\vdots$
176	$\tilde{x}_1 = (1.99999999999999, 0.499999999999999, 1.09999999999999)$	0.4E-014
	$\tilde{x}_2 = (1.50000000000000, 0.749999999999999, 0.249999999999999)$	
	$\tilde{x}_3 = (0.999999999999999, 0.500000000000000, 1.19999999999999)$	
	$\tilde{x}_4 = (1.24999999999999, 0.250000000000000, 1.00000000000000)$	
177	$\tilde{x}_1 = (2, 0.5, 1.1)$	@.0
	$\tilde{x}_2 = (1.5, 0.75, 0.25)$	
	$\tilde{x}_3 = (1, 0.5, 1.2)$	
	$\tilde{x}_4 = (1.25, 0.25, 1)$	

Table 6. Gauss-Seidel method

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (3.65688229194444, 1.23669110236666, 2.54319541212499)$	3.65688229194444
	$\tilde{x}_2 = (1.78429485916707, 1.12013743320461, 0.587902862511484)$	
	$\tilde{x}_3 = (1.55759705438353, 0.906209633320182, 1.911366014528020)$	
	$\tilde{x}_4 = (0.829843091661695, 0.15063481614000, 0.862852638868634)$	
2	$\tilde{x}_1 = (1.63716638626097, 0.12634625980984, 0.479817449745303)$	2.06337796237969
	$\tilde{x}_2 = (1.44928823554314, 0.566704774529940, 0.12988711292676)$	
	$\tilde{x}_3 = (1.22417591323634, 0.446504994724428, 1.22586612797774)$	
	$\tilde{x}_4 = (1.39015171184824, 0.338259054668160, 1.05579900971072)$	
$\vdots$	$\vdots$	$\vdots$
37	$\tilde{x}_1 = (1.99999999999999, 0.499999999999999, 1.09999999999999)$	0.4E-014
	$\tilde{x}_2 = (1.50000000000000, 0.749999999999999, 0.249999999999999)$	
	$\tilde{x}_3 = (0.999999999999999, 0.500000000000000, 1.19999999999999)$	
	$\tilde{x}_4 = (1.24999999999999, 0.250000000000000, 1.00000000000000)$	
38	$\tilde{x}_1 = (2, 0.5, 1.1)$ $\tilde{x}_2 = (1.5, 0.75, 0.25)$ $\tilde{x}_3 = (1, 0.5, 1.2)$ $\tilde{x}_4 = (1.25, 0.25, 1)$	@.0

Table 7. Richardson method with $\gamma = 0.097$

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = (2.66750000000000, 0.902100000000000, 1.855125000000000)$	2.66750000000000
	$\tilde{x}_2 = (1.72175000000000, 0.875425000000000, 0.826925000000000)$	
	$\tilde{x}_3 = (1.28525000000000, 0.669300000000000, 1.234325000000000)$	
	$\tilde{x}_4 = (2.37650000000000, 0.972424999999999, 1.792074999999999)$	
2	$\tilde{x}_1 = (1.154317300000000, 0.157624999999999, 0.781228299999999)$	1.36357750000000
	$\tilde{x}_2 = (1.27707774999999, 0.564845549999999, 0.153349725000000)$	
	$\tilde{x}_3 = (0.705165750000000, 0.249037800000000, 0.801845649999999)$	
	$\tilde{x}_4 = (1.01292249999999, 0.026119674999999, 0.544102100000000)$	
$\vdots$	$\vdots$	$\vdots$
48	$\tilde{x}_1 = (2.00000000000000, 0.500000000000000, 1.09999999999999)$	0.5E-014
	$\tilde{x}_2 = (1.50000000000000, 0.749999999999999, 0.249999999999999)$	
	$\tilde{x}_3 = (0.999999999999999, 0.499999999999999, 1.19999999999999)$	
	$\tilde{x}_4 = (1.25000000000000, 0.249999999999999, 0.99999999999999)$	
49	$\tilde{x}_1 = (2, 0.5, 1.1)$ $\tilde{x}_2 = (1.5, 0.75, 0.25)$ $\tilde{x}_3 = (1, 0.5, 1.2)$ $\tilde{x}_4 = (1.25, 0.25, 1)$	@.0

Example 2. Consider the FFLS with triangular fuzzy numbers as follows:

$$\begin{cases} (9,0.2,0.2)\tilde{x}_1 + (1,0.4,0.3)\tilde{x}_2 + (3,0.3,0.4)\tilde{x}_3 + (4,0.2,0.1)\tilde{x}_4 = (27.5,9.3,19.125), \\ (2,0.3,0.1)\tilde{x}_1 + (7,0.4,0.2)\tilde{x}_2 + (2,0.2,0.3)\tilde{x}_3 + (1,0.1,0.3)\tilde{x}_3 = (17.75,9.025,8.525), \\ (1,0.3,0.2)\tilde{x}_1 + (1,0.5,0.2)\tilde{x}_2 + (6,0.3,0.1)\tilde{x}_3 + (3,0.2,0.3)\tilde{x}_3 = (13.25,6.9,12.725), \\ (2,0.4,0.5)\tilde{x}_1 + (4,0.5,0.2)\tilde{x}_2 + (2,0.6,1.2)\tilde{x}_3 + (10,0.3,0.3)\tilde{x}_3 = (24.5,10.025,18.475). \end{cases} \quad (23)$$

By assumpsion $X^{(0)} = [0,0,0,0]^T, Y^{(0)} = [0,0,0,0]^T$, and $Z^{(0)} = [0,0,0,0]^T$, as the initial values and using Algorithm 2 in stochastic arithmetic, the Jacobi iterative method is converged after 177 iterations and the Gauss-seidel method is converged after 38 iterations. Also, in Richardson method with $\gamma = 0.097$, the number of iterations is optimized after 49 iterations and for the SOR iterative method with $\omega = 1.196797841$ after 29 iterations. The optimal solution in the stochastic arithmetic for this system listed in Tables 5, 6, 7 and 8 for each method.

Table 8. SOR method with $\omega = 1.196797841$

k	$\tilde{x}$	Distance
1	$\tilde{x}_1 = 3.05555555555555, 1.03333333333333, 2.12499999999999)$	3.05555555555555
	$\tilde{x}_2 = (1.66269841269841, 0.994047619047618, 0.610714285714285)$	
	$\tilde{x}_3 = (1.42195767195767, 0.812103174603174, 1.66488095238095)$	
	$\tilde{x}_4 = (0.889417989417989, 0.235793650793650, 0.845238095238095)$	
2	$\tilde{x}_1 = (2.00152851263962, 0.338422986478542, 0.930116108171663)$	1.19488389182833
	$\tilde{x}_2 = (1.43051566305534, 0.647582094566221, 0.16546569244981)$	
	$\tilde{x}_3 = (1.19161697600851, 0.475687865681251, 1.29017199266537)$	
	$\tilde{x}_4 = (1.23916463704823, 0.263287848884409, 0.953906896503457)$	
⋮	⋮	⋮
28	$\tilde{x}_1 = (1.99999999999999, 0.499999999999999, 1.09999999999999)$	0.4E-014
	$\tilde{x}_2 = (1.50000000000000, 0.749999999999999, 0.249999999999999)$	
	$\tilde{x}_3 = (0.999999999999999, 0.500000000000000, 1.19999999999999)$	
	$\tilde{x}_4 = (1.24999999999999, 0.250000000000000, 1.00000000000000)$	
29	$\tilde{x}_1 = (2, 0.5, 1.1)$ $\tilde{x}_2 = (1.5, 0.75, 0.25)$ $\tilde{x}_3 = (1, 0.5, 1.2)$ $\tilde{x}_4 = (1.25, 0.25, 1)$	@.0

6 Conclusion

In this paper, we proposed an algorithm in order to approximate the solution of a FFLS in the stochastic arithmetic. By applying the CESTAC method based on the stochastic arithmetic, and use the iterative method as Jacobi, Gauss-seidel, Richardson and SOR to approximate the solution of FFLS, the results are validated step by step. We determined the solution of FFLS with the least number of iterations on the stochastic arithmetic. Also, the useless iteration are omitted. By using the computational zero as the optimal termination criterion, the iterative process is stopped correctly and computation time is saved, because many useless operations and iterations are not performed. Finally, we show that by using the stochastic arithmetic is possible to increase the accuracy of the computed solution of FFLS, and the stochastic arithmetic can play an important role to rely the numerical algorithms.

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Some Remarks on an Order Induced by Uninorms

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Abstract. Recently an order induced by t-norms, uninorms and nullnorms have been investigated. This paper is mainly devoted to defining and investigating the set of incomparable elements with respect to the order induced by a uninorm. Also, by defining such an order, an equivalence relation on the class of uninorms is defined and this equivalence is deeply investigated.

1 Introduction

Uninorms are introduced by Yager and Rybalov [29]. Uninorms are special aggregation operators which have proven to be useful in many applications like fuzzy logic, expert systems, neural networks, fuzzy system modeling [14, 16, 30].

Uninorms on the real unit interval as generalizations of t-norms and t-conorms admit a neutral element e to be an arbitrary point from [0, 1].

In [27], a natural order for semigroups was defined. Similarly, in [20], a partial order defined by means of t-norms on a bounded lattice was introduced

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L,$$

where L is a bounded lattice, x, y of a bounded lattice L and T is a t-norm on L. This partial order $\preceq_T$ is called a T-partial order on L.

In [2], with the help of any t-norm T on $[0, 1]$, it was obtained that the family $(T_\lambda)_{\lambda\in(0,1)}$ of t-norms on $[0, 1]$. If T was a divisible t-norm, then it was obtained that $([0, 1], \preceq_{T_\lambda})$ was a lattice. The uninorms, t-norms and t-conorms were also studied by many other authors [1–3, 7, 9–13, 17–19, 23–26, 28, 31].

In the present paper, we introduce the set of incomparable elements with respect to the U-partial order for any uninorm on $[0, 1]$. The main aim is to investigate some properties of this set. The paper is organized as follows. We shortly recall some basic notions in Sect. 2 In Sect. 3, we define the set of incomparable elements with respect to the U-partial order for any uninorm on $[0, 1]$. Also, we determine the set of incomparable elements w.r.t. U-partial order for some special uninorms. Then, we define an equivalence on the class of uninorms on $[0, 1]$. In Sect. 4, we define that the set $\mathscr{I}_U^{(x)}$, consisting all incomparable elements with any $x \in (0, 1)$ according to $\preceq_U$. Finally we show that even if the uninorms are equivalent under this relation, it need not be the case that their partial orders coincide.

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2 Preliminaries

Let us now recall all necessary basic notions.

Definition 1. [22] Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A *triangular norm* T (briefly t-norm) is a binary operation on L which is commutative, associative, monotone and has neutral element 1.

Definition 2. [22] Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A *triangular conorm* S (briefly t-conorm) is a binary operation on L which is commutative, associative, monotone and has neutral element 0.

Example 1. [22] Well-known triangular norms and triangular conorms on $[0, 1]$ are:

$$
\begin{aligned}
&T_M(x, y) = min(x, y)\\
&T_P(x, y) = x.y\\
&T_L(x, y) = max(x + y - 1, 0)\\
&T_D(x, y) = \begin{cases} 0 & , (x, y) \in [0, 1)^2 \\ min(x, y) & , \text{otherwise} \end{cases}\\
&S_M(x, y) = max(x, y)\\
&S_P(x, y) = x + y - x.y\\
&S_L(x, y) = min(x + y, 1)\\
&S_D(x, y) = \begin{cases} 1 & , (x, y) \in (0, 1]^2 \\ max(x, y) & , \text{otherwise} \end{cases}
\end{aligned}
$$

Also, t-norms on a bounded lattice $(L, \leqslant, 0, 1)$ are defined in similar way, and then extremal t-norms $T_\wedge$ and T_W on L is defined as follows, respectively:

$$
\begin{aligned}
&T_\wedge(x, y) = x \wedge y\\
&T_W(x, y) = \begin{cases} x & , if\ \ y = 1 \\ y & , if\ \ x = 1 \\ 0 & , \text{ otherwise} \end{cases}
\end{aligned}
$$

Similarly it can be defined the t-conorms $S_\vee$ and S_W.
Especially we obtained that $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0, 1]$.

Definition 3. [6] A t-norm T on L is *divisible* if the following condition holds:

$$\forall\, x, y \in L \quad \text{with} \quad x \leqslant y \quad \text{there is a} \quad z \in L \quad \text{such that} \quad x = T(y, z).$$

A basic example of a non-divisible t-norm on an arbitrary lattice L (i.e., card$L > 3$) is the weakest t-norm T_W. Trivially, the infimum $T_\wedge$ is divisible: $x \leqslant y$ is equivalent to $x \wedge y = x$.

Definition 4. [4] Given a bounded lattice $(L, \leqslant, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case we use the notation $a \parallel b$.

Definition 5. [4] Given a bounded lattice $(L, \leqslant, 0, 1)$ and $a, b \in L$, $a \leqslant b$, a subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L \mid a \leqslant x \leqslant b\}$$

Similarly, $[a, b) = \{x \in L \mid a \leqslant x < b\}$, $(a, b] = \{x \in L \mid a < x \leqslant b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 6. [5] Let $(L, \leqslant, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

We denote by $\mathscr{U}(e)$ the set of all uninorms on L with the neutral element $e \in L$. $A(e) = (0, e] \times [e, 1) \cup [e, 1) \times (0, e]$ for $e \in L \setminus \{0, 1\}$.

Definition 7. [9] Let U be a uninorm on a bounded lattice L with a neutral element $e \in L$. An element $x \in L$ is called an idempotent element of U if $U(x, x) = x$.

Proposition 1. *[21] Let $(L, \leqslant, 0, 1)$ be a bounded lattice, $e \in L$ and U be a uninorm with the neutral element e on L. Then,*

(i) $T^* = U\mid_{[0,e]^2} : [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]$.
(ii) $S^* = U\mid_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$.

Definition 8. [20] Let L be a bounded lattice, T be a t-norm on L. The order defined as following is called a $T-$*partial order* (triangular order) for t-norm T:

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

Definition 9. [15] Let L be a bounded lattice, S be a t-conorm on L. The order defined as following is called a $S-$ *partial order* for t-conorm S:

$$x \preceq_S y :\Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L.$$

Definition 10. [15] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and U be a uninorm with neutral element e on L. Define the following relation, for $x, y \in L$, as

$$x \preceq_U y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0, e] \text{ and there exist } k \in [0, a] \text{ such that } U(k, y) = x \text{ or,} \\ \text{if } x, y \in [e, 1] \text{ and there exist } \ell \in [e, 1] \text{ such that } U(x, \ell) = y \text{ or,} \\ \text{if } (x, y) \in L^* \text{ and } x \leqslant y. \end{cases} \quad (1)$$

where $I_e = \{x \in L \mid x \parallel e\}$ and $L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times I_e \cup [e, 1] \times [0, e] \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e$.

Proposition 2. *[15] The relation $\preceq_U$ defined in (1) is a partial order on L.*

Proposition 3. *[15] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $U \in \mathscr{U}(e)$. If $x \preceq_U y$ for any $x, y \in L$, then $x \leqslant y$.*

Proposition 4. *[15] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $U \in \mathscr{U}(e)$. Then, $(L, \preceq_U)$ is a bounded partially ordered set.*

Lemma 1. *[15] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and U be a uninorm with neutral element e on L. The order $\preceq_U$ coincides with the order $\preceq_T$ ($\preceq_S$), when $e = 1$ ($e = 0$).*

Proposition 5. *[8] Let T be a t-norm on $[0, 1]$. T is divisible if and only if T is continuous.*

3 The Set $K_U \subset [0, 1]$ Consisting of Incomparable Elements with Respect to $\preceq_U$ on $[0, 1]$

Definition 11. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and U_1 and U_2 be two uninorms on L. If for all $x, y \in L$, $x \preceq_{U_1} y \Rightarrow x \preceq_{U_2} y$, then we say that U_2 is order-stronger than U_1, or equivalently, that U_1 is order-weaker than U_2.

In [2], it was shown that for the t-norms T_W and $T_\wedge$ on L, T_W is the order-weakest and $T_\wedge$ is the order-strongest t-norm, i.e., $\preceq_{T_W} \subseteq \preceq_T \subseteq \preceq_{T_\wedge}$. But for the uninorms, it need not be that case. Now, let us investigate the following example.

Example 2. Consider the uninorms $\underline{U_e} : [0, 1]^2 \to [0, 1]$ and $\overline{U}_e : [0, 1]^2 \to [0, 1]$ with neutral element $e \neq 0, 1$ defined by

$$\underline{U_e}(x, y) = \begin{cases} 0 & , (x, y) \in [0, e)^2 \\ max(x, y) & , (x, y) \in [e, 1]^2 \\ min(x, y) & , \text{otherwise} \end{cases}$$

and

$$\overline{U_e}(x, y) = \begin{cases} min(x, y) & , (x, y) \in [0, e]^2 \\ 1 & , (x, y) \in (e, 1]^2 \\ max(x, y) & , \text{otherwise} \end{cases}$$

We know that $\underline{U_e}$ is the smallest uninorm and $\overline{U_e}$ is the greatest uninorm on $[0, 1]$. We showed that $\underline{U_e}$ is not order-weakest and $\overline{U_e}$ is not order-strongest uninorm.

Now, we study the set of elements being incomparable with some other element with respect to the U-partial order $\preceq_U$ with U some uninorm on $[0, 1]$.

Definition 12. Let U be a uninorm on $[0, 1]$ and let K_U be defined by

$$K_U = \{x \in (0, 1) \mid for\ some\ y \in (0, 1),\ [x < y\ and\ x \not\preceq_U y]\ or\ [y < x\ and\ y \not\preceq_U x]\}$$

Note that an element $x \in K_U$ is not necessarily incomparable with all elements $y \in [0,1] \setminus \{0,1,x\}$.

Theorem 1. *[16] Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$. Then the sections $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ are continuous in each point except perhaps for e if and only if U is given by one of the following formulas.*

(a) If $U(0,1) = 0$, then

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & ,(x,y) \in [0,e]^2 \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & ,(x,y) \in [e,1]^2 \\ min(x,y) & ,(x,y) \in A(e). \end{cases} \quad (2)$$

where T is a t-norm and S is a t-conorm.

(b) If $U(0,1) = 1$, then the same structure holds, changing minimum by maximum in $A(e)$.

The set of uninorms as in case (a) will be denoted by $\mathscr{U}_{min}$ and the set of uninorms as in case (b) by $\mathscr{U}_{max}$. We will denote a uninorm U in $\mathscr{U}_{min}$ with underlying t-norm T, underlying t-conorm S and neutral element e by $U \equiv \langle T, e, S\rangle_{min}$ and in a similar way, a uninorm in $\mathscr{U}_{max}$ by $U \equiv \langle T, e, S\rangle_{max}$.

Proposition 6. *Let U be a uninorm such that $U \equiv \langle T, e, S\rangle_{min}$ or $U \equiv \langle T, e, S\rangle_{max}$. Then,*

$$K_U = eK_T \cup (e + (1-e)K_S).$$

Lemma 2. *Consider the smallest uninorm $\underline{U_e}$ on $[0,1]$ with neutral element $e \neq 0,1$ of Example 2. Then, we have that $K_{\underline{U_e}} = (0,e)$.*

Corollary 1. *For the drastic product t-norm T_D, $K_{T_D} = (0,1)$.*

Corollary 2. *For the t-conorm S_M, $K_{S_M} = \varnothing$.*

Lemma 3. *Consider the greatest uninorm $\overline{U}_e$ on $[0,1]$ with neutral element $e \neq 0,1$ of Example 2. Then, we have that $K_{\overline{U}_e} = (e,1)$.*

Corollary 3. *For the minimum t-norm T_M, $K_{T_M} = \varnothing$.*

Corollary 4. *For the t-conorm S_D, $K_{S_D} = (0,1)$.*

Lemma 4. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice. For all uninorms U and all $x \in L$ it holds that $0 \preceq_U x$, $x \preceq_U x$ and $x \preceq_U 1$.*

Definition 13. Define a relation β_U on the class of all uninorms on $[0,1]$ by $U_1 \beta_U U_2$,

$$U_1 \beta_U U_2 :\Leftrightarrow K_{U_1} = K_{U_2}.$$

Lemma 5. *The relation β_U given in Definition 13 is an equivalence relation.*

Definition 14. For a given uninorm U on $[0,1]$, we denote by $\overline{U}$ the β_U equivalence class linked to U, i.e.,

$$\overline{U} = \{U^{'} | \quad U^{'}\beta_U U\}.$$

Proposition 7. *The set* $[0,1]/\beta_U$*, is uncountably infinite.*

Theorem 2. *[16] Let* $e \in [0,1]$*.* $U \in \mathscr{U}(e)$ *if and only if*

$$U(x,y) = \begin{cases} T_U & ,(x,y) \in [0,e]^2 \\ S_U & ,(x,y) \in [e,1]^2 \\ C & ,(x,y) \in A(e) \end{cases}$$

where T_U *and* S_U *are operations respectively isomorphic with some triangular norm and triangular conorm and increasing operation* $C : A(e) \to [0,1]$ *fulfills*

$$min(x,y) \leqslant C(x,y) \leqslant max(x,y) \ for \ (x,y) \in A(e).$$

Proposition 8. *Let* U *be a uninorm on* $[0,1]$ *with neutral element* e *of Theorem 2. If* T_U *and* S_U *are continuous, then* $K_U = \varnothing$*.*

Corollary 5. *Let* $e \in [0,1]$*. Consider the uninorms* U^{min} *and* U^{max} *are unique idempotent uninorm* U_e^{min} *and* U_e^{max}*, respectively:*

$$U(x,y) = \begin{cases} max(x,y) & ,(x,y) \in [e,1]^2 \\ min(x,y) & ,otherwise \end{cases}$$

$$U(x,y) = \begin{cases} min(x,y) & ,(x,y) \in [0,e]^2 \\ max(x,y) & ,otherwise \end{cases}$$

Then, it is obtained that $K_U = \varnothing$*.*

4 About the Set $\mathscr{I}_U^{(x)}$ Consisting all Incomparable Elements with Any $x \in (0,1)$ According to $\preceq_U$

Definition 15. Let U be a uninorm on $[0,1]$ and let $\mathscr{I}_U^{(x)}$ for $x \in (0,1)$ be defined by

$$\mathscr{I}_U^{(x)} = \{y \in (0,1) \mid [x < y \ and \ x \npreceq_U y] \, or \, [y < x \ and \, y \npreceq_U x]\}$$

After that we will use the notation $\mathscr{I}_U^{(x)}$ to denote the set of all incomparable elements with $x \in (0,1)$ according to $\preceq_U$.

Example 3. The uninorm $U := U_{min(T^{nM},S_M,\frac{1}{2})} : [0,1]^2 \to [0,1]$ with neutral element $e = \frac{1}{2}$ defined as follows:

$$U_{min(T^{nM},S_M,\frac{1}{2})}(x,y) = \begin{cases} 0 & ,(x,y) \in [0,\frac{1}{2}]^2 \text{ } and \text{ } x+y \leqslant \frac{1}{2} \\ max(x,y) & ,(x,y) \in [\frac{1}{2},1]^2 \\ min(x,y) & ,\text{Otherwise} \end{cases}$$

Then,

(a) $\mathscr{I}_U^{(x)} = \{y \in (0, \frac{1}{2} - x] \mid x \neq y\}$ for $x \in (0, \frac{1}{2})$
(b) $\mathscr{I}_U^{(x)} = \varnothing$ for $x \geqslant \frac{1}{2}$.

Example 4. Consider the smallest uninorm $\underline{U_e}$ with neutral element e of Example 2. Then,

(a) $\mathscr{I}_{\underline{U_e}}^{(x)} = \{y \in (0, e) \mid x \neq y\}$ for $x \in (0, e)$ and
(b) $\mathscr{I}_{\underline{U_e}}^{(x)} = \varnothing$ for $x \geqslant e$.

Example 5. Consider the greatest uninorm $\overline{U_e}$ with neutral element e of Example 2. Then,

(a) $\mathscr{I}_{\overline{U_e}}^{(x)} = \{y \in (e, 1) \mid x \neq y\}$ for $x \in (e, 1)$ and
(b) $\mathscr{I}_{\overline{U_e}}^{(x)} = \varnothing$ for $x \leqslant e$.

Lemma 6. *Let U be a uninorm on $[0,1]$. Then $K_U = \bigcup_{x\in[0,1]} \mathscr{I}_U{}^{(x)}$.*

Proposition 9. *Let U_1 and U_2 be two uninorms on $[0,1]$. If for all $x \in [0,1]$, $\mathscr{I}_{U_1}{}^{(x)} = \mathscr{I}_{U_2}{}^{(x)}$, then the uninorms U_1 and U_2 are equivalent under the relation β_U.*

Remark 1. The converse of Proposition 9 may not be true.

Example 6. Consider the uninorms U on $[0,1]$ with neutral element $e = \frac{1}{2}$ of Example 3 and $\underline{U_e}$ on $[0,1]$ with neutral element $e = \frac{1}{2}$ of Example 2. We showed that the uninorms U and $\underline{U_e}$ are equivalent under the relation β_U. But, $\mathscr{I}_U{}^{(\frac{2}{5})} \neq \mathscr{I}_{\underline{U_e}}{}^{(\frac{2}{5})}$.

Corollary 6. *Although the uninorms U_1 and U_2 are equivalent under the relation β_U, it need not be the case that the U_1-partial order coincides with the U_2-partial order.*

5 Conclusion

We have defined the set of incomparable elements with respect to the U-partial order for any uninorm on $[0,1]$. Also we have introduced and studied an equivalence relation β_U defined on the class of all uninorms on $[0,1]$. We have defined that the set $\mathscr{I}_U^{(x)}$, consisting all incomparable elements with any $x \in (0,1)$ according to $\preceq_U$. Finally we have shown that even if the uninorms are equivalent under the this relation, it need not be the case that their partial orders coincide.

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Some Notes on the F-partial Order

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Abstract. Nullnorms have been produced from triangular norms and triangular conorms and they have several applications in fuzzy logic. The main purpose of this paper is to study the order induced by nullnorms on bounded lattices. We discuss the relationship between the natural order and the order induced by a nullnorm on bounded lattice.

1 Introduction

Menger, in [19], introduced triangular norms and triangular conorms. They have a lot of fields of mathematics, for instance in fuzzy logic and their applications.

Firstly t-operators were defined by Mas et al. in [17], and then in [5] Calvo et al. introduced a nullnorm. Also, in [5], they showed that nullnorms were equivalent to t-operators.

In [20], a natural order for semigroups was defined. Similarly, in [14], a partial order defined by means of t-norms on a bounded lattice was introduced.

In [2], with the help of any t-norm T on $[0,1]$, a family of t-norms on $[0,1]$, $(T_\lambda)_{\lambda\in(0,1)}$ was constructed. If T was a divisible t-norm, then it was obtained that $([0,1], \preceq_{T_\lambda})$ was a lattice. In [1], an order induced by nullnorms on bounded lattices was defined and discussed. The nullnorms and t-norms were also studied by many other authors [3,6,8–12,16,18,21,22].

In the present paper, we investigate some properties an order induced by nullnorms on bounded lattices. The paper is organized as follows. In Sect. 2, we will first recall all important notions and results. We will discuss the relationship between the order induced by a nullnorm and the natural order on the lattice in Sect. 3. Even if L is a chain, we will show that L need not be a chain with respect to the F-partial order. Similar arguments will be done for lattices. Finally, we will show that the set of all idempotent elements of F is a chain with respect to the F-partial order. We will give our concluding remarks in Sect. 4.

2 Basic Notions and Results

Let us now recall all necessary basic notions.

Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A triangular norm is a binary function $T : L^2 \to L$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element.

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Dual functions to t-norms are t-conorms. A triangular conorms is a binary function $S : L^2 \to L$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element.

Example 1. [15] Well-known triangular norms and triangular conorms on $[0, 1]$ are:

$$T_M(x, y) = min(x, y)$$
$$T_P(x, y) = x.y$$
$$T_L(x, y) = max(x + y - 1, 0)$$
$$T_D(x, y) = \begin{cases} 0 & , (x, y) \in [0, 1)^2 \\ min(x, y) & , \text{otherwise} \end{cases}$$
$$S_M(x, y) = max(x, y)$$
$$S_P(x, y) = x + y - x.y$$
$$S_L(x, y) = min(x + y, 1)$$
$$S_D(x, y) = \begin{cases} 1 & , (x, y) \in (0, 1]^2 \\ max(x, y) & , \text{otherwise} \end{cases}$$

Also, t-norms on a bounded lattice $(L, \leqslant, 0, 1)$ are defined in similar way, and then extremal t-norms $T_\wedge$ and T_W on L is defined as follows, respectively:

$$T_\wedge(x, y) = x \wedge y$$
$$T_W(x, y) = \begin{cases} x & , if\, y = 1 \\ y & , if\, x = 1 \\ 0 & , \text{otherwise} \end{cases}$$

Similarly it can be defined the t-conorms $S_\vee$ and S_W.

Especially we obtained that $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0, 1]$.

Definition 1. [6] A t-norm T on L is *divisible* if the following condition holds:

$$\forall\, x, y \in L \quad \text{with} \quad x \leqslant y \quad \text{there is a} \quad z \in L \quad \text{such that} \quad x = T(y, z).$$

A basic example of a non-divisible t-norm on an arbitrary lattice L (i.e., card$L > 3$) is the weakest t-norm T_W. Trivially, the infimum $T_\wedge$ is divisible: $x \leqslant y$ is equivalent to $x \wedge y = x$.

Proposition 1. *[7] Let T be a t-norm on $[0, 1]$. T is divisible if and only if T is continuous.*

Definition 2. [4] Given a bounded lattice $(L, \leqslant, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case we use the notation $a \parallel b$.

Definition 3. [4] Given a bounded lattice $(L, \leqslant, 0, 1)$ and $a, b \in L$, $a \leqslant b$, a subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L \mid a \leqslant x \leqslant b\}$$

Similarly, $[a, b) = \{x \in L \mid a \leqslant x < b\}$, $(a, b] = \{x \in L \mid a < x \leqslant b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 4. [5] Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $F : L^2 \to L$ is called a nullnorm if there is an element $a \in L$ such that $F(x, 0) = x$ for all $x \leqslant a$, $F(x, 1) = x$ for all $x \geqslant a$.

It can be easily obtained that $F(x, a) = a$ for all $x \in L$. So $a \in L$ is the zero (absorbing) element for F.

Consider the set $\mathscr{F}$ of all nullnorms on L with the following order: For $F_1, F_2 \in \mathscr{F}$,

$$F_1 \leqslant F_2 \Leftrightarrow F_1(x, y) \leqslant F_2(x, y) \text{ for all } (x, y) \in L^2.$$
$$D_a = [0, a) \times (a, 1] \cup (a, 1] \times [0, a) \text{ for } a \in L \backslash \{0, 1\}.$$

Definition 5. [13] An element $x \in L$ is called an idempotent element of a function $F : L^2 \to L$ if $F(x, x) = x$. The function F is called idempotent if all elements of L are idempotent.

Definition 6. [14] Let L be a bounded lattice, T be a t-norm on L. The order defined as following is called a $T-$ *partial order* (triangular order) for t-norm T:

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

The duality between t-norms and t-conorms is expressed by the fact that from T-partial order for any t-norm T we can obtain its dual S-partial order for any t-conorm S by the follows

$$x \preceq_S y :\Leftrightarrow S(\ell, x) = y \quad \text{for some } \ell \in L.$$

Definition 7. [1] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. Define the following relation, for $x, y \in L$, as

$$x \preceq_F y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0, a] & \text{and there exist } k \in [0, a] \text{ such that } F(x, k) = y \text{ or} \\ \text{if } x, y \in [a, 1] & \text{and there exist } \ell \in [a, 1] \text{ such that } F(y, \ell) = x \text{ or,} \\ \text{if } (x, y) \in L^* & \text{and } x \leqslant y. \end{cases} \tag{1}$$

where $I_a = \{x \in L \mid x \parallel a\}$ and $L^* = [0, a] \times [a, 1] \cup [0, a] \times I_a \cup [a, 1] \times I_a \cup [a, 1] \times [0, a] \cup I_a \times [0, a] \cup I_a \times [a, 1] \cup I_a \times I_a$.

Proposition 2. *[1] The relation $\preceq_F$ defined in (1) is a partial order on L.*

Note: The partial order $\preceq_F$ in (1) is called F-partial order on L.

Proposition 3. *[1] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm on L. If $x \preceq_F y$ for any $x, y \in L$, then $x \leqslant y$.*

Proposition 4. *[1] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a. Then, $(L, \preceq_F)$ is a bounded partially ordered set.*

3 On the F-partial Order

Lemma 1. *[1] Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. The order $\preceq_F$ coincides with the order $\preceq_T$ ($\preceq_S$), when $a = 0$ ($a = 1$).*

Remark 1. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. Even if $(L, \leqslant, 0, 1)$ is a chain, the partially ordered set $(L, \preceq_F)$ may not be a chain. To illustrate this claim we shall give the following example.

Example 2. Consider $L = [0, 1]$ and take the nullnorm $\overline{F_a} : [0, 1]^2 \to [0, 1]$ with the zero element $a \in (0, 1)$ defined as follows:

$$\overline{F_a}(x, y) = \begin{cases} min(x, y) & , (x, y) \in [a, 1]^2 \\ a & , (x, y) \in (0, a]^2 \cup D_a \\ max(x, y) & , \text{otherwise} \end{cases}$$

$\overline{F_a}$ is the greatest nullnorm on $[0, 1]$ by [12]. But L is not a chain with respect to the $\preceq_{\overline{F_a}}$.

In the paper, for any subset X of L, $\overline{X}_{\preceq_F}$ ($\underline{X}_{\preceq_F}$) denotes the set of the upper (lower) bounds of X with respect to $\preceq_F$. Also, for any $x, y \in L$, $x \wedge_F y$ ($x \vee_F y$) denotes the greatest (least) element of the lower (upper) bounds with respect to $\preceq_F$, if there exists.

Remark 2. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. Even if $(L, \leqslant, 0, 1)$ is a lattice, the partially ordered set $(L, \preceq_F)$ may not be a lattice. To illustrate this claim we shall give the following example.

Example 3. Consider the function $F := F_{(T^{nM}, S, \frac{1}{5})} : [0, 1]^2 \to [0, 1]$ defined as follows:

$$F_{(T^{nM}, S, \frac{1}{5})}(x, y) = \begin{cases} max(x, y) & , (x, y) \in [0, \frac{1}{5}]^2 \\ \frac{1}{5} & , ((x, y) \in [\frac{1}{5}, 1]^2 \text{ and } x + y \leqslant 1) \text{ or } (x, y) \in D_{\frac{1}{5}} \\ min(x, y) & , \text{otherwise} \end{cases}$$

The function F is a nullnorm with $\frac{1}{5}$ zero element by [1]. Then, $([0, 1], \preceq_F)$ is not join-semilattice. Because, we showed that $\overline{\{\frac{1}{4}, \frac{1}{2}\}}_{\preceq_F} = (\frac{3}{4}, 1]$. Since there does not exist the least element of $(\frac{3}{4}, 1]$ with respect to the $\preceq_F$, $([0, 1], \preceq_F)$ is not a join-semilattice. So, $([0, 1], \preceq_F)$ is not lattice.

Remark 3. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. If $(L, \leqslant, 0, 1)$ is a lattice, the partially ordered set $(L, \preceq_F)$ may be a lattice.

Example 4. Let $(L = \{0, x, y, a, z, t, 1\}, \leqslant, 0, 1)$ be a chain with $0 < x < y < a < z < t < 1$. Consider the function on L defined as follows:

$$F(x, y) = \begin{cases} x \wedge y & , (x, y) \in [a, 1]^2 \\ y & , x = 0 \text{ and } y \leqslant a \\ x & , y = 0 \text{ and } x \leqslant a \\ a & , \text{otherwise} \end{cases}$$

By [13], it can be easily seen that F is a nullnorm. The order $\preceq_F$ on L has its diagram as follows (see Fig. 1).

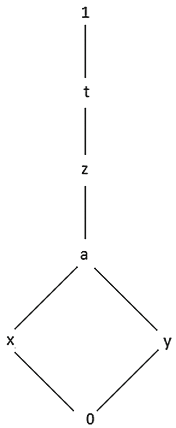

Fig. 1. The order $\preceq_F$ on L

Proposition 5. *[13] Let $(L, \leqslant, 0, 1)$ be a bounded lattice, $a \in L \backslash \{0, 1\}$ and F be a nullnorm with zero element a on L. Then,*

(i) $S^* = F\mid_{[0,a]^2}: [0, a]^2 \to [0, a]$ is a t-conorm on $[0, a]$.
(ii) $T^* = F\mid_{[a,1]^2}: [a, 1]^2 \to [a, 1]$ is a t-norm on $[a, 1]$.

Proposition 6. *[1] Let $(L, \leqslant, 0, 1)$ be a bounded lattice, F be a nullnorm with zero element a on L and $a \in L \backslash \{0, 1\}$. Then, S^* and T^* are divisible if and only if $\preceq_F = \leqslant$.*

Corollary 1. *Let $F : [0, 1]^2 \to [0, 1]$ be a nullnorm with zero element $a \in (0, 1)$. Then, S^* and T^* are continuous if and only if $\preceq_F = \leqslant$. Moreover, if F is an idempotent nullnorm on a bounded lattice L, then we have that $S^* = S_M$ and $T^* = T_M$. Thus, the order $\preceq_F$ coincides with the order $\leqslant$.*

Proposition 7. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and F be a nullnorm on L with zero element $a \in L \backslash \{0, 1\}$ such that a is comparable with all elements of L. Then, $([0, a], \preceq_{S^*})$ and $([a, 1], \preceq_{T^*})$ are lattices if and only if $(L, \preceq_F)$ is a lattice.*

Proposition 8. *[5,17] Let $F : [0,1]^2 \to [0,1]$ be a nullnorm with zero element $F(1,0) = k \notin \{0,1\}$. Then,*

$$F(x,y) = \begin{cases} kS(\frac{x}{k}, \frac{y}{k}) & , x,y \in [0,k] \\ k + (1-k)T(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & , x,y \in [k,1] \\ k & , \textit{otherwise} \end{cases}$$

where S is a t-conorm and T is a t-norm.

A nullnorm F with zero element k, underlying t-conorm S and underlying t-norm T will be denoted by $F =< S, k, T >$.

Proposition 9. *Let $F =< S, k, T >$ be a nullnorm with zero element $k \in (0,1)$. Then,*

(i) $x \preceq_F y$ for $x, y \in [0,k]$ if and only if $\frac{x}{k} \preceq_S \frac{y}{k}$ for $x, y \in [0,k]$.
(ii) $x \preceq_F y$ for $x, y \in [k,1]$ if and only if $\frac{x-k}{1-k} \preceq_T \frac{y-k}{1-k}$ for $x, y \in [k,1]$.

Proposition 10. *Let L be a chain, F be a nullnorm with zero element $a \in L \backslash \{0,1\}$ and H_F be the set of all idempotent elements of F. Then, $(H_F, \preceq_F)$ is a chain.*

4 Concluding Remarks

We have obtained some important results about F-partial order. We have determined the relationship between the order induced by a nullnorm and the order on the lattice. If L is a chain, we have shown that L need not be a chain with respect to the F-partial order. We have shown that the set of all idempotent elements of F is a chain with respect to the an order induced by nullnorms, denoted by $\preceq_F$.

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Two Intuitionistic Fuzzy Modal-Level Operators

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Abstract. Two new intuitionistic fuzzy operators are introduced. For them it is shown that they have a behaviour similar both to the modal, as well as to the level operators, defined over intuitionistic fuzzy sets and for this reason, they are called intuitionistic fuzzy modal-level operators. Their basic properties are discussed.

Keywords: Intuitionistic fuzzy set · Intuitionistic fuzzy operator

1 Introduction

The Intuitionistic Fuzzy sets (IFSs) were introduced 34 years ago in [1] and during this time, their theory was enriched with a lot of operators that do not have analogous in the standard fuzzy sets theory and in the rest of the fuzzy sets extensions. In the present paper, we introduce two operators that have behaviour similar to the modal, as well as to the level operators.

2 Preliminary Definitions

Following [3,4], we give the definitions of the basic concepts and the basic operations, relations and operators over IFSs.

Let us have a fixed universe E and its subset A. The set

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

is called IFS and functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ represent the *degree of membership (validity, etc.)* and *non-membership (non-validity, etc.).* Now, we can define also function $\pi_A : E \to [0,1]$by means of

$$\pi(x) = 1 - \mu(x) - \nu(x)$$

and it corresponds to *degree of indeterminacy (uncertainty, etc.).*

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For brevity, we shall write below A instead of A^*, whenever this is possible.

Obviously, for every ordinary fuzzy set A: $\pi_A(x) = 0$ for each $x \in E$ and these sets have the form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}$.

For every two IFSs A and B we can define (see [3,4]):

$$\begin{array}{lll}
A \subset B & \text{iff} & (\forall x \in E)((\mu_A(x) \le \mu_B(x) \ \&\ \nu_A(x) > \nu_B(x)) \\
 & & \quad \vee(\mu_A(x) < \mu_B(x) \ \&\ \nu_A(x) \ge \nu_B(x)) \\
 & & \quad \vee(\mu_A(x) < \mu_B(x) \ \&\ \nu_A(x) > \nu_B(x))); \\
A \subseteq B & \text{iff} & (\forall x \in E)(\mu_A(x) \le \mu_B(x) \ \&\ \nu_A(x) \ge \nu_B(x)); \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \&\ \nu_A(x) = \nu_B(x)); \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A + B & = & \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle \mid x \in E\}; \\
A \times B & = & \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \mid x \in E\}; \\
A@B & = & \{\langle x, (\frac{\mu_A(x)+\mu_B(x))}{2}, \frac{(\nu_A(x)+\nu_B(x))}{2}\rangle | x \in E\}; \\
A \to B & = & \{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle | x \in E\}.
\end{array}$$

In [4], 34 different operations intuitionistic fuzzy negation and 138 different operations intuitionistic fuzzy implications are described. They are analogous to the two classical operations, while the remaining operations have non-classical behaviour. In [6] their numbers have increased, respectively, to 53 and 185, while,

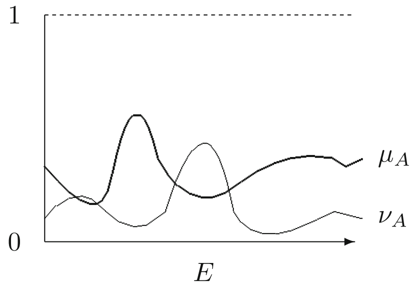

Fig. 1.

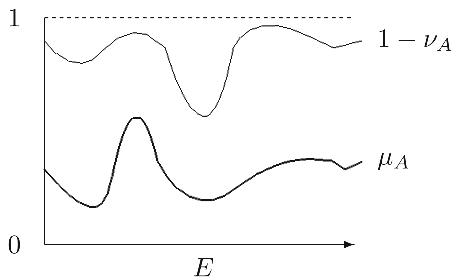

Fig. 2.

now there are already 189 intuitionistic fuzzy implications and probably, this number will continue to increase.

In IFS theory there are some other operations that we do not discuss here because (at least at the moment) they are not related to the operators introduced here.

The IFSs have different (more than 10) geometrical interpretations. The first of them (see Fig. 1) is a trivial modification of the fuzzy set geometrical interpretation and it was constructed in the beginning of the research of IFSs. Its analogue is given in Fig. 2. It is interesting to mention that some authors of papers over vague sets asserted that the vague sets are better than the IFSs because the geometrical interpretation of the IFSs is the first interpretation and of the vague sets – the second one. The truth is that both interpretations have existed already for 30 years (see [3,4]) in IFSs theory.

Another geometrical interpretation is shown on Fig. 3. It is worth mentioning that it played very important role in the development of the IFS theory.

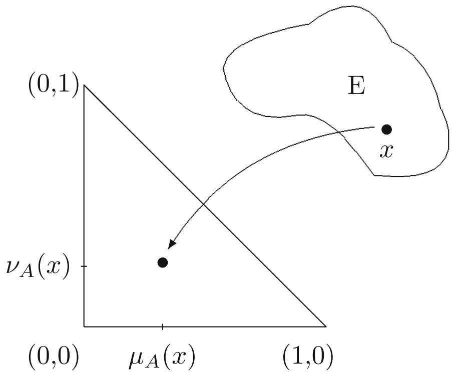

Fig. 3.

3 The New Operators

Now, we introduce the two new operators, defined over a given IFS A. They have the forms:

$$\overline{H}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta - \beta\nu_A(x)\rangle | x \in E\},$$

$$\overline{J}_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha - \alpha\mu_A(x), \beta\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

First, we check that

$$\alpha\mu_A(x) + \nu_A(x) + \beta - \beta\nu_A(x) = \alpha\mu_A(x) + (1 - \beta)\nu_A(x) + \beta$$

$$\leq (1 - \beta)(\mu_A(x) + \nu_A(x)) + \beta \leq 1 - \beta + \beta = 1,$$

i.e., the first definition is correct. Analogously, we check that the second definition is also correct.

After this, we see that for each IFS A and for every $\alpha, \beta, \gamma, \delta \in [0,1]$, so that $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$:

$$\overline{H}_{\alpha,\beta}(A) \subseteq A \subseteq \overline{J}_{\gamma,\delta}(A),$$

where $\subseteq$ is transformed to equality if $\alpha = \delta = 1$ and $\beta = \gamma = 0$.

Theorem 1. *For each IFS A and for $\alpha, \beta \in [0,1]$, so that $\alpha + \beta \leq 1$:*

$$\neg\overline{H}_{\alpha,\beta}(\neg A) = \overline{J}_{\beta,\alpha}(A),$$

$$\neg\overline{J}_{\alpha,\beta}(\neg A) = \overline{H}_{\beta,\alpha}(A).$$

Two analogues of the topological operators have been defined over the IFSs (see, e.g., [3,4]): operator *"closure"* C and operator *"intersection"* I:

$$C(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\},$$

$$I(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}.$$

In [4] a lot of other topological operators are introduced but here we will mention only one of them called a weight operator W:

$$W(A) = \{\langle x, \frac{\sum\limits_{y \in E} \mu_A(y)}{card(E)}, \frac{\sum\limits_{y \in E} \nu_A(y)}{card(E)}\rangle | x \in E\},$$

where $card(E)$ is the number of the elements of the (finite) set E.

Theorem 2. *For every two IFSs A and B, and for every $\alpha, \beta \in [0,1]$ so that $\alpha + \beta \leq 1$:*

$$C(\overline{H}_{\alpha,\beta}(A)) = \overline{H}_{\alpha,\beta}(C(A)),$$

$$I(\overline{H}_{\alpha,\beta}(A)) = \overline{H}_{\alpha,\beta}(I(A)),$$

$$W(\overline{H}_{\alpha,\beta}(A)) = \overline{H}_{\alpha,\beta}(W(A)),$$

$$C(\overline{J}_{\alpha,\beta}(A)) = \overline{J}_{\alpha,\beta}(C(A)),$$

$$I(\overline{J}_{\alpha,\beta}(A)) = \overline{J}_{\alpha,\beta}(I(A)),$$

$$W(\overline{J}_{\alpha,\beta}(A)) = \overline{J}_{\alpha,\beta}(W(A)).$$

Proof. Let A be a given IFS and let $\alpha, \beta \in [0,1]$ so that $\alpha + \beta \leq 1$. Then

$$C(\overline{H}_{\alpha,\beta}(A)) = C(\{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta - \beta\nu_A(x)\rangle | x \in E\})$$

$$= \{\langle x, \sup_{y \in E} \alpha\mu_A(x), \inf_{y \in E}(\nu_A(x) + \beta - \beta\nu_A(x))\rangle | x \in E\})$$

$$= \{\langle x, \alpha \sup_{y \in E} \mu_A(x), (1-\beta) \inf_{y \in E} \nu_A(x) + \beta\rangle | x \in E\})$$

$$= \{\langle x, \alpha \sup_{y \in E} \mu_A(x), \inf_{y \in E} \nu_A(x) + \beta - \beta \inf_{y \in E} \nu_A(x)\rangle | x \in E\})$$

$$= \overline{H}_{\alpha,\beta}(C(A)).$$

The other assertions are proved in the same way.

Theorem 3. *For every two IFSs A and B, and for every $\alpha, \beta \in [0,1]$ so that $\alpha + \beta \leq 1$:*

$$\overline{H}_{\alpha,\beta}(A \rightarrow B) = \overline{J}_{\beta,\alpha}(A) \rightarrow \overline{H}_{\alpha,\beta}(B),$$
$$\overline{J}_{\alpha,\beta}(A \rightarrow B) = \overline{H}_{\beta,\alpha}(A) \rightarrow \overline{J}_{\alpha,\beta}(B).$$

Proof. Let the IFSs A and B and the real numbers α, β are given. Then

$$\overline{H}_{\alpha,\beta}(A \rightarrow B)$$

$$= \overline{H}_{\alpha,\beta}(\{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle | x \in E\})$$

$$= \{\langle x, \alpha \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x)) + \beta$$
$$-\beta \min(\mu_A(x), \nu_B(x))\rangle | x \in E\})$$

$$= \{\langle x, \alpha \max(\nu_A(x), \mu_B(x)), (1-\beta) \min(\mu_A(x), \nu_B(x)) + \beta\rangle | x \in E\})$$

$$= \{\langle x, \max(\alpha\nu_A(x), \alpha\mu_B(x)), \min(\mu_A(x) + \beta - \beta\mu_A(x),$$
$$\nu_B(x) + \beta - \beta\nu_B(x))\rangle | x \in E\})$$

$$= \{\langle x, \mu_A(x) + \beta - \beta\mu_A(x), \alpha\nu_A(x)\rangle | x \in E\}$$
$$\rightarrow \{\langle x, \alpha\mu_B(x), \nu_B(x) + \beta - \beta\nu_B(x)\rangle | x \in E\})$$

$$= \overline{J}_{\beta,\alpha}(A) \rightarrow \overline{H}_{\alpha,\beta}(B).$$

The second equality is proved in the same way.

3.1 First Type of Intuitionistic Fuzzy Modal Operator's Point of View

The simplest intuitionistic fuzzy modal operators are

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

They are analogous to the modal logic operators *"necessity"* and *"possibility"*.

In the framework of the IFSs theory these operators are extended and modified in "a step by step"manner. The first group of extended modal operators are the following (see [4]):

$$\begin{array}{l} D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\}, \\ F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \\ \text{ where } \alpha + \beta \leq 1, \\ G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\}, \\ H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \\ H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}, \\ J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\}, \\ J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\}. \end{array}$$

where $\alpha, \beta \in [0, 1]$ are fixed numbers.

Comparing the two new operators with operator $F_{\alpha,\beta}$, we see that their parameters α and β satisfy the same conditions. Moreover, for the F- and G-operators are valid for each IFS A and for every $\alpha, \beta, \gamma, \delta \in]0, 1]$ the equalities:

$$F_{\alpha,\beta}(F_{\gamma,\delta}(A)) = F_{\alpha+\gamma-\alpha\gamma-\alpha\delta,\beta+\delta-\beta\delta-\beta\gamma}(A),$$

where $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$ and

$$G_{\alpha,\beta}(G_{\gamma,\delta}(A)) = G_{\alpha\gamma,\beta\delta}(A),$$

i.e., each one of these operators when applied to itself is represented by the same operator with parameters, being functions of the parameters of both identical operators.

This property is not valid for operators $H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}$, but now, we see that the following assertion is valid.

Theorem 4. *For each IFS A and for $\alpha, \beta, \gamma, \delta \in [0, 1]$, so that $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$:*

$$\overline{H}_{\alpha,\beta}(\overline{H}_{\gamma,\delta}(A)) = \overline{H}_{\alpha\gamma,\beta+\delta-\beta\delta}(A),$$
$$\overline{J}_{\alpha,\beta}(\overline{J}_{\gamma,\delta}(A)) = \overline{J}_{\alpha+\gamma-\alpha\gamma,\beta\delta}(A).$$

Proof. We check sequentially:

$$\begin{array}{c} \overline{H}_{\alpha,\beta}(\overline{H}_{\gamma,\delta}(A)) = \overline{H}_{\alpha,\beta}(\{\langle x, \gamma\mu_A(x), \nu_A(x) + \delta - \delta\nu_A(x)\rangle | x \in E\}) \\ = \{\langle x, \alpha\gamma\mu_A(x), \nu_A(x) + \delta - \delta\nu_A(x) + \beta - \beta(\nu_A(x) + \delta - \delta\nu_A(x))\rangle | x \in E\} \\ = \{\langle x, \alpha\gamma\mu_A(x), \nu_A(x) + (\delta + \beta - \beta\delta) - (\delta + \beta - \beta\delta)\nu_A(x)\rangle | x \in E\} \\ = \overline{H}_{\alpha\gamma,\beta+\delta-\beta\delta}(A) \end{array}$$

and

$$\overline{J}_{\alpha,\beta}(\overline{J}_{\gamma,\delta}(A)) = \overline{J}_{\alpha,\beta}(\{\langle x, \mu_A(x) + \gamma - \gamma\mu_A(x), \delta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \mu_A(x) + \gamma - \gamma\mu_A(x) + \alpha - \alpha(\mu_A(x) + \gamma - \gamma\mu_A(x)), \beta\delta\nu_A(x)\rangle | x \in E\}$$
$$= \{\langle x, \mu_A(x) + (\alpha + \gamma - \alpha\gamma) - (\alpha + \gamma - \alpha\gamma)\mu_A(x), \beta\delta\nu_A(x)\rangle | x \in E\}$$
$$= \overline{J}_{\alpha+\gamma-\alpha\gamma,\beta\delta}(A).$$

Therefore, these two operators satisfy the property of F- and G-operators.

Moreover, for each IFS A:

$$\overline{H}_{0,1}(A) = \{\langle x, 0, 1\rangle | x \in E\},$$
$$\overline{J}_{1,0}(A) = \{\langle x, 1, 0\rangle | x \in E\},$$

similarly to operators $H^*_{0,1}$ and $J^*_{1,0}$, respectively.

Theorem 5. *For every two IFSs A and B, and for every $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$:*

$$\overline{H}_{\alpha,\beta}(A \cap B) = \overline{H}_{\alpha,\beta}(A) \cap \overline{H}_{\alpha,\beta}(B),$$
$$\overline{H}_{\alpha,\beta}(A \cup B) = \overline{H}_{\alpha,\beta}(A) \cup \overline{H}_{\alpha,\beta}(B),$$
$$\overline{J}_{\alpha,\beta}(A \cap B) = \overline{J}_{\alpha,\beta}(A) \cap \overline{J}_{\alpha,\beta}(B),$$
$$\overline{J}_{\alpha,\beta}(A \cup B) = \overline{J}_{\alpha,\beta}(A) \cup \overline{J}_{\alpha,\beta}(B).$$

Proof. Let the IFSs A and B and real numbers $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$ be given. Then

$$\overline{H}_{\alpha,\beta}(A \cap B) = \overline{H}_{\alpha,\beta}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\})$$
$$= \{\langle x, \alpha \min(\mu_A(x), \mu_B(x)),$$
$$\max(\nu_A(x), \nu_B(x)) + \beta - \beta \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}$$
$$= \{\langle x, \min(\alpha\mu_A(x), \alpha\mu_B(x)), (1 - \beta) \max(\nu_A(x), \nu_B(x)) + \beta\rangle | x \in E\}$$
$$= \{\langle x, \min(\alpha\mu_A(x), \alpha\mu_B(x)),$$
$$\max((1 - \beta)\nu_A(x) + \beta, (1 - \beta)\nu_B(x)) + \beta)\rangle | x \in E\}$$
$$= \{\langle x, \alpha\mu_A(x), (1 - \beta)\nu_A(x) + \beta\rangle | x \in E\}$$
$$\cap\{\langle x, \alpha\mu_B(x), (1 - \beta)\nu_B(x) + \beta\rangle | x \in E\}$$
$$= \overline{H}_{\alpha,\beta}(A) \cap \overline{H}_{\alpha,\beta}(B).$$

The other three equalities can be checked analogously.

It is interesting to mention that similar equalities satisfies only operator $G_{\alpha,\beta}$, while for all other from the discussed operators, the equalities become inequalities.

The geometrical interpretations of the new operators are similar to these of operators $H^*_{\alpha,\beta}$ and $J^*_{\alpha,\beta}$ - see Figs. 4 and 5. Here, function f_X juxtaposes to each element x of IFS X in universe E, a point in the interpretation triangle.

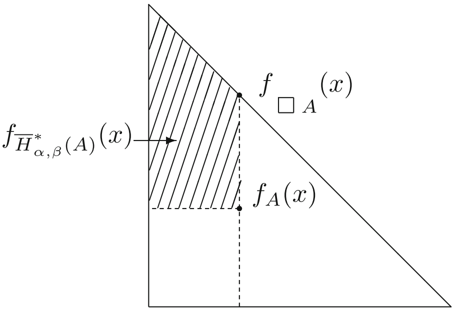

Fig. 4.

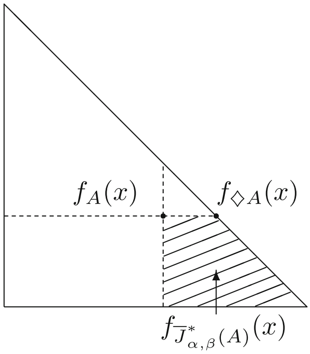

Fig. 5.

Theorem 6. *For every IFS A and for every* $\alpha, \beta \in [0,1]$ *so that* $al + \beta \leq 1$*:*

$$\square\,\overline{H}_{\alpha,\beta}(A) \subseteq \overline{H}_{\alpha,\beta}(\square\, A),$$
$$\overline{H}_{\alpha,\beta}(\Diamond A) \subseteq \Diamond\overline{H}_{\alpha,\beta}(A),$$
$$\square\,\overline{J}_{\alpha,\beta}(A) \subseteq \overline{J}_{\alpha,\beta}(\square\, A),$$
$$\overline{J}_{\alpha,\beta}(\Diamond A) \subseteq \Diamond\overline{J}_{\alpha,\beta}(A).$$

Proof. For the IFS A and its parameters, mentioned above, the equalities

$$\square\,\overline{H}_{\alpha,\beta}(A) = \square\,\{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta - \beta\nu_A(x)\rangle | x \in E\}$$
$$= \{\langle x, \alpha\mu_A(x), 1 - al\mu_A(x)\rangle | x \in E\}$$

and

$$\overline{H}_{\alpha,\beta}(\square\, A) = \overline{H}_{\alpha,\beta}(\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\})$$
$$= \{\langle x, \alpha\mu_A(x), 1 - \mu_A(x) + \beta - \beta(1 - \mu_A(x))\rangle | x \in E\}$$
$$= \{\langle x, \alpha\mu_A(x), 1 - \mu_A(x) + \beta\mu_A(x))\rangle | x \in E\}$$

hold. From

$$1 - \alpha\mu_A(x) - (1 - \mu_A(x) + \beta\mu_A(x)) = (1 - \alpha - \beta)\mu_A(x) \geq 0$$

it follows the validity of the first inequality. The others can be proved in the same manner.

Similarly to the research, presented in [4], we prove

Theorem 7. *For each IFS A and for* $\alpha, \beta, \gamma, \delta \in [0,1]$, *so that* $\alpha + \beta \leq 1$ *and* $\gamma + \delta \leq 1$*:*

$$H_{\alpha,\beta}(\overline{H}_{\gamma,\delta}(A)) \subseteq \overline{H}_{\gamma,\delta}(H_{\alpha,\beta}(A)),$$

$$H^*_{\alpha,\beta}(\overline{H}_{\gamma,\delta}(A)) \subseteq \overline{H}_{\gamma,\delta}(H^*_{\alpha,\beta}(A)),$$
$$\overline{J}_{\alpha,\beta}(J_{\gamma,\delta}(A)) \subseteq J_{\gamma,\delta}(\overline{J}_{\alpha,\beta}(A)),$$
$$\overline{J}_{\alpha,\beta}(J^*_{\gamma,\delta}(A)) \subseteq J^*_{\gamma,\delta}(\overline{J}_{\alpha,\beta}(A)).$$

Proof. The validity of the fourth inequality is checked as follows:

$$\overline{J}_{\alpha,\beta}(J^*_{\gamma,\delta}(A))$$
$$= \overline{J}_{\alpha,\beta}(\{\langle x, \mu_A(x) + \gamma(1 - \mu_A(x) - \delta.\nu_A(x)), \delta\nu_A(x)\rangle | x \in E\})$$
$$= \{\langle x, \mu_A(x) + \gamma(1 - \mu_A(x) - \delta\nu_A(x)) + \alpha - \alpha\mu_A(x) - \alpha\gamma(1 - \mu_A(x) - \delta\nu_A(x)),$$
$$\beta\delta\nu_A(x)\rangle | x \in E\})$$

(from

$$\mu_A(x) + \alpha - \alpha\mu_A(x) + \gamma(1 - \mu_A(x) - \alpha + \alpha\mu_A(x) - \beta\delta\nu_A(x))$$
$$-\mu_A(x) - \gamma(1 - \mu_A(x) - \delta\nu_A(x)) - \alpha + \alpha\mu_A(x) + \alpha\gamma(1 - \mu_A(x) - \delta\nu_A(x))$$
$$= \gamma\delta\nu_A(x) - \alpha\gamma\delta\nu_A(x) - \gamma\beta\delta\nu_A(x)$$
$$= \gamma\delta(1 - \alpha - \beta)\nu_A(x) \geq 0$$

it follows)

$$\subseteq \{\langle x, \mu_A(x) + \alpha - \alpha\mu_A(x) + \gamma(1 - \mu_A(x) - \alpha + \alpha\mu_A(x) - \beta\delta\nu_A(x)),$$
$$\beta\delta\nu_A(x)\rangle | x \in E\})$$
$$= J^*_{\gamma,\delta}(\{\langle x, \mu_A(x) + \alpha - \alpha\mu_A(x), \beta\nu_A(x)\rangle | x \in E\})$$
$$= J^*_{\gamma,\delta}(\overline{J}_{\alpha,\beta}(A)).$$

The remaining inequalities can be proved similarly.

As it is written in [4], *In 1991, during a lecture given by the author, a question was asked, whether the so far constructed operators (operators $D_\alpha, F_{\alpha,\beta}$, $G_{\alpha,\beta}, H_{\alpha,\beta}, J_{\alpha,\beta}, H^*_{\alpha,\beta}, J^*_{\alpha,\beta}$) can be derived as particular cases of one general operator? In the next lecture, the author gave a positive answer, preparing the following text, which was published in* [2]. In [5] an addition to the definition is given. Now, the final form of this operator has the form:

$$X_{a,b,c,d,e,f}(A) = \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)),$$
$$d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$

where $a, b, c, d, e, f \in [0, 1]$ and

$$a + e - e.f \leq 1,$$
$$b + d - b.c \leq 1,$$
$$b + e \leq 1.$$

Now, we see directly that

$$\overline{H}_{\alpha,\beta}(A) = X_{\alpha,0,r,1,\beta,0}(A),$$

$$\overline{J}_{\alpha,\beta}(A) = X_{1,\alpha,0,\beta,0,s}(A),$$

where $r, s \in [0, 1]$ are arbitrary numbers.

Therefore, the new operators have similar X-representation as the rest of the extended modal-type operators.

3.2 Second Type of Intuitionistic Fuzzy Modal Operator's Point of View

Following [4], we start with the first two simplest operators of the second type:

$$\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle | x \in E\}.$$

Let $\alpha \in [0, 1]$ and let A be an IFS. Then we can define the first extension:

$$\boxplus_\alpha A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + 1 - \alpha\rangle | x \in E\},$$

$$\boxtimes_\alpha A = \{\langle x, \alpha.\mu_A(x) + 1 - \alpha, \alpha.\nu_A(x)\rangle | x \in E\}.$$

The second extension of operators $\boxplus$ and $\boxtimes$ is introduced in [8] by Katerina Dencheva. She extended the last two operators to the forms:

$$\boxplus_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x) + \beta, \alpha.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

The third extension of the above operators has the forms:

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x) + \gamma\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

In [7] Gökhan Cuvalcioĝlu introduced operator $E_{\alpha,\beta}$ by

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha.\mu_A(x) + 1 - \alpha), \alpha(\beta.\nu_A(x) + 1 - \beta)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$ and studied some of its properties.

A natural extension of the three later operators is the operator

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0,1]$ and

$$\max(\alpha, \beta) + \gamma + \delta \leq 1.$$

A new (final?) extension of the above operators is the operator

$$\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A = \{\langle x, \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma, \beta.\nu_A(x) - \zeta.\mu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$ and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1,$$
$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0.$$

Now, we can check the validity of the following equalities (for fixed $\alpha, \beta \in [0,1]$, so that $\alpha + \beta \leq 1$:

$$\boxplus A = \overline{H}_{\frac{1}{2},\frac{1}{2}}(A),$$
$$\boxtimes A = \overline{J}_{\frac{1}{2},\frac{1}{2}}(A),$$
$$\boxplus_\alpha A = \overline{H}_{\alpha,1-\alpha}(A),$$
$$\boxtimes_\alpha A = \overline{J}_{1-\alpha,\alpha}(A),$$

i.e., the new operators can represent the first two second type of intuitionistic fuzzy modal operators.

The relation between operators $\boxplus_{\alpha,\beta}$ and $\boxtimes_{\alpha,\beta}$, and $\overline{H}_{\alpha,\beta}$ and $\overline{J}_{\alpha,\beta}$, respectively, is valid only in the following special cases:

$$\boxplus_{\alpha,1-\alpha} A = \overline{H}_{\alpha,1-\alpha}(A),$$
$$\boxtimes_{\alpha,1-\alpha} A = \overline{J}_{1-\alpha,\alpha}(A).$$

Similar is the situation with Guvalcioĝlu's operator:

$$E_{1,\beta}(A) = \overline{H}_{\beta,1-\beta}(A),$$
$$E_{\alpha,1}(A) = \overline{J}_{1-\alpha,\alpha}(A).$$

The other three second type of intuitionistic fuzzy modal operators can represent the new two operators as follows:

$$\overline{H}_{\alpha,\beta}(A) = \boxplus_{\alpha,1-\beta,\beta} A,$$
$$\overline{J}_{\alpha,\beta}(A) = \boxtimes_{1-\alpha,\beta,\alpha} A,$$
$$\overline{H}_{\alpha,\beta}(A) = \boxdot_{\alpha,1-\beta,0,\beta} A,$$
$$\overline{J}_{\alpha,\beta}(A) = \boxdot_{1-\alpha,\beta,\alpha,0} A,$$
$$\overline{H}_{\alpha,\beta}(A) = \boxcircle_{\alpha,1-\beta,0,\beta,0,0} A,$$
$$\overline{J}_{\alpha,\beta}(A) = \boxcircle_{1-\alpha,\beta,\alpha,0,0,0} A.$$

3.3 Intuitionistic Fuzzy Level Operator's Point of View

The basic intuitionistic fuzzy level operators are (see, e.g., [4]:

$$P_{\alpha,\beta}(A) = \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\},$$
$$Q_{\alpha,\beta}(A) = \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\},$$

for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

The degrees of membership and non-membership of the elements of a given universe to its subset can be directly changed by these operators.

Obviously, for every IFS A and for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$:

$$P_{\alpha,\beta}(A) = A \cup \{\langle x, \alpha, \beta\rangle | x \in E\},$$
$$Q_{\alpha,\beta}(A) = A \cap \{\langle x, \alpha, \beta\rangle | x \in E\},$$
$$Q_{\alpha,\beta}(A) \subset A \subset P_{\alpha,\beta}(A).$$

Therefore, it will be suitable to denote both operators as follows:

$$O^{\cup}_{\alpha,\beta}(A) = A \cup \{\langle x, \alpha, \beta\rangle | x \in E\},$$

$$O^{\cap}_{\alpha,\beta}(A) = A \cap \{\langle x, \alpha, \beta\rangle | x \in E\}.$$

Now, we can define two new intuitionistic fuzzy level operators on the basis of operations "+"and "×", defined in Sect. 2:

$$O^{+}_{\alpha,\beta}(A) = A + \{\langle x, \alpha, \beta\rangle | x \in E\},$$

$$O^{\times}_{\alpha,\beta}(A) = A \times \{\langle x, \alpha, \beta\rangle | x \in E\}.$$

We see immediately that:

$$O^{+}_{\alpha,\beta}(A) = \overline{J}_{\alpha,\beta}(A),$$

$$O^{\times}_{\alpha,\beta}(A) = \overline{H}_{\alpha,\beta}(A).$$

Therefore, the two new operators have level operator's behaviour.

Theorem 8. *For each IFS A and for $\alpha, \beta, \gamma, \delta \in [0, 1]$, so that $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$:*

$$\overline{H}_{\alpha,\beta}(P_{\gamma,\delta}(A)) \subseteq P_{\gamma,\delta}(\overline{H}_{\alpha,\beta}(A)),$$

$$\overline{H}_{\alpha,\beta}(Q_{\gamma,\delta}(A)) \subseteq Q_{\gamma,\delta}(\overline{H}_{\alpha,\beta}(A)),$$

$$P_{\alpha,\beta}(\overline{J}_{\gamma,\delta}(A)) \subseteq \overline{J}_{\gamma,\delta}(P_{\alpha,\beta}(A)),$$

$$Q_{\alpha,\beta}(\overline{J}_{\gamma,\delta}(A)) \subseteq \overline{J}_{\gamma,\delta}(Q_{\alpha,\beta}(A)).$$

It can be directly seen that for each IFS A and for $\alpha, \beta \in [0, 1]$, so that $\alpha + \beta \leq 1$:

$$\overline{H}_{\alpha,\beta}(A) \subseteq Q_{\alpha,\beta}(A) \subseteq A \subseteq P_{\alpha,\beta}(A) \subseteq \overline{J}_{\alpha,\beta}(A).$$

Now, we can calculate for every IFS A and for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$:

$$\overline{H}_{\alpha,\beta}(A)@\overline{J}_{\alpha,\beta}(A)$$

$$= \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta - \beta\nu_A(x)\rangle | x \in E\}$$

$$@\{\langle x, \mu_A(x) + \alpha - \alpha\mu_A(x), \beta\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \frac{\alpha\mu_A(x) + \mu_A(x) + \alpha - \alpha\mu_A(x)}{2},$$

$$\frac{\nu_A(x) + \beta - \beta\nu_A(x) + \beta\nu_A(x)}{2}\rangle | x \in E\}$$

$$= \{\langle x, \frac{\mu_A(x) + \alpha}{2}, \frac{\nu_A(x) + \beta}{2}\rangle | x \in E\}$$

$$A@\{\langle x, \alpha, \beta\rangle | x \in E\}.$$

From the above discussion we see that the two new operators are simultaneously modal as well as level operators. By this reason, we really can call them modal-level operators.

Finally, following the above notation, we can denote:

$$A@\{\langle x, \alpha, \beta\rangle | x \in E\} = O^{@}_{\alpha,\beta}(A).$$

4 Conclusion

In future, we will introduce other intuitionistic fuzzy modal-level operators. Each one of these new operators will be extended in the ways the standard modal and level operators are extended. For example, for each IFS A, defined over some universe E, their parameters α and β will be changed with the degrees of membership and of non-membership of the elements of another IFS B, defined over the same universe E.

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Generalized Net Model of Multicriteria Decision Making Procedure Using Intercriteria Analysis

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Abstract. The Generalized Nets (GNs) are extensions of the ordinary Petri nets and the other Petri net modifications. A GN-model of a multi-expert multi-criteria decision making process is described. It is extended with an intercriteria analysis of the criteria used by experts – an addition to the standard decision making procedure that changes in the end of a concrete procedure the criteria used by experts during it, so, in the next procedure they work with the modified set of criteria.

Keywords: Decision making · Generalized net · Intercriteria analysis

1 Introduction

In a series of papers of G. Pasi, R. Yager and the author, different multi-criteria decision making procedures are described, that contains essentially new ideas in this area. Here, we describe the process of functioning and the results of the work of such procedures. They are described by one of Petri Net extensions, called Generalized Net (GN; see [1–3]). Each GN has transitions, but now, they contain not only input and output places, but also, moments of activation, duration of the active state, predicates determining which token from input place to which output places can be transfered, capacities of transition arcs and a condition for activation of the transition, when the activation moment arises. Each token has initial and current characteristics, that it can keep and use during the GN functioning.

Below, we use the following three types of sets:

- $E = \{E_1, E_2, ..., E_m\}$ is the set of the measurement tools employed in the decision process;
- $A = \{A_1, A_2, ..., A_p\}$ is the set of the alternatives considered;

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- $C = \{C_1, C_2, ..., C_q\}$ is the set of the criteria used for evaluating the alternatives, which are ordered before their use.

The experts use as standard or given to them criteria, as well as new criteria suggested by the experts, participating in the current procedure. Each expert can use only those of the criteria, that he/she prefers. Each expert has own score in the form of Intuitionistic Fuzzy Pairs (IFPs; see [4,7]) $\langle a, b\rangle$, where $a, b \in [0, 1]$, so that $a+b \leq 1$ and a, b are respectively degree of validity, correctness, etc. and degree of non-validity, non-correctness, etc.

To illustrate the expert's reliability score we give the following example: a sports commentator made 10 prognoses for the results of 10 football matches. In 5 of the cases he predicted correctly the winner, in 3 of the cases he failed and in the remaining 2 cases he did not engage with final opinion about the result. That is why we determine his reliability score as $\langle 0.5, 0.3\rangle$.

When the i-th expert determines the criteria, which he/she likes to use, he orders them on the vertices of an oriented graph. As it is shown, e.g., in [5], each graph can be represent by an Index Matrix (IM, see [5]) in the form

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & & & & \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array},$$

where for a fixed set of indices I and for set $\mathcal{R}$ of numbers (0 and 1; natural, real, etc.), propositions, variables, predicates, IFPs, etc., $K = \{k_1, k_2, ..., k_m\} \subset I, L = \{l_1, l_2, ..., l_n\} \subset I$; for $1 \leq i \leq m$, and for $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R}$.

For two IMs different operations are defined, such as "addition", some types of "multiplication", "subtraction", "projection", "restriction", "substitution" and others; and some operators are defined, e.g., hierarchical operators.

When the elements of R are IFPs, the IM is Intuitionistic Fuzzy IM (IFIM). When some IFPs are associated to the arcs and/or vertices of a given graph, the graph becomes an Intuitionistic Fuzzy Graph (IFG).

2 Generalized Net Model

The present GN model (see Fig. 1) is an extension and modification of the model from [10], that is an extension of the GN-model from [9]. Since some of the places and transitions in the new model coincide in both models, we use the same notation. By this reason we use notation for transitions and places: $Y_1, Y_2, Y_3, Y_4, k_1, ..., k_8$ from the second model, $Z_1, Z_2, Z_3, l_1, ..., l_{14}$ from the first model and now for the new model: $X_1, X_2, X_3, X_4, m_1, ..., m_7$.

The GN that we describe below has seven types of tokens – α-, β-, γ-, δ-, ε, ζ- and η-tokens. The third, fourth, ..., and seventh tokens are unique, while α-tokens are m in number and they generate m in number β-tokens, that in place k_7 are united in one token β.

The α-tokens are ordered by some criterion (e.g., alphabetically following experts' names) and their order is not important. Each of the α-tokens has an initial characteristic: *"expert's name, his/her own (current) reliability score* $\langle \delta_i, \varepsilon_i \rangle \in [0,1]^2$ *such that* $\delta_i + \varepsilon_i \leq 1$*, and his/her own (current) number of participations in experts' investigations* γ_i" $(1 \leq i \leq m)$.

In the initial time-moment of the GN functioning, the first α-token, α_1 (let α_i denote the i-th α-token) and the γ-token enter places k_1 and l_3, respectively. The later token has initial characteristics *"list of the alternatives, i.e.* $A_1, A_2, \ldots, A_p$*"*.

The first GN-transition (as we noted above, it is not met in the GN from [10]) has the form:

$$Y_1 = \langle \{k_1\}, \{k_2, k_3\}, \begin{array}{c|cc} & k_2 & k_3 \\ \hline k_1 & true & true \end{array} \rangle.$$

The current token α_i splits into two tokens - the same token α_i that enters place k_2 without a new characteristic and the token β_i that enters place k_3 with a characteristic *"list of the estimation criteria that the i-th expert likes to use for his/her estimation, i.e.,* $C_{i,1}, C_{i,2}, \ldots, C_{i,q_i}$*".* $(1 \leq i \leq m$ and $1 \leq q_i \leq q_{cu})$, where q_{cu} is the current number of criteria that the experts can use.

$$Y_2 = \langle \{k_2, k_4\}, \{k_4, k_5\}, \begin{array}{c|cc} & k_4 & k_5 \\ \hline k_2 & true & false \\ k_4 & V_{4,4} & V_{4,5} \end{array} \rangle,$$

where
$V_{4,4} =$ "in the current step no token enters place k_1 and the current token has stayed the longest time in place k_4 in respect to all other tokens currently staying in the same place",
$V_{4,5} = \neg V_{4,4}$.

The α-tokens enter place k_4 without a new characteristic. They are collected there, waiting for the beginning of the process of experts' estimation. When predicate $V_{4,5}$ is valid, α-tokens enter place k_5 with the characteristic *"IFG* G_i *of the expert's opinion for the order among the criteria".* The vertices of the IFG G_i represent all or a part of the criteria from the last β-token characteristic. The arcs of this IFG are labeled by the i-th expert's scores.

Token ε permanently stays in place k_7, with an initial and current characteristic *"list of actual criteria (*q_{cu} *in number) that can be used for the current expertise".*

$$Y_3 = \langle \{k_3, k_7, k_{15}\}, \{k_6, k_7\}, \begin{array}{c|cc} & k_6 & k_7 \\ \hline k_3 & false & true \\ k_7 & V_{7,6} & V_{7,7} \\ k_{15} & false & true \end{array} \rangle,$$

where
$V_{7,6} =$ "in the current step no token enters place k_1",
$V_{7,7} = \neg V_{7,6}$.

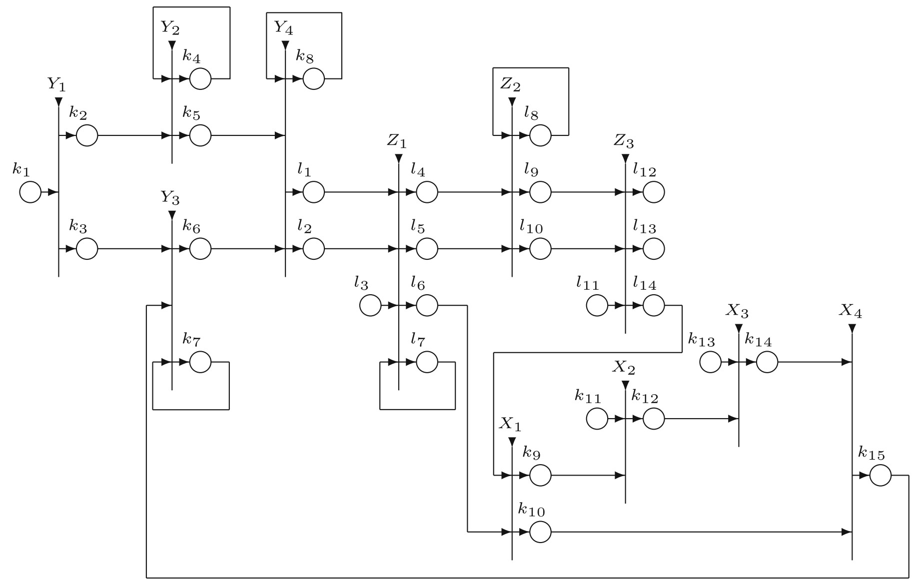

Fig. 1. GN-model

The i-th β-token (let us call it β_i) enters place k_7. It unites with the ε-token, that stays in place k_7. Let us call it β. It waits for the beginning of the process of experts' estimation and it obtains as a current characteristic

$$x^{\beta}_{cu} = x^{\varepsilon}_{cu-1} \cup x^{\beta_i}_{cu-1},$$

i.e., the list of all criteria that are formulated by the first i experts.

Token β from place k_{15} enters place k_7, where it is united with token ε, that obtain as a new (current) characteristic

$$x^{\varepsilon}_{cu} = x^{\varepsilon}_{cu-1} \cup x^{\beta}_{cu}.$$

When predicate $V_{7,6} = true$, then token ε splits into token ε that continues to stay in place k_7 and token β that enters place k_6 with the characteristic $x^{\beta_i}_{cu-1} \cup \lambda_i$, where $\lambda_i \subseteq x^{\varepsilon}_{cu}$ is the list of the criteria, different from these in the set $x^{\beta_i}_{cu-1}$ that the i-th expert likes and will use, too.

$$Y_4 = \langle \{k_5, k_6, k_8\}, \{k_8, l_1, l_2\}, \begin{array}{c|ccc} & k_8 & l_1 & l_2 \\ \hline k_5 & true & false & false \\ k_6 & false & false & V_{6,2} \\ k_8 & V_{8,8} & V_{8,1} & false \end{array} \rangle,$$

where
$V_{6,2} = V_{8,1} =$ "in the current step no token enters place k_5",
$V_{8,8} = \neg V_{8,1}$.

The α-tokens enter place l_1 without any new characteristic, while token β enters place l_2 with a characteristic *"IFG G, obtained by the procedure described below".*

The new IFG G is obtained by operation "$+_\circ$" over the IFGs G_i, which for two IMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$ has the form

$$A +_\circ B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}$$

and for two IFPs $\langle a, b\rangle$ and $\langle c, d\rangle$, operation $\circ$ can be, e.g.

$$\langle a, b\rangle \circ \langle c, d\rangle = \begin{cases} \langle \max(a, c), \min(b, d)\rangle, & \text{if } \circ \text{ is } \vee \\ \langle \min(a, c), \max(b, d)\rangle, & \text{if } \circ \text{ is } \wedge \\ \langle \frac{a+c}{2}, \frac{b+d}{2}\rangle, & \text{if } \circ \text{ is } @ \end{cases}$$

or others.

With this operation, we obtain an IM corresponding to the IFG G of all experts' opinions about the criteria ordering. Now, its arcs have intuitionistic fuzzy weights being the disjunctions of the weights, of the same arcs in the separate IFGs. Of course, the new graph may not be well ordered, while the expert graphs are well ordered. Now, we reconfigure IFG G as follows. If there is a cycle between two vertices V_1 and V_2, i.e., there are vertices $U_1, U_2, ..., U_u$ and vertices $W_1, W_2, ..., W_w$, such that $V_1, U_1, U_2, ..., U_u, V_2$ and $V_2, W_1, W_2, ..., W_w, V_1$ are simple paths in the graph, then we calculate the weights of both paths as conjunctions of the weights of the arcs which take part in the respective paths. The path that has smaller weight must be cut in two, removing its arc with smallest weight. If both arcs have equal weights, these arcs will be removed. Therefore, the new graph is already cycle-free. Now, we can determine the priorities of the vertices of the IFG, i.e., the priorities of the criteria. Let them be $\varphi_1, \varphi_2, ..., \varphi_{q_{cu}}$. For example, they can have values $\frac{s-1}{t}$ for the vertices from the s-th level bottom-up of the IFG with $t+1$ levels. We shall use these values below.

The first transition of the GN from [9] that now is in the subnet of the present GN, has the form:

$$Z_1 = \langle \{l_1, l_2, l_3, l_7\}, \{l_4, l_5, l_6, l_7\},$$

	l_4	l_5	l_6	l_7
l_1	$true$	$false$	$false$	$false$
l_2	$false$	$false$	$W_{2,6}$	$false$
l_3	$false$	$false$	$false$	$true$
l_7	$false$	$W_{7,5}$	$false$	$W_{7,7}$

$\rangle$,

where
$W_{7,7}$ = "there is a token in place l_1",
$W_{2,6} = W_{7,5} = \neg W_{7,7}$.

If predicate $W_{7,7} = true$, then token γ stays in place l_7 without a new characteristic. Only when predicate $W_{7,7} = false$, i.e., when predicates $W_{2,6} = W_{7,6} = true$, token γ enter place l_5 without any characteristic, too. Token α_i enters place l_4 and obtains as a next characteristic an IM that we shall describe in more details. Having in mind that the i-th expert can use only a part of the criteria and can estimate only a part of the alternatives, we can construct the IM of his/her estimations in the form

$$S_i = \begin{array}{c|cccc} & A_{l_1} & A_{l_2} & \dots & A_{l_{p_i}} \\ \hline C_{i_1} & & & & \\ & & \langle \alpha^i_{j,k} \beta^i_{j,k} \rangle & & \\ C_{i_2} & & & & \\ & & (1 \le j \le q_i \le q_{cu}, & & \\ \vdots & & & & \\ & & 1 \le k \le p_i \le p) & & \\ C_{i_{q_i}} & & & & \end{array}$$

where: $\alpha^i_{j,k}, \beta^i_{j,k} \in [0,1],$, $\alpha^i_{j,k} + \beta^i_{j,k} \le 1$ and $\langle \alpha^i_{j,k}, \beta^i_{j,k} \rangle$ is the i-th expert estimation for the k-th alternative about the j-th criterion; $C_{i_1}, ..., C_{i_{q_i}}$ and $A_{l_1}, ..., A_{l_{p_i}}$ are only those of the criteria and alternatives which the i-th expert prefers. In the cases when pair $\langle \alpha^i_{j,k}, \beta^i_{j,k} \rangle$ does not exist, we will work with pair $\langle 0, 1 \rangle$.

On the other hand, when all α-tokens transfered to place l_4, token β enters place l_6 with a characteristic, the set of all α-token's characteristic, i.e., "$\{S_1, S_2, ..., S_m\}$".

The GN-transition Z_2 has the form:

$$Z_2 = \langle \{l_4, l_5, l_8\}, \{l_8, l_9, l_{10}\}, \begin{array}{c|ccc} & l_8 & l_9 & l_{10} \\ \hline l_4 & true & false & false \\ l_5 & false & false & true \\ l_8 & false & W_{8,9} & false \end{array} \rangle,$$

where
$W_{8,9}$ = "there is a token in place l_{11}".

The α-tokens are collected without any characteristic in place l_8. They will continue their path to place l_9, without a characteristic, too, when there is an objective estimation of the alternatives (token δ in place l_{11}). Token γ obtains as a characteristic the following IM

$$S = \begin{array}{c|c} & A_1 \; A_2 \; \ldots \; A_p \\ \hline C_1 & \\ & \langle \alpha_{j,k}, \beta_{j,k} \rangle \\ C_2 & \\ & (1 \leq j \leq q_{cu}, \\ \vdots & \\ & 1 \leq k \leq p) \\ C_{q_{cu}} & \end{array}$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ can be calculated by different formulas, with respect to some specific aims. For example, such formulas are the following:

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_i . \alpha^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_i . \beta^i_{j,k}}{m} \end{cases}$$

(here the average degrees of experts' reliability are taken into account),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_{i,j} . \alpha^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_{i,j} . \beta^i_{j,k}}{m} \end{cases}$$

(here only the experts' degrees of reliability estimated by the corresponding criteria are taken into account).

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\alpha}^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\beta}^i_{j,k}}{m} \end{cases},$$

where $\overline{\alpha}^i_{j,k}$ and $\overline{\beta}^i_{j,k}$ can also be calculated by various formulas, according to particular goals and experts' knowledge. For example, such formulas can be:

$$\begin{cases} \overline{\alpha}^i_{j,k} = \gamma_i . \dfrac{\alpha^i_{j,k} . \delta_{i,j} + \beta^i_{j,k} . \varepsilon_{i,j}}{\gamma_i + 1} \\ \overline{\beta}^i_{j,k} = \gamma_i . \dfrac{\alpha^i_{j,k} . \varepsilon_{i,j} + \beta^i_{j,k} . \delta_{i,j}}{\gamma_i + 1} \end{cases}$$

or

$$\begin{cases} \overline{\alpha}^i_{j,k} = \alpha^i_{j,k}.\dfrac{\delta_{i,j}+1-\varepsilon_{i,j}}{2} \\ \overline{\beta}^i_{j,k} = \beta^i_{j,k}.\dfrac{\varepsilon_{i,j}+1-\delta_{i,j}}{2} \end{cases}.$$

The first formula takes into account not only the rating of each expert by the different criteria, but also the number of times he has given an opinion (the first time is neglected, since he does not have a rating then). Obviously, the so constructed elements of the IM satisfy the inequality: $\alpha_{j,k}+\beta_{j,k}\leq 1$. This IM contains the average experts estimations taking into account experts ratings. Let each one of the criteria $C_j(1\leq j\leq q_{cu})$ have a priority $\varphi_j\in[0,1]$. This information will be put in the initial characteristic of token β. We can determine for every alternative A_k the global estimation $\langle\alpha_k,\beta_k\rangle$, where

$$\begin{cases} \alpha_k = \dfrac{\sum_{j=1}^{q_{cu}} \varphi_j.\alpha_{j,k}}{q_{cu}} \\ \beta_k = \dfrac{\sum_{j=1}^{q_{cu}} \varphi_j.\beta_{j,k}}{q_{cu}} \end{cases}.$$

Transition Z_3 can be activated only when token δ enters place l_{11} with an initial (unique) characteristic in the form of an IM with elements (objective) values about the different criteria:

$$T = \begin{array}{c|c} & A_1\ A_2\ \ldots\ A_p \\ \hline C_1 & \\ & \langle a_{j,k}, b_{j,k}\rangle \\ \vdots & (1\leq j\leq q_{cu}, \\ & 1\leq k\leq p) \\ C_{q_{cu}} & \end{array}$$

where: $a_{j,k},b_{j,k}\in[0,1]$ and $a_{j,k}+b_{j,k}\leq 1$.

Here we mention a significant difference between the four examples. It constitutes in the time needed for the experts to understand how well they have made their evaluations and prognoses. In the case of election prognoses, the experts obtain their own score at the moment of the final announcement of the results of the vote. On the other hand, when a job candidate is evaluated, the experts can estimate his/her work in the company in a longer period of time, including periods of adaptation and training, and first finished projects. Moreover, in such case the expert's appraisal may be subjective and liable to refutation.

The transitions Z_2 and Z_3 will be active until all α-tokens from place l_8 go to place l_{12} through place l_9. Its form is

$$Z_3 =< \{l_9, l_{10}, l_{11}\}, \{l_{12}, l_{13}, l_{14}\},$$

$$\begin{array}{c|ccc} & l_{12} & l_{13} & l_{14} \\ \hline l_9 & true & false & false \\ l_{10} & false & false & true \\ l_{11} & false & true & false \end{array}.$$

In the present model, token δ leaves the GN via place l_{13} without a final characteristic, but in the future GN-models they will obtain characteristics related, e.g. to the behaviour of the process flow and GN functioning, to the alternatives and criteria, etc.

Token α_i enters place l_{12} with final characteristic *"expert's new rating, $\langle \delta_i, \varepsilon_i \rangle$, and new number of participances in expert investigations, γ_i'".* The values of this characteristic are estimated by formulas

$$\gamma_i' = \gamma_i + 1,$$

and

$$\begin{cases} \delta_i' = \dfrac{\gamma_i.\delta_i + \dfrac{c_M - c_i}{2}}{\gamma_i'}, \\ \varepsilon_i' = \dfrac{\gamma_i.\varepsilon_i - \dfrac{c_M - c_i}{2}}{\gamma_i'}, \end{cases},$$

where

$$c_i = \frac{\sum\limits_{j=1}^{q_{cu}} \sum\limits_{k=1}^{p} ((\alpha_{j,k} - a_{j,k})^2 + (\beta_{j,k} - b_{j,k})^2)^{1/2}}{p q_{cu}},$$

and

$$c_M = \frac{\sum\limits_{i=1}^{n} c_i}{n}.$$

Other formulas for the expert's rating are also possible and they will be discussed in a next research.

Token γ enters place l_{14} with a characteristic "IM T". We remind that its previous characteristic is IM S.

The GN-transition X_1 (the first of the new transitions) has the form:

$$X_1 = \langle \{l_6, l_{14}\}, \{k_9, k_{10}\}, \begin{array}{c|cc} & k_9 & k_{10} \\ \hline l_6 & true & false \\ l_{14} & false & true \end{array} \rangle.$$

Token γ from l_{14} enters place k_{10} with a characteristic "list of all used criteria in the process of decision making". Therefore, it contains the criteria, given to the experts, as well as the new criteria, introduced by the separate experts and used in the time of the process.

Token β from l_6 enters place k_9 with a characteristic "IM U", where

$$U = \begin{array}{c|c} & A_{1,1} \dots A_{1,m} A_{1,m+1} A_{1,m+2} \dots A_{p,1} \dots A_{p,m} A_{p,m+1} A_{p,m+2} \\ \hline C_1 & \\ & \langle \alpha_{k,i,j}, \beta_{k,i,j} \rangle \\ \vdots & (1 \le i \le m, \\ & 1 \le j \le q_{cu}, \\ C_{q_{cu}} & 1 \le k \le p) \end{array},$$

where $\langle \alpha_{k,i,j}, \beta_{k,i,j} \rangle$, $\langle \alpha_{k,m+1,j}, \beta_{k,m+1,j} \rangle$ and $\langle \alpha_{k,m+2,j}, \beta_{k,m+2,j} \rangle$ are, respectively, the evaluations of the i-th expert $(1 \le i \le m)$, aggregated evaluation of all experts (with index $i = m+1$) and objective result (with index $i = m+2$) for the k-th alternative about j-th criterion. Alternatives $A_{k,i}$ for each i coincide, but their evaluations are different and by this reason we can interprete them as different objects. IM U can have simpler form, if we include in it only experts evaluations, i.e., without the aggregated evaluations and objective results.

Token ζ enters place k_{11} with initial characteristic "(meta)criterion for a choice for near criteria". We discuss its meaning below.

$$X_2 = \langle \{k_9, k_{11}\}, \{k_{12}\}, \begin{array}{c|c} & k_{12} \\ \hline k_9 & true \\ k_{14} & true \end{array} \rangle.$$

In place k_{12}, tokens β and ζ unite in token β with a characteristic "IM V, list of near pair of criteria". The first component of this characteristic is obtained by the procedure, that we describe following [5,6,8]. It is the basic component of the so called intercriteria analysis. Here, it is described from intuitionistic fuzzy point of view (see, [8]).

Let us have the set of objects $O = \{O_1, O_2, ..., O_n\}$ that must be evaluated by criteria from the set $C = \{C_1, C_2, ..., C_m\}$.

Let us have an IM

$$A = \begin{array}{c|ccccccc} & O_1 & \cdots & O_i & \cdots & O_j & \cdots & O_n \\ \hline C_1 & a_{C_1,O_1} & \cdots & a_{C_1,O_i} & \cdots & a_{C_1,O_j} & \cdots & a_{C_1,O_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_k & a_{C_k,O_1} & \cdots & a_{C_k,O_i} & \cdots & a_{C_k,O_j} & \cdots & a_{C_k,O_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_l & a_{C_l,O_1} & \cdots & a_{C_l,O_i} & \cdots & a_{C_l,O_j} & \cdots & a_{C_l,O_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_m & a_{C_m,O_1} & \cdots & a_{C_m,O_i} & \cdots & a_{C_m,O_j} & \cdots & a_{C_m,O_n} \end{array},$$

where for every p, q $(1 \le p \le m,\ 1 \le q \le n)$:

(1) C_p is a criterion, taking part in the evaluation,
(2) O_q is an object, being evaluated.

(3) a_{C_p,O_q} is a variable, formula or $a_{C_p,O_q} = \langle \alpha_{C_p,O_q}, \beta_{C_p,O_q} \rangle$ is an intuitionistic fuzzy pair, that is comparable about relation R with the other a-objects, so that for each i, j, k: $R(a_{C_k,O_i}, a_{C_k,O_j})$ is defined. Let $\overline{R}$ be the dual relation of R in the sense that if R is satisfied, then $\overline{R}$ is not satisfied and vice versa. For example, if "R" is the relation "<", then $\overline{R}$ is the relation ">", and vice versa.

Let $S^{\mu}_{k,l}$ be the number of cases in which

$$\langle \alpha_{C_k,O_i}, \beta_{C_k,O_i} \rangle \leq \langle \alpha_{C_k,O_j}, \beta_{C_k,O_j} \rangle$$

and

$$\langle \alpha_{C_l,O_i}, \beta_{C_l,O_i} \rangle \leq \langle \alpha_{C_l,O_j}, \beta_{C_l,O_j} \rangle,$$

or

$$\langle \alpha_{C_k,O_i}, \beta_{C_k,O_i} \rangle \geq \langle \alpha_{C_k,O_j}, \beta_{C_k,O_j} \rangle$$

and

$$\langle \alpha_{C_l,O_i}, \beta_{C_l,O_i} \rangle \geq \langle \alpha_{C_l,O_j}, \beta_{C_l,O_j} \rangle$$

are simultaneously satisfied.

Let $S^{\nu}_{k,l}$ be the number of cases in which

$$\langle \alpha_{C_k,O_i}, \beta_{C_k,O_i} \rangle \geq \langle \alpha_{C_k,O_j}, \beta_{C_k,O_j} \rangle$$

and

$$\langle \alpha_{C_l,O_i}, \beta_{C_l,O_i} \rangle \leq \langle \alpha_{C_l,O_j}, \beta_{C_l,O_j} \rangle,$$

or

$$\langle \alpha_{C_k,O_i}, \beta_{C_k,O_i} \rangle \leq \langle \alpha_{C_k,O_j}, \beta_{C_k,O_j} \rangle$$

and

$$\langle \alpha_{C_l,O_i}, \beta_{C_l,O_i} \rangle \geq \langle \alpha_{C_l,O_j}, \beta_{C_l,O_j} \rangle$$

are simultaneously satisfied.

Obviously,

$$S^{\mu}_{k,l} + S^{\nu}_{k,l} \leq \frac{n(n-1)}{2}.$$

Now, for every k, l, such that $1 \leq k < l \leq m$ and for $n \geq 2$, we define

$$\mu_{C_k,C_l} = 2\frac{S^{\mu}_{k,l}}{n(n-1)}, \quad \nu_{C_k,C_l} = 2\frac{S^{\nu}_{k,l}}{n(n-1)}.$$

Hence,

$$\mu_{C_k,C_l} + \nu_{C_k,C_l} = 2\frac{S^{\mu}_{k,l}}{n(n-1)} + 2\frac{S^{\nu}_{k,l}}{n(n-1)} \leq 1.$$

Therefore, $\langle \mu_{C_k,C_l}, \nu_{C_k,C_l} \rangle$ is an IFP.

Now, we can construct the IM

$$\begin{array}{c|ccc} & C_1 & \cdots & C_m \\ \hline C_1 & \langle \mu_{C_1,C_1}, \nu_{C_1,C_1} \rangle & \cdots & \langle \mu_{C_1,C_m}, \nu_{C_1,C_m} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ C_m & \langle \mu_{C_m,C_1}, \nu_{C_m,C_1} \rangle & \cdots & \langle \mu_{C_m,C_m}, \nu_{C_m,C_m} \rangle \end{array},$$

that determines the degrees of correspondence between criteria $C_1, \ldots, C_m$.

When objects $O_1, ..., O_n$ and criteria $C_1, ..., C_m$ from the above procedure coincide with our alternatives and criteria, respectively, we obtain the IM V that is the first component of the token β characteristic. The second component of this characteristic is obtained on the basis of the token ζ characteristic. Using it, we obtain the list of the near criteria.

Token η enters place k_{13} with initial characteristic "(meta)criterion for a choice of a better criterion between two given ones". This (meta)criterion can determine the better criterion because it is easier for checking, chipper, requires less time for checking. etc.

$$X_3 = \langle \{k_{12}, k_{13}\}, \{k_{14}\}, \begin{array}{c|c} & k_{14} \\ \hline k_{12} & true \\ k_{13} & true \end{array} \rangle.$$

In place k_{14}, tokens β and η unite in token β with a characteristic "list of bad criteria".

$$X_4 = \langle \{k_{10}, k_{14}\}, \{k_{15}\}, \begin{array}{c|c} & k_{15} \\ \hline k_{10} & true \\ k_{14} & true \end{array} \rangle.$$

In place k_{15}, tokens β and γ unite in token β with a characteristic "list of good criteria", i.e., $x_{cu}^{\beta} = x_{cu}^{\gamma} - x_{cu-1}^{\beta}$, where operation "$-$" between both characteristics is in set-theoretical sense.

3 Conclusion

The paper is a first attempt to unite two mathematical objects - the GNs and intercriteria analysis, that is used for increasing the effectiveness of decision making processes.

In a next research we shall discuss a possible extensions of the so constructed GN-model including other decision making activities and other applications of the intercriteria analysis.

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From Semi-fuzzy to Fuzzy Quantifiers via Łukasiewicz Logic and Games

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Abstract. Various challenges for lifting semi-fuzzy quantifier models to fully fuzzy ones are discussed. The aim is to embed such models into Łukasiewicz logic in a systematic manner. Corresponding extensions of Giles' game with random choices of constants as well as precisifications of fuzzy models are introduced for this purpose.

1 Introduction

Fuzzy logic provides formal models of vague quantifier expressions like many, few, almost all, about half, etc. Following Zadeh [14], the literature on corresponding *fuzzy quantifiers* is huge: we refer to the monograph [12] and to the more recent survey article [2] for an verview of relevant literature. Following a useful and well argued suggestion by Glöckner [12], a truth function for a fuzzy quantifier should be determined in two separate steps: (1) define a suitable *semi-fuzzy* quantifier, where the (scope and range) predicates are crisp (i.e. classical 0/1-valued) and (2) lift the semi-fuzzy quantifier to a (fully) fuzzy quantifier in some systematic and uniform manner. Regarding step (1) we will refer to an approach based on extensions of Giles's game for Łukasiewicz logic [10] that involve random choices of witness elements. But in this paper we will focus on step (2). After reviewing various shortcomings of existing approaches, Glöckner proposed an axiomatic approach for this second step, arriving at a corresponding quantifier fuzzification mechanism (QFM). However, Glöckner's QFM is still unsatisfying in some respects. In particular it is incompatible with the paradigm of mathematical fuzzy logic [1], where implication is understood as the residuum of (strong) conjunction.

After reviewing some basic notions regarding quantifiers, Łukasiewicz logic Ł, and Giles's game for Ł, we will explain some problems that may arise for lifting semi-fuzzy quantifier models to fully fuzzy ones. We then discuss in a systematic manner various quantifier fuzzification methods that arise from considering *precisifications* of fuzzy interpretations. A central aim in this endeavor is to embed the quantifier models into (suitable extensions of) Łukasiewicz logic. Moreover, we want to avoid *ad hoc* definitions of truth functions. For this reason, our main tools are certain extensions of Giles game, where one considers random

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choices of domain elements (constants) as well as choices of precisifications, in addition to moves by the two strategic players of the game. We conclude with a brief summary and some hints on further topics for related research.

2 Types of Quantification

We are interested in models of vague quantifier expressions like almost all, about half, at least about a quarter. We focus on *unary* (also known as *monadic* or *type* $\langle 1 \rangle$) quantification, where the *scope* of the quantified statement consists of a single formula and where the quantifier binds a single object variable. Vagueness will be modeled by *fuzziness*. A fuzzy set $\widetilde{S}_D$ is a function of type $D \to [0, 1]$, where the (crisp) set D is the underlying *domain* or *universe*. Similarly, an n-ary fuzzy relation is a function of type $D^n \to [0, 1]$. Every interpretation $\mathcal{M}$ with domain D assigns an n-ary fuzzy relation over D to each n-ary predicate symbol. Any unary *fuzzy quantifier* $\widetilde{Q}$ is interpreted by a *truth function* which assigns a *truth degree* (*truth value*) in $[0, 1]$ to each fuzzy set over the domain. As a special case of fuzzy quantification, we obtain *semi-fuzzy quantifiers* by restricting the scope to classical predicates (corresponding to crisp sets).

Throughout this paper we will assume that the domain D is *finite*; an assumption that is justified by the intended application of modeling natural language expressions. We will focus on a specific, but very common type of quantifiers, namely *proportionality quantifiers*, where, in the (unary) semi-fuzzy case, the degree of truth of the quantified sentence depends only on the fraction of domain elements that satisfy the scope predicate. Given an interpretation $\mathcal{M}$ with domain D and a formula F we define[1]

$$Prop_{\mathcal{M}}(F) = \sum_{d \in D} \frac{v_{\mathcal{M}}(F(d))}{|D|}.$$

When F is a classical formula, then $Prop_{\mathcal{M}}(F)$ is $|\{d \in D : v_{\mathcal{M}}(F(d)) = 1\}|/|D|$ and denotes the proportion of elements of the domain satisfying F under $\mathcal{M}$. Hence, if Q is a semi-fuzzy proportionality quantifier, $v_{\mathcal{M}}(\mathsf{Q}xF(x))$ is uniquely determined by $Prop_{\mathcal{M}}(F)$. In the general fuzzy case, we can read $Prop_{\mathcal{M}}(F)$ as the *average truth value* of F under $\mathcal{M}$. It is much more straightforward to judge the linguistic adequateness of semi-fuzzy quantifiers as models of vague (proportional) quantification, than to deal directly with the general case, where the scope predicate may be vague as well. For this reason, as already mentioned in the introduction, Glöckner [12] suggested to split the design of adequate fuzzy models of vague quantifiers into two separate steps:

(1) specify the truth function for a semi-fuzzy quantifier,
(2) lift the function obtained in (1) to the fully fuzzy case.

[1] For convenience, we identify constant symbols with domain elements.

For step (2) Glöckner introduced the notion of a *quantifier fuzzification mechanism (QFM)* and presented a range of axioms that should be satisfied by a QFM that lifts a wide class of semi-fuzzy to fuzzy quantifiers in a uniform manner. While we definitely agree with the usefulness of splitting the task of designing fuzzy logic based quantifier models as indicated, there remains a number of challenges. In particular it is left unclear how task (1) can be accomplished without resorting to *ad hoc* decisions for selecting appropriate truth functions.

Of particular importance for the current paper, we moreover argue that the corresponding quantifiers should be embeddable into (full) Łukasiewicz logic or at least into some other t-norm based fuzzy logic, as suggested by the paradigm of Hájek [13], which provides the basis for contemporary Mathematical Fuzzy Logic [1]. The approach of Glöckner [12] as well as that of many others (see [2]) leaves much to be desired in this respect. Here, we will not deal directly with step (1), but rather rely on a framework for the systematic design of semi-fuzzy proportionality quantifiers, based on Giles's game for Łukasiewicz logic (see Sect. 4).

3 Łukasiewicz Logic

As already indicated, we do not want to consider fuzzy quantifiers in isolation, but rather suggest that such quantifiers should lead to natural generalizations of well understood deductive fuzzy logics, as investigated under the heading of contemporary Mathematical Fuzzy Logic [1]. Among the corresponding t-norm based logics, Łukasiewicz logic Ł can be singled out as particularly important, since it has the unique property that the truth functions of *all* logical connectives are continuous[2] functions [1]. The semantics of the propositional connectives of (full) Łukasiewicz logic is given by the following truth functions:

$$v_{\mathcal{M}}(F \wedge G) = \min(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G)) \quad v_{\mathcal{M}}(F \odot G) = \max(0, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G) - 1)$$
$$v_{\mathcal{M}}(F \vee G) = \max(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G)) \quad v_{\mathcal{M}}(F \oplus G) = \min(1, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G))$$
$$v_{\mathcal{M}}(F \to G) = \min(1, 1 - v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G))$$
$$v_{\mathcal{M}}(\bot) = 0 \qquad v_{\mathcal{M}}(\top) = 1 \qquad v_{\mathcal{M}}(\neg F) = 1 - v_{\mathcal{M}}(F)$$

Universal and existential quantification is specified as follows:

$$v_{\mathcal{M}}(\forall x F(x)) = \inf_{c \in D}(v_{\mathcal{M}}(F(c))) \qquad v_{\mathcal{M}}(\exists x F(x)) = \sup_{c \in D}(v_{\mathcal{M}}(F(c)))$$

There is a further reason for choosing Łukasiewicz logic as a frame for designing formal models of vague language: already in the 1970s Robin Giles [10,11] provided a game based semantics for Ł, that allows one to justify the particular choice of truth functions with respect to first principles about approximate reasoning. As we will see in the next section, Giles's game provides a suitable base for extending Ł with further quantifiers in a principled manner.

[2] In rival candidates, like Gödel logic or Product logic the truth function for implication is not continuous.

4 Giles's Game and Semi-fuzzy Quantifiers

In Giles's game for Ł, two players (You and Myself) stepwise reduce logically complex assertions (formulas) to their atomic components via systematic attack and corresponding defense moves. A state of the game is given by two multisets (*tenets*) of formulas, written as

$$[F_1, \dots, F_m \mid G_1, \dots, G_n],$$

where $F_1, \dots, F_m$ denotes the multiset of formulas currently asserted by You (your tenet), whereas $G_1, \dots, G_n$ denotes the multiset of formulas currently asserted by Myself (my tenet). The rules of the game specify how the player in role **P** ('proponent') may react to an attack by the player in role **O** ('opponent') on an occurrence of one the formulas asserted by **P**. For example, an attack (by **O**) on $\forall x F(x)$ has to be answered by **P** with the assertion of $F(c)$, where the constant c is chosen by **O**. Whereas in replying to an attack on $\exists x F(x)$, **P** chooses the instance $F(c)$ that replaces the attacked formula occurrence in the multiset of formulas currently asserted by her. Similar rules apply to propositional connectives: if a disjunctive formula $A \vee B$ is attacked, then it is replaced by either A or B, according to a choice by **P**, etc. In particular, implication and strong conjunction are specified by the following rules:

($R_\rightarrow$) If **P** asserts $F \rightarrow G$ then, if **O** chooses to attack this formula occurrence, it is replaced by G in **P**'s tenet and F is added to **O**'s tenet; otherwise, if **O** chooses not to attack this occurrence of $F \rightarrow G$, it is removed from **P**'s tenet.
($R_\odot$) If **P** asserts $F \odot G$ then **P** has to reply to **O**'s attack by either asserting F as well as G or else $\bot$ instead of $F \odot G$.

In any case, the successor state of the game is obtained by removing the attacked formula occurrence and adding zero or more immediate subformulas, or the logical constant $\bot$ to my or your tenet. This is repeated until a state is reached, where all asserted formulas are atomic. At such a final state the payoff for Myself is given by

$$m - n + 1 + \sum_{1 \le i \le n} v_{\mathcal{M}}(G_i) - \sum_{1 \le i \le m} v_{\mathcal{M}}(F_i),$$

where $v_{\mathcal{M}}(A)$ denotes the truth value[3] assigned to the atomic formula A by the given interpretation $\mathcal{M}$. Giles [10] (essentially) proved that for every formula F of Łukasiewicz logic, there is a strategy for Myself that guarantees a final payoff of $v_{\mathcal{M}}(F)$ if both players play rationally according to the rules of the outlined

[3] The payoff scheme may look arbitrary at a first glimpse. However it results from Giles's interpretation of the truth value of a given atom A in terms of the expected loss for a player, who has to pay a fixed amount of money (say 1 Euro) to the opposing player, if a certain experiment E_A associated with A fails. Such (binary) experiments may show dispersion, i.e. repeated executions of the same experiment E_A may show different results. However for each A a fixed *failure probability* (risk) is associated to E_A.

(finite, two person, perfect information) game. When this is the case, we say that the truth functions used in $v_{\mathcal{M}}(F)$ for interpreting the connectives and quantifiers match the corresponding game rules. Here we are interested in game rules—and the resulting truth functions—for proportional semi-fuzzy quantifiers. To obtain such rules Fermüller and Roschger [7,8] considered *uniformly random choices* of witnessing constants, in addition to the choices made by the two strategic players in roles **P** and **O**, as indicated above for the classical quantifiers $\forall$ and $\exists$. The most basic of such rules introduces a new *random choice quantifier* Π as follows.

(R_{Π}) If **P** asserts $\Pi x F(x)$ then this formula occurrence is replaced by $F(c)$, where c is a (uniformly) randomly chosen constant.

We will call Ł (Π) the expansion of Ł with the random choice quantifier Π. More generally, rules for a quantified formula $\mathsf{Q}xF(x)$ feature *bets for* and *bets against* instances $F(c)$ of its scope formula, where c is a randomly chosen constant. A bet for $F(c)$ is simply an assertion of $F(c)$ by the corresponding player, whereas a bet against $F(c)$ means that $\bot$ has to be asserted, while the opposing player asserts $F(c)$. Following [7,8], these notions allow us to formulate, e.g., the following families of rules for so-called *blind choice quantifiers.*

($R_{\mathsf{L}^k_m}$) If **P** asserts $\mathsf{L}^k_m xF(x)$ then **O** may attack by betting for k random instances of $F(x)$, while **P** bets against m random instances of $F(x)$.

($R_{\mathsf{G}^k_m}$) If **P** assert $\mathsf{G}^k_m xF(x)$ then **O** may attack by betting against m random instances of $F(x)$, while **P** bets for k random instances of $F(x)$.

Some clarifications are needed to render these rules intelligible:

1. *'Blind choice'* signifies that the identity of the randomly picked constants $c_1, \ldots, c_n$ used for the relevant random instances $F(c_1), \ldots, F(c_n)$ is revealed to the players only after they have placed their bets.
2. The choices of constants are *uniformly random* and *independent* of each other. In particular, the same constant may be picked more than once. Therefore the random instances form multisets, rather than sets of formulas.
3. Attacks are always optional in a Giles style game, which means that (the player in role) **O** can always decide that the attacked formula is simply removed from the current state. Giles speaks of a 'principle of limited liability' for attack (LLA) in such a situation.
4. A 'principle of limited liability' for defense (LLD) is also in place: if attacked by **O** then **P** may always decide to replace the attacked formula occurrence by $\bot$, rather than to continue the game as indicated in the above rules.

As shown in [8] the above rules, together with the just mentioned principles of limited liability, allow one to extract the following corresponding truth functions:

$$v_{\mathcal{M}}(\mathsf{L}^k_m xF(x)) = \min\{1, \max\{0, 1 + k - (m+k)Prop_{\mathcal{M}}(F)\}\} \quad (1)$$

$$v_{\mathcal{M}}(\mathsf{G}^k_m xF(x)) = \min\{1, \max\{0, 1 - k + (m+k)Prop_{\mathcal{M}}(F)\}\}. \quad (2)$$

These quantifiers are definable in Ł(Π) using additional truth constants [5].

5 Problems with Lifting

It is tempting to extend the above framework for semi-fuzzy quantifiers to fully fuzzy quantifiers by just applying the same functions and game rules to fuzzy predicates. From a purely mathematical point of view, no problem arises: for any formula F in the scope of a quantifier we can just compute $Prop_{\mathcal{M}}(F)$ and plug the obtained value into the corresponding truth functions. However, this leads to results that run counter to expectations on the behavior of vague quantifiers in natural language, as illustrated by the following example.

Example 1. Let F a predicate standing for "is tall". We want to evaluate the sentence About half (of the elements of the domain) are tall. For modeling About half we use the quantifier $\mathsf{H}_0^1 x$, introduced in [8] and shown there to be equivalent to $\mathsf{G}_1^1 x \wedge \mathsf{L}_1^1 x$. It is straightforward to see that

$$v_{\mathcal{M}}(\mathsf{H}_0^1 x(F(x))) = \max\{0, \min\{2Prop_{\mathcal{M}}(F), 2 - 2Prop_{\mathcal{M}}(F)\}\}.$$

We now consider the following two interpretations $\mathcal{M}_1$ and $\mathcal{M}_2$ under the same domain $D = \{d_1, d_2, d_3, d_4\}$. Under the interpretation $\mathcal{M}_1$ we let $v_{\mathcal{M}_1}(F(d_1)) = v_{\mathcal{M}_1}(F(d_2)) = 0.1$ and $v_{\mathcal{M}_1}(F(d_3)) = v_{\mathcal{M}_1}(F(d_4)) = 0.9$. Under the interpretation $\mathcal{M}_2$ we let instead $v_{\mathcal{M}_2}(F(d)) = 0.5$ for any $d \in D$. Note that $Prop_{\mathcal{M}_1}(F) = Prop_{\mathcal{M}_2}(F) = 0.5$, hence $v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF(x)) = v_{\mathcal{M}_2}(\mathsf{H}_0^1 xF(x)) = 1$.

In the first interpretation we have two almost clear cases of tall people and two almost clear cases of not tall people, and we correctly obtain a high value for $v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF(x))$. In the second interpretation instead, all individuals of the domain are meant to be of perfectly average height. Of course there is no clear fact in this situation which would determine the "correct" truth value, but we would expect it to be smaller than in the first interpretation. As we saw above, however, $v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF(x)) = v_{\mathcal{M}_2}(\mathsf{H}_0^1 xF(x))$. Informally, the approach is not sensitive to the difference between About half (of the people) are tall and All (of the people) are about half tall. Note that using any other truth function for the quantifier About half defined only in terms of the average truth value $Prop_{\mathcal{M}}(F)$ would not help. The example shows that, when evaluating a fuzzy quantifier over fuzzy predicates, one should also keep track of how the truth values are distributed over the elements of the domain.

6 Fuzzification via Random Precisification

Assume that we have a sentence $\tilde{\mathsf{Q}}xF(x)$, where $\tilde{\mathsf{Q}}$ is a fuzzy quantifier corresponding to a semi-fuzzy quantifier Q, and let $\mathcal{M}$ be an interpretation evaluating F over $[0, 1]$. As we saw before, we cannot interpret $\tilde{\mathsf{Q}}$ just in the same way as Q. To obtain more satisfactory models, we first need to associate to the interpretation $\mathcal{M}$ a set of interpretations evaluating F as a classical formula, so that the corresponding semi-fuzzy quantifier Q can be evaluated properly. Following the terminology used in supervaluationist accounts of vagueness [6,9], we call such a set of interpretations the *admissible precisifications*[4] of $\mathcal{M}$ and denote it by $C_{\mathcal{M}}$. Informally $C_{\mathcal{M}}$ collects the "reasonable" ways of making $\mathcal{M}$ precise (i.e. classical) over atomic formulas. A simple idea for the evaluation of fuzzy quantifiers via reduction to precisifications has been introduced in [4]. It can be formulated via the following *random-precisification* based rule, which extends Giles' game for $\text{Ł}(\mathsf{Q})$, where Q is a semi-fuzzy quantifier, to a corresponding fuzzy quantifier $\tilde{\mathsf{Q}}$:

($R^{RP}_{\tilde{\mathsf{Q}}}$) If **P** asserts $\tilde{\mathsf{Q}}xF(x)$ and **O** attacks the formula, a precisification $\mathcal{M}'$ is chosen randomly from $C_{\mathcal{M}}$ and **P** has to assert $\mathsf{Q}x(F(x))$, where this formula occurrence is evaluated over $\mathcal{M}'$.

A truth function for $\tilde{\mathsf{Q}}$ matching this rule is obtained as the *expected value* of $v_{\mathcal{M}'}(\mathsf{Q}xF(x))$, where $\mathcal{M}'$ ranges over the set of admissible precisifications $C_{\mathcal{M}}$. Even though not explicitly required by the general framework, we can assume that the random choice of a precisification from $C_{\mathcal{M}}$ follows a uniform distribution. A natural way to instantiate ($R^{RP}_{\tilde{\mathsf{Q}}}$) is by letting

$$C_{\mathcal{M}} = \{\mathcal{M}^{\geq\alpha} \mid \alpha \in [0,1]\},$$

where $\mathcal{M}^{\geq\alpha}$ denotes the interpretation such that, for any atomic formula A, $v_{\mathcal{M}^{\geq\alpha}}(A) = 1$ if $v_{\mathcal{M}}(A) \geq \alpha$ and $v_{\mathcal{M}^{\geq\alpha}}(A) = 0$ otherwise. Hence we can think of the random choice of a precisification as coinciding with the random choice of a value α acting as a threshold. The truth function matching ($R^{RP}_{\tilde{\mathsf{Q}}}$) is then obtained as:

$$v_{\mathcal{M}}(\widetilde{\mathsf{Q}}xF(x)) = \int_0^1 v_{\mathcal{M}^{\geq\alpha}}(\mathsf{Q}xF(x))d\alpha.$$

The same evaluation function and corresponding lifting mechanism for fuzzy quantifier is also obtained in [3], though motivated by a different semantics, based on voting models. In [3] it is also shown that the model satisfies many, though not all of Glöckner's desiderata for a quantifier fuzzification mechanism. Let us look now how this approach deals with the Example 1.

Example 1 (continued). Let $b_0 = 0, b_1 = v_{\mathcal{M}_1}(F(d_1)) = v_{\mathcal{M}_1}(F(d_2)) = 0.1$, $b_2 = v_{\mathcal{M}_1}(F(d_3)) = v_{\mathcal{M}_1}(F(d_4)) = 0.9$, $b_3 = 1$. Clearly, for any $b_{i-1} < \alpha \leq b_i$ we have $v_{\mathcal{M}_1^{\geq\alpha}}(\mathsf{H}^1_0 xF(x)) = v_{\mathcal{M}_1^{\geq b_i}}(\mathsf{H}^1_0 xF(x))$. As the domain D is finite, we get

$$v_{\mathcal{M}_1}(\widetilde{\mathsf{H}^1_0}xF(x)) = \sum_{i=1}^{3}(b_i - b_{i-1}) \cdot v_{\mathcal{M}_1^{\geq b_i}}(\mathsf{H}^1_0 xF(x))$$

$$= 0.1 \cdot v_{\mathcal{M}_1^{\geq 0.1}}(\mathsf{H}^1_0 xF(x)) + 0.8 \cdot v_{\mathcal{M}_1^{\geq 0.9}}(\mathsf{H}^1_0 xF(x)) + 0.1 \cdot v_{\mathcal{M}_1^{\geq 1}}(\mathsf{H}^1_0 xF(x)) = 0.8.$$

For the interpretation $\mathcal{M}_2$ we instead obtain

$$v_{\mathcal{M}_2}(\widetilde{\mathsf{H}^1_0}xF(x)) = 0.5 \cdot v_{\mathcal{M}_2^{\geq 0.5}}(\mathsf{H}^1_0 xF(x)) + 0.5 \cdot v_{\mathcal{M}_1^{\geq 1}}(\mathsf{H}^1_0 xF(x)) = 0.$$

[4] Note that, despite the fact that a precisification evaluates atomic formulas classically, the valuation under a precisification of a formula involving a semi-fuzzy quantifier might be an intermediate value in $[0,1]$.

This computation of the expected value of $\mathsf{H}_0^1 xF(x)$ over the admissible precisifications delivers more adequate results than the naive application of the function for H_0^1. From our perspective though, the fuzzification mechanism recalled here still poses a problem: we lose the possibility of expressing fuzzy quantifiers in the language of the logic Ł(Π) – in contrast to the case for the semi-fuzzy quantifiers introduced in [8]. To embed fuzzy quantifiers in a Łukasiewicz logic based framework, we have to consider appropriate expansions of Ł(Π). One way of doing so consists in introducing a random choice quantifier Πp operating over propositional variables rather than over elements of the domain. Game semantically, given a formula F containing occurrences of a propositional variable p, we can interpret $\Pi pF(p)$ via the following rule.

(R_{Π_p}) If **O** attacks $\Pi pF(p)$, a ropositional variable p' is randomly chosen and **P** has to assert $F(p')$.

The random choice of a propositional variable can be seen as a syntactic counterpart of the random choice of a threshold truth value. In addition, we need to expand Ł(Π) with the well-known unary connective Δ (see e.g. [1]), given by $v_{\mathcal{M}}(\Delta F) = 1$ if $v_{\mathcal{M}}(F) = 1$ and 0 otherwise. In what follows, Ł$^{\Delta}$(Π) denotes the corresponding expansion of Ł(Π). It is easy to see that, if $v_{\mathcal{M}}(p) = \alpha$, then $v_{\mathcal{M}^{\geq\alpha}}(F(x)) = v_{\mathcal{M}}(\Delta(p \to F(x))$. Hence in Ł$^{\Delta}$($\Pi$) extended with Πp we can express $\tilde{\mathsf{Q}}$ by $\tilde{\mathsf{Q}}xF(x) \equiv \Pi p\mathsf{Q}x(\Delta(p \to F(x)))$ for any fuzzy quantifier $\tilde{\mathsf{Q}}$.

7 A Closeness-Based Approach to Fuzzy Quantifiers

We will now introduce a different approach to lift semi-fuzzy to fuzzy quantifiers. As in the previous case the idea is rooted in Giles' game semantics setting, but has an important advantage: the resulting fuzzy quantifiers are already definable over the logic Ł$^{\Delta}$(Π). We start by presenting a more abstract framework.

Assume that, at a certain stage of a Giles' game, the player acting as **P** has in its tenet a fuzzy quantified sentence, say $\tilde{\mathsf{Q}}xF(x)$. If this assertion is attacked by **O**, the following two-step defense ensues:

(i) **P** adds $\mathsf{Q}x(F(x))$ to his tenet and evaluates this formula occurrence under a precisification $\mathcal{M}'$ of $\mathcal{M}$ of his choice.
(ii) **P** has to state that $v_{\mathcal{M}}(F(x))$ is "close" to $v_{\mathcal{M}'}(F(x))$.

Playing rationally, the proponent **P** will choose a precisification which maximizes the truth value of the semi-fuzzy quantified sentence, while staying as close as possible to the original truth values of the (fuzzy) predicate (a "reasonable" precisification). Let us consider some possible ways to instantiate the above abstract scheme in Giles' game in such a way as to obtain the expressibility of fuzzy quantifiers in Ł$^{\Delta}$(Π).

First, in addressing step (i) above, we may reduce the choice of a precisification to the choice of a certain element of the domain, say c, acting as a threshold. In other words, we take precisifications $\mathcal{M}^c$ of $\mathcal{M}$ such that

$v_{\mathcal{M}^c}(F(d)) = 1$ if $v_{\mathcal{M}}(F(d)) \geq v_{\mathcal{M}}(F(c))$, and $v_{\mathcal{M}^c}(F(d)) = 0$ otherwise. In this setting, we could actually even remove any explicit reference to $\mathcal{M}^c$: letting $F^c(x) \equiv \Delta(F(c) \to F(x))$, we easily see that $v_{\mathcal{M}^c}(F(x)) = v_{\mathcal{M}}(F^c(x))$. For ease of reference, in the following we also let $F^\top(x) \equiv \Delta F(x)$, where $F^\top$ stands for the choice of $\top$ as a threshold, instead of an element of the domain.

Let us turn now to step (ii). Given two formulas A and B in Łukasiewicz logic, the most obvious way of measuring the closeness of their truth values under a given interpretation is by evaluating $A \leftrightarrow B$, i.e. $(A \to B) \wedge (B \to A)$. Thus, a natural way to evaluate how close the formulas $F^c(x)$ and $F(x)$ are under the interpretation $\mathcal{M}$ is by computing $Prop_{\mathcal{M}}(F^c \leftrightarrow F)$. In the setting of Giles' game, the above ideas for (i) and (ii) result in the following *closeness-based* game rule for $\tilde{\mathsf{Q}}$.

($R^{Cl}_{\tilde{\mathsf{Q}}}$) If **P** asserts $\tilde{\mathsf{Q}}xF(x)$ and **O** attacks the formula, **P** can either invoke LLD (i.e. dismiss this formula occurrence) or add $\mathsf{Q}xF^c(x)$ to his tenet, where c is either an element of the domain of his choice or $\top$. An element d is then randomly chosen and **O** can then choose between the following:
1. **O** adds $F^c(d)$ to his tenet, thereby forcing **P** to add $F(d)$ to his tenet.
2. **O** adds $F(d)$ to his tenet, thereby forcing **P** to add $F^c(d)$ to his tenet.

The states of the game corresponding to **O**'s choices when **P** does not invoke LLD can be depicted as follows (Γ and Σ stand for arbitrary multisets of formulas):

$$\left[\Gamma \mid \Sigma, \tilde{\mathsf{Q}}xF(x)\right]$$

$$[\Gamma, F^c(d) \mid \Sigma, \mathsf{Q}xF^c(x), F(d)] \qquad [\Gamma, F(d) \mid \Sigma, \mathsf{Q}xF^c(x), F^c(d)]$$

Proposition 1. *Let us define the formula $Cl(QxF(x))$ as*

$$\exists z(QxF^z(x) \odot \Pi y(F^z(y) \leftrightarrow F(y))) \vee (Qx(\Delta F(x)) \odot \Pi y(F(y) \leftrightarrow \Delta F(y))).$$

The game rules for $Cl(QxF(x))$ in Giles' game for Łukasiewicz logic are essentially reducible to $(R^{Cl}_{\tilde{Q}})$, modulo some irrelevant change of order: letting $v_{\mathcal{M}}(\tilde{Q}xF(x)) = v_{\mathcal{M}}(Cl(QxF(x)))$, where

$$v_{\mathcal{M}}(Cl(QxF(x))) = \sup_{c \in D \cup \{\top\}} \left(\max\{0, Prop_{\mathcal{M}}(F^c \leftrightarrow F) + v_{\mathcal{M}}(QxF^c(x)) - 1\}\right),$$

we obtain an evaluation of $\tilde{Q}xF(x)$ matching the game rule $(R^{Cl}_{\tilde{Q}})$.

Note that the occurrence of $\Pi y(F^z(y) \leftrightarrow F(y))$ in $Cl(\mathsf{Q}xF(x))$ corresponds to asserting that $F^z(y)$ is "close" to $F(y)$, whereas the subformula $\mathsf{Q}x(\Delta F(x)) \odot \Pi y(F(y) \leftrightarrow \Delta F(y))$ reflects the choice of $\top$ as a threshold instead of an element of the domain.

Motivated by game semantics, we have thus obtained an abstract characterization of fuzzy quantifiers in terms of semi-fuzzy ones. We remark that simple changes in the choice of connectives and quantifiers in $Cl(\mathsf{Q}xF(x))$, lead to different fuzzification mechanisms, still expressible in $Ł^{\Delta}(\Pi)$. Let us check how the approach sketched here fares with respect to Example 1 of Sect. 5.

Example 1 (continued). Under the interpretation $\mathcal{M}_1$, the supremum in $v_{\mathcal{M}}(Cl(H_0^1 xF(x))$ is obtained equivalently by choosing d_3 or d_4 as a threshold element. Hence $v_{\mathcal{M}_1}(\widetilde{\mathsf{H}_0^1} xF(x)) = v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF^{d_4}(x) \odot \Pi y(F^{d_4}(y) \leftrightarrow F(y)))$. We have $Prop_{\mathcal{M}_1}(F^{d_4}) = 0.5$ and $Prop_{\mathcal{M}_1}\ (F^{d_4} \leftrightarrow F) = \frac{0.9+0.9+0.9+0.9}{4} = 0.9$. Hence $v_{\mathcal{M}_1}(\widetilde{\mathsf{H}_0^1} xF(x)) = 1 \odot 0.9 = 0.9$.

For the interpretation $\mathcal{M}_2$ it does not matter which element of the domain is taken as a threshold (the choice of $\top$ can only make the value of $v_{\mathcal{M}}(Cl(H_0^1 x))$ smaller). In any case $v_{\mathcal{M}_2}(\mathsf{H}_0^1 xF^{d_i}(x)) = 0$, hence $v_{\mathcal{M}_2}(\widetilde{\mathsf{H}_0^1} xF(x)) = 0$.

The method correctly determines two different truth values for $\widetilde{\mathsf{H}_0^1} xF(x)$ for the two interpretations $\mathcal{M}_1$ and $\mathcal{M}_2$. Note that a different value was obtained for $v_{\mathcal{M}_1}(\widetilde{\mathsf{H}_0^1} xF(x))$ under the random precisification mechanism in Sect. 6. The value 0.8 obtained for $\mathcal{M}_1$ in that approach points to a probabilistic interpretation of the truth values of $F(d_i)$. Indeed 0.8 stands for the probability that one picks the precisifications accepting half of the elements of the domain (d_3, d_4 but not d_1, d_2) as instances of F. What we present in this section instead follows a "metric" intuition: it determines how close the interpretation $\mathcal{M}_1$ is from one where exactly half of the elements of the domain fully satisfy $F(x)$; hence the value 0.9. We contend that both results are plausible and justifiable under the respective (different) underlying intuitions.

Some problems persist, due to our choice of the closeness measure: evaluating how close the truth values of all the elements of the domain are to a precisification can lead indeed to counterintuitive results, as illustrated in the following.

Example 2. Let us consider the quantifier G_1^1, which can be thought of as modeling $\mathsf{At\ least\ about\ half}$. Recall that $v_{\mathcal{M}}(\mathsf{G}_1^1 xF(x)) = \min\{1, 2Prop_{\mathcal{M}}(F)\}$. We compare the truth values of $\mathsf{G}_1^1 xF(x)$ under the interpretation $\mathcal{M}_1$ in Example 1 and under a new interpretation $\mathcal{M}_3$ over the same domain $D = \{d_1, \ldots, d_4\}$. As for the case of $\widetilde{\mathsf{H}_0^1} xF(x)$, we obtain $v_{\mathcal{M}_1}(\widetilde{\mathsf{G}_1^1} xF(x)) = 0.9$. Now let $v_{\mathcal{M}_3}(F(d_1)) = v_{\mathcal{M}_3}(F(d_2)) = 0.4$ and $v_{\mathcal{M}_3}(F(d_3)) = v_{\mathcal{M}_3}(F(d_4)) = 0.9$. For $\mathcal{M}_3$, the supremum of $v_{\mathcal{M}}(Cl(\mathsf{G}_1^1 xF(x)))$ is obtained by choosing the precisification determined by F^{d_4}. Again, we have $v_{\mathcal{M}_3}(\mathsf{G}_1^1 xF^{d_4}(x)) = 1$, but F^{d_4} is less close to F than in $\mathcal{M}_1$. Indeed, we have $v_{\mathcal{M}_3}(\Pi y(F^{d_4}(y) \leftrightarrow F(y))) = \frac{0.6+0.6+0.9+0.9}{4} = 0.75$ hence $v_{\mathcal{M}_3}(\widetilde{\mathsf{G}_1^1} xF(x)) = 1 \odot 0.75 = 0.75$.

Note that the semi-fuzzy quantifier G_1^1 is monotone increasing, which means that $v_{\mathcal{M}_1}(F(d)) \leq v_{\mathcal{M}_3}(F(d))$ for any $d \in D$, implies that $v_{\mathcal{M}_1}(\mathsf{G}_1^1 xF(x)) \leq v_{\mathcal{M}_3}(\mathsf{G}_1^1 xF(x))$. One would expect the same to happen also for the corresponding fuzzy quantifier, i.e. to have $v_{\mathcal{M}_1}(\widetilde{\mathsf{G}_1^1} xF(x)) \leq v_{\mathcal{M}_3}(\widetilde{\mathsf{G}_1^1} xF(x))$. But this is not the case, as shown above. The problem is that, when evaluating in $\mathcal{M}_3$ how close F is to the precisification F^{d_4}, we take into account also those elements (d_1 and d_2) for which $v_{\mathcal{M}_3}(F^{d_4}(d)) = 0$. These values should be indifferent for an increasing quantifier. A simple solution to address this problem is to replace in the subformula $\Pi y(F^z(y) \leftrightarrow F(y))$ of $Cl(\mathsf{Q}xF(x))$ (see Proposition 1) the quantifier occurrence Πy by $\forall y$ or $\exists y$, thus obtaining a stricter or looser measure of closeness, respectively. A more general – and hence more satisfactory – solution consists in replacing the rule $(R^{Cl}_{\tilde{\mathsf{Q}}})$ above by a simplified version, which reduces the choices available to **O**. In case we consider a monotone increasing semi-fuzzy quantifier, we allow the opponent **O** only the first choice, which is matched by the truth function $Prop_{\mathcal{M}}(F^c \to F)$. Similarly, for decreasing quantifiers we allow only the second option, corresponding to $Prop_{\mathcal{M}}(F \to F^c)$, while for other quantifiers we leave both options available to **O**. Let $Cl'(\mathsf{Q}xF(x))$ stand for

$$\exists z(\mathsf{Q}xF^z(x) \odot \Pi y(\circ(F^z(y), F(y))) \vee (\mathsf{Q}x(\Delta F(x) \odot \Pi y(\circ(\Delta F(y), F(y))))$$

where $\circ(F^z(y), F(y)) \equiv F^z(y) \to F(y)$ if Q is increasing, $F(y) \to F^z(y)$ if Q is decreasing, and $F(y) \leftrightarrow F^z(y)$ otherwise. The game rules for $Cl'(\mathsf{Q}xF(x))$ correspond to the refinement of $(R^{Cl}_{\tilde{\mathsf{Q}}})$ just discussed; i.e., we obtain the analogue of Proposition 1 for $v_{\mathcal{M}}(\tilde{\mathsf{Q}}xF(x)) = v_{\mathcal{M}}(Cl'(\mathsf{Q}xF(x)))$. We can now solve the problems with monotonicity in Example 2, while still retaining the important property of allowing to define fuzzy quantifiers over $\text{Ł}^{\Delta}(\Pi)$.

Example 2 (continued). For $\mathcal{M}_1$ the supremum in $v_{\mathcal{M}_1}(Cl'(\mathsf{Q}xF(x)))$ is obtained considering F^{d_4}. Since G^1_1 is a monotone increasing quantifier, the closeness of F to F^{d_4} is measured by $v_{\mathcal{M}_1}(\Pi y(F^{d_4}(y) \to F(y))) = \frac{1+1+0.9+0.9}{4} = 0.95$ and consequently we obtain $v_{\mathcal{M}_1}(\widetilde{\mathsf{G}^1_1}xF(x)) = 0.95$. The same value is obtained for $v_{\mathcal{M}_3}(\Pi y(F^{d_4}(y) \to F(y)))$, hence $v_{\mathcal{M}_3}(\widetilde{\mathsf{G}^1_1}xF(x)) = v_{\mathcal{M}_3}(\widetilde{\mathsf{G}^1_1}xF(x)) = 0.95$.

More generally, we obtain the following restricted preservation of monotonicity.

Proposition 2. *For any semi-fuzzy quantifier Q, let us interpret $\tilde{\mathsf{Q}}xF(x)$ as $Cl'(\mathsf{Q}xF(x))$; and let $\mathcal{M}$ and $\mathcal{M}'$ be two interpretations such that $v_{\mathcal{M}}(F(d_i)) \lhd v_{\mathcal{M}}(F(d_j))$ iff $v_{\mathcal{M}'}(F(d_i)) \lhd v_{\mathcal{M}'}(F(d_j))$ for arbitrary elements d_i, d_j of a finite domain D, where $\lhd$ is either $=$ or $<$. If Q is a monotone increasing quantifier and $v_{\mathcal{M}}(F(d)) \leq v_{\mathcal{M}'}(F(d))$ for any $d \in D$, then $v_{\mathcal{M}}(\widetilde{\mathsf{Q}}xF(x)) \leq v_{\mathcal{M}'}(\widetilde{\mathsf{Q}}xF(x))$. Analogously, for decreasing quantifiers.*

8 Conclusion

We have investigated different ways of lifting semi-fuzzy to fuzzy quantifiers. The two main approaches presented in Sects. 6 and 7 have the following advantages: (1) they have a clear semantic foundation, based on Giles' game and (2) they provide models of fuzzy quantifiers compatible with Łukasiewicz logic. The closeness-based method introduced in Sect. 7 fulfills (2) in an even stronger sense, by allowing for the definition of fuzzy quantifiers over $\text{Ł}^{\Delta}(\Pi)$. We partially departed from Glöckner's [12], as the set of axioms presented there for fuzzification mechanisms forces an interpretation of the connectives different from that of Łukasiewicz logic. Nevertheless we maintain that some of the properties listed by Glöckner are relevant for our purposes as well. Among them, we stress the preservation of monotonicity. In the closeness based approach we obtained only a restricted form of this property. Full preservation of monotonicity can be easily achieved if we drop the requirement that a precisification should be identified with the choice of a threshold element. This, however, results in losing the immediate expressibility of quantifiers in $\text{Ł}^{\Delta}(\Pi)$. Further refinements of the method yet to be explored can be obtained by changes to the closeness measure.

Another natural research direction is to extend the closeness based approach to binary, or more generally to n-ary vague quantifiers: linguistically adequate models of such quantifiers should also take into account general concerns regarding truth functionality, as already suggested in [4] for the random precisification-based approach.

Finally, we suggest to further explore the advantages of embedding fuzzy quantifiers models into logical calculi, in particular for t-norm based logics. An axiomatization and a proof-theoretic study of semi-fuzzy and fuzzy quantifiers is still lacking, even for the "basic" logic $\text{Ł}(\Pi)$. Promising steps in this direction consider modal counterparts of quantifiers, e.g. along the lines suggested in Chap. 8 of Hajek's ground breaking monograph [13].

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About Fisher-Tippett-Gnedenko Theorem for Intuitionistic Fuzzy Events

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Abstract. In the paper the space of observables with respect to a family of the intuitionistic fuzzy events is considered. We proved the modification of the Fisher-Tippet-Gnedenko theorem for sequence of independent intuitionistic fuzzy observables. It is the theorem of part of statistic, which is called the extreme value theory.

Keywords: Intuitionistic fuzzy set · Intuitionistic fuzzy state · The sequence of intuitionistic fuzzy observables · Independence · Joint intuitionistic fuzzy observable · Convergence in distribution · Fisher-Tippet-Gnedenko theorem · The extreme value theory

1 Introduction

The extreme value theory is a part of statistics, which deals with examination of probability of extreme and rare events with a large impact. The extreme value theory search endpoints of the distributions. The Fisher-Tippet-Gnedenko theorem says about convergence in probability distribution of maximums of independent, equally distributed random variables. In the paper we proved the modification of the Fisher-Tippet-Gnedenko theorem for sequence of independent intuitionistic fuzzy observables.

One of the preferences of the Kolmogorov concept of probability is the agreement of replacement the notion event with notion of a set. Therefore it seems to be important also in the intuitionistic fuzzy probability theory to work with the notion of an intuitionistic fuzzy event as an intuitionistic fuzzy set.

Recall that in the Kolmogorov theory a random variable is a measurable function $\xi : \Omega \to R$ defined on a space $(\Omega, \mathcal{S})$, i.e. such a function that the preimage $\xi^{-1}(I)$ of any interval $I \subset R$ belongs to the given σ-algebra $\mathcal{S}$ of subsets of Ω. It induces a mapping $I \to \xi^{-1}(I) \in \mathcal{S}$ from the family $\mathcal{J}$ of all intervals to a σ-algebra $\mathcal{S}$.

In the intuitionistic fuzzy probability theory instead of the probability $P : \mathcal{S} \to [0, 1]$ an intuitionistic fuzzy state $\mathbf{m} : \mathcal{F} \to [0, 1]$ is considered, where $\mathcal{F}$ is a family of intuitionistic fuzzy subsets of Ω. And instead of a random variable $\xi : \Omega \to R$ an intuitionistic fuzzy observable $x : \mathcal{B}(R) \to \mathcal{F}$ is considered.

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Our main idea is in a representation of a given sequence $(y_n)_n$ of intuitionistic fuzzy observables $y_n : \mathcal{B}(R) \to \mathcal{F}$ by a probability space $(\Omega, \mathcal{S}, P)$ and a sequence $(\eta_n)_n$ of random variables $\eta_n : \Omega \to R$. Then from the convergence of $(\eta_n)_n$ in distribution the convergence in distribution of $(y_n)_n$ follows. Of course to different sequences $(y_n)_n$ different probability spaces can be obtained. Anyway the transformation can be used for obtaining some new results about intuitionistic fuzzy states on $\mathcal{F}$.

Mention that the used Atanassov concept of intuitionistic fuzzy sets [1,2] is more general as the Zadeh notion of fuzzy sets [14,15]. Therefore in Sect. 2 some basic information about intuitionistic fuzzy states and intuitionistic fuzzy observables on families of intuitionistic fuzzy sets are presented [12]. Further in Sect. 3 the independence of intuitionistic fuzzy observables is studied. In Sect. 4 the convergence in $\mathbf{m}$ is studied and the Fisher-Tippet-Gnedenko theorem is proved.

Remark that in a whole text we use a notation "IF" for short a phrase "intuitionistic fuzzy".

2 IF-events, IF-states and IF-observables

Our main notion in the paper will be the notion of an IF-event, what is a pair of fuzzy events.

Definition 1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0,1]$ will be called the membership function, $\nu_A : \Omega \longrightarrow [0,1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0)),$$
$$\mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1))$$

and partial ordering is given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$$

and a unary operation $\neg$ by

$$\neg\mathbf{A} = (\nu_A, \mu_A).$$

Example 1. Fuzzy event $f : \Omega \longrightarrow [0,1]$ can be regarded as IF-event, if we put

$$\mathbf{A} = (f, 1-f).$$

If $f = \chi_A$, then the corresponding IF-event has the form

$$\mathbf{A} = (\chi_A, 1-\chi_A) = (\chi_A, \chi_{A'}).$$

In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the product of sets, $\neg\mathbf{A}$ to the complement, and $\leq$ to the set inclusion.

In the IF-probability theory [12] instead of the notion of probability we use the notion of state.

Definition 3. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$*;*
(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$*, then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$*;*
(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$*,* $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$*.*

Probably the most useful result in the IF-state theory is the following representation theorem [10]:

Theorem 1. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$\mathbf{m}(\mathbf{A}) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 4. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1,0)$, $x(\emptyset) = (0,1)$*;*
(ii) if $A \cap B = \emptyset$*, then* $x(A) \odot x(B) = (0,1)$ *and* $x(A \cup B) = x(A) \oplus x(B)$*;*
(iii) if $A_n \nearrow A$*, then* $x(A_n) \nearrow x(A)$*.*

If we denote $x(A) = \big(x^\flat(A), 1 - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - measurable\}$.

If $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ is an IF-observable, and $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ is an IF-state, then the IF-distribution function of x is the function $\mathbf{F} : R \longrightarrow [0,1]$ defined by the formula

$$\mathbf{F}(t) = \mathbf{m}\big(x((-\infty, t))\big)$$

for each $t \in R$.

Similarly as in the classical case the following two theorems can be proved [12].

Theorem 2. *Let $\mathbf{F} : R \longrightarrow [0,1]$ be the IF-distribution function of an IF-observable $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$. Then $\mathbf{F}$ is non-decreasing on R, left continuous in each point $t \in R$ and*

$$\lim_{n \to -\infty} \mathbf{F}(t) = 0, \quad \lim_{n \to \infty} \mathbf{F}(t) = 1.$$

Theorem 3. *Let $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ by the formula*

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ is a probability measure.

Theorem 2 enables us to define IF-expectation and IF-dispersion of an IF-observable.

Definition 5. *Let $\mathbf{F} : R \longrightarrow [0,1]$ be the IF-distribution function of an IF-observable $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$. If there exists $\int_R t \; d\mathbf{F}(t)$, then we define the IF-expectation of x by the formula*

$$\mathbf{E}(x) = \int_R t \; d\mathbf{F}(t).$$

Moreover if there exists $\int_R t^2 \; d\mathbf{F}(t)$, then we define the IF-dispersion $\mathbf{D}^2(x)$ by the formula

$$\mathbf{D}^2(x) = \int_R t^2 \; d\mathbf{F}(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \; d\mathbf{F}(t).$$

3 Independence

In the paper we shall work only with independent IF-observables. Of course first we must need the existence of the joint IF-observable. For this reason we shall define the product of IF-events [9].

Definition 6. *If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then their product $\mathbf{A}.\mathbf{B}$ is defined by the formula*

$$\mathbf{A}.\mathbf{B} = \big(\mu_A.\mu_B, 1 - (1 - \nu_A).(1 - \nu_B)\big) = \big(\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B\big).$$

The next important notion is a notion of a joint IF-observable and its existence (see [11]).

Definition 7. *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be two IF-observables. The joint IF-observable of the IF-observables x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h(R^2) = (1, 0)$, $h(\emptyset) = (0, 1)$;
(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) \oplus h(B)$ *and* $h(A) \odot h(B) = (0, 1)$;
(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$;
(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$.

Theorem 4. *For each two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their joint IF-observable.*

Proof. Put $x(A) = (x^\flat(A), 1 - x^\sharp(A)), y(B) = (y^\flat(B), 1 - y^\sharp(B))$. We want to construct $h(K) = (h^\flat(K), 1 - h^\sharp(K))$. Fix $\omega \in \Omega$ and put

$$\mu(A) = x^\flat(A)(\omega),$$
$$\nu(B) = y^\flat(B)(\omega).$$

It is not difficult to prove that $\mu, \nu : \mathcal{B}(R) \to [0, 1]$ are probability measures. Let $\mu \times \nu : \mathcal{B}(R^2) \to [0, 1]$ be the product of measures and define

$$h^\flat(K)(\omega) = \mu \times \nu(K).$$

Then $h^\flat : \mathcal{B}(R^2) \to \mathcal{T}$, where $\mathcal{T}$ is the family of all $\mathcal{S}$-measurable functions from Ω to [0,1]. If $C, D \in \mathcal{B}(R)$, then

$$h^\flat(C \times D)(\omega) = \mu \times \nu(C \times D) = \mu(C) \cdot \nu(D) = x^\flat(C)(\omega) \cdot y^\flat(D)(\omega),$$

hence

$$h^\flat(C \times D) = x^\flat(C) \cdot y^\flat(D).$$

Similarly $h^\sharp : \mathcal{B}(R^2) \to \mathcal{T}$ can be constructed such that

$$h^\sharp(C \times D) = x^\sharp(C) \cdot y^\sharp(D).$$

Put

$$h(A) = (h^\flat(A), 1 - h^\sharp(A)),$$

for $A \in \mathcal{B}(R^2)$.

Then we have for $C, D \in \mathcal{B}(R)$

$$\begin{aligned} x(C) \cdot y(D) &= (x^\flat(C), 1 - x^\sharp(C)) \cdot (y^\flat(D), 1 - y^\sharp(D)) \\ &= \Big(x^\flat(C) \cdot y^\flat(D), 1 - \big(1 - (1 - x^\sharp(C))\big) \cdot \big(1 - (1 - y^\sharp(D))\big)\Big) \\ &= (x^\flat(C) \cdot y^\flat(D), 1 - x^\sharp(C) \cdot y^\sharp(D)) \\ &= (h^\flat(C \times D), 1 - h^\sharp(C \times D)) = h(C \times D). \end{aligned}$$

Definition 8. *Let* $\mathbf{m}$ *be an IF-state. IF-observables* $x_1, x_2, ...x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *are independent if for* n*-dimensional IF-observable* $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ *there holds*

$$\mathbf{m}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \mathbf{m}\big(x_2(A_2)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, A_2, ..., A_n \in \mathcal{B}(R)$.

Theorem 5. *Let* R^N *be the set of all sequences* $(t_i)_i$ *of real numbers. Let* $(x_n)_n$ *be a sequence of independent IF-observables in* $(\mathcal{F}, \mathbf{m})$ *with the same IF-distribution function. Define for each* $n \in N$ *the mapping* $\xi_n : R^N \longrightarrow R$ *by the formula*

$$\xi_n((t_i)_i) = t_n.$$

Then $(\xi_n)_n$ *is a sequence of independent random variables in a space* $(R^N, \sigma(\mathcal{C}), P)$*. If there exists* $\mathbf{E}(x_n)$ *then* $E(\xi_n) = \mathbf{E}(x_n)$*. If there exists* $\mathbf{D}^2(x_n)$ *then* $D^2(\xi_n) = \mathbf{D}^2(x_n)$*.*

Proof. **Notation:** A set $C \subset R^N$ is called a cylinder, if there exists $n \in N$, and $D \in \mathcal{B}(R^n)$ such that

$$C = \{(t_i)_i : (t_1, ..., t_n) \in D\}.$$

By $\mathcal{C}$ we shall denote the family of all cylinders in R^N, by $\sigma(\mathcal{C})$ the σ-algebra generated by $\mathcal{C}$.

Construction: Consider the measurable space $(R^N, \sigma(\mathcal{C}))$ a sequence $(x_n)_n$ of independent IF-observables $x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ (i.e. $x_1, \ldots, x_n$ are independent for each $n \in N$), and the states $\mathbf{m}_n : \mathcal{B}(R^n) \longrightarrow [0,1]$ defined by

$$\mathbf{m}_n(B) = \mathbf{m}(h_n(B))$$

for each $B \in \mathcal{B}(R^n)$.

The states $\mathbf{m}_n$ are consisting, i.e.

$$\begin{aligned}\mathbf{m}_{n+1}(B \times R) &= \mathbf{m}\big(h_{n+1}(B \times R)\big) = (\mathbf{m} \circ h_{n+1})(B \times R) \\ &= (\mathbf{m}_{x_1} \times \ldots \times \mathbf{m}_{x_n} \times \mathbf{m}_{x_{n+1}})(B \times R) \\ &= \mathbf{m}\big(h_n(B)\big) \cdot \mathbf{m}\big(x(R)\big) = \mathbf{m}\big(h_n(B)\big) \cdot 1 = \mathbf{m}_n(B)\end{aligned}$$

for each $B \in \mathcal{B}(R^n)$.

Therefore by the Kolmogorov consistency theorem [13] there exists the probability measure $P : \sigma(\mathcal{C}) \longrightarrow [0,1]$ such that

$$P\big(\pi_n^{-1}(B)\big) = \mathbf{m}_n(B) = \mathbf{m}\big(h_n(B)\big)$$

for each $B \in \mathcal{C}$, where $\mathcal{C}$ is the family of all cylinders in R^N and $\pi_n : R^N \to R^n$ is a projection defined by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$.

Let $n \in N$, $A_1, ..., A_n \in \mathcal{B}(R)$. Then

$$\begin{aligned}
P\big(\xi_1^{-1}(A_1) \cap \ldots \cap \xi_n^{-1}(A_n)\big) &= P\big(\{(t_i)_1^{\infty} : t_i \in A_i, i = 1, 2, \ldots, n\}\big)\\
&= P\big(\pi_n^{-1}(A_1 \times \ldots \times A_n)\big)\\
&= \mathbf{m}\big(h_n(A_1 \times \ldots \times A_n)\big)\\
&= \mathbf{m}\big(x_1(A_1)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)\\
&= P\big(\pi_{\{1\}}^{-1}(A_1)\big) \cdot \ldots \cdot P\big(\pi_{\{n\}}^{-1}(A_n)\big)\\
&= P\big(\xi_1^{-1}(A_1)\big) \cdot \ldots \cdot P\big(\xi_n^{-1}(A_n)\big).
\end{aligned}$$

Let $\mathbf{F} : R \longrightarrow [0,1]$ be the IF-distribution function of IF-observables x_n, $G : R \longrightarrow [0,1]$ be the distribution function of random variables ξ_n. Then

$$\begin{aligned}
G(t) &= P\big(\xi_n^{-1}((-\infty, t))\big) = P\big(\pi_n^{-1}(R \times \ldots \times R \times (-\infty, t))\big)\\
&= \mathbf{m}\big(h_n(R \times \ ... \ \times R \times (-\infty, t))\big) = \mathbf{m}\big(x_n((-\infty, t))\big) = \mathbf{F}(t).
\end{aligned}$$

If there exists IF-mean value $\mathbf{E}(x_n)$, then

$$\mathbf{E}(x_n) = \int_R t \, d\mathbf{F}(t) = \int_R t \, dG(t) = E(\xi_n).$$

Similarly the equality $D^2(\xi_n) = \mathbf{D}^2(x_n)$ can be proved.

We need the notion of convergence IF-observables yet (see [8]).

Definition 9. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be independent IF-observables and $g_n : R^n \to R$ be a Borel measurable function. Then the IF-observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1}$$

where $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ is the n-dimensional IF-observable (joint IF-observable of $x_1, \ldots, x_n$).

Example 2. Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be independent IF-observables and $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be their joint IF-observable. Then

1. the IF-observable $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)$ is defined by the equality
$$y_n = h_n \circ g_n^{-1},$$
where $g_n(u_1, \ldots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} u_i - a\Big)$;
2. the IF-observable $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ is defined by the equality
$$y_n = h_n \circ g_n^{-1},$$
where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i$;

3. the IF-observable $y_n = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mathbf{E}(x_i))$ is defined by the equality
$$y_n = h_n \circ g_n^{-1},$$
where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum_{i=1}^{n}(u_i - \mathbf{E}(x_i))$;
4. the IF-observable $y_n = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big)$ is defined by the equality
$$y_n = h_n \circ g_n^{-1},$$
where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big)$.

Definition 10. *Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(y_n)_n$ converges in distribution to a function $\Psi : R \longrightarrow [0, 1]$, if for each $t \in R$*
$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\infty, t))\big) = \Psi(t).$$

4 Basic Theorem from the Extreme Value Theory

The next notions of the extreme value theory on real numbers we can find in works [3–7].

Let $X_1, X_2, ...$ be independent, equally distributed random variables of real numbers with a distribution function $F : R \to R$ defined by
$$F(x) = P(X_i < x),\ (i = 1, 2, ...),$$
where $x \in R$. Denote M_n maximum of n random variables
$$M_1 = X_1,\ M_n = \max(X_1, ..., X_n),$$
for $n \geq 2$.

Theorem 6 (Fisher-Tippett-Gnedenko). *Let $X_1, X_2, ...$ be a sequence of independent, equally distributed random variables. If there exists the sequences of real constant $a_n > 0$, b_n and a non-degenerate distribution function H, such that*
$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} < x\right) = H(x),$$
then H is the distribution function one of the following three types of distributions:

1. Gumbel
$$H_{\mu,\sigma}(x) = \exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right),\ \ x \in \mathbb{R},$$

2. Fréchet

$$H_{\mu,\sigma}(x) = \begin{cases} 0, & for\, x \le \mu, \\ \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right), & for\, x > \mu, \alpha > 0, \end{cases}$$

3. Weibull

$$H_{\mu,\sigma}(x) = \begin{cases} \exp\left(-\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{-\alpha}\right), & for\, x < \mu, \alpha < 0, \\ 1, & for\, x \ge \mu. \end{cases}$$

A parameter $\mu \in R$ is the **location parameter** and a parameter $\sigma > 0$ is the **scale parameter**. A parameter α is called the **shape parameter**.

Gumbel, Fréchet and Weiboll distribution from Theorem 6 we can write with using a generalized distribution of extreme values

$$H_{\mu,\sigma,\varepsilon}(x) = \begin{cases} \exp\left[-\left(1+\varepsilon\left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{\varepsilon}}\right], & 1+\varepsilon\left(\frac{x-\mu}{\sigma}\right) > 0, \varepsilon \ne 0, \\ exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right), & x \in \mathbb{R}, \varepsilon = 0. \end{cases}$$

Now we return to the IF-case. Let $x_1, x_2, ...$ be an independent, equally distributed IF-observables on $\mathcal{F}$. Denote $\mathbf{M}_n$ maximum of n IF-observables

$$\mathbf{M}_1 = x_1,\ \mathbf{M}_n = \max(x_1, ..., x_n),$$

for $n \ge 2$.

Theorem 7 (Fisher-Tippett-Gnedenko). *Let $x_1, x_2, ...$ be a sequence of independent, equally distributed IF-observables such that $\mathbf{D}^2(x_n) = \sigma^2, \mathbf{E}(x_n) = a, (n = 1, 2, \ldots)$. If there exists the sequences of real constant $a_n > 0$, b_n and a non-degenerate distribution function H, such that*

$$\lim_{n\to\infty} \mathbf{m}\left(\frac{1}{a_n}\left(\mathbf{M}_n - b_n\right)\left((-\infty, t)\right)\right) = H(t),$$

then H is the distribution function one of the following three types of distributions:

1. Gumbel

$$H_{\mu,\sigma}(t) = \exp\left(-e^{-\left(\frac{t-\mu}{\sigma}\right)}\right), \quad t \in \mathbb{R},$$

2. Fréchet

$$H_{\mu,\sigma}(t) = \begin{cases} 0, & for\, t \le \mu, \\ \exp\left(-\left(\frac{t-\mu}{\sigma}\right)^{-\alpha}\right), & for\, t > \mu, \alpha > 0, \end{cases}$$

3. Weibull

$$H_{\mu,\sigma}(t) = \begin{cases} \exp\left(-\left(-\left(\frac{t-\mu}{\sigma}\right)\right)^{-\alpha}\right), & for\, t < \mu, \alpha < 0, \\ 1, & for\, t \ge \mu. \end{cases}$$

Proof. For each $n = 1, 2, 3, \ldots$ let the Borel function $g_n : R^n \longrightarrow R$ be given by

$$g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big).$$

Let further the IF-observable $y_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be given by stipulation

$$y_n = h_n \circ g_n^{-1} = g_n(x_1, \ldots, x_n) = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big).$$

Consider the measure space $(R^N, \sigma(\mathcal{C}), P)$ and random variables

$$\xi_n((t_i)_i) = t_n, (n = 1, 2, ...).$$

Then by Theorem 5 the random variables ξ_n are independent. Moreover,

$$E(\xi_n) = \mathbf{E}(x_n) = a,\ D^2(\xi_n) = \mathbf{D}^2(x_n) = \sigma^2.$$

Therefore by the classical Fisher-Tippett-Gnedenko Theorem 6 we have

$$\lim_{n\to\infty} P\bigg(\Big\{(u_i)_1^\infty\,;\ \frac{1}{a_n}\bigg(\max\big(\xi_1((u_i)_1^\infty), \ldots, \xi_n((u_i)_1^\infty)\big) - b_n\bigg) < t\Big\}\bigg) = H(t).$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \quad & \mathbf{m}\bigg(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\bigg) = \mathbf{m}\big(y_n((-\infty, t))\big) \\
&= \lim_{n\to\infty} \mathbf{m}\big(h_n\big(g_n^{-1}((-\infty, t))\big)\big) = \lim_{n\to\infty} P\big(\pi_n^{-1}\big(g_n^{-1}((-\infty, t))\big)\big) \\
&= \lim_{n\to\infty} P\Big(\Big\{(u_i)_1^\infty; g_n\Big(\xi_1((u_i)_1^\infty), \ldots, \xi_n((u_i)_1^\infty)\Big) \in (-\infty, t)\Big\}\Big) \\
&= \lim_{n\to\infty} P\bigg(\Big\{(u_i)_1^\infty\,;\ \frac{1}{a_n}\bigg(\max\big(\xi_1((u_i)_1^\infty), \ldots, \xi_n((u_i)_1^\infty)\big) - b_n\bigg) < t\Big\}\bigg) \\
&= H(t).
\end{aligned}$$

5 Conclusion

We have proved a very important assertion of mathematical statistics for IF-observables in IF-theory. Evidently the results can be applied also to fuzzy sets theory. On the other hand families of IF-events may be embedded to suitable MV-algebras. Therefore it would be useful to try to extend the Fisher-Tippett-Gnedenko theorem to probability MV-algebras.

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Fuzzy Approaches in Forecasting Mortality Rates

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Abstract. Fundamental issues in the study of mortality rate modelling are goodness of fit and the quality of forecasts. These are still open questions despite the fact that dozens of mortality models have been formulated. Capturing all mortality patterns remains elusive for the proposed models. Nevertheless, there are models with better and worse abilities to explain historical mortality rates and to project accurate forecasts. This paper considers two fuzzy approaches for forecasting future mortality rates. First, the fuzzy autoregressive integrated moving average (ARIMA) method allows the making of fuzzy forecasts based on crisp estimates of mortality model parameters. Second, the fuzzy Lee-Carter method models past mortality rates as fuzzy numbers, and then allows the prediction of future fuzzy mortality rates. Numerical findings show that both methods may be useful tools for forecasting.

Keywords: Mortality rate · Lee-Carter model · Fuzzy ARIMA · Mortality forecasting

1 The Problem of Mortality Forecasting

1.1 Introduction

During the last two centuries developed countries have experienced a persistent increase in life expectancy. This increase, though a sign of social progress, poses a challenge for governments, private pension plans and life insurers because of its impact on pension and health costs. For example, in Poland life expectancy at birth for men has risen from 63 years in 1958 to 73.5 years in 2014. The increase for women was even more impressive during this period, from 68.5 years to 81.5 years.

Actuaries and demographers have recognised the problems caused by an ageing population and rising longevity and have thus devoted significant attention to the development of statistical techniques for the modeling and projection of mortality rates. The basic modeled variable is the central death rate. For a given population or cohort, the central death rate at age x during a given period of one year, is found by dividing the number of people, who died after they had reached the exact age x but before they reached the exact age $x + 1$, by the average number who were living in that age group during the period. Data are at the

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heart of mortality modelling, which is why it is important to understanding the real nature of the data used. Raw population data require various adjustments before being used as inputs for the demographic database. The most common adjustments are:

- to distribute persons of unknown age across the age range in proportion to the number of observed individuals, in each age group,
- to split data into finer or to aggregate into wider age categories,
- to smooth the observed values in order to obtain an improved representation of the given demographic ratio.

All of the above make data imprecise and so modelling using fuzzy methods is justified.

1.2 The Lee-Carter Method

One of the most influential approaches to the stochastic modeling of mortality rates is the mortality model proposed by Lee and Carter [9]. This model uses principal component analysis to decompose the age-time matrix of mortality rates into a bilinear combination of age and period parameters, with the latter being treated as time series to produce mortality projections. The mortality model for the central death rate $m_{x,t}$ is:

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}, \tag{1}$$

where: $m_{x,t}$ represents the matrix of the central death rates at age x in year t, α_x is a static age function capturing the general shape of mortality by age, κ_t is a time dependent parameter, an index of the level of mortality at time t, β_x age dependent parameter, is the relative speed of change at each age, $\varepsilon_{x,t}$ is an error term. The error term is expected to be Gaussian, $\varepsilon_{x,t} \sim N(0, \sigma_\varepsilon)$. Many empirical studies show that this requirement is often violated. To obtain a unique solution to (1) the following constraints are imposed:

$$\alpha_x = \frac{1}{T} \sum_{t=1}^{T} \ln(m_{x,t}), \tag{2}$$

$$\sum_{t=1}^{T} \kappa_t = 0, \tag{3}$$

$$\sum_{x=x_1}^{x_N} \beta_x = 1, \tag{4}$$

For forecasting mortality, the κ_t is projected into the future using $ARIMA$ (autoregressive integrated moving average) time series methods. Lee and Carter assume that α_x and β_x remains constant over time. Because of the linearity of κ_t, it is generally modeled as a random walk with trend. However several $ARIMA$

specification can be used. The Lee-Carter model has inspired numerous variants and extensions (see: [2,3]). Koissi and Shapiro [8] proposed a fuzzy approach for the Lee-Carter model. The advantage of the fuzzy approach is that the errors are viewed as fuzziness of the model structure. However, one of the disadvantages is the computational complexity. In particular, when we use standard fuzzy algebra we get non-differential functions to optimize.

1.3 Contribution

An imprecise forecast is better than a precise false forecast. Thus we test two approaches for fuzzy forecasting of mortality rates. First we apply the extended fuzzy $ARIMA$ method for forecast the κ parameter model and then we make predictions using it with the standard age specific coefficient of the Lee-Carter model. The second forecasting methods is the extended version of the fuzzy Lee-Carter model by Koissi and Shapiro proposed in [10]. The modification involves replacing the fuzzy numbers with oriented fuzzy numbers to obtain differentiated objective functions during looking for parameters. Thus the aim of this study is to investigate two different fuzzy approaches to morality rates' projections: the fuzzy Lee-Carter model [8] extended by [10] and the fuzzy $ARIMA$ model [11] extended by adding the fuzzy constant coefficient applied for forecasting in the Lee-Carter model. We test the efficiency of these two approaches efficiency in terms of the quality of forecasts.

The structure of the paper is as follows. In the next section, the fuzzy $ARIMA$ and fuzzy Lee-Carter models are described. Two ways of incorporating fuzziness in the Lee-Carter model are presented. In Sect. 3 an application for mortality data forecasting in Poland is made. The last section is a summary of the study.

2 Mortality Rate Projection Methods

2.1 Fuzzy *ARIMA* Method

The $ARIMA$ method presented by Box and Jenkins [1] is a class of statistical model for analyzing and forecasting time series data. $ARIMA$ generalizes the simpler autoregressive moving average and adds the notion of integration. A time series Z_t can be presented by the $ARIMA(p, d, q)$ process [4], if

$$\phi(B)(1-B)^d Z_t = c + \theta(B)a_t, \tag{5}$$

where

- $t = 1, 2, \ldots, k$,
- $\phi(B) = 1-\phi_1 B-\phi_2 B^2-\ldots-\phi_P B^P$ and $\theta(B) = 1-\theta_1 B-\theta_2 B^2-\ldots-\theta_q B^q$ are polynomials in B of degree p and q,
- c is a constant,
- B is the backward shift operator,

- p of $AR(p)$ is the integer derived by PACF (the partial autocorrelation function),
- q of $MA(q)$ is the integer derived by ACF (the autocorrelation function).

It is assumed that a_t are independent and identically distributed as normal random variables with mean 0 and variance σ^2, and the roots of $\phi(Z) = 0$ and $\theta(Z) = 0$ all lie outside the unit circle.

In [11] instead of using crisp fuzzy parameters $\tilde{\phi}_1, \ldots, \tilde{\phi}_p$ and $\tilde{\theta}_1, \ldots, \tilde{\theta}_q$ in the form of symmetric triangular fuzzy numbers are used. The algorithm can be described by following three steps:

1. Fitting the $ARIMA(p, d, q)$ with nonfuzzy input data to find crisp coefficients c, $\phi = \phi_1, \phi_2, \ldots, \phi_p$ and $\theta = \theta_1, \theta_2, \ldots, \theta_q$.
2. Determining the spreads $s = s_0, s_1, s_2, \ldots, s_{p+q}$ by the minimal fuzziness criterion (6).

$$\begin{gathered}
\sum_{i=1}^{p} \sum_{t=1}^{k} s_i |\xi_i| |W_{t-i}| + \sum_{i=p+1}^{p+q} \sum_{t=1}^{k} s_i |\rho_{i-p}| |a_{t+p-i}| + s_0 \rightarrow minimize \\
s.t.: \sum_{i=1}^{p} \phi_i W_{t-i} + a_t - \sum_{i=1}^{q} \theta_i a_{t-i} + \\
(1-h)(\sum_{i=1}^{p} s_i |W_{t-i}| + \sum_{i=p+1}^{p+q} s_i |a_{t+p-i}| + s_0) \geq W_t, \\
\sum_{i=1}^{p} \phi_i W_{t-i} + a_t - \sum_{i=1}^{q} \theta_i a_{t-i} - \\
(1-h)(\sum_{i=1}^{p} s_i |W_{t-i}| + \sum_{i=p+1}^{p+q} s_i |a_{t+p-i}| + s_0) \leq W_t, \\
t = 1, 2, \ldots, k, \\
s_i \geq 0 \; for \; all \; i = 0, 1, 2, \ldots, p+q,
\end{gathered} \tag{6}$$

   where:
   - $W_t = (1-B)^d Z_t$,
   - ξ_i is the partial autocorrelation coefficient of time lag i,
   - ρ_i is the autocorrelation coefficient of time lag i.
3. Optional phase 3: deleting the data around the model's upper boundary and lower boundary when the fuzzy $ARIMA$ model has outliers with a wide spread and back to step 1. In this case, this step can not be applied because unobservable variable κ_t is forecasted and the removal of outliers would be based on a forecast of mortality rate which is also affected by α_x and β_x.

2.2 Fuzzy Lee-Carter Model

The fuzzy formulation of the LC model [8] is:

$$\tilde{Y}_{x,t} = \tilde{A}_x \oplus \tilde{B}_x \odot \tilde{K}_t \tag{7}$$

for $x = x_1, x_2, \ldots, x_N$, $t = t_1, t_1 + 1, \ldots, T$, where

– $\tilde{Y}_{x,t} = (y_{x,t}, e_{x,t})$ are known fuzzy log-mortality rate of age group x at time t, presented by a symmetrical triangular fuzzy number,[1]
– $\tilde{A}_x = (\alpha_{x,t}, s_{\alpha_x})$ and $\tilde{B}_x = (\beta_x, s_{\beta_x})$ are the unknown fuzzy age-specific parameters,
– $\tilde{K}_t = (\kappa_t, s_{\kappa_t})$ is the unknown fuzzy time-variant mortality index.

Fuzzyfication. Koissi and Shapiro [8] used a fuzzy least-squares regression based on the minimum fuzziness criterion to fuzzify the crisp log-central death rates. It was emphasized that for simplicity, they used symmetric fuzzy numbers. Fuzzification of the given log-central death rates $y_{x,t}$ for age x in year t, comes down to finding the symmetric triangular fuzzy number (STFN): $Y_{x,t} = (y_{x,t}, e_{x,t})$, where $y_{x,t}$ is the central value and $e_{x,t}$ is the unknown spread. To find $e_{x,t}$ Koissi and Shapiro introduced STFN $C_0 = (c_{0x}, s_{0x}), C_1 = (c_{1x}, s_{1x})$, such that:

$$(y_{x,t}, e_{x,t}) = (c_{0x}, s_{0x}) \oplus (c_{1x}, s_{1x}) \odot t \tag{8}$$

for each age-group x. Based on fuzzy arithmetic operations [2] it is equivalent to:

$$y_{x,t} = c_0 x + c_1 xt, \; e_{x,t} = \max(s_{0x}, s_{1x} t). \tag{9}$$

In the following part, according to [10], the STFN are replaced by the Ordered Fuzzy Numbers. The Ordered Fuzzy Number (OFN), introduced by [7], is an ordered pair of two continuous functions $\overrightarrow{A} = (f_A, g_A)$, where f_A and g_A are called the up-part and the down-part, respectively, both defined on the closed interval $[0, 1]$ with values in R. Thus STFN $(y_{x,t}, e_{x,t})$, (c_{0x}, s_{0x}),(c_{1x}, s_{1x}) are replaced by their counterparts: $\overrightarrow{y_{x,t}} = (f_{Y_{x,t}}, g_{Y_{x,t}})$, $\overrightarrow{C_{0x}} = (f_{C_{0x}}, g_{C_{0x}})$, $\overrightarrow{C_{1x}} = (f_{C_{1x}}, g_{C_{1x}})$, where:

– $f_{Y_{x,t}}(z) = y_{x,t} - e_{x,t}(1-z)$,
– $g_{Y_{x,t}}(z) = y_{x,t} + e_{x,t}(1-z)$,
– $f_{C_{0x}}(z) = c_{0x} - s_{0x}(1-z)$,
– $g_{C_{0x}}(z) = c_{0x} + s_{0x}(1-z)$,
– $f_{C_{1x}}(z) = c_{1x} - s_{1x}(1-z)$,
– $g_{C_{1x}}(z) = c_{1x} + s_{1x}(1-z)$,
– $z \in [0, 1]$.

Therefore condition (9) can be expressed as follows:

$$(f_{Y_{x,t}}, g_{Y_{x,t}}) = (f_{C_{0x}}, g_{C_{0x}}) \oplus (f_{C_{1x}}, g_{C_{1x}}) \odot t, \tag{10}$$

for each age-group x. By definition of addition and multiplication of OFN it results that the following equations should be satisfied:

[1] we use notation $\tilde{A} = (\alpha, a)$, where α is the center and a is the spread of fuzzy number.

[2] Koissi and Shapiro used definition of multiplication and addition proposed by Hong [5] and Kolesarova [6] respectively.

$$
\begin{aligned}
&(y_{x,t} - e_{x,t}(1-z),\ y_{x,t} + e_{x,t}(1-z)) \\
&= (f_{C_{0x}}(z) + t f_{C_{1x}}(z),\ g_{C_{0x}}(z) + t g_{C_{1x}}(z)) \\
&= (c_{0x} + c_{1x}t - (s_{0x} + s_{1x}t)(1-z),\ c_{0x} + c_{1x}t + (s_{0x} + s_{1x}t)(1-z))
\end{aligned} \tag{11}
$$

for each age-group x, and $z \in [0,1]$. It means that:

$$y_{x,t} = c_{0x} + c_{1x}t, \tag{12}$$

$$e_{x,t} = s_{0x} + s_{1x}t, \tag{13}$$

$t = 1, 2, \ldots, T$.

With respect to (12) ordinary least-squares regression is used to find the centre values c_{0x} and c_{1x}. For each age-group x in the year t the estimates:

$$\hat{c}_{1x} = \frac{\bar{ty}_{x,t} - \bar{t}\,\bar{y}_{x,t}}{\bar{t^2} - \bar{t}^2} \tag{14}$$

$$\hat{c}_{0x} = \bar{y}_{x,t} - \hat{c}_{1x}\bar{t} \tag{15}$$

where $\bar{ty}_{x,t}, \bar{y}_{x,t}, \bar{t^2}, \bar{t}^2$ denotes the arithmetic mean of appropriate expressions.

Next, the spreads s_{0x} and s_{1x} are obtained by using the minimum fuzziness criterion (it is impossible to use least-squares regression, because the left side of Eq. (13) is unknown). Lets note that from the assumption $e_{x,t}$ are non-negative numbers. The smallest value is 0. Therefore, the minimum fuzziness criterion leads to $\hat{s}_{0x}$, $\hat{s}_{1x}$ which minimize the sum:

$$\sum_{t=1}^{T} e_{x,t} = T s_{0x} + s_{1x} \sum_{t=1}^{T} t, \tag{16}$$

subject to the constraints:

- $s_{0x}, s_{1x} \geq 0$,
- $\hat{c}_{0x} + \hat{c}_{1x}t + (s_{0x} + s_{1x}t)(1-z) \geq y_{x,t}$,
- $\hat{c}_{0x} + \hat{c}_{1x}t - (s_{0x} + s_{1x}t)(1-z) \leq y_{x,t}$

for each age x and for a fixed $z \in [0,1)$. Because the greater values of z lead to greater spreads, it is assumed in the sequel that $z = 0$. The criterion above is analogous to the criterion of Koissi and Shapiro. The difference concerns the manner of calculating spreads.

Estimating Fuzzy-LC. The search for the parameters is to minimize the Diamond distance between $\tilde{Y}_{x,t} = \tilde{A}_{x,t} \oplus \tilde{B}_{x,t} \odot \tilde{K}_{x,t}$ and $Y_{x,t}$. Thus for OFN, we have:

$$\sum_{x=0}^{x_N} \sum_{t_1}^{T} D^2(\overrightarrow{A}_x \oplus \overrightarrow{B}_x \odot \overrightarrow{K}_t, \overrightarrow{Y}_{x,t}) \rightarrow minimize \tag{17}$$

where

$$\sum_{x=0}^{x_N}\sum_{t_1}^{T} D^2(\overrightarrow{A}_x \oplus \overrightarrow{B}_x \odot \overrightarrow{K}_t, \overrightarrow{Y}_{x,t})$$
$$= \int_0^1 (f_{A_x}(z) + f_{B_x \odot K_t}(z) - f_{Y_{x,t}}(z))^2 dz \tag{18}$$
$$+ \int_0^1 (g_{A_x}(z) + g_{B_x \odot K_t}(z) - g_{Y_{x,t}}(z))^2 dz$$

One way of solution (17) is deriving the partial derivatives of (18) with respect to α_x, β_x, $\kappa(t)$, s_{α_x}, s_{β_x}, $s_{\kappa(t)}$, and next equaling them to 0.

2.3 Forecasting

In the first approach the algorithm for forecasting the morality rate for M years is the following:

1. calculate crisp Lee-Carter model coefficients
2. find the fuzzy $ARIMA$ coefficient for parameter κ
3. calculate κ for periods $t = T+1, T+2, ..., T+M$
4. calculate the fuzzy morality rate using the Lee-Carter formula with forecasted κ and coefficients α_x and β_x form step 1.

Despite the large number of parameters, this approach has low computational complexity. In step 1, by fitting the Lee-Carter model, the maximum likelihood estimation or SVD method can be used. Step 2 is reduced to a linear optimization problem.

The second forecasting approach consists of the following steps:

1. calculate crisp Lee-Carter model coefficients
2. fuzzification of log-central death rates
3. minimize the Diamond distance to obtain the fuzzy Lee-Carter model parameters
4. forecast (κ_t, s_{κ_t}) for periods $t = T+1, T+2, ..., T+M$ by linear trend
5. calculate the fuzzy morality rate using the Lee-Carter formula with forecasted (κ_t, s_{κ_t}) and coefficients (α_x, s_{α_x}) and (β_x, s_{β_x}) form step 3.

The second approach is associated with a large increase in computational complexity, since we have twice as many parameters to set.

3 The Empirical Study

The empirical example is conducted on data about central death rates for Poland[3] covering the years 1958–2014 and ages 0–100 (see Fig. 1). The sample is divided in training sub-sample - years 1958–2004 and test sub-sample - years 2005–2014. All calculations were performed separately for women and men.

[3] Source: Human Mortality Database http://www.mortality.org/.

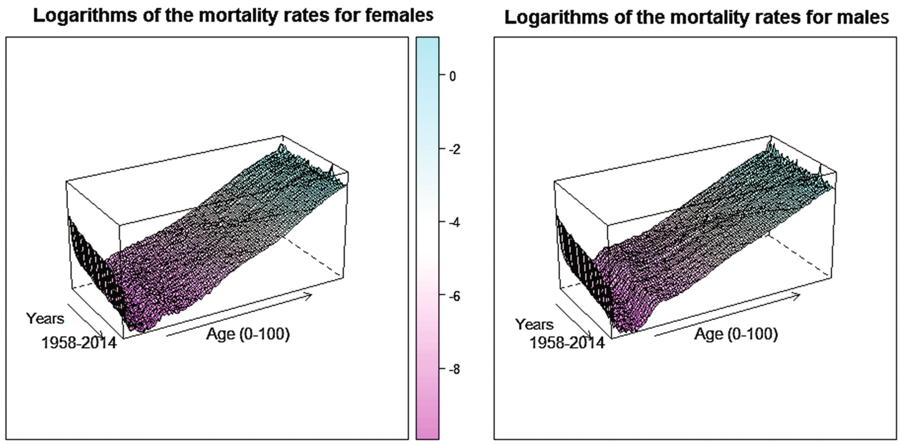

Fig. 1. Logarithms of central death rates for Poland in 1958–2014 and ages 0–100

The calculations were made in the R environment using packages *forecast* for $ARIMA$ fitting and *pracma* for solving equations. Using the revised fuzzy $ARIMA$ model (case $fARIMA$) and parameters from the fuzzy Lee-Carter method (case fLC), we forecast the future value of the κ_t (see Table 1) and then forecast mortality for the next 10 years. Results for 0, 20, 40 and 60-year-olds are presented in Figs. 2 and 3. A black vertical line on each chart separates fitted models from forecasts. As we can see, most problems with models fitting and forecasting are generated by subpopulations between the ages of 40 and 60. This is especially evident for men, for the above-mentioned ages are significantly different from the actual values of the mortality rates. The considered models do not cope with the change in mortality patterns that occurred in Poland in

Table 1. Projections of $\tilde{\kappa}$

	crisp LC		fARIMA		fLC	
Year	Females	Males	Females	Males	Females	Males
2005	−49.51	−22.58	(−51.14,−47.88)	(−23.94,−21.23)	(−31.98,−31.41)	(−13.71,−13.11)
2006	−49.74	−22.91	(−51.37,−48.11)	(−24.27,−21.56)	(−33.31,−32.73)	(−14.27,−13.67)
2007	−49.79	−23.03	(−51.42,−48.16)	(−24.39,−21.67)	(−34.63,−34.05)	(−14.83,−14.22)
2008	−49.80	−23.07	(−51.43,−48.17)	(−24.43,−21.71)	(−35.95,−35.37)	(−15.40,−14.78)
2009	−49.80	−23.09	(−51.43,−48.17)	(−24.45,−21.73)	(−37.28,−36.68)	(−15.96,−15.33)
2010	−49.80	−23.09	(−51.43,−48.17)	(−24.45,−21.73)	(−38.60,−38.00)	(−16.52,−15.89)
2011	−49.80	−23.09	(−51.43,−48.17)	(−24.45,−21.74)	(−39.93,−39.32)	(−17.09,−16.44)
2012	−49.80	−23.10	(−51.43,−48.17)	(−24.45,−21.74)	(−41.25,−40.64)	(−17.65,−17.00)
2013	−49.80	−23.10	(−51.43,−48.17)	(−24.45,−21.74)	(−42.57,−41.95)	(−18.21,−17.55)
2014	−49.80	−23.10	(−51.43,−48.17)	(−24.45,−21.74)	(−43.90,−43.27)	(−18.77,−18.12)

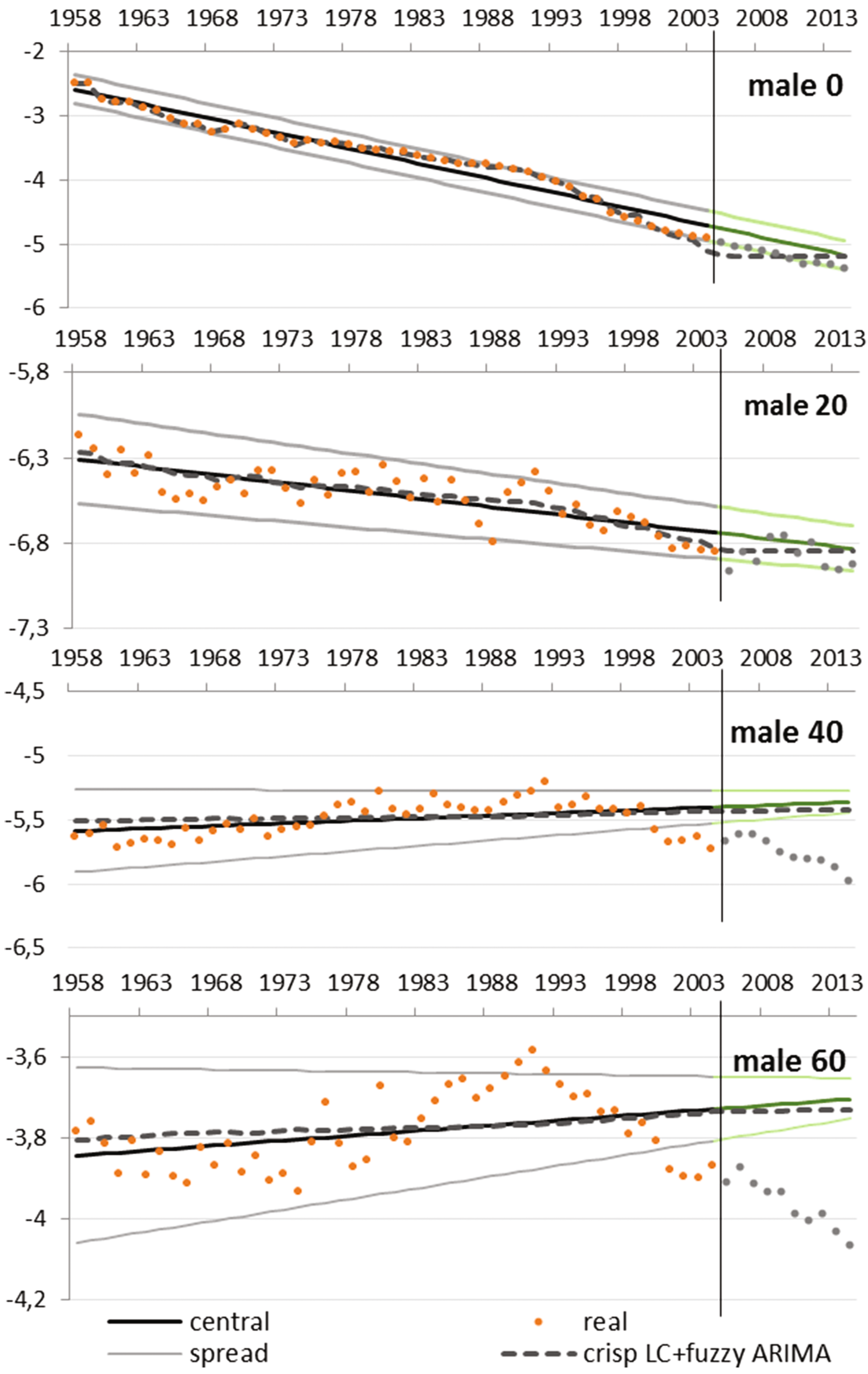

Fig. 2. Results of estimation and prediction for males

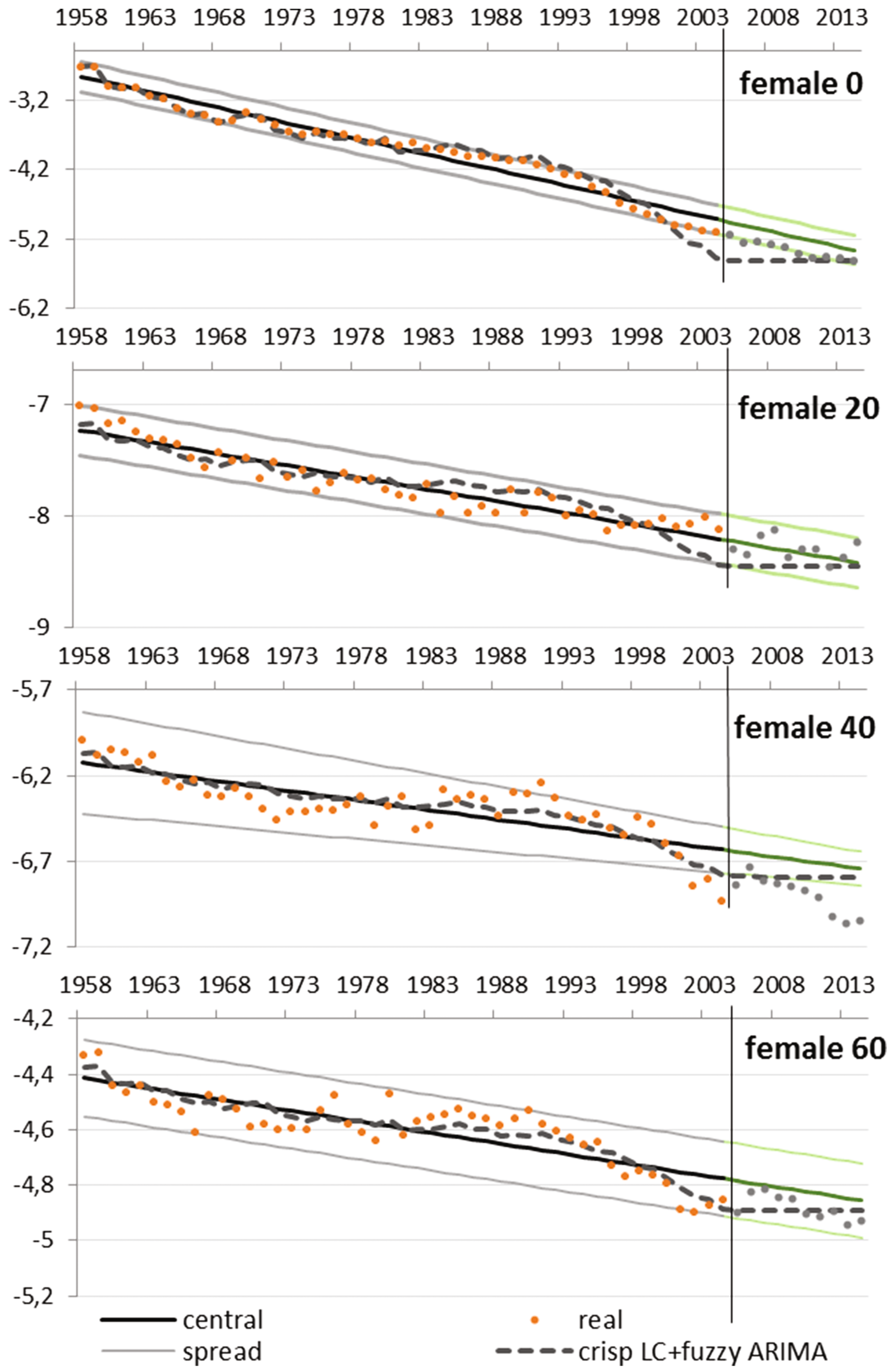

Fig. 3. Results of estimation and prediction for females

Table 2. Forecast errors

	MAE	MSE	RMSE	MAPE
Males				
fLC	0.239856	0.07623	0.276098	6.294489
fARIMA	**0.200425**	**0.055833**	**0.2316**	**5.035917**
Females				
fLC	0.161778	**0.036453**	**0.190927**	4.34994
fARIMA	**0.147042**	0.037737	0.2419	**3.205806**

the 1990s. Let's have a look at the forecasts for a 20-year-old females (Fig. 3, second panel). Because our forecasts are in the sample there are real values of the mortality rate (grey dots). The forecast obtained from crisp $ARIMA$ is a dashed line drawn. As STFN is used, central values got from fuzzy $ARIMA$ overlap crisp $ARIMA$. Finally, the forecast obtained from fuzzy Lee-Carter (as a linear trend) is a solid line. In this case, the real values are in the fuzziness area. For example, the forecast for year 2014 from crisp $ARIMA$ equals -8.46, and from fuzzy Lee-Carter we have an interval $(-8.64, -8.19)$, while the real value is equal to -8.23. For practical reasons, the logarithms of central mortality rates are modeled, but on this basis one can calculate the annual probability of dying or the annual probability of living, which can be used to calculate other demographic ratios.

The prediction errors, calculated for centre values of the fuzzy forecast, are presented in Table 2. All actual values fall within the fuzzy ARIMA predictive support, and 26.34% of fLC forecasts for men and for women 52.67%. However, the average forecast's spread for fLC is 0.1837 for females and 0.1638 for males. For fARIMA it is 1.6291 and 1.3582, respectively, so more than eight times bigger.

As we mentioned in the introduction, demographic data, of their nature, are imprecise, which in our opinion requires a fuzzy approach in modelling and forecasting. A comparison of crisp and two fuzzy methods for forecasting is shown in Table 3.

Table 3. Comparison of standard and fuzzy forcasting of mortality rates

LC + ARIMA	LC + fARIMA	fuzzy LC
Lack of vague	Crisp LC parameters and vague forecast	Vague parameters and forecast
Crisp numbers	Crisp and fuzzy numbers	OFN
Provide a confidence interval of at least 50 and preferably 100 observations or more	Provide a possibility interval less than ARIMA	Provide a possibility interval less than ARIMA
Homoscedasticity assumption	Homoscedasticity assumption	Lack of assumption

4 Conclusions

In the paper we test the prediction value of two fuzzy approaches for modelling mortality rates. The next natural step is checking models' performance for other data sets - different countries with different lengths of time series. The results for Poland have shown the fuzzy $ARIMA$ model to perform usually better than the fuzzy Lee-Carter model regarding the forecast accuracy. However, too wide support of fuzzy forecasting means there is no practical business use. Moreover, when using the fuzzy $ARIMA$ model, we do not avoid the difficult-to-meet assumption of the Lee-Carter model. The reason for the weaker results of the fuzzy Lee Carter model may be the linearity of the model, which we have already introduced by fuzzification steps. Therefore, in further investigations we plan to propose a new way of fuzzification for this model.

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Non-denoting Terms in Fuzzy Logic: An Initial Exploration

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Abstract. We introduce two variants of first-order fuzzy logic that can deal with non-denoting terms, or terms that lack existing referents, e.g., Pegasus, the current king of France, the largest number, or 0/0. Logics designed for this purpose in the classical setting are known as free logics. In this paper we discuss the features of free logics and select the options best suited for fuzzification, deciding on the so-called dual-domain semantics for positive free logic with truth-value gaps and outer quantifiers. We fuzzify the latter semantics in two levels of generality, first with a crisp and subsequently with a fuzzy predicate of existence. To accommodate truth-valueless statements about nonexistent objects, we employ a recently proposed first-order partial fuzzy logic with a single undefined truth value. Combining the dual-domain semantics with partial fuzzy logic, we define several kinds of 'inner-domain' quantifiers, relativized by the predicate of existence. Finally, we make a few observations on some of the resulting rules of free fuzzy quantification that illustrate the differences between the two proposed systems of free fuzzy logic and their well known non-free or non-fuzzy variants.

Keywords: Quantifier · Free logic · Existence · Referent · Partial fuzzy logic

1 Introduction

In both formal and natural languages there are terms with no existing referents. Classical examples include 1/0, $\sum_{n=0}^{\infty}(-1)^n$, the largest natural number, Pegasus, or the current king of France. In the classical setting, dealing with such non-denoting terms falls under the domain of *free logics,* or 'logics free of existential assumptions' [4–7]. Free logics differ from classical logic mainly in the conditional validity of certain inference rules for quantifiers. These differences ensue from the modifications free logics make to the classical first-order semantics in order to accommodate terms that either have no referents at all, or have referents that fall outside the domain of existential and universal quantification. Free logics find numerous applications in the logical analysis of natural language, esp. the theory of definite descriptions, temporal and fictional

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discourse, modal logics with non-constant domains (where possible worlds can differ in existent objects), computer science (for dealing with null objects and unassigned variables), or some areas of mathematics (algebra, foundations) and philosophy [5–7].

Non-denoting terms or terms denoting nonexistent objects can, obviously, be encountered in fuzzy contexts just like in crisp contexts, e.g., when a fuzzy property is predicated of a nonexistent object or in fuzzy definite or indefinite descriptions. However, like classical logic, known systems of predicate fuzzy logic all assume that each term in the language is evaluated within the domain of quantification, and so has an existent referent. To our knowledge, no attempt at developing free fuzzy logic has yet been undertaken.

This paper aims neither at providing a definite solution to the problem of handling non-denoting terms or nonexistent objects in fuzzy contexts, nor at deriving deep mathematical results on free quantification in fuzzy logic. Rather we make the first exploration into the landscape of viable variants of free fuzzy logic, pointing out some possible desiderata and design choices, and hint at a few features in which free fuzzy logic may differ from its non-free or non-fuzzy variants.

Possible applications of free quantification in fuzzy logic are envisaged wherever non-denoting terms might be encountered in fuzzy contexts, which includes fuzzy descriptions, fuzzy temporal, fictional, or modal discourse, as well as various fuzzy methods of computer science and engineering where variables may happen to lack referents. Naturally, these applications can only be developed after the sketched systems of free fuzzy logic are elaborated in more detail. Such an elaboration is a topic for future work.

2 Non-denoting Terms in the Classical Setting

As mentioned in Sect. 1, the treatment of non-denoting terms and nonexistent objects in the crisp setting is the domain of free logics. There are several variants of free logics known from the literature, which differ in various design choices for their semantics [5–7]. In this section, we review the main available options for the semantics of crisp free logics and justify the choice of one of them as our starting point for generalization to the fuzzy setting.

In free logics, singular terms may lack referents in the domain of quantification. Most variants of free logic contain the (primitive or defined) unary *existence predicate,* traditionally denoted by $E!$, where the atomic formula $E!t$ expresses the fact that the singular term t has a referent in the domain of quantification. Besides other things, $E!$ enables an explicit expression of existential presuppositions in inferences.

There are three main families of free logics, which differ in the way they assign truth values to empty-termed atomic formulae (i.e., atomic formulae containing terms that lack referents in the domain of quantification):

- *Negative:* All empty-termed atomic formulae are considered false.

- *Positive:* Some empty-termed atomic formulae can be true.
- *Neutral:* All empty-termed atomic formulae not of the form $E!t$ are considered truth-valueless.

A further distinction regards how non-denoting terms themselves are handled in the semantics. One option is to use a single domain D of referents; the Tarski conditions then need be modified to allow singular terms to have no value in D. Another option is the so-called *dual-domain semantics*. Here, models have two domains: the *outer domain* $D_0 \neq \emptyset$ and the *inner domain* $D_1 \subseteq D_0$. In D_1, which is the range of quantification, existent objects are collected. Singular terms with non-existing referents are assigned the elements of $D_0 \setminus D_1$. In the positive dual-domain semantics, the extensions of predicates can include objects from $D_0 \setminus D_1$; this makes it possible to assign truth values to claims about nonexistent objects (e.g., that Zeus $\neq$ Pegasus or that unicorns are animals). The appeal of the dual-domain semantics lies in its closeness to the classical semantics: since every singular term has a referent in D_0, there is no need to use some non-standard way of evaluation of empty-termed atomic formulae. The dual-domain semantics is also convenient for accommodating objects that exist in possible worlds other than the actual world: in modal logics with non-constant domains, each world w comes with its own inner domain (of objects existing in w), which is a subset of a common outer domain.

An attractive option in the positive dual-domain framework is to take the so-called *outer quantifiers,* which range over the outer domain D_0, as primitive. Since all terms have referents in D_0, these quantifiers behave as the standard quantifiers of classical first-order logic. The *inner quantifiers* (ranging over D_1) are then simply restrictions of the outer quantifiers to the inner domain (delimited by the existence predicate). In particular, if we denote the outer quantifiers by $\exists^0, \forall^0$, then the inner quantifiers $\exists^1, \forall^1$ are defined as

$$(\exists^1 x)\varphi \equiv_{\mathrm{df}} (\exists^0 x)(E!x \,\&\, \varphi) \tag{1}$$

$$(\forall^1 x)\varphi \equiv_{\mathrm{df}} (\forall^0 x)(E!x \rightarrow \varphi). \tag{2}$$

The ordinary meaning of the expressions "some" and "all" corresponds to *inner* quantification (over existing objects). The outer quantifiers, apart from their technical role in the semantics, are nevertheless useful in certain specific contexts: for instance, the statement "Some things do not exist", which is not straightforwardly formalizable by means of classical or inner quantification, can be expressed by the formula $(\exists^0 x)\neg E!x$. Since $D_1, \exists^1, \forall^1$ are definable from the (classically behaving) $D_0, \exists^0, \forall^0$ and $E!$, free logic with outer quantifiers is essentially the classical logic of restricted quantification.

More details on free logics can be found in [4–7]. It remains to decide which variant(s) from the rich landscape of free logics are best suited for generalization to fuzzy contexts.

As has been observed in the literature [6,7], each of the main variants (positive, negative, and neutral) comes with some problems. Neutral free logics tend to be rather weak; also, intuitively it seems strange for statements like "Zeus = Zeus" to lack a truth value (or be false, as in negative free logics). In negative free logics, the truth values of empty-termed formulae depend on the choice of primitive predicates. In bivalent positive free logics, we are often forced to assign truth values to empty-termed formulae without any clear reason.

Therefore, for our enterprise we favor a *non-bivalent* variant of *positive* free logic, which has also been studied in the literature (see [7]) and seems most flexible compared to alternatives. In non-bivalent positive free logics, some empty-termed propositions (such as "1/0 is prime") may lack truth values, while others (such as "Zeus = Zeus") can be true and yet others (such as "Zeus = Pegasus") false. The truth-value gaps, needed in non-bivalent positive semantics, can conveniently be handled within the framework of partial fuzzy logic, recently proposed in [2,3]. Since single-domain semantics require a non-standard evaluation of singular terms (and can anyway be emulated by a dual-domain semantics with a single element in $D_0 \setminus D_1$), our choice for fuzzification is that of *dual-domain non-bivalent positive* free logic.

3 Partial Fuzzy Logic

Partial fuzzy logic, suitable for dealing with truth-valueless propositions occurring in positive free fuzzy logic, has been proposed in a propositional form in [3] and extended to a first-order variant in [2]. It represents truth value gaps by an additional truth value $*$, added to the real unit interval $[0, 1]$ or another algebra $\boldsymbol{L}$ of truth degrees of an underlying fuzzy logic L. The underlying fuzzy logic L can be any implicative expansion of the logic $\text{MTL}_\triangle$ (i.e., an expansion of $\text{MTL}_\triangle$ where every connective is congruent w.r.t. fully true bi-implication), for instance, $\text{Ł}_\triangle$, $\text{BL}_\triangle$, ŁΠ, etc. For more information on these logics see, e.g., [1]; we assume the reader's familiarity with at least one such fuzzy logic, both propositional and first-order.

The semantics of the propositional partial fuzzy logic L^* based on the fuzzy logic L is defined as follows (for additional details see [3]):

- The primitive *propositional language* of L^* contains:
  - For each propositional connective c of L, the ('Bochvar-style') connective c_B of the same arity
  - The truth constant $*$ (representing an undefined truth degree)
  - The unary connective ! (for the crisp indicator of definedness)
  - The binary connective $\wedge_\text{K}$ (for 'Kleene-style' min-conjunction).
- The *intended algebras* of truth values for L^* are defined as expansions of the algebras for L by a dummy element $*$ (to be assigned to propositions with undefined truth). In the intended L^*-algebra $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$, for $\boldsymbol{L}$ an L-algebra, the connectives of L^* are interpreted by the following truth tables for each unary connective u_B, binary connective c_B (and similarly for higher arities), $\alpha, \beta \in \boldsymbol{L}$ and $\gamma, \delta \in \boldsymbol{L} \setminus \{0\}$:

	$!$
α	1
$*$	0

	u_{B}
α	$u\alpha$
$*$	$*$

c_{B}	β	$*$
α	$\alpha\, c\, \beta$	$*$
$*$	$*$	$*$

$\wedge_{\mathrm{K}}$	0	δ	$*$
0	0	0	0
γ	0	$\gamma \wedge \delta$	$*$
$*$	0	$*$	$*$

(3)

– The *tautologies* of L^* are defined as those L^*-formulae that are evaluated to 1 under all evaluations in all intended L^*-algebras. *Entailment* in L^* is defined as the transmission of the value 1 under all evaluations in all intended L^*-algebras. As usual, we write $\models \varphi$ if φ is a tautology of L^*, and $\Gamma \models \varphi$ if the set Γ of L^*-formulae entails the L^*-formula φ in L^*.

The primitive connectives of L^* make a broad class of derived connectives available in L^*. Besides the primitive *Bochvar-style* connectives c_{B}, which treat $*$ as the absorbing element, the following two important families of connectives are definable in L^*:

– The *Sobociński-style* connectives $c_{\mathrm{S}} \in \{\wedge_{\mathrm{S}}, \vee_{\mathrm{S}}, \&_{\mathrm{S}}\}$, which treat $*$ as the neutral element; and the Sobociński-style implication $\rightarrow_{\mathrm{S}}$, associated with $\&_{\mathrm{S}}$ via the residuation axiom $x \rightarrow_{\mathrm{S}} (y \rightarrow_{\mathrm{S}} z) = (x \&_{\mathrm{S}} y) \rightarrow_{\mathrm{S}} z$:

c_{S}	β	$*$
α	$\alpha\, c\, \beta$	α
$*$	β	$*$

$\rightarrow_{\mathrm{S}}$	β	$*$
α	$\alpha \rightarrow \beta$	$\neg\alpha$
$*$	β	$*$

– The *Kleene-style* connectives $c_{\mathrm{K}} \in \{\wedge_{\mathrm{K}}, \vee_{\mathrm{K}}, \&_{\mathrm{K}}, \rightarrow_{\mathrm{K}}\}$, which preserve the neutral and absorbing elements of the corresponding connectives of L, and otherwise are evaluated Bochvar-style. For the primitive connective $\wedge_{\mathrm{K}}$ see (3) above; the others are defined by the following truth tables:

$\&_{\mathrm{K}}$	0	β	$*$
0	0	0	0
α	0	$\alpha \,\&\, \beta$	$*$
$*$	0	$*$	$*$

$\vee_{\mathrm{K}}$	δ	1	$*$
γ	$\gamma \vee \delta$	1	$*$
1	1	1	1
$*$	$*$	1	$*$

$\rightarrow_{\mathrm{K}}$	δ	1	$*$
0	1	1	1
α	$\alpha \rightarrow \delta$	1	$*$
$*$	$*$	1	$*$

(4)

Moreover, several useful auxiliary connectives are definable in L^*, including those with the following truth tables (for $\alpha \in \boldsymbol{L}$ and $\gamma \in \boldsymbol{L} \setminus \{1\}$):

	$?$	$\downarrow$	$\uparrow$
α	0	α	α
$*$	1	0	1

	$\circledast$
γ	0
1	$*$
$*$	0

(5)

For examples of the logical laws governing the connectives of L^* see [3]. The semantics of the first-order extension of L^* introduced in [2] is defined as follows:

Let $\mathcal{L} = (\mathrm{Pred}_\mathcal{L}, \mathrm{Func}_\mathcal{L})$ be a first-order language with a non-empty set $\mathrm{Pred}_\mathcal{L}$ of predicate symbols and a set $\mathrm{Func}_\mathcal{L}$ of function symbols, each with an arity $n \geq 0$ (where predicate symbols of arity 0 are propositional constants and function symbols of arity 0 are object constants). Let Var be a set of object variables.

A *model* for a language $\mathcal{L}$ over an intended Ł*-algebra $\boldsymbol{L}_*$ is given as $\mathbf{M} = \big(D^\mathbf{M}, (P^\mathbf{M})_{P \in \mathrm{Pred}_\mathcal{L}}, (F^\mathbf{M})_{F \in \mathrm{Func}_\mathcal{L}}\big)$, where:

- $D^\mathbf{M}$ is a crisp non-empty set.
- $P^\mathbf{M} \colon (D^\mathbf{M})^n \to \boldsymbol{L}_*$ for each n-ary $P \in \mathrm{Pred}_\mathcal{L}$.
- $F^\mathbf{M} \colon (D^\mathbf{M})^n \to D^\mathbf{M}$ for each n-ary $F \in \mathrm{Func}_\mathcal{L}$.

The semantic values of a formula φ and a term t in a model $\mathbf{M}$ under an evaluation $e \colon \mathrm{Var} \to D^\mathbf{M}$ of object variables will be denoted by $\|\varphi\|^\mathbf{M}_e$ and $\|t\|^\mathbf{M}_e$, respectively. The evaluation that assigns $a \in D^\mathbf{M}$ to x and coincides with e on all other object variables will be denoted by $e[x \mapsto a]$.

The Tarski conditions for terms and atomic formulae are defined as in the first-order fuzzy logic Ł, and for propositional connectives by the truth tables (3) above. The primitive quantifiers $\exists_\mathrm{B}, \forall_\mathrm{B}$ of Ł* are interpreted *Bochvar-style,* i.e., yielding the 'undefined' value $*$ whenever an instance of the quantified formula is undefined:

$$\|(\exists_\mathrm{B} x)\varphi\|^\mathbf{M}_e = \begin{cases} * & \text{if } \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} = * \text{ for some } a \in D^\mathbf{M} \\ \sup_{a \in D^\mathbf{M}} \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} & \text{otherwise} \end{cases}$$

$$\|(\forall_\mathrm{B} x)\varphi\|^\mathbf{M}_e = \begin{cases} * & \text{if } \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} = * \text{ for some } a \in D^\mathbf{M} \\ \inf_{a \in D^\mathbf{M}} \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} & \text{otherwise.} \end{cases}$$

Like in the case of propositional connectives, further variants of universal and existential quantifiers are definable in Ł*, including the following important ones:

- The *Sobociński-style* quantifiers $\exists_\mathrm{S}, \forall_\mathrm{S}$, which ignore the undefined instances of the quantified formula:

$$\|(\exists_\mathrm{S} x)\varphi\|^\mathbf{M}_e = \begin{cases} * & \text{if } \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} = * \text{ for all } a \in D^\mathbf{M} \\ \sup\limits_{\|\varphi\|^\mathbf{M}_{e[x\mapsto a]} \neq *} \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} & \text{otherwise} \end{cases}$$

$$\|(\forall_\mathrm{S} x)\varphi\|^\mathbf{M}_e = \begin{cases} * & \text{if } \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} = * \text{ for all } a \in D^\mathbf{M} \\ \inf\limits_{\|\varphi\|^\mathbf{M}_{e[x\mapsto a]} \neq *} \|\varphi\|^\mathbf{M}_{e[x\mapsto a]} & \text{otherwise.} \end{cases}$$

They can be defined from $\exists_\mathrm{B}, \forall_\mathrm{B}$ by the Ł*-connectives (3)–(5) as follows:

$$(\exists_\mathrm{S} x)\varphi \equiv_\mathrm{df} (\exists_\mathrm{B} x){\downarrow}\varphi \vee_\mathrm{B} \circledast(\forall_\mathrm{B} x)?\varphi \tag{6}$$

$$(\forall_\mathrm{S} x)\varphi \equiv_\mathrm{df} (\forall_\mathrm{B} x){\uparrow}\varphi \vee_\mathrm{B} \circledast(\forall_\mathrm{B} x)?\varphi\ . \tag{7}$$

- The *Kleene-style* quantifiers $\exists_{\mathrm{K}}, \forall_{\mathrm{K}}$, respectively analogous to $\vee_{\mathrm{K}}$ and $\wedge_{\mathrm{K}}$, can be defined as:

$$(\exists_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}} x)\varphi \vee_{\mathrm{K}} (\exists_{\mathrm{S}} x)\varphi \tag{8}$$

$$(\forall_{\mathrm{K}} x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}} x)\varphi \wedge_{\mathrm{K}} (\forall_{\mathrm{S}} x)\varphi\ . \tag{9}$$

As usual, *validity* in a model is defined as truth to degree 1 under all evaluations of object variables in the model; *tautologicity* as validity in all models for the given language; and *entailment* as validity in all models validating all premises. We use the usual notation $\mathbf{M} \models \varphi$ for validity, $\models \varphi$ for tautologicity, and $\Gamma \models \varphi$ for entailment.

Observation 1. It can be easily verified that, e.g., the rule of generalization is sound for all the aforementioned quantifiers: $\varphi \models (\mathrm{Q}x)\varphi$ for $\mathrm{Q} \in \{\forall_{\mathrm{B}}, \forall_{\mathrm{S}}, \forall_{\mathrm{K}}, \exists_{\mathrm{B}}, \exists_{\mathrm{S}}, \exists_{\mathrm{K}}\}$. The rule of specification, on the other hand, only holds for Bochvar and Kleene universal quantifiers: $(\mathrm{Q}x)\varphi \models \varphi$ for $\mathrm{Q} \in \{\forall_{\mathrm{B}}, \forall_{\mathrm{K}}\}$. Sobociński-style universally quantified formulae may only be instantiated with terms that do not make them undefined: $(\forall_{\mathrm{S}} x)\varphi, !\varphi(t/x) \models \varphi(t/x)$.

4 Free Fuzzy Logic with a Crisp Existence Predicate

We have now collected all requisite ingredients to brew the first system of free fuzzy logic. By a design choice justified in Sect. 2, it is going to be a fuzzy variant of positive free logic with a dual-domain semantics admitting undefined truth degrees (represented by the dummy value $*$ of a partial fuzzy logic L^*). We shall start with the simpler case when the existence predicate $E!$ is *bivalent* (i.e., total and crisp). The more general case of a *fuzzy* existence predicate will be discussed later in Sect. 5.

Let L^* be a partial fuzzy logic based on a fuzzy logic L. The semantics for a free variant of L^* will only require a minor modification to the semantics of first-order L^* described in Sect. 3:

Let $\boldsymbol{L}_*$ be an intended L^*-algebra and $\mathcal{L}$ a first-order language as in Sect. 3. A *dual-domain model* for $\mathcal{L}$ over $\boldsymbol{L}_*$ is given as $\mathbf{M} = (D_0^{\mathbf{M}}, D_1^{\mathbf{M}}, (P^{\mathbf{M}})_{P \in \mathrm{Pred}_{\mathcal{L}}}, (F^{\mathbf{M}})_{F \in \mathrm{Func}_{\mathcal{L}}})$, where:

- $D_0^{\mathbf{M}}, D_1^{\mathbf{M}}$ are crisp sets such that $D_1^{\mathbf{M}} \subseteq D_0^{\mathbf{M}} \neq \emptyset$, respectively called the *outer* and *inner domain* of $\mathbf{M}$.

Predicate and function symbols are interpreted over the *outer* domain:

- $P^{\mathbf{M}} \colon (D_0^{\mathbf{M}})^n \to \boldsymbol{L}_*$ for each n-ary $P \in \mathrm{Pred}_{\mathcal{L}}$.
- $F^{\mathbf{M}} \colon (D_0^{\mathbf{M}})^n \to D_0^{\mathbf{M}}$ for each n-ary $F \in \mathrm{Func}_{\mathcal{L}}$.

The Tarski conditions for terms, atomic formulae, and propositional connectives in $\mathbf{M}$ under an evaluation $e \colon \mathrm{Var} \to D_0^{\mathbf{M}}$ are as in Sect. 3. The additional logical predicate symbols $=$ (identity) and $E!$ (existence) are interpreted in $\mathbf{M}$ as follows:

– $E!^{\mathbf{M}}$ indicates membership in the inner domain $D_1^{\mathbf{M}}$:

$$\|E!t\|_e^{\mathbf{M}} = \begin{cases} 1 & \text{if } \|t\|_e^{\mathbf{M}} \in D_1^{\mathbf{M}} \\ 0 & \text{otherwise.} \end{cases}$$

– $=^{\mathbf{M}}$ indicates the identity across the outer domain $D_0^{\mathbf{M}}$:

$$\|t = u\|_e^{\mathbf{M}} = \begin{cases} 1 & \text{if } \|t\|_e^{\mathbf{M}} = \|u\|_e^{\mathbf{M}} \\ 0 & \text{otherwise.} \end{cases}$$

Opting for free logic with outer quantifiers (see Sect. 2), we define the primitive Bochvar-style quantifiers $\forall_{\mathrm{B}}^0, \exists_{\mathrm{B}}^0$ as ranging over the *outer* domain $D_0^{\mathbf{M}}$:

$$\left\|(\exists_{\mathrm{B}}^0 x)\varphi\right\|_e^{\mathbf{M}} = \begin{cases} * & \text{if } \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} = * \text{ for some } a \in D_0^{\mathbf{M}} \\ \sup_{a\in D_0^{\mathbf{M}}} \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} & \text{otherwise} \end{cases}$$

$$\left\|(\forall_{\mathrm{B}}^0 x)\varphi\right\|_e^{\mathbf{M}} = \begin{cases} * & \text{if } \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} = * \text{ for some } a \in D_0^{\mathbf{M}} \\ \inf_{a\in D_0^{\mathbf{M}}} \|\varphi\|_{e[x\mapsto a]}^{\mathbf{M}} & \text{otherwise.} \end{cases}$$

The outer Sobociński and Kleene quantifiers can be defined from $\exists_{\mathrm{B}}^0, \forall_{\mathrm{B}}^0$ as in (6)–(9) of Sect. 3:

$$(\exists_{\mathrm{S}}^0 x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}}^0 x){\downarrow}\varphi \vee_{\mathrm{B}} \circledast(\forall_{\mathrm{B}}^0 x)?\varphi \qquad (\exists_{\mathrm{K}}^0 x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}}^0 x)\varphi \vee_{\mathrm{K}} (\exists_{\mathrm{S}}^0 x)\varphi$$
$$(\forall_{\mathrm{S}}^0 x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}}^0 x){\uparrow}\varphi \vee_{\mathrm{B}} \circledast(\forall_{\mathrm{B}}^0 x)?\varphi \qquad (\forall_{\mathrm{K}}^0 x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}}^0 x)\varphi \wedge_{\mathrm{K}} (\forall_{\mathrm{S}}^0 x)\varphi\ .$$

Analogously to (1) and (2) in Sect. 2, we would like to introduce *inner* (Bochvar-style) quantifiers $\exists_{\mathrm{B}}^1, \forall_{\mathrm{B}}^1$ by restricting the outer quantifiers $\exists_{\mathrm{B}}^0, \forall_{\mathrm{B}}^0$ to the inner domain $D_1^{\mathbf{M}}$ (delimited by $E!^{\mathbf{M}}$). In the partial fuzzy setting, there arises the question as to which of the available conjunctions and implications should be used in (1) and (2) for the relativization of quantifiers. The desired behavior of the inner quantifiers is such that they are only affected by the elements of the inner domain $D_1^{\mathbf{M}}$, i.e., iff $E!x$ evaluates to 1. In (1) and (2) we thus need to use a conjunction & and an implication → such that $0 \,\&\, \alpha = 0$ and $0 \rightarrow \alpha = 1$ (to screen off the elements outside $D_1^{\mathbf{M}}$), while $1 \,\&\, \alpha = \alpha$ and $1 \rightarrow \alpha = \alpha$ (not to affect the values for elements in $D_1^{\mathbf{M}}$), for all $\alpha \in \boldsymbol{L}_*$. This suggests the *Kleene* connectives $\&_{\mathrm{K}}$ and $\rightarrow_{\mathrm{K}}$ (cf. their truth tables (4) in Sect. 3) as the adequate choice for relativization. Therefore we define:

$$(\exists_{\mathrm{B}}^1 x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}}^0 x)(E!x \,\&_{\mathrm{K}}\, \varphi) \tag{10}$$
$$(\forall_{\mathrm{B}}^1 x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}}^0 x)(E!x \rightarrow_{\mathrm{K}} \varphi)\ . \tag{11}$$

The inner Sobociński and Kleene quantifiers can again be defined from $\exists_{\mathrm{B}}^1, \forall_{\mathrm{B}}^1$ just like in (6)–(9) of Sect. 3:

$$(\exists_{\mathrm{S}}^1 x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}}^1 x){\downarrow}\varphi \vee_{\mathrm{B}} \circledast(\forall_{\mathrm{B}}^1 x)?\varphi \qquad (\exists_{\mathrm{K}}^1 x)\varphi \equiv_{\mathrm{df}} (\exists_{\mathrm{B}}^1 x)\varphi \vee_{\mathrm{K}} (\exists_{\mathrm{S}}^1 x)\varphi$$
$$(\forall_{\mathrm{S}}^1 x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}}^1 x){\uparrow}\varphi \vee_{\mathrm{B}} \circledast(\forall_{\mathrm{B}}^1 x)?\varphi \qquad (\forall_{\mathrm{K}}^1 x)\varphi \equiv_{\mathrm{df}} (\forall_{\mathrm{B}}^1 x)\varphi \wedge_{\mathrm{K}} (\forall_{\mathrm{S}}^1 x)\varphi\ .$$

Finally, the notions of validity, tautologicity, and entailment are defined as in Sect. 3. Let us now give some observations on this version of free fuzzy logic.

Observation 2. First, it can be observed that the definition of $=^{\mathbf{M}}$ makes all self-identity statements true to degree 1, thus $\models t = t$ for all terms t.

Secondly, since all terms denote in $D_0^{\mathbf{M}}$, the outer quantifiers behave just like the non-free quantifiers of Sect. 3. Thus, for instance (cf. Sect. 3, Observation 1):

$$\begin{aligned} \varphi &\models (\mathrm{Q}x)\varphi && \text{for } \mathrm{Q} \in \{\forall^0_{\mathrm{B}}, \forall^0_{\mathrm{S}}, \forall^0_{\mathrm{K}}, \exists^0_{\mathrm{B}}, \exists^0_{\mathrm{S}}, \exists^0_{\mathrm{K}}\} \\ (\mathrm{Q}x)\varphi &\models \varphi && \text{for } \mathrm{Q} \in \{\forall^0_{\mathrm{B}}, \forall^0_{\mathrm{K}}\} \\ (\forall^0_{\mathrm{S}}x)\varphi, !\varphi(t/x) &\models \varphi(t/x)\ . \end{aligned} \tag{12}$$

However, the behavior of inner quantifiers, which only range over $D_1^{\mathbf{M}}$, differs. For example, unlike (12), in general $(\forall^1_{\mathrm{B}}x)\varphi \not\models \varphi$, since x can be evaluated outside the inner domain $D_1^{\mathbf{M}}$. The predicate $E!$ makes it possible to indicate the existence assumptions of inner quantification explicitly; for instance, the following rules are sound:

$$\begin{aligned} \varphi(t/x), E!t &\models (\mathrm{Q}x)\varphi && \text{for } \mathrm{Q} \in \{\exists^1_{\mathrm{S}}, \exists^1_{\mathrm{K}}\} \\ (\mathrm{Q}x)\varphi, E!t &\models \varphi(t/x) && \text{for } \mathrm{Q} \in \{\forall^1_{\mathrm{B}}, \forall^1_{\mathrm{K}}\}\ . \end{aligned}$$

For $\exists^1_{\mathrm{B}}, \forall^1_{\mathrm{S}}$, on the other hand, additional definedness assumptions are needed:

$$\begin{aligned} \varphi(t/x), E!t, (\forall^1_{\mathrm{B}}x)!\varphi &\models (\exists^1_{\mathrm{B}}x)\varphi \\ (\forall^1_{\mathrm{S}}x)\varphi, E!t, !\varphi(t/x) &\models \varphi(t/x)\ . \end{aligned}$$

5 Free Fuzzy Logic with a Fuzzy Existence Predicate

In this section, we outline a variant of free fuzzy logic in which the existence predicate $E!$ need not be bivalent as in Sect. 4, but can be fuzzy. In this more general setting, the existence of the referent of a singular term can be a matter of degree. This may be useful, e.g., for modeling definite or indefinite descriptions determined by a fuzzy condition: for instance, the referent of the term *the golden mountain* can be considered to exist in a possible world w to the degree to which the greatest lump of gold in w can be considered a mountain; or the degree of purity of gold in the mountain with the most content of gold in w; or a combination thereof.

The semantics described in Sect. 4 requires just a very minor adjustment in order to admit fuzzy existence. In fact, the only change required is to assume that the inner domain $D_1^{\mathbf{M}}$ is a fuzzy (rather than crisp) subset of the outer domain $D_0^{\mathbf{M}}$. As was already the case in Sect. 4, the existence predicate $E!$ is interpreted by the membership function of $D_1^{\mathbf{M}}$. Thus the only difference to the semantics of Sect. 4 consists in the following clauses:

- $D_1^{\mathbf{M}} \colon D_0^{\mathbf{M}} \to \boldsymbol{L}$.
- $E!^{\mathbf{M}} = D_1^{\mathbf{M}}$.

The Tarski condition for $E!$ thus reads: $\|E!t\|^{\mathbf{M}}_e = D^{\mathbf{M}}_1\bigl(\|t\|^{\mathbf{M}}_e\bigr)$. All the rest of the definitions of Sect. 4, including those of the inner and outer quantifiers, remain in place.

The bivalence of $E!$ (and so the setting of Sect. 4) can easily be enforced by adding the axiom $E!x \vee_{\mathrm{B}} \neg_{\mathrm{B}} E!x$, or equivalently, $(\forall^0_{\mathrm{B}} x)(E!x \vee_{\mathrm{B}} \neg_{\mathrm{B}} E!x)$. Note that using instead the axiom $(\forall^0_{\mathrm{S}} x)(E!x \vee_{\mathrm{B}} \neg_{\mathrm{B}} E!x)$ would enforce a crisp, but possibly not totally defined predicate of existence. The question whether a partial $E!$ is meaningful, i.e., whether we may want to admit referents whose existence has no truth value (i.e., is objectively undefined, rather than just unknown), is left aside here for space reasons.

Observation 3. Obviously, the fuzziness of $E!$ does not affect the behavior of the outer quantifiers, which remains the same as in Sects. 3 and 4. What differs is the behavior of the inner quantifiers, due to the relativization to a fuzzy rather than crisp inner domain in their definition; cf. (10) and (11) in Sect. 4. For example, the following rule is sound if $E!$ is crisp, but fails in general for fuzzy $E!$:

$$!\varphi, !\psi \models (\forall^1_{\mathrm{B}} x)(\varphi \to_{\mathrm{B}} \psi) \to_{\mathrm{B}} \bigl((\forall^1_{\mathrm{B}} x)\varphi \to_{\mathrm{B}} (\forall^1_{\mathrm{B}} x)\psi\bigr)\,. \tag{13}$$

In our present setting of Sect. 5, the rule (13) only holds if $E!$ is contractive, i.e., with the additional premise $E!x \to_{\mathrm{B}} (E!x \mathbin{\&_{\mathrm{B}}} E!x)$. (So in particular, it does hold if the underlying fuzzy logic L is Gödel or if $E!$ is crisp.)

As seen in Observation 3, the main culprit of the failure of (13), as well as many other rules for inner quantifiers, is the non-contractivity of fuzzy existence claims; i.e., the fact that $E!t$ is in general weaker than $E!t \mathbin{\&_{\mathrm{B}}} E!t$. Taking the non-contractivity of conjunction into account, we can obtain a more fine-grained analysis of the valid rules for inner quantifiers. Let us introduce the following notation:

$$\begin{aligned} \varphi^0 &= 1 \\ \varphi^{n+1} &= \varphi^n \mathbin{\&_{\mathrm{B}}} \varphi \\ \varphi^{\triangle} &= \triangle_{\mathrm{B}}\varphi\,. \end{aligned}$$

Then we can define the *inner quantifiers of grade n*, for $n \in \mathbb{N} \cup \{\triangle\}$, as follows:

$$\begin{aligned} (\exists^n_{\mathrm{B}} x)\varphi &\equiv_{\mathrm{df}} (\exists^0_{\mathrm{B}} x)\bigl((E!x)^n \mathbin{\&_{\mathrm{K}}} \varphi\bigr) \\ (\forall^n_{\mathrm{B}} x)\varphi &\equiv_{\mathrm{df}} (\forall^0_{\mathrm{B}} x)\bigl((E!x)^n \to_{\mathrm{K}} \varphi\bigr)\,. \end{aligned}$$

For $n \le 1$, the definition yields the usual outer and inner quantifiers, or the quantifiers respectively relativized to the outer and inner domain. The n-grade inner quantifiers can be viewed as relativized to the n-grade inner domain $D^{\mathbf{M}}_n$, defined as the fuzzy extension of the n-times iterated existence predicate:

$$D^{\mathbf{M}}_n(a) = \|(E!x)^n\|^{\mathbf{M}}_{e[x \mapsto a]}$$

for each $a \in D^{\mathbf{M}}_0$. Higher-grade inner domains are more restrictive for the existence degrees of referents: in terms of inclusion of fuzzy sets,

$$D^{\mathbf{M}}_{\triangle} \subseteq \cdots \subseteq D^{\mathbf{M}}_2 \subseteq D^{\mathbf{M}}_1 \subseteq D^{\mathbf{M}}_0\,.$$

Consequently, higher-grade existential quantifiers are stronger and higher-grade universal quantifiers weaker than lesser-grade ones. Since the strictest inner domain $D^{\mathbf{M}}_{\triangle}$ is bivalent, the $\triangle$-grade inner quantifiers $\exists^{\triangle}_{\mathrm{B}}, \forall^{\triangle}_{\mathrm{B}}$ behave like the inner quantifiers of Sect. 4.

The stratified hierarchy of inner quantifiers makes it possible to formulate sound versions of the rule (13) of Observation 3, as well as many other contraction-sensitive rules, even for a fuzzy (non-contractive) predicate of existence:

Observation 4. In the present setting, the following modifications of the rule (13) are sound for any $m, n \geq 0$:

$$!\varphi, !\psi \models (\forall^{m}_{\mathrm{B}} x)(\varphi \to_{\mathrm{B}} \psi) \to_{\mathrm{B}} \big((\forall^{n}_{\mathrm{B}} x)\varphi \to_{\mathrm{B}} (\forall^{m+n}_{\mathrm{B}} x)\psi\big)$$
$$!\varphi, !\psi \models (\forall^{\triangle}_{\mathrm{B}} x)(\varphi \to_{\mathrm{B}} \psi) \to_{\mathrm{B}} \big((\forall^{\triangle}_{\mathrm{B}} x)\varphi \to_{\mathrm{B}} (\forall^{\triangle}_{\mathrm{B}} x)\psi\big).$$

A detailed investigation of the two variants of free fuzzy logic outlined in Sects. 4 and 5, including an axiomatic treatment, is left for future work.

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On the Preservation of an Equivalence Relation Between Fuzzy Subgroups

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Abstract. Two fuzzy subgroups μ, η of a group G are said to be equivalent if they have the same family of level set subgroups. Although it is well known that given two fuzzy subgroups μ, η of a group G their maximum is not always a fuzzy subgroup, it is clear that the maximum of two equivalent fuzzy subgroups is a fuzzy subgroup. We prove that the composition of two equivalent fuzzy subgroups by means of an aggregation function is again a fuzzy subgroup. Moreover, we prove that if two equivalent subgroups have the *sup* property their corresponding compositions by any aggregation function also have the *sup* property. Finally, we characterize the aggregation functions such that when applied to two equivalent fuzzy subgroups, the obtained fuzzy subgroup is equivalent to both of them. These results extend the particular results given by Jain for the maximum and the minimum of two fuzzy subgroups.

Keywords: t-norm · t-conorm · Aggregation function · Fuzzy subgroup · Level fuzzy subset · Strong level fuzzy subset · Sup property · Equivalent fuzzy subgroups

1 Introduction

In the early eighties, Das [3] defined the level subgroup of a fuzzy subgroup μ of a group G as the classical subset $\mu_t = \{x \in G \mid \mu(x) \geq t\}$ for each $t \in [0, 1]$. This notion was a useful tool to develop and formulate many results in fuzzy set theory and their applications. Murali and Makamba [9] worked on fuzzy subgroups with finite images and on the equivalence of two fuzzy subgroups of a finite group G. Jain [6] worked with a weaker equivalence relation and obtained results under the condition of sup property, which is a generalization of the finite range property. The equivalence relation defined by Jain is the following: Let G be a group and μ, η fuzzy subgroups, we say μ and η are equivalent ($\mu \approx \eta$) if

$$\{\mu_t\}_{t \in \operatorname{Im} \mu} = \{\eta_s\}_{s \in \operatorname{Im} \eta}$$

where $\operatorname{Im} \mu$ is the range of μ.

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The purpose of aggregation functions is to combine inputs that are typically interpreted as degrees of membership in fuzzy sets, degrees of preference, strength of evidence, or support of a hypothesis, and so on [1]. A map $F : [0,1]^n \longrightarrow [0,1]$ is an aggregation function if satisfies:

1. If $x_i \leq y_i$ for all $i \in \{1,...,n\}$, then $F(x_1,...,x_n) \leq f(y_1,...,y_n)$. (Monotony)
2. $F(0,...,0) = 0$ and $F(1,....,1) = 1$ (Boundary conditions)

We extend some results which involve minimum and maximum operations to other results obtained for an aggregation function (hence, also for t-norms and t-conorm). Given a binary aggregation function F and two fuzzy equivalent subgroups μ and η of a group G, we define $F(\mu,\eta)$ as the fuzzy set given by $F(\mu,\eta)(x) = F(\mu(x),\eta(x))$. Our main purpose is to prove the following results.

1. $F(\mu,\eta)$ is again a fuzzy subgroup.
2. If μ and η have the sup property, then $F(\mu,\eta)$ also has the sup property.
3. $F(\mu,\eta)$ is in the same equivalence class of μ and η if and only if F is jointly strictly monotone.

Section 2 is devoted to recall the definitions and preliminary results that we need along this paper. Section 3 is devoted to the three main results detailed above. We end with some conclusions in Sect. 4.

2 Preliminaries

We give in this section a brief review of notions and results which are necessary in the following sections.

A n-ary operation $F : [0,1]^n \longrightarrow [0,1]$ is called a *aggregation function* [2,4] if it fulfills the following conditions:

1. If $x_i \leq y_i$ for all $i \in \{1,...,n\}$, then $F(x_1,...,x_n) \leq f(y_1,...,y_n)$. (Monotony)
2. $F(0,...,0) = 0$ and $F(1,....,1) = 1$ (Boundary conditions)

In the sequel, our aggregation functions will be binary operations, that is, $n = 2$.

Fuzzy set theory was formulated in terms of Zadeh's standard operations of intersection, union and complement. The axiomatic skeleton used for characterizing fuzzy intersection and fuzzy union are known as triangular norms (t-norms) and triangular conorms (t-conorms), respectively. Both notions are particular cases of aggregation function.

A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *t-norm* [5,7,8] if it is an aggregation function and fulfills the following conditions:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in [0,1]$. (Associativity)
2. $a * b = b * a$ for all $a, b \in [0,1]$. (Commutativity)
3. $1 * a = a$ for all $a \in [0,1]$. (1 is the neutral element)

A binary operation $\perp: [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *t-conorm* [5,7] if it is an aggregation function and fulfills the following conditions:

1. $(a \perp b) \perp c = a \perp (b \perp c)$ for all $a, b, c \in [0,1]$. (Associativity)
2. $a \perp b = b \perp a$ for all $a, b \in [0,1]$. (Commutativity)
3. $0 \perp a = a$ for all $a \in [0,1]$. (0 is the neutral element)

Given a t-norm $*$, $\perp_*$ is called the *dual t-conorm* of $*$ if it verifies that for all $x, y \in [0,1]$, $x \perp_* y = 1 - (1-x) * (1-y)$. T-norms and t-conorms are related in the following way.

Proposition 1 ([11]). *A binary operation* $\perp: [0,1] \times [0,1] \longrightarrow [0,1]$ *is a t-conorm if and only if there exists a t-norm* $*$ *such that for all* $x, y \in [0,1]$

$$x \perp y = 1 - (1-x) * (1-y)$$

Reciprocally, a binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a t-norm if and only if there exists a t-conorm* $\perp$ *such that for all* $x, y \in [0,1]$

$$x * y = 1 - (1-x) \perp (1-y)$$

Given an universal set X, the inclusion of fuzzy subsets is given by the pointwise order, that is, $\mu_1 \subset \mu_2$ if $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$ and $\underline{k}$ denotes the constant fuzzy set $\underline{k}(x) = k$ for all $x \in X$. We denote by *supp*μ the set defined by *supp*$\mu = \{x \in X \mid \mu(x) > 0\}$ and φ_a denotes the singleton a, that is,

$$\varphi_a(x) = \begin{cases} 1 & if \ x = a \\ 0 & if \ x \neq a \end{cases}$$

We recall the definition of fuzzy subgroup given by Rosenfeld.

Definition 1 ([10]). Let G be a group and μ a fuzzy subset of G. We say that μ is a fuzzy subgroup of G if:

- $(G1)$ $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$, where xy represents the value of the group law of G applied to x and y.
- $(G2)$ $\mu(x) \geq \mu(x^{-1})$ for all $x \in G$, where x^{-1} denotes the inverse element of any x.

Moreover, we have the following facts if μ is a fuzzy subgroup.

1. $\mu(x) = \mu(x^{-1})$
2. $\mu(e) \geq \mu(x)$, where e denotes the identity or neutral element of G.

In the sequel, $L(G)$ will denote the class of all fuzzy subgroup of G. For a fuzzy subset μ of X and $t \in [0,1]$, it is defined the level subset μ_t and the strong level subset $\mu_t^>$ as:

$$\mu_t = \{x \in X \mid \mu(x) \geq t\} \text{ and } \mu_t^> = \{x \in X \mid \mu(x) > t\}$$

The use of level sets is widely used in the context of fuzzy subgroups. If $\mu \in L(G)$, its non-empty level subsets are subgroups of G named level subgroups of μ. A particular case is $\sup \mu = \mu_0^>$. Moreover, we have the following result.

Proposition 2 ([3]). *Let G be a group and μ a fuzzy subset of G, then μ is a fuzzy subgroup of G if and only if its non-empty level subsets (strong level subsets) are subgroups of G.*

Definition 2. *Let G be a group and μ a fuzzy subgroup of G. We say that μ has the sup property if, for each non-empty subset A of G, there exists $a \in A$ such that*

$$\sup_{x \in A}\{\mu(x)\} = \mu(a)$$

Definition 3. *Let G be a group and $\mu, \eta \in L(G)$, then μ is said to be equivalent to η ($\mu \approx \eta$) if the following condition holds*

$$\mu(x) > \mu(y) \textit{ if and only if } \eta(x) > \eta(y)$$

for any $x, y \in G$.

It is easy to check that this defines an equivalence relation and the class of an element $\mu \in L(G)$ is denoted by $[\mu]$. The next characterization of $\approx$ will be used later on.

Proposition 3 ([6]). *Let G be a group and $\mu, \eta \in L(G)$. The following assertions are equivalents:*

1. $\mu(x) > \mu(y)$ *if and only if* $\eta(x) > \eta(y)$.
2. $\mu(x) \geq \mu(y)$ *if and only if* $\eta(x) \geq \eta(y)$.
3. $\{\mu_t\}_{t \in Im\ \mu} = \{\eta_s\}_{s \in Im\ \eta}$.
4. $\{\mu_t^{>}\}_{t \in Im\ \mu} = \{\eta_s^{>}\}_{s \in Im\ \eta}$.

3 The Preservation of the Equivalence Relation $\approx$ Between Fuzzy Subgroups

In the sequel, given a group G, an aggregation function F and $\mu, \eta \in L(G)$, $F(\mu, \eta)$ is defined by $F(\mu, \eta)(x) = F(\mu(x), \eta(x))$ for all $x \in G$. As particular cases, for a t-norm $*$ and a t-conorm $\perp$, $\mu * \eta$ and $\mu \perp \eta$ we have the operators:

$(\mu * \eta)(x) = \mu(x) * \eta(x)$ for all $x \in G$.
$(\mu \perp \eta)(x) = \mu(x) \perp \eta(x)$ for all $x \in G$.

From above definition, F defines a binary operation between fuzzy sets such that

- If $\mu \subset \eta$, then $F(\mu, \nu) \subset F(\eta, \nu)$ for all μ, η, ν fuzzy subgroups.

When the aggregation function is a t-norm $*$, we obtain new properties:

- $\mu * \eta = \eta * \mu$ for all μ, η fuzzy subgroups.
- $\underline{1} * \eta = \eta$ for all η fuzzy subgroup.
- $(\mu * \eta) * \nu = \mu * (\eta * \nu)$ for all μ, η, ν fuzzy subgroups.

Analogous, when the aggregation function is a t-conorm $\perp$, we obtain new properties:

- $\mu \perp \eta = \eta \perp \mu$ for all μ, η fuzzy subgroups.
- $\underline{0} \perp \eta = \eta$ for all η fuzzy subgroup.
- $(\mu \perp \eta) \perp \nu = \mu \perp (\eta \perp \nu)$ for all μ, η, ν fuzzy subgroups.

Proposition 4. *Let G be a group, $F : [0,1]^2 \longrightarrow [0,1]$ an aggregation function and $\mu, \eta \in L(G)$ with $\mu \approx \eta$, then $F(\mu, \eta)$ is a fuzzy subgroup.*

Proof. It is clear that $F(\mu, \eta)$ is a fuzzy subset of G. Now, using the monotony of F and the definition of fuzzy subgroup we obtain:

- $(G1)$ $F(\mu, \eta)(xy) = F(\mu(xy), \eta(xy)) \geq F(\min\{\mu(x), \mu(y)\}, \min\{\eta(x), \eta(y)\})$. We can consider two cases:

$$\mu(x) \leq \mu(y) \ (1) \qquad \text{or} \qquad \mu(x) > \mu(y) \ (2)$$

(1) Since $\mu \approx \eta$, $\eta(x) \leq \eta(y)$, hence

$$F(\min\{\mu(x), \mu(y)\}, \min\{\eta(x), \eta(y)\}) = F(\mu(x), \eta(x)) = F(\mu, \eta)(x)$$

Therefore, $F(\mu, \eta)(xy) \geq F(\mu, \eta)(x) \geq \min\{F(\mu, \eta)(x), F(\mu, \eta)(y)\}$.
(2) Since $\mu \approx \eta$, $\eta(x) > \eta(y)$, hence

$$F(\min\{\mu(x), \mu(y)\}, \min\{\eta(x), \eta(y)\}) = F(\mu(y), \eta(y)) = F(\mu, \eta)(y)$$

Therefore, $F(\mu, \eta)(xy) \geq F(\mu, \eta)(y) \geq \min\{F(\mu, \eta)(x), F(\mu, \eta)(y)\}$.
- $(G2)$ $F(\mu, \eta)(x) = F(\mu(x), \eta(x)) = F(\mu(x^{-1}), \eta(x^{-1})) = F(\mu, \eta)(x^{-1})$ □

Corollary 1. *Let G be a group, $*$ a t-norm, $\perp$ a t-conorm and μ, η fuzzy subgroup such that $\mu \approx \eta$, then $\mu * \eta$ and $\mu \perp \eta$ are fuzzy subgroups.*

Proposition 5. *Let G be a group and $F : [0,1]^2 \longrightarrow [0,1]$ an aggregation function. If $\mu, \eta \in L(G)$ are such that $\mu \approx \eta$ and both of them have the sup property, then the fuzzy subgroup $F(\mu, \eta)$ has the sup property.*

Proof. By Proposition 4, we know that $F(\mu, \eta) \in L(G)$. Now, we consider a non-empty subset A of the group G and $r, s \in A$ such that $\mu(r) = \sup_{x \in A}\{\mu(x)\}$ and $\eta(s) = \sup_{x \in A}\{\eta(x)\}$ (this is possible because μ and η have the sup property).

By the monotony of F, we have for each $x \in A$

$$F(\mu(x), \eta(x)) \leq F(\sup_{x \in A}\{\mu(x)\}, \sup_{x \in A}\{\eta(x)\}) = F(\mu(r), \eta(s))$$

Thus,

$$\sup_{x \in A}\{F(\mu(x), \eta(x))\} \leq F(\mu(r), \eta(s))$$

Now, we prove $\eta(r) = \eta(s)$. Since r verifies $\mu(r) \geq \mu(x)$ for all $x \in A$, we obtain $\mu(r) \geq \mu(s)$ and from $\mu \approx \eta$, $\eta(r) \geq \eta(s)$. In addition, $\eta(s)$ satisfies $\eta(s) \geq \eta(x)$ for all $x \in A$. Hence, $\eta(r) = \eta(s)$. We have

$$\sup_{x \in A}\{F(\mu(x), \eta(x))\} \leq F(\mu(r), \eta(r))$$

Since $F(\mu(r), \eta(r)) \in \{F(\mu(x), \eta(x)) \mid x \in A\}$ we conclude

$$\sup_{x \in A}\{F(\mu(x), \eta(x))\} = F(\mu(r), \eta(r))$$

Hence, $F(\mu, \eta)$ has the sup property. □

Corollary 2. *Let G be a group, $*$ a t-norm and $\perp$ a t-conorm. If μ, η are elements of $L(G)$ such that $\mu \approx \eta$ and both of them have the sup property, then the fuzzy subgroups $\mu * \eta$ and $\mu \perp \eta$ have the sup property.*

The following example shows that the hypothesis $\mu \approx \eta$ in Proposition 4 can not be deleted.

Example 1. Let $G = \{a, b, ab, e\}$ be the group with four elements such that $a^2 = b^2 = e$ and $ab = ba$, let $*$ be the product t-norm $t * s = ts$ for all $t, s \in [0, 1]$ and let μ and η be the following fuzzy subsets:

$$\begin{array}{rcl} \mu : G & \longrightarrow & [0,1] \\ e & \mapsto & 1 \\ a & \mapsto & 0.5 \\ b & \mapsto & 0.8 \\ ab & \mapsto & 0.5 \end{array} \qquad \begin{array}{rcl} \eta : G & \longrightarrow & [0,1] \\ e & \mapsto & 1 \\ a & \mapsto & 0.9 \\ b & \mapsto & 0.2 \\ ab & \mapsto & 0.2 \end{array}$$

We know by Proposition 2, that μ and η are fuzzy subgroup of G. Moreover, we have

$$(\mu * \eta)(ab) = \mu(ab) * \eta(ab) = \mu(ab)\eta(ab) = 0.5 \cdot 0.2 = 0.1$$

and,

$$\min\{(\mu * \eta)(a), (\mu * \eta)(b)\} = \min\{0.45, 0.4\} = 0.4$$

Hence $(\mu * \eta)(ab) < \min\{(\mu * \eta)(a), (\mu * \eta)(b)\}$. Therefore $\mu * \eta$ is not fuzzy subgroup.

Definition 4 ([2]). *Let $F : [0, 1]^n \longrightarrow [0, 1]$ an aggregation function. F is called jointly strictly monotone if $x_i < y_i$ for all $i \in \{1, ..., n\}$, then*

$$F(x_1, ..., x_n) < F(y_1, ..., y_n)$$

Proposition 6. *Let $*$ be a t-norm, then $*$ is jointly strictly monotone if and only if its dual t-conorm $\perp_*$ is jointly strictly monotone.*

Proof. Suppose that the t-norm $*$ is jointly strictly monotone. Consider $a_1 < a_2$ and $b_1 < b_2$ with $a_1, a_2, b_1, b_2 \in [0,1]$. This implies that $(1 - a_1) > (1 - a_2)$ and $(1 - b_1) > (1 - b_2)$. Since $*$ is jointly strictly monotone, we have

$$(1 - a_1) * (1 - b_1) > (1 - a_2) * (1 - b_2)$$

Therefore,

$$1 - (1 - a_1) * (1 - b_1) < 1 - (1 - a_2) * (1 - b_2)$$

This is, $\perp_*$ is jointly strictly monotone.

The converse is analogous. □

Theorem 1. *Let G be a non-trivial group and $F : [0,1]^2 \longrightarrow [0,1]$ be an aggregation function, then F is a binary operation on $[\mu]$ for every $\mu \in L(G)$, if and only if F is jointly strictly monotone.*

Proof. At first, we suppose F is jointly strictly monotone and we are going to prove that F is a binary operation on $[\mu]$ for every $\mu \in L(G)$. From Proposition 4, we obtain that given two elements μ, η in $[\mu]$, $F(\mu, \eta)$ is a fuzzy subgroup of G. Given $\mu, \eta \in [\mu]$, we need to prove the following assertion:

$$F(\mu, \eta) \approx \mu$$

We begin by proving that if $\mu(x) \geq \mu(y)$, then $F(\mu, \eta)(x) \geq F(\mu, \eta)(y)$. Given $x, y \in G$ such that $\mu(x) \geq \mu(y)$, since $\mu \approx \eta$, $\eta(x) \geq \eta(y)$. By the monotony of F, we have

$$F(\mu, \eta)(x) = F(\mu(x), \eta(x)) \geq F(\mu(y), \eta(y)) = F(\mu, \eta)(y)$$

For the other implication, given $x, y \in G$ with $F(\mu, \eta)(x) \geq F(\mu, \eta)(y)$, we need to prove $\mu(x) \geq \mu(y)$. For contradiction, there exists $x, y \in G$ such that $F(\mu, \eta)(x) \geq F(\mu, \eta)(y)$ and $\mu(x) < \mu(y)$. From $\mu \approx \eta$, $\eta(x) < \eta(y)$. Since F is jointly strictly monotone, we conclude

$$F(\mu(x), \eta(x)) < F(\mu(y), \eta(y))$$

Conversely, if F is not jointly strictly monotone, then there exists a_1, a_2, b_1, b_2 in the interval $[0,1]$ with $a_1 < a_2$ and $b_1 < b_2$ such that $F(a_1, b_1) \geq F(a_2, b_2)$ (by the monotony, we obtain $F(a_1, b_1) = F(a_2, b_2)$). We consider the following fuzzy subset of G:

$$\mu(x) = \begin{cases} a_2 & if\ x = e \\ a_1 & if\ x \neq e \end{cases} \qquad \eta(x) = \begin{cases} b_2 & if\ x = e \\ b_1 & if\ x \neq e \end{cases}$$

We know by Proposition 2 that μ and η are fuzzy subgroup of G. Since $\mu(x) > \mu(y)$ if and only if $\eta(x) > \eta(y)$, we have $\mu \approx \eta$. Finally, we prove that $F(\mu, \eta) \notin [\mu]$. Fix $x \in G$ with $x \neq e$ and suppose $F(\mu, \eta) \in [\mu]$. Since $\mu(e) > \mu(x)$, we have $F(\mu, \eta)(e) > F(\mu, \eta)(x)$, but

$$F(\mu, \eta)(e) = F(\mu(e), \eta(e)) = F(a_2, b_2) = F(a_1, b_1) = F(\mu(x), \eta(x)) = F(\mu, \eta)(x)$$

Therefore $F(\mu, \eta) \notin [\mu]$. □

Corollary 3. *Let G be a non-trivial group, $*$ be a t-norm and $\perp_*$ its dual t-conorm, then $*$ and $\perp_*$ are binary operations on $[\mu]$ for every $\mu \in L(G)$ if and only if $*$ is jointly strictly monotone.*

Notice that there are t-norms (t-conorms) that do not fulfill jointly strictly monotone property as, for instance, the drastic t-norm and the Lukasiewicz t-norm.

Corollary 4. *Let G be a non-trivial group and $F : [0,1]^2 \longrightarrow [0,1]$ be an aggregation function. If F verifies that for every $a_1, a_2, b \in [0,1]$ with $a_1 < a_2$ and $b \neq 0$ we have that $F(a_1, b) < F(a_2, b)$, then F is a binary operation on $[\mu]$ for every $\mu \in L(G)$.*

Proof. Fix x_1, x_2, y_1, y_2 in the interval $[0,1]$ with $x_1 < x_2$ and $y_1 < y_2$. Since $y_1 < y_2$ implies $y_2 \neq 0$, we have $F(x_1, y_2) < F(x_2, y_2)$, and by the monotony of F, we conclude:

$$F(x_1, y_1) \leq F(x_1, y_2) < F(x_2, y_2)$$

Hence F is jointly strictly monotone. By Theorem 1, F is a binary operation on $[\mu]$ for each $\mu \in L(G)$. □

Notice that the product t-norm is an example of aggregation function which satisfies Corollary 4. Moreover, we obtain the following result due to Jain.

Corollary 5 ([6]). *Let G be a non-trivial group, then the minimum t-norm and the maximum t-conorm are two binary operation on $[\mu]$ for every $\mu \in L(G)$.*

4 Concluding Remarks

It is well-known that the maximum of two fuzzy subgroups of a group G is not in general a fuzzy subgroup of G. However, if the fuzzy subgroups are equivalent then the maximum of two fuzzy subgroups so is it.

In this work we have proved that, for an arbitrary aggregation function F and two fuzzy subgroups μ, η of a group G, $F(\mu, \eta)$ given by $F(\mu, \eta)(x) = F(\mu(x), \eta(x))$ is again a fuzzy subgroup if μ is equivalent to η. Moreover we have proved that if two equivalent fuzzy subgroups μ and η have the sup property, then $F(\mu, \eta)$ also has the sup property.

Finally, for two equivalent fuzzy subgroup μ and η, we have characterized when $F(\mu, \eta)$ is in the same equivalence class of μ and η.

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Decision-Making on Flow Control Under Fuzzy Conditions in the Mechanical Transport System

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Abstract. The article deals with the problem of moving flows in mechanical transport systems suitable for prevention or greatly decreasing the probability of emergency situations. The solution is based on minimizing costs during transportation. Routing methods considering the specifics of the MTS are analyzed. It's developed routing algorithm with protective correction of flows with fuzzy temporal variability of adaptation. The algorithm consists in definition and establishment of high value of transportation cost on the particular segment of network on a fuzzy time interval. Methods for determining the parameters of protective correction of flows are studied. A structural diagram of the MTS, considering the protective correction, is presented. The diagram is implemented by introduction an intelligent module into the structure. Module operation feature is the use of case-based reasoning. The example of the implementation of protective correction of flows is given.

Keywords: Mechanical transport system (MTS) · Dynamic routing · Adaptive routing · Protective correction · Case-based reasoning (CBR)

1 Introduction

The mechanical transport system (MTS) represents a network built of elements of two types: conveyors and switches. Conveyors move the load placed on the belt. The switches play the role of network nodes in which the units of cargo are redirected from one conveyor to another. An example of MTS is the baggage handling system at airports. The control system for conveyors and switches is implemented as a local network of industrial PLC controllers (Programmable Logic Controllers) [1]. The controllers are assigned the switch control task of the load direction and the electric drive of the conveyor.

Each unit is provided with a cargo label. This label stores addresses initial and final nodes. These data permit in the intermediate network nodes to determine the direction of the cargo unit transfer, solving the routing problem [2–6]. Due to the fact, how is constructed routing algorithm, depend such important factors as the cost of transportation, the risk of damage or loss of cargo, delivery efficiency [1, 7].

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The problem of transportation management in the MTS can be formulated as follows:

$$\begin{cases} \sum_i (l_i + w_{f_i}) \to min, \\ t < t^*, \\ f_i \subseteq F, \end{cases} \quad (1)$$

where l_i is the transportation cost of each several cargo unit;

w_{f_i} are loss in the event of defect $f_i \subseteq F$;

F is set of possible defects that lead to emergency;

t^* is transportation time limit. The main means of solving the problem (1) is the routing. This is explained by the fact that:

- the transmission path of the cargo unit is not uniquely determined, and the route cost are different in MTS;
- the intensity of the cargo flow on an individual segment determines the possibility of defect occurrence.

Modern MTSs use dynamic routing [8, 9]. It helps to minimize the total cost of transportation. However, emergencies arising from overload segments are the result of an unacceptable increase flow intensity in some parts of the network. In this work, we propose a modification of the method of adaptive routing, allowing to solve this problem.

2 Routing Methods and Flow Control in MTS

The solution of the problem (1), using a fixed routing [10], is possible in the case of complete certainty the behavior of the MTS and outdoor environment. It means high reliability and stability of operational performance of MTS, strict conformance to the schedule of appearance and the completion of cargo flows, the stability properties of the cargo units. MTS in this case is described by a static model network [11], often used for calculations. The practical application of fixed routing is limited to the above conditions.

The fixed routing can be based on the dynamic network model [12]. Routing tables in the nodes are updated on a predetermined schedule. This approach is more adequate to the real situation, when the cargo is unstable and MTS parameters change over time. In this case, the complexity of the synthesis and analysis of dynamic models is much higher [5]. This creates difficulties in solving the problem (1) in real time scale.

Dynamic routing [13] solves problem (1) by adjusting of the routing tables in real time when changing the transport costs on individual segments. Cost is determined by the measured parameters network: direction selection speed, electric drive rod, elapsed time of device, etc. The disadvantage of dynamic routing is the inability to consider the transportation cost of the temporal parameters of traffic flows and properties of cargo units that are not available for the measurement. Because of that, it's possible overload. In addition, the significant role is playing by the lag of the mechanical part of the MTS. Routing table modification is completed much faster than the change in cargo flow value. And output stream may be unacceptably high in nodes, summing streams. The problem should be solved in advance decrease the flow value in dangerous situations.

The closest method of solution to the problem is to attract intelligent control mechanisms [14]. The incompleteness of the data, the time history of cargo flow sources make motivation for the use of intelligent observation over the MTS and predict of the cargo flow behavior. Attraction of expert-observer knowledge allows to generate dynamic routing strategy based on a holistic perception of the outdoor environment and MTS. The disadvantage of this approach is the lack of a protective mechanism of the flow correction to prevent accidents.

3 Protective Mechanism of the Flow Correction

Dynamic routing is implemented by changing the transfer value of individual segments of the MTS and warning about it neighboring nodes. Controller of each node corrects routing tables and send cargo units with the switch in the direction that minimizes the total cost of transportation. It will be observed, that dynamic routing does not control the flows, even though indirectly, affect their value. For instance, the low cost of transportation through the subnet stimulates the growth of the flow. Accordingly, a high value may lead to lower flow value. Since the danger an emergency is directly related to the flow value, we have an idea to use the cost of transportation as security facilities from accidents.

Protective flow correction is an artificial increase the transportation cost through the network segment to reduce the flow value. The parameters of the protective correction element are a pair (C_{s_i}, T_{s_i}), where C_{s_i} is cargo unit cost of transportation by the segment s_i; T_{s_i} is time window, during which the value of the cost is kept. Parameter determination of the protective correction is a non-trivial task, at the decision which found a few uncertainties:

- C_{s_i} should be chosen so as, don't to completely block flow through the segment and at the same time providing real decrease its rate. It requires an analysis of the number of cargo elements, that are in the MTS, the ways of their movement and changes in the rates of the input flows at the time of deciding upon the protective correction;
- T_{s_i} is determined in such a way so as don't to provoke an overload of other segments of the MTS. Figure 1 illustrates the general pattern of selection of value T_{s_i}. The greater its value is, the lower the probability P_{s_i} of occurrence of overload in the segment s_i. However, inevitably increases probability P_{N^-} of occurrence of overload on a subnet N^-, not consisting segment s_i. As follows from the qualitative analysis (Fig. 1), there is a compromise value $T^*_{s_i}$, deviation from which increases the probability of occurrence of accidents. As with the analysis of the parameters it need information about the network load, the response time to changes in the routing tables, variations of the rates of the input flows.

The need to consider the whole situation makes it unlikely the effective use of protective correction as the decentralized management tool of MTS. Means the following: the node controller of the MTS measures the flow rate v_{s_i}. If the threshold is

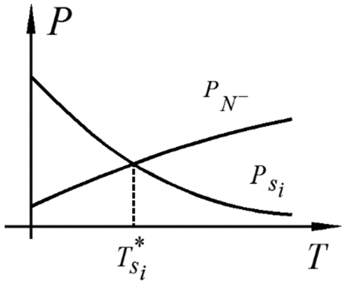

Fig. 1. Illustration of changes in the probability of overload

exceeded a predetermined value $\bar{v}_{s_i}$, then the controller sets protective correction with given parameters (C_{s_i}, T_{s_i}). In this case, there is dangers:

- accidents within the segment s_i due to the fact that the input flow is not reduced immediately, and flow may increase for some time inertia system;
- occurrence of deadlock, when the segment s_i may remove the protective correction, if the input flow rate v_{s_i} decrease. But the input flow rate v_{s_i} cannot be reduced as long as it will not remove the protective correction and the segment does not restore previous capacity.

Thus, we can conclude about the necessity of central determining the parameters of protective correction. Management is implemented in a lack of information about the behavior of the outdoor environment and this MTS, that indicate necessity of application of intelligent principles of management system. Figure 2 shows the structure of the system. Intelligent control module (ICM) is included in the control network PLC. PLC is associated with control devices of MTS. PLC implement dynamic routing algorithm, and can perform the ICM commands on the flow protective correction. The command contains the following fields:

1. timestamp start of protective correction;
2. timestamp end of protective correction;
3. address of output of directional switch;
4. value of the transportation cost to the specified output.

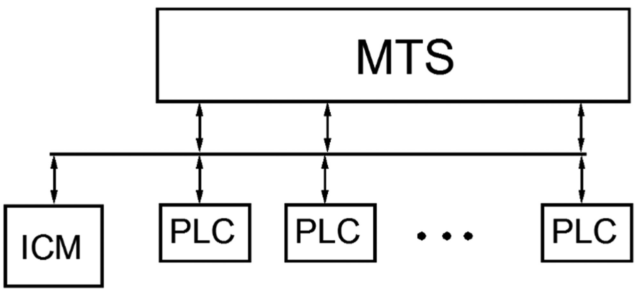

Fig. 2. Illustration of the structure of MTS with an intelligent control module.

The controller, receiving the command, implements the dynamic routing algorithm with fixed transmission cost by the output network segment.

ICM operates based on using the experience of observation of MTS [14]. Experience is represented by describing previously observed dangerous situations and decisions in these situations. The logical inference is based on the case-based reasoning [15]. Every new problem situation is compared with the known to find the nearest in meaning. The solution found situation is applied in the new situation. It should be highlighted that each of the known situations in the ICM knowledge base indicates a potential risk of accidents, i.e. it is forecast. The forecast is not absolutely reliable, so the "intelligence" of the system is manifested in the effort to prevent the occurrence of an abnormal situation and loss recovery.

4 Example of Implementation of Protective Correction Principle

This section examines a schematic application of the principle of flow protective correction. In fact, the consideration of this issue is accompanied by fuzzy values of network parameters. Therefore, actual use of fuzzy logic. The most illustrative way of presenting, in our opinion, is a fuzzy graph.

There is a mechanical transport network (Fig. 3).

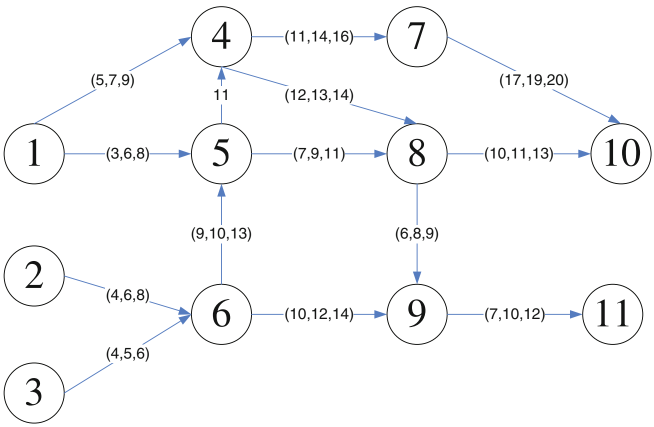

Fig. 3. Initial fuzzy graph illustrating the mechanical transport network

For ease of illustration we transform fuzzy graph (Fig. 3) to accurate form. Need to move cargo from point 2 to point 10. Based on the known routing algorithm finds the shortest route of transportation. This route is shown in Fig. 4.

$$(2, 10) = \{2, 6, 5, 8, 10\} = 36 \tag{2}$$

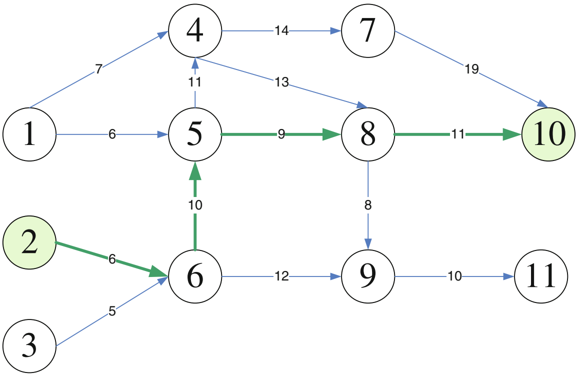

Fig. 4. Illustration of the route of minimum cost from 2 to point 10

At the time of cargo movement from point 2 to point 10 there is a need to move other cargo from point 1 to point 11. Excluding the first movement, the calculated route as follows (Fig. 5):

$$(1, 11) = \{1, 5, 8, 9, 11\} = 33 \tag{3}$$

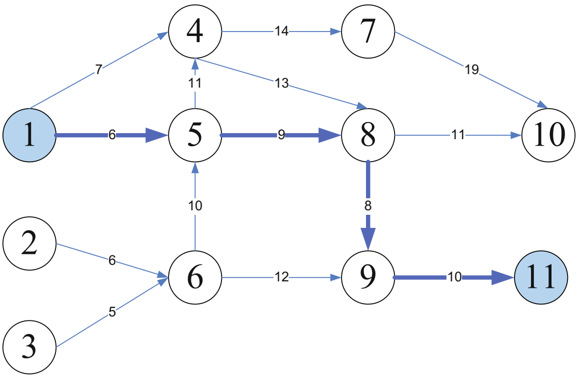

Fig. 5. Illustration of the route of minimum cost from 1 to point 11

Further there is the analysis of flow rate. After determining that the flow rate value v_{s_i} exceeds the threshold value $\bar{v}_{s_i}$, it triggers the principle of flow protective correction. Namely, there is an increase in the value segment $s_{58} = 15$. The new segment of the value does not lead to exclusion from the list of possible ways to move, and, accordingly, will not lead to network congestion.

Recalculates route.

$$(2, 10) = \{2, 6, 5, 8, 10\} = 42 \quad (4)$$

$$(1, 11) = \{1, 4, 8, 9, 11\} = 38 \quad (5)$$

Route from point 2 to point 10 is not changed, but there was an increase in the cost of transportation. But the route from point 1 to point 11 changed. The resulting routes are illustrated in Fig. 6.

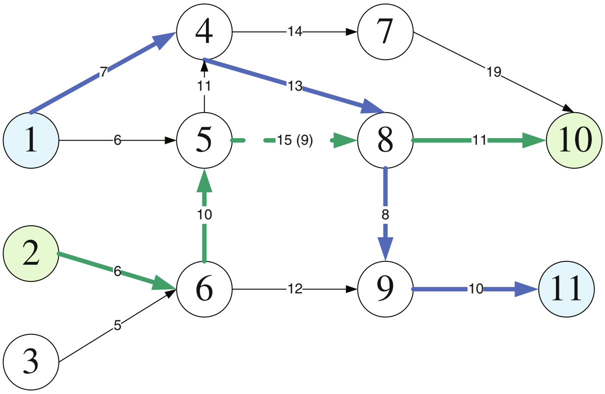

Fig. 6. Cargo moving considering flow protective correction

The time is determined by an expert or is taken from the base of precedent on the condition of this occurring. Time use of protective correction is always defined fuzzy, since it depends on network analysis, transportation time of cargo, the time of decision-making, and others.

5 Conclusion

The efficiency of flow protective correction is determined by the ratio of the loss on crash recovery and increased transportation costs. It follows from (1), protection from accidents makes sense if

$$\sum_i l_i \ll \sum_i w_{f_i} \quad (6)$$

Since protective correction leads to an increase transportation cost of the formula (2) can become the requirement to forecast reliability. Let P be the probability that the accident forecast is realized. Then the average loss from a wrong the forecast is

$$\bar{W} = (1 - P)\triangle L \quad (7)$$

where $\triangle$L is the increase in the transportation cost during runtime of protective correction. Transform the formula (3) in (2) we find that

$$P \gg 1 - \frac{\sum_i w_{f_i}}{\triangle L} \tag{8}$$

The resulting expression reflects the requirement to knowledge of the intelligent system in the problem of flow protective correction.

Further research, in our view, should be in the direction of improving the presentation and use of knowledge just as inside one of MTS, so when moving knowledge between different systems.

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Reducing Concept Lattices from Rough Set Theory

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Abstract. Due to real databases usually contain redundant information, reducing them preserving the main information is one of the most important branches of study within the theory of Formal Concept Analysis (FCA). Taking advantage of the close relationship between Rough Set Theory (RST) and FCA, in this work, we address the problem of attribute reduction in FCA using the reduction mechanism given in RST. We analyze the properties obtained from this kind of reduction and show an illustrative example.

Keywords: Attribute reduction · Formal concept analysis · Rough set theory

1 Introduction

Formal Concept Analysis (FCA) is a mathematical tool whose objective is to extract information collected in databases. With a similar goal, but considering a different philosophy, we find another interesting mathematical theory, Rough Sets Theory (RSC). Indeed, both theories are closely related and there are several papers which study the existing links between these two frameworks.

Both mathematical tools work with databases composed of a set of objects and a set of attributes related between them, but real databases usually contain redundant information. Therefore, knowledge reduction is one of the most important issues in both theories. Specifically, attribute reduction have been studied in some papers in these two frameworks [2,3,6–10]. However, the connections between the two theories from the point of view of the knowledge reduction have not yet been widely studied.

In this work, we will present a new mechanism to reduce formal contexts in FCA, based on the philosophy of attribute reduction in RST. In particular, we will propose to reduce a context in FCA considering the reducts of the associated context information system. We will show that this kind of reduction satisfies interesting properties and we will illustrate them by means of an example. Hence, in this paper we deal with the problem of attribute reduction in FCA considering the point of view of RST.

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We first recall in Sect. 2 the basic definitions and results of FCA and RST, in particular those notions related to the attribute reduction in both frameworks. Section 3 includes the reduction mechanism and some interesting results together with an example. The paper finishes with some conclusions and prospect for future work in Sect. 4.

2 Preliminary Notions in FCA and RST

In this section, we will recall the basic notions of FCA and RST needed to understand this work.

In FCA framework, the information is given by means of *contexts*, which are triples (A, B, R) composed of two non empty sets and a crisp relationship between them. That is, a set of attributes A, a set of objects B and the relation $R\colon A \times B \to \{0, 1\}$ defined, for each $a \in A$ and $b \in B$, as $R(a, b) = 1$, if a and b are related (we also write aRb) and $R(a, b) = 0$, otherwise. Fixed a context, we can define two concept-forming operators[1] ${}^{\uparrow}\colon 2^B \to 2^A$, ${}^{\downarrow}\colon 2^A \to 2^B$, for each $X \subseteq B$ and $Y \subseteq A$, as follows:

$$X^{\uparrow} = \{a \in A \mid \text{for all}\, b \in X, aRb\} = \{a \in A \mid \text{if } b \in X, \text{then } aRb\} \tag{1}$$
$$Y^{\downarrow} = \{b \in B \mid \text{for all}\, a \in Y, aRb\} = \{b \in B \mid \text{if } a \in Y, \text{then}\, aRb\} \tag{2}$$

Given $X \subseteq B$ and $Y \subseteq A$, we say that the pair (X, Y) is a *concept* in the context (A, B, R), if the equilities $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$ hold. The *extent* of the concept (X, Y) is the subset of objects X and the *intent* is the subset of attributes Y.

$\mathcal{B}(A, B, R)$ denotes the set of all the concepts, which has the structure of a complete lattice [4,7], when the inclusion ordering on the left argument is considered (equivalently, we can consider the opposite of the inclusion ordering on the right argument). For all $(X_1, Y_1), (X_2, Y_2) \in \mathcal{B}(A, B, R)$, the meet $(\wedge)$ and join $(\vee)$ operators are defined by:

$$(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \wedge X_2, (Y_1 \vee Y_2)^{\downarrow\uparrow})$$
$$(X_1, Y_1) \vee (X_2, Y_2) = ((X_1 \vee X_2)^{\uparrow\downarrow}, Y_1 \wedge Y_2)$$

In addition, given an attribute $a \in A$, *the concept generated by* a, that is $(a^{\downarrow}, a^{\downarrow\uparrow})$, is called *attribute-concept*. Due to the pair $({}^{\uparrow}, {}^{\downarrow})$ is a Galois connection [4,5], we can assure that this pair is really a concept. Analogously, given any object $b \in B$, the concept generated by b, $(b^{\uparrow\downarrow}, b^{\uparrow})$, is called *object-concept*.

Other necessary notions in this work are related to attribute reduction theory in formal concept analysis. Attribute reduction tries to reduce the set of attributes in a context, preserving the main information. That means, obtaining a new concept lattice isomorphic to the original one. Firstly, we need to recall the notion of isomorphism between concept lattices.

[1] Originally, Ganter and Wille denoted these operators as $'$ and they were called derivation operators. We have modified this notation to distinguish between the mapping defined on objects and on attributes.

Definition 1. *Given two concept lattices* $\mathcal{B}(A_1, B, R_1)$ *and* $\mathcal{B}(A_2, B, R_2)$*. If for any* $(X, Y) \in \mathcal{B}(A_2, B, R_2)$ *there exists* $(X', Y') \in \mathcal{B}(A_1, B, R_1)$ *such that* $X = X'$ *, then we say that* $\mathcal{B}(A_1, B, R_1)$ *is* finer *than* $\mathcal{B}(A_2, B, R_2)$ *and we will write:*

$$\mathcal{B}(A_1, B, R_1) \leq \mathcal{B}(A_2, B, R_2)$$

We said two concept lattices $\mathcal{B}(A_1, B, R_1), \mathcal{B}(A_2, B, R_2)$ are *isomorphic* if $\mathcal{B}(A_1, B, R_1) \leq \mathcal{B}(A_2, B, R_2)$ and $\mathcal{B}(A_2, B, R_2) \leq \mathcal{B}(A_1, B, R_1)$ hold; and we will write

$$\mathcal{B}(A_1, B, R_1) \cong \mathcal{B}(A_2, B, R_2)$$

Given a context (A, B, R), the triple $(Y, B, R_{|Y})$, where $Y \subseteq A$ and $R_{|Y}$ denotes the restricted relation $R_{|Y} = R \cap (Y \times B)$, is a formal context that can be seen as a *subcontext* of the original one. Obviously, for any $Y \subseteq A$, such that $Y \neq \varnothing$, $\mathcal{B}(A, B, R) \leq \mathcal{B}(Y, B, R_{|Y})$ holds.

The concept-forming operators in this subcontext are defined in a similar way to Eqs. (1) and (2) and are denoted as $^{\downarrow^Y}$ and $^{\uparrow_Y}$. In addition, the equality $X^{\uparrow_Y} = X^{\uparrow} \cap Y$ holds, for each $X \subseteq B$. In particular, if $Y_1 \subseteq Y$, then $Y_1^{\downarrow_Y} = Y_1^{\downarrow}$.

Definition 2. *Let* (A, B, R) *be a context, if there exists a set of attributes* $Y \subseteq A$ *such that* $\mathcal{B}(A, B, R) \cong \mathcal{B}(Y, B, R_{|Y})$*, then* Y *is called a* consistent *set of* (A, B, R)*. Moreover, if* $\mathcal{B}(Y \smallsetminus \{y\}, B, R_{|Y \smallsetminus \{y\}}) \ncong \mathcal{B}(A, B, R)$*, for all* $y \in Y$*, then* Y *is called* reduct *of* (A, B, R)*.*

The core *of* (A, B, R) *is the intersection of all the reducts of* (A, B, R)*.*

The last notion we need to recall is the definition of irreducible element of a lattice, which will be considered later.

Definition 3. *Given a lattice* $(L, \preceq)$*, such that* $\wedge, \vee$ *are the meet and the join operators, and an element* $x \in L$ *verifying*

1. *If* L *has a top element* $\top$*, then* $x \neq \top$*.*
2. *If* $x = y \wedge z$*, then* $x = y$ *or* $x = z$*, for all* $y, z \in L$*.*

we call x meet-irreducible ($\wedge$-irreducible) element *of* L*. Condition* (2) *is equivalent to*

$2'$*. If* $x < y$ *and* $x < z$*, then* $x < y \wedge z$*, for all* $y, z \in L$*.*

A join-irreducible ($\vee$-irreducible) element *of* L *is defined dually.*

Due to our goal is to apply the attribute reduction mechanism given in RST to FCA, we also need to recall several notions corresponding to the framework of RST. In this framework, relational databases are seen as decision systems or as information systems, depending on the problem we want to solve. Here, we only consider the notion of information systems since they are the natural systems used when RST and FCA are related.

Definition 4. *An* information system $(U, \mathcal{A})$ *is a tuple, satisfying that* $U = \{x_1, x_2, \ldots, x_n\}$ *and* $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ *are finite, non-empty sets of objects and attributes, respectively, in which, each* $a \in \mathcal{A}$ *corresponds to a mapping* $\bar{a} : U \to V_a$*, where* V_a *is the value set of* a *over* U*. For every subset* D *of* $\mathcal{A}$*, the* D-indiscernibility relation*,* $Ind(D)$*, is defined as the equivalence relation*

$$Ind(D) = \{(x_i, x_j) \in U \times U \mid \text{for all } a \in D, \bar{a}(x_i) = \bar{a}(x_j)\}$$

where each class given by this relation can be written as $[x]_D = \{x_i \mid (x, x_i) \in Ind(D)\}$*.* $Ind(D)$ *produces a partition on* U *denoted as* $U/Ind(D) = \{[x]_D \mid x \in U\}$*.*

If we have that the value set of a *is* $V_a = \{0, 1\}$*, for all* $a \in \mathcal{A}$*,* $(U, \mathcal{A})$ *is called a* boolean information system*.*

The notions of consistent set and reduct are considered in the reduction mechanism, so we recall them in the following definition.

Definition 5. *Let* $(U, \mathcal{A})$ *be an information system and a subset of attributes* $D \subseteq \mathcal{A}$*.* D *is a* consistent set *of* $(U, \mathcal{A})$ *if*

$$Ind(D) = Ind(\mathcal{A})$$

Moreover, if for each $a \in D$ *we have that* $Ind(D \setminus \{a\}) \neq Ind(\mathcal{A})$*, then* D *is called* reduct *of* $(U, \mathcal{A})$*.*

In the next definition, we present the notions of discernibility matrix and discernibility function [11]. We will use them in order to characterize the reducts in RST.

Definition 6. *Given an information system* $(U, \mathcal{A})$*, its* discernibility matrix *is a matrix with order* $|U| \times |U|$*, denoted as* $M_{\mathcal{A}}$*, in which the element* $M_{\mathcal{A}}(i, j)$ *for each pair of objects* (i, j) *is defined by:*

$$M_{\mathcal{A}}(i, j) = \{a \in \mathcal{A} \mid \bar{a}(i) \neq \bar{a}(j)\}$$

and the discernibility function *of* $(U, \mathcal{A})$ *is defined by:*

$$\tau_{\mathcal{A}} = \bigwedge \left\{ \bigvee (M_{\mathcal{A}}(i, j)) \mid i, j \in U \text{ and } M_{\mathcal{A}}(i, j) \neq \varnothing \right\}$$

The following result states a procedure to get reducts of an information system, using the discernibility function.

Theorem 1. *Given a boolean information system* $(U, \mathcal{A})$*. An arbitrary set* D*, where* $D \subseteq \mathcal{A}$*, is a reduct of the information system if and only if the cube* $\bigwedge_{a \in D} a$ *is a cube in the restricted disjunctive normal form (RDNF) of* $\tau_{\mathcal{A}}$*.*

Now, we can present the new mechanism to reduce a context in FCA based on RST.

3 Reducing a Context in FCA Based on RST

In this paper, we will use the reduction given in RST to reduce the set of attributes of a context in the FCA framework. Therefore, we are introducing a novel reduction mechanism in FCA, since we are considering the philosophy of RST. There exist some papers that study the relationship between FCA and RST, but only a small number of them analyze this relationship from the perspective of attribute reduction [1,12]. For example, in [12] was proven that the reduction in FCA in a certain way implies the reduction in RST. Specifically, they demonstrated that any consistent set in FCA is a consistent set in RST. However, the opposite statement does not hold, in general. Therefore, the reductions in both frameworks are not equivalent. For that reason, in this paper we are interested in studying the properties that we obtain when we reduce contexts in FCA from reducts of RST.

Hereafter, the set of attributes and the set of objects will be considered finite. In addition, consistent sets of a context (A, B, R) will be called *CL-consistent sets* and consistent sets of an information system $(U, \mathcal{A})$ as *RS-consistent sets*. Analogously, reducts of a context (A, B, R) will be called *CL-reducts* and reducts of an information system $(U, \mathcal{A})$ as *RS-reducts*.

Firstly, given a context we show how to define an information system.

Definition 7. *Let (A, B, R) be a context, a* context information system *is defined as the pair (B, A) where the mappings $\bar{a}: B \to V_a$, with $V_a = \{0, 1\}$, are defined as $\bar{a}(b) = R(a, b)$, for all $a \in A, b \in B$.*

The proposed mechanism will be explained in the following. Given a context (A, B, R), we consider the corresponding context information system and we compute the RS-reducts of this information system. We reduce the original context according to the obtained RS-reducts and analyze the properties satisfied by such reduction. The first one establishes the relationship between the operators in both frameworks.

Lemma 1. *Given a context (A, B, R) and the corresponding context information system (B, A), the following equality holds, for each $a \in A$:*

$$a^{\downarrow} = \bar{a}$$

On the other hand, the next proposition states that if two different objects generate two different concepts in the original context, then these objects also generate different concepts in the reduced one. Therefore, the number of different object-concepts is preserved.

Proposition 1. *Let (A, B, R) be a context and (B, A) the corresponding context information system. Considering $D \subseteq A$ a RS-consistent set of (B, A) and the objects $k, j \in B$, if $k^{\uparrow} \neq j^{\uparrow}$, then $k^{\uparrow_D} \neq j^{\uparrow_D}$.*

In addition, the consideration of RS-consistent set preserves the (strict) inequality between object-concepts.

Proposition 2. *Given a context (A, B, R) and its corresponding context information system (B, A). If $D \subseteq A$ is a RS-consistent set of (B, A) and we consider two objects $k, j \in B$ satisfying that $k^{\uparrow} < j^{\uparrow}$, then the inequality $k^{\uparrow_D} < j^{\uparrow_D}$ holds.*

Therefore, when this reduction is carried out, the ordering among the object-concepts is practically preserved.

Another interesting property states that if an object does not generate a join-irreducible concept in the original context, then it cannot generate a join-irreducible concept in the reduced one. Consequently, no new join-irreducible elements appear after the reduction using an RS-consistent set.

Theorem 2. *Given a context (A, B, R), the corresponding context information system (B, A) and $D \subseteq A$ a RS-consistent set. If an object $j \in B$ generates a join-irreducible concept in the concept lattice associated with the context (D, B, R), then it also generates a join-irreducible concept of the concept lattice associated with (A, B, R).*

Thus, from all the results previously presented this reduction mechanism has interesting properties to be considered in the FCA framework.

Now, we will explain how the proposed reduction can be applied in FCA, by means of the following example. Specifically, we will reduce a formal context, from the RS-reducts of the associated context information system.

Example 1. We fix a formal context (A, B, R), where the set of objects is formed by a group of cultived fields and the set A represents attributes as high temperature, high humidity, windy area or the use of, fertilizer and pesticide, corresponding to these seven different cultived fields. The relation R is given by Table 1. The concept lattice associated with this context is displayed in Fig. 1.

Table 1. Relation of example 1.

R	High temperature (ht)	High humidity (hh)	Windy area (wa)	Fertilizer (f)	Pesticide (p)
1	1	0	1	1	1
2	0	0	1	0	1
3	0	1	0	1	0
4	0	1	1	1	1
5	0	0	1	1	1
6	1	0	1	0	0
7	1	1	1	1	0

We want to reduce the context considering the RS-reducts that we obtain from the corresponding context information system. We will see that when this kind of reduction is applied, the structure of the original concept lattice cannot

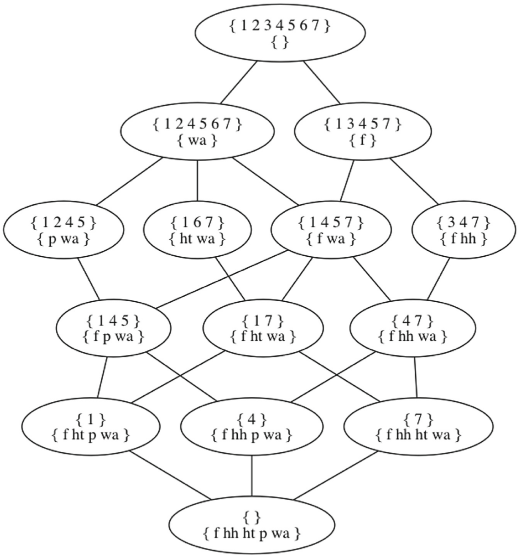

Fig. 1. Concept lattice of the context of Example 1

be necessarily preserved but, as we mentioned above, the obtained reduction satisfies interesting properties. In order to analyze these properties, it is important to observe in Fig. 1 that the set of join-irreducible elements of the concept lattice are composed of the following concepts, generated by objects 1, 2, 3, 4, 6 and 7:

$$\begin{array}{l}
(1^{\uparrow\downarrow}, 1^{\uparrow}) = (\{1\}, \{\text{ht, wa, f, p}\}) \\
(2^{\uparrow\downarrow}, 2^{\uparrow}) = (\{1, 2, 4, 5\}, \{\text{wa, p}\}) \\
(3^{\uparrow\downarrow}, 3^{\uparrow}) = (\{3, 4, 7\}, \{\text{hh, f}\}) \\
(4^{\uparrow\downarrow}, 4^{\uparrow}) = (\{4\}, \{\text{hh, wa, f, p}\}) \\
(6^{\uparrow\downarrow}, 6^{\uparrow}) = (\{1, 6, 7\}, \{\text{ht, wa}\}) \\
(7^{\uparrow\downarrow}, 7^{\uparrow}) = (\{7\}, \{\text{ht, hh, wa, f}\})
\end{array}$$

Now, we will take into account the associated context information system (B, A) and, applying Definition 6, we obtain the discernibility matrix and the corresponding discernibility function.

In this example, the discernibility matrix is the following[2]:

$$\begin{pmatrix}
\varnothing & & & & & & \\
\{\text{ht, f}\} & \varnothing & & & & & \\
\{\text{ht,hh,wa,p}\} & \{\text{hh,wa,f,p}\} & \varnothing & & & & \\
\{\text{ht,hh}\} & \{\text{hh,f}\} & \{\text{wa,p}\} & \varnothing & & & \\
\{\text{ht}\} & \{\text{f}\} & \{\text{hh,wa,p}\} & \{\text{hh}\} & \varnothing & & \\
\{\text{f,p}\} & \{\text{ht,p}\} & \{\text{ht,hh,wa,f}\} & \{\text{ht,hh,f,p}\} & \{\text{ht,f,p}\} & \varnothing & \\
\{\text{hh,p}\} & \{\text{ht,hh,f,p}\} & \{\text{ht,wa}\} & \{\text{ht,p}\} & \{\text{ht,hh,p}\} & \{\text{hh,f}\} & \varnothing
\end{pmatrix}$$

[2] Observe that the discernibility matrix is symmetric matrix since the discernibility relation is reflexive.

Therefore, the discernibility function is shown below:

$$\begin{aligned}\tau_A = &\{\mathrm{ht} \vee \mathrm{f}\} \wedge \{\mathrm{ht} \vee \mathrm{hh} \vee \mathrm{wa} \vee \mathrm{p}\} \wedge \{\mathrm{hh} \vee \mathrm{wa} \vee \mathrm{f} \vee \mathrm{p}\} \wedge \{\mathrm{ht} \vee \mathrm{hh}\}\\ &\wedge\{\mathrm{hh} \vee \mathrm{f}\} \wedge \{\mathrm{wa} \vee \mathrm{p}\} \wedge \{\mathrm{ht}\} \wedge \{\mathrm{f}\} \wedge \{\mathrm{hh} \vee \mathrm{wa} \vee \mathrm{p}\} \wedge \{\mathrm{hh}\}\\ &\wedge\{\mathrm{f} \vee \mathrm{p}\} \wedge \{\mathrm{ht} \vee \mathrm{p}\} \wedge \{\mathrm{ht} \vee \mathrm{hh} \vee \mathrm{wa} \vee \mathrm{f}\} \wedge \{\mathrm{ht} \vee \mathrm{hh} \vee \mathrm{f} \vee \mathrm{p}\}\\ &\wedge\{\mathrm{ht} \vee \mathrm{f} \vee \mathrm{p}\} \wedge \{\mathrm{hh} \vee \mathrm{p}\} \wedge \{\mathrm{ht} \vee \mathrm{hh} \vee \mathrm{f} \vee \mathrm{p}\} \wedge \{\mathrm{ht} \vee \mathrm{wa}\} \wedge \{\mathrm{ht} \vee \mathrm{p}\}\\ &\wedge\{\mathrm{ht} \vee \mathrm{hh} \vee \mathrm{p}\} \wedge \{\mathrm{hh} \vee \mathrm{f}\}\\ = &\{\mathrm{ht} \wedge \mathrm{hh} \wedge \mathrm{wa} \wedge \mathrm{f}\} \vee \{\mathrm{ht} \wedge \mathrm{hh} \wedge \mathrm{f} \wedge \mathrm{p}\}\end{aligned}$$

According to Theorem 1, two RS-reducts are obtained:

$$D_1 = \{\mathrm{ht}, \mathrm{hh}, \mathrm{wa}, \mathrm{f}\}$$
$$D_2 = \{\mathrm{ht}, \mathrm{hh}, \mathrm{f}, \mathrm{p}\}$$

Hence, D_1 and D_2 are the RS-reducts from which we are going to reduce the original context. The corresponding concept lattices obtained from these reductions are shown in Fig. 2. Observe that, the original structure has been modified in both cases.

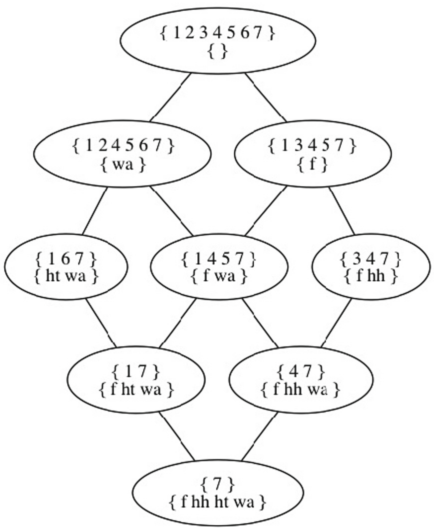

Fig. 2. Concept lattices built from the RS-reducts D_1 (left) and D_2 (right).

First of all, the reduction obtained from the second reduct will be analyzed. For that purpose, we list below the object-concepts generated after the reduction of the context from D_2[3]:

$$\begin{aligned}
(1^{\uparrow_2\downarrow^2}, 1^{\uparrow_2}) &= (\{1\}, \{\text{ht, f, p}\})\\
(2^{\uparrow_2\downarrow^2}, 2^{\uparrow_2}) &= (\{1,2,4,5\}, \{\text{p}\})\\
(4^{\uparrow_2\downarrow^2}, 4^{\uparrow_2}) &= (\{4\}, \{\text{hh, f, p}\})\\
(6^{\uparrow_2\downarrow^2}, 6^{\uparrow_2}) &= (\{1,6,7\}, \{\text{ht}\})\\
(7^{\uparrow_2\downarrow^2}, 7^{\uparrow_2}) &= (\{7\}, \{\text{ht, hh, f}\})
\end{aligned}$$

In this case, we can observe that 3 concepts have disappeared, hence the size of the original concept lattice have been reduced. However, according to Theorem 2, we have that all objects generating join-irreducible elements in the reducing concept lattice, that is, objects 1, 2, 4, 6 and 7 also generate join-irreducible concepts in the original one. Moreover, the object-concepts after the reduction preserve the same extensions as in the original concept lattice, which implies that Propositions 1 and 2 trivially hold. But this fact does not necessarily have to be satisfied, as we can see if we consider the reduct D_1.

The concept lattice obtained from the RS-reduct D_1 is displayed in left side of Fig. 2. Now, the set of join-irreducible concepts are obtained from the objects 1, 3, 4 and 6, and they are listened below:

$$\begin{aligned}
(1^{\uparrow_1\downarrow^1}, 1^{\uparrow_1}) &= (\{1,7\}, \{\text{ht, wa, f}\})\\
(3^{\uparrow_1\downarrow^1}, 3^{\uparrow_1}) &= (\{3,4,7\}, \{\text{hh, f}\})\\
(4^{\uparrow_1\downarrow^1}, 4^{\uparrow_1}) &= (\{4,7\}, \{\text{hh, wa, f}\})\\
(6^{\uparrow_1\downarrow^1}, 6^{\uparrow_1}) &= (\{1,6,7\}, \{\text{ht, wa}\})
\end{aligned}$$

From them, it is easy to see that Propositions 1, 2 and Theorem 2 also hold. In addition, observe that after the reduction, from the objects 3 and 6 we obtain join-irreducible concepts with the same extensions that in the initial context. Whereas, the extensions of the object-concepts from 1 and 4 have been modified. Specifically, the concepts whose extensions are $\{1,7\}$ and $\{4,7\}$ were not join-irreducible in the original context but they are irreducible concepts after the reduction.

The most interesting property that we obtain considering this mechanism to reduce the context, is that different object-concepts in the original concept lattice continue being different after the reduction. That is, given two objects if we have a concept in the original lattice which does not contain both objects, after the reduction, we also have a concept that does not contain these two objects. Therefore, we preserve the necessary information to distinguish the objects after using this reduction mechanism.

As a consequence, in this work an interesting way to reduce contexts in FCA have been presented, since useful properties are satisfied.

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[3] For the sake of simplicity, we will write $({}^{\uparrow_1}, {}^{\downarrow^1})$ and $({}^{\uparrow_2}, {}^{\downarrow^2})$, instead of $({}^{\uparrow_{D_1}}, {}^{\downarrow^{D_1}})$ and $({}^{\uparrow_{D_2}}, {}^{\downarrow^{D_2}})$ to denote the concept-forming operators in the reduced contexts by D_1 and D_2, respectively.

4 Conclusions and Future Work

In this paper, we have considered the philosophy of attribute reduction in RST in order to introduce a new mechanism for reducing context in FCA. We have shown that when we reduce the number of attributes of a context from an RS-reduct of the corresponding context information system, the obtained reduction satisfies interesting properties with respect to the object-concepts of the concept lattice. The most relevant characteristics of this new reduction procedure is that the number of different object-concepts is preserved and no new join-irreducible element is introduced after the reduction. We have illustrated the presented mechanism by an example.

As prospects for future work, we want to carry out a greater reduction of contexts in FCA, decreasing the number of objects. For this purpose, we will consider the notion of bireduct given in RST within the FCA framework.

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An Equivalence Relation and Admissible Linear Orders in Decision Making

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Abstract. In this paper interval-valued fuzzy relations in the context of decision making problems are studied. A new version of transitivity with admissible linear order involved in its notion is introduced. It is examined the connection of this new property and some equivalence relation for interval-valued fuzzy relations. There are also studied admissible linear orders generated by aggregation functions and their connection with the considered equivalence relation. Possible applications of the presented results in decision making are indicated.

Keywords: Interval-valued fuzzy relations · Admissible orders · Transitivity

1 Introduction

Interval-Valued Fuzzy Relations (IVFRs), which are extensions of fuzzy relations, are applied in databases, pattern recognition, neural networks, fuzzy modelling, economy, medicine or multicriteria decision making [13,18]. In recent applications to image processing [1] or classification [20] it has been proven that, under some circumstances, the use of IVFSs together with the total order defined by Xu and Yager [24] provide results that are better than their fuzzy counterparts.

In this paper we deal with an equivalence relation between aggregation functions and interval-valued fuzzy relations. Applying this equivalence relation is a method of comparison of diverse types of values. There are different methods of comparison of fuzzy quantities (cf. [21]). Some of them can generate equivalence relations between fuzzy sets (incomparable elements are similar). The first concept of a similarity between fuzzy sets was proposed by Warren [22]. It was connected with the normalization of fuzzy sets (cf. e.g. [25]). Two fuzzy sets were considered equivalent, if they produced the same normalized fuzzy set after the normalization. This relation can be considered to be the case of proportional membership functions. A more general relation was introduced by Bhattacharya [2] in the case of fuzzy groups. Two fuzzy groups were considered equivalent if they had a common family of level sets. Murali and Makamba [15] examined this relation in order to describe numerical dependence between membership functions of equivalent fuzzy groups. Murali [16] has generalized this

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equivalence for arbitrary fuzzy sets. The classification of fuzzy relations was introduced by Zadeh [25] and popularized by Kaufmann [12].

In this contribution we also discuss transitivity property and introduce a new type of this concept with respect to admissible linear orders. Transitivity is an important property of relations, since it may guarantee consistency of choices of decision makers. Diverse properties of IVFRs (also for the case of interval-valued fuzzy reciprocal relations IVFRRs), including transitivity, have been studied by a range of authors (cf. [1,9,14,23]).

This work is composed of the following parts. Firstly, some concepts and results useful in further considerations are recalled and new version of transitivity with an admissible linear order is introduced (Sect. 2). Next, an equivalence relation for aggregation functions and interval-valued fuzzy relations is considered (Sect. 3). Finally, (Sect. 4) it is provided information how obtained in the paper results may be applied in decision making problems.

2 Transitivity of Interval-Valued Fuzzy Relations with Respect to Admissible Linear Order

Firstly, we give the definition of an aggregation function on $[0, 1]$.

Definition 1 ([8], p. 6). *An increasing function* $A : [0, 1]^n \to [0, 1]$*,* $n \in \mathbb{N}$*,* $n \geqslant 2$*, is called an aggregation function if* $A(0, \dots, 0) = 0$*,* $A(1, \dots, 1) = 1$*.*

We recall the notion of the lattice operations and the order in the family of interval-valued fuzzy sets

$$L^I = \{[x_1, x_2] : x_1, x_2 \in [0, 1], x_1 \leqslant x_2\}.$$

Note that L^I endowed with the partial order $[x_1, x_2] \leq [y_1, y_2]$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$ is a complete, bounded lattice with the top element given by $\mathbf{1} = [1, 1]$ and the bottom element given by $\mathbf{0} = [0, 0]$. In this lattice, the supremum of any two elements is defined by $[x_1, x_2] \vee [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)]$, and the infimum is defined by $[x_1, x_2] \wedge [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)]$, respectively.

Note that, if we consider the partial order defined on L^I, we see that the family $(\mathcal{IVFR}(X \times Y),\vee,\wedge)$ is a complete and distributive lattice (see [7] for a study on the concept of lattices). The order $\leq$ is not linear. To overcome this problem the methods to obtain linear orders on L^I were introduced in [5].

Definition 2 ([5]). *An order* $\leq_{L^I}$ *on* L^I *is called admissible if it is linear and satisfies that, for all* $x, y \in L^I$*, such that if* $x \leq y$*, then* $x \leq_{L^I} y$*.*

In [6], this class of linear orders on L^I is used to extend the definition of OWA operators to interval-valued fuzzy setting.

Definition 3 ([6]). *Let $\leq_{L^I}$ be an admissible order on L^I, $w = (w_1, \ldots, w_n) \in [0,1]^n$ with $w_1 + \cdots + w_n = 1$. The Interval-Valued OWA operator (IVOWA) associated with $\leq_{L^I}$ and w is a mapping $IVOWA_{\leq_{L^I},w} : (L^I)^n \to L^I$, given by*

$$IVOWA_{\leq_{L^I},w}([a_1, b_1], \ldots, [a_n, b_n]) = \sum_{i=1}^{n} w_i \cdot [a_{(i)}, b_{(i)}],$$

where $[a_{(i)}, b_{(i)}]$, $i = 1, \ldots, n$, denotes the i-th greatest of the inputs with respect to the order $\leq_{L^I}$ and $w \cdot [a, b] = [wa, wb]$, $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$.

Similarly, we may consider ordered weighted geometric mean (cf. [11]) on L^I.

Definition 4. *Let $\leq_{L^I}$ be an admissible order on L^I, and let $w = (w_1, \ldots, w_n) \in [0,1]^n$, with $w_1 + \cdots + w_n = 1$. The Interval-Valued OWG operator (IVOWG) associated with $\leq_{L^I}$ and w is a mapping $IVOWG_{\leq_{L^I},w} : (L^I)^n \to L^I$, given by*

$$IVOWG_{\leq_{L^I},w}([a_1, b_1], \ldots, [a_n, b_n]) = \prod_{i=1}^{n} [a_{(i)}, b_{(i)}]^{w_i},$$

where $[a_{(i)}, b_{(i)}]$, $i = 1, \ldots, n$, denotes the i-th greatest of the inputs with respect to the order $\leq_{L^I}$ and $[a_1, b_1] \cdot [a_2, b_2] = [a_1 \cdot a_2, b_1 \cdot b_2]$, $[a, b]^w = [a^w, b^w]$.

There exist a construction of admissible orders with the use of aggregation functions.

Proposition 1 ([5]). *Let $B_1, B_2 : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that, for all $x = [\underline{x}, \overline{x}], y = [\underline{y}, \overline{y}] \in L^I$, the equalities $B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y})$ and $B_2(\underline{x}, \overline{x}) = B_2(\underline{y}, \overline{y})$ hold if and only if $x = y$. If the order $\leq_{B_{1,2}}$ on L^I is defined by $x \leq_{B_{1,2}} y$ if and only if*

$$B_1(\underline{x}, \overline{x}) < B_1(\underline{y}, \overline{y}) \text{ or } (B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y}) \text{ and } B_2(\underline{x}, \overline{x}) \leqslant B_2(\underline{y}, \overline{y})),$$

then $\leq_{B_{1,2}}$ is an admissible order on L^I.

Example 1 ([5]). Let $x = [\underline{x}, \overline{x}], y = [\underline{y}, \overline{y}] \in L^I$. Admissible orders on L^I are:
- the Xu and Yager order
$[\underline{x}, \overline{x}] \leq_{XY} [\underline{y}, \overline{y}]$ if and only if $\underline{x} + \overline{x} < \underline{y} + \overline{y}$ or $(\overline{x} + \underline{x} = \overline{y} + \underline{y}$ and $\overline{x} - \underline{x} \leq \overline{y} - \underline{y})$,
- the lexicographical order with respect to the first variable
$[\underline{x}, \overline{x}] \leq_{Lex1} [\underline{y}, \overline{y}]$ if and only if $\underline{x} < \underline{y}$ or $(\underline{x} = \underline{y}$ and $\overline{x} \leq \overline{y})$,
- the lexicographical order with respect to the second variable
$[\underline{x}, \overline{x}] \leq_{Lex2} [\underline{y}, \overline{y}]$ if and only if $\overline{x} < \overline{y}$ or $(\overline{x} = \overline{y}$ and $\underline{x} \leq \underline{y})$,
- let $K_\alpha : [0,1]^2 \to [0,1]$ be the function defined as $K_\alpha(x, y) = \alpha x + (1 - \alpha) y$ for some $\alpha \in [0,1]$, the order defined as
$[\underline{x}, \overline{x}] \leq_{\alpha,\beta} [\underline{y}, \overline{y}]$ if and only if $K_\alpha(\underline{x}, \overline{x}) < K_\alpha(\underline{y}, \overline{y})$ or $(K_\alpha(\underline{x}, \overline{x}) = K_\alpha(\underline{y}, \overline{y})$ and $K_\beta(\underline{x}, \overline{x}) \leq K_\beta(\underline{y}, \overline{y}))$ is an admissible order for $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$.

The orders $\leq_{XY}$, $\leq_{Lex1}$ and $\leq_{Lex2}$ are special cases of the order $\leq_{\alpha,\beta}$ with $\leq_{0.5,\beta}$ (for $\beta > 0.5$), $\leq_{1,0}$, $\leq_{0,1}$, respectively. The orders $\leq_{XY}$, $\leq_{Lex1}$, $\leq_{Lex2}$, and $\leq_{\alpha,\beta}$ are admissible linear orders defined by pairs of aggregation functions (cf. Proposition 1), namely weighted means. In the case of the orders $\leq_{Lex1}$ and $\leq_{Lex2}$ these means are reduced to the pairs of projections: P_1, P_2 and P_2, P_1, respectively.

Definition 5 (cf. [19,25]). *An IVFR R between universes X, Y is a mapping $R : X \times Y \to L^I$ such that*

$$R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \textit{for all pairs} (x,y) \in X \times Y.$$

The class of all IVFRs between universes X, Y is denoted by $\mathcal{IVFR}(X \times Y)$, or $\mathcal{IVFR}(X)$ for $X = Y$.

We recall the notion of transitivity (in its basic form for operation $\wedge$) which is an important measure of consistency of decision makers.

Definition 6 (cf. [3]). *$R \in \mathcal{IVFR}(X)$ is said to be transitive, if*

$$\underset{x,y,z\in X}{\forall} R(x,y) \wedge R(y,z) \leq R(x,z). \tag{1}$$

Remark 1. Transitivity of $R \in \mathcal{IVFR}(X)$ described by (1) may be characterized by the property involving max-min-composition, namely $R^2 \leq R$ (cf. [4]). In the context of preference relations, for $X = \{x_1, \ldots, x_n\}$, transitivity captures the fact that, if the alternative x_i is preferred to x_k and x_k is preferred to x_j, then x_i should be preferred to x_j.

Definition 7. *Let $\leq_{L^I}$ be an admissible order. $R \in \mathcal{IVFR}(X)$ is said to be transitive with respect to $\leq_{L^I}$, if*

$$\underset{x,y,z\in X}{\forall} R(x,y) \wedge R(y,z) \leq_{L^I} R(x,z). \tag{2}$$

Example 2. Relation $R \in \mathcal{IVFR}(X)$ is not transitive but it is transitive with respect to $\leq_{Lex1}$, where

$$R = \begin{bmatrix} [0.1, 0.2] & [0.3, 0.4] \\ [0, 0.3] & [0.1, 0.3] \end{bmatrix}.$$

Remark 2. Transitivity with respect to $\leq_{L^I}$ may not be characterized by composition in a similar way to the one presented for transitivity in Remark 1. Relation R from Example 2 is transitive with respect to $\leq_{Lex1}$ but it is not true that $R^2 \leq_{Lex1} R$, where

$$R^2 = \begin{bmatrix} [0.1, 0.3] & [0.1, 0.3] \\ [0, 0.3] & [0.1, 0.3] \end{bmatrix}.$$

In the next section we will consider this new concept of transitivity in connection with some equivalence relation.

3 Equivalence Relation

We will consider an equivalence relation which was originally introduced for fuzzy groups [2] and then generalized to fuzzy sets [16]. We will use this notion for aggregation functions (similarly to [10]), but apply it for interval-valued fuzzy relations. Since the considered relation is an equivalence it can be useful in the classification of fuzzy information.

Definition 8. *Let $A, B : [0,1]^2 \to [0,1]$ be aggregation functions. Aggregation functions A and B are equivalent ($A \sim B$) if*

$$\forall_{x,y,z,t \in [0,1]} A(x,y) \leqslant A(z,t) \Leftrightarrow B(x,y) \leqslant B(z,t).$$

Example 3. If we take two aggregation functions $K_{0.3}$ and $K_{0.2}$ we observe that, $K_{0.3} \sim K_{0.2}$. For example we calculate:

x	y	z	t	$K_{0.3}(x,y)$	$K_{0.3}(z,t)$	$K_{0.2}(x,y)$	$K_{0.2}(z,t)$
0,466079512	0,695313702	0,199977086	0,82690187	0,626543445	0,638824437	0,649466864	0,701516916
0,699377369	0,081830674	0,51141676	0,437083250	0,267094683	0,459383303	0,205340013	0,451949952
0,858878619	0,106145726	0,400104100	0,150400847	0,331965594	0,225311823	0,256692305	0,200341498
0,652742085	0,352831954	0,941989642	0,41648294	0,442804994	0,574134954	0,41281398	0,521584284
0,473595392	0,927195804	0,026486991	0,81285943	0,791115680	0,576947703	0,836475721	0,655584947
0,099923048	0,304847811	0,16435941	0,58175802	0,243370381	0,456538443	0,263862857	0,498278304
0,612547796	0,942061811	0,896368801	0,19422334	0,84320760	0,404866982	0,876159008	0,334652436
0,597453753	0,783110941	0,305027938	0,843501889	0,727413789	0,681959704	0,745979509	0,735807099
0,895580406	0,969919426	0,687527348	0,037429380	0,947617720	0,23245877	0,955051622	0,167448974
0,497693834	0,519135501	0,672966809	0,497403272	0,512703001	0,550072333	0,514847167	0,532515980
0,596747066	0,130312935	0,292547222	0,940475743	0,270243175	0,746097187	0,223599762	0,810890039

It is clear that relation $\sim$ from Definition 8 is an equivalence relation. Now, we will recall definition and some results for this equivalence relation used for interval-valued fuzzy relations. We also introduce the notion of equivalence relation with respect to an admissible order.

Definition 9 (cf. [4]). *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. We say that relations R and S are equivalent ($R \sim S$), if for all $x, y, u, v \in X$*

$$R(x,y) \leq R(u,v) \;\Leftrightarrow\; S(x,y) \leq S(u,v).$$

By definition of the partial order $\leq$ we obtain the following result.

Corollary 1. *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. Relations R and S are equivalent ($R \sim S$) if and only if for all $x, y, u, v \in X$*

$$\underline{R}(x,y) \leqslant \underline{R}(u,v) \;\Leftrightarrow\; \underline{S}(x,y) \leqslant \underline{S}(u,v)$$

and

$$\overline{R}(x,y) \leqslant \overline{R}(u,v) \;\Leftrightarrow\; \overline{S}(x,y) \leqslant \overline{S}(u,v).$$

Definition 10. *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. We say that relations R and S are equivalent with respect to $\leq_{L^I}$ ($R \sim_{L^I} S$), if for all $x, y, u, v \in X$*

$$R(x,y) \leq_{L^I} R(u,v) \;\Leftrightarrow\; S(x,y) \leq_{L^I} S(u,v).$$

In this paper we will apply the concept defined in Definition 10 to admissible orders described in Proposition 1. This is why we will specify what we mean by such equivalence in this case.

Definition 11. *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. We say that relations R and S are equivalent with respect to $\leq_{B_{1,2}}$ ($R \sim_{B_{1,2}} S$), if for all $x, y, u, v \in X$ one of the following equivalences is fulfilled*

$$B_1(\underline{R}(x,y), \overline{R}(x,y)) < B_1(\underline{R}(u,v), \overline{R}(u,v)) \Leftrightarrow B_1(\underline{S}(x,y), \overline{S}(x,y)) < B_1(\underline{S}(u,v), \overline{S}(u,v)) \quad (3)$$

or

$$B_1(\underline{R}(x,y), \overline{R}(x,y)) = B_1(\underline{R}(u,v), \overline{R}(u,v)) \text{ and } B_2(\underline{R}(x,y), \overline{R}(x,y)) \leqslant B_2(\underline{R}(u,v), \overline{R}(u,v))$$

$$\Leftrightarrow$$

$$B_1(\underline{S}(x,y), \overline{S}(x,y)) = B_1(\underline{S}(u,v), \overline{S}(u,v)) \text{ and } B_2(\underline{S}(x,y), \overline{S}(x,y)) \leqslant B_2(\underline{S}(u,v), \overline{S}(u,v)). \quad (4)$$

The approach proposed in Definition 11 gives us more precise information about the behaviour of elements in both compared interval-valued fuzzy relations than just straight applying Definition 10 to the order $\leq_{B_{1,2}}$.

Example 4. Let us consider relations $R, S \in \mathcal{IVFR}(X)$, where

$$R = \begin{bmatrix} [0.2, 0.4] & [0.3, 0.6] \\ [0, 0] & [0, 0] \end{bmatrix}, \qquad S = \begin{bmatrix} [0.3, 0.5] & [0.4, 0.4] \\ [0.1, 0.1] & [0.1, 0.1] \end{bmatrix}.$$

We see that it is not true that $R \sim S$ but it is true that $R \sim_{Lex1} S$. Moreover, it is neither true that $R \sim_{Lex2} S$ nor that $R \sim_{XY} S$ (cf. Definition 11).

Example 5. Let us consider relations $R, S \in \mathcal{IVFR}(X)$, where

$$R = \begin{bmatrix} [0.2, 0.4] & [0.3, 0.6] \\ [0, 0] & [0, 0] \end{bmatrix}, \qquad S = \begin{bmatrix} [0.2, 0.5] & [0.4, 0.4] \\ [0, 0] & [0, 0] \end{bmatrix}.$$

It holds that $R \sim_{XY} S$ and $R \sim_{Lex1} S$ but it neither holds that $R \sim_{Lex2} S$ nor $R \sim S$ (cf. Definition 11).

Example 6. Let us now consider the following relations $R, S \in \mathcal{IVFR}(X)$:

$$R = \begin{bmatrix} [0.3, 0.4] & [0.2, 0.6] \\ [0, 0] & [0, 0] \end{bmatrix}, \qquad S = \begin{bmatrix} [0.1, 0.5] & [0.4, 0.6] \\ [0, 0] & [0, 0] \end{bmatrix}.$$

It holds both $R \sim_{XY} S$ and $R \sim_{Lex2} S$ but it neither holds that $R \sim_{Lex1} S$ nor $R \sim S$ (cf. Definition 11).

Some results related to the operations supremum and infimum may be applied in verifying the equivalence between two given IVFRs. Firstly, we present them for relation $\sim$ and then for $\sim_{L^I}$.

Proposition 2 ([4]). *Let* $R = [\underline{R}, \overline{R}]$*,* $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$*. If* $R \sim S$*, then for every non-empty subset* P *of* $X \times X$ *and each* $x, y, z, t \in P$*, the following conditions are fulfilled*

$$\begin{cases} \underline{R}(x,y) = \sup\limits_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \sup\limits_{(u,v)\in P} \underline{S}(u,v) \\ \\ \overline{R}(z,t) = \sup\limits_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(z,t) = \sup\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}, \tag{5}$$

$$\begin{cases} \underline{R}(x,y) = \inf\limits_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \inf\limits_{(u,v)\in P} \underline{S}(u,v) \\ \\ \overline{R}(z,t) = \inf\limits_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(z,t) = \inf\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}, \tag{6}$$

$$\begin{cases} \underline{R}(x,y) = \sup\limits_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \sup\limits_{(u,v)\in P} \underline{S}(u,v) \\ \\ \overline{R}(z,t) = \inf\limits_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(z,t) = \inf\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}, \tag{7}$$

$$\begin{cases} \underline{R}(x,y) = \inf\limits_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \inf\limits_{(u,v)\in P} \underline{S}(u,v) \\ \\ \overline{R}(z,t) = \sup\limits_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(z,t) = \sup\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}. \tag{8}$$

The converse statement to Proposition 2 is true, and it is enough to assume that only one of the conditions in Eqs. (5)–(8) is fulfilled and that set P is finite. Equivalent relations have connection with the transitivity property. We can obtain for IVFRs and the partial order $\leq$ the following property.

Proposition 3 ([4]). *Let* $R = [\underline{R}, \overline{R}]$*,* $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$*. If* $R \sim S$*, then* R *is transitive if and only if* S *is transitive.*

Now we will present the analogous results to the ones from Proposition 2 and Proposition 3 but for admissible orders $\leq_{L^I}$. We will analyze concrete orders, namely $\leq_{Lex1}$, $\leq_{Lex2}$, $\leq_{XY}$.

Proposition 4. *Let* $R = [\underline{R}, \overline{R}]$*,* $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$*. If* $R \sim_{Lex1} S$*, then for every non-empty subset* P *of* $X \times X$ *and each* $x, y \in P$*, the following conditions are fulfilled:*

$$\underline{R}(x,y) = \inf_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \inf_{(u,v)\in P} \underline{S}(u,v), \tag{9}$$

$$\underline{R}(x,y) = \sup_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \sup_{(u,v)\in P} \underline{S}(u,v). \tag{10}$$

Proof. We will prove condition (9). Let $R \sim_{Lex1} S$, $x, y \in X$, $P \subset X \times X$ and $\underline{R}(x,y) = \inf\limits_{(u,v)\in P} \underline{R}(u,v)$. It means that for every $u, v \in P$ we have $\underline{R}(x,y) \leqslant \underline{R}(u,v)$. It means that for every $u, v \in P$ we have $\underline{R}(x,y) < \underline{R}(u,v)$ or $\underline{R}(x,y) =$

$\underline{R}(u,v)$. By Definition 11 applied to $\leq_{Lex1}$ and assumption $R \sim_{Lex1} S$ we have for every $u, v \in P$ the following cases:

(1) $\underline{R}(x,y) < \underline{R}(u,v) \Leftrightarrow \underline{S}(x,y) < \underline{S}(u,v)$,
(2) $(\underline{R}(x,y) = \underline{R}(u,v),\ \overline{R}(x,y) \leqslant \overline{R}(u,v)) \Leftrightarrow (\underline{S}(x,y) = \underline{S}(u,v),\ \overline{S}(x,y) \leqslant \overline{S}(u,v))$,
(3) $(\underline{R}(x,y) = \underline{R}(u,v),\ \overline{R}(x,y) > \overline{R}(u,v)) \Leftrightarrow (\underline{S}(x,y) = \underline{S}(u,v),\ \overline{S}(x,y) > \overline{S}(u,v))$.

In each case we have: $\underline{S}(x,y) < \underline{S}(u,v)$ or $\underline{S}(x,y) = \underline{S}(u,v)$ for every $u, v \in P$. As a result for every $u, v \in P$ we have $\underline{S}(x,y) \leqslant \underline{S}(u,v)$ which proves that $\underline{S}(x,y)$ is a lower bound of values $\underline{S}(u,v)$ for $u, v \in P$. Using analogous methods we may prove that $\underline{S}(x,y)$ is the greatest lower bound of values $\underline{S}(u,v)$ for $u, v \in P$, i.e. $\underline{S}(x,y) = \inf\limits_{(u,v)\in P} \underline{S}(u,v)$.

Condition (10) may be proven analogously.

By Example 4 we see that supremum and infimum for the remaining ends of intervals are not 'preserved' by the equivalence $\sim_{Lex1}$. Similarly to Proposition 4 we may prove the following result

Proposition 5. *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. If $R \sim_{Lex2} S$, then for every non-empty subset P of $X \times X$ and each $x, y \in P$, the following conditions are fulfilled:*

$$\overline{R}(x,y) = \inf_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(x,y) = \inf_{(u,v)\in P} \overline{S}(u,v), \tag{11}$$

$$\overline{R}(x,y) = \sup_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(x,y) = \sup_{(u,v)\in P} \overline{S}(u,v). \tag{12}$$

By Example 6 we see that supremum and infimum for the remaining ends of intervals are not 'preserved' by $\sim_{Lex2}$.

Remark 3. Analyzing interval-valued fuzzy relations from Examples 5 and 6 we see that for the admissible order $\leq_{XY}$ we do not have similar results to the ones presented in Propositions 4 and 5.

Now we will consider 'preservation' of transitivity with respect to $\leq_{L^I}$ by the equivalence relation $\sim_{L^I}$ for the orders $\leq_{Lex1}$ and $\leq_{Lex2}$.

Proposition 6. *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. If $R \sim_{Lex1} S$, then R is transitive with respect to $\leq_{Lex1}$ if and only if S is transitive with respect to $\leq_{Lex1}$.*

Proof. Let $R \sim_{Lex1} S$ and R be transitive with respect to $\leq_{Lex1}$. Transitivity of R with respect to $\leq_{Lex1}$ means that for $x, y, z \in X$ we have one of the following conditions fulfilled:
(a) $\min(\underline{R}(x,y), \underline{R}(y,z)) < \underline{R}(x,z)$ or
(b) $\min(\underline{R}(x,y), \underline{R}(y,z)) = \underline{R}(x,z)$ and $\min(\overline{R}(x,y), \overline{R}(y,z)) \leqslant \overline{R}(x,z)$.

As a result, by Proposition 4 and the fact that ends of intervals are just real numbers linearly ordered by $\leqslant$, we have the following cases for (a):
(1) If $\underline{R}(x,y) < \underline{R}(y,z)$, then with $P = \{(x,y),(y,z)\}$ we get

$$\underline{R}(x,y) = \min(\underline{R}(x,y), \underline{R}(y,z)) \ \Leftrightarrow\ \underline{S}(x,y) = \min(\underline{S}(x,y), \underline{S}(y,z)),$$

$$\underline{R}(x,y) < \underline{R}(x,z) \ \Leftrightarrow\ \underline{S}(x,y) < \underline{S}(x,z),$$

which proves that

$$\min(\underline{R}(x,y), \underline{R}(y,z)) < \underline{R}(x,z) \ \Leftrightarrow\ \min(\underline{S}(x,y), \underline{S}(y,z)) < \underline{S}(x,z).$$

(2) If $\underline{R}(x,y) > \underline{R}(y,z)$, then similarly

$$\underline{R}(y,z) = \min(\underline{R}(x,y), \underline{R}(y,z)) \ \Leftrightarrow\ \underline{S}(y,z) = \min(\underline{S}(x,y), \underline{S}(y,z)),$$

$$\underline{R}(y,z) < \underline{R}(x,z) \ \Leftrightarrow\ \underline{S}(y,z) < \underline{S}(x,z),$$

(3) If $\underline{R}(x,y) = \underline{R}(y,z)$, then we may prove it analogously to the cases (1) or (2).
For the case b) and condition $\min(\underline{R}(x,y), \underline{R}(y,z)) = \underline{R}(x,z)$, by Definition 11, we obtain the formula

$$\min(\underline{R}(x,y), \underline{R}(y,z)) = \underline{R}(x,z) \Leftrightarrow \min(\underline{S}(x,y), \underline{S}(y,z)) = \underline{S}(x,z).$$

For the case b) and condition $\min(\overline{R}(x,y), \overline{R}(y,z)) \leqslant \overline{R}(x,z)$ we also consider three cases and by Definition 11 we conclude:
(1) $\overline{R}(x,y) < \overline{R}(y,z)$, then $\min(\overline{R}(x,y), \overline{R}(y,z)) = \overline{R}(x,y)$ and since

$$\overline{R}(x,y) < \overline{R}(y,z) \Leftrightarrow \overline{S}(x,y) < \overline{S}(y,z),$$

as a consequence $\min(\overline{S}(x,y), \overline{S}(y,z)) = \overline{S}(x,y)$, so by the second part of Definition 11 we have $\min(\overline{S}(x,y), \overline{S}(y,z)) \leqslant \overline{S}(x,z)$. Similar considerations we may obtain for the remaining cases:

(2) $\overline{R}(x,y) > \overline{R}(y,z)$
(3) $\overline{R}(x,y) = \overline{R}(y,z)$.

As a result S is transitive with respect to $\leq_{Lex1}$, which finishes the proof.

Similarly to Proposition 6 we may prove the analogous statement for the order $\leq_{Lex2}$.

Proposition 7. *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. If $R \sim_{Lex2} S$, then R is transitive with respect to $\leq_{Lex2}$ if and only if S is transitive with respect to $\leq_{Lex2}$.*

We will now apply the results considered in this paper which are related to the equivalence relation of admissible orders described in Proposition 1.

Definition 12. *Admissible orders* $\le_{A_{1,2}}$ *and* $\le_{B_{1,2}}$ *are equivalent* ($\le_{A_{1,2}} \sim \le_{B_{1,2}}$), *if*

$$x \le_{A_{1,2}} y \Leftrightarrow x \le_{B_{1,2}} y, \text{ where } x, y \in L^I. \tag{13}$$

Equivalent pairs of aggregation functions, which generate admissible orders (cf. Proposition 1), generate equivalent admissible orders.

Proposition 8. *Let* A_i, B_i *be aggregation functions for* $i = \{1, 2\}$ *and* $A_1 \sim B_1$, $A_2 \sim B_2$. *If* $\le_{A_{1,2}}$ *(respectively* $\le_{B_{1,2}}$*) is an admissible order, then* $\le_{A_{1,2}} \sim \le_{B_{1,2}}$ *and* $\le_{B_{1,2}}$ *(respectively* $\le_{A_{1,2}}$*) is an admissible order.*

Proof. Let $\le_{A_{1,2}}$ be an admissible order, $x \le_{A_{1,2}} y$ and $x = [\underline{x}, \overline{x}]$, $y = [\underline{y}, \overline{y}]$. By condition (13), interpreted similarly to the way presented in Definition 11, we obtain: if $A_1(\underline{x}, \overline{x}) < A_1(\underline{y}, \overline{y})$, then

$$A_1(\underline{x}, \overline{x}) < A_1(\underline{y}, \overline{y}) \;\Leftrightarrow\; \neg(A_1(\underline{x}, \overline{x}) \geqslant A_1(\underline{y}, \overline{y}))$$

$$\Leftrightarrow \neg(B_1(\underline{x}, \overline{x}) \geqslant B_1(\underline{y}, \overline{y})) \;\Leftrightarrow\; B_1(\underline{x}, \overline{x}) < B_1(\underline{y}, \overline{y}).$$

If $A_1(\underline{x}, \overline{x}) = A_1(\underline{y}, \overline{y})$, then $B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y})$ since

$$A_1(\underline{x}, \overline{x}) = A_1(\underline{y}, \overline{y}) \Leftrightarrow (A_1(\underline{x}, \overline{x}) \leqslant A_1(\underline{y}, \overline{y}) \text{ and } A_1(\underline{y}, \overline{y}) \leqslant A_1(\underline{x}, \overline{x}))$$

and it is enough to use twice definition of $\sim$ relation for B_1, B_2. If $A_2(\underline{x}, \overline{x}) \leqslant A_2(\underline{y}, \overline{y})$, then by definition of $\sim$ we get $B_2(\underline{x}, \overline{x}) \leqslant B_2(\underline{y}, \overline{y})$, so $x \le_{A_{1,2}} y \Leftrightarrow x \le_{B_{1,2}} y$ and $\le_{B_{1,2}}$ is an admissible order.

In the next section we will indicate the application of Proposition 8 in decision making algorithms.

4 Decision Making Algorithm

We consider interval-valued fuzzy relations on $X = \{x_1, \ldots, x_n\}$ (set of alternatives). The preferences over these alternatives will be represented with respect to a finite number of criteria, by relations $R_1, \ldots, R_n \in \mathcal{IVFR}(X)$. We will apply IVOWA (or IVOWG) to aggregate these relations in order to obtain the final result. In definition of IVOWA (IVOWG) we use linear order $\le_{A_{1,2}}$ generated by aggregation functions (cf. Proposition 1). The set of weights w represents the importance of criteria. IVOWA (IVOWG) are widely used in computational intelligence because of their ability to model linguistically expressed aggregation instructions. To find the solution alternative we may apply diverse methods, for example nondominance method or voting methods (cf. [4,17]) but here we will not focus on this issue in detail. To find the selection alternative, in the final step linear orders generated by aggregation functions $\le_{B_{1,2}}$, but not necessarily with the same pair of aggregation functions which are used in IVOWA (IVOWG), may be applied. The following algorithm (general steps) gives a solution alternative.

Algorithm

Inputs : $X = \{x_1, \ldots, x_n\}$ set of alternatives; $R_1, \ldots, R_n \in \mathcal{IVFR}(X)$; linear orders $\leq_{A_{1,2}}$, $\leq_{B_{1,2}}$ generated by $A_1, A_2, B_1, B_2 : [0,1]^2 \to [0,1]$; aggregation operator $IVOWA_{\leq_{A_{1,2}},w}$ (respectively $IVOWG_{\leq_{A_{1,2}},w}$).

Output : Solution alternative: $x_{selection}$ the best alternative with respect to given criteria.

(**Step 1**) Aggregation of given $R_1, \ldots, R_n \in \mathcal{IVFR}(X)$ by the use of $IVOWA_{\leq_{A_{1,2}},w}$ ($IVOWG_{\leq_{A_{1,2}},w}$) obtaining aggregated fuzzy relation R_A

(**Step 2**) Applying the nondominance or voting methods to R_A

(**Step 3**) Ordering the alternatives in a non-increasing way using a linear order $\leq_{B_{1,2}}$

In (Step 3) of the given algorithm we may apply diverse linear orders of the type $\leq_{B_{1,2}}$. If we compare the orders of alternatives obtained with the use of two different but equivalent admissible orders $\leq_{A_{1,2}} \sim \leq_{B_{1,2}}$, then we obtain for both of them the same orders of alternatives (cf. Proposition 8).

5 Conclusion

We introduced the notion of transitivity with respect to an admissible order for interval-valued fuzzy relations. We also discussed its connection with some equivalence relation and consequences of applying both the equivalence relation and the new type of transitivity. For future work it would be interesting to find another (stronger) sufficient condition for equivalence of admissible orders (cf. Proposition 8).

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A Fuzzy Linguistics Supported Model to Measure the Contextual Bias in Sentiment Polarity

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Abstract. The polarity detection problem typically relies on experimental dictionaries, where terms are assigned polarity scores lacking contextual information. As a matter of fact, the polarity is highly dependant on the domain or community it is analysed, so we can speak of a contextual bias. We propose a method supported by fuzzy linguistic modelling to quantify this contextual bias and to enable the bias-aware sentiment analysis. To show how our approach work, we measure the bias of common concepts in two different domains and discuss the results.

Keywords: Sentiment analysis · Polarity · Linguistic modelling · Fuzzy logic · Contextual bias

1 Introduction

The identification of positive and negative opinions, emotions and evaluations summarises the scope of the sentiment analysis task. The use of sentiment analysis is widely spread in many industries to support use cases such as understanding how customers react to product offerings and campaigns based on their social media interactions [3], to make the patient experience after a treatment measurable [9], to predict the outcome of political elections [24].

The analysis of sentiments involves determining whether the sentiment expressed in a phrase or a document is positive or negative. This task, also known as *Polarity detection*, has become more and more relevant with the advent of the web 2.0 and the new technologies supporting user generated content (Forums, Blogosphere, Microblogging, Rating and Reviews, etc.). Thus, sentiment analysis is widely applied to obtain the unfiltered voice of the customer about products, services, companies, politicians, etc. [3,26].

Polarity detection usually relies on polarity values that are defined according to a lookup procedure at term or lemma level. A lexicon of positive and negative words and phrases consists of entries that have been tagged with their prior

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polarity, which means without any context, to which degree the word seems to evoke something positive or something negative. When we for example talk about something like "football", it might have a neutral polarity, but if we are in a domain (e.g.: "sport press" or "sport bets") where we consistently use the word "football" with other words such as "thrilling", "exciting", "miracle" or "fascinating", individuals who are part of this domain tend to associate these positive feelings to the world "football". Bias intuitively refers to this positive "load". Thus, in this domain, when we compute the sentiment of a sentence where the term "football" is present, we should have a mechanism to include this bias in the computation. The value of the bias itself depends on the neighbouring terms' polarity (the near context).

The novelty of the method we propose in this paper tackles following aspects:

– We aim at extracting the bias of a particular term at domain level, unlike other polarity bias studies that focus on the atomic bias modelling (e.g.: in [27]). It requires a high number of occurrences of the particular term within the domain corpus to guarantee certain stability and introduces the need for (a) a threshold and (b) a second indicator to assess how stable (or how volatile) the bias is.
– We express the bias and stability bias using a multi-granular fuzzy linguistic approach, to make sure our indicators remain as generic and human-readable as possible. On the other hand, our linguistic modelling approach allows for the usage of different linguistic hierarchies to incorporate the bias modelling into the different methods of sentiment evaluation without incurring in information loss, which is one of the most useful advantages of our proposal.

This paper is structured as follows: after having introduced the problem and explained at high-level our motivation to solve it, we provide the background references our work builds upon. Then we present our model and the supporting techniques, for example fuzzy linguistic modelling. After discussing the results obtained for bias modelling in 2 different domains, we share our concluding remarks and point to further research directions based on our work.

2 Background

In this section we provide the background information required to describe our model. It is divided in three parts: an introduction to polarity detection, a discussion about the bias modelling applied to polarity and the description of the fuzzy linguistic approach we followed.

2.1 On Polarity Detection

Polarity detection for a sentence or a document can be performed by a variety of techniques, but almost each of them relies on the existence of a pre-trained or manually labelled polarity lexicon or dictionary. In [14], the authors created a dictionary of 6779 terms (4776 being tagged with -1 and 2003 being tagged with 1) to support the task of opinion mining in customer reviews- The well-known Lexicoder [6] aimed at a similar purpose and is even offered in a commercial way. The more recent *Syuzhet* dictionary [17] provides over 10K entries, with scoring ranging from -1 to 1. The Positive Affect Negative Affect Scale technique (PANAS) [25] consists of a psychometric scale for detecting mood fluctuations. In [1], the authors suggest a mapping to negative and positive polarities.

SentiWordNet [2] implements a dictionary based approach to sentiment extraction. Similar to our approach, Part of Speech labelling is used to apply the lexical dictionary to *synsets* or synonym set groups (adjectives, nouns, verbs, and other grammatical classes). The polarity computation of a given text is an aggregation operation across all the existing synsets, each one contributing with their own positive or negative affect score.

In [7], the authors present a dictionary based method to compute the Happiness Index, for which they use the Affective Norms for English Words (ANEW). The values of the happiness index can be mapped to positive or negative polarity values, as shown in [1]. ANEW has been used for many applications, such as extraction of emotional profiles for locations [4]. SentiStrength [22] relies on the existing Linguistic Inquiry and Word Count dictionary [18] to implement supervised and unsupervised classification methods and extract the strength of the sentiments, including polarity. SenticNet [5] applies classification techniques to Natural Language Processing structures to infer the polarity for nearly 14K concepts.

2.2 On Bias Modelling

Bias modelling has been also extensively researched. In our research, bias manifests as a polarity shift. There are plenty of factors that might influence a polarity shift given a particular context, as thoroughly explained in [19]. In [8] the authors demonstrated the presence of a positive bias using an annotated ground truth for 6 different polarity detection methods. In [15], the authors discussed the so called domain search bias, which states that a user's propensity to believe that a page is more relevant might be influenced by the fact that the page is hosted in a particular domain.

In [23], the authors recognized the bias introduced by each user in the product reviews and suggested a method to measure it at user level. Aligned with this research, a machine learning supported classifying approach was taken in [20] to solve the problem of the bias introduced by manual annotations created by non-experts.

In [16] the authors suggests a Bias-Aware Thresholding approach that turns any lexicon-based method into bias-aware by minimizing the so called Polarity Bias Rate ($PBR = \frac{FP+FN}{N}$) by changing the prediction and therefore changing the cost associated with making one type of error over the other.

2.3 On Fuzzy Linguistic Modelling

The fuzzy linguistic approach is a tool based on the concept of linguistic variable proposed by Zadeh [28]. This theory has given very good results to model qualitative information and it has been proven to be useful in many problems.

The 2-Tuple Fuzzy Linguistic Approach. The 2-Tuple Fuzzy Linguistic Approach [11] is a continuous model of information representation that allows reduction in the loss of information that typically arises when using other fuzzy linguistic approaches, both classical and ordinal [10,13,28]. To define it both the 2-tuple representation model and the 2-tuple computational model to represent and aggregate the linguistic information have to be established.

Let $\mathcal{S} = \{s_0, ..., s_g\}$ be a linguistic term set with odd cardinality. We assume that the semantics of labels is given by means of triangular membership functions and consider all terms distributed on a scale on which a total order is defined. In this fuzzy linguistic context, if a symbolic method aggregating linguistic information obtains a value $\beta \in [0, g]$, and $\beta \notin \{0, ..., g\}$, we can represent β as a 2-tuple (s_i, α_i), where s_i represents the linguistic label, and α_i is a numerical value expressing the value of the translation between numerical values and 2-tuple: $\Delta(\beta) = (s_i, \alpha)$ y $\Delta^{-1}(s_i, \alpha) = \beta \in [0, g]$ [11].

In order to establish the computational model negation, comparison and aggregation operators are defined. Using functions Δ and Δ^{-1}, any of the existing aggregation operators can be easily be extended for dealing with linguistic 2-tuples without loss of information [11]. Some examples are:

Definition 1. *Arithmetic mean. Let* $x = \{(r_1, \alpha_1), \ldots, (r_n, \alpha_n)\}$ *be a set of linguistic 2-tuples, the 2-tuple arithmetic mean* $\overline{x}^e$ *is computed as:*

$$\overline{x}^e[(r_1, \alpha_1), \ldots, (r_n, \alpha_n)] = \Delta(\sum_{i=1}^{n} \frac{1}{n} \Delta^{-1}(r_i, \alpha_i)) \\ = \Delta(\frac{1}{n} \sum_{i=1}^{n} \beta_i).$$

Definition 2. *Weighted Average Operator. Let* $x = \{(r_1, \alpha_1), \ldots, (r_n, \alpha_n)\}$ *be a set of linguistic 2-tuples and* $W = \{(w_1, \alpha_1^w), ..., (w_n, \alpha_n^w)\}$ *be their associated weights. The 2-tuple weighted average* $\overline{x}_l^w$ *is:*

$$\overline{x}_l^w[((r_1, \alpha_1), (w_1, \alpha_1^w))...((r_n, \alpha_n), (w_n, \alpha_n^w))] \\ = \Delta(\frac{\sum_{i=1}^{n} \beta_i \cdot \beta_{W_i}}{\sum_{i=1}^{n} \beta_{W_i}}),$$

with $\beta_i = \Delta^{-1}(r_i, \alpha_i)$ *and* $\beta_{W_i} = \Delta^{-1}(w_i, \alpha_i^w)$.

Multi-granular Linguistic Information Approach. To accommodate the requirements of the different sentiment analysis methods, it's important to support different "granularity levels". Certain methods could for example only deal with yes/no values and direction only (e.g.: *"Negative Bias", "No Bias", "Positive Bias"*). Other methods might be able to incorporate higher granularity values in the aggregation operation for the sentiment computation (e.g.: *"Lowest", "Low", "Normal", "High", "Highest"*).

To enable the compatibility of sentiment analysis methods, we need to support the different granularities and provide tools to manage the multi-granular linguistic information. In [12] a multi-granular 2-tuple fuzzy linguistic modelling based on the concept of linguistic hierarchy is proposed.

A *Linguistic Hierarchy*, LH, is a set of levels $l(t, n(t))$, where each level t is a linguistic term set with different granularity $n(t)$. The levels are ordered according to their granularity, so that we can distinguish a level from the previous one, i.e., a level $t+1$ provides a linguistic refinement of the previous level t. We can define a level from its predecessor level as: $l(t, n(t)) \rightarrow l(t+1, 2 \cdot n(t) - 1)$. In [12] a family of transformation functions between labels from different levels was introduced. To establish the computational model we select a level that we use to make the information uniform and thereby we can use the defined operator in the 2-tuple model. This result guarantees that the transformations between levels of a linguistic hierarchy are carried out without loss of information. Using this LH, the linguistic terms in each level are the following:

- $S^3 = \{b_0 = None = N,\ b_1 = Medium = M,\ b_2 = Total = T\}$
- $S^5 = \{c_0 = Lowest = LE,\ c_1 = Low = L,\ c_2 = Normal = N,\ c_3 = High = H,\ c_4 = Highest = HE\}$
- $S^9 = \{c_0 = None = N,\ c_1 = Very\_Low = VL,\ c_2 = Low = L,\ c_3 = More\_Less\_Low = MLL,\ c_4 = Medium = M,\ c_5 = More\_Less\_High = MLH,\ c_6 = High = H,\ c_7 = Very\_High = VH,\ c_8 = Total = T\}$

A graphical example of a linguistic hierarchy is shown in Fig. 1.

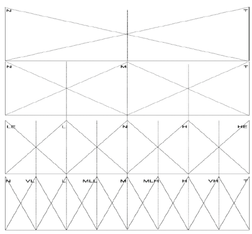

Fig. 1. Linguistic Hierarchy of 1, 3, 5 and 9 labels

3 Modelling the Polarity Bias for a Particular Domain

In this section we describe how our bias model is built up. First we provide the definitions required to explain our model, then we show how to apply fuzzy linguistic modelling to express the bias itself and its stability or volatility.

3.1 Domain Specific Bias and Volatility

We start introducing some preliminary definitions our model builds upon.

Definition 3. *Atomic Polarity Bias. Let N be the set of terms in the semantic neighbourhood of t_i. For the definition of semantic neighbourhood, we recommend using standard natural language processing scopes, such as sentence, paragraph or even document. Let $PS(t_i, D)$ is the polarity score of the term t_i according to the dictionary D.*

We define Atomic Polarity Bias as follows:

$$APB(t_i, D, N) = \sum_{j=1}^{\#N} PS(t_j, D) * \omega(t_j), t_i \neq t_j$$

where $\omega(t_j)$ is a function to provide the a specific weight to the polarity score of the term (t_j). This function can defined depending on different criteria (e.g.: based on the part of speech tag of the term t_j to for example give more importance to the polarity of adjectives or adverbs, based on some quantification of the distance between t_j and t_i to for example assign more weight to closer terms, or just equally distributed for all terms having a polarity score in the dictionary D).

Definition 4. *Bias Computing Threshold. This is the minimum number of documents with occurrences of any term t_i in a Domain Corpus, so that the bias quantification makes sense. It is established for a particular Domain Corpus C and is a constant value $BCT(C) = K$.*

Definition 5. *Polarity Bias. Based on the Atomic Polarity Bias, the Polarity Bias is defined as an aggregation over all documents in the Domain Corpus of the Atomic Polarity Bias*

$$PB(t_i, D) = \frac{1}{\#M} \sum_{j=1}^{\#M} APB(t_j, D, N_j) \quad (1)$$

where M represents the set of documents in the domain corpus, where the term t_i is present. As we are trying to quantify the contextual bias introduced by the repeatedly use of the term in the domain, we impose the condition that $\#M >= K$, being K the established value for the $BiasComputingThreshold$ in the Domain under analysis.

Definition 6. *Bias Stability. This is an indicator for how stable the Bias computation for a particular term is. The minimum value can be the imposed $BCT(C)$ and the maximum of $\#C$. To standardize this value, we define a normalizing* *function* ϵ*, defined as* $\epsilon : [BCT(C), \#C] \longrightarrow [0, 1]$*, which makes the Bias Stability value range between 0 and 1:*

$$BS(t_i, C) = \epsilon(\frac{\#M}{\#C}) \quad (2)$$

where M represents the set of documents in the domain corpus, where the term t_i is present and C the set of all documents in the Corpus.

3.2 Fuzzy Language Bias Modelling

After establishing the fundamentals of fuzzy linguistic modelling and defining the 2-tuple based supporting arithmetic operations to enable the bias-aware computing of sentiment analysis tasks, we define the label sets for concepts of *polarity bias* (Definition 5) and *bias stability* (Definition 6).

For both cases, we are going to use different label sets (S_1, S_2) selected from a *LH* [12]:

- **Polarity Bias indicator** of a term in a particular domain, which is assessed in S_1.
- **Bias Stability indicator** applied to the previous indicator, which is assessed in S_2.

Although this framework guarantees the flexibility in the choice of the *LH*, we suggest using a 2 level *LH* with 3 and 5 labels each one for the *Bias Model stability indicator* and a 2 level *LH* with 5 and 9 labels for the *Polarity Bias indicator* itself. Our suggestion is motivated by the intent of making it more tangible for the reader, but the choice of (S_1, S_2) remains generic and shall be taken depending on the nature of the problem or convenience for further operations.

The *Polarity Bias indicator* allows for example in combination with the *Bias Stability indicator* the correction of the sentiment for a particular sentiment extraction based on the presence of highly-biased terms, as we are going to show in the next section.

4 Experimentation

For our experimentation, we gather up to 460 transcripts from the website TED Talks.[1] The reason why we used this data source is many folded: (a) easy and clean "spoken English" (b) appropriate length of the transcripts (unlike for example micro blogs) and (c) the talks are quite actual.

In addition, each talk has different topic labels assigned, which makes the separation in domains feasible (in fact, it is possible filtering talks that belong to an specific domain). For our experiment, we focused on two domains: *environment* (with topics such as *sustainability, energy, climate change, environment itself, green*) and *education.*

[1] http://www.ted.org.

For the design of our experiment, we just focused on nouns (which required a previous Part of Speech tagger [21]), set the Bias Computing Threshold to 20 (at least a minimum of 20 documents with one or more occurrences of the term) and used a linguistic granularity level of 5 labels in both polarity senses (positive and negative) for the *Bias* and just positive for the *Volatility*.

Discussing the Results. From the terms that satisfied the aforementioned conditions (Bias Computing Threshold = 20) we took the subset of nouns that were present in both domains, resulting in a final number of 744 terms. In the Table 1 we show the distribution of Bias and Volatility for this experiments; this table allows for observing general trends in the bias for the studied domains, for example, we observed as general trend a rather positive bias and quite volatile results -which might point us to increasing the corpus size to have more evidence about the bias quantification-. In Tables 2 and 3 we provide the top 20 most positive and negative biased terms respectively, each one with both indicators (Bias and Volatility) computed for both domains (Education and Environment).

Figure 2 (a) shows the computed crisp value for the top most and least positive terms per domain. Having a look at Fig. 2 (b), we see how the same term can be assigned different bias labels depending on the domain (e.g.: *dead*, or *disease*)... It can be possible, that the same term is assigned a positive bias in a domain but a negative one in the other, which we can observe in terms such as *kill* and *societies*.

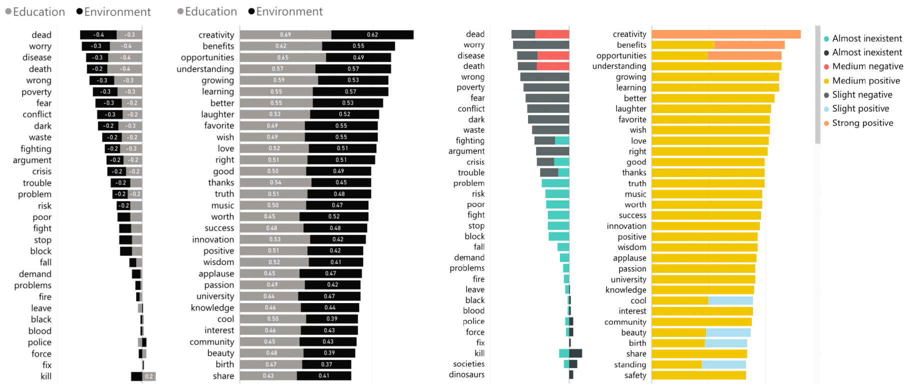

Fig. 2. (a) Top negative and positive biased terms by domain and (b) fuzzy bias quantification labels

For the sake of showing how the polarity bias is computed, we provide in the Fig. 3 a histogram of all atomic polarity bias values for the term *difference* in both domains.

Our approach is to our knowledge, the first attempt to quantify the polarity bias in a domain specific manner. In the Subsect. 3.2 we explained why we

Table 1. Bias-volatility indicators distribution for environment and education domakins

Environment					Education			
Bias\Volatility	Medium	Slightly	Strong	Very strong	Medium	Slightly	Strong	Very strong
Slightly	56	3	243	23	109	7	316	19
Almost inexistent	96	3	262	22	28	0	197	16
Medium	7	0	25	3	10	0	37	2
Strong	0	0	0	1	0	0	2	1

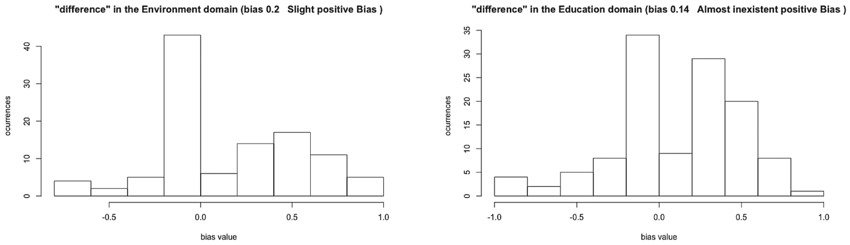

Fig. 3. Histogram of the domain bias for (a) Environment and (b) Education for the term "difference"

Table 2. Top 20 positive biased concepts by domain

	term	domain	occurences	bias.val	fuzzy.label	fuzzy.label.complete	docs	volatility.label
1	people	Environment	1664	0.16	Almost inexistent	Almost inexistent positive	33655	Medium
2	people	Education	1428	0.20	Slightly	Slightly positive	26886	Slightly
3	world	Environment	1366	0.18	Almost inexistent	Almost inexistent positive	33655	Medium
4	world	Education	896	0.24	Slightly	Slightly positive	26886	Medium
5	think	Environment	1371	0.20	Slightly	Slightly positive	33655	Medium
6	think	Education	1187	0.21	Slightly	Slightly positive	26886	Medium
7	time	Environment	1389	0.15	Almost inexistent	Almost inexistent positive	33655	Medium
8	time	Education	1198	0.20	Slightly	Slightly positive	26886	Medium
9	years	Environment	1134	0.16	Almost inexistent	Almost inexistent positive	33655	Medium
10	years	Education	681	0.18	Almost inexistent	Almost inexistent positive	26886	Medium
11	things	Environment	877	0.22	Slightly	Slightly positive	33655	Medium
12	things	Education	775	0.28	Slightly	Slightly positive	26886	Medium
13	laughter	Environment	754	0.52	Medium	Medium positive	33655	Medium
14	laughter	Education	906	0.53	Medium	Medium positive	26886	Medium
15	way	Environment	1654	0.18	Almost inexistent	Almost inexistent positive	33655	Medium
16	way	Education	1271	0.25	Slightly	Slightly positive	26886	Medium
17	right	Environment	810	0.51	Medium	Medium positive	33655	Medium
18	right	Education	762	0.51	Medium	Medium positive	26886	Medium
19	need	Environment	1023	0.23	Slightly	Slightly positive	33655	Medium
20	need	Education	720	0.23	Slightly	Slightly positive	26886	Medium
21	look	Environment	1225	0.21	Slightly	Slightly positive	33655	Medium
22	look	Education	849	0.24	Slightly	Slightly positive	26886	Medium
23	say	Environment	735	0.18	Almost inexistent	Almost inexistent positive	33655	Medium
24	say	Education	893	0.23	Slightly	Slightly positive	26886	Medium
25	thing	Environment	2576	0.21	Slightly	Slightly positive	33655	Slightly
26	thing	Education	2309	0.22	Slightly	Slightly positive	26886	Slightly
27	school	Environment	221	0.24	Slightly	Slightly positive	33655	Strong
28	school	Education	1046	0.22	Slightly	Slightly positive	26886	Medium
29	water	Environment	974	0.14	Almost inexistent	Almost inexistent positive	33655	Medium
30	water	Education	118	0.12	Almost inexistent	Almost inexistent positive	26886	Strong
31	good	Environment	563	0.49	Medium	Medium positive	33655	Medium
32	good	Education	491	0.50	Medium	Medium positive	26886	Medium
33	percent	Environment	573	0.14	Almost inexistent	Almost inexistent positive	33655	Medium
34	percent	Education	357	0.15	Almost inexistent	Almost inexistent positive	26886	Medium
35	year	Environment	1662	0.14	Almost inexistent	Almost inexistent positive	33655	Medium
36	year	Education	1074	0.19	Almost inexistent	Almost inexistent positive	26886	Medium
37	work	Environment	1058	0.27	Slightly	Slightly positive	33655	Medium
38	work	Education	1034	0.28	Slightly	Slightly positive	26886	Medium
39	applause	Environment	501	0.47	Medium	Medium positive	33655	Medium
40	applause	Education	512	0.45	Medium	Medium positive	26886	Medium

Table 3. Top 15 negative biased concepts by domain

	term	domain	occurences	bias.val	fuzzy.label	fuzzy.label.complete	docs	volatility.label
1	dead	Environment	102	-0.43	Medium	Medium negative	33655	Strong
2	death	Education	52	-0.41	Medium	Medium negative	26886	Strong
3	disease	Education	54	-0.40	Medium	Medium negative	26886	Strong
4	worry	Education	32	-0.38	Slightly	Slightly negative	26886	Strong
5	wrong	Education	163	-0.33	Slightly	Slightly negative	26886	Strong
6	worry	Environment	37	-0.33	Slightly	Slightly negative	33655	Strong
7	fear	Environment	64	-0.31	Slightly	Slightly negative	33655	Strong
8	dead	Education	31	-0.30	Slightly	Slightly negative	26886	Strong
9	conflict	Environment	44	-0.30	Slightly	Slightly negative	33655	Strong
10	wrong	Environment	84	-0.29	Slightly	Slightly negative	33655	Strong
11	poverty	Environment	49	-0.29	Slightly	Slightly negative	33655	Strong
12	poverty	Education	52	-0.29	Slightly	Slightly negative	26886	Strong
13	dark	Education	80	-0.28	Slightly	Slightly negative	26886	Strong
14	disease	Environment	107	-0.26	Slightly	Slightly negative	33655	Strong
15	fighting	Education	23	-0.26	Slightly	Slightly negative	26886	Very Strong
16	waste	Education	27	-0.24	Slightly	Slightly negative	26886	Strong
17	dark	Environment	106	-0.24	Slightly	Slightly negative	33655	Strong
18	death	Environment	67	-0.24	Slightly	Slightly negative	33655	Strong
19	fear	Education	47	-0.24	Slightly	Slightly negative	26886	Strong
20	waste	Environment	239	-0.23	Slightly	Slightly negative	33655	Medium
21	trouble	Environment	33	-0.23	Slightly	Slightly negative	33655	Strong
22	conflict	Education	33	-0.23	Slightly	Slightly negative	26886	Strong
23	crisis	Environment	100	-0.22	Slightly	Slightly negative	33655	Strong
24	argument	Education	35	-0.22	Slightly	Slightly negative	26886	Strong
25	argument	Environment	33	-0.20	Slightly	Slightly negative	33655	Strong
26	crisis	Education	34	-0.19	Almost inexistent	Almost inexistent negative	26886	Strong
27	problem	Environment	638	-0.18	Almost inexistent	Almost inexistent negative	33655	Medium
28	fighting	Environment	22	-0.18	Almost inexistent	Almost inexistent negative	33655	Very Strong
29	problem	Education	422	-0.17	Almost inexistent	Almost inexistent negative	26886	Medium
30	risk	Environment	95	-0.16	Almost inexistent	Almost inexistent negative	33655	Strong

employed Fuzzy Language Modelling to guarantee the interoperability across well-established sentiment computation methods using different polarity dictionaries.

4.1 Bias-Aware Sentiment Analysis

After understanding the results, we show a possible way of incorporating the bias modelling might impact/improve the computation of the sentiment for a phrase using well-known schemas. Using the Syuzhet [17] dictionary, we compute for examples the sentiment for the sentence S, *"Unfortunately it is going to end up in a big crisis"*. With the standard implementation, the quantified sentiment is -1.5 *Unfortunately* contributes with -1.00, big with 0.25 and crisis with -0.75. In the *Education* domain, crisis presents an *Almost inexistent negative* bias, while in the *Environment* domain the bias is a bit higher *Slightly negative*. Both values are quite volatile, but a bias correction might make sense for the later domain. It would mean that for the sentence S, $sentiment_{Environment}(S) < sentiment_{Education}(S)$, being the correction the 2-tuple representation of the level *"Slightly negative"*, as explained in the Subsect. 2.3.

5 Concluding Remarks

In this article we propose a method to quantify the polarity bias inherent to a particular domain within the context of sentiment analysis. For a given domain specific corpus, our approach extracts first the terms that are eligible for bias modelling. The eligibility is given by an occurrences number above a given threshold -as semantic bias requires repetitiveness-. These terms are then filtered by Part of Speech label (usually only verbs and nouns are taken, as the polarity of adverbs, adjectives, etc. doesn't make much sense).

Following on that, each remaining topic inherits a polarity scores from neighbouring terms for each occurrence of the term in the domain corpus. These values are then aggregated and normalized and assigned a fuzzy level for a particular granularity level. An additional indicator, stability (or volatility) is also compute for each term to express how "reliable" is the fuzzy indicator of the bias for the term.

Our linguistic modelling approach allows for the usage of different linguistic hierarchies to incorporate the bias modelling in the different methods of sentiment evaluation without incurring in information loss, which is one of the most useful advantage of our proposal. To show the performance of our approach, we modelled the bias of the top 100 nouns in the *Environment* and *Education* domains based on over 400 recent Ted Talks transcripts.

Further research lines might focus on the development of lexical distance specific weighting for the computation of the polarity inheritance at document level. A corpus length and document length dependent modelling might also bring some improvements to our idea, in particular in combination with the analysis of optimal values for the Bias Computing Threshold.

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Generating Load Profiles Using Smart Metering Time Series

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Abstract. In this work we present a practice-oriented approach for generating *load profiles* as a means to forecast energy demand by using *smart metering time series*. The general idea is to apply *fuzzy clustering* on historic consumption time series. The segmentation yielded helps electricity companies to identify customers with similar consumption behavior. This knowledge can be used to plan available energy capacities in advance. What makes this approach special is that this approach segments consumption time series by time in addition to identifying customer groups. This is done not only to accommodate for customers potentially behaving completely different on working days than on local holidays for example, but also to build the resulting load profiles in a way the electricity companies can adapt with minimal adjustments. We also evaluate our approach using two real world smart metering datasets and discuss potential improvements.

Keywords: Big data · Data mining · Knowledge discovery · Clustering · Time series · Smart metering · Load profiles

1 Introduction

When electricity companies are confronted with the task to ensure the availability of energy for every customer, they are faced with the important obligation to announce the demand to the energy producers in advance. This is due to the fact that energy producers require lead time to ensure the availability of necessary capacities. Thus, in order to be able to announce the demand in advance, electricity companies require a means to forecast the total energy demand of their customers.

In general however, the electricity demand is not known in advance. Among other things, this is because not every customer behaves the same on every day; for example, it seems reasonable that during working days energy consumption is different from weekends. In addition, customers like families, singles, agricultural organizations and businesses typically each have different daily routines and thus might genuinely differ in their consumer behavior. To forecast the energy demand

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for all customers, electricity companies use so called *load profiles*. In the case of Germany, the specific profiles used are usually the *standard load profiles* released by the *"Bundesverband der Energie- und Wasserwirtschaft" (Federal association of the energy and water economy)*. These profiles, having been created during the 1990s, have not been adjusted to the technological advances and are thus deemed as an increasingly bad model to forecast the energy demand [28]. Due to the growing competition and legal obligations like the liberalization of the energy market and the policies towards green energy usage, there is an increasingly high interest and necessity to convert the information available thanks to modern technologies into valuable knowledge [15]. In this paper we focus on applying *fuzzy clustering* [5] on *smart metering time series* to generate load profiles as a means to provide such knowledge. We also assess our approach in the way electricity companies in Germany implement load profiles; we do think however that assessments in other regions might be comparable.

2 Related Work

Along with the increasing amount of data electricity companies and regulators can measure, process and analyze during the past decades, research towards understanding and making use of this kind of data gained significant attention [10,19,24]. These fields of research include, but are not limited to, outlier detection [29], marketing and tariff optimizations [9,21]. [2,22] present methods to predict long-term changes of the aggregated energy load. Publications that rely on clustering often choose *K-Means* to analyze the data [4,11,16,23,25] or approaches based on it [20]. Newer publications also increasingly use *fuzzy clustering* [19,27,29] and include newer technologies like *Smart Metering* [3,12].

In this paper, we continue our previous work [6] and present an approach based on *fuzzy clustering* [5] for electricity companies to generate load profiles using smart metering time series. We focus on not only identifying *customer groups* to aggregate customers with similar consumption behavior, but also to dynamically find *day types*, which help to model long- and short-term periodic changes in consumer behavior. We also assess our approach using two real-world smart metering datasets and grade the quality of the energy forecast in a way closely related to how electricity companies would evaluate the generated load profile in a production environment.

3 Structure of Load Profiles

When electricity companies forecast the energy demand and negotiate corresponding capacities with producers, they typically assign one *load profile* and a *year consumption forecast (YCF)* to each customer. Load profiles work by segmenting all calendar days into groups of so called *day types* and by segmenting all customers into *customer groups*. The idea behind the usage of day types is that customers tend to have a finite set of genuinely different daily routines. Examples for this type of behavior are that office employees typically consume less energy at home on working days and more on weekends, while seasonal events like funfairs orientate themselves by local holidays. Load profiles allow for different day types to accommodate for these trends. However, each customer group gets assigned exactly one consumption pattern on a given day type, which is why this model expects a customer to behave the same for all calendar days belonging to the same day type. This is a rather strong simplification as customers rarely behave exactly the same, even on days with the same daily routine. Thus, load profiles expect a consumption pattern assigned to a set of customers to be representative and to even out deviations between individual consumer behaviors and their associated consumption pattern. This characteristic makes load profiles bad at forecasting the energy demand of a single user, but useful at the use case relevant for electricity companies, which is predicting the aggregated total demand. Thus, a good day type segmentation groups calendar days in a way such that the total demand over the course of a day on calendar days belonging to the same day type are as similar as possible while aiming to make the total demand on calendar days belonging to different day types as dissimilar as possible. This task description closely matches the goal of generic clustering algorithms which is why we opted to rely on *fuzzy clustering* for our approach.

Another important aspect of load profiles is the aforementioned *year consumption forecast (YCF)* assigned to each customer. This *YCF* is needed because the consumption patterns used by load profiles are represented by one normalized time series. The normalization is done to ensure that the consumption patterns describe only the shape of the consumer behavior, rather than the total amount used. When used in practice, the model will first determine what day type the calendar day for which a prognosis shall be generated belongs to. Depending on the regularity and complexity of day type patterns, this can be done via rule sets or using the help of an analyst. The appropriate consumption pattern of each load profile is then chosen depending on the day type. The aforementioned *YCF* of each customer is then used to scale the applicable consumption pattern to match the estimated demand. Specifically, scaling is done so that the estimated total energy demand from the beginning to the end of a given year equals the *YCF*, which in turn requires the model to be able to forecast the day type segmentation for an entire year. Including the *YCF* as part of the model accommodates for customers with similar daily routines but different total consumption. In the next section, we describe an approach to generate such load profiles.

4 Generating Load Profiles

Due to the way load profiles are structured when used in practice, our approach for building them consists of three stages:

1. Determine the optimal number of day types as well as their segmentation onto the individual calendar days. In addition, compose rules to classify future calendar days.

2. For each day type, determine the optimal number of consumption patterns and their characteristics.
3. Compile load profiles by combining former results and assign a profile to each customer.

4.1 Day Type Segmentation

The most important criteria for the day type segmentation is finding calendar days where the total energy consumption is sufficiently dissimilar. In addition, the quality of the process of building the partitions must satisfy the properties of being independent of the number of measurements available and focussing on the *shape* of the consumption time series rather than on the *amount* of energy consumed. Requiring these properties is based on multiple reasons. First and foremost, independence from the number of customers is desirable because it enables electricity companies with both small and big customer bases to use the approach. It also helps to diminish the impact of missing values introduced by temporary malfunctions in the way measurements are gathered, transmitted and processed. As we demonstrate in Sect. 5.1, management of missing values in the available time series is a non-negligible task in real-world datasets. In addition, focusing on the *shape* of the consumption time series is required because the daily routine that will be predicted by the load profiles is scaled using the *year consumption forecast (YCF)* as mentioned in Sect. 3. Thus, we want customers whose consumption behaviors differ almost only by a scalar to be assigned to the same load profile. The *YCF* of a given customer for the current year itself is usually known in advance; common ways electricity companies calculate the *YCF* include setting it equal to the total energy consumed in the previous year by said customer, or using a moving average of the total consumption over the last periods. Because of this, even though the *YCF* is used in combination with the load profiles when forecasting the energy demand, it is not part of the process of building the load profiles themselves. As a result, for the purpose of this paper, we assume the *YCF* to be known when evaluating our approach.

To construct the day type segmentation, we first build a new time series $X = \{x_1, x_2, ..., x_T\}$ for each point in time $t_j, 1 \leq j \leq T$ using the smart metering time series $S_i, 1 \leq i \leq N$ as follows:

$$x_j := \frac{1}{N_j} \sum_{i=1}^{N} \frac{s_{i,j}}{YCF_{i,j}} \tag{1}$$

Here, N stands for the number of distinct customers and N_j represents the total number of measurements available for t_j. The term $YCF_{i,j}$ is a time series specific scalar we use to normalize each customer; it represents the aforementioned *year consumption forecast* for the customer associated with S_i for the year that t_j belongs to. This enables us to solely concentrate on the *shape* of the consumption time series during clustering. The time series X can be vividly descriped as an average time series of all normalized smart metering time series. Using X we subsequently construct dataset D with elements d_l as follows:

$$D := \left\{ d_l := (x_j, ..., x_{j+m}) \,\middle|\, \begin{array}{c} \forall a \text{ with } 1 \leq j \leq a \leq j+m \leq T : \\ t_a \text{ belongs to the } l\text{-th calendar day} \end{array} \right\} \quad (2)$$

The term $(m + 1)$ describes the number of measurements per day. Since smart metering time series are typically measured at fixed points in time, e.g. every 15, 30 or 60 min, each d_l corresponds to a 96-, 48- or 24-tuple, respectively. Afterwards we apply clustering on the dataset to retrieve a good day type segmentation. In principle, an arbitrary clustering algorithm can be used for this task. For our purposes, we have opted to use *Fuzzy-C-Means* [5] as the clustering algorithm and repeat the clustering process with different values for the number of clusters c; the optimal value for c is then determined using a variety of *Cluster Validity Indices* [7]. The reason we chose Fuzzy-C-Means is its tendency to build spherical clusters [7] as this better conforms to the way load profiles are expected to even out derivations between the individual consumption time series belonging to the same customer group by using a representative consumption pattern as outlined in Sect. 3. The optimal clustering yielded by this procedure is the desired day type segmentation. By knowing which d_l got assigned to the same cluster and which calendar days they represent, it is possible to determine which calendar days belong to the same day type. For categorizing future calendar days we rely on the expertise of an analyst to review the day type segmentation and derive rulesets based upon obeserved regularities.

4.2 Identifying Typical Consumption Patterns

To determine the optimal number and characteristics of customer groups and their corresponding consumption patterns, we look at the smart metering data available for each day type separately. For this purpose, let $K_n, 1 \leq n \leq L$ be the sets of day types built in Sect. 4.1 where each K_n contains its matching t_j. We then construct the disjoint sets $P_n, 1 \leq n \leq L$ with elements $p_{e,n}$ as follows:

$$P_n := \left\{ (y_{i,j}, ..., y_{i,j+m}) \,\middle|\, \begin{array}{c} \forall a, b \text{ with } 1 \leq j \leq a, b \leq j+m \leq T : \\ t_a, t_b \in K_n \text{ and } y_{i,a}, y_{i,b} \text{ belong to} \\ \text{the same calendar day} \end{array} \right\} \quad (3)$$
$$\text{with } y_{i,j} := \frac{s_{i,j}}{YCF_{i,j}}$$

Each P_n is, similar to D, a dataset where smart metering measurements have been aggregated to $(m+1)$-tuples. P_n however does only contain data belonging to the day type K_n and contains the individual normalized measurements $y_{i,j}$ rather than the average of the normalized measurements. Each P_n is then individually segmented using clustering. Contrary to identifying the day type segmentation however, this time we are restricted to centroid-based clustering algorithms. The reason for this is that the cluster prototypes $C_{q,n}, 1 \leq q \leq c_{n,optimal}$

for a given P_n that have been deemed optimal by the algorithm directly correspond to the desired typical consumption patterns for the day type K_n. For our experiments, similar to the day type segmentation, we have opted to use *Fuzzy-C-Means*, try different values for the number of clusters c and evaluate each segmentation using *Cluster Validity Indices*.

4.3 Compiling Load Profiles

Load profiles as outlined in Sect. 3 can be represented as a set of L-tuples where the n-th entry contains the consumption pattern to use for calendar days assigned to K_n. In order for a set of load profiles to be usable however, we also require a way to assign each customer exactly one load profile; ideally the one that best suits him. Thus we propose to individually build the optimal load profile for a given customer based on the available consumption patterns and assign the constructed load profile to the customer in the process. For this purpose, let each customer be represented by its smart metering time series S_i. We then propose that for the day type K_n a given customer gets assigned to the consumption pattern $C_{q,n}$ if the highest membership degrees of S_i-based elements of P_n most commonly point to $C_{q,n}$.

Algorithm 1. Compiling load profiles

Input: $S_i, P_n, K_n, C_{q,n}, U_n$
Output: set of all load profiles G, set of profile assignments Z

$G \leftarrow \emptyset$
$Z \leftarrow \emptyset$
for $i = 1$ **to** N **do**
  for $n = 1$ **to** L **do**
    $H[n] \leftarrow C_{q,n}$ with $q := \arg\max_{q'} \left| \left\{ p_{e,n} \,\middle|\, \begin{array}{c} \exists j : (y_{i,j}, ..., y_{i,j+m}) = p_{e,n} \\ \wedge \nexists q'' : u_{q'',e,n} > u_{q',e,n} \end{array} \right\} \right|$
  end for
  $G \leftarrow G \cup H$
  $Z \leftarrow Z \cup (S_i, H)$
end for
return G, Z

This procedure is illustrated in Algorithm 1. Here, the profile assignments Z are described by a set of 2-tuples where the first entry contains the customer S_i and the second entry the load profile H constructed for him. H itself is a L-dimensional array with $H[n]$ containing the consumption pattern for the day type K_n. The $u_{q,e,n} \in U_n$ used in Algorithm 1 correspond to the final membership degree of the dataset-tuple $p_{e,n}$ towards the consumption pattern $C_{q,n}$ we determined in Sect. 4.2 via *Fuzzy-C-Means*.

5 Evaluation

5.1 Description of Datasets

To evaluate the performance of our approach we have used two real world smart metering datasets which are both visualized in Fig. 1.

The first one, which we will call the *BTU-Dataset*, contains a total of 7668 distinct customers with a resolution of 1 measurement every hour over the course of 26 months. Because this dataset is provided in cooperation with a German electricity company who had a complete rollout of smart meters, we are able to test our approach under realistic conditions. The dataset is maintained and made available by the *BTU EVU Beratung GmbH* [1].

The second dataset, which we will refer to as the *CER-Dataset*, consists of 6445 distinct Irish customers with 1 measurement every 30 min over the course of 18 months. It is provided by the *Irish CER (Commission for Energy Regulation)* and accessed via the *Irish Social Science Data Archive (ISSDA)* [8].

Since both datasets contain real world data they are also subject to temporary technical failures, e.g. by any of the smart metering devices installed in the homes of the consumers or by network transmission errors. In either of these cases, a *missing value* is introduced into the respective dataset.

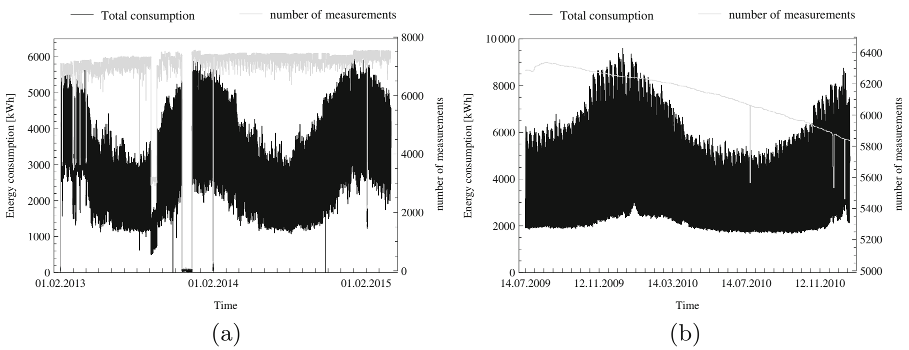

Fig. 1. Overview of (a) the *BTU-Dataset* and (b) the *CER-Dataset*. The black colored graphs show the sum of the energy consumption of all customers (applied on the primary axis); the grey colored graphs show the number of non-missing values from distinct customers available for a given point in time (applied on the secondary axis).

5.2 Results

To derive the optimal segmentation of the day types and consumption patterns for both datasets, we have preprocessed each dataset according to Eqs. 2 and 3. As for the *year consumption forecast* required to normalize each time series, we used the total energy consumed per customer per year:

$$YCF_{i,j} := \sum_{j' \in Z_j} s_{i,j'} \quad \text{with} \quad Z_j := \left\{ j' \,\middle|\, t_{j'} \text{ belongs to the same year as } t_j \right\} \tag{4}$$

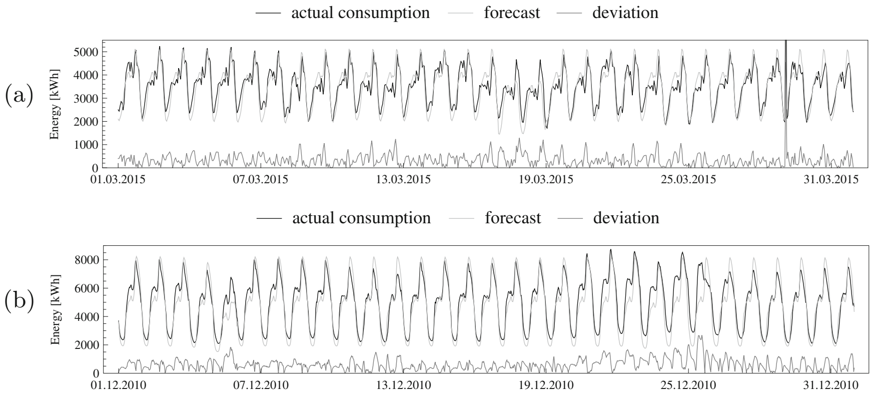

Fig. 2. Comparison of the actual consumption (black graph) and the consumption predicted using the load profiles based on 2 day types and 2 consumption patterns per day type (light gray graph) for (a) the *BTU-Dataset* and (b) the *CER-Dataset*. The dark gray graph shows the absolute difference of the actual consumption and the forecast. The apparent outlier on march 29 is due to a switch from *standard time* to *daylight saving time*.

To handle the *missing values* present in both datasets, we have incorporated the *Partial Distance Strategy* [14] adaptation of *Fuzzy-C-Means*, which has shown a solid performance in experimental evaluations [17], using different values for c $(2 \leq c \leq 25)$. Since *Fuzzy-C-Means* uses random coordinates as the starting configuration of the cluster prototypes, we have opted to improve the statistical significance of our results by independently repeating the clustering process for every value of c a total of 100 times each. We have then used the *Cluster-Validity-Indices Partition Coefficient* [7], *Normalized Partition Coefficient* [17], *Compactness & Separation by Xie and Beni* [7], *Compactness & Separation by Bouguessa, Wang and Sun* [7], *Fuzzy Hypervolume* [13] and *Partition Density* [13] to evaluate each clustering and choose the best day type segmentation as the basis for the following consumption pattern analysis, as well as choose the best consumption pattern segmentation as the basis for the following load profile compilation.

Some of these day type segmentations are visualized in Fig. 4 and are further discussed in Sect. 5.3. In order to test the accuracy of the profiles, we have excluded the last month of smart metering data from both the *BTU-Dataset* and the *CER-Dataset* while building the load profiles; we have then used the formerly excluded month to compare the actual consumption in that month with the one predicted by the profiles. Some of our results are exemplary shown in Fig. 2. If an electricity company were to use these load profiles, they would plan their buy-in of energy according to the *forecast*-graph. Because the forecast is known and necessary capacities can be planned for in advance, they are relatively cheap from a business standpoint. Deviations from the actual total consumption however, both by overestimating and underestimating the actual demand, require extremely short-term adjustments in

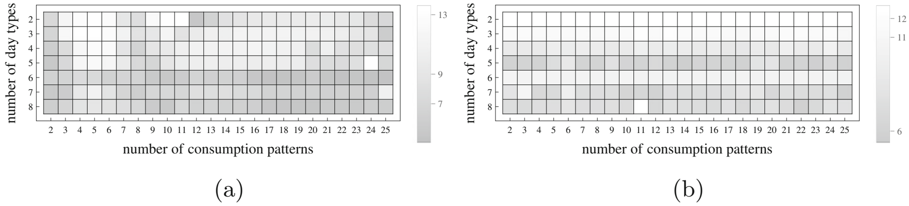

Fig. 3. Ratio of the deviations and the actual consumption in percent yielded by the load profiles generated using different values for the number of day types and the number of consumption patterns. The graphs visualize the results for (a) the *BTU-Dataset* and (b) the *CER-Dataset*.

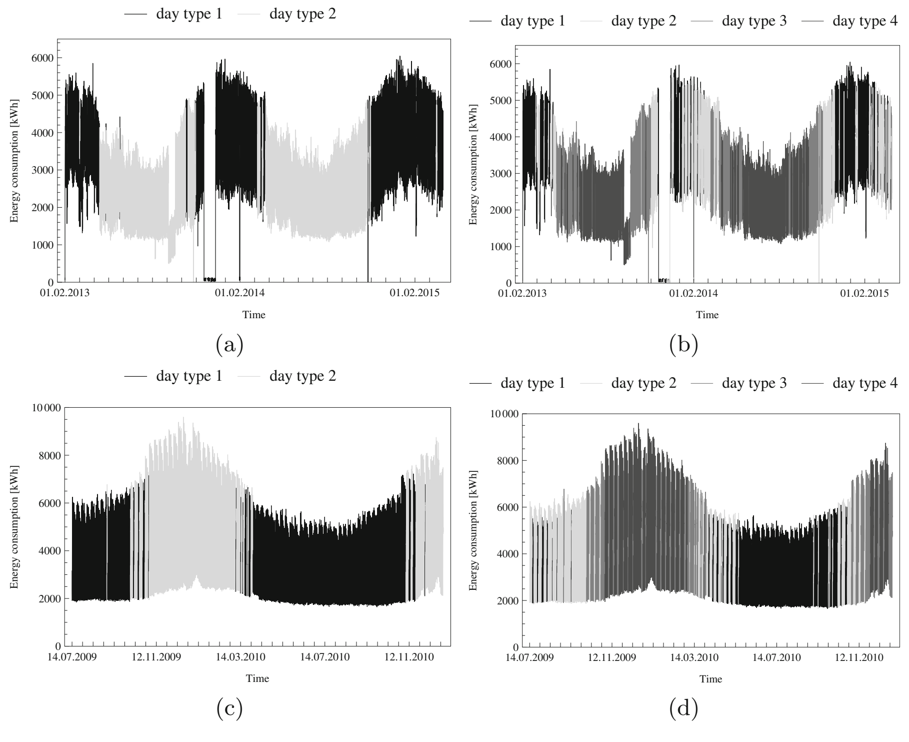

Fig. 4. Overview of the segmentations of day types for the (a) (b) *BTU-Dataset* and (c) (d) *CER-Dataset* yielded by using different values for the numbers of day types. The graphs have been colorized depending on which cluster the total consumption time series has been assigned to on a given day.

the amount of energy circulating in the energy grid. Because of the ad hoc nature, their limited availability and the importance of these adjustments in terms of preventing electricity outages, the cost of these reserves are generally much higher than long-term agreements with producers. Thus, electricity companies typically aim to keep deviations to a minimum and assess load profiles by the amount of energy they are required to trade using the aforementioned short-term reserves to meet the actual demand. This performance can be made comparable between electricity companies of different sizes by looking at the ratio of the deviations and the actual consumption [18]. The results of our approach are shown in Fig. 3. These results pose a significant improvement over the standard load profiles that are in use by most electricity companies today. Electricity companies using standard load profiles typically achieve ratios of roughly 14% [18]. In the next section, we will present and discuss ideas to further improve the accuracy of load profiles generated by our approach.

5.3 Improvements

The idea behind the segmentation of the total energy consumption is to identify periods of time in which consumption patterns genuinely differ from one another. Since we are only interested in accurately forecasting the total energy demand rather than focussing on predicting the consumption of individual customers, it is reasonable to use the same consumption patterns for each customer on days where the total consumption is not expected to be significantly dissimilar. Figure 4 shows a subset of the segmentation we got for the day types for the *BTU-Dataset* and the *CER-Dataset*. One striking property of the clustering using four or less day types is that it roughly resembles the seasonal segmentation manually chosen in other publications, e.g. in [26,27]. Increasing the number of day types further however, it becomes progressively clear that, for both datasets, the identification of day types has approximately resulted in segmenting the data according to a threshold filter. This fact by itself does not necessarily mean the segmentation is flawed; however, while the load profiles based on these day type segmentations have resulted in a significant performance improvement compared to the standard load profiles as pointed out in Sect. 5.2, the repeating sine-shaped pattern visible in the graphs has prevented our approach to compare the daily consumption tuples based on their shape.

This yearly periodic pattern with its peak near the end of december is something we also see in a similar fashion in many other (non smart metering based) total consumption time series. Because of this, we propose to apply a *high-pass filter* on the dataset to create the load profiles based only on the daily consumption behavior of the individual customers. We propose this high-pass filter to be used during both the day type segmentation as well as the identification of typical consumption patterns. The yearly periodic pattern is then reapplied onto the time series when the load profiles have been later used to forecast the energy demand. A candidate to fulfill these requirements is the *Fourier transformation*, where the lowest-frequency terms from the total consumption time series can be used to describe and filter the yearly periodic patterns from the dataset.

Another optimization we propose is to change the function for computing the dissimilarity between tuples. For our experimental results presented in Sect. 5.2 we have used the *partial distance*, an adaptation of the *euclidean distance* for missing values originally introduced in [14]. However, to make the process of generating load profiles more sensitive to minimize these costly deviations between the total energy consumption and its forecast, we propose to adapt the *manhattan distance* to handle missing values. Using this distance function and the high-pass filter derived by using the *Fourier transformation* more closely complies to our original vision to compare consumer behavior based on their shape and optimize for low deviations between forecast and the actual total consumption.

6 Conclusion and Future Work

In this paper we have introduced a clustering method for generating load profiles using smart metering time series. In order to tailor our approach to the specific needs of electricity companies we have incorporated the use of consumption patterns and day types the same way they are treated by the industry. Furthermore, our method does predetermine neither the number nor the shape of the consumption patterns or day types. Our findings show that using the presented approach results in a significant improvement regarding the deviations between the forecasted total demand and the actual energy consumption compared to the standard load profiles typically in use. This helps electricity companies to better plan the buy-in of energy ahead of time, which lowers costs and improves the security of the energy supply. In addition, we have presented and discussed possible enhancements for our method which we plan to further investigate in future publications.

As of now, our approach requires the load profiles to be generated from scratch each time new smart metering data is available. While this is not a major concern for typical use cases since load profiles are usually changed at most once per year, there is an interest to reduce the required computation time, e.g. by looking into ways incrementally add new smart metering data as it becomes available. This might be one potential area for future research.

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Uninorms on Bounded Lattices – Recent Development

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Abstract. The main goal of this paper is to explore whether on every bonded lattice L, possessing incomparable elements, one can choose incomparable elements $\mathbf{e}$ and $\mathbf{a}$ and then to construct a uninorm on L having $\mathbf{e}$ and $\mathbf{a}$ as its neutral and absorbing elements, respectively. Some necessary and some sufficient conditions for construction of uninorms on L which are neither conjunctive nor disjunctive, are given. Example of an infinite bounded lattice on which only conjunctive and disjunctive uninorms exist is presented.

Keywords: Bounded lattice · Conjunctive uninorm · Disjunctive uninorm · Uninorm on bounded lattice · Uninorm which is neither conjunctive nor disjunctive

1 Introduction

Uninorms on the unit interval are special types of aggregation functions since, due to their associativity they can be straightforwardly extended to n-ary operations for arbitrary $n \in \mathbb{N}$. They are important in various fields of applications, e.g., neuron nets, fuzzy decision making and fuzzy modelling. They are interesting also from a theoretical point of view. Recently they have been studied on bounded lattices (see, e.g., [2,6,7,14,15]).

Uninorms were introduced by Yager and Rybalov [18]. Special types of associative, commutative and monotone operations with neutral elements had been already studied in [5,8,9]. Deschrijver [6,7] has shown that on the lattice $L^{[0,1]}$ of closed subintervals of the unit interval there exist uninorms which are neither conjunctive nor disjunctive (i.e., whose absorbing element is different from both, $\mathbf{0}$ and $\mathbf{1}$). Particularly, he constructed uninorms having the neutral element $\mathbf{e} = [e, e]$, where $e \in]0, 1[$. In [15] the authors have shown that on arbitrary bounded lattice L it is possible to construct a uninorm regardless which element of L is chosen to be the neutral one. A different type of construction of uninorms on bounded lattices was presented in [2].

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In [13] construction of a uninorm for arbitrary pair $(\mathbf{e}, \mathbf{a})$ of incomparable elements such that $\mathbf{e}$ is the neutral element and $\mathbf{a}$ the absorbing one, was presented. In [12] the author showed that on some special bounded lattices, one can construct operations which are both, proper uninorms and nullnorms, meaning that their neutral, as well as absorbing elements are different from both, $\mathbf{0}$ and $\mathbf{1}$.

At the International Symposium on Aggregation and Structures ISAS 2016, Ince et al. [11] presented a finite lattice where it is not possible to construct a uninorm whose pair of neutral and absorbing element is chosen arbitrarily. On Fig. 1 a lattice L_1 is depicted which is a slightly modified version of that one, presented by Ince et al. [11]. It is not possible to choose, say, element α_3 as the neutral one and β_3 as the absorbing one. However, one can choose, e.g., α_4 to be the neutral element, and α_5 to be the absorbing element.

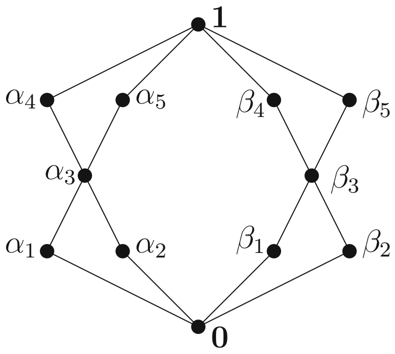

Fig. 1. Lattice L_1

The aim of this contribution is to answer the question whether, on arbitrary bounded lattice L possessing incomparable elements, it is possible to find incomparable elements $\mathbf{a}, \mathbf{e} \in L$ and then to construct a uninorm on L such that $\mathbf{e}$ would be its neutral element and $\mathbf{a}$ its absorbing element.

2 A Short Review of Known Notions and Properties

This paragraph consists of two parts. In the first part operations on $[0, 1]$ are reviewed. The other part is devoted to operations on bonded lattices.

2.1 Uninorms on $[0, 1]$

Several types of associative, commutative and monotone (isotone) operations are known. An operation $*\colon [0, 1]^2 \to [0, 1]$ possesses:

(NE) neutral element $e \in [0, 1]$, if for every $x \in [0, 1]$

$$x * e = e * x = x,$$

(AE) absorbing element (called also annihilator) $a \in [0,1]$, if for every $x \in [0,1]$

$$x * a = a * x = a,$$

(IE) idempotent element $i \in [0,1]$, if $i * i = i$,
(0D) zero-divisor $z \in]0,1[$, if there exists $x \in]0,1[$ such that

$$z * x = 0,$$

(1D) one-divisor $z \in]0,1[$, if there exists $x \in]0,1[$ such that

$$z * x = 1.$$

Lemma 1. *Let* $*\colon [0,1]^2 \to [0,1]$ *be an associative commutative and monotone operation. Then* $*$ *has an idempotent element* i *which is also absorbing element. If* $*$ *has a neutral element* e *then* 0, 1 *and* e *are idempotent elements. Further,* $a = 0 * 1$ *is the absorbing element of* $*$.

Schweizer and Sklar [17] introduced the notion of a triangular norm (t-norm for brevity).

Definition 1 ([17]). *An operation* $T\colon [0,1]^2 \to [0,1]$ *is a t-norm if it is associative, commutative, monotone, and* 1 *is its neutral element.*

T-norms and t-conorms are dual to each other. If $T\colon [0,1]^2 \to [0,1]$ is a t-norm, then

$$S(x,y) = 1 - T(1-x, 1-y)$$

is the dual t-conorm to T. For details on t-norms and t-conorms see, e.g., [16].

As a generalization of both t-norms and t-conorms Yager and Rybalov [18] proposed the notion of uninorm.

Definition 2 ([18]). *An operation* $U\colon [0,1]^2 \to [0,1]$ *is a uninorm if it is associative, commutative, monotone, and if it possesses a neutral element* $e \in [0,1]$.

A uninorm U is *proper* if its neutral element $e \in]0,1[$.

Every uninorm has an absorbing element. A uninorm with the absorbing element 0 is conjunctive, and a uninorm with absorbing element 1 is disjunctive.

Lemma 2 ([18]). *Let* $U\colon [0,1]^2 \to [0,1]$ *be a uninorm whose neutral element is* e. *Then its dual operation*

$$U^d(x,y) = 1 - U(1-x, 1-y)$$

is a uninorm whose neutral element is $1-e$. *Moreover,* U *is conjunctive if and only if* U^d *is disjunctive.*

Results in paper [18] imply the following assertion.

Lemma 3. *Let* $U\colon [0,1]^2 \to [0,1]$ *be a uninorm whose neutral element is* e*. Then there exists a t-norm* $T_U\colon [0,1]^2 \to [0,1]$ *and a t-conorm* $S_U\colon [0,1]^2 \to [0,1]$ *such that*

$$(\forall x, y \in [0,e]^2)(U(x,y) = T_U(\tfrac{x}{e}, \tfrac{y}{e})),$$
$$(\forall x, y \in [e,1]^2)(U(x,y) = S_U(\tfrac{x-e}{1-e}, \tfrac{y-e}{1-e})).$$

Lemma 4 ([18]). *Assume* U *is a uninorm with neutral element* e*. Then:*

1. *for any* x *and all* $y > e$ *we get* $U(x,y) \geq x$*,*
2. *for any* x *and all* $y < e$ *we get* $U(x,y) \leq x$*.*

Nullnorms as operations were proposed in the paper by Calvo et al. [3].

Definition 3 ([3]). *An operation* $V\colon [0,1]^2 \to [0,1]$ *is a nullnorm if it is associative, commutative, monotone and with an absorbing element* $a \in [0,1]$ *and moreover*

$$(\forall x \leq a)\; V(x,0) = x,$$
$$(\forall x \geq a)\; V(x,1) = x.$$

We say that a nullnorm V is *proper* if its absorbing element $a \in]0,1[$.

For more details on associative monotone operations on $[0,1]$ see, e.g., monographs [4,10].

2.2 Uninorms on Bounded Lattices

Detailed information on bounded lattices can be found in the monograph [1]. Recall that on every lattice $(L, \leq_L)$ there exists a partial order $\leq_L$. This order induces two binary operations *meet*, $\wedge$, and *join*, $\vee$. For every $x, y \in L$, $x \vee y$ is the greatest lower bound of x, y, and $x \wedge y$ is the lowest upper bound of x, y.

For incomparable elements $x, y \in L$ the notation will be $x \parallel y$.

On every bounded lattice $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ one can define *t-norms* (*t-conorms*, *proper uninorms*) as associative commutative and monotone operations having $\mathbf{1}$ ($\mathbf{0}$, an element $\mathbf{e} \in L$, respectively) as neutral element. Some examples of t-norms and t-conorms are

$$T_\perp(x,y) = \begin{cases} x \wedge y & \text{if } x = \mathbf{1} \text{ or } y = \mathbf{1}, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$
$$T_M(x,y) = x \wedge y,$$
$$S_\top(x,y) = \begin{cases} x \vee y & \text{if } x = \mathbf{0} \text{ or } y = \mathbf{0}, \\ \mathbf{1} & \text{otherwise,} \end{cases}$$
$$S_M(x,y) = x \vee y.$$

T_M and $T_\perp$ are the greatest and the least t-norm, respectively. S_M and $S_\top$ are the least and the greatest t-conorm, respectively.

Based on an arbitrary t-norm and a t-conorm defined on intervals $[\mathbf{0}, \mathbf{e}]$ and $[\mathbf{e}, \mathbf{1}]$, respectively, two uninorms can be constructed [2]:

$$U_c(x,y) = \begin{cases} S(x,y) & \text{if } x > \mathbf{e}, y > \mathbf{e}, \\ x & \text{if } y \geq \mathbf{e}, x \not> \mathbf{e}, \\ y & \text{if } x \geq \mathbf{e}, y \not> \mathbf{e}, \\ T(x,y) & \text{if } x < \mathbf{e}, y < \mathbf{e}, \\ T(x \wedge \mathbf{e}, y \wedge \mathbf{e}) & \text{otherwise,} \end{cases}$$

$$U_d(x,y) = \begin{cases} T(x,y) & \text{if } x < \mathbf{e}, y < \mathbf{e}, \\ x & \text{if } y \leq \mathbf{e}, x \not< \mathbf{e}, \\ y & \text{if } x \leq \mathbf{e}, y \not< \mathbf{e}, \\ S(x,y) & \text{if } x > \mathbf{e}, y > \mathbf{e}, \\ S(x \vee \mathbf{e}, y \vee \mathbf{e}) & \text{otherwise.} \end{cases}$$

Then U_c is a conjunctive uninorms and U_d is a disjunctive uninorm, each of them with its neutral element $\mathbf{e}$.

For arbitrary uninorm $U\colon L \times L \to L$ on L with a neutral element $\mathbf{e}$ the operation $T_U = U \restriction [\mathbf{0}, \mathbf{e}]^2$ is the *underlying t-norm of* U, and the operation $S_U = U \restriction [\mathbf{e}, \mathbf{1}]^2$ is the *underlying t-conorm of* U.

3 Uninorms Which Are Neither Conjunctive nor Disjunctive

In this section some conditions, under which it is possible to construct uninorms on a bounded lattice L which are neither conjunctive nor disjunctive, will be explored.

Definition 4. *Let L be a bounded lattice and $U\colon L \times L \to L$ be a uninorm whose absorbing element is $\mathbf{a} \in L \setminus \{\mathbf{0}, \mathbf{1}\}$. Then U is called the* third type uninorm.

First, consider again the lattice L_1 from Fig. 1.

Lemma 5. *Let $U\colon L_1 \times L_1 \to L_1$ be a uninorm whose neutral element is $\alpha_3 \in L_1$. Then U is either conjunctive or disjunctive.*

Proof. First, observe that, regardless which uninorm U is constructed, elements α_4, α_5 are one-divisors, and elements α_1, α_2 are zero-divisors. Monotonicity of U implies that $U(\mathbf{0}, \mathbf{1}) = \mathbf{a}$ where $\mathbf{a} \in L_1$ is the absorbing element of U. Assume that U is neither conjunctive nor disjunctive. Then $\mathbf{a} \notin \{\mathbf{0}, \mathbf{1}\}$, and

$$\mathbf{a} = U(\mathbf{0}, U(\alpha_4, \alpha_5)) = U(U(\mathbf{0}, \alpha_4), \alpha_5).$$

There are two possibilities. First, when $U(\mathbf{0}, \alpha_4) = \alpha_4$, then

$$U(U(\mathbf{0}, \alpha_4), \alpha_5) = U(\alpha_4, \alpha_5) = \mathbf{1} \neq \mathbf{a}. \tag{1}$$

Second, $U(\mathbf{0}, \alpha_4) < \alpha_4$, then

$$U(U(\mathbf{0}, \alpha_4), \alpha_5) \in [\mathbf{0}, \alpha_5]$$

and hence

$$U(U(\mathbf{0}, \alpha_4), \alpha_5) \neq \mathbf{a}. \tag{2}$$

Formulae (1) and (2) imply that associativity is violated, and therefore U is either conjunctive or disjunctive. □

Now, a uninorm on L_1 which is neither conjunctive nor disjunctive, will be presented.

Example 1. Observe that $\alpha_3 = \alpha_4 \wedge \alpha_5$, and for every element $z \in L_1$, if $z \parallel \alpha_3$, then $z \parallel \alpha_4$ and $z \parallel \alpha_5$. Consider an operation U on L_1 defined as follows:

$$U(x,y) = \begin{cases} x \wedge y & \text{if } x \leq \alpha_4 \text{ and } y \leq \alpha_4, \\ & \quad \text{or if } \beta_1 \leq x < \mathbf{1} \text{ and } \beta_1 \leq y < \mathbf{1}, \\ \alpha_5 & \text{if } x \leq \alpha_5 \text{ and } y \geq \alpha_5 \\ & \quad \text{or if } x = \alpha_5 \text{ or } y = \alpha_5, \\ x & \text{if } y = \alpha_4, \\ y & \text{if } x = \alpha_4, \\ \mathbf{1} & \text{if } x = \mathbf{1} \text{ and } (y \geq \alpha_4 \text{ or } y \parallel \alpha_3) \\ & \quad \text{or if } y = \mathbf{1} \text{ and } (x \geq \alpha_4 \text{ or } x \parallel \alpha_3), \\ \mathbf{0} & \text{if } x \leq \alpha_3 \text{ and } (y = \mathbf{0} \text{ or } y \parallel \alpha_3) \\ & \quad \text{or if } x \leq \alpha_3 \text{ and } (y = \mathbf{0} \text{ or } y \parallel \alpha_3), \\ \beta_1 & \text{if } x = \beta_2 \text{ and } \beta_1 \leq y < \mathbf{1}, \\ & \quad \text{or if } y = \beta_2 \text{ and } \beta_1 \leq x < \mathbf{1}. \end{cases} \tag{3}$$

Then U is a uninorm on L_1 whose absorbing element is α_5 and neutral element is α_4. Because of the lack of space, the detailed proof of the fact that U is a uninorm is omitted. The fact that α_4 and α_5 are the neutral and absorbing elements of U, respectively, follows directly from (3).

3.1 Conditions Under Which It Is Possible to Construct a Uninorm of the Third Type

Further in this paper the following assumption and notation are adopted:

Assumption (A). Let $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be a bounded lattice with two distinguished elements, $\mathbf{a}$, and $\mathbf{e}$, such that $\mathbf{a} \parallel \mathbf{e}$.

Notation (P). Elements of $L \backslash \{\mathbf{a}, \mathbf{e}\}$ are partitioned into 9 different subsets as follows:

1. $P_1 = \{x \in L; x \leq \mathbf{a} \wedge \mathbf{e}\}$,
2. $P_2 = \{x \in L \backslash P_1; x < \mathbf{e}\}$,

3. $P_3 = \{x \in L \backslash P_1;\, x < \mathbf{a}\}$,
4. $P_4 = \{x \in L;\, x \geq \mathbf{a} \vee \mathbf{e}\}$,
5. $P_5 = \{x \in L \backslash P_4;\, x > \mathbf{e}\}$,
6. $P_6 = \{x \in L \backslash P_4;\, x > \mathbf{a}\}$,
7. $P_7 = \{x \in L;\, x \parallel \mathbf{e},\, x \parallel \mathbf{a}\ \&\ (\mathbf{a} \wedge \mathbf{e}) < x < (\mathbf{a} \vee \mathbf{e})\}$,
8. $P_8 = \{x \in L \backslash P_7;\, x \parallel \mathbf{e},\, x \parallel \mathbf{a}\ \&\ ((\exists z \in P_2)(z < x) \text{ or } (\exists z \in P_5)(z > x))\}$,
9. $P_9 = L \backslash \{\mathbf{a}, \mathbf{e}\} \backslash (P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8)$.

In what follows the Notation (P) will be used. The next proposition presents some necessary conditions to be fulfilled by a bounded lattice L, if there exists a uninorm of the third type on L.

Proposition 1. *Let L be a lattice fulfilling Assumption (A), and let $U\colon L \times L \to L$ be a uninorm of the third type with the neutral element $\mathbf{e}$ and absorbing element $\mathbf{a}$. Then:*

(i) for all $x, y \in P_2$ we have $U(x, y) \in P_2$,
(ii) for all $x, y \in P_4$ we have $U(x, y) \in P_5$,
(iii) if $P_3 = \emptyset$, then for all $x, y \parallel \mathbf{a}$ we have $U(x, y) \nleq \mathbf{a}$,
(iv) if $P_6 = \emptyset$, then for all $x, y \parallel \mathbf{a}$ we have $U(x, y) \ngeq \mathbf{a}$.

Proof. We present only the proofs of items (i) and (iii). The proofs of items (ii) and (iv) are just simple modifications of those of (i) and (iii).

(i) Assume $x, y \in P_2$ such that $U(x, y) = \xi \notin P_2$. Then necessarily $\xi \notin P_1$. Let $z \in P_4$ be arbitrarily chosen. By monotonicity, since $\xi < \mathbf{a}$, we get $U(\xi, z) = \mathbf{a}$. By Lemma 4 we get

$$U(x, z) = \zeta \in [x, z] \quad \text{and} \quad U(y, \zeta) \in [y, \zeta].$$

Since $y \nleq \mathbf{a}$, $U(y, \zeta) \neq \mathbf{a}$ which is a contradiction with $U(\xi, z) = \mathbf{a}$.

(iii) Assume there exist elements $x, y \parallel \mathbf{a}$ such that $U(x, y) \leq \mathbf{a}$. Then for any $z \in P_4$ we have $U(z, U(x, y)) = \mathbf{a}$. By associativity $U(z, U(x, y)) = U(U(z, x), y)$ and by monotonicity $U(z, x) \geq x$, but also $U(z, x) \geq \mathbf{a}$, i.e., $U(z, x) \in P_4$. Similarly we get $U(U(z, x), y) \in P_4$, and this contradicts the assumption $U(z, U(x, y)) = \mathbf{a}$. Similarly we could show a contradiction in the case $U(x, y) \geq \mathbf{a}$. □

Corollary 1. *Let L be a lattice fulfilling Assumption (A), and let U be a uninorm on L of the third type with the neutral and absorbing elements $\mathbf{e}$ and $\mathbf{a}$, respectively. Then:*

(i) for all $x, y \in P_2$ we have $U(x, y) \neq \mathbf{0}$,
(ii) for all $x, y \in P_4$ we have $U(x, y) \neq \mathbf{1}$.

Remark 1.

(i) If $P_3 \neq \emptyset$ as well as $P_6 \neq \emptyset$, we may have $U(x, y) = \mathbf{a}$ for $x, y \in P_7 \cup P_8 \cup P_9$.
(ii) Assertions (i) and (ii) of Corollary 1 do not imply that P_2 and P_5 contain no zero-divisors and one-divisors, respectively.

Some uninorms of the third type were already presented in papers [7,12,13]. Now, we show some other possibilities when, for suitably chosen $\mathbf{e}$ and $\mathbf{a}$, we are able to construct a uninorm U on a lattice L, whose neutral and absorbing elements are $\mathbf{e}$ and $\mathbf{a}$, respectively.

Proposition 2. *Let L be a lattice fulfilling Assumption (A). Further, assume that there exists a t-norm $T\colon [\mathbf{0},\mathbf{e}]^2 \to [\mathbf{0},\mathbf{e}]$ and a t-conorm $S\colon [\mathbf{e},\mathbf{1}]^2 \to [\mathbf{e},\mathbf{1}]$ fulfilling properties (i) and (ii) from Proposition 1, respectively. Let at least one of the following three conditions be fulfilled:*

(C1) $P_7 \cup P_8 = \emptyset$ and for any $x, y \in P_9$ either $x \wedge y \in P_9$ or $x \vee y \in P_9$,
(C2) $P_7 \cup P_8 = \{\xi\}$ is a one element set such that

$$(\forall x \in P_2)(\forall y \in P_5)(x < \xi < y),$$

and for any $x, y \in P_9$ either $x \wedge y \in P_9$ or $x \vee y \in P_9$,
(C3) the lattice L is the horizontal sum of lattices L_1, L_2, and $\mathbf{e} \in L_1$, $\mathbf{a} \in L_2$. Moreover, assume that for all $x, y \in L_1 \setminus \{\mathbf{0},\mathbf{1}\}$ the following inequality holds

$$\mathbf{0} < x \wedge y \leq x \vee y < \mathbf{1}.$$

Then there exists a uninorm U on L, whose neutral element is $\mathbf{e}$ and absorbing element is $\mathbf{a}$.

Proof. A possible construction of uninorms for cases (C1), (C2) and (C3) is presented below.

(C1) We construct uninorm U_1 for the case that $x \wedge y \in P_9$ holds for all $x, y \in P_9$:

$$U_1(x,y) = \begin{cases} \mathbf{0} & \text{if } x \in P_1 \cup P_3,\ y \in P_1 \cup P_2 \cup P_3 \cup P_9, \\ & \text{ or if } y \in P_1 \cup P_3,\ x \in P_1 \cup P_2 \cup P_3 \cup P_9, \\ \mathbf{1} & \text{if } x \in P_4 \cup P_6,\ y \in P_4 \cup P_5 \cup P_6 \cup P_9, \\ & \text{ or if } y \in P_4 \cup P_6,\ x \in P_4 \cup P_5 \cup P_6 \cup P_9, \\ x & \text{if } x \in P_1 \cup P_2 \cup P_3,\ y \in P_5, \\ & \text{ or if } x \in P_4 \cup P_6,\ y \in P_2, \\ & \text{ or if } y = \mathbf{e}, \text{ or } x \in P_9 \text{ and } y \in P_2 \cup P_5, \\ y & \text{if } y \in P_1 \cup P_2 \cup P_3,\ x \in P_5, \\ & \text{ or if } y \in P_4 \cup P_6,\ x \in P_2, \\ & \text{ or if } x = \mathbf{e}, \text{ or } y \in P_9 \text{ and } x \in P_2 \cup P_5, \\ \mathbf{a} & \text{if } x \in P_1 \cup P_3,\ y \in P_4 \cup P_6, \\ & \text{ or if } y \in P_1 \cup P_3,\ x \in P_4 \cup P_6, \\ & \text{ or if } x = \mathbf{a} \text{ or } y = \mathbf{a}, \\ w \wedge y & \text{if } x, y \in P_2 \text{ or if } x, y \in P_9, \\ x \vee y & x, y \in P_5, \end{cases}$$

(C2) Also in this case we assume that $x \wedge y \in P_9$ holds for all $x, y \in P_9$. Then the construction of a uninorm U_2 is just a slight modification of the uninorm U_1 when we define $U_2(x, y) = \xi$ if $x \in P_2 \cup \{\xi\}$ and $y \in P_5 \cup \{\xi\}$, or vice versa, and $U_2(\xi, x) = x$ for $x \in P_9$, $U_2(\xi, x) = \mathbf{0}$ for $x \in P_1 \cup P_3$, $U_2(\xi, x) = \mathbf{1}$ for $x \in P_4 \cup P_6$.

(C3) We choose some elements $\mathbf{e} \in L_1 \setminus \{\mathbf{0}, \mathbf{1}\}$ and $\mathbf{a} \in L_2 \setminus \{\mathbf{0}, \mathbf{1}\}$. First, define a nullnorm V on L_2 by

$$V(x, y) = \begin{cases} x \vee y & \text{or if } x, y \leq \mathbf{a}, \\ x \wedge y & \text{if } x, y \in L_2,\ x, y \geq \mathbf{a}, \\ \mathbf{a} & \text{otherwise,} \end{cases}$$

if $x, y \in L_2$.

Further, define an 'almost uninorm' $\tilde{U}_3$. This means that $\tilde{U}_3$ is an associative, commutative operation with the neutral element $\mathbf{e}$. But values when one of the inputs is $\mathbf{0}$ or $\mathbf{1}$, are not defined. Recall that L is the horizontal sum of L_1 and L_2, i.e., $L_1 \cap L_2 = \{\mathbf{0}, \mathbf{1}\}$. Assume that $x, y \in L_1 \setminus \{\mathbf{0}, \mathbf{1}\}$. Then

$$\tilde{U}_3(x, y) = \begin{cases} x \vee y & \text{if } x, y \geq \mathbf{e}, \\ x \wedge y \wedge \mathbf{e} & \text{if } x < \mathbf{e} \text{ or } y < \mathbf{e}, \\ x & \text{if } y \geq \mathbf{e} \text{ and } x \parallel \mathbf{e}, \\ y & \text{if } x \geq \mathbf{e} \text{ and } y \parallel \mathbf{e}. \end{cases}$$

Finally, the uninorm $U_3 \colon L \times L \to L$ is defined by

$$U_3(x, y) = \begin{cases} \tilde{U}_3(x, y) & \text{if } x, y \in L_1 \setminus \{\mathbf{0}, \mathbf{1}\}, \\ V(x, y) & \text{if } x, y \in L_2, \\ x & \text{if } x \in L_2 \text{ and } y \in L_1 \setminus \{\mathbf{0}, \mathbf{1}\}, \\ y & \text{if } y \in L_2 \text{ and } x \in L_1 \setminus \{\mathbf{0}, \mathbf{1}\}. \end{cases} \tag{4}$$

□

Remark 2. Note that the uninorm U_3 given by formula (4), is also a (proper) nullnorm. This means that this is another example (besides those presented in paper [12]) of a proper nullnorm with neutral element.

3.2 Example of a Bounded Lattice with only Conjunctive and Disjunctive Uninorms

In this paragraph, we present an infinite lattice with incomparable elements on which only conjunctive or disjunctive uninorms can be constructed.

Example 2. Consider the lattice $\tilde{L}$ depicted on Fig. 2, fulfilling assumption (A). Every element of $\tilde{L}$, up to $\mathbf{0}$, has two immediate predecessors.

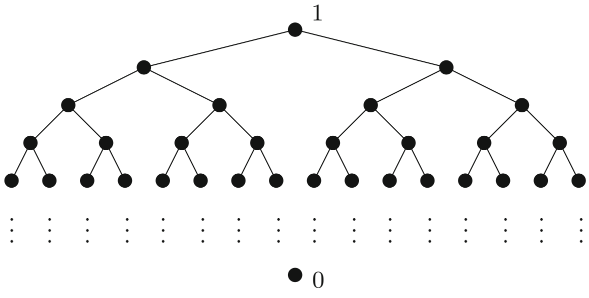

Fig. 2. Infinite lattice $\tilde{L}$

Regardless how neutral element $\mathbf{e} \in \tilde{L}\backslash\{\mathbf{0},\mathbf{1}\}$ is chosen, all elements $x \in]\mathbf{0},\mathbf{e}[$ are zero-divisors. On the other hand, if $\mathbf{a} \parallel \mathbf{e}$ then $x \parallel \mathbf{a}$ for all $x \in [\mathbf{0},\mathbf{e}[$, i.e., $x \in P_2$. This violates Proposition 1 and also Corollary 1. Therefore the lattice L is an example of a lattice with incomparable elements, on which only conjunctive or disjunctive uninorms can be constructed.

4 Conclusion

Work presented in this paper is a contribution to theoretical aspects on construction of uninorms on bounded lattices. It was demonstrated that on a bounded lattice L with incomparable elements it is impossible to choose incomparable elements $\mathbf{e}$ and $\mathbf{a}$, and then to construct a uninorm on L whose neutral and absorbing elements are $\mathbf{e}$ and $\mathbf{a}$, respectively.

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Kleene Algebras as Sequences of Orthopairs

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Abstract. We study sequences of approximations of sets given by refining tolerance relations on the universe, and we show that such sequences can be equipped with a structure of finite centered Kleene algebra satisfying the interpolation property. We further show that every such Kleene algebra is isomorphic to the algebra of sequences of approximations of subsets of a suitable universe.

1 Introduction

An equivalence relation R on a universe U (hence every partition of U) determines for every $X \subseteq U$ the orthopair $(\mathcal{L}(X), \mathcal{E}(X))$ consisting respectively of the union of all equivalence classes fully contained in X (lower approximation) and the union of all the equivalence classes disjoint with X (impossibility domain). On the set $OP(U)$ of all orthopairs of U it is possible to defined operations $\wedge$, $\vee$ and $\neg$ by the following stipulations: let $X, Y \subseteq U$

$$(\mathcal{L}(X), \mathcal{E}(X)) \wedge (\mathcal{L}(Y), \mathcal{E}(Y)) = (\mathcal{L}(X) \cap \mathcal{L}(Y), \mathcal{E}(X) \cup \mathcal{E}(Y))$$
$$(\mathcal{L}(X), \mathcal{E}(X)) \vee (\mathcal{L}(Y), \mathcal{E}(Y)) = (\mathcal{L}(X) \cup \mathcal{L}(Y), \mathcal{E}(X) \cap \mathcal{E}(Y))$$
$$\neg(\mathcal{L}(X), \mathcal{E}(X)) = (\mathcal{E}(X), \mathcal{L}(X))$$

and it can be shown that $\mathbb{OP}(U) = (OP(U), \wedge, \vee, \neg, (\emptyset, U), (U, \emptyset))$ is a Kleene algebra.

If R is not an equivalence relation, orthopairs can still be defined by adapting the notion of equivalence classes. In [14] it is proved that if R is a tolerance, i.e., R is a reflexive and symmetric binary relation, then $\mathbb{OP}(U)$ in general is not a lattice [13], but if R is induced by an *irredundant covering* of U, then $\mathbb{OP}(U)$ is a Kleene algebra that is also an algebraic and completely distributive lattice. By an irredundant covering C of U we mean a covering (set of subsets of U whose union is U) such that $C \backslash \{b\}$ is not a covering of U for any $b \in C$.

In this paper we consider a sequence $R_1, \ldots, R_n$ of tolerances induced by a refinement sequence of coverings. Namely, by refinement sequence we mean a sequence $\mathcal{C} = C_1, \ldots, C_n$ of coverings of a universe U such that every block of C_i is contained in a block of C_{i-1} for each i from 2 to n.

Example 1. We consider the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ of objects and the set $Y = \{y_1, y_2, y_3, y_4\}$ of attributes that represent 6 athletes and 4 athletic disciplines in a race. The fuzzy formal context $K = (X, Y, I)$ (see [4]), given by

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the following table, reports the score of each athlete in each discipline in the interval $[0, 1]$.

I	y_1	y_2	y_3	y_4
x_1	0.2	1	0	0.5
x_2	0.3	0.8	0	0.7
x_3	1	0.3	0.8	0.25
x_4	0.95	0.4	0.9	0
x_5	0.7	0	0.3	0.95
x_6	0.7	0.1	0.4	1

For each $\alpha \in [0, 1]$, we define the relation R_α on X as follow: let $x_i, x_j \in X$,

$$x_i\ R_\alpha\ x_j \quad \text{if and only if} \quad |I(x_i, y) - I(x_j, y)| \leq \alpha \text{ for each } y \in Y.$$

Note that R_α is a tolerance and $C_\alpha = \{\{x' \in X \mid xR_\alpha x'\} \mid x \in X\}$ is a covering of X. Moreover, if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, then each block of C_{α_1} is included in a block of C_{α_2}. For example if $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$, we obtain $C_{0.7} = \{\{x_1, x_2\}, \{x_1, x_2, x_6\}, \{x_3, x_4, x_5\}, \{x_3, x_4\}, \{x_3, x_5, x_6\}, \{x_2, x_5, x_6\}\}$ and $C_{0.3} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$.

We shall consider *partial coverings* of the universe U meaning that the union of all blocks of each covering is not always equal to U. We generalize the definition of irredundant covering in the case of partial covering: a partial covering C is irredundant if and only if each block of C is not included in the union of other blocks of C that are different to it.

Therefore, let $\mathcal{C} = C_1, \dots, C_n$ be a refinement sequence of U, we assign a sequence of orthopairs to $\mathcal{C}$ for each $X \subseteq U$, since we consider the orthopair $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ for each covering C_i of $\mathcal{C}$. We call $\mathcal{O}_\mathcal{C}$ the set of all sequences of orthopairs of $\mathcal{C}$. In [1] sequences of orthopairs given by refinement of partitions have been put in correspondence with finite IUML-algebras. In this paper we show how to equip $\mathcal{O}_\mathcal{C}$ with a structure of finite centered Kleene algebra with interpolation property, starting by hypothesis that $\mathcal{C}$ is made by irredundant partial coverings of a given universe. Further, any finite centered Kleene algebra A with interpolation property is associated with an universe U and a refinement sequence $\mathcal{C}$ of irredundant coverings such that A is isomorphic to the algebraic structure on $\mathcal{O}_\mathcal{C}$. Some of the proofs in this paper are adapted from the ones in [1] and will be omitted here.

2 Preliminaries

We summarize here some notion and results on De Morgan and Kleene algebras (see [9] for further references).

Definition 1. *A* De Morgan algebra *is a bounded distributive lattice* $(A, \wedge, \vee, \neg, 0, 1)$ *such that for each* $x, y \in A$, $\neg(x \vee y) = \neg x \wedge \neg y$ *and* $\neg\neg x = x$.

A Kleene algebra *([7,15])* $(A, \wedge, \vee, \neg, 0, 1)$ *is a De Morgan algebra such that for each* $x, y \in A$ *the Kleene property holds:* $x \wedge \neg x \leq y \vee \neg y$.

Definition 2. *An element c of a Kleene algebra A is called a* center *of A in case $c = \neg c$. We say that A is a* centered *Kleene algebra if A has a center.*

Note that the Kleene property implies that if a center c of A exists then it is unique. As in [9], for each Kleene algebra A, we set

$$A^+ = \{x \in A \mid \neg x \leq x\} \quad \text{and} \quad A^- = \{x \in A \mid x \leq \neg x\}.$$

We have that A^+ is a sublattice of A containing 1.

The following costruction is due to Kalman [15]: let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice and let $K(L) = \{(x, y) \in L \times L \mid x \wedge y = 0\}$ with the operations $\sqcap$, $\sqcup$ and $\neg$ defined as follow:

$$(x, y) \sqcap (u, v) = (x \wedge u, y \vee v) \tag{1}$$

$$(x, y) \sqcup (u, v) = (x \vee u, y \wedge v) \tag{2}$$

$$\neg(x, y) = (y, x) \tag{3}$$

for each $(x, y), (u, v) \in K(L)$. Then, $(K(L), \sqcap, \sqcup, \neg, (0, 1), (1, 0))$ is a centered Kleene algebra, with center $(0, 0)$. Moreover, $K(L)^+ = \{(x, 0) \mid x \in L\}$.

Definition 3. *Let $(A, \wedge, \vee, \neg, 0, 1)$ be a centered Kleene algebra. Let c be the center of A. We say that A has the* interpolation property *if and only if for every $x, y \geq c$ such that $x \wedge y \leq c$ there exists z such that $z \vee c = x$ and $\neg z \vee c = y$.*

In [8] the above definition is called (CK) property, but it is also noted that it coincides with the interpolation property described in [9], so we will use this last name. Not every centered Kleene algebra has the interpolation property, see Example 5 in [8].

Theorem 1. *[9] A Kleene algebra A is isomorphic to $K(L)$ for some bounded distributive lattice L if and only if A is centered and satisfies the interpolation property. In this case L is isomorphic to the lattice A^+.*

Birkhoff representation theorem states that given a partially ordered set P, the structure $(U(P), \cap, \cup, \emptyset, P)$ where $U(P)$ is the set of upsets of P, $\cap$ and $\cup$ are respectively the set intersection and union, is a bounded distributive lattice. Vice-versa, if L is a bounded distributive lattice, then there is a partially ordered set P such that $L \cong U(P)$.

In this paper, we focus on $K(U(P)) = \{(A, B) \in U(P) \times U(P) \mid A \cap B = \emptyset\}$, with $\sqcap$, $\sqcup$ and $\neg$ defined respectively as (1), (2) and (3).

3 Refinement Sequences and Orthopairs

We call *partial covering* of U any subset of 2^U, i.e. any set of subsets of U.

Definition 4. *A sequence $\mathcal{C} = C_1, \ldots, C_n$ of partial coverings of U is a* refinement sequence *if each element of C_i is contained in an element of C_{i-1}, for $i = 2, \ldots, n$.*

From now on, by refinement sequence of U, we mean a refinement sequence of partial coverings of U.

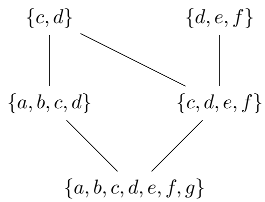
Example 2. If $U = \{a, b, c, d, e, f, g\}$, then (C_1, C_2), where $C_1 = \{\{a, b, c, d\}, \{d, e, f, g\}\}$ and $C_2 = \{\{a, b, c\}, \{c, d\}, \{d, e\}, \{f, g\}\}$, is a refinement sequence of U.

In order to prove our results, we do not consider coverings containing singletons.

Definition 5. *Given a refinement sequence $\mathcal{C} = C_1, \ldots, C_n$ of U, the set of all blocks of $\mathcal{C}$ equipped with the reverse inclusion is a partially ordered set. In particular, we associate with $\mathcal{C}$ the partially ordered set $(P_\mathcal{C}, \leq_\mathcal{C})$ where*

- *$P_\mathcal{C} = \bigcup_{i=1}^{n} C_i$ (the set of nodes is the set of all subsets of U belonging to the coverings $C_1, \ldots, C_n$), and*
- *for $N, M \in P_\mathcal{C}$, $N \leq_\mathcal{C} M$ if and only if $M \subseteq N$.*

Example 3. Let (C_1, C_2, C_3) be a refinement sequence of $\{a, b, c, d, e, f, g, h\}$, where $C_1 = \{\{a, b, c, d, e, f, g\}\}, C_2 = \{\{a, b, c, d\}, \{c, d, e, f\}\}$ and $C_3 = \{\{c, d\}, \{d, e, f\}\}$. Then the poset assigned to (C_1, C_2, C_3) is shown in the following figure:

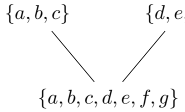


If $\mathcal{C}$ is a refinement sequence and $X \subseteq P_\mathcal{C}$, we denote by X^* the union of all nodes belonging to X.

Definition 6. *A refinement sequence $\mathcal{C}$ is* complete *if $A^* \cap B^* = \emptyset$ for each $(A, B) \in K(U(P_\mathcal{C}))$.*

Example 4. Let us consider the refinement sequence $\mathcal{C}$ given by $C_1 = \{\{a, b, c, d, e, f\}\}$, $C_2 = \{\{a, b, c, d\}, \{d, e, f\}\}$ and $C_3 = \{\{a, b\}\}$. Then set $A = \{\{a, b, c, d\}, \{a, b\}\}$ and $B = \{\{d, e, f\}\}$. Then $(A, B) \in K(U(P_C)$ but $A^* = \{a, b, c, d\}$ and $B^* = \{d, e, f\}$ hence $A^* \cap B^* = \{d\}$ and $\mathcal{C}$ is not complete.

On the other side, the following refinement sequence $\mathcal{C}$ is complete:



Definition 7. *A refinement sequence* $\mathcal{C} = C_1, \ldots, C_n$ *is* safe *if for each* $N \in P_{\mathcal{C}}$*, if* $N \in C_i$ *with* $i \in \{1, \ldots, n\}$ *and* $N \subseteq N_1 \cup \ldots \cup N_r$ *with* $N_1, \ldots, N_r \in P_{\mathcal{C}}$*, then there exists* $j \in \{1, \ldots r\}$ *such that* $N \subseteq N_j$ *and* $N \in C_k$ *with* $j < k$*.*

Example 5. Let $C_1 = \{\{a, b, c, d, e\}, \{a, f, g, h\}\}$ and $C_2 = \{\{a, b, c\}, \{c, d\}, \{f, g\}\}$, then (C_1, C_2) is safe. Instead, the refinement sequence $(\tilde{C}_1, \tilde{C}_2)$ with $\tilde{C}_1 = \{\{a, b, c, d, e\}, \{c, d, e, f, g, h\}\}$ and $\tilde{C}_2 = \{\{a, b, c\}, \{c, d\}, \{e, f, g\}\}$, is not safe, since $\{a, b, c, d, e\} \subseteq \{a, b, c\} \cup \{c, d\} \cup \{e, f, g\}$.

Definition 8. *A refinement sequence* $\mathcal{C} = C_1, \ldots, C_n$ *is* pairwisely overlapping *if there are no disjoint blocks in* C_i*, for each* $i \in \{1, \ldots, n\}$*.*

Example 6. The refinement sequence (C_1, C_2) of Example 3 is pairwisely overlapping, since d belongs to each block of C_1, C_2 and C_3. Trivially, all other refinement sequences define in the above examples are not pairwisely overlapping.

Definition 9. *A refinement sequence* $\mathcal{C}$ *is* irredundant *if each its coverings is irredundant.*

Proposition 1. *Let* $\mathcal{C} = C_1, \ldots, C_n$ *be a refinement sequence of* U*. If* $\mathcal{C}$ *is safe, then* $\mathcal{C}$ *is irredundant.*

Proof. The proof follows by Definition 7: let us fix $i \in \{1, \ldots, n\}$ and $N \in C_i$, then there are not $N_1, \ldots, N_r \in P_{\mathcal{C}}$ such that $N \subseteq N_1 \cup \ldots \cup N_r$ and $N \nsubseteq N_i$ for each $i \in \{1, \ldots, r\}$. Then, there are not nodes of C_i such that their union includes N, hence C_i is irredundant.

Remark 1. On the contrary, a refinement sequence $\mathcal{C}$ made by all irredundant coverings is not always safe.

Example 7. Suppose that $C_1 = \{a, b, c, d, e, f, g, h\}$, $C_2 = \{\{a, b, c\}, \{d, e, f, g\}\}$ and $C_3 = \{\{a, b\}, \{d, e\}, \{f, g\}\}$. Then, (C_1, C_2, C_3) is not safe, but C_1, C_2 and C_3 are irredundant.

Given a covering C of U and $X \subseteq U$ we consider $\mathcal{L}(X) = \{N \in C \mid N \subseteq X\}$ and $\mathcal{E}(X) = \{N \in C \mid N \cap X = \emptyset\}$. Given a refinement sequence $\mathcal{C} = C_1, \ldots, C_n$ of U, for any $X \subseteq U$ and for every $i = 1, \ldots, n$, we consider the *orthopair* $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ determined by C_i. Then we let

$$\mathcal{O}_{\mathcal{C}}(X) = ((\mathcal{L}_1(X), \mathcal{E}_1(X)), \ldots, (\mathcal{L}_n(X), \mathcal{E}_n(X))),$$

and $\mathcal{O}_{\mathcal{C}} = \{\mathcal{O}_{\mathcal{C}}(X) \mid X \subseteq U\}$.

Example 8. Let $U = \{a, b, c, d, e, f, g, h, i, j\}$ and $X = \{a, b, c, d, e\}$. If $\mathcal{C}$ is the refinement sequence of U made by $C_1 = \{\{a, b, c, d, e, f, g, h, i, j\}\}$, $C_2 = \{\{a, b, c, d, e\}, \{e, f, g, h, i\}\}$, $C_3 = \{\{a, b, c\}, \{c, d\}, \{e, f, g\}, \{g, h\}\}$, then

$$\mathcal{O}_{\mathcal{C}}(X) = ((\emptyset, \emptyset), (\{\{a, b, c, d, e\}\}, \emptyset), (\{\{a, b, c\}, \{c, d\}\}, \{\{g, h\}\}))\,.$$

4 Sequences of Orthopairs as Pairs of Upsets

Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of a universe U. For any $X \subseteq U$ the sequence $\mathcal{O}_{\mathcal{C}}(X)$ of orthopairs with respect to $\mathcal{C}$ determines two subsets of the poset $P_{\mathcal{C}}$, obtained by considering for every $i = 1, \ldots, n$, the blocks contained in $\mathcal{L}_i(X)$ and the blocks contained in $\mathcal{E}_i(X)$. This observation leads to the following definition (see [1]).

Definition 10. *For every refinement sequence $\mathcal{C} = C_1, \ldots, C_n$ of U and any $X \subseteq U$, we let $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}})$ be such that*

$$X^1_{\mathcal{C}} = \{N \in P_{\mathcal{C}} \mid N \subseteq X\} \quad \textit{and} \quad X^2_{\mathcal{C}} = \{N \in P_{\mathcal{C}} \mid N \cap X = \emptyset\}.$$

Moreover, we let $SO(\mathcal{C}) = \{(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \mid X \subseteq U\}$.

Example 9. Given U and $\mathcal{C}$ of Example 3, if $X = \{b, c, d\}$ then $X^1_{\mathcal{C}} = \{\{c, d\}\}$ and $X^2_{\mathcal{C}} = \emptyset$.

Following [1] we can prove that given a set U and a refinement sequence $\mathcal{C}$ of U, the map

$$h : \mathcal{O}_{\mathcal{C}}(X) \in \mathcal{O}_{\mathcal{C}} \mapsto (X^1, X^2) \in SO(\mathcal{C})$$

is a bijection. We write (X^1, X^2) instead of $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}})$, when $\mathcal{C}$ is clear from the context. The following proposition shows that, given a refinement sequence $\mathcal{C}$, the set $SO(\mathcal{C})$ is made by pairs of disjoint upsets of $P_{\mathcal{C}}$.

Proposition 2. *Let $\mathcal{C}$ be a refinement sequence, then $SO(\mathcal{C}) \subseteq K(U(P_{\mathcal{C}}))$.*

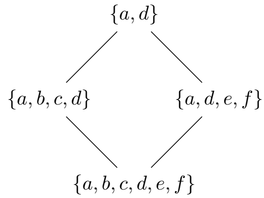
On the contrary, given a refinement sequence $\mathcal{C}$, a pair of disjoint upsets of $P_{\mathcal{C}}$ is not always an element of $SO(\mathcal{C})$.

Example 10. Given the refinement sequence $\mathcal{C}$ of Example 3, the pair $(\{\{c, d\}, \{d, e, f\}\}, \emptyset)$ of disjoint upsets of $P_{\mathcal{C}}$ does not belong to $SO(\mathcal{C})$, since if a set contains $\{c, d\}$ and $\{d, e, f\}$, then it contains also the block $\{c, d, e, f\}$.

In this section, we want to investigate the set $SO(\mathcal{C})$ equipped with operations defined on $K(U(P_{\mathcal{C}}))$.

First we note that, as shown in the following example, $(SO(\mathcal{C}), \sqcap, \sqcup, \neg, (\emptyset, P_{\mathcal{C}}), (P_{\mathcal{C}}, \emptyset))$ is not necessarily even a lattice.

Example 11. Let us consider the refinement sequence $\mathcal{C}$, where $P_{\mathcal{C}}$ is depicted in the following figure:



If we consider $(\emptyset, \{\{a, b, c, d\}, \{a, d\}\}), (\emptyset, \{\{a, d, e, f\}, \{a, d\}\}) \in SO(\mathcal{C})$, then
$(\emptyset, \{\{a, b, c, d\}, \{a, d\}\}) \sqcap (\emptyset, \{\{a, d, e, f\}, \{a, d\}\})$ is equal to the pair

$$(\emptyset, \{\{a, b, c, d\}, \{a, d, e, f\}, \{a, d\}\})$$

that does not belong to $SO(\mathcal{C})$, since it would mean that there is a set X that is disjoint with $\{a, b, c, d\}$, $\{a, d, e, f\}$ and $\{a, d\}$ but it is not disjoint with $\{a, b, c, d, e, f\}$, which is clearly impossible.

Proofs of next propositions are similar to the analogous in [1].

Proposition 3. *Let $\mathcal{C}$ be a safe refinement sequence of U. Suppose that A is an upset of $P_{\mathcal{C}}$. Then, $N \in A$ if and only if $N \subseteq \mathcal{A}$.*

Proposition 4. *Let $\mathcal{C}$ be a safe refinement sequence of U and $(A, B) \in K(U(P_{\mathcal{C}}))$. If $A^* \cap B^* = \emptyset$, then $(A, B) \in SO(\mathcal{C})$.*

Theorem 2. *Let $\mathcal{C}$ be a safe refinement sequence of U. Then, $SO(\mathcal{C})$ contains $K(U(P_{\mathcal{C}}))^+$ and $\mathbb{SO}_{\mathcal{C}} = (SO(\mathcal{C}), \sqcap, \sqcup, \backsim, (\emptyset, P_{\mathcal{C}}), (P_{\mathcal{C}}, \emptyset))$ is a centered Kleene subalgebra of $K(U(P_{\mathcal{C}}))$.*

Proof. Let $(A, B) \in K(U(P_{\mathcal{C}}))^+$, then $B = \emptyset$. By Proposition 4, $(A, B) \in SO(\mathcal{C})$ since $A^* \cap \emptyset = \emptyset$. $SO(\mathcal{C})$ is closed under all operations of $K(U(P_{\mathcal{C}}))$, since, for any $X, Y \subseteq U$, $(X^1 \cap Y^1, X^2 \cap Y^2)$ and $(X^1 \cup Y^1, X^2 \cap Y^2)$ are pairs of disjoint upsets of $P_{\mathcal{C}}$ satisfying the condition of Proposition 4.

Moreover, by Proposition 4 $(\emptyset, \emptyset) \in SO(\mathcal{C})$, hence $SO(\mathcal{C})$ is a centered Kleene algebra.

Remark 2. It is easy to observe that, when $\mathcal{C}$ is a safe refinement sequence of U, also $K(U(P_{\mathcal{C}}))^-$ is included in $SO(\mathcal{C})$.

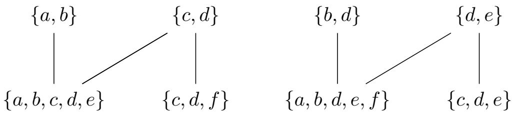
Theorem 3. *Let $\mathcal{C}$ be a complete and safe refinement sequence of U. Then, $SO(\mathcal{C}) = K(U(P_{\mathcal{C}}))$. Hence, $\mathbb{SO}_{\mathcal{C}} = (SO(\mathcal{C}), \sqcap, \sqcup, \neg, (\emptyset, P_{\mathcal{C}}), (P_{\mathcal{C}}, \emptyset))$ is a finite centered Kleene algebra with interpolation property.*

Proof. By Proposition 2, $SO(\mathcal{C}) \subseteq K(U(P_{\mathcal{C}}))$. By hypothesis $\mathcal{C}$ is complete, hence $A^* \cap B^* = \emptyset$ for each $(A, B) \in K(U(P_{\mathcal{C}}))$. Then, $(A, B) \in SO(\mathcal{C})$ since $\mathcal{C}$ is safe by hypothesis.

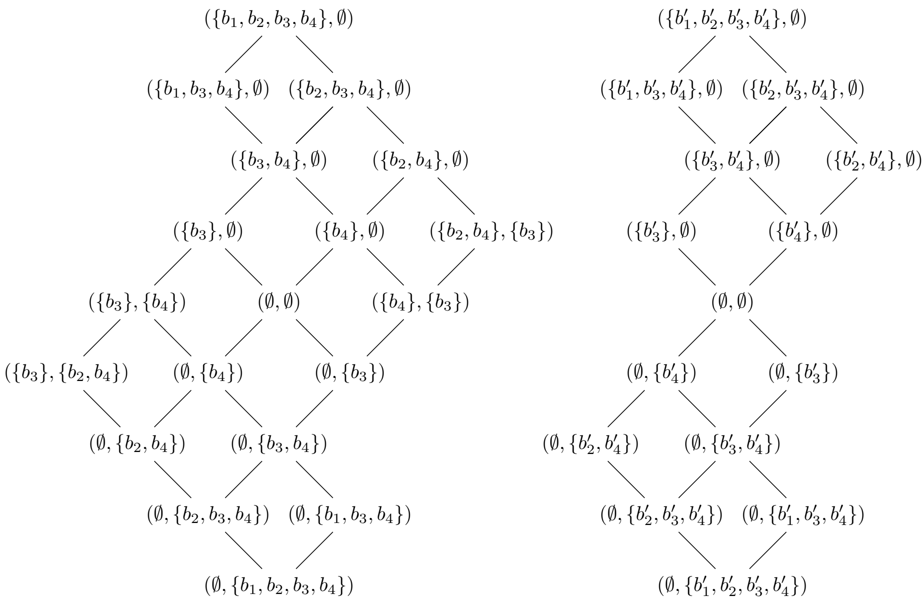
Theorem 4. *Let $\mathcal{C}$ be a safe and pairwisely overlapping refinement sequence of U, then $SO(\mathcal{C}) = K(U(P_{\mathcal{C}}))^- \cup K(U(P_{\mathcal{C}}))^+$.*

Proof. By Theorem 2, $SO(\mathcal{C}) \supseteq K(U(P_{\mathcal{C}}))^- \cup K(U(P_{\mathcal{C}}))^+$. Vice versa, let $(X^1, X^2) \in SO(\mathcal{C})$. If $X^1 \neq \emptyset$ and $X^2 \neq \emptyset$, then there exist $N, M \in P_{\mathcal{C}}$ such that $N \in X^1$ and $M \in X^2$. Therefore, $N \cap M = \emptyset$. Suppose that $N \in C_i$ and $M \in C_j$ with $i, j \in \{1, \dots, n\}$ and $i \leq j$. By Definition 4, there exists $N' \in C_j$ such that $N' \subseteq N$. Then, N' and M are two disjoint blocks of C_j. This is an absurd, since, by hypothesis $\mathcal{C}$ is pairwisely overlapping.

Example 12. We consider the refinement sequences $\mathcal{C} = (C_1, C_2)$ and $\tilde{\mathcal{C}} = (\tilde{C}_1, \tilde{C}_2)$ of $\{a, b, c, d, e, f\}$, where $C_1 = \{\{a, b, c, d, e\}, \{c, d, f\}\}$, $C_2 = \{\{a, b\}, \{c, d\}\}$, $\tilde{C}_1 = \{\{a, b, d, e, f\}, \{c, d, e\}\}$ and $\tilde{C}_2 = \{\{b, d\}, \{d, e\}\}$. As shown in the following two figures, $P_{\mathcal{C}}$ and $P_{\tilde{\mathcal{C}}}$ have the same Hasse diagram. Then, $K(U(P_{\mathcal{C}})) \cong K(U(P_{\tilde{\mathcal{C}}}))$.



We set $b_1 = \{a, b, c, d, e\}$, $b_2 = \{c, d, f\}$, $b_3 = \{a, b\}$, $b_4 = \{c, d\}$, $b'_1 = \{a, b, d, e, f\}$, $b'_2 = \{c, d, e\}$, $b'_3 = \{b, d\}$ and $b'_4 = \{d, e\}$. Then, $\mathbb{SO}_{\mathcal{C}}$ and $\mathbb{SO}_{\tilde{\mathcal{C}}}$ have the following Hasse diagrams.



Note that $\mathcal{SO}(\mathcal{C}) = K(U(P_{\mathcal{C}}))$, since $\mathcal{C}$ is safe and complete. Instead, since $\tilde{\mathcal{C}}$ is safe and pairwisely overlapping, $\mathcal{SO}_{\tilde{\mathcal{C}}} \subset K(U(P_{\tilde{\mathcal{C}}}))$ and $(\{b'_3\},\{b'_4\}),(\{b'_4\},\{b'_3\}),(\{b'_3\},\{b'_2,b'_4\}),(\{b'_2,b'_4\},\{b'_3\}) \notin SO(\tilde{\mathcal{C}})$.

Remark 3. When $\mathcal{C}$ is a safe refinement sequence, it is not always true that $\mathbb{SO}_{\mathcal{C}}$ has the interpolation property: if $\mathcal{C}$ is not complete, then there exists $(A,B) \in K(U(P_{\mathcal{C}}))$ such that $A^* \cap B^* \neq \emptyset$, hence $(A,B) \notin SO(\mathcal{C})$; on the other hand $(A,\emptyset),(B,\emptyset) \in SO(\mathcal{C})$ and trivially $(X,Y) \sqcup (\emptyset,\emptyset) = (A,\emptyset)$ and $\neg(X,Y) \sqcup (\emptyset,\emptyset) = (B,\emptyset)$ if and only if $(X,Y) = (A,B)$.

Corollary 1. *Let $\mathcal{C}$ be a safe refinement sequence. Then, $U(P_{\mathcal{C}})$ is isomorphic to the lattice $SO(\mathcal{C})^+$.*

The following proposition shows that we can equip $SO(\mathcal{C})$ with a structure of finite Kleene algebra also when each covering of $\mathcal{C}$ is irredundant.

Theorem 5. *Let $\mathcal{C} = C_1, \dots, C_n$ be an irredundant refinement sequence of U. Then, there exists a safe refinement sequence $\mathcal{C}'$ of universe U such that the function $g : SO(\mathcal{C}) \mapsto SO(\mathcal{C}')$ where $g((X^1_{\mathcal{C}}, X^2_{\mathcal{C}})) = (X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'})$ for each $X \subseteq U$ is bijective. Hence,*

$$\mathbb{SO}'_{\mathcal{C}} = (SO(\mathcal{C}), \sqcap_{\mathcal{C}}, \sqcup_{\mathcal{C}}, \neg, (\emptyset, P_{\mathcal{C}}), (P_{\mathcal{C}}, \emptyset))$$

where

- $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \sqcap_{\mathcal{C}} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) = g^{-1}((X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \sqcap (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}))$ *and*
- $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \sqcup_{\mathcal{C}} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) = g^{-1}((X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \sqcup (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}))$ *for each $X, Y \subseteq U$*

is a centered Kleene algebra. Moreover,

- *if $\mathcal{C}$ is complete, then $\mathbb{SO}'_{\mathcal{C}}$ satisfies the interpolation property, and*
- *if $\mathcal{C}$ is pairwisely overlapping, then $SO(\mathcal{C}) \cong K(U(P_{\mathcal{C}'}))^+ \cup K(U(P_{\mathcal{C}'}))^-$.*

Proof. We obtain $\mathcal{C}'$ by $\mathcal{C}$ with same construction made in Sect. 4 of [1] in the case of partial partitions of a given universe.

Remark 4. If $\mathcal{C}$ is safe, then $\mathcal{C} = \mathcal{C}'$ and $\mathbb{SO}'_{\mathcal{C}} = \mathbb{SO}_{\mathcal{C}}$.

Let $\mathcal{C}$ be a refinement sequence of U, through function h given in this section, we define on $\mathcal{O}_{\mathcal{C}}$ the following operations:

1. $\mathcal{O}_{\mathcal{C}}(X) \wedge_{SO} \mathcal{O}_{\mathcal{C}}(Y) = h^{-1}(h(\mathcal{O}_{\mathcal{C}}(X)) \sqcap h(\mathcal{O}_{\mathcal{C}}(Y)))$,
2. $\mathcal{O}_{\mathcal{C}}(X) \vee_{SO} \mathcal{O}_{\mathcal{C}}(Y) = h^{-1}(h(\mathcal{O}_{\mathcal{C}}(X)) \sqcup h(\mathcal{O}_{\mathcal{C}}(Y)))$,
3. $\neg_{SO}\mathcal{O}_{\mathcal{C}}(X) = h^{-1}(h(\neg\mathcal{O}_{\mathcal{C}}(X)))$.

Therefore, each result obtain for $\mathbb{SO}'_{\mathcal{C}}$ holds also for the structure

$$(\mathcal{O}_{\mathcal{C}}, \wedge_{SO}, \vee_{SO}, \neg_{SO}, (M_{\mathcal{C}}, \emptyset), (\emptyset, M_{\mathcal{C}}))$$

where $M_{\mathcal{C}} = \{x \in N \mid N \in P_{\mathcal{C}}\}$.

Remark 5. If $\mathcal{C} = C_1, \ldots, C_n$ is a safe refinement sequence of U and $X, Y \subseteq U$, then $\mathcal{O}_\mathcal{C}(X) \wedge_{SO} \mathcal{O}_\mathcal{C}(Y)$ is obtained by applying $\wedge$ between $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ and $(\mathcal{L}_i(Y), \mathcal{E}_i(Y))$ for each $i \in \{1, \ldots, n\}$. In the same way we obtain $\vee_{SO}$ and $\neg_{SO}$ from $\vee$ and $\neg$ on single orthopairs of the same level.

Example 13. We consider the safe refinement sequence $\mathcal{C} = (C_1, C_2)$ of $\{a, b, c, d, e\}$, where $C_1 = \{\{a, b, c, d, e\}\}$ and $C_2 = \{\{a, b\}, \{c, d\}\}$. Then, $\mathcal{O}_\mathcal{C}(\{a, b\}) \wedge_{SO} \mathcal{O}_\mathcal{C}(\{a, b, c\}) = h^{-1}(h(\mathcal{O}_\mathcal{C}(\{a, b\})) \sqcap h(\mathcal{O}_\mathcal{C}(\{a, b, c\}))) = h^{-1}((\{\{a, b\}\}, \{\{c, d\}\}) \sqcap (\{\{a, b\}\}, \emptyset)) = h^{-1}(\{\{a, b\}\}, \{\{c, d\}\}) = ((\emptyset, \emptyset), (\{a, b\}, \{c, d\}))$.

Moreover, $(\mathcal{L}_1(\{a, b\}), \mathcal{E}_1(\{a, b\})) \wedge (\mathcal{L}_1(\{a, b, c\}), \mathcal{E}_1(\{a, b, c\})) = (\emptyset, \emptyset) \wedge (\emptyset, \emptyset) = (\emptyset, \emptyset)$ and $(\mathcal{L}_2(\{a, b\}), \mathcal{E}_2(\{a, b\})) \wedge (\mathcal{L}_2(\{a, b, c\}), \mathcal{E}_2(\{a, b, c\})) = (\{a, b\}, \{c, d\}) \wedge (\{a, b\}, \emptyset) = (\{a, b\}, \{c, d\}))$.

Example 14. Let $\mathcal{C} = (C_1, C_2)$ be a refinement sequence, where $C_1 = \{\{a, b, c, d\}\}$ and $C_2 = \{\{a, b\}, \{c, d\}\}$. $\mathcal{C}$ is not safe and $\mathcal{C}'$ is only made by covering $\{\{a, b\}, \{c, d\}\}$ (see the construction of $\mathcal{C}'$ in [1]).

We have that $\mathcal{O}_\mathcal{C}(\{a, b\}) \wedge_{SO} \mathcal{O}_\mathcal{C}(\{c, d\}) = h^{-1}(h(\mathcal{O}_\mathcal{C}(\{a, b\})) \sqcap_\mathcal{C} h(\mathcal{O}_\mathcal{C}(\{c, d\}))) = h^{-1}(\{\{a, b\}\}, \{\{c, d\}\}) \sqcap_\mathcal{C} (\{\{c, d\}\}, \{\{a, b\}\}) = h^{-1}(g^{-1} (\emptyset, \{\{a, b\}, \{c, d\}\})) = h^{-1}((\emptyset, \{\{a, b\}, \{c, d\}, \{a, b, c, d\}\})) = ((\emptyset, \{a, b, c, d\}), (\emptyset, \{a, b, c, d\}))$.

On the other hand, $(\mathcal{L}_1(\{a, b\}), \mathcal{E}_1(\{a, b\})) \wedge (\mathcal{L}_1(\{c, d\}), \mathcal{E}_1(\{c, d\})) = (\emptyset, \emptyset)$ and $(\mathcal{L}_2(\{a, b\}), \mathcal{E}_2(\{a, b\})) \wedge (\mathcal{L}_2(\{c, d\}), \mathcal{E}_2(\{c, d\})) = (\emptyset, \{a, b, c, d\})$.

5 Representation Theorem

In this section we associate a sequence of orthopairs with any finite poset.

Let $(P, \leq)$ be a finite partially ordered set and let n be the maximum number of elements of a chain in P. For each $i \in \{1, \ldots, n\}$ we define the i-th level of P as

$$P^i = \{\, N \in P \mid i = max\{|h| \mid h \text{ is a chain of } \downarrow N\}\,\}. \tag{4}$$

We denote by $\mathcal{M}(P)$ the set of maximal element of P and we set $U_P = \{x_1, \ldots, x_m\}$, where $m = |P| + |\mathcal{M}(P)|$. We call *maximal sequence* of P the sequence $\mathcal{C} = C_1, \ldots, C_n$ built as follows. Suppose $\mathcal{M}(P)$ consists of nodes $N_1, \ldots, N_u$, where $u = |\mathcal{M}(P)| \leq \lfloor m/2 \rfloor$ since $u < 2u \leq |\mathcal{M}(P)| + |P| = m$. We set

$$b_{N_i} = \{x_{2i-1}, x_{2i}\} \tag{5}$$

for every $i = 1, \ldots, u$ and

$$C_n = \{b_{N_i} \mid N_i \in \mathcal{M}(P)\}. \tag{6}$$

Since $|P \backslash \mathcal{M}(P)| = m - 2u$, we denote by $N_{u+1}, \ldots, N_{m-u}$ the nodes of $P \backslash \mathcal{M}(P)$ and we set $\alpha_P(N_i) = x_{i+u}$ for any $i \in \{u+1, \ldots, m-u\}$.

For each $N \notin \mathcal{M}(P)$, let

$$b_N = \bigcup_{M > N} b_M \cup \{\alpha_P(N)\} \tag{7}$$

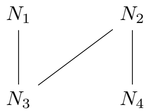
and, for each $j \in \{1, \ldots, n-1\}$,

$$C_j = \{b_N \mid N \in P^j\} \cup \{b_M \mid M \in \mathcal{M}(P) \text{ and } \downarrow M \cap P^j = \emptyset\}. \tag{8}$$

It is trivial to see that for each $N, M \in P$

$$b_N \cap b_M = \cup\{\, b_L \mid L \in \uparrow N \cap \uparrow M \,\}. \tag{9}$$

Example 15. Let P be the partially ordered set with the following Hasse diagram:



$U_P = \{x_1, \ldots, x_6\}$, since $6 = 4 + 2$, where $|P| = 4$ and $|\mathcal{M}(P)| = 2$. We have $\alpha_P(N_3) = x_5$ and $\alpha_P(N_4) = x_6$. Then, we have $b_{N_1} = \{x_1, x_2\}$, $b_{N_2} = \{x_3, x_4\}$, $b_{N_3} = \{x_1, x_2\} \cup \{x_3, x_4\} \cup \{\alpha_P(N_3)\} = \{x_1, x_2, x_3, x_4, x_5\}$ and $b_{N_4} = \{x_3, x_4\} \cup \{\alpha_P(N_4)\} = \{x_3, x_4, x_6\}$. Moreover, $n = 2$, then the maximal sequence is made by two partial coverings of $\{x_1, \ldots, x_6\}$ that are $C_1 = \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_6\}\}$ and $C_2 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$.

Proposition 5. *Let P be a finite partially ordered set. Then, the maximal sequence $\mathcal{C}$ of P is a complete and safe refinement sequence of U_P and $SO(\mathcal{C}) \cong K(U(P))$.*

Proof. Firstly, we prove that $\mathcal{C}$ is a refinement sequence of U_P. Then, suppose that $b \in C^i$ with $i > 1$, we have $b = b_N$ where $N \in P$. Since $b_N \in C^i$, two cases are possible: if $N \in P^i$, then there exists at least a node M of P^{i-1} such that $M < N$ (see (4)), hence $b_M \in C^{i-1}$ (see (8)) and $b_N \subset b_M$ (see (7)); if $N \notin P^i$, then $N \in \mathcal{M}(P)$ and $\downarrow N \cap P^i = \emptyset$. In this latter case, we have two subcases to consider: $\downarrow N \cap P^{i-1} = \emptyset$ which implies $b_N \in C^{i-1}$ and $\downarrow N \cap P^{i-1} \neq \emptyset$ which implies that there exists $M \in P^{i-1}$ with $M \leq N$, hence $b_N \subseteq b_M$ where $b_M \in C^{i-1}$.

$\mathcal{C}$ is complete, since if $b_N \cap b_M \neq \emptyset$ with $b_N, b_M \in P_{\mathcal{C}}$, then $b_N \cap b_M \supseteq b_L$ with $L \in \uparrow N \cap \uparrow M$ (see (9)), hence b_N and b_M can not belong to two upsets that are disjoint.

To prove that $\mathcal{C}$ is safe, we consider the blocks $b_N, b_{N_1}, \ldots, b_{N_k}$ of coverings of $\mathcal{C}$ with $b_N \subseteq b_{N_1} \cup \ldots \cup b_{N_k}$. Then, we pick a subset $\{b_{N'_1}, \ldots, b_{N'_h}\}$ of $\{b_{N_1}, \ldots, b_{N_k}\}$ such that $b_N \subseteq b_{N'_1} \cup \ldots \cup b_{N'_h}$ and $b_N \cap b_{N'_i} \neq \emptyset$ for each $i \in \{1, \ldots, h\}$. Trivially, $b_N \cap b \neq \emptyset$ if and only if $b_N \subseteq b$, when $N \in \mathcal{M}(P)$. Otherwise, if $N \notin \mathcal{M}(P)$, by (7) we have that $\alpha_P(N) \in b_N$, hence $\alpha_P(N)$ belongs to b'_{N_i} for some $i \in \{1, \ldots, h\}$, then $b_N \subseteq b_{N'_i}$ since $N'_i \leq N$ (see (7)).

Trivially, the poset $P_{\mathcal{C}}$ associate with $\mathcal{C}$ is $\{b_N \mid N \in P\}$ (see Definition 5) and it is isomorphic to P. Then, it is sufficient to show that $SO(\mathcal{C}) = K(U(P_{\mathcal{C}}))$.

By Theorem 2, $SO(\mathcal{C}) \subseteq K(U(P_{\mathcal{C}}))$. Viceversa, let $(A, B) \in K(U(P_{\mathcal{C}}))$, then $A^* \cap B^* = \emptyset$, since otherwise, by (9), there exist $N, M, L \in P$ such that

$b_L \subseteq b_N \cap b_M$, then $b_L \in A \cap B$ that is an absurd. By Theorem 3, $(A, B) \in SO(\mathcal{C})$. Therefore, $K(U(P_{\mathcal{C}})) \subseteq SO(\mathcal{C})$.

Furthermore, note that if $\mathcal{C} = C_1, \ldots, C_n$ is the maximal sequence of the poset P, then C_n is a partial partition of the respective universe U_P.

By Theorem 1 and Proposition 5, the following Theorem holds:

Theorem 6. *Let P be a partially ordered set and $\mathcal{C}$ its maximal sequence. Then, $SO(\mathcal{C})$ is a centered Kleene algebra that satisfies the interpolation property.*

Theorem 7. *Let $\mathcal{S}$ be a Kleene algebra. S is a finite centered Kleene algebra with interpolation property if and only if $\mathcal{S} \cong \mathbb{SO}'_{\mathcal{C}}$, where $\mathcal{C}$ is a complete and irridundant refinement sequence of a finite universe U.*

Proof. ($\Rightarrow$). If $\mathcal{T}$ is a centered Kleene algebra with interpolation property, then there exists a bounded distributive lattice $L_{\mathcal{T}}$ such that $\mathcal{T} \cong K(L_{\mathcal{T}})$, by Theorem 1. By Birkhoff representation theorem, there exists a poset $P_{L_{\mathcal{T}}}$ such that $L_{\mathcal{T}} \cong U(P_{L_{\mathcal{T}}})$. Consequently, $\mathcal{T} \cong K(U(P))$, where P is a partially ordered set.

By Proposition 5, $\mathcal{S} \cong SO(P_{\mathcal{C}})$ where $\mathcal{C}$ is the maximal sequence of P.

($\Leftarrow$). By Theorem 5, $\mathbb{SO}'_{\mathcal{C}}$ is a finite centered Kleene algebra with interpolation property.

Proposition 6. *Let $\mathcal{S}$ be a Kleene algebra. $\mathcal{S} \cong \mathcal{T}^+ \cup \mathcal{T}^-$ where $\mathcal{T}$ is a finite centered Kleene algebra with interpolation property if and only if $\mathcal{S} \cong \mathbb{SO}'_{\mathcal{C}}$, where $\mathcal{C}$ is a safe and pairwisely overlapping refinement sequence of a finite universe U.*

Proof. ($\Rightarrow$). Let $\mathcal{T}$ be a finite centered Kleene algebra with interpolation property, then $\mathcal{T} \cong K(U(P))$, where P is a partially ordered set. We consider the maximal sequence $\mathcal{C}$ of P. We call $\mathcal{C}_x$ the sequence obtained by adding x to each block of $\mathcal{C}$. Trivially $\mathcal{C}_x$ is a safe and pairwisely overlapping refinement sequence of the universe $U \cup \{x\}$ and $\mathcal{S} \cong \mathbb{SO}'_{\mathcal{C}}$.

($\Leftarrow$). By Theorem 5, if $\mathcal{C}$ is irridundant and pairwisely overlapping, then $SO(\mathcal{C}) \cong K(U(P_{\mathcal{C}'}))^+ \cup K(U(P_{\mathcal{C}'}))^-$.

6 Conclusions and Further Works

In this paper we have shown that sequences of orthopairs $SO(\mathcal{C})$ determined by refinement of coverings $\mathcal{C}$ of a universe U can be equipped with a structure of Kleene algebra $\mathbb{SO}_{\mathcal{C}}$. In particular, if the refinement sequence $\mathcal{C}$ is safe, then $\mathbb{SO}_{\mathcal{C}}$ is a subalgebra of a Kalman rotation $K(U(P_{\mathcal{C}}))$; if $\mathcal{C}$ is safe and pairwisely overlapping, then $\mathbb{SO}_{\mathcal{C}}$ coincides with the union of positive and negative elements of $K(U(P_{\mathcal{C}}))$; if $\mathcal{C}$ is safe and complete then it is isomorphic with $K(U(P_{\mathcal{C}}))$. Generalizing the assumption that $\mathcal{C}$ is safe, we have the weaker condition of irredundancy: if $\mathcal{C}$ is irredundant then we can find a safe and complete $\mathcal{C}'$ and define over $SO(\mathcal{C})$ a structure of Kleene algebra in such a way that it is isomorphic with $\mathbb{SO}_{\mathcal{C}'}$. While both for safe and pairwisely overlapping and for safe and complete refinement sequence we have a characterization of the associated

Kleene algebras, a characterization of Kleene algebras given by safe refinement sequence is not presented in this paper and this problem will be faced in the future. On the other side, we shown that for any finite centered Kleene algebra K with interpolation property, that is, for every $K \cong K(L)$, there is a safe and complete refinement sequence $\mathcal{C}$ such that $K \cong \mathbb{SO}_{\mathcal{C}}$.

This work can be framed in the context of generalization of operations on orthopairs to sequences of orthopairs, as done for example in [1,6]. This gives a way to interpret the operations in the considered algebraic structures (centered Kleene algebras here and IUML algebras in [1]) in terms of approximation of sets. Refinement of orthopairs can be also interpreted by a temporal semantics, as done for example in [5] for NM-algebras and this will be the topic of future works.

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Method of Maximum Two-Commodity Flow Search in a Fuzzy Temporal Graph

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Abstract. This paper is devoted to the task of the two-commodity maximum flow finding in a fuzzy temporal graph. Arcs of the network are assigned by the fuzzy arc capacities and crisp transit times. All network's parameters can vary over time, therefore, it allows to consider network as dynamic one. The task is to maximize total flow passing through the network, considering temporal nature of the network. Such methods can be applied in the real railways, roads, when it is necessary to take into account the commodities of two types solving the task of the optimal cargo transportation, for example, passenger and cargo trains or motor cars and lorries Method of operating fuzzy numbers for flow tasks is proposed that doesn't lead to the blurring of the resulting number.

Keywords: Fuzzy dynamic graph · Two-commodity fuzzy flow

1 Introduction

Considering transportation networks, the situation of two commodities, simultaneously passing along the network is often occurs. It is valid for passenger and cargo trains simultaneously moving along the railways, or motor cars and lorries etc. Therefore, we turn to the two-commodity flow network. This task is a variety of the multi-commodity flow problem, when different commodities travel along the network and have their own sources and sinks [1, 2]. The main difficulty of the multi-commodity flow problems (3 or more simultaneously passing commodities) is that the min-cut theorem does not hold for these tasks [3]. However, the mentioned theorem is valid for the two-commodity flow problems in undirected graphs [4, 5].

The relevant task while analyzing these networks is the maximum two-commodity flow-finding task, that consists in finding the maximum value of the flow of two types in the network. Its peculiarity is in possibility of the first flow to block the second one in such a way, that we do not obtain the optimal value. Hu [4] devised algorithm for such a problem and obtained optimal results.

Solving flow tasks in networks it is important to take into account either that the flow needs time to pass among adjacent nodes or that different departure time can influence the travel and arrival time. Literature analysis showed that dynamic networks

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[6] considered in the literature took into account no instant flow departure, while network's parameters hold constant. Multi-commodity flow tasks analysis is presented in [7, 8]. Therefore, such networks can not be treated as fully temporal. Temporal networks should be examined as networks with no instant flow transitions and time-varying parameters depend on flow departure time.

Transportation networks are usually connected with uncertainty of any kind. There are errors in measurements, changes in environment, road, weather conditions, the lack of information about road parameters and so on. Thus, it is necessary to consider uncertainty peculiar to network's parameters and represent edges of the network as fuzzy triangular numbers, therefore, we turn problem statements of flow tasks in fuzzy networks [9, 10]. The task of the maximum flow finding with fuzzy arc capacities was presented in [11, 12] and didn't consider the dynamic nature of the graph and the possibility of multiple flow passing.

Summarizing, we obtain the problem of the maximum two-commodity flow finding in the fuzzy temporal network, that was not described in the literature and propose algorithm for its solution.

The paper is organized as follows. In the Sect. 2 we give basic definitions and rules, underlying the proposed method. Section 3 presents the method of the maximum two-commodity dynamic flow finding in fuzzy temporal graph. Section 4 provides numerical example illustrating the main steps of the proposed method. Section 5 is conclusion and future work.

2 Definitions and Rules

Peculiarity of the two-commodity dynamic maximum flow finding method is that calculated maximum flow can block the flow of the second commodity. It is necessary to find the double path and redistribute the flow, if it is possible to avoid blocking. If the path doesn't exist, the corresponding maximum flow is found. The proposed method is based on following rules.

Rule 1. Transition to the fuzzy static two-commodity graph from the dynamic one.

Fuzzy time-expanded static two-commodity graph $\tilde{G}_p = (X_p, \tilde{A}_p)$ is constructed from the original fuzzy two-commodity dynamic graph $\tilde{G}$ creating a copy of each node $x_i \in X$ at each time period $\theta \in T$. The set of nodes X_p of the graph $\tilde{G}_p$ is defined as $X_p = \{(x_i, \theta) : (x_i, \theta) \in X \times T\}$. Expand node-sources and sinks for each commodity at each time period. Add artificial sources and sinks for each commodity and connect them by edges of infinite capacity with true sources and sinks. The set of edges $\tilde{A}_p$ includes edges from each node-time pair $(x_i, \theta) \in X_p$ to every node-time pair $(x_j, \vartheta = \theta + \tau_{ij}(\theta))$, where $x_j \in (x_i)$ and $\theta + \tau_{ij}(\theta) \leq p$. Fuzzy arc capacities $\tilde{u}(x_i, x_j, \theta, \vartheta)$ are equal to $\tilde{u}_{ij}(\theta)$. Transit times $\tau(x_i, x_j, \theta, \vartheta)$ are equal to $\tau_{ij}(\theta)$.

Rule 2. of transmission flow along the double path [3].

Define flows of the backward and forward paths depending on the flow values:
$\tilde{\xi}_f^1 = \min \tilde{\xi}^1 (x_i, x_j, \theta, \vartheta), \tilde{\xi}^1 (x_i, x_j, \theta, \vartheta) > \tilde{0}$,
$\tilde{\xi}_b^1 = \min \tilde{\xi}^1 (x_j, x_i, \vartheta, \theta), \tilde{\xi}^1 (x_j, x_i, \vartheta, \theta) > \tilde{0}$.

Thus, the flow along the double path is defined as the minimum flow value from the backward and forward paths: $\tilde{\xi}_{bf} = \min(\tilde{\xi}_f^1, \tilde{\xi}_b^1)$.

Edge capacity of the double path is determined as minimum from the residual capacities of the backward and forward paths: $\tilde{u}_{bf} = \min(\tilde{u}_b, \tilde{u}_f)$.

As the double path consists of two paths we pass $\tilde{r} = \min(\tilde{\xi}_{bf}^1, 0.5\tilde{u}_{bf})$ flow units of each commodity along the double path as follows: push $\tilde{r}$ units of the first commodity from the source to the sink along the backward path, then pass $\tilde{r}$ units of the first commodity from the sink to the source along the forward path.

After that push $\tilde{r}$ flow units of the second commodity along the double path as follows: transit $\tilde{r}$ units of the second commodity from the source to the sink along the backward path, then pass $\tilde{r}$ units of the second commodity from the source to the sink along the forward path.

3 Presented Method of the Maximum Two-Commodity Flow Finding Task in the Fuzzy Dynamic Graph

3.1 Problem Statement

Let us consider the proposed method for the maximum two-commodity flow finding in dynamic undirected graph in fuzzy terms, presented as a model (1)–(5):

$$\mathit{Maximize} \sum_{\theta=0}^{p} \sum_{s=1}^{2} \tilde{v}\left(s, s'\right), \tag{1}$$

$$\sum_{i} \sum_{\theta=0}^{p} \left(\sum_{x_j \in \Gamma(x_i)} \tilde{\xi}_{ij}^s(\theta) - \sum_{x_j \in \Gamma^{-1}(x_i)} \tilde{\xi}_{ji}^s(\theta - \tau_{ji}(\theta)) \right) = \tilde{v}(s, s')(p),\ x_i = s, \tag{2}$$

$$\sum_{i} \sum_{\theta=0}^{p} \left(\sum_{x_j \in \Gamma(x_i)} \tilde{\xi}_{ij}^s(\theta) - \sum_{x_j \in \Gamma^{-1}(x_i)} \tilde{\xi}_{ji}^s(\theta - \tau_{ji}(\theta)) \right) = \tilde{0},\ x_i \neq s, s';\ \theta \in T, \tag{3}$$

$$\sum_{i} \sum_{\theta=0}^{p} \left(\sum_{x_j \in \Gamma(x_i)} \tilde{\xi}_{ij}^s(\theta) - \sum_{x_j \in \Gamma^{-1}(x_i)} \tilde{\xi}_{ji}^s(\theta - \tau_{ji}(\theta)) \right) = -\tilde{v}(s, s')(p),\ x_i = t, \tag{4}$$

$$\sum_{s=1}^{2} \left| \tilde{\xi}_{ij}^s(\theta) \right| \leq \tilde{u}_{ij}(\theta),\ \forall\, (x_i, x_j) \in \tilde{A},\ \theta \in T. \tag{5}$$

In the model (1)–(5) $\tilde{\xi}_{ij}^s(\theta)$ – the s-th flow along the arc (x_i, x_j) at time period θ, $\tilde{v}(s, s')$ – the value s-th flow from the source s to the sink s', $\tilde{u}_{ij}(\theta)$ – capacity of the arc (x_i, x_j) at time period θ.

3.2 Problem Statement

Step 1. Turn to the time-expanded two-commodity fuzzy static graph $\tilde{G}_p = (X_p, \tilde{A}_p)$ from the original fuzzy two-commodity dynamic graph $\tilde{G}$ according to the rule 1.

Step 2. Search the maximum flow of the first commodity, considering the second commodity as zero by selecting the augmenting shortest path (in terms of the number of arcs) $\tilde{P}_p^{*\mu}$ from the artificial source 1^s to the artificial sink 1^t in the time-expanded graph according to the breadth-first-search.

Step 3. Find the maximum flow of the second commodity considering edge capacities as $\tilde{u}^1(x_i, x_j, \theta, \vartheta) = \tilde{u}(x_i, x_j, \theta, \vartheta) - \left|\xi^1(x_i, x_j, \theta, \vartheta)\right| \pm \xi^2(x_i, x_j, \theta, \vartheta)$. Search the augmenting shortest path (in terms of the number of arcs) $\tilde{P}_p^{*\mu}$ from the artificial source s^* to the artificial sink t^* in the constructed fuzzy residual network according to the breadth-first-search.

Step 4. Find the double path from 2^s to 2^t. There are two paths in the double path: backward and forward. Backward path consists of edges with first commodity passing in the opposite direction (negative value) and unsaturated edges. Residual edge capacity of the backward path is $\tilde{u}_b = \tilde{u}(x_i, x_j, \theta, \vartheta) + \tilde{\xi}^1(x_j, x_i, \vartheta, \theta) - \tilde{\xi}^2(x_i, x_j, \theta, \vartheta) > \tilde{0}$. Forward path is the path with nonzero positive value of the first commodity or negative value of the second commodity. Residual edge capacity of the forward path is $\tilde{u}_f = \tilde{u}(x_i, x_j, \theta, \vartheta) + \tilde{\xi}^1(x_i, x_j, \theta, \vartheta) - \tilde{\xi}^2(x_i, x_j, \theta, \vartheta) > \tilde{0}$.

4.1. If the whole double path or one of the constituent paths doesn't exist, the obtained flow $\tilde{r} = (\max[\tilde{\xi}^1(1^s, 1^t, \theta, \theta + \tau_{1^s1^t}(\theta)) + \tilde{\xi}^2(2^s, 2^t, \theta, \theta + \tau_{2^s2^t}(\theta))]$ is the maximum flow in the time-expanded graph, turn to the step 6.

4.2. If the path is found, pass the value $\tilde{r} = \min(\tilde{\xi}^1_{bf}, 0.5\tilde{u}_{bf})$ of each commodity along the double paths, where $\tilde{\xi}^1_{bf}$ and $\tilde{u}_{bf}$ are calculated due to the rule 2.

Step 5. Update the flow values in the graph $\tilde{G}_p(\tilde{\xi})$ and turn to the step 3.

Step 6. Turn to the initial dynamic graph $\tilde{G}$ as follows: reject the artificial nodes s', t' and arcs, connecting them with other nodes.

Two-commodity flow task is NP-hard and polynomial in the case of even arc capacities. The time-complexity is based on the time-complexity of the shortest path finding algorithm.

4 Numerical Example

Consider numerical example, illustrating fuzzy maximum two-commodity dynamic flow problem. The task is to find the maximum flow of the total commodity of two types from the source to the sink in the graph, shown in the Fig. 1.

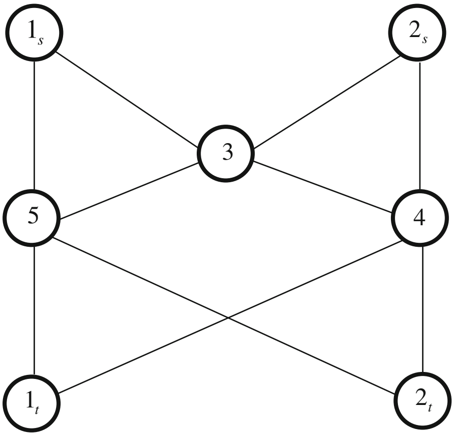

Fig. 1. Initial dynamic graph $\tilde{G}$

The values of fuzzy arc capacities depending on the flow departure time are given in the Table 1. Table 2 illustrates the values of transit times, depending on the flow departure time.

Table 1. Fuzzy arc capacities $\tilde{u}_{ij}$, depending on the flow departure time θ

Edges of the graph	Fuzzy edge capacities $\tilde{u}_{ij}$ at time periods θ, time units				
	0	1	2	3	4
$(1^s, 3)$	$\tilde{60}$	$\tilde{35}$	$\tilde{35}$	$\tilde{40}$	$\tilde{25}$
$(1^s, 5)$	$\tilde{20}$	$\tilde{20}$	$\tilde{25}$	$\tilde{15}$	$\tilde{15}$
$(2^s, 3)$	$\tilde{38}$	$\tilde{10}$	$\tilde{30}$	$\tilde{18}$	$\tilde{25}$
$(2^s, 4)$	$\tilde{55}$	$\tilde{45}$	$\tilde{52}$	$\tilde{60}$	$\tilde{60}$
$(3, 4)$	$\tilde{15}$	$\tilde{40}$	$\tilde{24}$	$\tilde{24}$	$\tilde{20}$
$(3, 5)$	$\tilde{55}$	$\tilde{55}$	$\tilde{16}$	$\tilde{30}$	$\tilde{30}$
$(4, 1^t)$	$\tilde{20}$	$\tilde{20}$	$\tilde{30}$	$\tilde{40}$	$\tilde{55}$
$(4, 2^t)$	$\tilde{40}$	$\tilde{40}$	$\tilde{30}$	$\tilde{35}$	$\tilde{52}$
$(5, 1^t)$	$\tilde{40}$	$\tilde{45}$	$\tilde{15}$	$\tilde{30}$	$\tilde{30}$
$(5, 2^t)$	$\tilde{16}$	$\tilde{20}$	$\tilde{18}$	$\tilde{20}$	$\tilde{40}$

Table 2. Time parameters τ_{ij} depending on the flow departure time θ

Edges of the graph	Time parameters τ_{ij} at time periods θ, time units				
	0	1	2	3	4
$(1^s, 3)$	1	1	3	2	2
$(1^s, 5)$	1	4	4	3	3
$(2^s, 3)$	1	1	3	1	2
$(2^s, 4)$	5	5	1	4	3
$(3, 4)$	1	1	1	4	4
$(3, 5)$	5	1	1	2	3
$(4, 1^t)$	5	5	1	1	1
$(4, 2^t)$	6	6	1	1	4
$(5, 1^t)$	5	2	1	2	2
$(5, 2^t)$	5	4	1	1	2

Step 1. Turn to the time-expanded static version of the initial dynamic graph, as shown in the Fig. 2.

Step 2. Find the augmenting paths, considering the flow of the second commodity as zero, therefore, the maximum flow of the first commodity is obtained, as presented in the Fig. 3. Maximum flow includes paths $1^s \rightarrow 1_s^0 \rightarrow 5^1 \rightarrow 1_t^3 \rightarrow 1_t$ with $\tilde{20}$ flow inits, $1^s \rightarrow 1_s^0 \rightarrow 3^1 \rightarrow 4^2 \rightarrow 1_t^3 \rightarrow 1_t$ with $\tilde{30}$ flow inits, $1^s \rightarrow 1_s^0 \rightarrow 3^1 \rightarrow 5^2 \rightarrow 1_t^3 \rightarrow 1_t$ with $\tilde{15}$ flow inits, $1^s \rightarrow 1_s^1 \rightarrow 3^2 \rightarrow 4^3 \rightarrow 1_t^4 \rightarrow 1_t$ with $\tilde{24}$ flow inits and it is equal to $\tilde{89}$ units.

Step 3. Search the maximum flow of the second commodity, considering already found flow of the first commodity $\tilde{u}^1(x_i, x_j, \theta, \vartheta) = \tilde{u}(x_i, x_j, \theta, \vartheta) - \left|\xi^1(x_i, x_j, \theta, \vartheta)\right| \pm \xi^2(x_i, x_j, \theta, \vartheta)$ in such a way that the total edge capacity constraint will be valid. Thus, we receive the following paths: $2^s \rightarrow 2_s^2 \rightarrow 4^3 \rightarrow 2_t^4 \rightarrow 2_t$ and pass $\tilde{35}$ units along it, $2^s \rightarrow 2_s^1 \rightarrow 3^2 \rightarrow 5^3 \rightarrow 2_t^4 \rightarrow 2_t$ and pass $\tilde{10}$ units along it, $2^s \rightarrow 2_s^0 \rightarrow 3^1 \rightarrow 4^2 \rightarrow 2_t^3 \rightarrow 2_t$ with $\tilde{10}$ units, as $(3^1, 4^2)$ is a joint edge with the maximum residual capacity $\tilde{10}$ units and $2^s \rightarrow 2_s^0 \rightarrow 3^1 \rightarrow 5^2 \rightarrow 2_t^3 \rightarrow 2_t$ with $\tilde{18}$ units along it. This flow distribution is presented in the Fig. 4.

Step 4.1. Find the double path from 2^s to 2^t.

Backward path is $2^s \rightarrow 2_s^2 \rightarrow 4^3 \rightarrow 3^2 \rightarrow 5^3 \rightarrow 2_t^4 \rightarrow 2_t$. It includes edge $(4^3, 3^2)$ with the flow of the first commodity, passing in the opposite direction and other edges that are unsaturated. The flow of this path is defined according to the rule 2 and it is equal to $\tilde{\xi}_b^1 = \min \tilde{\xi}_{3^2 4^3}^1 = \tilde{24}$ units. Edge capacity of the residual path is determined as minimum of the residual capacities of the backward path, i.e. $\tilde{u}_b = \min(\infty, \tilde{17}, \tilde{48}, \tilde{6}, \tilde{10}, \infty) = \tilde{6}$.

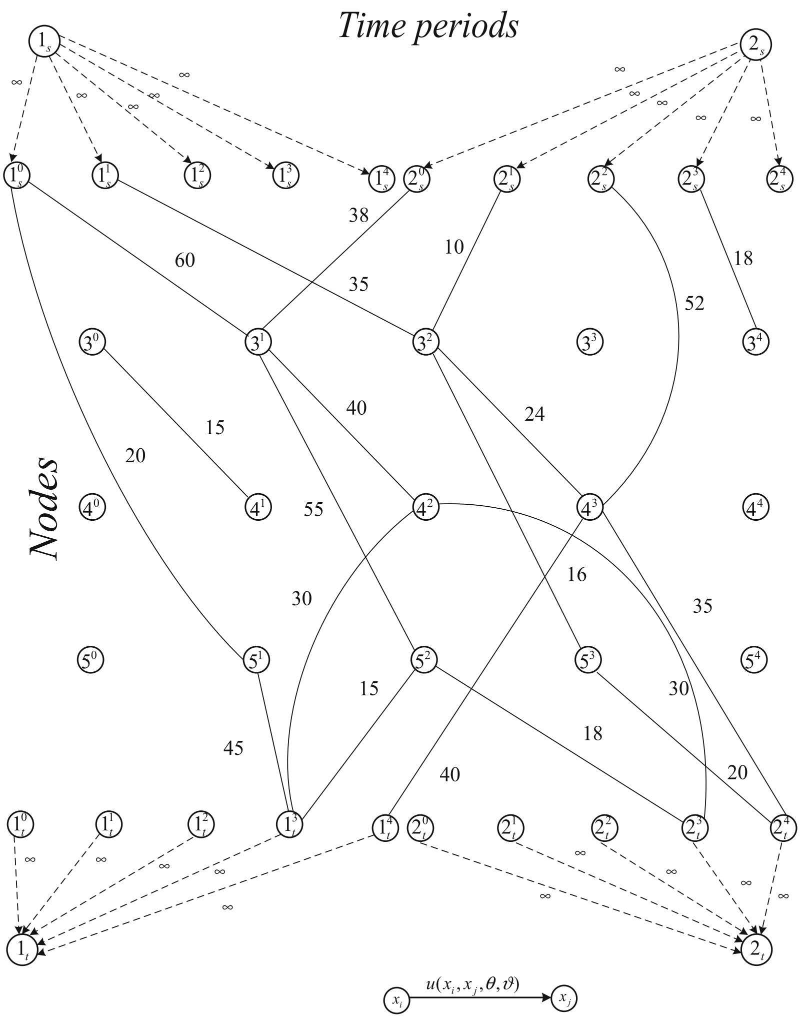

Fig. 2. Time-expanded static graph $\tilde{G}_p$

Forward path is $2^s \to 2_s^0 \to 3^1 \to 4^2 \to 2_t^3 \to 2_t$ and it consists of unsaturated edges and the edge $(3^2, 4^3)$ with positive flow passing in the forward direction. The flow of this path is defined according to the rule 2 and it is equal to $\tilde{\xi}_f^1 = \min \tilde{\xi}^1_{3^1 4^2} = \tilde{30}$ units. Edge capacity of the residual path is determined as minimum of the residual capacities of the forward path, i.e. $\tilde{u}_f = \min(\infty, \tilde{10}, \tilde{60}, \tilde{20}, \infty) = \tilde{10}$.

Therefore, $\tilde{\xi}_{bf}^1 = \min(\tilde{24}, \tilde{30}) = \tilde{24}$ units, $\tilde{u}_{bf} = \min(\tilde{6}, \tilde{10}) = \tilde{6}$ units, $\tilde{r} = \min(\tilde{24}, \tilde{3}) = \tilde{3}$ units.

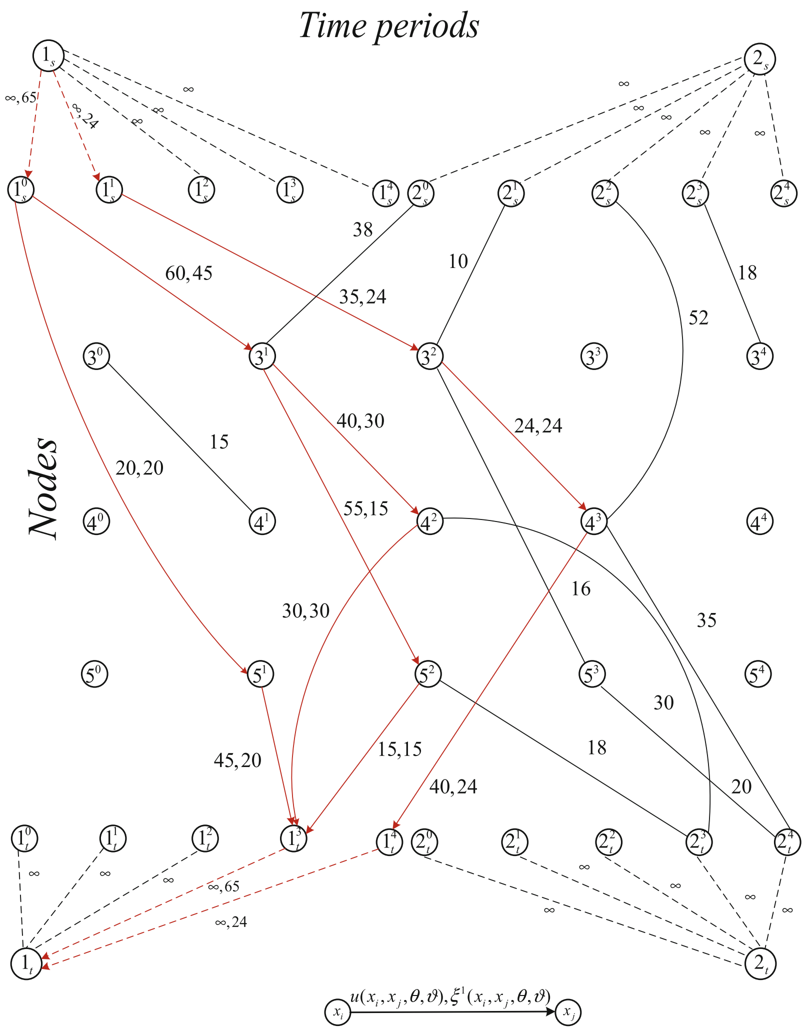

Fig. 3. $\tilde{G}_p$ with the maximum flow of the first commodity

Step 4.2. Pass $\tilde{3}$ flow units of the first commodity and second commodity along the double path, as presented in the Fig. 5.

Step 5. The flow of the second commodity cannot be increased and there are no augmenting paths. Therefore, the obtained maximum flow is $8\tilde{9}+7\tilde{9}=16\tilde{8}$ units.

The fuzzy maximum flow is obtained and it is necessary to present it as fuzzy triangular number, determining its deviation borders.

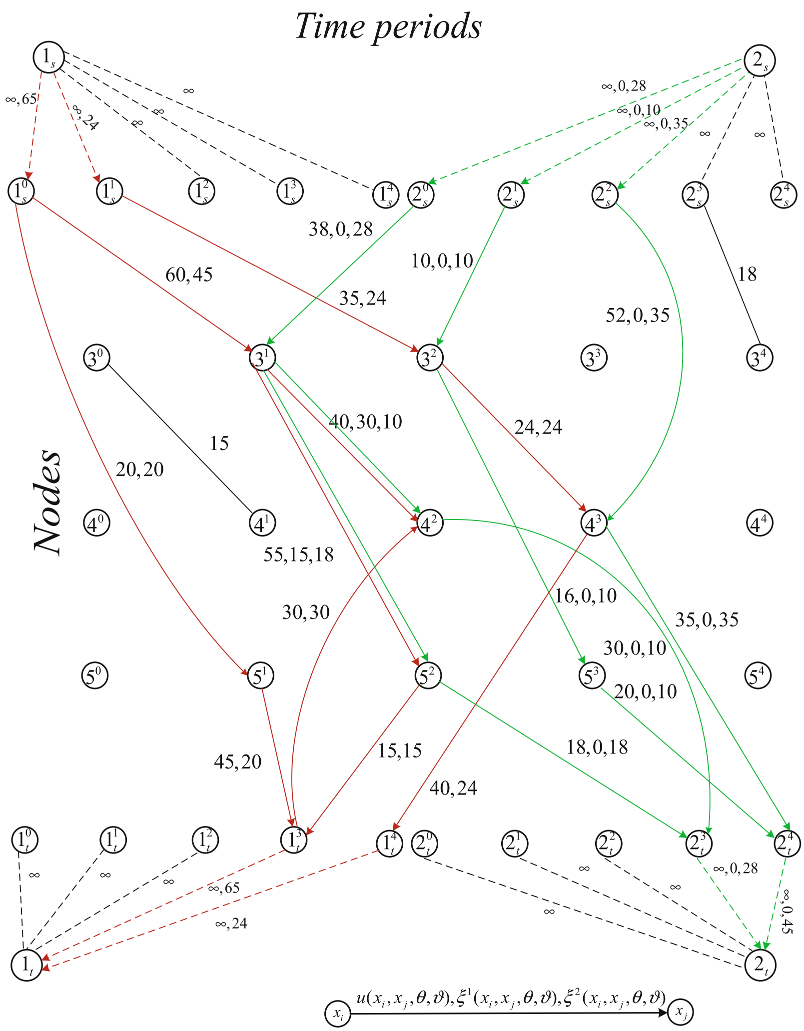

Fig. 4. $\tilde{G}_p$ with the maximum flow of the first commodity and corresponding maximum flow of the second commodity

We can utilize the approach, described in [9, 10]. Its main idea is that it is no need to operate deviations of fuzzy numbers at the first step of calculations: it is sufficiently to operate centers of fuzzy numbers, blurring them at the final step. Advantages of such a method is that final fuzzy number preserves its value, avoiding strong blurring of borders.

Thus, Fig. 6 shows deviation borders of the fuzzy basic capacities, set by experts.

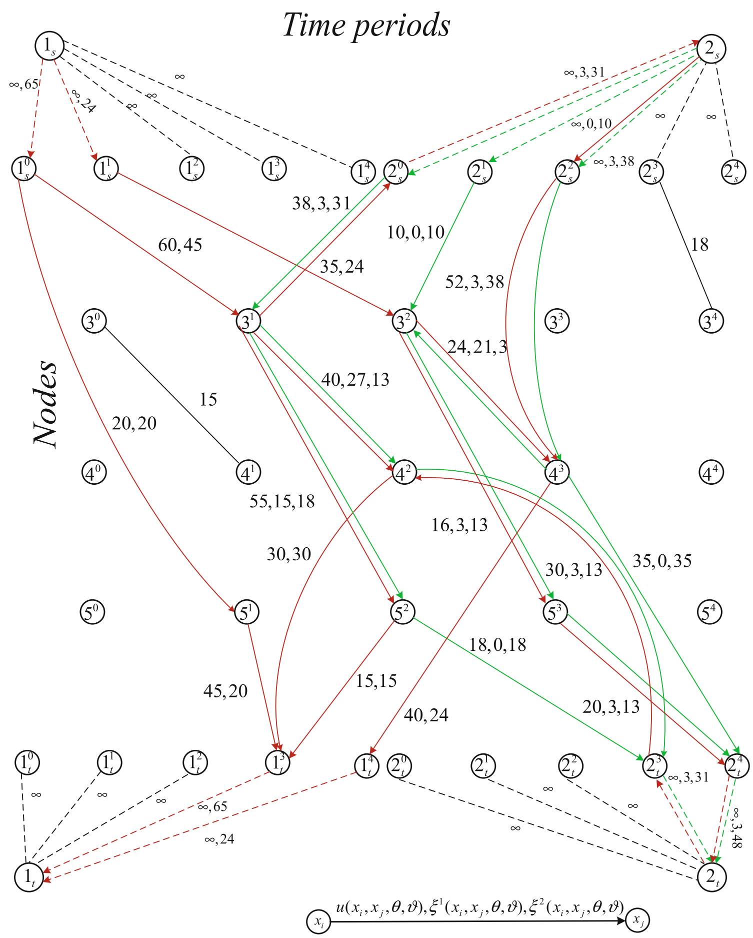

Fig. 5. $\tilde{G}_p$ with $\tilde{\xi} = 3$ of each commodity sent along the double path

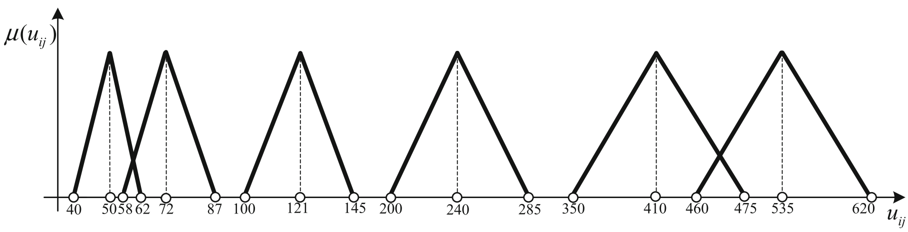

Fig. 6. Membership functions of the basic values of edge capacities of the network $\tilde{G}$

The result is between two adjacent basic values of the arc capacities: $\tilde{121}$ with the left deviation $l_1^L = 21$, right deviation – $l_1^R = 24$ and $\tilde{240}$ with the left deviation $l_2^L = 40$, right deviation – $l_2^R = 45$. We obtain deviations: $l_1^L \approx 28$, $l_1^R \approx 32$.

Finally, the maximum flow in the fuzzy two-commodity dynamic graph can be represented by fuzzy triangular number (140, 168, 200) units.

5 Conclusion and Future Work

Present paper illustrates the approach to the maximum flow determining in the fuzzy two-commodity dynamic flow graph. Underlying undirected graph has fuzzy values of arc capacities and two distinguished sources and sinks for each commodity. The parameters of the graph are transit and depend on the flow departure time. The considered approach is based on the formulated rules of the time-expanded graph and transmission the flow along the double path. The proposed approach has important practical value in transportation planning and optimization the flows on the real types of roads in the tasks, where it is important to take into account the commodities of two types, for example, passenger and cargo trains or motor cars and lorries. In the future methods of the minimum cost multi-commodity flow determining will be solved in the fuzzy dynamic networks.

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Allocation Method for Fuzzy Interval Graph Centers Based on Strong Connectivity

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Abstract. In this paper the problem of defining service centres optimum allocation in transportation network is observed. It is supposed that transportation network is described by a fuzzy interval graph. The notion of fuzzy set of strong connectivity is introduced. It is shown that the problem of service centers location can be reduced to a problem of finding fuzzy set of strong connectivity. The method and algorithm of finding fuzzy set of strong connectivity is considered in this paper. The example of finding optimum allocation of centers in fuzzy interval graph is considered.

Keywords: Fuzzy interval · Fuzzy interval graph · Service centers · Strong connectivity fuzzy set

1 Introduction

The worldwide expansion and diversified implementation of geographic information systems (GIS) are largely appearing due to the need to improve information systems that support decision-making. Application spheres of GIS are huge, thus geoinformation technologies become leaders in information retrieval, display, analytical tools and decision support [1, 2].

However, geographic data are often associated with significant uncertainty. Errors in data that are often used without considering their inherent uncertainties lead to a high probability of obtaining information of doubtful value. Uncertainty presents throughout the process of geographical abstraction: from acquiring data to using them [3].

Data modeling [4] is the process of abstraction and generation of real forms of geographic data. This process provides a conceptual model of the real world. It is doubtful that the geographical complexity can be reduced in models of perfect accuracy. So, the imminent contradiction between the real world and the model is the inaccuracy and uncertainty that can lead to the wrong decision making.

Allocation of centers [5] is the optimization problem that is effectively solved by GIS. This problem includes the tasks of optimum allocation of extremely important services, such as hospitals, police stations, fire brigades etc. In some tasks, the optimality criterion can considered as distance minimization (travel time) from the service

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center to the most remote service point, therefore, the problem is in optimization of the "worst case" [6]. At the same time, the information, presented in GIS, can be approximate or insufficiently reliable [7]. Therefore, utilization of subjunctive estimates dealing with the distances between the parts of the considered area or travel time, based on the expert's experience using linguistic variables, can be a convenient way for the formalization task of the centers allocation.

Thus, the linguistic variable "distance" can take arbitrary values of the type «about 50 km», «near 20–25 min» , that can be considered as fuzzy number $\tilde{a}$ and fuzzy interval $[\tilde{a}, \tilde{b}]$ with triangular and trapezoidal membership functions, respectively [8, 9].

In the present work, the approach to service centers allocation is proposed. The peculiarity of the problem is that distance between the parts of the area or travel time is fuzzy interval, thus, the model of the territory is a fuzzy interval graph.

2 Basic Concepts and Definitions

We consider some territory which is divided into n areas (set X). There are k service centres, which may be placed into these areas (set V, $k < n$). It is supposed that service centre may be placed into some stationary place of each area. It is necessary for the given number of the service centers to define the places of their best allocation. In other words, it is necessary to define the places of k service centers into n areas such that the service of all territory was carried out on is minimum possible time or distance at least to one service center.

The task of the best allocation of centers on the fuzzy graph can be limited to the problem of finding a subset of vertices V, which all the other vertices X/V of the fuzzy graph are attained in the best way according to a given criterion. Three strategies of the selection of vertices V can be proposed [10]:

- We "go" from each vertex of subset X/V, and arrive at a vertex of V;
- We "come out" of any of the vertices of V, and reach all vertices of subset X/V;
- We "come out" of any of the vertices of V, reach all vertices of subset X/V and come back.

In this paper, the third strategy is considered, and the criterion of optimality is the minimization of the total fuzzy interval (distance or time) from the service center to the most remote vertex and back.

Let's consider that the information received from GIS is presented in the form of the fuzzy interval graph $\tilde{G} = (X, \tilde{U})$ [11]. A set $X = \{x_i\}$, $i \in I = \{1, 2, \ldots, n\}$ is the set of vertices. The vertices represent areas of some territory. A set $\tilde{U} = \{\langle \tilde{l}_{ij}/(x_i, x_j)\rangle\}$ is the set of the fuzzy directed edges. A value $\tilde{l}_{ij} = [\tilde{a}_{ij}, \tilde{b}_{ij}]$ is fuzzy interval «approximately $[a_{ij}, b_{ij}]$». It is a meaning of linguistic variable «time of journey from vertex x_i to vertex x_j». Here a_{ij}, $b_{ij} \in R^1$, and $a_{ij} \leq b_{ij}$. Let's believe, that the interval $\tilde{l}_{ii} = [0, 0]$, $\forall i \in \{1, 2, \ldots, n\}$.

For the decision of this problem we will consider the concept of strong connectivity fuzzy set of the fuzzy interval graph.

Let $\tilde{l}_1 = [\tilde{a}_1, \tilde{b}_1]$ and $\tilde{l}_2 = [\tilde{a}_2, \tilde{b}_2]$ are any fuzzy intervals. If $a_1 > a_2$ and $b_1 < b_2$, then we set that $\tilde{l}_1 < \tilde{l}_2$. Otherwise, we can set naturally relations >, <, $\leq$, and $\geq$ between fuzzy intervals [11].

The sum of fuzzy intervals $\tilde{l}_1$ and $\tilde{l}_2$ we call an interval $\tilde{l} = [\tilde{a}, \tilde{b}]$, in which borders $\tilde{a} = \tilde{a}_1 + \tilde{a}_2$ and $\tilde{b} = \tilde{b}_1 + \tilde{b}_2$ [12].

Let $\tilde{L}_1 = \{[\tilde{a}_{1i}, \tilde{b}_{1i}]\}$ and $\tilde{L}_2 = \{[\tilde{a}_{2j}, \tilde{b}_{2j}]\}$, $i = \overline{1, p_1}$, $j = \overline{1, p_2}$ are two sets of fuzzy intervals. We denote by $\tilde{L}_1 + \tilde{L}_2$ the sum defined as $\tilde{L}_1 + \tilde{L}_2 = \{[\tilde{a}_k, \tilde{b}_k]\}$, $\tilde{a}_k = \tilde{a}_{1i} + \tilde{a}_{2j}$, $\tilde{b}_k = \tilde{b}_{1i} + \tilde{b}_{2j}$, $k = p_1 \times p_2$.

Example 1. Let sets of intervals $\tilde{L}_1 = \{[\tilde{10}, \tilde{15}], [\tilde{12}, \tilde{14}]\}$ and $\tilde{L}_2 = \{[\tilde{7}, \tilde{9}], [\tilde{8}, \tilde{10}]\}$, then $\tilde{L}_1 + \tilde{L}_2 = \{[\tilde{17}, \tilde{24}], [\tilde{18}, \tilde{25}], [\tilde{19}, \tilde{23}], [\tilde{20}, \tilde{24}]\}$. Here $p_1 = p_2 = 2$, $k = 4$.

We consider two operations $MIN(\tilde{L})$ and $MAX(\tilde{L})$ on set of intervals $\tilde{L}$ [11]. These operations are an estimation of subsets of the least and the greatest intervals from set of intervals $\tilde{L}$.

Example 2. Let set of intervals $\tilde{L} = \{[10, 15], [12, 14], [12, 17], [15, 18]\}$, then $MIN(\tilde{L}) = \{[12, 14]\}$, and $MAX(\tilde{L}) = \{[15, 18]\}$.

Property 1.

$$MIN(\tilde{L}_1 + \tilde{L}_2) = MIN(\tilde{L}_1) + MIN(\tilde{L}_2),$$
$$MAX(\tilde{L}_1 + \tilde{L}_2) = MAX(\tilde{L}_1) + MAX(\tilde{L}_2).$$

Let x and y are any vertices of fuzzy interval graph $\tilde{G} = (X, \tilde{U})$. We will define through $\tilde{L}_{xy}$ a set of fuzzy intervals by means of which the vertex y is achievable from the vertex x. Then for each pair of vertices (x, y) we can put a fuzzy interval $MIN(\tilde{L}_{xy} + \tilde{L}_{yx})$ in conformity.

Definition 1. A strong connectivity subset is called subset vertices $V_{\tilde{\Lambda}} \subseteq X$ with interval $\tilde{\Lambda} = \underset{\forall y \in X \setminus V}{MAX}(\underset{\forall x \in V}{MIN}(\tilde{L}_{xy} + \tilde{L}_{yx}))$ from which any vertex of fuzzy graph is accessible with an interval not more $\tilde{\Lambda}$ and which is minimal in the sense that there is no subset $V'_{\tilde{\Lambda}} \subseteq V_{\tilde{\Lambda}}$, having the same accessible property.

Among all strong connectivity subsets consisting of 1 vertex, we select such subset in which fuzzy interval is the least. We designate them as $\tilde{\Lambda}_1$. Among all strong connectivity subsets consisting of two vertices we select such subset in which fuzzy interval also is the least among themselves and we will designate them as $\tilde{\Lambda}_2$, and etc.

Definition 2. A fuzzy set

$$\tilde{V} = \{\langle\tilde{\Lambda}_1/1\rangle, \langle\tilde{\Lambda}_2/2\rangle, \ldots, \langle\tilde{\Lambda}_n/n\rangle\}$$

is called a strong connectivity fuzzy set of fuzzy interval graph $\tilde{G}$.

Fuzzy interval $\tilde{\Lambda}_k$ ($k \in \{1, 2, \ldots, n\}$) signify that we can place k-centers in graph $\tilde{G}$ so that there is a route from at least one center to any vertex of graph $\tilde{G}$ and back. The length of this cyclic route will be not more than interval $\tilde{\Lambda}_k$.

Example 3. Consider the fuzzy interval graph presented in Fig. 1:

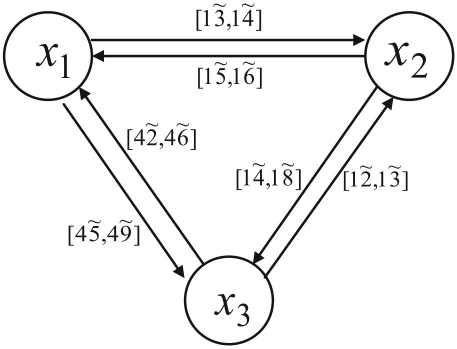

Fig. 1. Fuzzy interval graph (n = 3)

For the strong connectivity subsets, consisting of 1, 2, and 3 vertices fuzzy intervals families have view:

if $V = \{x_1\}$ or $V = \{x_3\}$ then $\tilde{\Lambda} = [5\tilde{4}, 6\tilde{2}]$; if $V = \{x_2\}$ then $\tilde{\Lambda} = [2\tilde{8}, 3\tilde{1}]$;
if $V = \{x_1, x_3\}$ or $V = \{x_1, x_2\}$ then $\tilde{\Lambda} = [2\tilde{6}, 3\tilde{1}]$; if $V = \{x_2, x_3\}$ then $\tilde{\Lambda} = \{[2\tilde{8}, 3\tilde{1}]\}$;
if $V = \{x_1, x_2, x_3\}$ then $\tilde{\Lambda} = \{[0, 0]\}$.

Hence, the strong connectivity fuzzy set is defined as:

$$\tilde{V} = \{\langle[2\tilde{8}, 3\tilde{1}]/1\rangle, \langle[2\tilde{6}, 3\tilde{1}]/2\rangle, \langle[0, 0]/3\rangle\}.$$

3 Method and Algorithm for Finding Strong Connectivity Fuzzy Set

We will consider the method of finding a strong connectivity fuzzy set in the fuzzy interval graph. The given method is an analogue Maghout's method for definition of all fuzzy interval base sets [11], and Maghout's method for the definition of fuzzy vitality sets for fuzzy no interval graphs [13].

Let's consider some strong connectivity fuzzy set $V_{\tilde{\Lambda}} \subseteq X$ with fuzzy interval $\tilde{\Lambda}$. Then for an arbitrary vertex $x_i \in X$, one of the following conditions must be true:

(a) $x_i \in V_{\tilde{\Lambda}}$;
(b) if $x_i \notin V_{\tilde{\Lambda}}$, then there is a vertex $x_j \in V_{\tilde{\Lambda}}$ such that fuzzy interval $\tilde{l}_{ji} = \tilde{l}(x_j, x_i)$ is no more $\tilde{\Lambda}$.

In other words, the following statement is true:

$$(\forall x_i \in X)[x_i \in V_{\tilde{\Lambda}} \vee (\exists x_j \in V_{\tilde{\Lambda}} | \tilde{l}_{ji} \leq \tilde{\Lambda})]. \quad (1)$$

To each vertex $x_i \in X$ we assign Boolean variable p_i that takes the value 1, if $x_i \in V_{\tilde{\Lambda}}$ and 0 otherwise. Let's enter the predicate form $Q(\tilde{l}_{ji})$ that takes the value 1, if $\tilde{l}_{ji} \leq \tilde{\Lambda}$ and 0 otherwise. Using analogy between generality and existence quantifiers on the one hand, both operations conjunction and disjunction with another, we obtain a true logical proposition:

$$\Phi_V = \underset{i=\overline{1,n}}{\&} (p_i \vee \underset{j=\overline{1,n}}{\vee} (p_j \& Q(\tilde{l}_{ji})) = 1. \quad (2)$$

Believing, that $Q(\tilde{l}_{ii}) = Q([0,0]) = 1$, an expression (2) may be rewrite as:

$$\Phi_V = \underset{i=\overline{1,n}}{\&} \left(\underset{j=\overline{1,n}}{\vee} (p_j \& Q(\tilde{l}_{ji})) \right) = 1. \quad (3)$$

We open the parentheses in the expression (3) and reduce the similar terms the following rules:

$$\begin{cases} Q(\tilde{l}_1)\&Q(\tilde{l}_2) = Q(\tilde{l}_1), \ if \ \tilde{l}_1 \geq \tilde{l}_2; \\ p_1\&p_2\&Q(\tilde{l}_1)\&Q(\tilde{l}_2) \vee p_1\&p_2\&Q(\tilde{l}_3) \\ = p_1\&p_2\&Q(\tilde{l}_1)\&Q(\tilde{l}_2) \ if \ \tilde{l}_1 < \tilde{l}_3\&\tilde{l}_2 < \tilde{l}_3; \\ p_1\&p_2\&Q(\tilde{l}_1) \vee p_1\&Q(\tilde{l}_2) = p_1\&Q(\tilde{l}_2), \ if \ \tilde{l}_1 \geq \tilde{l}_2. \end{cases} \quad (4)$$

As a result expression (3) will become:

$$\Phi_V = \underset{i=\overline{1,t}}{\vee} (p_{1i}\&p_{2i}\&\ldots\&p_{ki}\&Q(\tilde{l}_{1i})\&Q(\tilde{l}_{2i})\&\ldots\&Q(\tilde{l}_{ri})) = 1. \quad (5)$$

Property 2. If in expression (5) further simplification on the basis of rules (4) is impossible, then everyone disjunctive member *i* defines strong connectivity fuzzy set with the least interval.

The following method of foundation of a fuzzy set of strong connectivity fuzzy set may be propose on the base of property:

- write proposition (3) for given fuzzy interval graph $\tilde{G}$;
- simplify proposition (3) by proposition (4) and present it as proposition (5);
- define fuzzy set of strong connectivity, which correspond to the disjunctive members of proposition (5).

To construct the expression (5) we rewrite expression (3) as follows:

$$\Phi_V = \underset{i=\overline{1,n}}{\&}(a_{i1}p_1 \vee a_{i2}p_2 \vee \ldots \vee a_{in}p_n). \quad (6)$$

To perform absorption, and as a result reduction the number of elements in target formula, we have to transform expression (6) to another form.

An element $a_{ij}p$ can be converted to a vector with coefficient:

$$a_{ij}\bar{P}_j, \text{ where } \bar{P}_j = ||p_i^{(j)}|| \text{ and } p_i^{(j)} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Using this notation, let's show the conjunction rule:

$$a_1\bar{P}_1 \& a_2\bar{P}_2 = a\bar{P}, \text{ where } a = \min\{a_1, a_2\}, \bar{P} = ||p_i|| p_i = \max\left\{p_i^{(1)}, p_i^{(2)}\right\}, i = \overline{1,n}$$

Also, we need an auxiliary operation of comparing two vectors:

$$(\bar{P}_1 \leq \bar{P}_2) \leftrightarrow (\forall i = \overline{1,n})[p_i^{(1)} \leq p_i^{(2)}].$$

Considering the operations described above, the rule of absorption would be as follows:

$$a_1\bar{P}_1 \vee a_2\bar{P}_2 = a_1\bar{P}_1, \text{ if} a_1 \geq a_2 \text{ and } \bar{P}_1 \leq \bar{P}_2. \quad (7)$$

The correspondent pseudo algorithm below is used to reduce the number of operands and can be presented as follows:

1. Each element of the first (j = 1) multiplication operand $(a_{11}p_1 \vee a_{12}p_2 \vee \ldots \vee a_{1n}p_n)$ of expression (6) is converted to the vector. The result is to be written in the first n elements of the buffer vector $\bar{V}_1 = ||v_i^{(1)}||, i = \overline{1, n^2}$.
2. Increase $j := j + 1$.
3. Convert each element of the operand $(a_{j1}p_1 \vee a_{j2}p_2 \vee \ldots \vee a_{jn}p_n)$ to the vector. The result is to be written in the first n elements of the buffer vector $\bar{V}_2 = ||v_i^{(2)}||, \ i = \overline{1,n}$.
4. The next stage consists of the conjunction of two vectors $\bar{V}_1$ and $\bar{V}_2$. The result is placed into $\bar{V}_3 = ||v_i^{(3)}||, \ i = \overline{1, n^2}$. While placing elements into $\bar{V}_3$, absorption is made using rule (7).
5. Copy all the elements of vector $\bar{V}_3$ to vector $\bar{V}_1$ $(v_i^{(1)} := v_i^{(3)}, \ i = \overline{1, n^2})$.
6. Increase $j := j+1$.
7. If $j \leq n$ then go to step 3, otherwise go to step 8.
8. Expression (5) is built from the elements in $\bar{V}_1$. This way we have strong connectivity fuzzy set $\tilde{V}$ of graph $\tilde{G} = (X, \tilde{U})$.

4 Numerical Example

Let's consider the given method on an example of the fuzzy interval graph presented in Fig. 2:

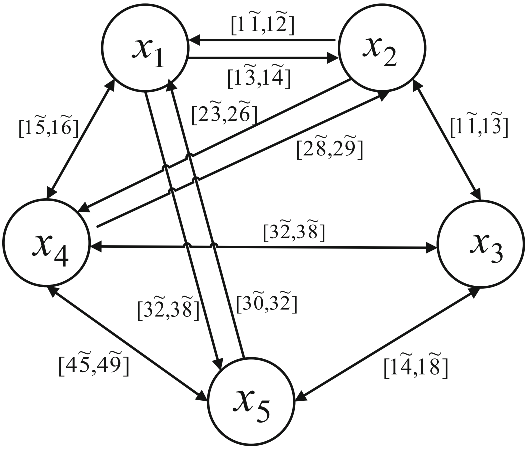

Fig. 2. Fuzzy interval graph (n = 5)

The adjacency matrix for this graph has the following form:

$$R_X = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc} \infty & [1\tilde{1},1\tilde{2}] & \infty & [1\tilde{5},1\tilde{6}] & [3\tilde{0},3\tilde{5}] \\ [1\tilde{3},1\tilde{4}] & \infty & [1\tilde{1},1\tilde{3}] & [2\tilde{8},2\tilde{9}] & \infty \\ \infty & [1\tilde{1},1\tilde{3}] & \infty & [1\tilde{1},1\tilde{2}] & [1\tilde{2},1\tilde{3}] \\ [1\tilde{5},1\tilde{6}] & [2\tilde{3},2\tilde{6}] & [1\tilde{1},1\tilde{2}] & \infty & [1\tilde{4},1\tilde{5}] \\ [3\tilde{2},3\tilde{8}] & \infty & [1\tilde{2},1\tilde{3}] & [1\tilde{4},1\tilde{5}] & \infty \end{array} \right) \end{array}.$$

For a finding of a reachability matrix of the graph, we will define operation of adjacency matrix exponentiation as:

– zero degree -

$$R_X^0 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc} [0,0] & \infty & \infty & \infty & \infty \\ \infty & [0,0] & \infty & \infty & \infty \\ \infty & \infty & [0,0] & \infty & \infty \\ \infty & \infty & \infty & [0,0] & \infty \\ \infty & \infty & \infty & \infty & [0,0] \end{array} \right) \end{array}.$$

– second degree - $R_X^2 = R_X \times R_X = ||\tilde{l}_{ik}^{(2)}||$, where matrix elements are defined as

$$\tilde{l}_{ik}^{(2)} = \underset{j=\overline{1,n}}{MIN}(\tilde{l}_{ij}^{(1)} + \tilde{l}_{jk}^{(1)})$$

– degree t - $R_X^t = R_X^{t-1} \times R_X$.

We define matrices $R_X^0, R_X^1, R_X^2, R_X^3, R_X^4$. Then we find their crossing. As a result we receive a reachability matrix $N_X = R_X^0 \cap R_X^1 \cap R_X^2 \cap R_X^3 \cap R_X^4$:

$$N_x = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} [0,0] & [1\tilde{1},1\tilde{2}] & [2\tilde{2},2\tilde{5}] & [1\tilde{5},1\tilde{6}] & [2\tilde{9},3\tilde{1}] \\ [1\tilde{3},1\tilde{4}] & [0,0] & [1\tilde{1},1\tilde{3}] & [2\tilde{2},2\tilde{5}] & [2\tilde{3},2\tilde{6}] \\ [2\tilde{4},2\tilde{7}] & [1\tilde{1},1\tilde{3}] & [0,0] & [1\tilde{1},1\tilde{2}] & [1\tilde{2},1\tilde{3}] \\ [1\tilde{5},1\tilde{6}] & [2\tilde{2},2\tilde{5}] & [1\tilde{1},1\tilde{2}] & [0,0] & [1\tilde{4},1\tilde{5}] \\ [2\tilde{9},3\tilde{1}] & [2\tilde{3},2\tilde{6}] & [1\tilde{2},1\tilde{3}] & [1\tilde{4},1\tilde{5}] & [0,0] \end{pmatrix} \end{array}.$$

Let's make an expression (2) on the received reachability matrix:

$$\Phi_V = \begin{array}{c} \{Q(0,0)p_1 \vee Q(24,26)p_2 \vee Q(46,52)p_3 \vee Q(30,32)p_4 \vee Q(58,62)p_5\}\& \\ \&\{Q(24,26)p_1 \vee Q(0,0)p_2 \vee Q(22,26)p_3 \vee Q(44,50)p_4 \vee Q(46,52)p_5\}\& \\ \&\{Q(46,52)p_1 \vee Q(22,26)p_2 \vee Q(0,0)p_3 \vee Q(22,24)p_4 \vee Q(24,26)p_5\}\& \\ \&\{Q(30,32)p_1 \vee Q(44,50)p_2 \vee Q(22,24)p_3 \vee Q(0,0)p_4 \vee Q(28,30)p_5\}\& \\ \&\{Q(58,62)p_1 \vee Q(46,52)p_2 \vee Q(24,26)p_3 \vee Q(28,30)p_4 \vee Q(0,0)p_5\}. \end{array}$$

Before the first iteration of algorithm vectors $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$, $\bar{V}_4$ and $\bar{V}_5$ have forms:

$$\bar{V}_1 = \begin{pmatrix} [0,0](10000) \\ [24,26](01000) \\ [46,52](00100) \\ [30,32](00010) \\ [58,62](00001) \end{pmatrix}, \ \bar{V}_2 = \begin{pmatrix} [24,26](10000) \\ [0,0](01000) \\ [22,26](00100) \\ [44,50](00010) \\ [46,52](00001) \end{pmatrix}, \ \bar{V}_3 = \begin{pmatrix} [46,52](10000) \\ [22,26](01000) \\ [0,0](00100) \\ [22,24](00010) \\ [24,26](00001) \end{pmatrix},$$

$$\bar{V}_4 = \begin{pmatrix} [30,32](10000) \\ [44,50](01000) \\ [22,24](00100) \\ [0,0](00010) \\ [28,20](00001) \end{pmatrix}, \ \bar{V}_5 = \begin{pmatrix} [58,62](10000) \\ [46,52](01000) \\ [24,26](00100) \\ [28,30](00010) \\ [0,0](00001) \end{pmatrix}.$$

After the first iteration of the algorithm vector $\bar{V}_1 := \bar{V}_3 = \bar{V}_1 \& \bar{V}_2$ has the following form:

$$\bar{V}_1^T = ([24,26](10000),\ [0,0](11000),\ [22,26](10100),\ [24,26](01000),\ [46,52](00100),\ [30,32](00110),\ [44,50](00010),\ [58,62](00001))$$

After completing the iterations, finally we have:

$$\bar{V}_1^T = \begin{array}{l} ([58,62](10000),\ [46,52](01000),\ [46,52](00100),\ [44,50](00010), \\ [58,62](00001),\ [24,26](10100),\ [24,26](01100),\ [30,32](00110),\ [28,30](10010), \\ [46,52](10001),\ [28,30](01001),\ [46,52](00101),\ [44,50](00011),\ [22,26](11001), \\ [22,26](10101),\ [24,26](01011),\ [30,32](00111),\ [22,24](11101),\ [22,24](11011), \\ [0,0](11111)) \end{array}.$$

So, the formula (6) for this graph has the form:

$$\Phi_{\mathrm{B}} = Q(58,62)p_1 \vee Q(46,52)p_2 \vee Q(46,52)p_3 \vee \underline{Q(44,50)p_4} \vee Q(58,62)p_5 \vee Q(24,26)p_1p_3 \vee \underline{Q(24,26)p_2p_3} \vee Q(30,32)p_3p_4 \vee Q(28,30)p_1p_4 \vee Q(46,52)p_1p_5 \vee Q(28,30)p_2p_5 \vee Q(46,52)p_3p_5 \vee Q(44,50)p_4p_5 \vee \underline{Q(22,26)p_1p_2p_5} \vee Q(22,26)p_1p_3p_5 \vee Q(24,26)p_2p_4p_5 \vee Q(30,32)p_3p_4p_5 \vee \underline{Q(22,24)p_1p_2p_3p_5} \vee Q(22,24)p_1p_2p_4p_5 \vee \underline{Q(0;0)p_1p_2p_3p_4p_5}.$$

From this equation follows, that the strong connectivity fuzzy set is:

$$\tilde{V} = \{\langle [4\tilde{4}, 5\tilde{0}]/1\rangle, \langle [2\tilde{4}, 2\tilde{6}]/2\rangle, \langle [2\tilde{2}, 2\tilde{6}/3]\rangle, \langle [2\tilde{2}, 2\tilde{4}]/4\rangle, \langle [0;0]/5\rangle\}.$$

The strong connectivity fuzzy set defines following optimum allocation of the centers: If we have 5 centers we place them in each vertex. In this case any expenses for achievement of other areas do not required (time equally 0). If we have 4 centers they should be placed in the vertices 1, 2, 3, and 5. In this case the least time is placed into fuzzy interval $[2\tilde{2}, 2\tilde{4}]$. If we have 3 centers they should be placed in the vertices 1, 2, and 5. In this case the least time is placed into fuzzy interval $[2\tilde{2}, 2\tilde{6}]$. If we have 2 centers they should be placed in the vertices 2 and 3. In this case the least time is placed into fuzzy interval $[2\tilde{4}, 2\tilde{6}]$. And at last if we have only 1 center it should be placed in the vertex 4. In this case the least time is placed in fuzzy interval $[4\tilde{4}, 5\tilde{0}]$.

A fuzzy set of strong connectivity also gives some information about the possible selection of number centers. Since the allocation of centers is associated with material costs, for the given fuzzy graph we can conclude that there is not much point in placing three or four centers in comparison with the two centers.

Let us introduce the approach to obtaining of membership functions of found fuzzy intervals according to the method, presented in [11, 14]. Present the parameter of fuzzy time "near $\tilde{x}'$", which is among the neighboring basic values "near $\tilde{x}_1$" and "near $\tilde{x}_2$" with triangular membership functions $\mu_{\tilde{x}_1}(x)$ and $\mu_{\tilde{x}_2}(x)$. Therefore, the borders of membership function $\mu_{\tilde{x}'}(x)$ of fuzzy number «near $\tilde{x}'$» can be presented as linear combination of the left and right basic values' parameters:

$$\begin{cases} l^L = \frac{(x_2 - x')}{(x_2 - x_1)} \times l_1^L + (1 - \frac{(x_2 - x')}{(x_2 - x_1)}) \times l_2^L, \\ l^R = \frac{(x_2 - x')}{(x_2 - x_1)} \times l_1^R + (1 - \frac{(x_2 - x')}{(x_2 - x_1)}) \times l_2^R. \end{cases} \quad (8)$$

Neighboring membership functions are represented in Fig. 3:

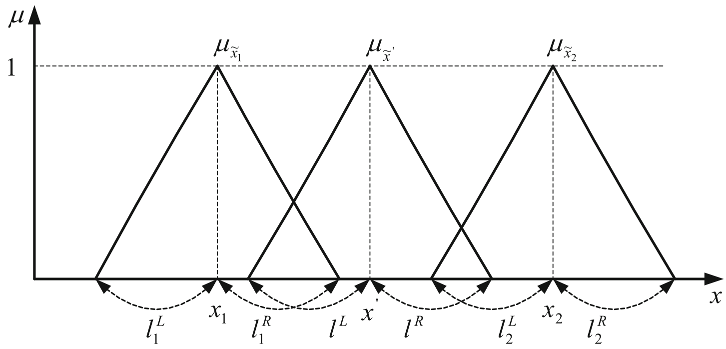

Fig. 3. Definition of the triangular membership function

Let Fig. 4 shows the membership functions of basic values of linguistic variable «travel time» in the form of fuzzy number «near 9», «near 12», «near 18», «near 23», «near 25» and «near 30».

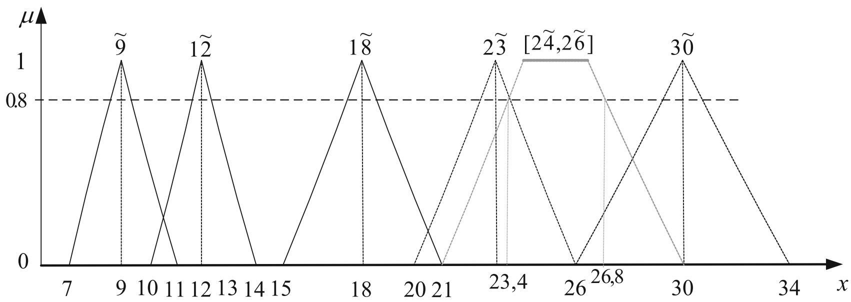

Fig. 4. Linguistic variable «travel time», its basic values' membership function

Let us found deviation borders for the number $\tilde{24}$: $l^L = 3$, $l^R = 3$, and for number $\tilde{26}$: $l^L = 4$, $l^R = 4$, according to expressions (8).

The proposed method can calculate membership function parameters $\mu_{[\tilde{x}_l,\tilde{x}_r]}(x)$ of arbitrary fuzzy interval $[\tilde{x}_l, \tilde{x}_r]$ in the form of trapezium:

$$\mu_{[\tilde{x}_l,\tilde{x}_r]}(x) = \begin{cases} \mu_{\tilde{x}_l}(x), & \text{if } x \le x_l; \\ 1, & \text{if } x_l \le x \le x_r; \\ \mu_{\tilde{x}_r}(x), & \text{if } x \ge x_r. \end{cases}$$

Figure 4 shows trapezoidal membership function, corresponding to the fuzzy interval $[\tilde{24}, \tilde{26}]$.

We can conclude, that interval metrics between objects with the assigned values of membership function (time or distance) is obtained. Thus, if the membership function value is more than 0.8, the service time is within the interval [23.4, 26.8] with two service centers location, as shown in Fig. 4.

5 Conclusion and Future Work

The task of defining of optimal allocation of centres as the task of definition fuzzy set of strong connectivity of fuzzy interval graphs was considered. The definition method of strong connectivity fuzzy set is the generalization of Maghout's method for fuzzy no interval graphs. This method is effective for the graphs which have no homogeneous structure and no large dimensionality. The results of this method were used to solve the problem of choosing the location of centers on railway networks. The implementation of this method was carried out on the railway map of the Russian Federation. The data are taken from GIS «Object Land» [15]. It is necessary to notice, that the considered method allows define the best places of allocations of the service centers in case of their placing only into vertices of the graph (instead of on edges with generation of new vertices). In our future work, we are going to investigate the problem of placing absolute centers, that is, the problem of finding service centers in the case where it is possible to place centers on the edges of a fuzzy graph.

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Measuring Uncertainty for Interval Belief Structures and its Application for Analyzing Weather Forecasts

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Abstract. While analyzing statistical data we face with a problem of modeling uncertainty. One among well justified models is based on belief structures that allow us to describe imprecision and conflict in information. We use this model for analyzing contradiction in weather forecasts. For this aim we build several measures of contradiction based on the introduced imprecision index and the disjunctive aggregation rule for interval belief structures. We use these characteristics for analyzing weather forecasts.

Keywords: Interval belief structures · Inclusion indices · Wasserstein metric · Measures of contradiction

1 Introduction

Weather forecast is an interval of possible temperature changes during a day (or more exactly during night, morning, midday, or evening). These forecasts are made several days before the target day. This information can be easily aggregated using belief structures: forecasts are focal elements and the value of the mass function gives us the degree of our beliefs that a given forecast is true.

Because belief functions can describe information with different nature, the methods of evaluating uncertainty can be also different. Thus, we should adopt them for a solving problem. In our investigation we evaluate the amount of contradiction among information sources based on the introduced imprecision index and the disjunctive rule for interval belief structures, and on related to them inclusion indices and the introduced distances between interval belief structures.

The paper has the following structure. We remind at first the basic definitions from the theory of belief structures, and then we discuss types of uncertainty and their evaluation for interval belief structures. The remained part of the paper is devoted to analyzing weather forecasts and our conclusions.

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2 Interval Belief Structures for Describing Weather Forecasts

Let X be the universal set and let $\mathcal{A}$ be a system of its subsets. By definition [15], a tuple $BS = (\mathcal{A}, m)$, where $m : \mathcal{A} \to [0,1]$, is called a *finitely defined belief structure*, if $m(A) > 0$ for a finite number of elements A in $\mathcal{A}$ and $\sum\limits_{A\in\mathcal{A}} m(A) = 1$. The function m is called the *basic belief assignment* (bba) or *mass function*. If $m(A) > 0$ for $A \in \mathcal{A}$, then the set A is called a *focal element*. Because in this paper only finitely defined belief structures are considered, the term "finitely defined" will be often omitted in the sequel. The set of all focal elements for a belief structure is called the *body of evidence*. We usually associate with any belief structure set functions

$$Bel(B) = \sum_{A\in\mathcal{A}|A\subseteq B} m(A), Pl(B) = \sum_{A\in\mathcal{A}|A\cap B=\emptyset} m(A),$$

where B can be an arbitrary subset of X, called the *belief function* and the *plausibility function* respectively.

In the next we will consider *interval finitely defined belief structures*. In this case $X = \mathbb{R}$ and $\mathcal{A}$ consists of segments, i.e. every $A \in \mathcal{A}$ is of the type $A = [a, b]$, where $a \leqslant b$.[1] In the paper the set of all possible finitely defined interval belief structures is denoted by $\mathcal{BS}$. Assume that every focal element in a $BS = (\mathcal{A}, m)$ is of the type $[a, b]$, where $a = b$, then BS is called *Bayesian*, and the set of all such belief structures is denoted by $\mathcal{BBS}$.

Assume that $BS_1 = (\mathcal{A}, m_1)$ and $BS_2 = (\mathcal{A}, m_2)$ in $\mathcal{BS}$, then their convex sum $aBS_1 + (1-a)BS_2$ for $a \in [0,1]$ is the belief structure $BS = (\mathcal{A}, m)$ with bba $m(A) = am_1(A) + (1-a)m_2(A)$ for all $A \in \mathcal{A}$.

Assume that we have the forecast of temperature for the next midday, and it certifies that the temperature will be between 12 °C and 16 °C. Then we can describe this information by the segment $[12, 16]$. In the next we will have forecasts for a target day made during a week. For example, assume that we should analyze forecasts of the weather in Moscow on the 1st of December, 2016, made during seven days before the target date, which are shown in Table 1.

Table 1. Temperature forecasts

A_7	A_6	A_5	A_4	A_3	A_2	A_1	Actual temperature
$[-10, -9]$	$[-11, -6]$	$[-11, -4]$	$[-8, -3]$	$[-9, -3]$	$[-10, -5]$	$[-9, -4]$	$[-7, -6]$

Obviously, forecast A_7 made seven days before the target day should be less precise than A_1 made one day before the target day. We can aggregate this

[1] Smets considers also in [17] interval belief structures, but they are not finitely defined.

information using the belief structure assigning values $m(A_i)$, $i = 1, ..., 7$. The simplest way is to define $m(A_i) = 1/7$, $i = 1, ..., 7$. If we try to take in account our beliefs in forecasts, then obviously $m(A_1) > m(A_2) > ... > m(A_7)$. For example, let $m(A_{i+1}) = qm(A_i)$, $i = 1, ..., 6$, where $q \in [0, 1)$ is a discount of our beliefs, then $\sum_{i=1}^{7} m(A_i) = m(A_1) \sum_{i=0}^{6} q^i = 1$ and $m(A_i) = (1-q)q^i / (1-q^7)$.

3 Types of Uncertainty Described by Interval Belief Structures

It is well known [1,11] that belief structures model two types of uncertainty: conflict and imprecision (non-specificity). In our problem we observe conflict if we can find contradictory forecasts in the body of evidence. For example, forecasts $A_7 = [-10, -9]$ and $A_4 = [-8, -3]$ are contradictory because they give us absolutely different intervals of temperature. Thus, one can decide that a belief structure $(\mathcal{A}, m)$ with a body of evidence $\mathfrak{M}$ does not contain conflict iff $\bigcap_{A_i \in \mathfrak{M}} A_i \neq \emptyset$. This definition seems to be not good for weather forecasts, because intuitively forecasts $A_3 = [-9, -3]$ and $A_2 = [-10, -5]$ are also contradictory, because A_2 predicts lower temperature than A_3. Thus, it seems that we don't observe conflict in the body of evidence $\mathfrak{M}$ iff $A_i \subseteq A_j$ or $A_j \subseteq A_i$ for every $A_i, A_j \in \mathfrak{M}$, i.e. the set $\mathfrak{M}$ is a linear ordered w.r.t. inclusion relation $\subseteq$. The third possible interpretation is that each forecast A_i gives us the exact interval of temperatures during the whole day (24 h). In this case predictions $A_1, ..., A_7$ are not contradictory iff $A_1 = ... = A_7$. We will see later how these three interpretations of conflict can be implemented for defining different conflict measures.

In our problem imprecision is explained by the fact that forecasts usually do not give us the exact value of temperature, but the interval of its possible values. The simplest idea for measuring imprecision of forecasts is to use the Lebesgue measure V for measurable subsets of $\mathbb{R}$. It gives us for a segment $[a, b]$ the value $V([a, b]) = b - a$. Following the idea for constructing linear imprecision indices [2] or the generalized Hartley measure [5], we can define the imprecision index ν_I for a interval belief structure $BS = (\mathcal{A}, m)$ as

$$\nu_I(BS) = \sum_{A_i \in \mathcal{A}} m(A_i)V(A_i).$$

Let us consider the problem of measuring contradiction between belief structures. Obviously, we can define any belief structure $BS = (\mathcal{A}, m)$ describing its body of evidence $\mathfrak{M}$ and giving values $m(A)$, $A \in \mathfrak{M}$. Thus, any $BS \in \mathcal{BS}$ can be defined by $BS = (\mathfrak{M}, m)$. Assume that we have two belief structures $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$. For example, these belief structures can aggregate forecasts for the same target day but obtained from different research organizations or these structures may describe forecasts for the same day but made in different intervals of time. For example, BS_1 may be the aggregation of forecasts A_7, A_6, A_5, and BS_2 may be the aggregation of next ones. Let us answer the question, when there is no contradiction between BS_1 and BS_2? Consider first the simplest case, when $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are singletons, i.e. $\mathfrak{M}_1 = \{A\}$ and $\mathfrak{M}_2 = \{B\}$. Following the same arguments, as we discuss conflict within belief structures, we can come to the following possibilities: there is no contradiction iff (a) $A = B$; (b) $A \subseteq B$ or $B \subseteq A$; (c) $A \cap B \neq \emptyset$. Let us recognize that any among given definitions can be appropriate for a solving problem. To generalize given definitions for any pair BS_1 and BS_2, we should extend the relation $\subseteq$ to the set $\mathcal{BS}$. Let us remind that such relation was introduced in [6] and called specialization.

Definition 1. *Let $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$ be belief structures then we say that BS_1 is the specialization of BS_2 and write $BS_1 \subseteq BS_2$ if there is a mapping $m : \mathfrak{M}_1 \times \mathfrak{M}_2 \to [0, 1]$ such that*

(1) $\sum\limits_{A \in \mathfrak{M}_1} m(A, B) = m_2(B)$ *for every* $B \in \mathfrak{M}_2$*;*
(2) $\sum\limits_{B \in \mathfrak{M}_2} m(A, B) = m_1(A)$ *for every* $A \in \mathfrak{M}_1$*;*
(3) $m(A, B) = 0$ *if* $A \not\subseteq B$*.*

Using Definition 1, we can give the following possible descriptions, when there is no contradiction between $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$: (a) $BS_1 \subseteq BS_2$ and $BS_2 \subseteq BS_1$; (b) $BS_1 \subseteq BS_2$ or $BS_2 \subseteq BS_1$; (c) there is a belief structure $BS_3 = (\mathfrak{M}_3, m_3)$ such that $BS_3 \subseteq BS_1$ and $BS_3 \subseteq BS_2$.

We will describe and analize some known measures of contradiction in the next section.

4 Basic Contradiction Measures in the Theory of Belief Functions

There are many approaches for evaluating contradiction between belief functions, but, in our opinion, since most of them are defined for a finite universal set X, they are not well suited for interval belief structures. In this case the set $\mathcal{A}$ consists of all possible subsets of X also denoted by 2^X. In the next we will consider belief structures $BS_i = (2^X, m_i)$ with their corresponding belief Bel_i and plausibility functions Pl_i, $i = 1, 2$.

Aggregation based approaches use some aggregation rules for belief structures. The firstly introduced contradiction measure is derived from the classical *conjunctive rule* [16]: $BS_3 = (2^X, m_3)$ is the conjunction of BS_1 and BS_2 if $m_3(C) = \sum\limits_{A \cap B = C} m(A)m(B)$, $C \in 2^X$, and we evaluate the contradiction using the value $m_3(\emptyset)$. This conjunction is used in the theory of belief functions if sources of information are independent. If we cannot use this assumption, then it is justifiable to use conjunctive rules of the type $m_3(C) = \sum\limits_{A \cap B = C} m(A, B)$, where the joint belief assignment $m : 2^X \times 2^X \to [0, 1]$ obeys the conditions: (1) $\sum\limits_{A \in 2^X} m(A, B) = m_2(B)$ for every $B \in 2^X$; (2) $\sum\limits_{B \in 2^X} m(A, B) = m_1(A)$ for every $A \in 2^X$, and if we don't know the interaction between information sources, we take the smallest possible value $m_3(\emptyset)$ as the amount of contradiction.

It is also possible to evaluate contradiction using the *disjunctive rule* [8]. Let $BS_3 = (2^X, m_3)$ be the belief structure generated by the classical disjunctive rule, then $m_3(C) = \sum_{A \cup B = C} m(A)m(B)$, $C \in 2^X$. Assume that contradiction between belief structures is transformed to non-specificity in BS_i, and we evaluate the amount of non-specificity in BS_i by an imprecision index $\nu_I(BS_i) = \sum_{A \in 2^X} m(A)\,|A|$. Then the amount of contradiction between BS_1 and BS_2 is not higher than $\max_{i=1,2}(\nu_I(BS_3) - \nu_I(BS_i))$. Analogously, the classical disjunctive rule is used, when sources of information are independent. In general case we use the disjunctive rule based on joint belief assignment m of BS_1 and BS_2. In this case we define $m_3(C) = \sum_{A \cup B = C} m(A, B)$. In case of unknown interaction between belief structures the disjunction BS_3 can be chosen with the smallest non-specificity.

Metric based approaches evaluate contradiction using some metric on belief structures [9]. One among popular metrics is

$$d_J(BS_1, BS_2) = \sqrt{0.5 \sum_{A,B \in 2^X} d_{A,B}\,(m_1(A) - m_2(A))\,(m_1(B) - m_2(B))},$$

where $d_{A,B}$ are called Jaccard indices and defined by $d_{A,B} = |A \cap B| / |A \cup B|$ for $A, B \neq \emptyset$, and $d_{\emptyset,\emptyset} = 0$. Another metric is based on plausibility functions

$$d_{Pl}(BS_1, BS_2) = 1 - \frac{1}{r_1 r_2} \sum_{A \in 2^X} Pl_1(A) Pl_2(A),$$

where $r_i = \sqrt{\sum_{A \in 2^x} Pl_i^2(A)}$, $i = 1, 2$.

Lui [13] proposes a contradiction measure, in which the distance between pignistic probabilities is used. For belief structures BS_i, $i = 1, 2$, pignistic probabilities are defined by $P_i(\{x\}) = \sum_{x \in B} m(B)/\,|B|$, where $x \in X$. Then for evaluating contradiction we use the distance $d(P_1, P_2) = 0.5 \sum_{x \in X} |P_1(\{x\}) - P_2(\{x\})|$. Analogously, we can evaluate contradiction [4] using values of plausibility functions on singletones: $Con_{cf}(BS_1, BS_2) = \sum_{x \in X} |Pl_1(\{x\}) - Pl_2(\{x\})|$.

Let us analyze when according to above contradiction measures belief structures $BS_i = (\mathfrak{M}_i, m_i)$ are assumed to be non-contradictory. If we use the contradiction measure based on the classical conjunctive rule, then the amount of contradiction is equal to zero iff $A \cap B \neq \emptyset$ for every $A \in \mathfrak{M}_1$ and $B \in \mathfrak{M}_2$, thus, in this case the amount of contradiction is not necessarily equal to zero if $BS_1 = BS_2$. If we measure contradiction based on all possible conjunctive rules, then sources of information are non-contradictory iff there is a belief structure BS with $BS \subseteq BS_1$ and $BS \subseteq BS_2$. If we measure contradiction using disjunctive rules by $\max_{i=1,2}(\nu_I(BS_3) - \nu_I(BS_i))$, then we have non-contradictory sources of information iff $BS_1 = BS_2$. Obviously, we have the same property if we measure contradiction by some metric on the set of all belief structures. Of course, the choice of a contradiction measure should be justified by an application. Some of above measures can be generalized for interval belief structures, but they do not look as good measures of contradiction. For example, if we use the contradiction measure based on the classical conjunctive rule for intervals, then this measure is equal to zero if intervals have empty intersection, and it is equal to one otherwise. This means that this measure is not stable to small changes in processing data (see [12] how to handle this problem for finite case).

5 The Disjunction and Inclusion Indices of Interval Belief Structures

The disjunction of interval belief structures looks like their union. The formal definition of this concept is given in [8]. We will give the definition of disjunction of belief structures assuming that the resulting disjunction should have the smallest imprecision. Because the disjunction of interval belief structures has to be also an interval belief structure, we define the disjunction of intervals $[a_i, b_i]$, where $a_i \leqslant b_i$, $i = 1, 2$, as $[a_1, b_1] \cup_I [a_2, b_2] = [\min\{a_1, a_2\}, \max\{b_1, b_2\}]$.

Definition 2. *Let $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$ be in $\mathcal{BS}$, then the disjunction $BS_3 = BS_1 \cup BS_2$ of BS_1 and BS_2 is an interval belief structure $BS_3 = (\mathfrak{M}_3, m_3)$ with the body of evidence $\mathfrak{M}_3 = \{A \cup_I B | A \in \mathfrak{M}_1, B \in \mathfrak{M}_2\}$ and m_3 is defined as a solution of the linear programming problem:*

$$\nu_I(BS_3) \to \min, \tag{1}$$

where $m_3(C) = \sum\limits_{A \cup_I B = C} m(A, B)$, $C \in \mathfrak{M}_3$, and the set function m obeys conditions (1), (2), and (3) from Definition 1.

Remark 1. The interval belief structure BS_3 from Definition 2 is not uniquely defined in general. But this does not influence on our conclusions, because we will use in the next only the value $\nu_I(BS_3)$.

Lemma 1. *Let $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$ be in $\mathcal{BS}$, and let $BS_3 = BS_1 \cup BS_2$. Then (1) $BS_i \subseteq BS_3$, $i = 1, 2$; (2) $\nu_I(BS_i) \leqslant \nu_I(BS_3)$, $i = 1, 2$; (3) $\nu_I(BS_2) = \nu_I(BS_3)$ iff $BS_1 \subseteq BS_2$.*

Definition 3. *Let $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$ be in $\mathcal{BS}$. Then the inclusion index $\nu(BS_1 \subseteq BS_2)$ is defined by*

$$\nu(BS_1 \subseteq BS_2) = \nu_I(BS_1 \cup BS_2) - \nu_I(BS_2).$$

Let us analyze first the properties of the introduced inclusion index, when bodies of evidence of $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$ are singletons, i.e.

$\mathfrak{M}_i = \{[a_i, b_i]\}$, $i = 1, 2$. Then we identify BS_1 and BS_2 with segments $[a_1, b_1]$ and $[a_2, b_2]$. Thus, we can write

$$\nu([a_1, b_1] \subseteq [a_2, b_2]) = \max\{b_1, b_2\} - \min\{a_1, a_2\} - (b_2 - a_2). \tag{2}$$

Because

$$\max\{b_1, b_2\} = 0.5\left(|b_1 - b_2| + b_1 + b_2\right), \;\; \min\{a_1, a_2\} = 0.5\left(a_1 + a_2 - |a_1 - a_2|\right),$$

the formula (2) can be rewritten as

$$\nu([a_1, b_1] \subseteq [a_2, b_2]) = 0.5\left(|b_1 - b_2| + |a_1 - a_2| + (b_1 - b_2) - (a_1 - a_2)\right).$$

Lemma 2. *Let $d_I : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ be defined by*

$$d_I([a_1, b_1], [a_2, b_2]) = \nu([a_1, b_1] \subseteq [a_2, b_2]) + \nu([a_2, b_2] \subseteq [a_1, b_1]).$$

Then d_I is a metric on $\mathcal{A}$, and

$$d_I([a_1, b_1], [a_2, b_2]) = |b_1 - b_2| + |a_1 - a_2|.$$

Proposition 1. *Let $d : \mathcal{BS} \times \mathcal{BS} \to \mathbb{R}$ be defined by*

$$d(BS_1, BS_2) = \nu(BS_1 \subseteq BS_2) + \nu(BS_2 \subseteq BS_1),$$

Then d is a metric on $\mathcal{BS}$, and its values for any $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_2 = (\mathfrak{M}_2, m_2)$ can be found as a solution of the linear programming problem:

$$d(BS_1, BS_2) = \sum_{A \in \mathfrak{M}_1} \sum_{B \in \mathfrak{M}_2} m(A, B) d_I(A, B) \to \min, \tag{3}$$

where the set function m obeys conditions (1), (2), and (3) from Definition 1.

6 Measuring Uncertainty Based on the Inclusion Index

In this section we propose functionals for measuring contradiction between interval belief structures based on the considered inclusion index. These functionals reflect three types of possible interpretations of contradiction in decision problems. We will introduce these functionals and illustrate their properties for the case when the considered belief structures can be identified with segments. These functionals are

(1) $Con^{(1)}(BS_1, BS_2) = 0.5\left(\nu(BS_1 \subseteq BS_2) + \nu(BS_2 \subseteq BS_1)\right)$;
(2) $Con^{(2)}(BS_1, BS_2) = \max\left\{\nu(BS_1 \subseteq BS_2), \nu(BS_2 \subseteq BS_1)\right\}$;
(3) $Con^{(3)}(BS_1, BS_2) = \min\left\{\nu(BS_1 \subseteq BS_2), \nu(BS_2 \subseteq BS_1)\right\}$;
(4) $Con^{(4)}(BS_1, BS_2) = 2 \inf_{BS \in \mathcal{BS}} \max\left\{\nu(BS \subseteq BS_1), \nu(BS \subseteq BS_2)\right\}$.

Noticing that $\nu(BS_1 \subseteq BS_2) = 0.5d(BS_1, BS_2) - 0.5(\nu_I(BS_2) - \nu_I(BS_1))$ we can express $Con^{(k)}(BS_1, BS_2)$, $k = 1, 2, 3$ through the metric d on $\mathcal{BS}$ as

$$\begin{aligned} Con^{(1)}(BS_1, BS_2) &= 0.5d(BS_1, BS_2), \\ Con^{(2)}(BS_1, BS_2) &= 0.5d(BS_1, BS_2) + 0.5\left|\nu_I(BS_1) - \nu_I(BS_2)\right|, \\ Con^{(3)}(BS_1, BS_2) &= 0.5d(BS_1, BS_2) - 0.5\left|\nu_I(BS_1) - \nu_I(BS_2)\right|. \end{aligned}$$

The computation of $Con^{(4)}(BS_1, BS_2)$ can be simplified by using the following lemma.

Lemma 3. $Con^{(4)}(BS_1, BS_2) = 2 \inf_{BS \in \mathcal{BBS}} \max\{\nu(BS \subseteq BS_1), \nu(BS \subseteq BS_2)\}$.

The next result shows how introduced functionals recognize types of contradiction.

Proposition 2. *The following statements are true:*

(1) $Con^{(1)}$ *and* $Con^{(2)}$ *are metrics on* $\mathcal{BS}$*, in particular,* $Con^{(i)}(BS_1, BS_2) = 0$ *for* $BS_1, BS_2 \in \mathcal{BS}$*,* $i = 1, 2$*, iff* $BS_1 = BS_2$*;*
(2) $Con^{(3)}(BS_1, BS_2) = 0$ *for* $BS_1, BS_2 \in \mathcal{BS}$ *iff* $BS_1 \subseteq BS_2$ *or* $BS_2 \subseteq BS_1$*;*
(3) $Con^{(4)}(BS_1, BS_2) = 0$ *for* $BS_1, BS_2 \in \mathcal{BS}$ *iff there is a* $BS \in \mathcal{BS}$ *such that* $BS \subseteq BS_1$ *and* $BS \subseteq BS_2$*.*

Remark 2. Notice that bbas m_1 and m_2 in (1) can be viewed as probability distributions on $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively. Therefore, d is the Wasserstein metric [14] on $\mathcal{BS}$. The metric (1) has many names in the literature [9] like Kantorovich-Rubinshtein metric, Wasserstein metric, Mallows metric, Earth Mover's Distance, but it has been firstly introduced in [10]. Metrics or distances play an important role in the theory of belief functions (see the detail review in [2]).

7 Measuring Conflict Within an Interval Belief Structure

We will propose here the general approach for defining conflict measures based on contradiction measures. We should define first the set of all possible interval belief structures without conflict. For example, assume that we have the following possible choices for sets of interval belief structures without conflict:

(a) $\mathcal{BS}^{(1)}$ consists of all possible belief structures $BS = (\mathfrak{M}, m)$ in $\mathcal{BS}$ whose $\mathfrak{M} = \{[a, b]\}$;
(b) $\mathcal{BS}^{(2)}$ consists of all possible belief structures $BS = (\mathfrak{M}, m)$ in $\mathcal{BS}$ whose $\mathfrak{M}$ is such that $[a_1, b_1] \subseteq [a_2, b_2]$ or $[a_2, b_2] \subseteq [a_1, b_1]$ for every $[a_1, b_1], [a_2, b_2] \in \mathfrak{M}$;
(c) $\mathcal{BS}^{(3)}$ consists of all possible belief structures $B = (\mathfrak{M}, m)$ in $\mathcal{BS}$ whose $\mathfrak{M}$ is such that $\bigcap_{[a_i, b_i] \in \mathfrak{M}} [a_i, b_i] \neq \emptyset$.

Then we can use the metric d^* on $\mathcal{BS}$ ($Con^{(1)}$ or $Con^{(2)}$) to find the closest interval belief structure in $\mathcal{BS}^{(k)}$, $k = 1, 2, 3$, to a chosen $BS \in \mathcal{BS}$, and the distance gives us the evaluation of conflict in BS. Thus, it is possible to define indices of conflict by

$$\nu_C^{(k)}(BS) = \inf\left\{d^*(BS, BS')|BS' \in \mathcal{BS}^{(k)}\right\}. \tag{4}$$

Obviously, the optimization problem (4) depends on the choice of the set $\mathcal{BS}^{(k)}$ and the metric d^*. This explains that there are many possible ways to solve it. In this paper we don't try to analyze it in detail, but give some hints how these problems can be solved.

(a) if we compute $\nu_C^{(1)}(BS)$ and $BS = (\mathfrak{M}, m)$, then a good approximation for $BS' \in \mathcal{BS}^{(1)}$ that gives the infinum in (4) is $[a^*, b^*]$, in which $a^* = \sum\limits_{[a_i,b_i]\in\mathfrak{M}} a_i m([a_i, b_i])$ and $b^* = \sum\limits_{[a_i,b_i]\in\mathfrak{M}} b_i m([a_i, b_i])$. Then it is possible to apply any search method for finding two optimal parameters.

(b) if we compute $\nu_C^{(2)}(BS)$, then the set $\mathcal{BS}^{(k)}$ consists of so called consonant belief structures. In this case the corresponding belief (plausibility) function is called the necessity (possibility) measure. The problem of finding the optimal approximation of belief structures by consonant belief structures has been intensively analyzed in the literature, so we can try to use known approaches [3,7] for solving (4).

(c) In this case the optimization problem can be simplified as follows:

$$Q(BS) = \inf\left\{\nu(BS^* \subseteq BS)|BS^* \in \mathcal{BS}^{(4)}\right\},$$

where $\mathcal{BS}^{(4)}$ consists of all possible belief structures $BS^* = (\{[x, x]\}, m^*)$, and $\nu_C^{(3)}(BS) = 0.5\, Q(BS)$ for $d^* = Con^{(1)}$, and $\nu_C^{(3)}(BS) = Q(BS)$ if we take $d^* = Con^{(2)}$.

8 Testing Contradictory Measures on Actual Weather Forecasts

Let us notice first that every weather forecast in our data set is precise, i.e. the left point a_i of the segment $A_i = [a_i, b_i]$ should give us the lowest night temperature, and the point b_i should give us the highest day temperature. This implies that in computations we can use contradiction measures $Con^{(1)}$ and $Con^{(2)}$, i.e. forecasts A_i and A_j are not contradictory iff $A_i = A_j$. Synopticians say that the forecast $A_i = [a_i, b_i]$ for the target day $A_0 = [a_0, b_0]$ is accurate if predicted temperatures differ from actual temperatures not greater than three degrees. Thus, the forecast A_i is accurate if $Con^{(1)}(A_i, A_0) \leqslant 3$, i.e. we can evaluate the accuracy of forecasts by contradiction measures $Con^{(1)}$ and $Con^{(2)}$. Additionally, we have prior information that the forecast A_1 is accurate approximately in 95% of observations, forecasts A_2 and A_3 are accurate approximately in 90% of observations, and forecasts $A_4, ..., A_7$ are accurate approximately in 80% of observations. This information should be also taken into consideration for aggregating forecasts in belief structures. One way consists in the following. At first consider the case when we measure contradiction between the actual temperature described by A_0 and forecasts $A_1, ..., A_7$, assume in addition that our confidence in each forecast is the same, i.e. each forecast is accurate with the same probability. Then we can aggregate them to a belief structure $BS_1 = (\mathfrak{M}_1, m_1)$, where $\mathfrak{M}_1 = \{A_1, ..., A_7\}$ and $m_1(A_i) = 1/7$, $i = 1, ..., 7$, and we can measure the contradiction between $BS_1 = (\mathfrak{M}_1, m_1)$ and $BS_0 = (\mathfrak{M}_0, m_0)$, where $\mathfrak{M}_0 = \{A_0\}$ and $m_0(A_0) = 1$, using

$$Con^{(1)}(BS_0, BS_1) = 0.5 \sum_{i=1}^{7} m_1(A_i) d_I(A_0, A_1).$$

Consider the general case, when our confidence in forecasts A_i are described by probabilities p_i, $i = 1, ..., 7$, and assume that $1 > p_1 \geqslant p_2 \geqslant ... \geqslant p_7$. In this case consider auxiliary forecasts A'_i consisting of two segments A_i and A_0 in which the forecasts A_i and A_0 are chosen randomly: A_i with probability α_i and A_0 with probability $(1 - \alpha_i)$. Such a forecast can be modeled by a belief structure $(\{A_i, A_0\}, m^{(i)})$, where $m^{(i)}(A_i) = \alpha_i$ and $m^{(i)}(A_0) = 1 - \alpha_i$. Then the probability of accuracy for this forecast is $\alpha_i p_i + (1 - \alpha_i)$. It is possible to choose α_i such that $\alpha_i p_i + (1 - \alpha_i) = q$, for example, if $q = p_1$, then $\alpha_i = (1 - p_1)/(1 - p_i)$, $i = 1, ..., 7$. We can aggregate such auxiliary forecasts by a belief structure $BS = (\mathfrak{M}, m)$, where $\mathfrak{M} = \{A_0, A_1, ..., A_7)$ and $m(A) = \frac{1}{7} \sum_{i=1}^{7} m^{(i)}(A)$, in which each forecast A'_i has the same confidence. Let us observe that BS can be represented as a convex sum

$$BS = \alpha BS_1 + (1 - \alpha) BS_0,$$

in which $\alpha = \frac{1}{7} \sum_{i=1}^{7} \alpha_i$, BS_0 is the belief structure described above, and $BS_1 = (\mathfrak{M}_1, m_1)$ is the belief structure with the body of evidence $\mathfrak{M}_1 = \{A_1, ..., A_7\}$ and

$$m_1(A_i) = \frac{1}{1 - p_i} \Bigg/ \sum_{k=1}^{7} \frac{1}{1 - p_i}.$$

Obviously, BS_1 can be considered as the aggregation of forecasts $A_1, ..., A_7$ for general case, when probabilities p_i can be different. Assume that $p_1 = 0.95$, $p_2 = p_3 = 0.9$, and $p_i = 0.8$, $i = 4, ..., 7$. Then the obtained formula for m_1 gives us values:

$$m_1(A_1) = 1/3, \;\; m_1(A_2) = m_1(A_3) = 1/6, \;\; m_1(A_i) = 1/12, \;\; i = 4, ..., 7. \quad (5)$$

When we analyze our statistical data, we observe the following regularity. If we have the high conflict between forecasts made during the week before the target day, then the last forecast was very often not accurate. We have checked this hypothesis using the following functionals:

1. Contradiction between the last forecast and previous forecasts. In this case we compute contradiction measures $\xi_{2,3}^{(k)} = Con^{(k)}(BS_2, BS_3)$, where $k = 1, 2$, and the belief structure $BS_2 = (\mathfrak{M}_2, m_2)$ aggregates forecasts $A_2, ..., A_7$, and the belief structure BS_3 describes the last forecast. Then $\mathfrak{M}_2 = \{A_2, ..., A_7\}$ and

$$m_2(A_2) = m_2(A_3) = 1/4, m_2(A_i) = 1/8, i = 4, ..., 7.$$

2. Contradiction among forecasts $A_1, ..., A_7$. In this case we describe these forecasts by the belief structure $BS_1 = (\mathfrak{M}_1, m_1)$, where $\mathfrak{M}_1 = \{A_1, ..., A_7\}$ and m_1 is defined by (4). After that we compute the functional $\xi_1^{(k)} = \nu_C^{(1)}(BS_1)$, where k shows the choice of $d^* = Con^{(k)}$, $k = 1, 2$.
3. We measure the contradiction $\xi_{0,3}^{(k)} = Con^{(k)}(BS_0, BS_3)$, $k = 1, 2$, between the actual temperature and the forecast made one day before the target day.
4. We estimate correlation coefficients between characteristics, described in items 1–2, and accuracy of the last forecast defined by measures $Con^{(k)}(BS_0, BS_2)$, $k = 1, 2$. These correlation coefficients are given in Table 2. We see that all estimated correlation coefficients are positive; this means that higher contradiction among forecasts implies lower probability of accuracy for the last forecast. This dependence is detected more explicitly by the functional $\xi_1^{(2)}$, that gives the greatest correlation coefficient.

Table 2. Correlation coefficients

	$\xi_{2,3}^{(1)}$	$\xi_{2,3}^{(2)}$	$\xi_1^{(1)}$	$\xi_1^{(2)}$
$\xi_{0,3}^{(1)}$	0,462	0,506	0,526	0,613
$\xi_{0,3}^{(2)}$	0,373	0,443	0,464	0,700

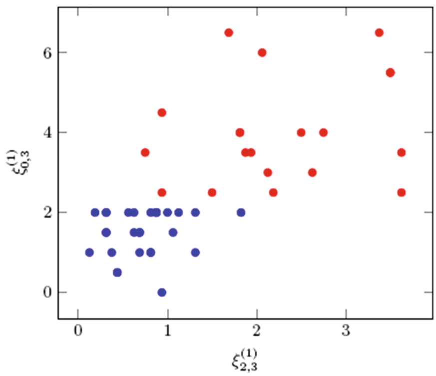

Fig. 1. The joint distribution of $\xi_{0,3}^{(1)}$ and $\xi_{2,3}^{(1)}$

Let us notice that it is possible to use these functionals in order to detect if the probability that the forecasts will be accurate is high. As shown in Fig. 1, even the plot built not on the optimal functional $\xi_{2,3}^{(1)}$ represents evidentially the dependence of contradiction between the last forecast and the actual weather on the contradiction between the last forecast and previous forecasts.

9 Conclusion

Finitely defined interval belief structures potentially have many applications for interval data processing. We have shown how they can be used for analyzing weather forecasts. For this purpose, it can be very helpful to apply these functionals for measuring conflict, contradiction with the underlying inclusion indices and metrics introduced in the paper.

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Generalized Net Model of Fingerprint Recognition with Intuitionistic Fuzzy Evaluations

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Abstract. In the paper, a method for evaluation of fingerprint equivalence obtained in a fingerprint recognition system is proposed. For the assessment of the equivalence of the respective assessment units, the theory of intuitionistic fuzzy sets is used. The obtained intuitionistic fuzzy estimations reflect on the recognition of the system. We also consider a degree of uncertainty when the information is not enough. In this case we use threshold values for the minimum and maximum of the degree of membership and non-membership. For the description of the entire process, we use generalized nets model.

Keywords: Intuitionistic fuzzy sets · Fingerprints · Fingerprint system · Generalized nets

1 Introduction

The most popular biometric method is based on comparing fingerprints, using two wide spread techniques. The first takes and compares the details of lines crests, where they diverge and interrupt. The second technique measures and compares the directions of the lines of the fingerprint, e.g. changes, crests and arches, their width and depth and the gnarls also [13, 18]. Some difficulties are possible to occur in automatic recognition, when the fingers are dirty, wet or oil covered. Besides, the fingerprint depends on the specific usage of the hand in some professions, such as building, where wounding, abrasion or some other similar changes of the hand's skin are frequently occurred. Also when the skin is getting older some changes are possible in the lines of the fingerprint. However, this is fixable if periodically the data base is updated, which is important for comparing identification. For the recognition to be sure, the necessary data base, which the comparison will be made with, needs to be of very high quality.

Intuitionistic fuzzy sets defined by Atanassov [2, 4] represent an extension of the concept of fuzzy sets, exhibiting function $\mu_A(x)$ defining the membership of an element x to set A, evaluated in the interval [0; 1]. The difference between fuzzy sets and intuitionistic fuzzy sets (IFSs) is in the presence of a second function $\nu_A(x)$ defining the non-membership of element x to set A, where $\mu_A(x) \in [0; 1]$, $\nu_A(x) \in [0; 1]$, under the condition of $\mu_A(x) + \nu_A(x) \in [0; 1]$. The IFS itself is formally denoted by:

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$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}.$$

Comparison between elements of any two IFSs, say A and B, involves pairwise comparisons between their respective elements' degrees of membership and non-membership to both sets.

Generalized nets (GNs) [1, 3, 5] are defined in a way that is principally different from the ways of defining the other types of Petri nets. The first basic difference between GNs and ordinary Petri nets is the "place – transition" relation. Here the transitions are objects of a more complex nature. A transition may contain m input places and n output places where $m, n \geq 1$.

Formally, every transition is described by a seven-tuple:

$$z = <L', L'', t_1, t_2, r, M, \square>,$$

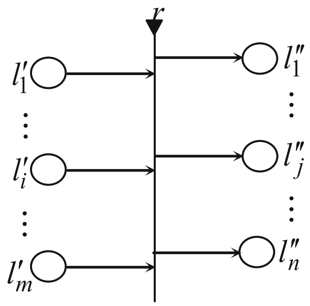

Fig. 1. A GN-transition

where

(a) L' and L'' are finite, non-empty sets of places (the transition's input and output places, respectively). For the transition in Fig. 1 these are

$$L' = \{l'_1, l'_2, \ldots, l'_m\}$$

and

$$L'' = \{l''_1, l''_2, \ldots, l''_n\};$$

(b) t_1 is the current time-moment of the transition's firing;
(c) t_2 is the current value of the duration of its active state;
(d) r is the *condition* of the transition to determine which tokens will pass (or *transfer)* from the inputs to the outputs of the transition; it has the form of an Index Matrix:

$$
r = \begin{array}{c|ccccc}
 & l''_1 & \cdots & l''_j & \cdots & l''_n \\
\hline
l'_1 & & & & & \\
\vdots & & & r_{i,j} & & \\
l'_i & & & (r_{i,j} - \text{predicate}) & & \\
\vdots & & & (1 \le i \le m,\ 1 \le j \le n) & & \\
l'_m & & & & &
\end{array},
$$

$r_{i,j}$ is the predicate that corresponds to the i-th input and j-th output place. When its truth value is "*true*", a token from the i-th input place transfers to the j-th output place; otherwise, this is not possible;

(e) M is an IM of the capacities of transition's arcs:

$$
M = \begin{array}{c|ccccc}
 & l''_1 & \cdots & l''_j & \cdots & l''_n \\
\hline
l'_1 & & & & & \\
\vdots & & & m_{i,j} & & \\
 & & & (m_{i,j} \ge 0 - \text{natural number}) & & \\
l'_i & & & (1 \le i \le m,\ 1 \le j \le n) & & \\
\vdots & & & & & \\
l'_m & & & & &
\end{array};
$$

(f) $\square$ is an object of a form similar to a Boolean expression. It may contain as variables the symbols that serve as labels for a transition's input places, and $\square$ is an expression built up from variables and the Boolean connectives $\wedge$ and $\vee$ and the semantics of which is defined as follows:

$\wedge(l_{i_1}, l_{i_2}, \ldots, l_{i_u})$ - every place $(l_{i_1}, l_{i_2}, \ldots, l_{i_u})$ must contain at least one token, $\vee(l_{i_1}, l_{i_2}, \ldots, l_{i_u})$ - there must be at least one token in all places $(l_{i_1}, l_{i_2}, \ldots, l_{i_u})$, where $\{l_{i_1}, l_{i_2}, \ldots, l_{i_u}\} \subset L'$.

When the value of a type (calculated as a Boolean expression) is "*true*", the transition can become active, otherwise it cannot.

In this paper we propose a method for evaluation of the fingerprint equivalence of two fingerprints – one saved in the system and one scanned from a fingerprint recognition system. For the assessment of the equivalence of the respective assessment units we use the theory of intuitionistic fuzzy sets. The obtained intuitionistic fuzzy estimations can affect or not the recognition in the system. We also consider a degree of uncertainty when the information is not enough. In this case we use threshold values

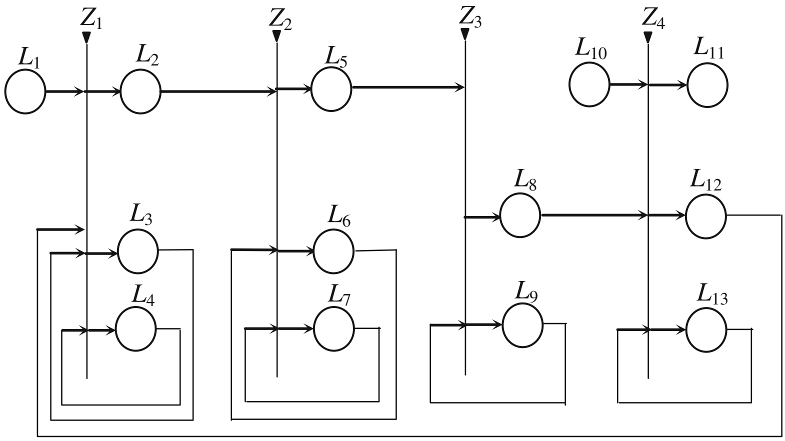

Fig. 2. Generalized net of fingerprints recognition with intuitionistic fuzzy estimations

for the minimum and maximum of the degree of membership and non-membership. For the describtion of the entire process we use generalized nets model (Fig. 2).

2 Generalized Nets Model

Many data mining, decision making and pattern recognition tools are modeled with generalized nets [6–12, 14–17, 19, 20]. There are many papers describing models of different kinds of data mining process.

A Generalized Net includes the set of transitions:

$$A = \{Z_1,\ Z_2,\ Z_3,\ Z_4,\ Z_5\},$$

where the transitions describe the following processes:

Z_1 – Check if the finger is in right position;
Z_2 – Check if the finger has right rotation;
Z_3 – Correction of intensity;
Z_4 – Fingerprint intuitionistic fuzzy estimation.

A token enters the net from place L_1 with initial characteristic: "*new scanned fingerprint*".

Transition Z_1 has the following form:

$$Z_1 = \langle \{L_1, L_3, L_4, L_{12}\}, \{L_2, L_3, L_4\}, R_1, \vee(L_1, L_3, L_4, L_{12}) \rangle,$$

where

$$R_1 = \begin{array}{c|ccc} & L_2 & L_3 & L_4 \\ \hline L_1 & false & false & true \\ L_3 & W_{3,2} & W_{3,3} & false \\ L_4 & false & W_{4,3} & W_{4,4} \\ L_{12} & false & false & true \end{array}$$

$W_{3,2}$ = "The fingerprint has a correct vertical positioning";
$W_{3,3} = \neg W_{3,2}$;
$W_{4,3}$ = "The fingerprint has a correct horizontal positioning";
$W_{4,4} = \neg W_{4,3}$.

The tokens that enters the place L_4 from place L_1 do not obtain new characteristics. They generate a new tokens that enter in place L_3 with characteristics:

"*fingerprint with a correct vertical positioning*".

At the second activation of the transition the tokens from place L_3 generate new tokens that enter in place L_2

"*fingerprint with a correct horizontal positioning*".

Transition Z_2 has the following form:

$$Z_2 = \langle \{L_2, L_6, L_7\}, \{L_5, L_6, L_7\}, R_2, \vee(L_2, L_6, L_7) \rangle,$$

where

$$R_2 = \begin{array}{c|ccc} & L_5 & L_6 & L_7 \\ \hline L_2 & false & false & true \\ L_6 & W_{6,5} & W_{6,6} & false \\ L_7 & false & W_{7,6} & W_{7,7} \end{array}$$

and:

$W_{6,5}$ = "The fingerprint is not turned counter clockwise";
$W_{6,6} = \neg W_{6,5}$;
$W_{7,6}$ = "The fingerprint is not turned clockwise";
$W_{7,7} = \neg W_{7,6}$.

The tokens that enter places L_7 from place L_2 do not obtain new characteristics. They generate a new tokens that enter in place L_6 with the characteristics:

"*fingerprint not turned counter clockwise*".

At the second activation of the transition the tokens from place L_6 generate new tokens that enter in place L_5

"*fingerprint not turned clockwise*".

Transition Z_3 has the following form:

$$Z_3 = \langle \{L_5, L_9\},\ \{L_8, L_9\}, R_3, \vee(L_5, L_9) \rangle$$

where

$$R_3 = \begin{array}{c|cc} & L_8 & L_9 \\ \hline L_5 & false & true \\ L_9 & W_{9,8} & W_{9,9} \end{array}$$

and:

$W_{9,8}$ = "The intensity of the images is standardized";
$W_{9,9}$ = ¬ $W_{9,8}$.

The tokens that enter the places L_9 (from place L_5) do not obtain new characteristic. The tokens that enter in places L_8 obtain the characteristic:

"*Standardized intensity of the images*".

Tokens enters the net from place L_{10} with initial characteristics:

" *fingerprints from the database and minimal threshold of equivalence*".

Transition Z_4 has the following form:

$$Z_4 = \langle \{L_8, L_{10}, L_{13}\}, \{L_{11}, L_{12}, L_{13}\}, R_4, \vee(\wedge(L_8, L_{10}), L_{13}) \rangle,$$

where

$$R_4 = \begin{array}{c|ccc} & L_{11} & L_{12} & L_{13} \\ \hline L_8 & false & false & true \\ L_{10} & false & false & true \\ L_{13} & W_{13,11} & W_{13,12} & W_{13,13} \end{array}$$

and:

$W_{13,11}$ = "Recognized fingerprint".
$W_{13,12}$ = "Not enough fingerprint matches"
$W_{13,13}$ = ¬ $W_{13,11}$

The tokens that enter place L_{13} (from places L_8, L_{10}) do not obtain new characteristics. They generate a new tokens that enter in places L_{11} and L_{12}, with the characteristics:

"*Recognized fingerprint*" in place L_{11}

and "*Not enough fingerprint matches*" in place L_{12}.

Let us examine a monochrome image. This image has size *m*x*n*. The pixels have values from 0 to 255. The evaluation is of several images by calculating their intuitionistic fuzzy estimations. Let us have the following sets for an image:

- total number of the pixels - e;
- number of assigned values for all pixels in the image - s;
- number of common values of the pixels for the images - n;
- number of pixels having the value greater than the average value for the image - m;
- number of pixels having the value smaller than the average value for the image - f.

Initially we calculate average value for an image dividing the sum of all the pixels in the image to the number of the pixels in the image.

$$S_{avg} = \frac{s}{e}.$$

Then, we can find the degree of membership of the image dividing the sum of the common pixels for the two images to the sum of all the pixels in the image.

$$\mu_{image} = \frac{n}{e}.$$

In the next step we compare the last pixels in the image with the average value for the image.

In the case when $s - n > S_{avg}$, we obtain the degree of non-membership having the following form:

$$v_{image} = \frac{m}{e},$$

and in the case when $s - n < S_{avg}$, we obtain the uncertainty:

$$\pi_{image} = \frac{f}{e}$$

The calculated final part based on all assessment units for fingerprints has to satisfy the necessary "minimal threshold of equivalence". To check this, we introduce threshold values: M_{max}, M_{min}, N_{max}, N_{min}.

If

$$\mu_{\text{image}} > M_{\max} \ \& \ V_{\text{image}} < N_{\min},$$

then the fingerprints satisfy the "minimal threshold of equivalence" for the current estimation and $W_{13,11}$ = "Recognized fingerprint".

If

$$\mu_{\text{image}} > M_{\min} \ \& \ V_{\text{image}} < N_{\max},$$

then the fingerprints do not satisfy the "minimal threshold of equivalence" for the current estimation and it has to be evaluated for all assessment units again and $W_{13,13} = \neg W_{13,11}$.

In the rest of the cases the "minimal threshold of equivalence" is undefined and fingerprints have to be evaluated again for the assessment units for which: $\mu_{image} \leq M_{max}$ & $V_{image} \geq N_{min}$, is valid and $W_{13,12}$ = "Not enough fingerprint matches".

3 Conclusion

This article elaborates on the main stages of fingerprint recognition. Solving the problems of such a system requires a total analysis to be done. The model gives the opportunity to consider the different stages of fingerprint identification. The fingerprint is one of the most unique parts of the human body and that is why it is used in devices for identification. For the purpose, we use assessment with intuitionistic fuzzy sets. The obtained intuitionistic fuzzy estimations reflect on the recognition of the system. The obtained intuitionistic fuzzy estimations can affect or not the recognition in the system. We also consider a degree of uncertainty when the information is not enough. In this case we use threshold values for the minimum and maximum of the degree of membership and non-membership. For the description of the entire process, we use generalized nets model.

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A New Extension of Monotonicity: Ordered Directional Monotonicity

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Abstract. In this work, we discuss a recent generalization of the classical notion of monotonicity, with a special focus on the idea of directional monotonicity. This idea leads to the concepts of pre-aggregation functions and of ordered directional monotonicity. For the former, the direction along which monotonicity is considered is the same for all the points of the domain and the same boundary conditions as for aggregation functions are imposed. For the latter, different directions of monotonicity may be considered at different points.

Keywords: Aggregation function · Pre-aggregation function · Directional monotonicity · Ordered-directional monotonicity

1 Introduction

Aggregation functions [1,13] are increasingly relevant nowadays to deal with a type of problems where some kind of information fusion is required [4–7,10,14]. But some operators which are used for this type of problems do not fall into the scope of aggregation function due to the lack of monotonicity. This is specially the case of the mode. These operators have caused a growing interest on defining generalized forms of monotonicity which, while covering the usual case, may provide a general framework to include other functions. A first step in this sense was given in [16], where the notion of weak monotonicity was discussed. Weak monotonicity requires increasingness (or decreasingness) only along the fixed ray defined by the first quadrant diagonal. The consideration of increasingness (decreasingness) along general rays was proposed by Bustince et al. [3] and corresponds to the idea of directional monotonicity. Note that monotone functions in the usual sense are both weakly monotone and directionally monotone.

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However, these extensions may not be sufficient for some applications. In some cases, the problem lays on the fact that the direction along which monotonicity is needed may vary from point to point, depending on the specific inputs to be fused. This is specially the case in some edge detection problems in image processing [2, 11]. But it is also the situation in those problems where extensions of operators such as the OWA should be considered in order to deal with generalized forms of monotonicity.

For this reason, and following the developments in [2], in this work we present the concept of ordered directionally monotone functions, as functions such that the direction along which monotinicity is considered at each point depends on the relative size of the inputs at that point. In this way, at different points monotonicity along a different direction may be considered. Such a possibility has led to interesting applications in image processing problems [2], which we not discuss here due to the lack of space.

The structure of this work is the following: in Sect. 2 we present some preliminary definitions and concepts. Section 3 is devoted to the notion of ordered directionally monotone functions. In Sect. 4 we present some construction methods of such functions. We finish with some conclusions and references.

2 Preliminaries

Let $n \in \mathbb{N}, n > 1$. Bold letters will be used for points in $[0,1]^n$, that is, $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$. By abuse of notation, we denote $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$. We also consider the partial order on $\mathbb{R}^n$, defined as: given $\mathbf{x}, \mathbf{y} \in [0,1]^n$, we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for every $i \in \{1, \ldots, n\}$.

The symbol $(\cdot)$ denotes n-dimensional vectors in the Euclidean space $\mathbb{R}^n$. We denote by S_n the set of permutations of $\{1, \ldots, n\}$.

Let $\mathcal{S}_n$ be the set of all permutations of the set $\{1, \ldots, n\}$ (i.e., bijective functions from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$). Given $\sigma \in S_n$, $\mathbf{x} \in [0,1]^n$ and $\mathbf{r} \in \mathbb{R}^n$, we use the notation:

$$\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

and

$$\mathbf{r}_\sigma = (r_{\sigma(1)}, \ldots, r_{\sigma(n)}).$$

Definition 1 ([3])**.** *A (n-dimensional) fusion function is any function* $F\colon [0,1]^n \to [0,1]$.

An outstanding class of fusion functions is that of aggregation functions [1, 9, 13].

Definition 2. *An aggregation function is a fusion function* $A\colon [0,1]^n \to [0,1]$ *such that*

(i) A *is increasing; that is,* $A(\mathbf{x}) \leq A(\mathbf{y})$ *for every* $\mathbf{x}, \mathbf{y} \in [0,1]^n$ *such that* $\mathbf{x} \leq \mathbf{y}$*;*
(ii) $A(\mathbf{0}) = 0$*;*
(iii) $A(\mathbf{1}) = 1$.

Choquet integrals [12] will be of special interest for us. As a first step to present them, we recall here the definition of fuzzy measure.

Definition 3. *Let $N = \{1, 2, \ldots, n\}$. A function $\mathfrak{m} : 2^N \to [0, 1]$ is a fuzzy measure if it satisfies the following properties:*

($\mathfrak{m}$1) *Increasingness: for all $X, Y \subseteq N$, if $X \subseteq Y$, then $\mathfrak{m}(X) \leq \mathfrak{m}(Y)$;*
($\mathfrak{m}$2) *Boundary conditions: $\mathfrak{m}(\emptyset) = 0$ and $\mathfrak{m}(N) = 1$.*

The Choquet integral is a generalization of the Lebesgue integral, where additive measures are replaced by fuzzy measures.

Definition 4 ([1,13])**.** *Let $\mathfrak{m}\colon 2^N \to [0, 1]$ be a fuzzy measure. The discrete Choquet integral with respect to $\mathfrak{m}$ is defined as the function $C_{\mathfrak{m}}\colon [0, 1]^n \to [0, 1]$, given, for all $\mathbf{x} = (x_1, \ldots, x_n) \in [0, 1]^n$, by*

$$C_{\mathfrak{m}}(\mathbf{x}) = \sum_{i=1}^{n} \left(x_{(i)} - x_{(i-1)}\right) \cdot \mathfrak{m}\left(A_{(i)}\right), \tag{1}$$

where $\left(x_{(1)}, \ldots, x_{(n)}\right)$ is an increasing permutation of the input n-tuple $\mathbf{x}$, that is, $x_{(1)} \leq \ldots \leq x_{(n)}$, with the convention that $x_{(0)} = 0$, and $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices of $n - i + 1$ largest components of $\mathbf{x}$.

Finally we also recall here several weaker forms of monotonicity, which are at the origin of the present work.

Definition 5 ([16])**.** *A function $F\colon [0, 1]^n \to [0, 1]$ is said to be weakly monotone increasing if the inequality*

$$F(x_1 + h, \ldots, x_n + h) \geq F(x_1, \ldots, x_n) \tag{2}$$

holds for every $x_1, \ldots, x_n, h \in [0, 1]$ such that $x_i + h \leq 1$, $i \in \{1, \ldots, n\}$.

Analogously, weakly monotone decreasing functions can be defined.

The concept of weak monotonicity can be further extended if we consider monotonicity along general rays. This idea has led to the notion of directional monotonicity introduced in [3].

Definition 6 ([3])**.** *Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a real n-dimensional vector, $\mathbf{r} \neq \mathbf{0}$. A fusion function $F\colon [0, 1]^n \to [0, 1]$ is $\mathbf{r}$-increasing if for all points $(x_1, \ldots, x_n) \in [0, 1]^n$ and for all $c > 0$ such that $(x_1 + cr_1, \ldots, x_n + cr_n) \in [0, 1]^n$ we have*

$$F(x_1 + cr_1, \ldots, x_n + cr_n) \geq F(x_1, \ldots, x_n).$$

The notion of **r**-decreasing fusion function is defined analogously, reversing the previous inequality.

Directional monotonicity has lead to considering the idea of pre-aggregation function as a generalization of usual aggregation functions [15].

Definition 7. *A function $F : [0,1]^n \to [0,1]$ is said to be an n-ary pre-aggregation function if the following conditions hold:*

(PA1) *There exists a real vector $\mathbf{r} \in [0,1]^n$ ($\mathbf{r} \neq \mathbf{0}$) such that F is $\mathbf{r}$- increasing.*
(PA2) *F satisfies the boundary conditions: $F(0,\ldots,0) = 0$ and $F(1,\ldots,1) = 1$.*

Example 1. Some examples of pre-aggregation functions are the following.

(i) Consider the mode, $Mod(x_1,\ldots,x_n)$, defined as the function that gives back the value which appears most times in the considered n-tuple, or the smallest of the values that appears most times, in case there are more than one. Then, the mode is $(1,\ldots,1)$-increasing, and it is a particular case of pre-aggregation function. Note that if $\mathbf{r}$ is not a constant vector then the mode is not $\mathbf{r}$-monotone.
(ii) $F(x,y) = x - (\max\{0, x-y\})^2$ is, for instance, $(0,1)$-increasing, and then it is an example of pre-aggregation function. It is also $(1,1)$-increasing but not $(1,0)$-monotone and thus not an aggregation function.

Pre-aggregation function can be built in different ways. In particular, we recall here the following two methods [3].

Proposition 1. *Let $A : [0,1]^m \to [0,1]$ be an aggregation function. Let $F_i : [0,1]^n \to [0,1]$ ($i \in \{1,\ldots,m\}$) be a family of m $\mathbf{r}$-pre-aggregation functions for the same vector $\mathbf{r} \in [0,1]^n$. Then, the function $A(F_1,\ldots,F_m) : [0,1]^n \to [0,1]$, defined as*

$$A(F_1,\ldots,F_m)(x_1,\ldots,x_n) = A(F_1(x_1,\ldots,x_n),\ldots,F_m(x_1,\ldots,x_n))$$

is also an $\mathbf{r}$-pre-aggregation function.

In order to present the second construction method, let $\mathfrak{m} : 2^N \to [0,1]$ be a fuzzy measure and $M : [0,1]^2 \to [0,1]$ be a function such that $M(0,x) = 0$ for every $x \in [0,1]$. Taking as basis the Choquet integral, we define the function $C_{\mathfrak{m}}^M : [0,1]^n \to [0,n]$ by

$$C_{\mathfrak{m}}^M(\mathbf{x}) = \sum_{i=1}^{n} M\left(x_{(i)} - x_{(i-1)}, \mathfrak{m}\left(A_{(i)}\right)\right), \qquad (3)$$

where $N = \{1,\ldots,n\}$, $(x_{(1)},\ldots,x_{(n)})$ is an increasing permutation on the input $\mathbf{x}$, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)}$, with the convention that $x_{(0)} = 0$, and $A_{(i)} = \{(i),\ldots,(n)\}$ is the subset of indices of $n-i+1$ largest components of $\mathbf{x}$. Note that $C_{\mathfrak{m}}^M$ is well defined by (3) even if the permutation is not unique.

We have the following result.

Theorem 1. *Let $M : [0,1]^2 \to [0,1]$ be a function such that for all $x,y \in [0,1]$ it satisfies $M(x,y) \leq x$, $M(x,1) = x$, $M(0,y) = 0$ and M is (1,0)-increasing. Then, for any fuzzy measure $\mathfrak{m}$, $C_{\mathfrak{m}}^M$ is a pre-aggregation function which is idempotent and averaging, i.e., $\min(x_1,\ldots,x_n) \leq C_{\mathfrak{m}}^M(x_1,\ldots,x_n) \leq \max(x_1,\ldots,x_n)$.*

Example 2. Taking into account that a semi-copula is an aggregation function M such that $M(1,x) = M(x,1) = x$ for every $x \in [0,1]$, we have that, if $M : [0,1]^2 \to [0,1]$ is a semi-copula, then, for any measure $\mathfrak{m}$, $C_{\mathfrak{m}}^{M}$ is a pre-aggregation function which is idempotent and averaging.

3 Ordered Directional Monotonicity

In this section, following the developments in [2], we discuss the main notion of this paper: ordered directional monotonicity.

To motivate the introduction of this concept, note that by means of directional monotonicity, usual monotonicity may be relaxed, in order to require increasingness along some fixed ray. However, the direction along which monotonicity is demanded is the same for every point in the domain $[0,1]^n$, and it is independent of which particular point is considered. This can be a problem for some applications such as image processing [2].

For ordered directionally monotone functions, on the contrary, the direction along which monotonicity is required varies depending on the ordinal size of the coordinates of the considered input. The formal definition reads as follows.

Definition 8. *Let $F\colon [0,1]^n \to [0,1]$ be a fusion function and let $\mathbf{r} \neq \mathbf{0}$ be an n-dimensional real vector. F is said to be ordered directionally (OD) $\mathbf{r}$-increasing if for each $\mathbf{x} \in [0,1]^n$, and any permutation $\sigma \in S_n$ with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ and any $c > 0$ such that*

$$1 \geq x_{\sigma(1)} + cr_1 \geq \cdots \geq x_{\sigma(n)} + cr_n \geq 0$$

it holds that

$$F(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) \geq F(\mathbf{x}), \tag{4}$$

where $\mathbf{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$.

Analogously, F is said to be ordered directionally $\mathbf{r}$-decreasing if for each $\mathbf{x} \in [0,1]^n$, and any permutation $\sigma \in S_n$ with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ and any $c > 0$ such that

$$1 \geq x_{\sigma(1)} + cr_1 \geq \cdots \geq x_{\sigma(n)} + cr_n \geq 0$$

it holds that

$$F(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) \leq F(\mathbf{x}). \tag{5}$$

Observe that, in general, the $\mathbf{r}$-directional increasingness (decreasingness) is equivalent to the OD $\mathbf{r}$-increasingness (decreasingness) if and only if $\mathbf{r}_\sigma = \mathbf{r}$ for any permutation $\sigma \in S_n$, i.e., if $\mathbf{r}$ is a constant vector. Obviously, if the considered function F is symmetric, then directional and OD $\mathbf{r}$-increasingness (decreasingness) are equivalent for any vector $\mathbf{r}$.

Proposition 2. *Let $F\colon [0,1]^n \to [0,1]$ be an OD $\mathbf{r}$-monotone function. Then for any increasing (decreasing) function $\varphi\colon [0,1] \to [0,1]$, the function $\varphi \circ F$ is a fusion function and has the same (reversed) type of OD $\mathbf{r}$-monotonicity as F.*

Proof. The claim is a trivial consequence of the definition of the OD $\mathbf{r}$-monotonicity of F and the standard monotonicity of φ. ∎

Example 3.

(i) Let $F : [0,1]^n \to [0,1]$ be a constant fusion function, i.e., $F(x_1, \ldots, x_n) = c$ for some $c \in [0,1]$ and for every $(x_1, \ldots, x_n) \in [0,1]^n$. Then, for every vector $\mathbf{r} \in \mathbb{R}^n$, $\mathbf{r} \neq \mathbf{0}$, F is OD $\mathbf{r}$-increasing and also OD $\mathbf{r}$-decreasing.
(ii) Let $p > 0$. Then the function $G(x,y) = |x-y|^p$ is OD $\mathbf{r}$-increasing if and only if $\mathbf{r} = (r_1, r_2)$, $r_1 \geq r_2$.
(iii) Let $F_{\mathbf{w}} : [0,1]^n \to [0,1]$ be an OWA operator [17] related to the normed weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$. Then F_w is OD $\mathbf{r}$-increasing if and only if $\mathbf{w} \cdot \mathbf{r} = \sum w_i r_i \geq 0$.

Example 4. Let $p > 0$. Consider the function $F : [0,1]^n \to [0,1]$ given by

$$F(\mathbf{x}) = \frac{1}{n} \sum_{j=2}^{n} |x_1 - x_j|^p .$$

Then F is OD $\mathbf{r}$-increasing for every $\mathbf{r}$ of the type $\mathbf{r} = (t, \ldots, t, s)$ with $t \geq s$.

Note that as a consequence of the previous example, it follows that penalty functions [8] of the type

$$P(x_1, \ldots, x_n, y) = \frac{1}{n} \sum |x_i - y|^p$$

are OD $\mathbf{r}$-increasing for every $\mathbf{r}$ of the type $\mathbf{r} = (t, \ldots, t, s)$ with $t \geq s$.

Let us introduce some properties of OD monotone functions. First of all, it comes out that OD-increasingness and OD-decreasingness are closely related, as the following result shows.

Proposition 3. *A function $F \colon [0,1]^n \to [0,1]$ is OD $\mathbf{r}$-increasing if and only if F is OD $-\mathbf{r}$-decreasing.*

Proof. It follows from a straightforward calculation. ∎

The following result is also straightforward.

Proposition 4. *Let $F \colon [0,1]^n \to [0,1]$ be an OD $\mathbf{r}$-increasing function. Then,*

(i) for every $\alpha > 0$, F is OD $\alpha\mathbf{r}$-increasing;
(ii) for every $\alpha < 0$, F is OD $\alpha\mathbf{r}$-decreasing.

That is, vectors determining the direction for OD-increasingness or decreasingness can be normalized.

Now we consider what happens if a fusion function F is OD increasing with respect to two different vectors $\mathbf{r}$ and $\mathbf{s}$. We show that OD increasingness of F with respect to a non-negative linear combination of both vectors can be considered under appropriate conditions.

Theorem 2. *Let* $\mathbf{r}$ *and* $\mathbf{s}$ *be* n*-dimensional non-null vectors. Consider any* $a, b \geq 0$ *with* $a + b > 0$*, such that for each point* $\mathbf{x} \in [0,1]^n$*, and for any permutation* $\sigma \in S_n$ *with* $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ *and* $c > 0$ *such that* $\mathbf{x}_\sigma + c\mathbf{u}$ *is in* $[0,1]^n$ *and comonotone with* $\mathbf{x}_\sigma$*, where* $\mathbf{u} = a\mathbf{r} + b\mathbf{s}$*, it holds that the points* $\mathbf{x}_\sigma + ca\mathbf{r}$ *or* $\mathbf{x}_\sigma + cb\mathbf{s}$ *are also in* $[0,1]^n$ *and comonotone with* $\mathbf{x}_\sigma$*. Then, if a function* $F : [0,1]^n \to [0,1]$ *is both OD* $\mathbf{r}$*- and OD* $\mathbf{s}$*-increasing, it is also OD* $\mathbf{u}$*-increasing.*

Proof. Note that for each $\sigma \in S_n$, if $\mathbf{u} = a\mathbf{r} + b\mathbf{s}$, then $\mathbf{u}_{\sigma^{-1}} = a\mathbf{r}_{\sigma^{-1}} + b\mathbf{s}_{\sigma^{-1}}$, and thus, for each $\mathbf{x} \in [0,1]^n$, (under the given assumptions), using the OD $\mathbf{s}$- and $\mathbf{r}$-increasingness of F, we can write

$$\begin{aligned} F(\mathbf{x} + c\mathbf{u}_{\sigma^{-1}}) &= F(\mathbf{x} + ca\mathbf{r}_{\sigma^{-1}} + cb\mathbf{s}_{\sigma^{-1}}) \\ &\geq F(\mathbf{x} + ca\mathbf{r}_{\sigma^{-1}}) \geq F(\mathbf{x}) \end{aligned}$$

and, hence, the statement holds. ■

The different types of monotonicity are related as follows.

Proposition 5. *Let* $F : [0,1]^n \to [0,1]$ *be a fusion function and let* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *be the canonical basis in* $\mathbb{R}^n$*. Then the following statements are equivalent.*

(i) F *is increasing.*
(ii) F *is* $\mathbf{e}_i$*-increasing for every* $i \in \{1 \ldots, n\}$*.*
(iii) F *is OD* $\mathbf{e}_i$*-increasing for every* $i \in \{1 \ldots, n\}$*.*

Proof. (i) ⇔ (ii): Let F be an increasing function. Fix any element $i \in \{1 \ldots, n\}$. Consider any $\mathbf{x} \in [0,1]^n$ and any $c > 0$ such that $\mathbf{x} + c\mathbf{e}_i \in [0,1]^n$. As $\mathbf{x} + c\mathbf{e}_i \geq \mathbf{x}$, due to the increasing monotonicity of F, we obtain $F(\mathbf{x} + c\mathbf{e}_i) \geq F(\mathbf{x})$, i.e., F is $\mathbf{e}_i$-increasing. On the other hand, the $\mathbf{e}_i$-increasingness of F for each i implies that F is increasing in each coordinate, which means that F is increasing.
(ii) ⇔ (iii): Suppose that F is $\mathbf{e}_i$-increasing for every $i \in \{1 \ldots, n\}$. Fix any $i \in \{1 \ldots, n\}$ and put $\mathbf{r} = \mathbf{e}_i$. Let $\mathbf{x}$ be any element in $[0,1]^n$. Consider any $\sigma \in \mathcal{S}_n$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(i)} \geq \ldots \geq x_{\sigma(n)}$, and any $c > 0$ such that also $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(i)} + c \geq \ldots \geq x_{\sigma(n)}$. Suppose that $\sigma(i) = j$. Then $r_{\sigma^{-1}(j)} = r_i = 1$ and $\mathbf{r}_{\sigma^{-1}} = \mathbf{e}_j$, so, due to the $\mathbf{e}_j$-increasingness of F, we have

$$\begin{aligned} &F(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) = F\left(\mathbf{x} + c\left(\mathbf{e}_i\right)_{\sigma^{-1}}\right) \\ &= F(\mathbf{x} + c\mathbf{e}_j) \geq F(\mathbf{x}), \end{aligned}$$

which proves the OD $\mathbf{e}_i$-increasingness of F.

On the other hand, suppose that F is OD $\mathbf{e}_i$-increasing for each i. Fix $i \in \{1, \ldots, n\}$. Consider any $\mathbf{x} \in [0,1]^n$ and any $c > 0$ such that $\mathbf{x} + c\mathbf{e}_i \in [0,1]^n$.

If $x_i = \max\{x_1, \ldots, x_n\}$, then there is a permutation σ such that $\sigma(1) = i$ and $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$. We have $\mathbf{e}_i = (\mathbf{e}_1)_{\sigma^{-1}}$ and due to the OD $\mathbf{e}_1$-increasingness of F, as $\mathbf{x}_\sigma$ and $\mathbf{x}_\sigma + c\mathbf{e}_1$ are comonotone, we can write

$$F(\mathbf{x} + c\mathbf{e}_i) = F(\mathbf{x} + c(\mathbf{e}_1)_{\sigma^{-1}}) \geq F(\mathbf{x}).$$

Now, suppose that $x_i \neq \max\{x_1, \ldots, x_n\}$. Let σ be a permutation such that $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$. Let $i = \sigma(j)$ which implies $\mathbf{e}_i = (\mathbf{e}_j)_{\sigma^{-1}}$. Clearly, $j > 1$. If $x_i + c = x_{\sigma(j)} + c \leq x_{\sigma(j-1)}$ then $\mathbf{x}_\sigma$ and $\mathbf{x}_\sigma + c\mathbf{e}_j$ are comonotone, and thus the OD $\mathbf{e}_j$-increasingness of F implies $F(\mathbf{x} + c\mathbf{e}_i) \geq F(\mathbf{x})$.

If $x_i + c > x_{\sigma(j-1)}$, the mentioned points are not more comonotone. However, the point $\mathbf{x}_\sigma$ is comonotone with $\mathbf{x}_\sigma + \left(x_{\sigma(j-1)} - x_i\right)\mathbf{e}_j$, and due to the OD $\mathbf{e}_j$-increasingness of F we can deduce

$$F(\mathbf{x} + \left(x_{\sigma(j-1)} - x_i\right)\mathbf{e}_i) \geq F(\mathbf{x}).$$

Clearly, then

$$\mathbf{x} + c\mathbf{e}_i = \mathbf{y} + d\mathbf{e}_i,$$

where $\mathbf{y} = \mathbf{x} + \left(x_{\sigma(j-1)} - x_i\right)\mathbf{e}_i$ and $d = c + x_i - x_{\sigma(j-1)} > 0$, and we can continue by (finite) induction. This follows from the fact that there is permutation τ such $y_{\tau(1)} \geq \ldots \geq y_{\tau(n)}$ and $\tau^{-1}(i) = j - 1$. Thus, in at most j steps we obtain our result.

The equivalence of (i) and (iii) already follows. ■

4 Construction Methods for OD Monotone Functions

In this section we discuss different methods for construction of OD $\mathbf{r}$-increasing functions from other types of fusion functions.

The first method is based on composition. We first provide a sufficient condition for OD $\mathbf{r}$-increasingness of composition of OD $\mathbf{r}$-increasing fusion functions.

Proposition 6. *Let $\mathbf{r} \in \mathbb{R}^n$ be a non-null vector and let $F_i : [0,1]^n \to [0,1]$, $i \in \{1, \ldots, m\}$, be OD $\mathbf{r}$-increasing functions. If $A : [0,1]^m \to [0,1]$ is an increasing function then the function $A(F_1, \ldots, F_m) \colon [0,1]^n \to [0,1]$ defined by*

$$\begin{aligned} &A(F_1, \ldots, F_m)(x_1, \ldots, x_n) \\ &= A(F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n)), \end{aligned}$$

is also OD $\mathbf{r}$-increasing.

Proof. Let $\mathbf{x}$ be any element in $[0,1]^n$. Consider any permutation $\sigma \in \mathcal{S}_n$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and $c > 0$ such that $\mathbf{x}_\sigma + c\mathbf{r}$ is in $[0,1]^n$ and comonotone with $\mathbf{x}_\sigma$. From the definition of OD $\mathbf{r}$-monotonicity, we have that

$$F_i(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) \geq F_i(\mathbf{x}), \quad i \in \{1, \ldots, m\}.$$

Moreover, from the increasing monotonicity of A, it follows that

$$A(F_1, \ldots, F_m)(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) \geq A(F_1, \ldots, F_m)(\mathbf{x}),$$

and, hence, the result holds. ■

Ordered directionally increasing functions can also be built from directionally increasing functions by ordering the inputs in a decreasing way.

Proposition 7. *Let $F : [0,1]^n \to [0,1]$ be an* $\mathbf{r}$*-increasing function. Then the function* $G\colon [0,1]^n \to [0,1]$*, given by*

$$G(x_1, \ldots, x_n) = F(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where a permutation $\sigma \in \mathcal{S}_n$ satisfies the property $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$, is OD $\mathbf{r}$*-increasing.*

Proof. Let $\mathbf{x}$ be any element in $[0,1]^n$. Consider any permutation $\sigma \in \mathcal{S}_n$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and any $c > 0$ such that $\mathbf{x}_\sigma + c\mathbf{r}$ is in $[0,1]^n$ and comonotone with $\mathbf{x}_\sigma$. Then, from the $\mathbf{r}$-increasingness of F, we have

$$\begin{aligned} G(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}) &= F\left((\mathbf{x} + c\mathbf{r}_{\sigma^{-1}})_\sigma\right) \\ &= F(\mathbf{x}_\sigma + c\mathbf{r}) \geq F(\mathbf{x}_\sigma) = G(\mathbf{x}), \end{aligned}$$

which proves the claim. ■

Example 5. Consider the weighted Lehmer mean $L_\lambda\colon [0,1]^2 \to [0,1]$, given by

$$L_\lambda(x,y) = \frac{\lambda x^2 + (1-\lambda)y^2}{\lambda x + (1-\lambda)y}$$

(with the convention $0/0 = 0$), which is $(1-\lambda, \lambda)$-increasing (but not $\mathbf{r}$-increasing if $\mathbf{r} \neq k(1-\lambda,\lambda)$, $k > 0$) [3]. By Proposition 7, the function

$$G_\lambda(x,y) = \frac{\lambda(\max(x,y))^2 + (1-\lambda)(\min(x,y))^2}{\lambda \max(x,y) + (1-\lambda)\min(x,y)}$$

is OD $(1-\lambda,\lambda)$-increasing. In fact, G_λ is not OD $\mathbf{r}$-increasing with respect to any vector $\mathbf{r} \neq k(1-\lambda,\lambda)$, $k > 0$.

Note that L_λ is symmetric only if $\lambda = \frac{1}{2}$. Then $L_\lambda = G_\lambda$ is $\left(\frac{1}{2}, \frac{1}{2}\right)$-increasing and thus, also OD $\left(\frac{1}{2}, \frac{1}{2}\right)$-increasing, and also weakly increasing.

Finally, we analyze the construction of OD-monotone functions using the discrete Choquet integral.

Theorem 3. *Let $\mathfrak{m}\colon 2^N \to [0,1]$ be a fuzzy measure and let $\mathbf{r} = (r_1, \ldots, r_n)$ be a non-null real vector. Then the Choquet integral $C_\mathfrak{m}\colon [0,1]^n \to [0,1]$ is an OD* $\mathbf{r}$*-increasing fusion function if and only if for each permutation $\tau \in \mathcal{S}_n$ it holds that*

$$\sum_{i=1}^{n} r_i \mathfrak{m}_\tau(i) \geq 0,$$

where $\mathfrak{m}_\tau(1) = \mathfrak{m}(\{\tau(n)\})$, and for each $i \in \{2, \ldots, n\}$, $\mathfrak{m}_\tau(i) = \mathfrak{m}(\{\tau(n-i+1), \ldots, \tau(n)\}) - \mathfrak{m}(\{\tau(n-i+2), \ldots, \tau(n)\})$.

Proof. Recall that for any $\mathbf{x} \in [0,1]^n$ the Choquet integral introduced in Definition 4 can be written equivalently as

$$C_\mathfrak{m}(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} \left(\mathfrak{m}\left(A_{(i)}\right) - \mathfrak{m}\left(A_{(i+1)}\right)\right),$$

where $(\cdot)$ is a permutation in $\mathcal{S}_n$, such that $x_{(1)} \leq \ldots \leq x_{(n)}$, and for each $i \in \{1, \ldots, n\}$, $A_{(i)} = \{(i), \ldots, (n)\}$, with the convention $A_{(n+1)} = \emptyset$. If $\mathbf{x}_\sigma + c\mathbf{r} \in [0,1]^n$ and $\mathbf{x}_\sigma$ and $\mathbf{x}_\sigma + c\mathbf{r}$ are comonotone for some permutation $\sigma \in \mathcal{S}_n$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$ and $c > 0$, then if $C_\mathfrak{m}$ is OD $\mathbf{r}$-increasing, necessarily $C_\mathfrak{m}(\mathbf{x}+c\mathbf{r}_{\sigma^{-1}}) \geq C_\mathfrak{m}(\mathbf{x})$. Clearly, one can assume that $\sigma(1) = (n), \ldots, \sigma(n) = (1)$, i.e., $\sigma(i) = (n - i + 1)$. Thus

$$\begin{aligned} & C_\mathfrak{m}\left(\mathbf{x} + c\mathbf{r}_{\sigma^{-1}}\right) \\ &= \sum_{i=1}^{n} \left(x_{(i)} + c r_{n-i+1}\right) \left(\mathfrak{m}\left(A_{(i)}\right) - \mathfrak{m}\left(A_{(i+1)}\right)\right) \\ &\geq C_\mathfrak{m}(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} \left(\mathfrak{m}\left(A_{(i)}\right) - \mathfrak{m}\left(A_{(i+1)}\right)\right), \end{aligned}$$

which implies that

$$\begin{aligned} & \sum_{i=1}^{n} r_i \left(\mathfrak{m}\left(A_{(n-i+1)}\right) - \mathfrak{m}\left(A_{(n-i+2)}\right)\right) \\ &= \sum_{i=1}^{n} r_i \mathfrak{m}_{(\cdot)}(i) \geq 0. \end{aligned}$$

As any permutation τ can be obtained from a permutation $(\cdot)$ related to some input $\mathbf{x} \in [0,1]^n$, the necessity of our result is proved.

The sufficiency can be shown similarly. ∎

5 Conclusions

In this paper we have discussed the notion of ordered directionally monotonicity, presenting some properties and construction methods. Ordered directionally monotone functions allow for considering different directions for monotonicity for different inputs, according to the relative size of the latter. This is specially of interest in image processing problems or in those settings where extensions of operators such as the OWA must be considered.

In future work, we will carry on an analysis of the possible applications of this new concept, specially in the fields of image processing and edge detection. In particular, we intend to discuss how edge detectors may be built making use of this new notion.

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Smart Medical Device Selection Based on Interval Valued Intuitionistic Fuzzy VIKOR

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Abstract. Advances in wireless communication technologies and the internet of things are leading to new developments in the domain of wearable, smart medical devices (SMDs) as a major disruptive trend for the medical industry. Wearable smart sensor technology with non-invasive or invasive implantable materials has a great potential for interfacing with the human body, thanks to low-power silicon-based electronics that are very efficient in data processing and transmission. Novel SMDs are designed for monitoring living being's vital signs, such as blood pressure, cardiac monitoring, respiration rate, body temperature, etc. in either medical diagnostic or health monitoring. Considering various smart devices in the medical industry, a key decision is which device to choose and apply on the patient. The decision on the evaluation of SMDs is a complicated problem that needs to be assessed from different perspectives. This study guides decision makers on the selection of SMDs of wearable vital sign sensors under different evaluation criteria. A multi criteria decision making approach is proposed to support the SMD selection process under group decision making (GDM) in an uncertain environment. A significant feature of this analysis is the complexity of the selected decision criteria for the SMD evaluation. To simulate these processes, a methodology that combines interval valued intuitionistic fuzzy (IVIF) with Višekriterijumsko kompromisno rangiranje (VIKOR) under GDM is proposed. This methodology is then used to measure the assessment of four SMDs using five evaluation criteria. To validate the proposed approach, the selection methodology for wearable vital sign monitoring devices is applied on a case study.

Keywords: Smart medical devices (SMDs) · Multi criteria decision making (MCDM) · Group decision making (GDM) · Interval valued intuitionistic fuzzy (IVIF) · Višekriterijumsko kompromisno rangiranje (VIKOR)

1 Introduction

The healthcare industry is revolutionizing how patients are treated by using technological tools and devices helping medical staff as well as patients. These innovative ways of treatment support handling health situations outside of a healthcare facility. This growth

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in the number of novel medical devices is closely associated with digital technologies and their applications in medical systems. The first driving factor in this progress is due to the developments in the sensors and actuators technology. Interface of electronic devices with chemical elements is improving, and micro and nano-technological substances make it easier to detect and inject more and more substances directly into the human body. These minimally-invasive or non-invasive small-scale medical tools present important opportunities for developing dexterous, smart, and robust devices. There are novel portable or wearable devices to monitor and measure the vital sign of living beings. These wearable vital sign measurement devices use biometric information from human body to continuously measure real-time heart rate, body temperature, speed, cadence and distance data. Several types of SMD are created to perform different medical tasks. Digital evolution in current era revolutionizes the delivery of healthcare in homes and as well as at hospitals by introducing either surgical robots to assist complex procedures or to simply support routine tasks, e.g. increasing medical staff's efficiency and administration of medicine to patients. As the use of SMDs at homes increase, people find it less necessary to physically visit health facilities. New smartphone apps and wireless connection among different devices with locally available processing power permit communication of remote units, i.e. hospitals, homes, staff, etc.

New SMDs can be fixed on the human body as skin patches or smart woven textiles or they could also be implanted in smaller SMDs to make the process simpler. Current trend in the medical industry is to keep tabs on patients' vital information to aid medical staff. Thanks to the wearable monitoring technology, the dreaded traditional ward rounds can soon become a thing of the past. Instead of any medical staff going from patient to patient taking note of their vital information, SMDs can collect and report on these data uniformly and surreptitiously on their behalf. For instance, an SMD can be a biosensor embedded in a wearable patch band with electrocardiography electrodes with axis accelerometer to detect and record the breathing rate, temperature, and heart rate, etc. There has been an increase in the demand for wearable devices. Further, these smart, wearable and connected devices can also be linked to external smartphones or tablets for the analysis of the results collected by their biosensors. The SMDs' literature review presents that there is big gap between the practice in industry and scientific theory with a limited number of studies. As far as the authors are aware of, there are few studies that discuss SMD technologies or their applications but no academic study that explicitly deals with the concept of SMDs. Assenting this assumption. Stoppa and Chiolerio [1] review the recent advances in the field of smart textiles and their manufacturing process. Vashist et al. [2] review the widely used personalized smartphone based healthcare monitoring and management devices. Walsh et al. [3] review the novel wireless cardiac monitoring devices. Khan et al. [4] review the latest development in flexible and wearable human vitals sensors.

Decision making processes are usually linked with the selection of the best among the pre-defined set of alternatives by considering the impact of multiple criteria. Since the early emergence of multi criteria decision making (MCDM), this methodology has been evolving and has been one of the key research areas in solving complex decision problems in the presence of multiple objectives or criteria [5]. As a result of this revolution, many types of MCDM methodologies are proposed which are being successfully used in solving numerous types of scientific and industrial decision making problems. MCDM methodology can be used in evaluation and ranking problems that integrate several conflicting criteria. For decision makers (DMs), this powerful theory often entails quantitative and qualitative data which are used in the measurement of available alternatives performance in terms of relevant decision criteria. Many decision problems faced with in real life also necessitate the contribution of more than one DM in decision-making processes. Thus, most of the MCDM approaches are applied with a group decision making (GDM) structure.

Many real problems are mostly characterized by conflicting and noncommensurable criteria with no solution that simultaneously satisfies all criteria. An MCDM technique, Višekriterijumsko kompromisno rangiranje (VIKOR) [6], is first proposed by Opricovic as an efficient methodology to solve these kind of problems. Most of the ordinary MCDM approaches, however, run on crisp values. Therefore, it is mostly inefficient to solve these problems using one of these ordinary methods. The so-called fuzzy logic can address these challenges, which can be combined with many MCDM approaches. Zadeh [7] generalized the concept of the fuzzy set theory in which a membership value is linked to every element in a set. Nevertheless, gathered information might not always be adequate to define exact values, and there can sometimes be lack of precision due to conventional fuzzy or crisp sets. To address such problems, Atanassov [8] developed a substitute approach, called the intuitionistic fuzzy (IF) set. Intuitionistic fuzzy sets can also be extended to IVIF sets [9]. This method is widely studied in the last decades by numerous researchers. Many different types of MCDM approaches are integrated by IVIF sets. Characterized by a membership degree, non-membership degree and hesitancy degree parameters, the IVIF sets are very strong and successful in handling the situations under vagueness, uncertainty, and imprecision. Thus, IVIF sets present an appropriate tool to express DMs' preferences and defining its membership function properly that are subject to hesitation or lack of expertise. There are some studies with IVIF VIKOR in literature [10, 11]. The proposed method is different than others as it is easily comprehensible and computationally simple.

To the best of authors' knowledge, the methodology in this study has not yet been proposed in any literature at this extent. The originality of this methodology comes from its ability to present a new methodology which applies VIKOR with GDM under IVIF set theory for selecting SMDs. This research contributes to literature by providing a framework on the selection of SMDs with IVIF VIKOR under GDM.

Organization of this paper is summarized as follows. Section 2 provides a detailed explanation of problem definition and evaluation criteria for SMDs. In Sect. 3, a simple introduction to IVIF set and the detailed steps of the proposed methodology structure is given. Section 4 presents a sample case with numerical results in which a suitable alternative of wearable SMD is selected to present the performance of the applied methodology. Finally, the last section discusses the results and limitations of this study.

2 Problem Definition of SMD Selection

The way people live is being altered by the digital revolution and the rapid development of social networking, mobile connectivity and smart phones. Any average person in a developed country is constantly connected to a vast amount of information.

This revolutionary era on digitalization has conspicuously transformed every industry as well as every facet of people's personal lives. The reflections of this revolution are also evident in the medical industry by quickly adopting smart, digital devices. Many different types of wearable SMDs emerge to monitor patients' vitals. The authors believe that the evolution of these SMDS marks a new era in the healthcare industry. Currently, there are numerous companies developing new wearable vital sign monitoring SMDs, with different strengths and objectives. This differentiation in SMDs make it necessary to view them in the light of different criteria. Table 1 presents the evaluation criteria for SMDs, its description and related weights.

Table 1. Evaluation criteria

Criteria	W_j	Description
Safety (C_1)	0.2170	Helping the implementation of safety principles and requirements should be a priority for any SMDs [1–3]
Cost (C_2)	0.2393	Economic, cost-effective SMDs should be in the center [1–3]
Ease of use (C_3)	0.1580	The SMDs themselves and their user interfaces should be user-friendly [3]
Service life (C_4)	0.2148	Prolonged duration of SMDs should be the priority for any SMD [1, 3]
Quality (C_5)	0.1709	Quality should be the top priority for any SMD. While operating, every stage should be ensuring highest reliability [1–3]

3 Proposed Methodology

The aim of this proposed methodology is to develop a framework to be used in the evaluation of prioritization of wearable vital sign monitoring SMDs for achieving various objectives. The next subsection gives the preliminary explanations about IVIF values and then, the proposed methodology will be introduced.

3.1 Preliminaries

Here, X is a given fixed set. An IVIF set in x in $\tilde{A}$ is defined in Eq. (1) as the basic component of an IVIF set that is an ordered pair, characterized by an interval valued membership value and an interval valued non membership value $\tilde{A}$ is called IVIF set, where $\tilde{\mu}_{\tilde{A}}(x) \subset [0, 1]$ and $\tilde{v}_{\tilde{A}}(x) \subset [0, 1], x \in X$ with the condition of $\sup \tilde{\mu}_{\tilde{A}}(x) + \sup \tilde{v}_{\tilde{A}}(x) \leq 1$.

$$\tilde{A} = \left\{ \left\langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{v}_{\tilde{A}}(x) \right\rangle | x \in X \right\} \quad (1)$$

For convenience, the IVIF set lower and upper end points are denoted by $\tilde{A} = \left[\mu^L_{\tilde{A}}, \mu^U_{\tilde{A}}\right], \left[v^L_{\tilde{A}}, v^U_{\tilde{A}}\right]$ or $\tilde{B} = \left[\mu^L_{\tilde{B}}, \mu^U_{\tilde{B}}\right], \left[v^L_{\tilde{B}}, v^U_{\tilde{B}}\right]$. Using these two IVIF numbers, the following expressions are defined [12]:

$$\tilde{A} \leq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}^{L} \leq \mu_{\tilde{B}}^{L}, \mu_{\tilde{A}}^{U} \leq \mu_{\tilde{B}}^{U}, v_{\tilde{B}}^{L} \leq v_{\tilde{A}}^{L}, v_{\tilde{B}}^{L} \leq v_{\tilde{A}}^{U} \tag{2}$$

$$\tilde{A} + \tilde{B} = \left(\left[\mu_{\tilde{A}}^{L} + \mu_{\tilde{B}}^{L} - \mu_{\tilde{B}}^{L}\mu_{\tilde{A}}^{L}, \mu_{\tilde{A}}^{U} + \mu_{\tilde{B}}^{U} - \mu_{\tilde{A}}^{U}\mu_{\tilde{B}}^{U}\right], \left[v_{\tilde{A}}^{L}v_{\tilde{B}}^{L}, v_{\tilde{A}}^{U}v_{\tilde{B}}^{U}\right]\right) \tag{3}$$

$$\tilde{A} * \tilde{B} = \left(\left[\mu_{\tilde{A}}^{L}\mu_{\tilde{B}}^{L}, \mu_{\tilde{A}}^{U}\mu_{\tilde{B}}^{U}\right], \left[v_{\tilde{A}}^{L} + v_{\tilde{B}}^{L} - v_{\tilde{B}}^{L} - v_{\tilde{A}}^{L}v_{\tilde{B}}^{L}, v_{\tilde{A}}^{U} + v_{\tilde{B}}^{U} - v_{\tilde{A}}^{U}v_{\tilde{B}}^{U}\right]\right) \tag{4}$$

$$\lambda\tilde{A} = \left(\left[1 - (1 - \mu_{\tilde{A}}^{L})^{\lambda}\right], \left[1 - \left(1 - \mu_{\tilde{A}}^{U}\right)^{\lambda}\right], \left[(v_{\tilde{A}}^{L})^{\lambda}\right], \left[\left(v_{\tilde{A}}^{U}\right)^{\lambda}\right]\right) \tag{5}$$

$$\tilde{A}^{\lambda} = \left((\mu_{\tilde{A}}^{L})^{\lambda}, \left(\mu_{\tilde{A}}^{U}\right)^{\lambda}, \left[1 - (1 - v_{\tilde{A}}^{L})^{\lambda}\right], \left[1 - \left(1 - v_{\tilde{A}}^{U}\right)^{\lambda}\right]\right) \tag{6}$$

$$\tilde{A} - \tilde{B} = \left(\left[\frac{\mu_{\tilde{A}}^{L} - \mu_{\tilde{B}}^{L}}{1 - \mu_{\tilde{B}}^{L}}, \frac{\mu_{\tilde{A}}^{U} - \mu_{\tilde{B}}^{U}}{1 - \mu_{\tilde{B}}^{U}}\right], \left[\frac{v_{\tilde{A}}^{L}}{v_{\tilde{B}}^{L}}, \frac{v_{\tilde{A}}^{U}}{v_{\tilde{B}}^{U}}\right]\right), \frac{\tilde{A}}{\tilde{B}} = \left(\left[\frac{\mu_{\tilde{A}}^{L}}{\mu_{\tilde{B}}^{L}}, \frac{\mu_{\tilde{A}}^{U}}{\mu_{\tilde{B}}^{U}}\right], \left[\frac{v_{\tilde{A}}^{L} - v_{\tilde{B}}^{L}}{1 - v_{\tilde{B}}^{L}}, \frac{v_{\tilde{A}}^{U} - v_{\tilde{B}}^{U}}{1 - v_{\tilde{B}}^{U}}\right]\right) \tag{7}$$

3.2 IVIF VIKOR Methodology

The steps of the IVIF VIKOR methodology is as follow:

Step 1. Get the judgments of DMs

DMs are asked to express their opinions on the alternative A_i over criterion C_j from the viewpoint of the k^{th} DM based on their prior knowledge and their field of experience.

Step 2. Transform the linguistic variables into IVIF values

The DMs apply linguistic variables listed in Table 2 for voicing their judgments about the alternatives for each of the criteria.

Table 2. Linguistic variables for rating alternatives [14].

Linguistic terms		$[\mu^L, \mu^U]$	$[v^L, v^U]$
Extremely good	EG	0.00, 0.20	0.50, 0.80
Very good	VG	0.10, 0.30	0.40, 0.70
Medium good	MG	0.20, 0.40	0.30, 0.60
Good	G	0.30, 0.50	0.20, 0.50
Approximately equal	AE	0.40, 0.60	0.20, 0.40
Bad	B	0.50, 0.70	0.10, 0.30
Medium bad	MB	0.60, 0.80	0.00, 0.20
Very bad	VB	0.70, 0.90	0.00, 0.10
Extremely bad	EB	0.80, 1.00	0.00, 0.00

Step 3. Determine DMs Weights

Equation (8) is used to determine K DMs' weights [13]. Linguistic importance scale in Table 2 is used to assess the importance of DMs. Where $[\pi^L, \pi^U]$ represents the hesitancy degree (unknown degree) of an IVIF sets and $\pi^L = 1 - \nu^U - \mu^U$, $\pi^U = 1 - \nu^L - \mu^L$.

$$\lambda^k = \frac{\sqrt{\frac{1}{2}\left[\left(1-\pi_{\tilde{A}}^{L^k}\right)^2 + \left(1-\pi_{\tilde{A}}^{U^k}\right)^2\right]}}{\sum_{l=1}^{K}\sqrt{\frac{1}{2}\left[\left(1-\pi_{\tilde{A}}^{L^l}\right)^2 + \left(1-\pi_{\tilde{A}}^{U^l}\right)^2\right]}} \tag{8}$$

Step 4. Calculate the criteria weights

The weight vectors $w_1, w_2, \ldots, w_n$ with $w_j \geq 0, j = 0, j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$ are determined that give the relative significance of different criterion by using Eq. (9).

$$\mathrm{W_j} = \frac{1 - \breve{\mathrm{w}}_\mathrm{j}}{\mathrm{n} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \breve{\mathrm{w}}_\mathrm{j}}, \breve{\mathrm{w}}_\mathrm{j} = 1 - \frac{\sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\mathrm{w_j}\left(\mu_{\tilde{\mathrm{A}}\mathrm{ij}}^{\mathrm{L}} + \mu_{\tilde{\mathrm{A}}\mathrm{ij}}^{\mathrm{U}}\right)}{2}}{\sqrt{\sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\mathrm{w_j}\left(\mu_{\tilde{\mathrm{A}}\mathrm{ij}}^{\mathrm{L2}} + \mu_{\tilde{\mathrm{A}}\mathrm{ij}}^{\mathrm{U2}} + \mathrm{V}_{\tilde{\mathrm{A}}\mathrm{ij}}^{\mathrm{L2}} + \mathrm{V}_{\tilde{\mathrm{A}}\mathrm{ij}}^{\mathrm{U2}}\right)}{2}}} \tag{9}$$

Step 5. Construct the aggregated matrix

Using DMs' weights $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ calculated in step 3, individual opinions of DMs are aggregated to evaluate alternatives by using the IVIF weighted averaging (IIFWA) operator [15], as shown in Eq. (10).

Let $X_{(k)} = (x_{ij}^{(k)})_{m \times n}$ be an IVIF decision matrix of the k^{th} DM for the alternatives.

$$\mathrm{IIFWA} = \left(\left[1 - \prod_{\mathrm{j}=1}^{\mathrm{n}} \left(1-\mu_{\tilde{A}}^{L}\right)^{\lambda^k}, 1 - \prod_{\mathrm{j}=1}^{\mathrm{n}}\left(1-\mu_{\tilde{A}}^{U}\right)^{\lambda^k}\right], \left[\prod_{\mathrm{j}=1}^{\mathrm{n}}\left(\nu_{\tilde{A}}^{L}\right)^{\lambda^k}, \prod_{\mathrm{j}=1}^{\mathrm{n}}\left(\nu_{\tilde{A}}^{U}\right)^{\lambda^k}\right]\right) \tag{10}$$

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{21} & & \tilde{x}_{i1} & \cdots & \tilde{x}_{m1} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \tilde{x}_{12} & \cdots & \tilde{x}_{22} & & \tilde{x}_{i2} & \cdots & \tilde{x}_{m2} \\ & \vdots & & \ddots & & \vdots & \\ \tilde{x}_{1j} & \cdots & \tilde{x}_{2j} & & \tilde{x}_{ij} & \cdots & \tilde{x}_{mj} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \tilde{x}_{1n} & \cdots & \tilde{x}_{2n} & & \tilde{x}_{in} & \cdots & \tilde{x}_{mn} \end{bmatrix} \tag{11}$$

Here, $\tilde{x}_{ij} = \left[\tilde{\mu}_{\tilde{A}}^{L}(\tilde{x}_{ij}), \tilde{\mu}_{\tilde{A}}^{U}(\tilde{x}_{ij})\right], \left[\tilde{\nu}_{\tilde{A}}^{L}(\tilde{x}_{ij}), \tilde{\nu}_{\tilde{A}}^{U}(\tilde{x}_{ij})\right]$ denotes an IVIF value.

Step 6. Calculate the positive and negative ideal solutions

The positive and negative ideal solutions are found by Eqs. (12 and 13) [10, 11].

$$\tilde{f}_j^* = \left\langle \left[\tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^*\right), \tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^*\right)\right], \left[\tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^*\right), \tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^*\right)\right] \right\rangle$$

$$\begin{array}{c} \left[\tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) = \max_i \tilde{\mu}_{\tilde{A}}^L(\tilde{x}_{ij}), \tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) = \max_i \tilde{\mu}_{\tilde{A}}^U(\tilde{x}_{ij})\right], \\ \left[\tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) = \min_i \tilde{v}_{\tilde{A}}^L(\tilde{x}_{ij}), \tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) = \min_i \tilde{v}_{\tilde{A}}^U(\tilde{x}_{ij})\right] \end{array} \tag{12}$$

$$\tilde{f}_j^- = \left\langle \left[\tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^-\right), \tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^-\right)\right], \left[\tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^-\right), \tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^-\right)\right] \right\rangle$$

$$\begin{array}{c} \left[\tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^-\right) = \min_i \tilde{\mu}_{\tilde{A}}^L(\tilde{x}_{ij}), \tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^-\right) = \min_i \tilde{\mu}_{\tilde{A}}^U(\tilde{x}_{ij})\right], \\ \left[\tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^-\right) = \max_i \tilde{v}_{\tilde{A}}^L(\tilde{x}_{ij}), \tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^-\right) = \max_i \tilde{v}_{\tilde{A}}^U(\tilde{x}_{ij})\right] \end{array} \tag{13}$$

Step 7. Calculate the group utility value and the individual regret value

The group utility value $S(A_i)$ and individual regret value $R(A_i)$ for alternative A_i are found by the Eq. (14) [10, 11].

$$S(A_i) = \sum_{j=1}^{n} \left[w_j \frac{d\left(\tilde{f}_j^*, \tilde{x}_{ij}\right)}{d\left(\tilde{f}_j^*, \tilde{f}_j^-\right)} \right], R(A_i) = \max_j \left[w_j \frac{d\left(\tilde{f}_j^*, \tilde{x}_{ij}\right)}{d\left(\tilde{f}_j^*, \tilde{f}_j^-\right)} \right]. \tag{14}$$

$$d\left(\tilde{f}_j^*, \tilde{x}_{ij}\right) = \frac{1}{4} \left(\begin{array}{c} \left|\tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) - \tilde{\mu}_{\tilde{A}}^L(\tilde{x}_{ij})\right| + \left|\tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) - \tilde{\mu}_{\tilde{A}}^U(\tilde{x}_{ij})\right| \\ + \left|\tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) - \tilde{v}_{\tilde{A}}^L(\tilde{x}_{ij})\right| + \left|\tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) - \tilde{v}_{\tilde{A}}^U(\tilde{x}_{ij})\right| \\ + \left|\tilde{\pi}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) - \tilde{\pi}_{\tilde{A}}^L(\tilde{x}_{ij})\right| + \left|\tilde{\pi}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) - \tilde{\pi}_{\tilde{A}}^U(\tilde{x}_{ij})\right| \end{array} \right) \tag{15}$$

$$d\left(\tilde{f}_j^*, \tilde{f}_j^-\right) = \frac{1}{4} \left(\begin{array}{c} \left|\tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) - \tilde{\mu}_{\tilde{A}}^L\left(\tilde{x}_j^-\right)\right| + \left|\tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) - \tilde{\mu}_{\tilde{A}}^U\left(\tilde{x}_j^-\right)\right| \\ + \left|\tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) - \tilde{v}_{\tilde{A}}^L\left(\tilde{x}_j^-\right)\right| + \left|\tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) - \tilde{v}_{\tilde{A}}^U\left(\tilde{x}_j^-\right)\right| \\ + \left|\tilde{\pi}_{\tilde{A}}^L\left(\tilde{x}_j^*\right) - \tilde{\pi}_{\tilde{A}}^L\left(\tilde{x}_j^-\right)\right| + \left|\tilde{\pi}_{\tilde{A}}^U\left(\tilde{x}_j^*\right) - \tilde{\pi}_{\tilde{A}}^U\left(\tilde{x}_j^-\right)\right| \end{array} \right). \tag{16}$$

Step 8. Compute the values $Q(A_i)$ using the $S^*(A_i)$, $S^-(A_i)$ and $R^*(A_i)$, $R^-(A_i)$ values.

$$S^*(A_i) = \min_i S(A_i),\ S^-(A_i) = \max_i S(A_i) \tag{17}$$

$$R^*(A_i) = \min_i R(A_i),\ R^-(A_i) = \max_i R(A_i) \tag{18}$$

$$\mathrm{Q(A_i)} = \upsilon\left(\frac{S(A_i) - S^*(A_i)}{S^-(A_i) - S^*(A_i)}\right) + (1 - \upsilon)\left(\frac{R(A_i) - R^*(A_i)}{R^-(A_i) - R^*(A_i)}\right) \quad (19)$$

where, "υ" is presented as a weight of that defines "the majority of criteria". Usually υ value is taken as 0.5.

Step 9. The ranking order of alternatives is determined

Alternatives are ranked by sorting each of the $S(A_i)$, $R(A_i)$, and $Q(A_i)$ index values. These values are sorted in ascending order as in the original VIKOR method [6]. The outcome is a set of ranking lists denoted as $S_{[i]}$, $R_{[i]}$ and $Q_{[i]}$.

Step 10. The alternative A_i related to $Q_{[1]}$, that is the smallest in ranking of $Q(A_i)$ values, is proposed as a compromise solution if:

1. The alternative A_i has an acceptable advantage, in other words $Q_{[2]} - Q_{[1]} \geq DQ$ where $DQ = 1/(z - 1)$ and z is the number of the alternatives.
2. The alternative A_i is constant within the decision-making process, that is, it is also the best ranked among $S_{[i]}$ and $R_{[i]}$ lists. If none of the above state is satisfied, then it is a set of compromise solutions, which consists of:
   a. The alternatives A_i and A_{i+2} where $Q(A_{i+2}) = Q_{[2]}$ if only the condition 2 is not satisfied, or
   b. The alternatives $A_1, A_2, \ldots, A_z$ if 1. state is not satisfied; and A_z is determined by the relation $Q_{[z]} - Q_{[1]} < DQ$ for the maximum z where $Q(A_z) = Q_{[z]}$ (the ranking of these alternatives are in closeness).

4 Practical Case

This section presents a practical case to select wearable vital sign monitoring SMDs. There are three DMs; DM_1, DM_2, and DM_3 and four candidate SMD vendors; A_1, A_2, A_3, and A_4 for the final evaluation. In order to evaluate candidate SMD vendors, five criteria are considered as evaluation factors; Safety (C_1), Cost (C_2), Ease of Use (C_3), Service Life (C_4), and Quality (C_5), as presented in Sect. 2.

Step 1: DMs opinions are displayed in Table 3. DMs give their judgment on each factor as linguistic terms.

Table 3. Ratings of alternatives and criteria by DMs

	DM_1					DM_2					DM_3				
	C_1	C_2	C_3	C_4	C_5	C_1	C_2	C_3	C_4	C_5	C_1	C_2	C_3	C_4	C_5
A_1	MB	EG	VG	VG	MG	B	EG	VG	G	VG	MB	VG	G	MB	VB
A_2	VG	G	MB	VG	VG	VG	EG	EG	EG	G	VG	G	EG	B	EG
A_3	G	G	B	VG	B	VG	G	VB	VG	G	VG	G	VG	G	G
A_4	VG	VG	G	VG	VG	VG	B	VG	G	MG	VG	MG	G	MB	MG
C_j	MG	VG	B	G	G	MG	VG	MB	G	VB	G	VG	MG	B	B

Step 2: DMs opinions are transformed into IVIF values using the linguistic variables in Table 2. Due to space limitation, the following Table 4 only displays the transformed IVIF values of A_1 and A_2.
Step 3: DMs' weights are determined by the Eq. (8), as shown in Table 6. Nine-point IVIF preference scale in Table 2 is used to weight DMs and the highest importance is given to the first DM and the least importance to the third DM.

Table 4. Transformed IVIF values

		DM1		DM2		DM3	
		$[\mu^L, \mu^U]$	$[v^L, v^U]$	$[\mu^L, \mu^U]$	$[v^L, v^U]$	$[\mu^L, \mu^U]$	$[v^L, v^U]$
A_1	C_1	0.600, 0.800	0.000, 0.200	0.500, 0.700	0.100, 0.300	0.600, 0.800	0.000, 0.200
	C_2	0.000, 0.200	0.500, 0.800	0.000, 0.200	0.500, 0.800	0.100, 0.300	0.400, 0.700
	C_3	0.100, 0.300	0.400, 0.700	0.100, 0.300	0.400, 0.700	0.300, 0.500	0.200, 0.500
	C_4	0.100, 0.300	0.400, 0.700	0.300, 0.500	0.200, 0.500	0.600, 0.800	0.000, 0.200
	C_5	0.200, 0.400	0.300, 0.600	0.100, 0.300	0.400, 0.700	0.700, 0.900	0.000, 0.100
A_2	C_1	0.100, 0.300	0.400, 0.700	0.100, 0.300	0.400, 0.700	0.100, 0.300	0.400, 0.700
	C_2	0.300, 0.500	0.200, 0.500	0.000, 0.200	0.500, 0.800	0.300, 0.500	0.200, 0.500
	C_3	0.600, 0.800	0.000, 0.200	0.000, 0.200	0.500, 0.800	0.000, 0.200	0.500, 0.800
	C_4	0.100, 0.300	0.400, 0.700	0.000, 0.200	0.500, 0.800	0.500, 0.700	0.100, 0.300
	C_5	0.100, 0.300	0.400, 0.700	0.300, 0.500	0.200, 0.500	0.000, 0.200	0.500, 0.800

Step 4: Criteria weights are calculated using Eq. (9) and the results are displayed in Table 1. Nine-point IVIF preference scale in Table 2 is used to assess the alternatives. The highest importance is given to C_2 and the least importance is given to C_3.

Table 5. Aggregated decision matrix

	A_1		A_2	
	$[\mu^L, \mu^U]$	$[v^L, v^U]$	$[\mu^L, \mu^U]$	$[v^L, v^U]$
C_1	0.568, 0.770	0.000, 0.230	0.100, 0.300	0.400, 0.700
C_2	0.030, 0.230	0.469, 0.770	0.209, 0.413	0.274, 0.587
C_3	0.162, 0.364	0.328, 0.636	0.288, 0.522	0.000, 0.478
C_4	0.345, 0.564	0.000, 0.436	0.211, 0.425	0.290, 0.575
C_5	0.371, 0.621	0.000, 0.379	0.149, 0.352	0.336, 0.648

Table 6. Weights of each DMs

DM	DM_1	DM_2	DM_3
Weights	0.3711	0.3429	0.2860

Step 5: Aggregation for GDM is done in this step with Eq. (10). Due to space limitations, the aggregated decision matrix is displayed in Table 5 only for A_1 and A_2.
Step 6: The positive and negative ideal solutions are determined by Eqs. (12 and 13), as shown in Table 7.

Table 7. The positive and negative ideal solutions

	A_1		A_2		A_3		A_4	
	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$
$\tilde{f}_j^*$	0.568, 0.770	0.000, 0.230	0.288, 0.522	0.000, 0.478	0.504, 0.738	0.000, 0.262	0.345, 0.564	0.000, 0.436
$\tilde{f}_j^-$	0.030, 0.230	0.469, 0.770	0.100, 0.300	0.400, 0.700	0.162, 0.364	0.328, 0.636	0.100, 0.300	0.400, 0.700

Step 7: The distances from the ideal and the negative ideal solution for each alternative are computed using Eqs. (15 and 16), as given in Table 8. By the help of these distances, the group utility value and the individual regret value are determined with the Eq. (14). The results are listed in Table 9.

Table 8. The distances from the ideal and negative ideal solutions

	$d\left(\tilde{f}_j^*, \tilde{x}_{ij}\right)$				$d\left(\tilde{f}_j^*, \tilde{f}_j^-\right)$
	A_1	A_2	A_3	A_4	
C_1	0.0000	0.4692	0.388	0.4692	0.4692
C_2	0.0000	0.1889	0.2702	0.2641	0.2702
C_3	0.3573	0.2157	0.0000	0.2827	0.3573
C_4	0.0000	0.2148	0.264	0.0000	0.264
C_5	0.1602	0.4055	0.1718	0.3916	0.2768

Step 8: $Q(A_i)$ values are determined by the Eq. (19) using the $S^*(A_i)$, $S^-(A_i)$ and $R^*(A_i)$, $R^-(A_i)$ values determined by the Eqs. (17 and 18). Table 9 show these results.
Step 9: The ranking order of alternatives is determined with the procedure presented in the step. Alternatives are ranked by sorting each $S(A_i)$, $R(A_i)$, and $Q(A_i)$ index values in increasing order as in the original VIKOR method. The result is a set of three ranking lists denoted as $S_{[i]}$, $R_{[i]}$ and $Q_{[i]}$ and displayed in Table 9.

Table 9. Ranking results and relevant parameters

	$S(A_i)$	Rank	$R(A_i)$	Rank	$Q(A_i)$	Rank
A_1	0.409	4	0.256	3	0.096	4
A_2	0.952	1	0.388	1	1.000	1
A_3	0.615	3	0.225	4	0.190	3
A_4	0.830	2	0.375	2	0.846	2

Step 10: The alternative A_1 corresponds to the smallest value $Q_{[1]}$ among $Q(A_i)$ values. Therefore, A_1 is proposed as a compromise solution since it is also best among the ranking lists of $S_{[i]}$ and $R_{[i]}$.

The result according to the performance index indicates that A_2 is the best one, A_4 is the second, A_3 is the third and A_1 is the last one in ranking;

$$\mathbf{A}_2 > \mathbf{A}_4 > \mathbf{A}_3 > \mathbf{A}_1.$$

5 Concluding Remarks

SMD vendors notice that patients desire to measure their activity themselves. Medical devices are not just for healthcare specialists anymore. Every day SMDs are becoming smarter and more incorporated, such as smart watches, wearable monitoring sensors, smartphones and even contact lenses. The authors introduce a combined IVIF VIKOR methodology for an MCDM problem for evaluating and selecting the best wearable vital sign monitoring SMD. Their evaluation criteria are compiled through an extensive literature review and experts' views. Since VIKOR contemplates the complexity of decision criteria, the ranking outcome of the proposed methodology is more accurate and realistic than other MCDM techniques. The partiality and bias of individual opinions are reduced by a group of DMs, which is preferable over a single DM. The evaluation process is enriched by the use of IVIF values. The IVIF theory is able to prevent data loss and to assist with the integration of linguistic non-numerical statements into analytic numerical models. The information gathered this way are then evaluated by means of the IVIF VIKOR methodology, which is a powerful combined technique for complete or partial rankings. Even though the introduced methodology is applied for the sake of wearable vital sign monitoring evaluation, it can also be utilized for other SMD evaluations.

As a further research direction, the proposed methodology can be extended for other types of SMDs selection processes. Another area to be the examined could be the further comparison of the proposed methodology with different MCDM problems based on classical fuzzy, IF or IVIF sets.

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Cloud Computing Technology Selection Based on Interval Valued Intuitionistic Fuzzy COPRAS

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Abstract. Cloud computing technology provides virtual services based on subscriptions with an associated cost that is accessible to its users from anywhere, wherever they are. Technology brings many different benefits to companies as well as to the public by reducing the time and resources for them which would be needed for establishing and operating their own Information Technology infrastructure. The main aim in this study is to identify significant decision criteria that are relevant to the cloud computing technology selection problem among 'Infrastructure as a Service' cloud providers, to provide an effective framework to evaluate and select the most appropriate 'Infrastructure as a Service' providers and also to apply the proposed approach through an empirical study. Technology selection essentially is a difficult multi-criteria problem that deals with both quantitative and qualitative parameters, which are usually conflicting and uncertain. Interval valued intuitionistic fuzzy set is a powerful method to cope with uncertainty by taking both degree of membership and non-membership function in an interval. A multi-criteria approach based on the combination of interval valued intuitionistic fuzzy set theory and complex proportional assessment is proposed to deal with cloud computing technology selection problem in uncertain and ambiguous environment. Finally, in order to illustrate the procedure thoroughly, an application of the proposed approach is considered.

Keywords: Cloud computing technology · Infrastructure as a service provider selection · Interval valued intuitionistic fuzzy · Multi-criteria · Complex proportional assessment

1 Introduction

In recent years, cloud computing technology (CCT) is gaining high attention by bringing next-generation access to infrastructure and application services for the Information Technology (IT) industry. Following the emergence of this evolution in technological development, several companies have initiated their own CCT services to serve their customers. This offering of cloud services online over the internet has generated many decision-making problems from the view point of customers. Not every CCT service provider is the same and considering a multitude of criteria concurrently makes it crucial

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to have a CCT service provider selection framework so that customers are best able to decide on the desired service that fits their expectations. CCT has numerous advantages over conventional technologies [1], since CCT enables companies to develop and transfer their applications easily in a flexible manner. In order to extract the full potential, a key issue for customers is to ensure the fulfillment of their requirements and to fully utilize the features of the applications from CCT providers. In parallel to these recent developments in CCT, Infrastructure as a Service (IaaS) is one of the CCT models that allow companies to outsource resources and computing equipment. Several service providers, such as Microsoft, IBM, and Google, have started offering similar services with different features and prices, leading to a differentiation in the quality and the level of services. IaaS cloud providers own and maintain the equipment while their clients rent out the specific services they desire, usually on a subscription basis with a fee. Nowadays, the question is less about whether or not to use IaaS cloud services, but rather which IaaS providers to choose. This diversity in the CCT makes it a challenge for customers to discover the “right” technology that satisfies the desires and needs. Acknowledging this CCT selection problem, an effective decision support framework can benefit CCT users and create value for them.

Decision-making is quite often associated with the process of selecting the best among the set of available alternatives. In many cases when selecting the best available alternative, it is essential to consider the impact of many criteria at the same time. Since the early 1970s, multiple-criteria decision-making (MCDM) methods are being developed. MCDM today has become the main area of research in dealing with complex problems in case of multiple objectives or criteria. As a result, many different types of MCDM methodologies have been proposed, such as AHP [2], TOPSIS [3] and VIKOR [4], among others. These approaches are being successfully applied in solving many types of decision-making problems. However, most of these so-called ordinary MCDM methods are run with crisp numbers, which prove inadequate in many real-life problems. Therefore, most of MCDM methods are extended so that they can be used with fuzzy numbers. Even fuzzy numbers can be insufficient in certain environments. Given information can be limited leading to inexactness and lack of precision. To address these weaknesses of the conventional fuzzy or crisp sets, the intuitionistic fuzzy or interval valued intuitionistic fuzzy (IVIF) concept is developed. In addition, most of the decision-making problems also require the participation of more than one decision maker (DM) in decision-making processes. Hence, many MCDM methods are also extended to a group decision making (GDM) environment. The major advantage of IVIF sets over the crisp or classical fuzzy sets is that IVIF sets differentiate the positive and the negative indication for an element’s interval membership and non-membership in the set.

The Multi-Attribute Complex Proportional Assessment (COPRAS) approach was first introduced by Zavadskas et al. [5]. When it is compared to other methods, it has slightly more complex aggregation procedure which does not require transformation of benefit to cost type criteria [6]. IVIF COPRAS methodology is an MDCM method to optimize complicated systems with inconsistent criteria and focuses on the ranking of distinctive alternatives among a set of other options. IVIF numbers provide an opportunity for a much more adequate modelling and solving complex problems. The originality of the paper comes from its strength in presenting an extension of COPRAS methodology, an IVIF MCDM technique based on GDM, where the ratings of alternatives are voiced in IVIF values. There are several studies in the literature considering COPRAS methodology under intuitionistic fuzzy environment [7–9]. The proposed method has advantage over others as it is easily comprehensible and computationally simple. Even though the combination of IVIF and COPRAS has been studied already, this method differs from those approaches in a way that utilizes GDM approach. It seems that the proposed framework can be satisfactorily implemented in MCDM problems under vague and imprecise conditions. GDM is also involved in the proposed extension, in which multiple DMs can express individual ratings using IVIF values. Group performance ratings provided in this approach have more adequate determination strength and greater flexibility. This new methodology is then applied on a practical case. Doing so, this study contributes to the state of research by providing a methodology for the first time by developing an evaluation model to assess applicable CCT providers in order to handle the inexact and vague information that is fundamental in criteria.

This paper is structured as follows. The next section discusses the role of MCDM methodologies on the CCT provider selection problem. The following section present the CCT selection framework to guide customers in decision processes for the most suitable IaaS service provider among a set of available alternatives by applying the IVIF COPRAS methodology. A practical case is presented in Sect. 4 to validate the proposed methodology. The last section concludes the paper.

2 Problem Definition of CCT Selection in MCDM

MCDM techniques are used for ranking, comparing, and selecting alternatives using multiple criteria. When assessing the decision-making problems, MCDM techniques are used extensively. When there is more than one alternative available, MCDM approach is used and a decision needs to favored over others. In case of real world conditions, this type of problems frequently occur when a decision needs to be taken in the presence of multiple criteria. Deciding on a proper alternative requires available alternatives to be judged, whereby compromise or trade-offs shall also be considered. There exist numerous different types of MCDM methodologies in literature. A variety of different approaches to CCT selection have been proposed by scholars and practitioners using different factors and techniques. This study proposes an IVIF COPRAS methodology for CCT IaaS provider selection in which the criteria are determined with the help of an expert team and extensive survey of literature. Accordingly, these criteria affecting the provider selection are determined to be Agility, Accountability, Cost, Reliability of Service, Response Speed, Performance, Latency, Usability, Security and Privacy. Agility of an organization is one of the most important advantages of CCT. The organization can adapt and change quickly without excessive costs. A flexible and portable CCT service is considered agile [10]. Accountability measures various CCT providers' specific characteristics. This factor is important as it translates into establishing trust with the customer for any CCT provider [10]. Cost criterion arises as a first question in mind before switching to CCT. Therefore, cost is clearly the vital attribute for CCT selection [10]. Reliability stands for the likelihood of a CCT service performing as expected [11]. Response Speed is a criterion that measures the efficiency of service availability. The response time depends on various factors and sub-factors [11]. Performance is needed to understand how successfully their applications will perform and whether it will meet their expectations [12]. Latency criterion is described as the time it takes for any requests to reach and get back to the user from the virtual machine in the cloud [13]. Usability plays an important role for the rapid adoption of CCT services [10]. Security and Privacy is one of the most important criteria that every organization is concerned with about data protection and privacy [10].

3 Proposed Methodology

The IVIF COPRAS methodology employs a stepwise ordering and assessing procedure of the alternatives with respect to evaluation criteria.

3.1 Preliminaries

Zadeh [14] proposed ordinary fuzzy sets in which a membership value is given to each element of a set. Later, Atanassov [15] introduced the concept of intuitionistic fuzzy sets, where a non-membership value is assigned to each element of the set alongside its membership value. IVIF sets the extended versions of intuitionistic fuzzy sets [16]. Here, X is a given fixed set. An IVIF set in x in $\tilde{A}$ defined in (Eq. 1) as the basic component of an IVIF set that is an ordered pair, characterized by an interval valued membership value and an interval valued non-membership value. $\tilde{A}$ is called IVIF set, where $\tilde{\mu}_{\tilde{A}}(x) \subset [0,1]$ and $\tilde{v}_{\tilde{A}}(x) \subset [0,1]$, $x \in X$ with the condition of sup $\tilde{\mu}_{\tilde{A}}(x) +$ sup $\tilde{v}_{\tilde{A}}(x) \leq 1$.

$$\tilde{A} = \left\{ \left\langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{v}_{\tilde{A}}(x) \right\rangle | x \in X \right\} \tag{1}$$

For convenience, the IVIF set lower and upper end points are denoted by $\tilde{A} = \left[\mu_{\tilde{A}}^L, \mu_{\tilde{A}}^U\right], \left[v_{\tilde{A}}^L, v_{\tilde{A}}^U\right]$ or $\tilde{B} = \left[\mu_{\tilde{B}}^L, \mu_{\tilde{A}}^U \mu_{\tilde{B}}^U\right], \left[v_{\tilde{B}}^L, v_{\tilde{B}}^U\right]$. Using these two IVIF numbers, the following expressions are defined [17]:

$$\tilde{A} \leq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}^L \leq \mu_{\tilde{B}}^L, \mu_{\tilde{A}}^U \leq \mu_{\tilde{B}}^U, v_{\tilde{B}}^L \leq v_{\tilde{A}}^L, v_{\tilde{B}}^U \leq v_{\tilde{A}}^U \tag{2}$$

$$\tilde{A} + \tilde{B} = \left(\left[\mu_{\tilde{A}}^L + \mu_{\tilde{B}}^L - \mu_{\tilde{A}}^L \mu_{\tilde{B}}^L, \mu_{\tilde{A}}^U + \mu_{\tilde{B}}^U - \mu_{\tilde{A}}^U \mu_{\tilde{B}}^U\right], \left[v_{\tilde{A}}^L v_{\tilde{B}}^L, v_{\tilde{A}}^U v_{\tilde{B}}^U\right] \right) \tag{3}$$

$$\tilde{A} * \tilde{B} = \left(\left[\mu_{\tilde{A}}^L \mu_{\tilde{B}}^L, \mu_{\tilde{A}}^U \mu_{\tilde{B}}^U\right], \left[v_{\tilde{A}}^L + v_{\tilde{B}}^L - v_{\tilde{A}}^L v_{\tilde{B}}^L, v_{\tilde{A}}^U + v_{\tilde{B}}^U - v_{\tilde{A}}^U v_{\tilde{B}}^U\right] \right) \tag{4}$$

$$\lambda\tilde{A} = \left(\left[1 - (1 - \mu_{\tilde{A}}^L)^{\lambda}\right], \left[1 - \left(1 - \mu_{\tilde{A}}^U\right)^{\lambda}\right], \left[v_{\tilde{A}}^{L^{\lambda}}\right], \left[v_{\tilde{A}}^{U^{\lambda}}\right] \right) \tag{5}$$

$$\tilde{A}^{\lambda} = \left((\mu_{\tilde{A}}^L)^{\lambda}, \left(\mu_{\tilde{A}}^U\right)^{\lambda}, \left[1 - (1 - v_{\tilde{A}}^L)^{\lambda}, \left[1 - \left(1 - v_{\tilde{A}}^U\right)\right]^{\lambda}\right] \right) \tag{6}$$

$$\tilde{A}-\tilde{B}=\left(\left[\frac{\mu_{\tilde{A}}^{L}-\mu_{\tilde{B}}^{L}}{1-\mu_{\tilde{B}}^{L}},\frac{\mu_{\tilde{A}}^{U}-\mu_{\tilde{B}}^{U}}{1-\mu_{\tilde{B}}^{U}}\right],\left[\frac{v_{\tilde{A}}^{L}}{v_{\tilde{B}}^{L}},\frac{v_{\tilde{A}}^{U}}{v_{\tilde{B}}^{U}}\right]\right),\frac{\tilde{A}}{\tilde{B}}=\left(\left[\frac{\mu_{\tilde{A}}^{L}}{\mu_{\tilde{B}}^{L}},\frac{\mu_{\tilde{A}}^{U}}{\mu_{\tilde{B}}^{U}}\right],\left[\frac{v_{\tilde{A}}^{L}-v_{\tilde{B}}^{L}}{1-v_{\tilde{B}}^{L}},\frac{v_{\tilde{A}}^{U}-v_{\tilde{B}}^{U}}{1-v_{\tilde{B}}^{U}}\right]\right) \quad (7)$$

3.2 IVIF COPRAS Methodology

The steps of the IVIF COPRAS methodology is as follow:

Step 1: Get Judgments of DMs.

DMs are asked to express their opinions on the alternative A_i over criterion C_j from the viewpoint of the k^{th} DM, based on their prior knowledge and expertise on the topics.

Step 2: Transform linguistic variables into IVIF numbers.

Since linguistic variables are not mathematically operable, DMs apply these linguistic variables that are listed in Table 1 for voicing their judgments about the alternatives for each of the criteria, as well as about the weights of each criterion in the overall decision.

Table 1. Linguistic variables for rating alternatives [19].

Linguistic terms		$[\mu^L, \mu^U]$	$[\nu^L, \nu^U]$
Extremely good	EG	0.00, 0.20	0.50, 0.80
Very good	VG	0.10, 0.30	0.40, 0.70
Medium good	MG	0.20, 0.40	0.30, 0.60
Good	G	0.30, 0.50	0.20, 0.50
Approximately equal	AE	0.40, 0.60	0.20, 0.40
Bad	B	0.50, 0.70	0.10, 0.30
Medium bad	MB	0.60, 0.80	0.00, 0.20
Very bad	VB	0.70, 0.90	0.00, 0.10
Extremely bad	EB	0.80, 1.00	0.00, 0.00

Step 3: Determine DMs' weights.

Equation (8) is used to determine K DMs' weights [18]. Linguistic importance scale in Table 1 is used to assess the importance of DMs, $\lambda^k(1<k<K)$ Where $[\pi^L, \pi^U]$ represents the hesitancy degree (unknown degree) of an IVIF sets and $\pi^L = 1 - \nu^U - \mu^U$, $\pi^U = 1 - \nu^L - \mu^L$.

$$\lambda^k=\frac{\sqrt{\frac{1}{2}\left[\left(1-\pi_{\tilde{A}}^{L^k}\right)^2+\left(1-\pi_{\tilde{A}}^{U^k}\right)^2\right]}}{\sum_{l=1}^{K}\sqrt{\frac{1}{2}\left[\left(1-\pi_{\tilde{A}}^{L^l}\right)^2+\left(1-\pi_{\tilde{A}}^{U^l}\right)^2\right]}} \quad (8)$$

Step 4: Calculate the criteria weights.

The weight vectors $w_1, w_2, \ldots w_n$ with $w_j \geq 0, j = 0, j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$ are determined to capture the relative significance of different criteria by using Eq. (9).

$$\mathrm{w_j} = \frac{1 - \breve{\mathrm{w}}_\mathrm{j}}{\mathrm{n} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \breve{\mathrm{w}}_\mathrm{j}}, \quad \breve{\mathrm{w}}_\mathrm{j} = 1 - \frac{\sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\mathrm{w_j}\left(\mu^{\mathrm{L}}_{\tilde{\mathrm{A}}_{\mathrm{ij}}} + \mu^{\mathrm{U}}_{\tilde{\mathrm{A}}_{\mathrm{ij}}}\right)}{2}}{\sqrt{\sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\mathrm{w_j}\left(\mu^{\mathrm{L}2}_{\tilde{\mathrm{A}}_{\mathrm{ij}}} + \mu^{\mathrm{U}2}_{\tilde{\mathrm{A}}_{\mathrm{ij}}} + \nu^{\mathrm{L}2}_{\tilde{\mathrm{A}}_{\mathrm{ij}}} + \nu^{\mathrm{U}2}_{\tilde{\mathrm{A}}_{\mathrm{ij}}}\right)}{2}}} \tag{9}$$

Step 5: Construct the aggregated matrix.

Aggregate IVIF numbers into group TIFNs to evaluate alternatives $A_1, A_2, \ldots, A_m$ based on criteria $C_1, C_2, \ldots, C_n$ IVIF weighted averaging (IIFWA) operator [20] is used for aggregation as shown in Eq. (10). The merged IVIF decision matrix is formed for each alternative based on opinions collected from all experts. Let $\mathrm{X}_{(\mathrm{k})} = \left(\mathrm{x}_{\mathrm{ij}}^{(\mathrm{k})}\right)_{\mathrm{m}\times\mathrm{n}}$ be an IVIF decision matrix of the k^{th} DM for the alternatives. Using DMs' weights $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$ calculated in step 3, individual opinions of DMs are merged into the IVIF decision matrix.

$$\mathrm{IIFWA} = \left(\left[1 - \prod_{\mathrm{j}=1}^{\mathrm{n}} \left(1 - \mu^L_{\tilde{A}}\right)^{\lambda^k}, 1 - \prod_{\mathrm{j}=1}^{\mathrm{n}} \left(1 - \mu^U_{\tilde{A}}\right)^{\lambda^k}\right], \left[\prod_{\mathrm{j}=1}^{\mathrm{n}} \left(\nu^L_{\tilde{A}}\right)^{\lambda^k}, \prod_{\mathrm{j}=1}^{\mathrm{n}} \left(\nu^U_{\tilde{A}}\right)^{\lambda^k}\right]\right) \tag{10}$$

$$\tilde{X} = \begin{pmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{21} & & \tilde{x}_{i1} & \cdots & \tilde{x}_{m1} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \tilde{x}_{12} & \cdots & \tilde{x}_{22} & & \tilde{x}_{i2} & \cdots & \tilde{x}_{m2} \\ & \vdots & & \ddots & & \vdots & \\ \tilde{x}_{1j} & \cdots & \tilde{x}_{2j} & & \tilde{x}_{ij} & \cdots & \tilde{x}_{mj} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \tilde{x}_{1n} & \cdots & \tilde{x}_{2n} & & \tilde{x}_{in} & \cdots & \tilde{x}_{mn} \end{pmatrix} \tag{11}$$

where, $\tilde{x}_{ij} = \left[\tilde{\mu}^L_{\tilde{A}}\left(\tilde{x}_{ij}\right), \tilde{\mu}^U_{\tilde{A}}\left(\tilde{x}_{ij}\right)\right], \left[\tilde{\nu}^L_{\tilde{A}}\left(\tilde{x}_{ij}\right), \tilde{\nu}^U_{\tilde{A}}\left(\tilde{x}_{ij}\right)\right]$ denote IVIF values representing the aggregated performance value of the i alternative in terms of the j criterion.

Step 6: Establish the normalized decision matrix.

Normalize the IVIF decision matrix using the Eqs. (12 and 13).

Establish normalized matrix $\tilde{R} = \left[\tilde{r}_{ij}\right]_{m\times n}$ with $\tilde{r}_{ij} = \left[a^L_{ij}, a^U_{ij}\right], \left[b^L_{ij}, b^U_{ij}\right]$.

$$a^L_{ij} = \frac{\mu^L_{ij}}{\left(\sum_{l=1}^{m}\left(\left(\mu^L_{lj}\right)^2 + \left(\mu^U_{lj}\right)^2\right)\right)^{\frac{1}{2}}} \quad \text{and} \quad a^U_{ij} = \frac{\mu^U_{ij}}{\left(\sum_{l=1}^{m}\left(\left(\mu^L_{lj}\right)^2 + \left(\mu^U_{lj}\right)^2\right)\right)^{\frac{1}{2}}} \tag{12}$$

$$b_{ij}^{L} = \frac{v_{ij}^{L}}{\left(\sum_{l=1}^{m}\left(\left(v_{lj}^{L}\right)^{2} + \left(v_{lj}^{U}\right)^{2}\right)\right)^{\frac{1}{2}}} \quad \text{and} \quad b_{ij}^{U} = \frac{v_{ij}^{U}}{\left(\sum_{l=1}^{m}\left(\left(v_{lj}^{L}\right)^{2} + \left(v_{lj}^{U}\right)^{2}\right)\right)^{\frac{1}{2}}} \tag{13}$$

Step 7: Establish the weighted-normalized decision matrix.

Use the criteria weights calculated in step 4, and construct the normalized weighted values of the criteria with Eq. (14). The weighted normalized decision matrix $\dot{\tilde{R}} = \left[\dot{\tilde{r}}_{ij}\right]_{m \times n}$ is formed.

$$\dot{\tilde{R}} = \tilde{r}_{ij} * w_i. \tag{14}$$

Step 8: Calculate the sum of all criteria for each alternative where larger values are more preferable.

$$\tilde{Z}_{\mathrm{i}} = \sum_{\mathrm{j}=1}^{\mathrm{g}} \dot{\tilde{r}}_{ij}. \tag{15}$$

where j = 1, 2, …, g.

Step 9: Calculate the sums of all criteria for each alternative where smaller values are more preferable.

$$\tilde{\mathrm{T}}_{\mathrm{i}} = \sum_{j=\mathrm{g}+1}^{\mathrm{n}} \dot{\tilde{r}}_{ij} \tag{16}$$

where g = g + 1, g + 2, …, n.

Step 10: Defuzzify $\tilde{Z}_{\mathrm{i}}$ and $\tilde{\mathrm{T}}_{\mathrm{i}}$ values [21] using the Eq. (17).

$$1 - \frac{\mu_{\tilde{A}}^{L} + \mu_{\tilde{A}}^{U} + \left(1 - \nu_{\tilde{A}}^{L}\right) + \left(1 - \nu_{\tilde{A}}^{U}\right) + \mu_{\tilde{A}}^{L} * \mu_{\tilde{A}}^{U} - \sqrt{\left(1 - \nu_{\tilde{A}}^{L}\right) + \left(1 - \nu_{\tilde{A}}^{U}\right)}}{4} \tag{17}$$

Step 11: Calculate the relative importance of each alternative.

$$\mathrm{Q}_{\mathrm{i}} = \mathrm{Z}_{\mathrm{i}} + \frac{\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{T}_{\mathrm{i}}}{\mathrm{T}_{\mathrm{i}} \sum_{\mathrm{i}=1}^{\mathrm{m}} \frac{1}{\mathrm{T}_{\mathrm{i}}}} \tag{18}$$

Step 12: Find the maximum relative importance.

$$\mathrm{Q}_{\max} = \max(\mathrm{Q}_{\mathrm{i}}) \forall \mathrm{i} = 1, 2, \ldots \mathrm{m} \tag{19}$$

Step 13: Rank alternatives based on the performance index for each alternative.

The alternative with a value of 100 in the performance index "$I_i = 100$" is the best one. The ranking is completed in descending order of the performance index for each alternative.

$$I_i = \frac{Q_i}{Q_{max}}.100\% \tag{20}$$

4 Practical Case

This section presents a practical case for a company, hereafter called ABC, which plans to select IaaS cloud service provider for supporting its operations. There are three DMs; DM_1, DM_2, and DM_3 and four candidate CCT service providers; A_1, A_2, A_3, and A_4 for the final evaluation. In order to evaluate candidate CCTs, the following nine criteria are considered as evaluation factors, Agility (C_1), Accountability (C_2), Cost (C_3), Reliability of Service (C_4), Response Speed (C_5), Performance (C_6), Latency (C_7), Usability (C_8), Security and Privacy (C_9) as presented in Sect. 3.

Step 1: DMs' opinions are displayed in Table 2. DMs give their judgment on each factor in linguistic terms.

Step 2: DMs opinions are transformed into IVIF values using the linguistic variables in Table 1. Due to space limitations, the following Table 3 displays the transformed IVIF values of A_1 and A_2 only.

Table 2. Ratings of alternatives and criteria by DMs

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	MB	VG	G	VG	G	B	VG	VG	VG
A_2	MB	VG	MB	VG	MB	B	G	MB	VG
A_3	EG	G	G	VG	VG	EG	EG	G	B
A_4	G	VG	VG	MG	VG	VB	VG	G	VB
C_j	VG	MB	B	G	MG	VG	EG	VB	VG
A_1	VG	EG	MB	VG	EG	G	G	B	VG
A_2	VG	VG	VG	VG	B	G	EG	VG	G
A_3	MB	VG	VG	VG	VB	G	G	MB	VG
A_4	MG	VG	B	VG	EG	VG	G	G	MG
C_j	MB	VG	G	VG	G	B	VG	VG	VG
A_1	MB	VG	MB	VG	MB	B	G	MB	VG
A_2	EG	G	G	VG	VG	EG	EG	G	B
A_3	G	VG	VG	MG	VG	VB	VG	G	VB
A_4	VG	MB	B	G	MG	VG	EG	VB	VG
C_j	VG	EG	MB	VG	EG	G	G	B	VG

Table 3. Transformed IVIF values

		DM1		DM2		DM3	
		$[\mu^L, \mu^U]$	$[\nu^L, \nu^U]$	$[\mu^L, \mu^U]$	$[\nu^L, \nu^U]$	$[\mu^L, \mu^U]$	$[\nu^L, \nu^U]$
A_1	C_1	0.35, 0.44	0.45, 0.60	0.25, 0.34	0.55, 0.70	0.35, 0.44	0.45, 0.60
	C_2	0.35, 0.44	0.45, 0.60	0.25, 0.34	0.55, 0.70	0.55, 0.64	0.25, 0.40
	C_3	0.45, 0.54	0.35, 0.50	0.45, 0.54	0.35, 0.50	0.85, 0.95	0.04, 0.15
	C_4	0.65, 0.74	0.15, 0.30	0.15, 0.24	0.65, 0.80	0.75, 0.84	0.05, 0.20
	C_5	0.75, 0.84	0.05, 0.20	0.75, 0.84	0.05, 0.20	0.65, 0.74	0.15, 0.30
	C_6	0.75, 0.84	0.05, 0.20	0.65, 0.74	0.15, 0.30	0.65, 0.74	0.15, 0.30
	C_7	0.75, 0.84	0.05, 0.20	0.65, 0.74	0.15, 0.30	0.35, 0.44	0.45, 0.60
	C_8	0.35, 0.44	0.45, 0.60	0.65, 0.74	0.15, 0.30	0.25, 0.34	0.55, 0.70
	C_9	0.55, 0.64	0.25, 0.40	0.85, 0.95	0.04, 0.15	0.15, 0.24	0.65, 0.80
A_2	C_1	0.75, 0.84	0.05, 0.20	0.85, 0.95	0.04, 0.15	0.75, 0.84	0.05, 0.20
	C_2	0.75, 0.84	0.05, 0.20	0.65, 0.74	0.15, 0.30	0.85, 0.95	0.04, 0.15
	C_3	0.65, 0.74	0.15, 0.30	0.45, 0.54	0.35, 0.50	0.65, 0.74	0.15, 0.30
	C_4	0.85, 0.95	0.04, 0.15	0.75, 0.84	0.05, 0.20	0.75, 0.84	0.05, 0.20
	C_5	0.35, 0.44	0.45, 0.60	0.45, 0.54	0.35, 0.50	0.45, 0.54	0.35, 0.50
	C_6	0.45, 0.54	0.35, 0.50	0.65, 0.74	0.15, 0.30	0.65, 0.74	0.15, 0.30
	C_7	0.75, 0.84	0.05, 0.20	0.45, 0.54	0.35, 0.50	0.25, 0.34	0.55, 0.70
	C_8	0.75, 0.84	0.05, 0.20	0.65, 0.74	0.15, 0.30	0.15, 0.24	0.65, 0.80
	C_9	0.85, 0.95	0.04, 0.15	0.65, 0.74	0.15, 0.30	0.45, 0.54	0.35, 0.50

Step 3: DMs' weights are determined by the Eq. (8), as displayed in Table 4. The highest importance is given to the first DM and the least importance is assigned to the third DM.
Step 4: Criteria weights are calculated using Eq. (9) and the results are displayed in Table 4. The highest importance is given to C_9 and the least importance is given to C_2.

Table 4. DMs and criteria weights

Weights		Weights		Weights	
DM_1	0.3711	DM_2	0.3429	DM_3	0.2860
C_1	0.0997	C_2	0.0643	C_3	0.1264
C_4	0.1362	C_5	0.1302	C_6	0.1085
C_7	0.0874	C_8	0.1109	C_9	0.1364

Step 5: The GDM-aggregation is done in this step with Eq. (10). Due to space limitations, only the aggregated decision matrix for A_1 and A_2 is displayed in Table 5.
Step 6: Normalization of the aggregated IVIF values is done based on Eqs. (12 and 13). Table 6 displays the normalized decision matrix.
Step 7: Weighted-Normalized decision matrix is done by the Eq. (14) using the criteria weights calculated in Step 4. The result is shown in Table 6.

Table 5. Aggregated decision matrix

	A_1		A_2	
	$[\mu^L, \mu^U]$	$[\nu^L, \nu^U]$	$[\mu^L, \mu^U]$	$[\nu^L, \nu^U]$
C_1	0.568, 0.770	0.000, 0.230	0.100, 0.300	0.400, 0.700
C_2	0.474, 0.685	0.000, 0.315	0.174, 0.376	0.315, 0.624
C_3	0.030, 0.230	0.469, 0.770	0.209, 0.413	0.274, 0.587
C_4	0.438, 0.683	0.000, 0.317	0.100, 0.300	0.400, 0.700
C_5	0.162, 0.364	0.328, 0.636	0.288, 0.522	0.000, 0.478
C_6	0.232, 0.434	0.259, 0.566	0.201, 0.405	0.281, 0.595
C_7	0.345, 0.564	0.000, 0.436	0.211, 0.425	0.290, 0.575
C_8	0.483, 0.692	0.000, 0.308	0.397, 0.642	0.000, 0.358
C_9	0.371, 0.621	0.000, 0.379	0.149, 0.352	0.336, 0.648

Table 6. The decision matrix

	Normalized				Weighted-Normalized			
	A_1		A_2		A_1		A_2	
	$[\mu^L,\mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L,\mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L,\mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L,\mu^Y]$	$[\nu^L, \nu^Y]$
C_1	0.27, 0.37	0.00, 0.15	0.07, 0.21	0.20, 0.35	0.00, 0.03	0.81, 0.85	0.04, 0.07	0.65, 0.78
C_2	0.23, 0.33	0.00, 0.20	0.12, 0.26	0.16, 0.31	0.00, 0.04	0.79, 0.84	0.03, 0.06	0.71, 0.81
C_3	0.01, 0.11	0.30, 0.50	0.14, 0.29	0.14, 0.30	0.05, 0.10	0.53, 0.72	0.02, 0.05	0.75, 0.83
C_4	0.21, 0.33	0.00, 0.20	0.07, 0.21	0.20, 0.35	0.00, 0.01	0.92, 0.94	0.01, 0.02	0.80, 0.92
C_5	0.08, 0.17	0.21, 0.41	0.20, 0.36	0.00, 0.24	0.02, 0.05	0.80, 0.86	0.00, 0.02	0.87, 0.91
C_6	0.11, 0.21	0.17, 0.37	0.14, 0.28	0.14, 0.30	0.01, 0.02	0.90, 0.93	0.01, 0.02	0.91, 0.94
C_7	0.16, 0.27	0.00, 0.28	0.15, 0.29	0.15, 0.29	0.00, 0.05	0.76, 0.82	0.02, 0.05	0.74, 0.83
C_8	0.23, 0.33	0.00, 0.20	0.27, 0.44	0.00, 0.18	0.00, 0.03	0.81, 0.85	0.00, 0.03	0.83, 0.89
C_9	0.18, 0.30	0.00, 0.24	0.10, 0.24	0.17, 0.33	0.00, 0.01	0.93, 0.95	0.01, 0.02	0.91, 0.94

Table 7. The decision matrix

	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$	$[\mu^L, \mu^Y]$	$[\nu^L, \nu^Y]$
$\tilde{Z}_i$	06, 0.20	0.23, 0.39	0.03, 0.07	0.69, 0.78	0.02, 0.07	0.70, 0.77	0.03, 0.06	0.69, 0.79
$\tilde{T}_i$	02, 0.10	0.56, 0.66	0.04, 0.10	0.56, 0.70	0.05, 0.12	0.52, 0.63	0.01, 0.07	0.63, 0.74

Step 8: The sums of all criteria for each alternative are calculated by the Eq. (15), where the larger values are preferable. Table 7 displays the calculated values.
Step 9: The sums of all criteria for each alternative are calculated by the Eq. (16), where the smaller values are preferable. Table 7 displays the calculated values.
Step 10: The $\tilde{Z}_i$ and $\tilde{T}_i$ values are defuzzified with Eq. (17).
Step 11: The relative importance of each alternative is calculated with Eq. (18). $\sum_{i=1}^{4} T_i = 3.487$, related values are given in Table 8.
Step 12: The maximum relative importance is calculated with Eq. (19) as 14.978.
Step 13: The performance index for each alternative is calculated with Eq. (20) as shown in Table 8.

Table 8. Ranking results and relevant parameters

	Z_i	T_i	Q_i	I_i	Rank
A_1	0.757	0.869	14.748	96.620	3
A_2	0.908	0.871	14.866	97.388	2
A_3	0.911	0.847	15.264	100.000	1
A_4	0.912	0.899	14.432	94.548	4

The result according to the performance index indicates that $\mathbf{A_3}$ is the best one, $\mathbf{A_2}$ is the second, $\mathbf{A_1}$ is the third and $\mathbf{A_4}$ is the last one in ranking;

$$\mathbf{A_3} > \mathbf{A_2} > \mathbf{A_1} > \mathbf{A_4}.$$

5 Concluding Remarks

This study applies the IVIF COPRAS methodology. The main aim of this study is to use an MCDM method which combines COPRAS technique with IVIF sets to evaluate a set of qualified alternatives for the sake of deciding on a best IaaS service provider in the context of CCT. In the evaluation process, each distinctive alternative is voiced using linguistic terms which were represented as IVIF values. The success of the provider selection depends on taking the precise strategic decisions. In this IaaS cloud service selection problem case, taking decisions can be difficult and costly if the activity would have to be reversed afterwards. This suggests that the method used for assessment of the most suitable IaaS cloud service provider under CCT is quite useful. An extensive literature review and a committee of DMs are used to identify the decision criteria that need to be taken into account when selecting the most appropriate CCT. In the light of these attributes, all candidate providers are evaluated with the IVIF COPRAS method. To derive the significance of the selection, the method has been applied in the IVIF environment for assessment and ranking of alternatives. Noting that IVIF MCDM is quite successful in dealing with vague and imprecise information, the method proposed in this paper can also be used for handling uncertainty in complex decision problems. A practical case is also illustrated to validate the proposed approach. This method is capable of dealing with similar types of uncertain situations in MCDM problems. For future research, other MCDM methods such as VIKOR, TOPSIS, etc. can be combined with IVIF preference relation to solve CCT selection problems. The paper presents a MCDM methodology for solving the problem considered, which is a usual practice in the cloud computing area. However, it might be interesting to include strict methodologies (e.g., optimization).

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A Hesitant Fuzzy Based TOPSIS Approach for Smart Glass Evaluation

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Abstract. Hesitant fuzzy linguistic term sets (HFLTS) are applied to better represent decision maker's (DMs') preferences in complex situations such as uncertainty in DMs' opinions and the difficulty about expressing thoughts by numbers. As an important tool, HFLTS presents a novel and strong approach for processing qualitative judgments of DMs. Therefore, this paper develops an approach based on HFLTS, ordered weighted averaging (OWA) operator and hesitant fuzzy technique for order performance by similarity to ideal solution (TOPSIS). A case study about smart glass (SG) evaluation is given to demonstrate the potential of the approach. The originality of the work comes from its evaluation methodology and its use on a case study for a logistics company. The study contributes the smart glass (SG) evaluation literature by introducing the integrated OWA Operator-Hesitant TOPSIS methodology. Since technology selection is an important subject for managers, the proposed methodology can be guided managers for an effective SG evaluation process.

Keywords: Hesitant fuzzy linguistic term sets · Multi criteria decision making · Smart glasses · TOPSIS · OWA

1 Introduction

The selection of a most suitable technology is not a simple judgment. Multiple influencing factors should be considered in an evaluation process. These factors positively affect the evaluation process. If there are a lot of factors, however, then the problem and its solution become more complex. Multi-criteria Decision Making (MCDM) techniques provide researchers with effective tools for considering these factors. In order to begin the process, evaluation criteria and alternatives should be constructed and evaluated by decision makers (DMs).

In general, DMs are inclined to voice their preferences quantitatively in wordings. Sometimes these quantitative measurements may result in misconceptions or biased results that do not completely overlap with the actual reality of events. Since it is hard to evaluate thoughts by numbers, the hesitancy concept is developed. By using hesitant fuzzy linguistic term sets, DMs have a chance to express their opinions by quantitative measurements which are closer to the truth. Answering the question of the degree of a relationship between criteria or alternatives may be "absolutely high importance" or "weakly high importance", for example. The term "high importance" is classified and

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expressed by hesitant terms. As a result, in order to obtain reliable results, the consistent hesitant fuzzy MCDM technique is generated. This study proposes a hesitant fuzzy MCDM approach.

Decision making activities in general aim to select the best available option from two or more of alternatives. There may be many factors that affect the final decision. MCDM problems deal with a certain number of alternatives that are clearly known and available to the DMs. There may be a better choice which was not considered or additional information that was not available at that time. The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) is an MCDM technique. The basic idea behind TOPSIS is that the best option is the one that is closest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS). Hesitant fuzzy TOPSIS is a useful approach that works fine with limited subjective information from DMs [1].

This article aims to seek the most feasible option for smart glasses (SGs) selection process under hesitant fuzzy environment. SGs are expected to be a daily used item in the future, also providing efficiency gains to workspace, for instance through hands-free working, real time work-data and augmented reality [2]. In related literature, there are a number of studies integrating wearable technology and MCDM tools. In contrast to literature, this paper introduces a TOPSIS approach based on hesitant fuzzy concept. To the best of the authors' knowledge, no study so far has applied the Hesitant Fuzzy TOPSIS approach on the SG selection problem. This study aims to address this research gap by using hesitant fuzzy linguistic term sets (HFLTS) to handle the hesitancy and difficulty of expressing thoughts by numbers on the SG selection problem. Briefly, the study proposes a hesitant fuzzy MCDM approach for an effective SG evaluation problem where the main objective is to decide on the most suitable SG alternative. The methodology intends to function as a decision support system for business managers who seek to identify the best SG for their needs. This integrated selection method will also be tested on a case study.

The structure of this paper is designed as follows. The next section summarizes precedent research. In Sect. 3, the proposed methods are explained in detail. In Sect. 4, an application is given to demonstrate the potential of the approach. Finally, in Sect. 5, the consequences of this research and guidance for future work will be given.

2 Literature Survey

2.1 Hesitant Fuzzy TOPSIS

In recent years, several studies have contemplated with the concept of Hesitant TOPSIS. Boltürk et al. [9] applied Hesitant Fuzzy TOPSIS to select the best hospital site. Beg and Rashid [10] proposed Hesitant Fuzzy Linguistic TOPSIS for aggregating the opinions of experts and DMs on various criteria by group decision making. Liao et al. [11] integrated Hesitant Fuzzy Sets and VIKOR to propose a combined approach to solve complex problems with HFLTS in the presence of conflicting criteria.

Zhang et al. [12] utilized Hesitant Fuzzy TOPSIS and linear programming for selecting the best supplier. Büyüközkan et al. [13] proposed Hesitant Fuzzy Linguistic AHP, Hesitant Fuzzy Linguistic TOPSIS and QFD methods and explored the applicability and effectiveness of their approach by a case study. Kahraman et al. [14] developed a Hesitant Fuzzy TOPSIS model that considers the complexity and imprecision of strategic decisions and presented a case study for an electronics company.

Zhou et al. [15] proposed Hesitant TOPSIS and Hesitant TODIM and combined it with linguistic hesitant fuzzy sets (LHFS) with the evidential reasoning (ER) method. Zhang and Wei [16] applied Hesitant Fuzzy VIKOR and Hesitant Fuzzy TOPSIS methods to select the best project alternative. Xu and Zhang [17] applied Hesitant Fuzzy TOPSIS with max deviation methods for selecting the energy policy. Rodriguez and Liu [18] combined HFLTS, OWA and Fuzzy TOPSIS in the material supplier selection problem. Liao et al. [19] applied Hesitant VIKOR to develop HFL cosine distance measure for dealing with HFL MCDM problems and applied HFL-VIKOR method on a ERP system selection case. Joshi and Kumar [20] proposed Hesitant Fuzzy TOPSIS and Hesitant Fuzzy Choquet Integral methodology. Although there are many studies and applications, Hesitant Fuzzy TOPSIS methodology integrated with the OWA operator is still the missing link in literature.

The wearable technology selection process is usually defined by different factors and brings complexity to the subject. Therefore, the proposed Hesitant MCDM methodology provides distinct benefits in evaluating different alternatives by linguistic expressions with regard to different criteria.

2.2 Evaluation Criteria for Smart Glasses

Evaluation criteria for this problem are identified with the help of a detailed literature survey about existing models and consultation with experts in the industry.

Three main criteria are identified, which are technology (C1), ergonomics (C2) and privacy (C3). Since the technological components of a SG are the key factors for its information processing capability, technology is an indispensable parameter for a SG to determine if it is good or sufficient. Practicality is also important for SG users; therefore, ergonomics is another key factor. In the modern world, technology causes several problems concerning the personal life. Therefore, the security of information and privacy is the other main decision criterion.

There are nine sub-criteria below these main criteria. They are summarized as follows:

- High performance on ubiquitous computing (C11): Personal computers and ubiquitous computing have become a part of our daily life. This can be linked with the rapid growth and development of virtual reality applications and their associated hardware [3].
- Information provision (C12) and communication network (C13): The literature search concluded that smart 3C products should be equipped with information provision, intuitive interaction, communication network, and automation [3].
- Mobility and lightness (C21): More opportunities arise through smart clothing and miniaturization of electronic devices for better mobility and higher comfort thanks to continuously improving technical functions [3].

- Aesthetic appearance (C22): The growth of the wearables market depends on its ability to balance the aesthetics of existing consumer tastes and preferences through proper styling and by overcoming design hurdles [4].
- Facility of use (user-friendly) (C23): Unless the device interface and apps are intuitive and user friendly, only eager adopters will likely use them, while the mainstream market will remain untapped [5].
- Protection and security of information (C31): Personal health information is more sensitive for human beings for a number of reasons. Therefore, privacy and consumers' acceptance for healthcare wearable devices are essential [6].
- Defense against malware (C32): Digital security is a major concern for technology users. However, mobile devices are more susceptible to changing circumstances, such as different networks, internet connection, new applications etc. in addition to being connected over the air [7].
- Seamless life integration (C33): Consumers do not wish to change their lifestyle; they instead want access to more particular information and customized outputs that enhance their activities [8].

3 Research Methodology

MCDM methods can provide alternative approaches to perceive problems in a distinct and systematic way. Experts have the option to investigate the problem and adapt the methods to their own needs [21].

The proposed approach is based on HFLTS, OWA operator and hesitant fuzzy TOPSIS as indicated in Fig. 1.

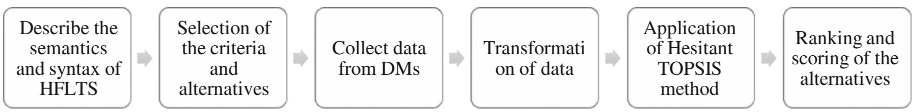

Fig. 1. The methodology of proposed approach

The approach is described in following sub-sections.

3.1 The Technique Hesitant Fuzzy Set

Hesitant Fuzzy Sets (HFS) are first presented in 2009 by Torra and Narukawa [22]. HFS describe the degree of adhesion of an element in terms of a set of possible values between 0 and 1. They are generally helpful for decision problems for processing DMs' existing hesitation during the evaluation exercise. HFS have attracted great scientific interest [22], the functioning of which is explained below with definitions.

Definition 1: X is defined as a universal set. HFS over X are defined as a function that will render a subset between 0 and 1. When applied to X, it can be presented as [23]:

$$E = \{\langle x,\ h_E(x)\rangle\ x \in X\} \tag{1}$$

Here, $h_E(x)$ is called a hesitant fuzzy element (HFE) and is defined as a set with values between [0, 1]. Possible degrees of adhesion of the element $x \in X$ to the set E are specified. H is the set of all HFE.

Rodriguez et al. [24] presents an MCDM model where DMs express their evaluations with linguistic expressions. This model presents these expressions by representing a set of hesitant fuzzy linguistic terms (HFLTS) [24].

Definition 2: X is defined as a reference set. Let HFS over X be a function h which returns values between [0, 1]:

$$h:\ X \rightarrow \{[0, 1]\} \tag{2}$$

Subsequently, an HFS is described as the union of their membership functions.

Definition 3: M = {μ_1, μ_2,..., μ_n} is defined as a set of membership functions n. HFS are linked to M. Here, h_M is defined as:

$$h_M:\ M \rightarrow \{[0, 1]\} \tag{3}$$

$$h_M(x) = \bigcup\nolimits_{\mu \in M} \{\mu(x)\} \tag{4}$$

Definition 4: The lower and upper boundaries of *h*, an HFS, are [25]:

$$h^-(x) = \min h(x) \tag{5}$$

$$h^+(x) = \max h(x) \tag{6}$$

Definition 5: When h s defined as an HFS, the envelope of *h*, $A_{env(h)}$, is defined as:

$$A_{env(h)} = \{x, \mu_A(x), \nu_A(x)\} \tag{7}$$

Here, $A_{env(h)}$ is an intuitionistic fuzzy set of *h.* The factors μ and v can be formulated as:

$$\mu_A(x) = h^-(x) \tag{8}$$

$$v_A(x) = 1 - h^+(x) \tag{9}$$

Definition 6: *S* is defined as a set of linguistic terms, $S = \{s_0, \cdots, S_g\}$. An HFLTS, *Hs,* is an ordered finite subset of the consecutive linguistic terms of S.

Definition 7: The upper and lower bounds of HFLTS *Hs*, *Hs*+ and *Hs*− respectively, are defined as:

$$H_{s+} = \max(s_i) = s_j,\ s_i \in H_S \text{ et } s_i \le s_j \forall i \tag{10}$$

$$H_{s-} = \min(s_i) = s_j,\ s_i \in H_S \text{ et } s_i \le s_j \forall i \tag{11}$$

Definition 8: Suppose that E_{GH} is a function that transforms linguistic phrases into HFLTS, $H_{S.}$ Let G_H be an out-of-context grammar that makes use of the linguistic term set in *S*. Let S_{ll} be the expression domain generated by G_H. This relationship is expressed as:

$$E_{GH}: \quad S_{ll} \to H_s \tag{12}$$

The following conversions are applied to convert comparative linguistic phrases into HFLTS;

$$E_{GH}(s_i) = \{s_i | s_i \in S\} \tag{13}$$

$$E_{GH}(\text{at most } s_i) = \{s_j | s_j \in S \text{ et } s_j \le s_i\} \tag{14}$$

$$E_{GH}(\text{lower than } s_i) = \{s_j | s_j \in S \text{ et } s_j < s_i\} \tag{15}$$

$$E_{GH}(\text{at least } s_i) = \{s_j | s_j \in S \text{ et } s_j \ge s_i\} \tag{16}$$

$$E_{GH}(\text{greater than } s_i) = \{s_j | s_j \in S \text{ et } s_j > s_i\} \tag{17}$$

$$E_{GH}(\text{between } s_i \text{ and } s_j) = \{s_k | s_k \in S \text{ et } s_i \le s_k \le s_j\} \tag{18}$$

Definition 9: An HFLTS's envelope *env*(H_S) is a linguistic with the following upper and lower values:

$$\text{env}(H_S) = [H_{s-},\ H_{s+}],\ H_{s-} \le H_{s+} \tag{19}$$

3.2 OWA Operator

This operator is used for aggregating purposes. It is between the two of 'AND' which requires compliance for all the criteria, and 'OR' which requires compliance of at least one of the criteria [26].

In order to obtain a fuzzy membership function, the fuzzy membership functions of the HFLTS are merged with this OWA operator [27].

Definition 10: $A = \{a_1, a_2, \cdots, a_n\}$ is defined as a value set for aggregation. The OWA operator F can be presented as:

$$F(a_1, a_2, \ldots, a_n) = wb^T = \sum_{i=1}^{n} w_i b_i \quad (20)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector such that $w_i \in [0, 1]$ *et* $\sum_{i=1}^{n} w_i = 1$. *Also,* b is defined as the corporate ordered value vector, where b_i represents the n^{th} highest value in A.

We compute the weight of the OWA operator with the help of fuzzy linguistic quantifiers. For a non-decreasing quantifier with respect to Q, they are formulated as:

$$w_i = Q(i/m) - Q((i-1)/m), \quad i = 1, \ldots, m \quad (21)$$

The non-decreasing relative quantifier Q can be found as [27]:

$$Q(y) = \begin{cases} 0, & y < a, \\ \frac{y-a}{b-a}, & a \le y \le b, \\ 1, & y > b, \end{cases} \quad (22)$$

Here, a, b and y are values between 0 and 1. $Q(y)$ indicates the extent to which the proportion y is compatible with the direction of the quantifier it represents.

Some non-decreasing relative quantifiers are identified by the terms “most”, “at least half” and “as far as possible”, with parameters *(a, b)* as (0.3, 0.8) (0.5, 1), respectively [28].

3.3 Hesitant Fuzzy TOPSIS Method

DMs first used hesitant fuzzy set values to evaluate the scenarios of the alternatives. Then, the matrix of the results is multiplied with the criteria weights vector and then added for the global criteria. Consequently, an order of estimation is obtained for scenarios [1].

Definition 11:

$$\tilde{X}_{ij} = \frac{1}{N}\left[\widetilde{z_{ij}^{(1)}} + \widetilde{z_{ij}^{(2)}} + \ldots + \widetilde{z_{ij}^{(N)}}\right] \quad (23)$$

where $\tilde{X}ij$ are hesitant fuzzy values assigned by the k^{th} DM. $(\widetilde{z_{ij}})_{m \times n}$ is a decision matrix qualified by hesitant fuzzy numerical values.

Definition 12: The standardized decision matrix is defined as $\tilde{R}_{ij}$ presented in (24) and (25) for alternatives on the criteria.

$$\tilde{R}_{ij} = [\tilde{r}_{ij}]_{m \times n}, i = 1, 2, \ldots m; \; j = 1, 2, \ldots n \quad (24)$$

where

$$\widetilde{r_{ij}} = \left(r_{ij}^{l}, r_{ij}^{m}, r_{ij}^{u}\right) = \left(\frac{z_{ij}^{l}}{c_j^*}, \frac{z_{ij}^{m}}{c_j^*}, \frac{z_{ij}^{n}}{c_j^*}\right), \quad i = 1, 2, 3, \ldots, m, j \in B \tag{25}$$

$$c_j^* = \max_t\left[z_{ij}^{u}\right],\ j = 1, 2, 3, \ldots, n \tag{26}$$

$$c_j^- = \min_t\left[z_{ij}^{u}\right],\ j = 1, 2, 3, \ldots, n \tag{27}$$

Definition 13: The weighted standardized decision matrix ($\tilde{V}$) can then be computed as (28).

$$\widetilde{v_{ij}} = (\widetilde{v_{ij}^{l}}, \widetilde{v_{ij}^{m}}, v_{ij}^{u}) = \widetilde{w_j} \otimes \widetilde{r_{ij}} = (w_j^l r_{ij}^l, w_j^m r_{ij}^m, w_j^u r_{ij}^u) \tag{28}$$

Definition 14: The ideal ideal solution (PIS) and the ideal negative solution (NIS) are denoted by $A\,*$ and $A-$ which can be presented in (29) and (30), respectively.

$$A^* = (v_1^*, v_2^*, \ldots v_n^*) \tag{29}$$

$$A^- = (v_1^-, v_2^-, \ldots v_n^-) \tag{30}$$

$$v_j^* = \left(\max\left\{v_{ij}^{l}\right\}, \max\left\{v_{ij}^{m}\right\}\max\left\{v_{ij}^{u}\right\}\right)\ j = 1, 2, \ldots, n \tag{31}$$

$$v_j^- = \left(\min\left\{v_{ij}^{l}\right\}, \min\left\{v_{ij}^{m}\right\}\min\left\{v_{ij}^{u}\right\}\right)\ j = 1, 2, \ldots, n \tag{32}$$

Definition 15: The vertex approach is used to calculate the distance between the two fuzzy hesitant numbers. The PIS and NIS values of the alternatives are calculated in (33) and (34), respectively.

$$d_i^* = \sum\nolimits_{j=1}^{n} d(\widetilde{v_{ij}}, \widetilde{v_j^*}), \quad i = 1, 2, \ldots, m \tag{33}$$

$$d_i^- = \sum\nolimits_{j=1}^{n} d(\widetilde{v_{ij}}, \widetilde{v_j^-}), \quad i = 1, 2, \ldots, m \tag{34}$$

$$\text{Alternative scenario}_{(1,\ldots,6)}^{*} = [(1,1,1), (1,1,1), (1,1,1), (1,1,1), (1,1,1), (1,1,1)] \tag{35}$$

$$\text{Alternative scenario}_{(1,\ldots,6)}^{-} = [(0,0,0), (0,0,0), (0,0,0), (0,0,0), (0,0,0), (0,0,0)] \tag{36}$$

Definition 16: $d(d_i^*, d_i^-)$ is defined as the distance between two hesitant numbers which help to compute the proximity coefficient (CC_i).

$$CC_i = \frac{d_i^-}{d_i^* + d_i^-} \quad i = 1, 2, \ldots, m \tag{37}$$

4 Implementation of the Proposed Model to Smart Glass Selection

The model's applicability will be tested on a logistics company, called hereafter as ABC, that intends to find the best technologic wearable product. Sometimes, operations in the warehouse decelerate the work flow. The warehouse manager decides to solve this problem and start a new project with the main objective to augment the efficiency of warehouse operations. After a series of meetings, the team decides to implement a smart glass technology for warehouse operations. In order to select the best SG in the market, the Research & Development department analyses different types of smart glasses in the market and decides to evaluate them by using scientific methods. Evaluation criteria are identified based on academic research and thoughts of industrial experts. The criteria and sub-criteria are determined. Accordingly, C1 is technology (sub-criteria: high performance on ubiquitous computing (C11), information provision (C12) and communication network (C13)); C2 is ergonomics (sub-criteria: mobility and lightness (C21), esthetic appearance (C22), facility of using (user-friendly) (C23)) and C3 is privacy (sub-criteria: protection & security of information (C31), defense against malware (C32), seamless life integration (to fit into user's life) (C33)). After a long market research, the company finds four possible alternatives: A1 is Google Glass, A2 is KiSoft Vision, A3 is Vuzix M100 Smart Glasses and A4 is Epson Moverio BT200. There are three DMs; the warehouse manager, IT manager and technology & innovation department manager of the company. Computational steps are as follows:

Step 1. In the first stage, three DMs evaluated the criteria by using linguistic expressions. The evaluations the first DM with linguistic expressions are shown in Table 1.

Step 2. Tables with linguistic hesitant expressions are transformed to fuzzy values on a scale as given in Fig. 2.

Step 3. OWA operator is used to form the decision matrix. Using (21) and (22), the matrix is obtained as a function of aggregation. In addition, quantifiers are identified by the terms "most" with parameters (a, b): (0.3, 0.8). Then, this matrix is normalized with the multiplication of weights.

Step 4. The normalized matrix is constructed using (22) so that TOPSIS can be implemented. Then, standardized decision matrix $(\tilde{R}_{ij})$ that defined as presented in (24) is created using (25–27).

Step 5. The PIS and the NIS values of the scenarios are computed by (33–36), respectively. The point (1, 1, 1, 1) is used for calculating the ideal distances, and the point (0, 0, 0, 0) is used for calculating the ideal negative distances.

Table 1. The linguistic decision matrix of DM1

Alternatives	C11	C12	C13
Google Glass (A1)	At least G	Between M and VG	At least VG
KiSoft Vision (A2)	At least VG	At least G	At least VG
Vuzix M100 (A3)	At least VG	At least G	At least G
Epson Moverio BT200 (A4)	At least G	At least VG	At least G
Alternatives	C21	C22	C23
Google Glass (A1)	Between M and VG	Between VB and M	Between M and VG
KiSoft Vision (A2)	Between M and VG	Between B and G	Between B and G
Vuzix M100 (A3)	Between M and VG	Between B and G	Between B and G
Epson Moverio BT200 (A4)	Between B and G	At most VB	Between Vb and M
Alternatives	C31	C32	C33
Google Glass (A1)	At least VG	At least G	At least VG
KiSoft Vision (A2)	At least VG	At least G	At least VG
Vuzix M100 (A3)	At least VG	Between M and VG	Between M and VG
Epson Moverio BT200 (A4)	At least G	At least VG	Between M and VG

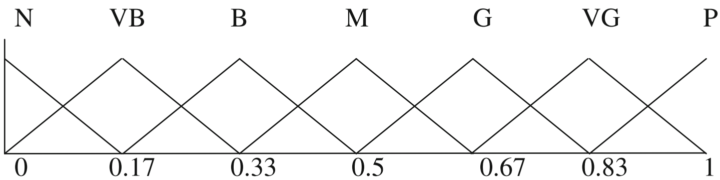

Fig. 2. Table of seven boundaries with its semantics [10]

Step 6. The proximity coefficient *(CCi)* for each alternative is computed as a function of (37). *(CCi)* measures the effectiveness of each alternative. The best alternative and the order of the alternatives are obtained according to this measure.

The results about the alternatives give an idea about the best technologic wearable product. As a result, Google Glass (A1) (CC_1:0.802) is the most desirable product among these alternatives, slightly ahead of its nearest competitor, Kisoft Vision (A2) (CC_2:0.788). Vuzix M100 Smart Glasses (A3) (CC_3:0.746) ranks the third, and the last one is Epson Moverio BT200 (A4) (CC_4:0.692).

$$A1 > A2 > A3 > A4$$

5 Conclusion

This article presents a novel integrated hesitant fuzzy MCDM approach that is based on HFLTS, OWA operator, hesitant fuzzy TOPSIS. A case study is used to demonstrate the applicability of the method for selecting the most suitable SG alternative.

There is a lack of SG selection applications in the literature. Evaluating the influencing factors for the SG selection is difficult when comparing different alternatives. This is because expressing human thoughts in the form of numbers can introduce complexity and hesitancy for DMs. Complexity and hesitancy of this type of problems makes decision making even more difficult. As an advantage, HFLTS allow us to express DMs' opinions with the help of linguistic term sets. The use of HFLTS facilitates such expressions that include hesitancy and provides the possibility of giving preference information by words. The main contribution of this study to literature is its application of hesitant fuzzy TOPSIS methodology with OWA operator and its application on SG selection area for the first time. Therefore, this study fills the gap by using hesitant fuzzy MCDM methodology in this specific SG field.

This problem is illustrated by a case study in the logistics sector and its results are presented. The evaluation criteria and SG alternatives are identified with a literature and product review. This study can be useful to researchers to better understand the hesitant fuzzy MCDM problem practically. Companies can also benefit from this study for their decision problems about wearable devices. The number of criteria or alternatives can differ for different purposes. In the future, research can be directed to the use of other MCDM methods such as Hesitant Fuzzy AHP and Hesitant Fuzzy VIKOR for solving this problem and comparing the results with the presented approach.

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On Topological Entropy of Zadeh's Extension Defined on Piecewise Convex Fuzzy Sets

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Abstract. As the main result of this article we prove that a given continuous interval map and its Zadeh's extension (fuzzification) to the space of fuzzy sets with the property that α-cuts have at most m convex (topologically connected) components, for m being an arbitrary natural number, have both positive (resp. zero) topological entropy. Presented topics are studied also for set-valued (induced) discrete dynamical systems. The main results are proved due to variational principle describing relations between topological and measure-theoretical entropy, respectively.

1 Introduction

Recently, many authors have been interested in analyzing dynamics of extensions of discrete dynamical systems (see e.g. [1,6,7,15,18]). The motivation for this interest comes from several facts. On one hand, numerical simulations that sometimes are used to describe the dynamics can be affected by various round off effects or estimates of initial states or parameters. On the other hand, the behavior of discrete dynamical systems can exhibit several "sensitive dependencies to initial conditions", that is, close initial conditions may lead to different dynamical behaviors. Hence, it is interesting to analyze the dynamics not on single points, but on suitable subsets or other objects generated over the phase space. For instance, nowadays there are many objects (soft sets, fuzzy sets, raw sets etc.) which allows to work with different kinds of uncertainty on the state space.

To fix these ideas, let $f : X \to X$ be a continuous map on a compact metric space X. The idea is to consider a suitable family $\mathcal{K}$ of subsets of X and then to extend the map f naturally to $\mathcal{K}$ (we will give more precise definitions in the next section). Then, it is quite interesting and natural to ask what are dynamical properties of this extension, e.g. topological entropy, chaoticity, rigidity,..., related to the same property or similar dynamical properties of the original crisp map f. Analogous question is natural for other systems, e.g., of soft, fuzzy, raw sets, generated over the space X. In principle, other natural families to be considered are compact or connected subsets, convex subsets (if the space X is

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convex) and, in general, any topologically relevant family of subsets of X. Some papers in this direction are for instance [6,7,15,16,19,20]. Many relevant papers are mentioned therein as well.

The question above was stated for a compact metric space. Although properties of general dynamical systems are studied since H. Poincaré's work from early 19th century [2]. Approximately in the last 30 years interest in dynamics of low-dimensional spaces and their extensions increased rapidly. There is a natural need to study these systems. Low-dimensional systems (like intervals, graphs, circles etc.) often allow to study phenomena which appear in high-dimensional systems - for instance, transitive one-dimensional maps are models of attractors of arbitrary maps of positive topological entropy [4]. Low-dimensional systems are often used to create more complicated dynamical systems on spaces of higher dimensions - probably the most simple example are skew-product (triangular) maps intensively studied since Kolyada's paper [14]. Although we mention some general facts, main results are stated for the simplest one-dimensional maps, namely for interval maps, which is the first step of understanding dynamics of fuzzy dynamical systems generated by them.

This topic (studying topological entropy of fuzzy dynamical systems) is authors' long-time project. Topological entropy is a classical property in topological dynamics (e.g. [3,5]). There are several attempts how to study this concept in fuzzy setting (see e.g. [12,13]). It is worth mentioning that our papers are the first in which the size of topological entropy of various fuzzy dynamical systems is studied. For example, we found a difference in chaotic behavior of induced and fuzzy dynamical systems [6]. In the same paper we showed that, on general compact metric spaces, topological entropy can easily reach its maximal value. In contrast to this, we also studied the simplest fuzzy dynamical systems with nontrivial fuzzy numbers, i.e. fuzzy dynamical systems on the space of fuzzy numbers generated by interval maps, and showed that their size of topological entropy is reasonable [7]. Moreover, we studied influence of degree of fuzziness on the size of topological entropy [8] which lead to better understanding of behavior of such systems. This contribution is a natural continuation of this work. The main aim of this paper is to extend this study to more general family of fuzzy sets on a compact interval, i.e. we study fuzzy sets for which their level sets (α-cuts) have at most m connected (convex) components, for some positive integer m. We will study the topological entropy of the Zadeh's extension induced by the crisp interval map, finding that it is positive if and only if the topological entropy of the crisp map it is positive. However, we will show that both entropies are not the same when they are positive, providing a formula for computing the topological entropy of the Zadeh's extension from that of the crisp map.

The paper is organized as follows. The next section is devoted to introduce the basic definitions and notation which are necessary to understand the paper. The main results can be found in Sect. 3. Finally, we finish this paper by obtaining some conclusions on variation of topological entropy called dynamical fuzzy entropy.

2 Basic Notions

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map. We note $f \in C(X)$. As usually, a pair (X, f) forms a *discrete dynamical system.* In the case of need we emphasize the dependence on X by the notation d_X for the metric on X. Further, we define systems $\mathcal{K}(X) = \{K \subseteq X | K$ is compact and nonempty$\}$ endowed with the Hausdorff metric D_X given by

$$D_X(A, B) = \max\{d(A, B), d(B, A)\},$$

where

$$d(A, B) = \max\{d(a, B) : a \in A\},$$

and

$$d(a, B) = \min\{d(a, b) : b \in B\}$$

for any $A, B \in \mathcal{K}(X)$. It is well known that the space $(\mathcal{K}(X), D_X)$ equipped with the metric (Vietoris) topology induced by D_X is compact, complete and separable if the original space X has the same three properties.

In this paper we distinguish several maps induced by f. First, we deal with a set-valued extension - a continuous map $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined in a very natural way by $\bar{f}(K) = f(K)$ for any $K \in \mathcal{K}(X)$.

For a discrete dynamical system (X, f) and a given point $x \in X$, an n-th *iteration* of the point x is defined inductively by

$$f^0(x) = x \text{ and } f^{n+1}(x) = f(f^n(x))$$

for any $n \in \mathbb{N}$. Then, the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ of all iterations of x is called a *trajectory* and points of this trajectory form an *orbit* of the point x. Limit points of $\{\varphi^n(x)\}_{n\in\mathbb{N}}$ are *ω-limit points* of the point x, and their union $\omega_f(x)$ (resp. $\omega(x, f)$) of all ω-limit points of the point x is the *ω-limit set* of the point x with respect to the map f. Finally, an ω-limit set $\omega(f)$ of the map f we mean $\omega(f) = \bigcup_{x\in X} \omega_f(x)$. We say that $A \subseteq X$ is *invariant* if $f(A) \subseteq A$ It is well known that every ω-limit set is nonempty and invariant (even $f(\omega_f(x)) = \omega_f(x)$).

Below we distinguish several types of ω-limit points - we say that the point $x \in X$ is *fixed* if $f(x) = x$ or *periodic* if $f^k(x) = x$ for some $k \in \mathbb{N}$. Then the minimal integer k satisfying this condition is a *period* of x. Moreover, $P(f)$ denotes the set of all periodic points of f, respectively. Finally, we say that two discrete dynamical systems (X, f), (Y, g) are *conjugate* if there exists a continuous bijection $h : X \to Y$ (called a *conjugacy*) for which $h \circ f = g \circ h$. If h is a continuous surjection only then we speak about a *semiconjugacy*. It is well known that the topological entropy defined below is a conjugacy invariant, i.e. two conjugate dynamical systems have the same topological entropy.

2.1 Topological and Measure-Theoretical Entropy

Let (X, f) be a discrete dynamical system. In this subsection we introduce the Bowen's definition of *topological entropy* (see [5]). Let $K \subseteq X$, $\varepsilon > 0$ and $n \in \mathbb{N}$ be fixed. We say that a set $E \subseteq K$ is (n, ε, K, f)*-separated* (by the map f) if for any $x, y \in E$, $x \neq y$, there exists $k \in \{0, 1, ..., n-1\}$ such that $d(f^k(x), f^k(y)) > \varepsilon$. We denote by $s_n(\varepsilon, K, f)$ the cardinality of the maximal (n, ε, K, f) -separated set in K and define

$$s(\varepsilon, K, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, K, f).$$

The *topological entropy* of f is

$$h_d(f) = \lim_{\varepsilon\to 0} s(\varepsilon, X, f).$$

If the space X is not compact, we consider the following definition [9] of topological entropy

$$\mathrm{ent}(f) = \sup\{h(f|_K) : K \in \mathcal{K}_f(X)\}, \tag{1}$$

where $\mathcal{K}_f(X)$ denotes the set of all f*-invariant* compact subsets of X. It is easy to see that the size of the topological entropy depends on the choice of a metric d. It is also easy to see that, e.g., the topological entropy of an isometry is necessarily equal to zero, and topological entropy is monotonous, i.e. $h(f|_A) \leq h(f)$ for any f-invariant $A \subseteq X$.

Further, $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of X. A probability measure $\mu : \beta(X) \to [0, 1]$ is *invariant* by f (shortly f*-invariant*) if $\mu(A) = \mu(f^{-1}(A))$ for any $A \in \mathcal{B}(X)$. It is well-known that when X is compact the set of all f-invariant measures is non-empty, convex and compact. Moreover, its extremal points are called *ergodic* measures of f. Ergodic measures are those satisfying $\mu(A) \in \{0, 1\}$ for each A with $\mu(A) = \mu(f^{-1}A)$.

Now we are ready to recall the definition of a *measure theoretic entropy* (also known as *Kolmogorov-Sinai entropy*) of f with respect to a probability measure μ on X. For a given finite partition $A := \{A_1, A_2, \ldots, A_k\}$ put

$$f^{-1}(A) = \{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_k)\}.$$

Further, for two partitions A and $B := \{B_1, B_2, \ldots, B_m\}$ we define a *refinement* by

$$A \vee B := \{A_i \cap B_j \,| i = 1, 2, \ldots k,\ j = 1, 2, \ldots, m\},$$

and analogously $\bigvee_{i=0}^{N} f^{-i}(A)$ is defined. Then an *entropy of a partition* A is defined as

$$H(A) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

Then, a *measure theoretic entropy* of a dynamical system (X, f) *with respect to a partition* A is defined by the following expression

$$h_\mu(f, A) = \lim_{N\to\infty} H\left(\bigvee_{i=0}^{N} f^{-i}(A)\right).$$

Finally, a *measure theoretic entropy* of a map f (resp. of a dynamical system (X, f)) is

$$h_\mu(f) = \sup_A h_\mu(f, A)$$

where the supremum is taken over all finite measurable partitions in X.

The following relationship (so-called *variational principle*) between measure-theoretical and topological entropy can be seen in many books, e.g. in [10]. The variational principle for topological entropy states that, in the case of X being a compact metric space, we have the following expression

$$h(f) = \sup_{\mu\in\mathcal{E}(X,f)} h_\mu(f). \quad (2)$$

where $\mathcal{E}(X, f)$ denotes the set of ergodic measures of f. If there is no risk of confusion we put $\mathcal{E}(X) := \mathcal{E}(X, f)$.

2.2 Zadeh's Extension, Spaces of Fuzzy Sets

In this subsection we introduce some spaces of fuzzy sets and their topological structures. Formally, a fuzzy set A on the space X is a function $A : X \to I$. The *α-cuts* (or the *α-level sets*), $[A]_\alpha$, and the *support*, *supp*(A), of a given fuzzy set A are defined as

$$[A]_\alpha = \{x \in X | A(x) \geq \alpha\}, \text{ for } \alpha \in [0, 1],$$

and

$$supp(A) = \overline{\{x \in X | A(x) > 0\}}.$$

In this contribution we consider the system $\mathcal{F}(X)$ of all upper semi-continuous fuzzy sets A on X. Moreover, let

$$\mathcal{F}^1(X) = \left\{A \in \mathcal{F}(X) | \max_{x\in X}\{A(x)\} = 1\right\}$$

denote the system of *normal* fuzzy sets on X. In this contribution we mainly consider a system $\mathcal{F}^1_m(X) \subseteq \mathcal{F}^1(X)$ consisting of fuzzy sets whose each α cut consists of at most m (topologically) connected components. Then for $m = 1$ we obtain a special class $\mathcal{F}_1(X)$ of so-called *fuzzy numbers*. Finally, by an *empty* fuzzy set $\emptyset_X$ we call a map $\emptyset_X(x) = 0$ for each $x \in X$, $\mathcal{F}_0(X)$ denotes the system of all nonempty fuzzy sets.

The following metrics are usually taken on the space of nonempty fuzzy sets. A *levelwise* metric d_∞ on $\mathcal{F}_0(X)$ is given by

$$d_\infty(A,B) = \sup_{\alpha\in(0,1]} D_X([A]_\alpha,[B]_\alpha). \tag{3}$$

It is known that considered spaces of fuzzy sets ($\mathcal{F}(X)$, $\mathcal{F}^1(X)$ and $\mathcal{F}^1_m(X)$) equipped with the levelwise topology, i.e. the metric topology induced by d_∞, are complete but are not compact and are not separable (see [15] and references therein) in general.

For a fuzzy set $A \in \mathcal{F}^1(X)$, its *endograph* $end(A)$ and *sendograph* $send(A)$ of A are defined by

$$end(A) = \{(x,\alpha) \in X \times I | A(x) \geq \alpha\},\ \ send(A) = end(A) \cap (supp(A) \times I),$$

respectively. Then

$$d_E(A,B) := H_{X\times I}(end(A), end(B))$$

and

$$d_S(A,B) := H_{X\times I}(send(A), send(B))$$

define the endograph and sendograph metrics on the relevant families of fuzzy sets. The metric topologies induced by d_E and d_S are denoted by τ_E and τ_S. It is worth noticing that the three metrics above are the most commonly used metrics in fuzzy topological dynamics and that the levelwise topology induced by d_∞ is stronger than the others. From this point of view it is natural to study various dynamical properties in the levelwise topology since many properties can be naturally shifted to the remaining topological structures. However, for more details and properties we again refer to [15] and references therein.

Now we are ready to define a self-map on spaces of fuzzy sets. Any map $f \in C(X)$ can be naturally extended to the space of fuzzy sets on X. Namely, a *fuzzification* (or *Zadeh's extension*) of the dynamical system (X, f) is a map $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ defined by the expression

$$(\tilde{f}(A))(x) = \sup_{y\in f^{-1}(x)} \{A(y)\}$$

for arbitrary $A \in \mathcal{F}(X)$ and $x \in X$. It is known that $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is continuous if $f : X \to X$ is continuous as well. Therefore, it is obvious that the continuity is preserved for any restriction of $\tilde{f}$, especially, for $\tilde{f}_m := \tilde{f}|_{\mathcal{F}^1_m(X)}$.

It is known [11] that the original map and its fuzzification are related via α-cuts, i.e., for any $\alpha \in (0,1]$ and any $A \in \mathcal{F}(X)$,

$$f([A]_\alpha) = [\tilde{f}(A)]_\alpha. \tag{4}$$

Similarly, the equality $f(supp(A)) = supp(f\varphi(A))$ can be proven. Consequently, we often deal with a set-valued extension of $f : X \to X$. This *set-valued* (or *induced*) extension $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is naturally defined by $f(B) = \bar{f}(B)$ for any $B \in \mathcal{K}(X)$. By $\mathcal{K}_m(X) \subseteq \mathcal{K}(X)$ we denote the subsystem of $\mathcal{K}(X)$ consisting of at most m connected components and $\bar{f}_m := \bar{f}|_{\mathcal{K}(X)}$.

3 On Topological Entropy

In this section we study the size of topological entropy on the space $\mathcal{F}^1_m(X)$. First, let us give some results on the set-valued case. Let X^{*k} a system of all nonempty subset of X consisting of at most k singleton sets. The following result was mentioned in the proof of Theorem 9 in [18].

Lemma 1. *For any dynamical system (X, f), $h(\bar{f}|_{X^{*k}}) = k \cdot h(f)$.*

Applying this result to the induced system $(\mathcal{K}(X), \bar{f})$ we immediately obtain the following result.

Proposition 1. *Let $f \in C(X)$ and $m \in \mathbb{N}$ be fixed. Then*

$$h(\bar{f}|_{\mathcal{K}_m(X)}) = m \cdot h(f).$$

This statement has an interesting consequence, namely, the main theorem from [6] claiming that $h(\bar{f}) > 0$ implies $ent(\tilde{f}) = \infty$ for some dynamical systems.

Let us study the case of $\mathcal{F}^1_m(X)$. It is easy to see that, for any $m \in \mathbb{N}$, there exists a continuous injective map $i : \mathcal{K}_m \to \mathcal{F}^1_m$ (regardless to any chosen metrics) such that $\bar{f}_m \circ i = i \circ \tilde{f}_m$ and hence, by [9],

$$h(\bar{f}_m) \leq ent(\tilde{f}_m). \tag{5}$$

Let us discuss the converse inequality for the case of continuous interval map, that is, for the space $\mathcal{F}^1_m(I)$. Note that $L \in \mathcal{F}^1_m(I)$ can be written as a union of m fuzzy numbers, where $\bigcup$ is represented by minimum t-norm. In addition, we have the following description of ω-limit sets of $\bar{f}$ (see [7]).

Lemma 2. *Let $f \in C(I)$ and $B \subseteq \mathcal{K}_1(I)$. Then, $\omega_{\bar{f}}(B)$ is either equal to an ω-limit set of the original map f, or a union of finitely many cyclically permuted intervals.*

Then, we can prove the following converse inequality.

Lemma 3. *Let (I, f) be a discrete dynamical system and $m \in \mathbb{N}$. Then*

$$h(\bar{f}_m) \geq ent(\tilde{f}_m). \tag{6}$$

Proof. Since the set $\mathcal{F}^1_m(I)$ is not compact, we must show that (6) holds on arbitrary compact $\tilde{f}$-invariant $K \subseteq I$. Due to the variational principle, which is valid on compact spaces, we have

$$h(\tilde{f}_m|_K) = \sup\{h_\mu(\tilde{f}_m)|\mu \in \mathcal{E}(K)\}.$$

Further, it is well known that the support of any ergodic measure is a subset of some ω-limit set. Consequently, it is sufficient to prove that

$$h(\tilde{f}_m|_A) \leq h(\bar{f}_m) \tag{7}$$

on an arbitrary ω-limit set A.

Let $A \in \omega(\tilde{f}_m) \cap K$ be fixed. Since A is invariant (see also (4)) any $L \in A$ can be written as a union of at most m fuzzy numbers. Let us assume that L is expressed by just m fuzzy numbers. More precisely,

$$L = \bigcup_{i=1}^{m} L_i, \text{ where } L_i \in \mathcal{F}_1^1(I).$$

We distinguish several cases according to different shapes of L_i's. By Lemma 2 and (4), α-cuts can be either singletons or nondegenerated periodic intervals. Two boundary cases are simple to describe. Really (Case I), if $[L_i]_1$ is a nondegenerated and hence periodic interval, for $i = 1, 2, \ldots, m$, then L is a periodic point in $\tilde{f}_m$. Then obviously $h(\tilde{f}_m|_A) = 0$ and (7) is satisfied.

Further (Case II) we can assume that $[L_i]_\alpha$ is a singleton for any $\alpha \in (0, 1]$ and $i = 1, 2, \ldots, m$. In this case $\tilde{f}_m|_A$ and $\bar{f}_m|_{supp(A)}$ are conjugate and hence $h(\tilde{f}_m|_A) = h(\bar{f}_m|_{supp(A)})$, i.e. (7) is satisfied due to the monotonicity of topological entropy.

Now (Case III), to finish this proof, we may assume without loss of generality that supports of L_i's are nondegenerated intervals and $[L_i]_1$'s are singletons. More precisely, for fixed $\varepsilon > 0$, there are α_i for $i \in \{1, 2, \ldots, m\}$ such that

$$diam([L_i]_\alpha) < \varepsilon \text{ for } \alpha > \alpha_i$$

and $[L_i]_\alpha$ is a nondegenerated periodic interval for $\alpha \le \alpha_i$.

It is easy to see that $A = \bigcup_i A_i$ where $L_i \in A_i \in \omega(\tilde{f}_1)$. Then, for any $B, C \in A_i$, $\beta \in (\alpha_i, 1]$ and $j \ge 0$ we have

$$\begin{aligned} D_I(\bar{f}^j([B]_\beta), \bar{f}^j([C]_\beta)) &\le D_I(\bar{f}^j([B]_\beta), \bar{f}^j([B]_1)) \\ &\quad + D_I(\bar{f}^j([B]_1), \bar{f}^j([C]_1)) \\ &\quad + D_I(\bar{f}^j([C]_1), \bar{f}^j([C]_\beta)) \\ &\le 2\varepsilon + D_I(\bar{f}^j([B]_1), \bar{f}^j([C]_1)), \end{aligned} \tag{8}$$

and, consequently,

$$s_n(3\varepsilon, A_i, \tilde{f}_1) \le s_n(\varepsilon, D(A_i), \bar{f}_1)$$

where $D(A_i) := \{[B]_1 \mid B \in A_i\}$. Since (8) can be done for any A_i we immediately have

$$s_n(3\varepsilon, A, \tilde{f}_n) \le s_n(\varepsilon, D(A), \bar{f}_n)$$

Thus, by the definition of topological entropy, we get (7). □

Remark 1. In fact, the same result seems to be valid for all compact metric spaces X and maps f for which each ω-limit set of $\bar{f}$ is like in Lemma 2, that is, it is either an ω-limit of the original map f or a periodic (and hence finite) orbit consisting of nondegerated intervals.

Now (5) together with Lemma 3 implies the following statement.

Theorem 1. *Let us consider a discrete dynamical system* (I, f) *and its fuzzy extension* $(\mathcal{F}^1_m(I), \tau_\infty)$ *for* $m \in \mathbb{N}$. *Then*

$$h(f) = m \cdot ent(\tilde{f}_m).$$

Let us study what happens in other topological structures.

Theorem 2. *Let us consider a discrete dynamical system* (I, f) *and its fuzzy extension* $(\mathcal{F}^1_m(I), \tau)$, $\tau \in \{\tau_E, \tau_S\}$, *for* $m \in \mathbb{N}$. *Then*

$$ent(\tilde{f}_m) = m \cdot h(f).$$

Proof. Let us prove this statement for $d = d_E$ first. Let us recall that $m \cdot h(f) = h(\bar{f}_m)$ by Proposition 1. As $h(\bar{f}_m) \leq ent(\tilde{f}_m)$ is clear, it remains to show

$$h(\bar{f}_m) \geq ent_d(\tilde{f}_m). \tag{9}$$

Let $\varepsilon > 0$ be fixed and $n_0 \in \mathbb{N}$ be chosen such that $1/n_0 < \varepsilon$. According to the definition of d_E, if $d_E(A, B) > \varepsilon$ for $A, B \in \mathcal{F}^1_m(I)$ then there exists $i \in \{1, 2, \ldots, n_0\}$ such that

$$D_I([A]_\alpha, [B]_\alpha) > \frac{\varepsilon}{2} \text{ for any } \alpha \in J_i := [1 - (i+1)n_0, 1 - in_0]. \tag{10}$$

Thus, for fixed $n \in \mathbb{N}$ such that $n \geq n_0$, let $k_n := s_n(\varepsilon, \mathcal{F}^1_m(I), \tilde{f}_m)$ and $\mathcal{A}$ denote (n, ε)-separated set of cardinality k_n with respect to $\tilde{f}_m$. According to (4) and (10), there exists $i \in \{1, 2, \ldots, n_0\}$ such that, for any $\alpha \in J_i$, the set $\{[A]_\alpha \mid A \in \mathcal{A}\}$ forms an $(n, \varepsilon/2)$-separated set of cardinality k_n/n_0 with respect to $\bar{f}_m$. Consequently,

$$s_n(\varepsilon/2, \mathcal{K}_m, \bar{f}_m) \geq \frac{k_n}{n_0}.$$

Thus

$$\log s_n(\varepsilon/2, \mathcal{K}_m, \bar{f}_m) \geq \log \frac{k_n}{n_0} = \log k_n - \log n_0 = \log s_n(\varepsilon, \mathcal{F}^1_m(I), \tilde{f}_m).$$

Consequently, by the definitions of relevant entropies, (9) is proved.

The proof for τ_S is analogous. It is true that $(\mathcal{F}^1_m(I), \tau_S)$ is not compact and some fuzzy points need not have ω-limit sets [7,15]. But we have dealt neither with compactness of $(\mathcal{F}^1_m(I), \tau_E)$ nor with ω-limit points of fuzzy sets with m connected components. It is true that $d_E(\tilde{f}^k_m(A), \tilde{f}^k_m(B)) \geq d_S(\tilde{f}^k_m(A), \tilde{f}^k_m(B))$ for any $k \in \mathbb{N}$ and $A, B \in \mathcal{F}^1_m(I)$ in general, but (10) is still valid. □

4 Relations to Dynamical Fuzzy Entropy

We have proved that for continuous interval maps the equality $ent(\tilde{f}_m) = m \cdot h(f)$ holds. A natural question can be stated, namely, what is the influence of fuzzy and non–fuzzy sets in the dynamics of $\tilde{f}_m$. For that end, we consider the notion of degree of fuzziness of a fuzzy set $A \in \mathcal{F}_m$ (see [8] and the references therein).

According to [17], in order to define a degree of fuzziness on X, we need a measure space $(X, \beta(X), \mu)$, where $\beta(X)$ is the Borel σ-algebra, μ is a nonzero finite measure, and $\mathbb{F}(X)$ consists of μ-measurable functions. Then, for any real-valued function $g : I \to \mathbb{R}$ such that

- $g(0) = g(1) = 0$,
- $g(\alpha) = g(1 - \alpha)$ for any $\alpha \in I$,
- g is strictly increasing on $[0, 1/2]$,

the expression

$$e_\mu(A) = \frac{1}{\mu(X)} \int g(A(x)) d\mu(x) \tag{11}$$

defines a *degree of fuzziness* $e_\mu(A)$ (or $e(A)$) of a fuzzy set $A \in \mathbb{F}(X)$. As usually, χ_B denotes the characteristic function of a set B. Then, if $A = \sum_{i=1}^{k} a_i \chi_{B_i}$, with $a_i \in (0, 1]$ and $B_i \in \beta(X)$, $i = 1, ..., k$, then

$$e_\mu(A) = \sum_{i=1}^{k} g(a_i)\mu(X_i).$$

The following lemma describes basic properties of this notion. Let us recall that $\overset{a.e.}{\circ}$ denotes that the relation $\circ$ holds almost everywhere with respect to a given measure.

Lemma 4 ([17]). *The degree of fuzziness $e : \mathbb{F}(X) \to \mathbb{R}$ given by (11) has the following properties:*

A1. $e(A) = 0$ *if and only either* $A \overset{a.e.}{=} \chi_C$ *for some* $C \subseteq X$ *or* $A \overset{a.e.}{=} \emptyset_X$, *where* χ_C *is the characteristic function on* C,
A2. $e(A)$ *is maximal if and only if* $A \overset{a.e.}{=} \frac{1}{2}\chi_X$,
A3. $e(A) \leq e(B)$ *whenever* A *is less fuzzy than* B, *that is* $A(x) \leq B(x) \leq 1/2$ *or* $A(x) \geq B(x) \geq 1/2$ *for almost all* $x \in X$,
A4. $e(A) = e(A^c)$,
A5. e *is continuous with respect to the supremum metric on* $\mathbb{F}(X)$.

We demonstrated in [8] that the degree of fuzziness need not be a continuous function when the metrics d_∞, d_E and d_S are considered. Given $\alpha \in [0, 1]$, we define

$$\mathcal{F}_\alpha(e_\mu) = \left\{ A \in \mathbb{F}(X) | \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} e_\mu \circ \tilde{f}_m^i(A) \right\}.$$

The set $\mathcal{F}_0(e_\mu)$ contains all the fuzzy sets that are essentially non–fuzzy along its orbit, while the set $\mathcal{F}(e_\mu) = \bigcup_{\alpha>0} \mathcal{F}_\alpha(e_\mu)$ contains all the fuzzy sets whose orbit is fuzzy with respect to degree of fuzziness. In [8] we have defined several concepts helping us to study relations between topological entropy and degree of fuzziness. For instance, we define the fuzzy entropy of $\tilde{f}_m$ as

$$fuzzent(\tilde{f}_m) = ent(\tilde{f}_m|_{\mathcal{F}(e_\mu)}).$$

It is clear from the definition that $fuzzent(\tilde{f}_m) \leq ent(\tilde{f}_m)$. However for interval maps we can state the following result. The proof follows the arguments contained in [8], so we skip it.

Theorem 3. *Let $f \in C(I)$ and let e_μ be a degree of fuzziness as above. Then, for any metric d_∞, d_E and d_S it holds that*

$$fuzzent(\tilde{f}_m) = h(\bar{f}_m) = m \cdot h(f) = ent(\tilde{f}_m).$$

5 Conclusion and Open Questions

We have studied the topological entropy of the Zadeh's extension on the space of fuzzy sets with the property that every α–cut has at most m convex (resp. topologically connected) components, for some fixed positive integer m. This is a natural continuation of our previous work (see [6–8]) In particular, we prove that for extensions of continuous interval maps the topological entropy is positive if and only if the topological entropy of the crisp map is positive. Moreover, for extensions of continuous interval maps, we showed that fuzzy sets which are essentially fuzzy, that is, with positive degree of fuzziness along the orbit, have full topological entropy. There are still some open questions which, under our point of view, are interesting.

First, it is an open problem for which compact metric spaces equations

$$m \cdot h(f) = h(\bar{f}_m) = fuzzent(\tilde{f}_m) = ent(\tilde{f}_m)$$

are satisfied. We have shown that it holds for dynamical systems on intervals. But it is also known that the formula $m \cdot h(f) = h(\bar{f}_m)$ doesn't hold for some special dendrites [1]. Consequently, the above chain of equalities has to be analyzed individually.

On the other hand, it is also interesting to study in detail what the topological structure of sets $\mathcal{F}_\alpha(e_\mu)$ for $\alpha \in [0, 1]$ is. This question can be stated not only for the set $\mathcal{F}^1_m(X)$ but also for general normal fuzzy sets $\mathcal{F}^1(X)$.

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Fuzzy Relation Equations with Fuzzy Quantifiers

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Abstract. In this paper, we follow the previous works on fuzzy relation compositions based on fuzzy quantifiers and we introduce systems of fuzzy relation equations stemming from compositions based on fuzzy quantifiers. We address the question, whether such systems under some specific conditions may become solvable, and we provide a positive answer. Based on the computational forms of the compositions using fuzzy quantifiers, we explain a way of getting solutions of the systems. In addition to showing some new properties and theoretical results, we provide readers with illustrative examples.

Keywords: Fuzzy relation equations · Mamdani-Assilian model · Implicative model · Fuzzy (generalized) quantifiers

1 Introduction and Preliminaries

1.1 Introduction

Systems of fuzzy relation equations have a very important role in many areas of fuzzy mathematics, especially in fuzzy control and approximate reasoning in general. The first work on this field was done by Sanchez [19]. Numerous authors have deeply studied and developed the topic, mainly the problems of finding solvability criterions of the systems of fuzzy relation equations was focused on. For the most valuable results, we refer readers to relevant literature [5,6,10,12, 15–17,21]. Furthermore, the interest of the scientific community in this filed does not seem to decrease, see e.g. [11,18,20,24].

Standardly, two types of compositions, namely the sup-T composition and the inf-R composition (also called Bandler-Kohout subproduct) are considered in the investigated systems of equations. These compositions model fuzzy inference mechanisms in the terminology of fuzzy rule based systems. In this paper, we follow this standard setting however, with a significant difference consisting in the employment of fuzzy quantifiers that replace the standard ones – the existential and the universal. So, the used compositions appearing in the systems of fuzzy relation equations will be based on generalized intermediate quantifiers [2–4].

Fuzzy relations solving the systems of fuzzy relation equations are considered as correct models of fuzzy rule bases [16]. In the standard systems, the Mamdani-Assilian model [14] and the implicative model do possess unique

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positions among other solutions, in particular, if the sup-T system is solvable, the implicative model is among the solutions, and similarly, if the inf-R system is solvable, the Mamdani-Assilian model is among the solutions. Note, that the implicative model aggregates rules by the minimum, which represents the universal quantifier, and the Mamdani-Assilian model aggregates rules by the maximum, which represents the existential quantifier. The natural questions are as follows. If we replace the original compositions sup-T and inf-R by compositions based on generalized quantifiers, will we be still able to obtain solvable systems? And in the case of a positive answer, will the solutions remain in the Mamdani-Assilian-like and the implicative-like shape with the only difference: the quantifiers used for aggregation of rules will be replaced by the generalized ones? The answer will be positive and it will be provided in a constructive way, i.e., we will show how to modify the fuzzy sets in order to satisfy the equalities. Furthermore, we show some valid properties for the new systems with fuzzy quantifiers.

1.2 Systems of Fuzzy Relation Equations

Let us fix a residuated lattice $\mathcal{L} = \langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1 \rangle$[1] as the background algebraic structure and by $\mathcal{F}(U)$ let us denote the set of all fuzzy sets on a given universe U. Let $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y)$ be the antecedent and consequent fuzzy sets, respectively. There are two standard systems of fuzzy relation equations considered with respect to an unknown fuzzy relation $R \in \mathcal{F}(X \times Y)$:

$$A_i \circ R = B_i, \quad i = 1, 2, \ldots, m \qquad \text{(sup-T system)}, \tag{1}$$

$$A_i \lhd R = B_i, \quad i = 1, 2, \ldots, m \qquad \text{(inf-R system)} \tag{2}$$

where the used compositions (images) $\circ$ and $\lhd$ may be expanded as follows:

$$(A_i \circ R)(y) = \bigvee_{x \in X} (A_i(x) * R(x,y)) \ , \quad (A_i \lhd R)(y) = \bigwedge_{x \in X} (A_i(x) \rightarrow R(x,y)) \ .$$

There are two models with priority positions among other potential solutions, namely the *Mamdani-Assilian model* [14] and the *implicative model*:

$$R_{MA}(x,y) = \bigvee_{i=1}^{m} (A_i(x) * B_i(y)) \ , \quad R_{IMP}(x,y) = \bigwedge_{i=1}^{m} (A_i(x) \rightarrow B_i(y)) \ .$$

Let us recall fundamental solvability criterions demonstrating the priority positions for the above models of fuzzy rule bases [5,12,13,15].

Theorem 1. *The system (1) [(2)] is solvable if and only if fuzzy relation R_{IMP} [R_{MA}] is its solution and then R_{IMP} [R_{MA}] is its greatest [least] solution.*

[1] The operations $\wedge, \vee, *, \rightarrow$ stand for meet (infimum), join (supremum), multiplication (left-continuous t-norm) and its residual implications, respectively.

Theorem 2. *Let fuzzy sets $A_i \in \mathcal{F}(X)$ be normal (with non-zero support) and $B_i \in \mathcal{F}(Y)$, $i = 1, 2, \dots, m$. Then fuzzy relation R_{IMP} $[R_{MA}]$ is a solution to (2) [(1)] if and only if for all $i, j = 1, 2, \dots, m$ the following inequality holds:*

$$\bigvee_{x \in X} (A_i(x) * A_j(x)) \leq \bigwedge_{y \in Y} (B_i(y) \leftrightarrow B_j(y)) \ . \tag{3}$$

Definition 1 [16,23]**.** Fuzzy relation $R \in \mathcal{F}(X \times Y)$ is said to be a *continuous model of fuzzy rules w.r.t.* $@ \in \{\circ, \lhd\}$ if for each $i \in \{1, \dots, m\}$ and for $A \in \mathcal{F}(X)$ the following inequality holds

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A@R)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \ . \tag{4}$$

Theorem 3 [16]**.** *Let $R \in \mathcal{F}(X \times Y)$ be a fuzzy relation. Then for any $A \in \mathcal{F}(X)$ and all $i = 1, \dots, m$ and $y \in Y$ it is true that*

$$B_i(y) \leftrightarrow (A \circ R)(y) \geq \delta_{R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \, , \tag{5}$$

where $\delta_{R,i}(y) = B_i(y) \leftrightarrow (A_i \circ R)(y)$ is called degree of solvability *of the system.*

A similar result has been proven in [23] also for the inf-R system, in particular:

$$B_i(y) \leftrightarrow (A \lhd R)(y) \geq \sigma_{R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \, , \tag{6}$$

where $\sigma_{R,i}(y) = B_i(y) \leftrightarrow (A_i \lhd R)(y)$.

Theorem 4 [16,23]**.** *R is a solution of system (1) [(2)] if and only if R is continuous.*

1.3 Fuzzy Relational Compositions Based on Fuzzy Quantifiers

In this section, we recall some basic definitions of fuzzy (generalized) quantifiers determined by fuzzy measures [7–9] that were used e.g. in [2,3,22].

Definition 2. Let $U = \{u_1, \dots, u_n\}$ be a finite universe, let $\mathcal{P}(U)$ denotes the power set of U. A mapping $\mu : \mathcal{P}(U) \to [0, 1]$ is called a fuzzy measure on U if: $\mu(\emptyset) = 0$ and $\mu(U) = 1$, and if $\forall C, D \in \mathcal{P}(U), C \subseteq D$ then $\mu(C) \leq \mu(D)$. Fuzzy measure μ is called *invariant with respect to cardinality* if: $\forall C, D \in \mathcal{P}(U) : |C| = |D| \Rightarrow \mu(C) = \mu(D)$ where $|\cdot|$ denotes the cardinality of a set.

Unless specified differently, for the rest of the paper, we will consider only measures invariant w.r.t. cardinality, i.e., measure, that modify the relative cardinality, see Example 1.

Example 1. Fuzzy measure $\mu_{rc}(C) = \frac{|C|}{|U|}$ is called *relative cardinality* and it is invariant w.r.t. cardinality. Let $f : [0, 1] \to [0, 1]$ be a non-decreasing mapping with $f(0) = 0$ and $f(1) = 1$ then μ^f defined as $\mu^f(C) = f(\mu_{rc}(C))$ is again a fuzzy measure that is invariant w.r.t. cardinality.

Example 2. Consider the fuzzy measure $\mu_{rc}^{50\%}$ on U defined as follows:

$$\mu_{rc}^{50\%}(A) = \begin{cases} 1 & if \quad \mu_{rc}(A) \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

for any $A \in \mathcal{P}(U)$. Such measure is used to construct a quantifier "*at least half*". Analogously, one can define a measure "*at least x %*" for any $x \in [0, 100]$.

Definition 3. A mapping $Q : \mathcal{F}(U) \to [0,1]$ defined by

$$Q(C) = \bigvee_{D \in \mathcal{P}(U) \setminus \{\emptyset\}} \left(\left(\bigwedge_{u \in D} C(u) \right) * \mu(D) \right) , \quad C \in \mathcal{F}(U) \tag{7}$$

is called *fuzzy (generalized) quantifier determined by fuzzy measure* μ on U.

Example 3. Fuzzy measures $\mu^{\forall}(D) = \begin{cases} 1 & D \equiv U \\ 0 & \text{otherwise} \end{cases}$ and $\mu^{\exists}(D) = \begin{cases} 0 & D \equiv \emptyset \\ 1 & \text{otherwise} \end{cases}$ construct the classical *universal* ($Q^{\forall}$) and *existential* ($Q^{\exists}$) *quantifiers*. It should be noted that $\mu^{\forall} \leq \mu \leq \mu^{\exists}$ for any fuzzy measure μ.

Formula (7) can be rewritten into a computationally cheaper form:

$$Q(C) = \bigvee_{i=1}^{n} C(u_{\pi(i)}) * f(i/n) , \quad C \in \mathcal{F}(U) \tag{8}$$

where π is a permutation on $\{1, \ldots, n\}$ such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \cdots \geq C(u_{\pi(n)})$.

Definition 4. Let X, Y, Z be non-empty finite universes, let $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$. Let Q be a fuzzy quantifier on Y determined by a fuzzy measure μ. Then, the compositions $R@^Q S$ where $@ \in \{\circ, \triangleleft\}$ is defined as follows:

$$(R@^Q S)(x,z) = \bigvee_{D \in \mathcal{P}(Y) \setminus \{\emptyset\}} \left(\left(\bigwedge_{y \in D} R(x,y) \circledast S(y,z) \right) * \mu(D) \right) , \tag{9}$$

where $\circledast \in \{*, \to\}$ corresponds to the composition and $x \in X$ and $z \in Z$.

By (8), these compositions can be rewritten into computationally cheap form:

$$(R@^Q S)(x,z) = \bigvee_{i=1}^{n} \left(\left(R(x, y_{\pi(i)}) \circledast S(y_{\pi(i)}, z) \right) * f(i/n) \right) . \tag{10}$$

where π is a permutation on $\{1, \ldots, n\}$ such that $R(x, y_{\pi(i)}) \circledast S(y_{\pi(i)}, z) \geq R(x, y_{\pi(i+1)}) \circledast S(y_{\pi(i+1)}, z)$ for any $i = 1, \ldots, n-1$ where n denotes the cardinality of Y.

2 Fuzzy Relation Equations with Fuzzy Quantifiers

This Section introduces new systems of fuzzy relation equations and presents some related theoretical results. Let us fix the index set $I_m = \{1, \ldots, m\}$ and $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y), i \in I_m$. Let Q be a fuzzy quantifier on X determined by a fuzzy measure μ. The considered systems solved w.r.t. an unknown $R \in \mathcal{F}(X \times Y)$ are given as follows:

$$A_i \circ^Q R = B_i \ , \quad i = 1, \ldots, m \ , \tag{11}$$

$$A_i \lhd^Q R = B_i \ , \quad i = 1, \ldots, m \ . \tag{12}$$

First of all, let us present two results stemming from (and generalizing) the results provided by I. Perfilieva in [16] for the standard systems of fuzzy relation equations.

Lemma 1. *Let Q be a fuzzy quantifier determined by a fuzzy measure μ on X and let $R \in \mathcal{F}(X \times Y)$ be a fuzzy relation. Then for any $A \in \mathcal{F}(X)$ and all $i = 1, \ldots, m$ and $y \in Y$ it is true that*

$$B_i(y) \leftrightarrow (A \circ^Q R)(y) \geq \varepsilon_{Q,R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \, , \tag{13}$$

where $\varepsilon_{Q,R,i}(y) = B_i(y) \leftrightarrow (A_i \circ^Q R)(y)$.

Sketch of the proof: The proof is similar to the proof of inequality (5) provided in [16], mainly the following basic properties are used:

$$\bigwedge_{i \in I} (a_i \leftrightarrow b_i) \leq \left(\bigvee_{i \in I} a_i \right) \leftrightarrow \left(\bigvee_{i \in I} b_i \right) , \tag{14}$$

$$\bigwedge_{i \in I} (a_i \leftrightarrow b_i) \leq \left(\bigwedge_{i \in I} a_i \right) \leftrightarrow \left(\bigwedge_{i \in I} b_i \right) , \tag{15}$$

$$(a \leftrightarrow b) * (c \leftrightarrow d) \leq (a * c) \leftrightarrow (b * d). \tag{16}$$

□

Lemma 2. *Let Q be a fuzzy quantifier determined by a fuzzy measure μ on X and let $R \in \mathcal{F}(X \times Y)$ be a fuzzy relation. Then for any $A \in \mathcal{F}(X)$ and all $i = 1, \ldots, m$ and $y \in Y$ it is true that*

$$B_i(y) \leftrightarrow (A \lhd^Q R)(y) \geq \xi_{Q,R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \, , \tag{17}$$

where $\xi_{Q,R,i}(y) = B_i(y) \leftrightarrow (A_i \lhd^Q R)(y)$.

Sketch of the proof: The proof is similar to the proof of inequality (6) provided in [23], mainly properties (14)–(16) jointly together with the facts that $a \to (b \to c) = (a * b) \to c = (b * a) \to c = b \to (a \to c), (a \to b) \to b \geq a \vee b$ are used. □

Lemmas 1 and 2 lead to the following important results.

Proposition 1. *A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a solution of system (11) if and only if the following inequality holds for any $A \in \mathcal{F}(X)$ and for all $i = 1, \ldots, m$ and $y \in Y$.*

$$\bigwedge_{y \in Y} \big(B_i(y) \leftrightarrow (A \circ^Q R)(y)\big) \geq \bigwedge_{x \in X} \left(A_i(x) \leftrightarrow A(x)\right). \tag{18}$$

Sketch of the proof: The proof can be provided only by using inequality (13). □

Proposition 2. *A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a solution of system (12) if and only if the following inequality holds for any $A \in \mathcal{F}(X)$ and for all $i = 1, \ldots, m$ and $y \in Y$.*

$$\bigwedge_{y \in Y} \big(B_i(y) \leftrightarrow (A \lhd^Q R)(y)\big) \geq \bigwedge_{x \in X} \left(A_i(x) \leftrightarrow A(x)\right). \tag{19}$$

Sketch of the proof: The proof can be provided only by using inequality (17) only. □

Lemmata 1 and 2 and Propositions 1 and 2 actually show what was provided already in [16] for the standard type of fuzzy relation equations, in particular, that the solvability is identical to a sort of (Lipschitz-like) "continuity". In other words, the inferred outputs have to be at least as close[2] to the consequent fuzzy sets as are the input fuzzy sets close to the antecedent fuzzy sets. This consequently means, that we should have consequents at least as "close" to each other as the antecedents, having in mind again the construction of the closeness.

The above results did not employ the particular shape of the expected solutions, i.e., we have not consider the Mamdani-Assilian nor the implicative models of fuzzy rules bases with the modification in quantifiers aggregating the rules. So, let us consider such fuzzy relations possibly serving as solutions of given systems of fuzzy relation equations and then, let us introduce natural results.

Let Q be fuzzy quantifier on I_m determined by a fuzzy measure μ. By formula (9), we define fuzzy relations R^Q_{IMP}, R^Q_{MA} as follows:

$$R^Q_{IMP}(x,y) = \bigvee_{D \in \mathcal{P}(I_m) \setminus \emptyset} \left(\left(\bigwedge_{i \in D} A_i(x) \to B_i(y) \right) * \mu(D) \right),$$

$$R^Q_{MA}(x,y) = \bigvee_{D \in \mathcal{P}(I_m) \setminus \emptyset} \left(\left(\bigwedge_{i \in D} A_i(x) * B_i(y) \right) * \mu(D) \right).$$

Proposition 3. *Let Q_1, Q_2 be fuzzy quantifiers on X determined by fuzzy measures μ_1, μ_2, respectively. Let Q_3, Q_4 be fuzzy quantifiers on I_m determined by*

[2] The closeness is here given by the similarity measure using the fuzzy equivalence from the underlying algebraic structure. In the case of a continuous Archimedean t-norm, the similarity is a dual notion to the metric function induced by the additive generator, which justifies the continuity point of view as well as the terminology [16].

fuzzy measures μ_3, μ_4*, respectively. Furthermore, let* $R^{Q_3}_{IMP}$ *solves the system* $A_i \circ^{Q_1} R = B_i$ *and let* $R^{Q_4}_{IMP}$ *solves the system* $A_i \circ^{Q_2} R = B_i$*. If* $\mu_1 \leq \mu_2$ *then* $\mu_4 \leq \mu_3$*.*

Sketch of the proof: The proof is based on the monotonicity property preserved by the compositions based on fuzzy quantifiers [2, Lemma 1]. □

Proposition 3 actually confirms a natural monotonic behaviour w.r.t. change of the used quantifiers. The more the quantifiers replacing the existential one in the used composition "moves" to the right, i.e. requires more than a single point but rather a few points, the more the quantifiers replacing the universal one in the aggregation of implicative rules "moves" to the left, i.e. requiring to meet less then all rules, for example only most or many of them. An analogous result may by obtained also for the inf-R equations and the Mamdani-Assilian rules aggregated by a fuzzy quantifier.

Proposition 4. *Let* Q_1, Q_2 *be fuzzy quantifiers on* X *determined by fuzzy measures* μ_1, μ_2*, respectively. Let* Q_3, Q_4 *be fuzzy quantifiers on* I_m *determined by fuzzy measures* μ_3, μ_4*, respectively. Furthermore, let* $R^{Q_3}_{MA}$ *solves the system* $A_i \lhd^{Q_1} R = B_i$ *and let* $R^{Q_4}_{MA}$ *solves the system* $A_i \lhd^{Q_2} R = B_i$*. If* $\mu_1 \leq \mu_2$ *then* $\mu_4 \leq \mu_3$*.*

Sketch of the proof: The proof is based on the monotonicity property preserved by the compositions based on fuzzy quantifiers [2, Lemma 1]. □

3 First Observations on Solvability

Assume that systems (1) and (2) are solvable and thus, R_{IMP} solves the first system and R_{MA}, solves the second system. The question addressed in this Section is, whether there exist fuzzy measures μ_1 on X and μ on I_m such that the following equalities

$$A_i \circ^{Q_1} R^{Q}_{IMP} = B_i \ , \quad A_i \lhd^{Q_1} R^{Q}_{MA} = B_i \ , \tag{20}$$

where Q_1, Q are fuzzy quantifiers determined by μ_1 and μ, respectively, hold? As we will show, under some assumptions, the answer will be positive.

3.1 Sup-T System

For the sake of comprehension and the convenience of calculation, let us consider the computational form of the compositions (10) and finite universes $I_m = \{1, \ldots, m\}$, $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_s\}$. Then $R^{Q}_{IMP}(x, y)$ and $(A \circ^{Q_1} R)(y)$ are given by

$$R^{Q}_{IMP}(x, y) = \bigvee_{j=1}^{m} \left(\left(A_{\pi(j)}(x) \to B_{\pi(j)}(y)\right) * f\left(\frac{j}{m}\right)\right), \tag{21}$$

$$(A \circ^{Q_1} R)(y) = \bigvee_{k=1}^{n} \left(\left(A(x_{\pi_1(k)}) * R(x_{\pi_1(k)}, y)\right) * f_1\left(\frac{k}{n}\right)\right) \tag{22}$$

where π, π_1 are permutations such that

$$A_{\pi(j)}(x) \rightarrow B_{\pi(j)}(y) \geq A_{\pi(j+1)}(x) \rightarrow B_{\pi(j+1)}(y), j = 1, 2, \ldots, m-1, \text{and}$$
$$A(x_{\pi_1(k)}) * R(x_{\pi_1(k)}, y) \geq A(x_{\pi_1(k+1)}) * R(x_{\pi_1(k+1)}, y), k = 1, 2, \ldots, n-1.$$

Before showing some examples, we start with a simple but important observation. Assume that for each $x \in X$, there is only $i \in I_m$ such that $A_i(x) > 0$ and $A_j(x) = 0$ for any $j \neq i$. Then, by formula (21), we have $A_{\pi(j)}(x) \rightarrow B_{\pi(j)}(y) = 1$ for any $j = 1, \ldots, m-1$. Now, let us consider μ to be strictly greater than $\mu^{\forall}$ on I_m. We may consider, e.g., the measure μ such that $f(\frac{j}{m}) = 0$ for $j \in \{1, \ldots, m-2\}$, $f(\frac{m-1}{m}) = a_{m-1} > 0$ and $f(1) = 1$. Then $R^Q_{IMP}(x, y) = a_{m-1} \vee \big(A_{\pi(m)}(x) \rightarrow B_{\pi(m)}(y)\big)$. The smaller the value a_{m-1}, the closer is the value $R^Q_{IMP}(x, y)$ to the value $R_{IMP}(x, y)$, and vise-versa. In particular, if $a_{m-1} = 1$ then $R^Q_{IMP}(x, y) = 1$, for all pairs (x, y). This implies that $(A_i \circ R^Q_{IMP})(y) = 1$ for all $y \in Y$ assuming normality of all A_i's . If we define similarity degrees between membership degrees to $A_i \circ R^Q_{IMP}$ and to B_i as

$$\varepsilon_{i,Q,Q^{\exists}}(y) = (A_i \circ R^Q_{IMP})(y) \leftrightarrow B_i(y),\ y \in Y$$

and the overall similarity degree as $\varepsilon_{Q,Q^{\exists}} = \bigwedge_{i \in I_m} \bigwedge_{y \in Y} (\varepsilon_{i,Q,Q^{\exists}}(y))$, then this similarity $\varepsilon_{Q,Q^{\exists}}$ will be equal to 0 whenever there will be some $y \in Y$ and some $i \in I_m$ such that $B_i(y) = 0$.

What if we replace $\mu^{\exists}$ used for the composition $\circ$ to a measure μ_1 that is strictly lower than the $\mu^{\exists}$ on X? This approach would be natural as it would decrease the values of $A_i \circ^{Q_1} R^Q_{IMP}$ closer to the membership degrees of y's to B_i.

However, we have assumed that $A_i(x) > 0$ just for one $i \in I_m$ for each $x \in X$. Now if we consider μ_1 such that $f(\frac{1}{n}) = b_1 < 1$ and $f(\frac{k}{n}) = 1$ for $k \in \{2, \ldots, n\}$ then, by formula (22) we get $\left(A_i \circ^{Q_1} R^Q_{IMP}\right)(y) = b_1$ for any $y \in Y$. This implies that the similarity degree

$$\varepsilon_{i,Q,Q_1} = \bigwedge_{y \in Y} \left(\left(A_i \circ^{Q_1} R^Q_{IMP}\right)(y) \leftrightarrow B_i(y)\right)$$

is rather low. In particular, for the Łukasiewicz algebra and the consequent fuzzy sets B_i taking all values from the unit interval [0, 1], the similarity will be always lower or equal to 0.5, which derives the overall similarity degree ε_{Q,Q_1},

$$\varepsilon_{Q,Q_1} = \bigwedge_{i \in I_m} \bigwedge_{y \in Y} \left((A_i \circ^{Q_1} R^Q_{IMP})(y) \leftrightarrow B_i(y)\right) \tag{23}$$

to be low too.

Analogously, if for each $x \in X$ there will be only two indexes $i, j \in I_m$ such that $A_i(x) > 0, A_j(x) > 0$ and if we consider measure μ determined by $f(\frac{j}{m}) = 0$ for $j \in \{1, \ldots, m-3\}$, and by $f(\frac{m-2}{m}) = a_{m-2} > 0$ then we would encounter the same situation.

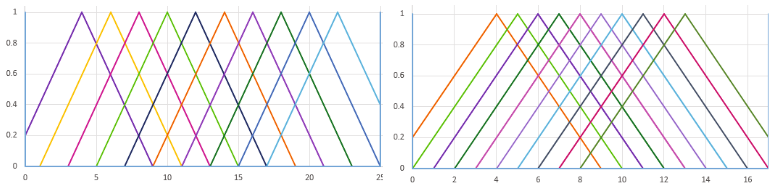
Thus, we may formulate the following observation.

Observation 5. *For a higher similarity degree, the antecedent fuzzy sets A_i have to be modified in order to overlap in a denser way. And, the higher and closer the membership degrees at each overlapping point, the higher similarity is obtained.*

It should be noted that while adjusting A_i, we have to take into account condition (4) in order to ensure that the both standard systems are solvable. By Irina's and Lehmke's results [16] on the continuity of a model of fuzzy IF-THEN rules, the condition means that consequents B_i have to be "closer" than antecedents A_i. With such a modifying of fuzzy sets, we expect that, in a certain sense, the systems of equations with fuzzy quantifiers are also solvable.

Let us demonstrate the observation on examples. For the technical reasons, let us restrict our choice of Q_1, Q to fuzzy quantifiers of the type "*at least x %*" mentioned in Example 2. Let us consider the systems based on fuzzy rule bases with m rules. Then Q will be used to model expressions of the type "at least J rules" and Q_1 will be used to model expressions of the type "at least K elements" from X.

Example 4. Assume that $I_m = \{1, \ldots, 10\}$, $X = \{x_1, \ldots, x_{26}\}$, $Y = \{y_1, \ldots, y_{18}\}$ and consider $A_i, B_i, i = 1, \ldots, 10$ as depicted in Fig. 1. Let us run computations for the combinations of fuzzy quantifier Q "at least J rules" with J ranging form 1 to 9, and the fuzzy quantifier Q_1 "at least K elements" with K ranging from 2 to 26. The goal is to find the combination with the maximal overall similarity degree, ε_{Q,Q_1}.

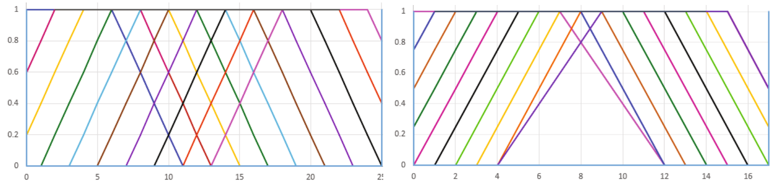


(a) Fuzzy sets on X (b) Fuzzy sets on Y

Fig. 1. Depiction of fuzzy sets in the case that the overall similarity degree reaches the maximum value $\varepsilon_{Q,Q_1} = \mathbf{0.6}$ for the combinations of $Q =$ "at least J rules" where $J = 9$ and $Q_1 =$ "at least K elements from X" for $K = 4$.

As we can see from Fig. 1, the overlapping area of any two antecedent fuzzy sets is rather narrow. Therefore, only the standard sup-T system is solvable. For the system with fuzzy quantifiers, the calculations provided the maximum value of the similarity degree equal to 0.6, which occurred for $J = 9$ and for $K = 4$ representing $Q_1 =$ "at least 4 elements" from 26 elements of X.

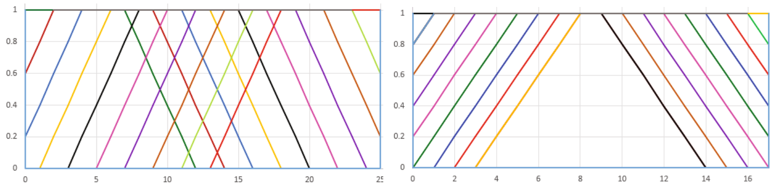
Example 5. In Fig. 2, we expanded the areas covered by each A_i's support as well as kernel and modified corresponding consequents fuzzy sets B_i accordingly. Then the highest overall similarity degree (0.75) is obtained for $Q =$ "at least 9 rules" and for $Q_1 =$ "at least 4 elements from X".



(a) Fuzzy sets on X (b) Fuzzy sets on Y

Fig. 2. Depiction of fuzzy sets in the case when the overall similarity degree reaches the maximum value $\varepsilon_{Q,Q_1} = \mathbf{0.75}$ for the combinations of $Q =$ "at least 9 rules" and $Q_1 =$ "at least 4 elements from X".

Example 6. Finally, we will continue in the process in order to reach the overall similarity degree $\varepsilon_{Q,Q_1} = 1$. After a sufficient expansion of the antecedent fuzzy sets' supports and kernels and the corresponding modification of the consequent fuzzy sets in the same way, see Fig. 3, we obtain the desirable setting.



(a) Fuzzy sets on X (b) Fuzzy sets on Y

Fig. 3. Depiction of fuzzy sets in the case when the overall similarity degree equals to one for the combination of $Q =$ "at least 9 rules" and $Q_1 =$ "at least 3 elements".

3.2 Inf-R System

Let us again consider the computational form of the compositions (10) and finite universes $I_m = \{1, \ldots, m\}$, $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_s\}$. Then $R^Q_{MA}(x, y)$ and $(A \lhd^{Q_1} R)(y)$ are given by

$$R^Q_{MA}(x, y) = \bigvee_{j=1}^{m} \left(\left(A_{\pi(j)}(x) * B_{\pi(j)}(y)\right) * f\left(\frac{j}{m}\right)\right), \tag{24}$$

$$(A \lhd^{Q_1} R)(y) = \bigvee_{k=1}^{n} \left(\left(A(x_{\pi_1(k)}) \rightarrow R(x_{\pi_1(k)}, y)\right) * f_1\left(\frac{k}{n}\right)\right) \tag{25}$$

where π, π_1 are permutations such that

$A_{\pi(j)}(x) * B_{\pi(j)}(y) \geq A_{\pi(j+1)}(x) * B_{\pi(j+1)}(y), j = 1, 2, \ldots, m-1,$ and
$A(x_{\pi_1(k)}) \to R(x_{\pi_1(k)}, y) \geq A(x_{\pi_1(k+1)}) \to R(x_{\pi_1(k+1)}, y), k = 1, 2, \ldots, n-1.$

Before showing some examples, we start with a simple but important observation. Assume that there is $i \in I_m$ such that $A_i(x) > 0$ for only one $x \in X$. Now, let us consider measure μ_1 to be strictly greater than $\mu^\forall$ on X. We may consider, e.g., the measure μ_1 such that $f(\frac{k}{n}) = 0$ for $k \in \{1, \ldots, n-2\}$, $f(\frac{n-1}{n}) = a_{n-1} > 0$ and $f(1) = 1$. Then $(A_i \lhd^{Q_1} R)(y) = a_{n-1} \vee \big(A_i(x_{\pi(m)}) \to R(x_{\pi(m)}, y)\big)$. The smaller the value a_{n-1}, the closer is the value $(A_i \lhd^{Q_1} R)(y)$ to the value $(A_i \lhd R)(y)$ and vice-versa. In particular, if $a_{n-1} = 1$ then $(A_i \lhd^{Q_1} R)(y) = 1$ for any $R \in \mathcal{F}(X \times Y)$ and for any $y \in Y$. If we define the similarity degree between membership degrees to $A_i \lhd^{Q_1} R$ and to B_i as

$$\xi_{Q_1,i} = \bigwedge_{y \in Y} (A_i \lhd^{Q_1} R)(y) \leftrightarrow B_i(y)$$

and the overall similarity degree $\xi_{Q_1} = \bigwedge_{i \in I_m} \xi_{Q_1,i}$, then $\xi_{Q_1} = 0$ whenever there will be some $y \in Y$ such that $B_i(y) = 0$.

Thus, consider more than only one $x \in X$ such that $A_i(x) > 0$ for all $i \in I_m$, however, still without any overlapping areas of the antecedent fuzzy sets A_i. In order to make an observation, let us consider a particular measure $\mu < \mu^\exists$ such that $f(\frac{1}{m}) = b_1 < 1$ and $f(\frac{j}{m}) = 1$ for $j \in \{2, \ldots, m\}$. Then, by formula (24), we have $R^Q_{MA}(x, y) = (A_{\pi(1)} * B_{\pi(1)}) * b_1$. The higher the value b_1, the closer is the value $R^Q_{MA}(x, y)$ to the value R_{MA} and vise-versa. In particular, if $b_1 = 0$ then $R^Q_{MA}(x, y) = 0$ for all pairs (x, y). This implies that $(A_i \lhd^{Q_1} R^Q_{MA})(y) = c$ for any $i \in I_m$ and for any $y \in Y$ which leads to the similarity degree at i

$$\xi_{Q,Q_1,i} = \bigwedge_{y \in Y} (A_i \lhd^{Q_1} R^Q_{MA})(y) \leftrightarrow B_i(y)$$

that is rather low. In particular, for the Łukasiewicz algebra and the consequent fuzzy sets B_i taking all values from the unit interval [0, 1], the similarity will be always lower or equal to 0.5, which derives the overall similarity degree ξ_{Q,Q_1},

$$\xi_{Q,Q_1} = \bigwedge_{i \in I_m} \bigwedge_{y \in Y} (A_i \lhd^{Q_1} R^Q_{MA})(y) \leftrightarrow B_i(y). \tag{26}$$

to be low too.

Analogously, if there is $i \in I_m$ such that $A_i(x) > 0, A_i(x') > 0$ for only two elements $x, x' \in X$ and if we consider measure μ_1 determined by $f(\frac{k}{n}) = 0$ for $k \in \{1, \ldots, n-3\}$, and by $f(\frac{n-2}{n}) = a_{n-2} > 0$ then we would encounter the same situation.

Thus, the analysis leads to the following observation which is the same as in the case of sup-T system:

Observation 6. *For a higher similarity degree, the antecedent fuzzy sets A_i have to be modified in order to overlap in a denser way. And, the higher and closer the membership degrees at each overlapping point, the higher similarity is obtained.*

Example 7. Let us consider fuzzy sets $A_i, B_i, i = 1, \ldots, 10$ depicted in Fig. 1. Now let us run computations for the combinations of fuzzy quantifier Q "at least J rules" with J ranging form 2 to 10, and the fuzzy quantifier Q_1 "at least K elements" with K ranging from 1 to 25. The goal is to find the combination with the maximal overall similarity degree, ξ_{Q,Q_1}.

As the overlapping area of any two antecedent fuzzy sets is rather narrow, only the standard inf-R system is solvable. For the system with fuzzy quantifiers, the calculations provided the maximum value of the similarity degree equal to **0.6**, which occurred for $J = 2$ and $K = 23$ representing $Q_1 =$ "at least 23 elements" from the 26 elements of X.

Example 8. Fuzzy sets constructed in Fig. 2 provided the maximum value of the overall similarity degree equal to **0.75** which is obtained for $Q_1 =$ "at least 24 elements" from the 26 elements of X and $Q =$ "at least 2 rules".

Example 9. In Fig. 3, if we slightly modify fuzzy sets B_1, B_{10} so that $B_1 \equiv B_2, B_9 \equiv B_{10}$, then, the maximum of the overall similarity degree $\xi_{Q,Q_1} = \mathbf{1}$. This result is obtained for $Q_1 =$ "at least 24 elements" from the 26 elements of X and $Q =$ "at least 2 rules".

Let us mention that the experiment has been computed using the "Linguistic Fuzzy Logic" *lfl* v1.3 R-package [1] which contains all necessary functions for the fuzzy relational calculus.

4 Conclusion

The problem of solvability of fuzzy relation equations with fuzzy quantifiers has been firstly considered. Under the assumption that the two standard systems are solvable, we have shown that the two systems of fuzzy relation equations with fuzzy quantifiers are possibly also solvable. However, this holds only in the case of a "sufficient" overlap of antecedent fuzzy sets and, consequently, also of the consequent fuzzy sets. Only in that case we can find appropriate fuzzy quantifiers Q, Q_1 determined by fuzzy measures μ, μ_1 such that $\mu > \mu^{\forall}$ on I_m and $\mu_1 < \mu^{\exists}$ on X so that the equality of $A_i \circ^{Q_1} R^{Q}_{IMP} = B_i$ is preserved.

These results may lead to a construction of generalized systems of fuzzy inference systems that closely represent reasoning in natural language. For example, the implicative rules are aggregated with the universal quantifier which means, that "all" rules have to be met at the same time. With help of fuzzy quantifiers, we may model situations when the reasoning is made, e.g., on "Majority" or "Most" of the rules in the given fuzzy rule base. This provide us with a wider choice of models that may better fit for particular problems incorporated human-like reasoning with natural language that often encompasses a sort of "tolerance".

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Incorporation of Excluding Features in Fuzzy Relational Compositions Based on Generalized Quantifiers

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Abstract. The concepts of incorporation of excluding features in fuzzy relational compositions and the compositions based on generalized quantifiers are useful tools for improving relevance and precision of the suspicion provided by the standard fuzzy relational compositions initial studied by Willis Bandler and Ladislav Kohout. They are independently extended from the standard compositions. However, it may become a very effective tool if they are used together. Taking this natural motivation leads us to introduce the concept of incorporation of excluding features in fuzzy relational compositions based on generalized quantifiers. Most of valid properties preserved for the two mentioned approaches will be proved for the new concept as well. Furthermore, an illustrative example will be presented for showing the usefulness of the approach.

Keywords: Fuzzy relational compositions · Fuzzy relational products · Bandler-Kohout products · Fuzzy measures · Generalized (fuzzy) quantifiers · Medical diagnosis · Classification

1 Introduction

In the late 70's and the early 80's, fuzzy relational compositions (or fuzzy relational products) are firstly studied by Willis Bandler and Ladislav Kohout [1]. After that, they have been studied, extended and developed on various aspects by numerous authors, see e.g. [3,5,13]. They have an important role in many areas of fuzzy mathematics, including the formal constructions of fuzzy inference systems [19,25] and related systems of fuzzy relational equations [12,14,18,21], medical diagnosis [1], architectures of information processing [2] or in flexible queries to relational databases [15].

During such a long time and up to the present time, the development on this topic has been made. It is demonstrated by numerous recent works, for instance, on flexible query answering systems [20], inference systems [16,17,23] or modeling monotone fuzzy rule bases [22]. Other recently interesting directions of the research are the incorporation of excluding features in fuzzy relational compositions [6–8] and the employment of generalized quantifiers on the compositions [9–11,24].

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This paper provides an investigation stemming from the two previous directions on the incorporation of excluding features in fuzzy relational compositions [6–8] and the compositions based on generalized quantifiers [9–11]. The first one is motivated by the existence of excluding symptoms for some particular diseases in the medical problem and it has been applied to the classification problems of animals in biology. The second one is motivated by the possibility to fill a big gap between the basic composition that uses existential quantifier and the Bandler-Kohout products that uses universal quantifier. These approaches both are helpful for improving relevance and precision of the suspicions provided by the standard compositions, i.e., they both help to reducing number of initial suspicions given by the basic "circlet" composition without losing the possibly correct suspicion that often happens when we use Bandler-Kohout products. Furthermore, the problems, that are motivated for forming the two approaches, simultaneously happen in the medical problem or classification problem. Because of this, we propose the concept of *incorporation of excluding features in fuzzy relational compositions based on generalized quantifiers*. The contribution of this extension will be demonstrated on an illustrative example.

2 Preliminaries

We recall some basic definitions of the incorporation of excluding features in fuzzy relational composition [6–8] and the compositions based on generalized quantifiers [9,11,24]. In the sequel, we fix a residuated lattice $\mathcal{L} = \langle [0,1], \wedge, \vee, \otimes, \rightarrow 0, 1\rangle$ as the underlying algebraic structure and we denote by $\mathcal{F}(U)$ the set of all fuzzy sets on a given universe U.

2.1 Excluding Features in Fuzzy Relational Compositions

Definition 1. Let X, Y, Z be non-empty finite universes, let $R \in \mathcal{F}(X \times Y)$, $S, E \in \mathcal{F}(Y \times Z)$. Then the composition $R \circ S`E \in \mathcal{F}(X \times Z)$ is defined:

$$(R \circ S`E)(x,z) = \bigvee_{y\in Y} (R(x,y) \otimes S(y,z)) \otimes \neg \bigvee_{y\in Y} (R(x,y) \otimes E(y,z)) \,.$$

We recall that the semantic that $E(x, y)$ means that y is an excluding feature for object (class) z. There are two other ways to define the composition:

$$(R \circ S`E)^{\lhd}(x,z) = \bigvee_{y\in Y} (R(x,y) \otimes S(y,z)) \otimes \bigwedge_{y\in Y} (R(x,y) \rightarrow \neg E(y,z)) \;,$$

$$(R \circ S`E)^{\triangledown}(x,z) = \bigvee_{y\in Y} (R(x,y) \otimes S(y,z)) \otimes \bigwedge_{y\in Y} (\neg R(x,y) \oplus \neg E(y,z)) \;.$$

Simplified formulas may be given as follows:

$$(R \circ S`E)(x,z) = (R \circ S)(x,z) \otimes \neg(R \circ E)(x,z), \tag{1}$$

$$(R \circ S`E)^{\lhd}(x,z) = (R \circ S)(x,z) \otimes (R \lhd \neg E)(x,z), \tag{2}$$

$$(R \circ S`E)^{\triangledown}(x,z) = (R \circ S)(x,z) \otimes (\neg R \triangledown \neg E)(x,z), \tag{3}$$

where the so-called inf-S composition [18] is defined by

$$(R \,\triangledown\, S)(x,z) = \bigwedge_{y \in Y} \left(R(x,y) \oplus S(y,z)\right).$$

Regarding the conditions for which the three distinct definitions (1)–(3) are equivalent, we recall the following result:

Lemma 1. *[8] Let the underlying algebraic structure* $\langle [0,1], \wedge, \vee, \rightarrow, \otimes, 0, 1\rangle$ *be a complete residuated lattice such that the negation* $\neg a = a \rightarrow 0$ *is strict and* $\otimes$ *has no zero divisors or it is an MV-algebra. Then*

$$(R \circ S' E)(x,z) = (R \circ S' E)^{\triangleleft}(x,z) = (R \circ S' E)^{\triangledown}(x,z).$$

2.2 Fuzzy Relational Compositions Based on Generalized Quantifiers

Definition 2. [9,11,24] Let $U = \{u_1, \ldots, u_n\}$ be a finite universe, let $\mathcal{P}(U)$ denote the power set of U. A mapping $\mu : \mathcal{P}(U) \rightarrow [0,1]$ is called a fuzzy measure on U if $\mu(\emptyset) = 0$ and $\mu(U) = 1$ and if $\forall C, D \in \mathcal{P}(U), C \subseteq D$ then $\mu(C) \leq \mu(D)$. Fuzzy measure μ is called *invariant with respect to cardinality* (w.r.t) if $\forall C, D \in \mathcal{P}(U) : |C| = |D| \Rightarrow \mu(C) = \mu(D)$ where $|\cdot|$ denotes the cardinality of a set.

Definition 3. [9,11,24] A mapping $Q : \mathcal{F}(U) \rightarrow [0,1]$ defined by

$$Q(C) = \bigvee_{D \in \mathcal{P}(U) \setminus \{\emptyset\}} \left(\left(\bigwedge_{u \in D} C(u) \right) \otimes \mu(D) \right), C \in \mathcal{F}(U)$$

is called *generalized (fuzzy) quantifier determined by a fuzzy measure* μ on U.

In case of fuzzy measure μ is invariant w.r.t cardinality, for the simplicity of calculation, the quantifier can be rewritten into a simpler form:

$$Q(C) = \bigvee_{i=1}^{n} C(u_{\pi(i)}) \otimes \mu(\{u_1, \ldots, u_i\}),\, C \in \mathcal{F}(U)$$

where π is a permutation on $\{1, \ldots, n\}$ such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \cdots \geq C(u_{\pi(n)})$.

Definition 4. [9,11,24] Let X, Y, Z be non-empty finite universes, let $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$. Let Q be a quantifier on Y determined by a fuzzy measure μ. Then, the compositions $R@^Q S$ where $@ \in \{\circ, \triangleleft, \triangleright, \square\}$ are defined as follows:

$$(R@^Q S)(x,z) = \bigvee_{D \in \mathcal{P}(Y) \setminus \{\emptyset\}} \left(\left(\bigwedge_{y \in D} R(x,y) \circledast S(y,z) \right) \otimes \mu(D) \right)$$

for all $x \in X$, $z \in Z$ and for $\circledast \in \{\otimes, \rightarrow, \leftarrow, \leftrightarrow\}$ corresponding to the composition @.

3 Excluding Features in Fuzzy Relational Compositions Based on Generalized Quantifiers

In this section, we directly combine Definitions 1 and 4 to define the new concept of excluding features in fuzzy relational compositions based on generalized quantifiers and to study its properties.

Definition 5. Let X, Y, Z be non-empty finite universes, let $R \in \mathcal{F}(X \times Y)$, $S, E \in \mathcal{F}(Y \times Z)$. Let Q be a quantifier on Y determined by a fuzzy measure μ. Then, $(R \circ^Q S^{\backprime} E), (R \lhd^Q S^{\backprime} E), (R \rhd^Q S^{\backprime} E), (R \square^Q S^{\backprime} E)$ are fuzzy relations on $X \times Z$ defined as follows:

$$(R \circ^Q S^{\backprime} E)(x, z) = (R \circ^Q S)(x, z) \otimes \neg(R \circ E)(x, z), \tag{4}$$

$$(R \lhd^Q S^{\backprime} E)(x, z) = (R \lhd^Q S)(x, z) \otimes \neg(R \circ E)(x, z), \tag{5}$$

$$(R \rhd^Q S^{\backprime} E)(x, z) = (R \rhd^Q S)(x, z) \otimes \neg(R \circ E)(x, z), \tag{6}$$

$$(R \square^Q S^{\backprime} E)(x, z) = (R \square^Q S)(x, z) \otimes \neg(R \circ E)(x, z), \tag{7}$$

for all $x \in X$ and $z \in Z$.

The proposed compositions provide desirable meanings as we use the generalized quantifiers such as ‘Most’. For example, in the classification problem of animals in biology, $(R \square^Q S^{\backprime} E)(x, z)$ means that animal x has ‘Most’ of the features of a given family and ‘Most’ of the features of x is related to a given family and at the same time there is no excluding features related to family z carried by the animal.

Note, that each expression in Definition 5 can be rewritten in two other ways. For examples, (7) can be rewritten as follows:

$$(R \square^Q S^{\backprime} E)(x, z) = (R \square^Q S)(x, z) \otimes \neg(R \circ E)(x, z),$$
$$(R \square^Q S^{\backprime} E)^{\lhd}(x, z) = (R \square^Q S)(x, z) \otimes (R \lhd \neg E)(x, z),$$
$$(R \square^Q S^{\backprime} E)^{\triangledown}(x, z) = (R \square^Q S)(x, z) \otimes (\neg R \triangledown \neg E)(x, z).$$

And expressions (4)–(6) can be rewritten analogously.

Regarding the relationship of the three distinct definitions of each expression in Definition 5, we have the following results:

Lemma 2.

$$R \circ^Q S^{\backprime} E = (R \circ^Q S^{\backprime} E)^{\lhd}, \qquad R \circ^Q S^{\backprime} E \supseteq (R \circ^Q S^{\backprime} E)^{\triangledown}, \tag{8}$$

$$R \lhd^Q S^{\backprime} E = (R \lhd^Q S^{\backprime} E)^{\lhd}, \qquad R \lhd^Q S^{\backprime} E \supseteq (R \lhd^Q S^{\backprime} E)^{\triangledown}, \tag{9}$$

$$R \rhd^Q S^{\backprime} E = (R \rhd^Q S^{\backprime} E)^{\lhd}, \qquad R \rhd^Q S^{\backprime} E \supseteq (R \rhd^Q S^{\backprime} E)^{\triangledown} \tag{10}$$

$$R \square^Q S^{\backprime} E = (R \square^Q S^{\backprime} E)^{\lhd}, \qquad R \square^Q S^{\backprime} E \supseteq (R \rhd^Q S^{\backprime} E)^{\triangledown}. \tag{11}$$

Sketch of the proof: The proof is similar to the proof of Lemma 11 [8]. For example, the first equality of property (11) is proved based on the facts that $\neg \bigvee_{i\in I} a_i = \bigwedge_{i\in I} \neg a_i$ for an arbitrary index set I and $(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c)$, and the second one is derived from $\neg(a \otimes b) \geq \neg a \oplus \neg b$ in the residuated lattice. □

Another raised question is, under which conditions, the inclusions shown above become the equalities. Similarly to the investigation of excluding features in standard fuzzy relational compositions, we obtain the following result.

Lemma 3. *Let the underlying algebraic structure $\langle [0,1], \wedge, \vee, \rightarrow, \otimes, 0, 1\rangle$ be a complete residuated lattice such that the negation $\neg a = a \rightarrow 0$ is strict and $\otimes$ has no zero divisors or it is an MV-algebra. Then*

$$R \circ^Q S^{\backslash} E = (R \circ^Q S^{\backslash} E)^{\triangledown}\,, \tag{12}$$

$$R \lhd^Q S^{\backslash} E = (R \lhd^Q S^{\backslash} E)^{\triangledown}\,, \tag{13}$$

$$R \rhd^Q S^{\backslash} E = (R \rhd^Q S^{\backslash} E)^{\triangledown}\,, \tag{14}$$

$$R \square^Q S^{\backslash} E = (R \square^Q S^{\backslash} E)^{\triangledown}\,. \tag{15}$$

Sketch of the proof: The proof is similar to the proof of Lemma 12 [8]. □

4 Properties

In this section, we present some valid properties of the proposed compositions which might be very useful. As will be shown, many properties of the fuzzy relational compositions using generalized quantifiers and the compositions incorporating excluding features are still preserved for the proposed compositions.

Let us fix the notation, let $R \in \mathcal{F}(X \times Y)$, let $S, S_1, S_2, E, E_1, E_2 \in \mathcal{F}(Y \times Z)$ and furthermore, let $\cup, \cap$ denote the Gödel union and intersection, respectively.

Proposition 1 *(Containment).*

$$R \circ^Q S^{\backslash} E \subseteq R \circ^Q S\,, \qquad R \circ^Q S^{\backslash} E \subseteq R \lhd \neg E \tag{16}$$

$$R \lhd^Q S^{\backslash} E \subseteq R \lhd^Q S\,, \qquad R \lhd^Q S^{\backslash} E \subseteq R \lhd \neg E \tag{17}$$

$$R \rhd^Q S^{\backslash} E \subseteq R \rhd^Q S\,, \qquad R \rhd^Q S^{\backslash} E \subseteq R \lhd \neg E \tag{18}$$

$$R \square^Q S^{\backslash} E \subseteq R \square^Q S\,, \qquad R \square^Q S^{\backslash} E \subseteq R \lhd \neg E. \tag{19}$$

Sketch of the proof: Using the properties $a \otimes b \leq a, a \otimes b \leq b$, we get

$$(R \circ^Q S^{\backslash} E)(x,z) \leq (R \circ^Q S)(x,z)$$

for all $(x,z) \in X \times Z$. Thus, $R \circ^Q S^{\backslash} E \subseteq R \circ^Q S$. The other inclusions are proved analogously. □

Proposition 2 *(Monotonicity).*

$$S_1 \subseteq S_2 \Rightarrow R \circ^Q S_1\lq E \subseteq R \circ^Q S_2\lq E\,, \tag{20}$$
$$E_1 \subseteq E_2 \Rightarrow R \circ^Q S\lq E_1 \supseteq R \circ^Q S\lq E_2\,, \tag{21}$$
$$S_1 \subseteq S_2 \Rightarrow R \lhd^Q S_1\lq E \subseteq R \lhd^Q S_2\lq E\,, \tag{22}$$
$$E_1 \subseteq E_2 \Rightarrow R \lhd^Q S\lq E_1 \supseteq R \lhd^Q S\lq E_2\,, \tag{23}$$
$$R_1 \subseteq R_2 \Rightarrow R_1 \lhd^Q S\lq E \supseteq R_2 \lhd^Q S\lq E\,, \tag{24}$$
$$S_1 \subseteq S_2 \Rightarrow R \rhd^Q S_1\lq E \supseteq R \rhd^Q S_2\lq E\,, \tag{25}$$
$$E_1 \subseteq E_2 \Rightarrow R \rhd^Q S\lq E_1 \supseteq R \rhd^Q S\lq E_2\,, \tag{26}$$
$$E_1 \subseteq E_2 \Rightarrow R \,\square^Q S\lq E_1 \supseteq R \,\square^Q S\lq E_2\,. \tag{27}$$

Sketch of the proof: The proof is based on properties $a \to c \geq b \to c$ whenever $a \leq b$ and $a \to b \leq a \to c$ whenever $b \leq c$. Indeed,

$$\begin{aligned}(R \circ^Q S_1\lq E)(x,z) &= (R \circ^Q S_1)(x,z) \otimes \neg(R \circ E)(x,z)\\ &\leq (R \circ^Q S_2)(x,z) \otimes \neg(R \circ E)(x,z)\end{aligned}$$

and

$$\begin{aligned}(R \circ^Q S\lq E_1)(x,z) &= (R \circ^Q S)(x,z) \otimes \neg(R \circ E_1)(x,z)\\ &\geq (R \circ^Q S)(x,z) \otimes \neg(R \circ E_2)(x,z).\end{aligned}$$

prove (20)–(21). Properties (22)–(27) are proved similarly. □

Proposition 3 *(Interaction with union).*

$$R \circ^Q (S_1 \cup S_2)\lq E \supseteq (R \circ^Q S_1\lq E) \cup (R \circ^Q S_2\lq E), \tag{28}$$
$$R \lhd^Q (S_1 \cup S_2)\lq E \supseteq (R \lhd^Q S_1\lq E) \cup (R \lhd^Q S_2\lq E), \tag{29}$$
$$R \rhd^Q (S_1 \cup S_2)\lq E \subseteq (R \rhd^Q S_1\lq E) \cap (R \rhd^Q S_2\lq E), \tag{30}$$
$$R \circ^Q S\lq(E_1 \cup E_2) = (R \circ^Q S\lq E_1) \cap (R \circ^Q S\lq E_2), \tag{31}$$
$$R \lhd^Q S\lq(E_1 \cup E_2) = (R \lhd^Q S\lq E_1) \cap (R \lhd^Q S\lq E_2), \tag{32}$$
$$R \rhd^Q S\lq(E_1 \cup E_2) = (R \rhd^Q S\lq E_1) \cap (R \rhd^Q S\lq E_2), \tag{33}$$
$$R \,\square^Q S\lq(E_1 \cup E_2) = (R \,\square^Q S\lq E_1) \cap (R \,\square^Q S\lq E_2), \tag{34}$$

Sketch of the proof:

$$\begin{aligned}(R \circ^Q (S_1 \cup S_2)\lq E)(x,z) &= (R \circ^Q (S_1 \cup S_2))(x,z) \otimes \neg(R \circ E)(x,z)\\ &\geq \left((R \circ^Q S_1) \cup (R \circ^Q S_2)\right)(x,z) \otimes \neg(R \circ E)(x,z)\\ &= \left((R \circ^Q S_1\lq E) \cup (R \circ^Q S_2\lq E)\right)(x,z)\end{aligned}$$

for all $(x,z) \in X \times Z$ which proves (28). Property (31) is due to

$$\begin{aligned}(R \circ^Q S\lq(E_1 \cup E_2))(x,z) &= (R \circ^Q S)(x,z) \otimes \neg(R \circ (E_1 \cup E_2))(x,z)\\ &= (R \circ^Q S)(x,z) \otimes (\neg(R \circ E_1)(x,z) \wedge \neg(R \circ E_2)(x,z))\\ &= ((R \circ^Q S\lq E_1) \cap (R \circ^Q S\lq E_2))(x,z).\end{aligned}$$

The other properties are proved analogously. □

Proposition 4. *(Interaction with intersection).*

$$R \circ^Q (S_1 \cap S_2){\backprime}E \subseteq (R \circ^Q S_1{\backprime}E) \cap (R \circ^Q S_2{\backprime}E), \tag{35}$$
$$R \lhd^Q (S_1 \cap S_2){\backprime}E \subseteq (R \lhd^Q S_1{\backprime}E) \cap (R \lhd^Q S_2{\backprime}E), \tag{36}$$
$$R \rhd^Q (S_1 \cap S_2){\backprime}E \supseteq (R \rhd^Q S_1{\backprime}E) \cup (R \rhd^Q S_2{\backprime}E), \tag{37}$$
$$R \circ^Q S{\backprime}(E_1 \cap E_2) = (R \circ^Q S{\backprime}E_1) \cup (R \circ^Q S{\backprime}E_2), \tag{38}$$
$$R \lhd^Q S{\backprime}(E_1 \cap E_2) = (R \lhd^Q S{\backprime}E_1) \cup (R \lhd^Q S{\backprime}E_2), \tag{39}$$
$$R \rhd^Q S{\backprime}(E_1 \cap E_2) = (R \rhd^Q S{\backprime}E_1) \cup (R \rhd^Q S{\backprime}E_2), \tag{40}$$
$$R \square^Q S{\backprime}(E_1 \cap E_2) = (R \square^Q S{\backprime}E_1) \cup (R \square^Q S{\backprime}E_2), \tag{41}$$

Sketch of the proof:

$$\begin{aligned}(R \circ^Q (S_1 \cap S_2){\backprime}E)(x,z) &= (R \circ^Q (S_1 \cap S_2))(x,z) \otimes \neg(R \circ E)(x,z)\\ &\leq \big((R \circ^Q S_1) \cap (R \circ^Q S_2)\big)(x,z) \otimes \neg(R \circ E)(x,z)\\ &= \big((R \circ^Q S_1{\backprime}E) \cap (R \circ S_2{\backprime}E)\big)(x,z).\end{aligned}$$

which proves (35). And (38) is implied from

$$\begin{aligned}(R \circ^Q S{\backprime}(E_1 \cap E_2))(x,z) &= (R \circ^Q S)(x,z) \otimes \neg(R \circ (E_1 \cap E_2))(x,z)\\ &= (R \circ^Q S)(x,z) \otimes (\neg(R \circ E_1)(x,z) \vee \neg(R \circ E_2)(x,z))\\ &= ((R \circ^Q S{\backprime}E_1) \cup (R \circ^Q S{\backprime}E_2))(x,z).\end{aligned}$$

The other properties are proved similarly. □

Proposition 5 *(Interdefinability).*

$$R \square^Q S{\backprime}E \subseteq (R \lhd^Q S{\backprime}E) \cap (R \rhd^Q S{\backprime}E). \tag{42}$$

Sketch of the proof: The proof is derived from the property $R \square^Q S \subseteq (R \lhd^Q S) \cap (R \rhd^Q S)$. Indeed,

$$\begin{aligned}(R \square^Q S{\backprime}E)(x,z) &= (R \square^Q E)(x,z) \otimes \neg(R \circ E)(x,z)\\ &\leq \big((R \lhd^Q S)(x,z) \wedge (R \rhd^Q S)(x,z)\big) \otimes \neg(R \circ E)(x,z)\\ &= (R \lhd^Q S{\backprime}E)(x,z) \wedge (R \rhd^Q S{\backprime}E)(x,z).\end{aligned}$$

□

Proposition 6. *Let Q_1, Q_2 be quantifiers determined by fuzzy measures μ_1, μ_2, respectively, such that $\mu_1 \leq \mu_2$. Then,*

$$R \circ^{Q_1} S{\backprime}E \subseteq R \circ^{Q_2} S{\backprime}E\,, \tag{43}$$
$$R \lhd^{Q_1} S{\backprime}E \subseteq R \lhd^{Q_2} S{\backprime}E\,, \tag{44}$$
$$R \rhd^{Q_1} S{\backprime}E \subseteq R \rhd^{Q_2} S{\backprime}E\,, \tag{45}$$
$$R \square^{Q_1} S{\backprime}E \subseteq R \square^{Q_2} S{\backprime}E\,. \tag{46}$$

Sketch of the proof: The proof is based on Lemma 1 in [9]. For example, property (43) is due to

$$\begin{aligned}(R \circ^{Q_1} S^{\backprime} E)(x,z) &= (R \circ^{Q_1} S)(x,z) \otimes \neg(R \circ E)(x,z)\\ &\leq (R \circ^{Q_2} S)(x,z) \otimes \neg(R \circ E)(x,z)\\ &= (R \circ^{Q_2} S^{\backprime} E)(x,z).\end{aligned}$$

□

5 Illustrative Example

Let us demonstrate the influence of the incorporation of the excluding features into the fuzzy relational compositions based on generalized quantifiers on an illustrative example. All the calculations are computed using the "Linguistic Fuzzy Logic" *lfl* v1.3 R-package [4].

For the convenience of comparing with the former methods, let us consider the problem of classification of animals mentioned in [6] or in [8]. Let Z be a set of families of animals (z_1 - Bird, z_2 - Fish, z_3 - Dog, z_4 - Equidae, z_5 - Mosquito, z_6 - Monotreme, z_7 - Reptile), Y be a set of animal features (y_1 - flies, y_2 - feathers, y_3 - fins, y_4 - claws, y_5 - hair, y_6 - teeth, y_7 - beak, y_8 - scales, y_9 - swims) and let X be a set of particular animals (x_1 - Platypus, x_2 - Emu, x_3 - Hairless dog, x_4 - Aligator, x_5 - Parrotfish, x_6 - Puffin). Let $R \in \mathcal{F}(X \times Y)$, $S, E \in \mathcal{F}(Y \times Z)$. Our task is to classify the animals to their families. Let $S, E \in \mathcal{F}(Y \times Z)$ and $R \in \mathcal{F}(X \times Y)$ be given as follows

S	z_1	z_2	z_3	z_4	z_5	z_6	z_7
y_1	0.8	0	0	0	1	0	0
y_2	1	0	0	0	0	0	0
y_3	0	1	0	0	0	0.5	0
y_4	0.9	0	1	0	0	0.8	0.3
y_5	0	0	0.8	1	0	0.9	0
y_6	0	0.6	1	1	0	0	0.7
y_7	1	0.1	0	0	0	0.5	0
y_8	0.7	0.9	0	0	0	0	1
y_9	0.5	1	0.8	0.6	0.1	0.7	0.8

E	z_1	z_2	z_3	z_4	z_5	z_6	z_7
y_1	0	1	1	1	0	1	1
y_2	0	1	1	1	1	1	1
y_3	1	0	1	1	1	0	1
y_4	0	1	0	1	1	0	0
y_5	0.8	1	0	0	1	0	1
y_6	1	0	0	0	1	1	0
y_7	0	0.1	1	1	1	0	1
y_8	0	0	1	0	1	1	0
y_9	0	0	0	0	0.8	0	0

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
x_1	0	0	0	1	1	0	1	0	0.9
x_2	0	1	0	1	0	0	1	0.5	0.4
x_3	0	0	0	1	0.2	1	0	0	0.7
x_4	0	0	0	1	0	1	0	1	0.9
x_5	0	0	1	0	0	0.9	0.8	1	1
x_6	1	1	0	1	0	0	1	0.4	0.9

We use the Łukasiweicz algebra as the underlying algebraic structure. The basic compositions $\circ$ gives us too much suspicions as we see below and if we calculate the square product $R \square S$, we get nearly no more suspicion.

$R \circ S$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	1	0.9	1	1	0	0.9	0.7
x_2	1	0.4	1	0	0	0.8	0.5
x_3	0.9	0.7	1	1	0	0.8	0.7
x_4	0.9	0.9	1	1	0	0.8	1
x_5	0.8	1	0.9	0.9	0.1	0.7	1
x_6	1	0.9	1	0.5	1	0.8	0.7

$R \square S$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0	0	0	0	0	0.5	0
x_2	0.2	0	0	0	0	0	0
x_3	0	0	0.4	0	0	0	0
x_4	0	0	0	0	0	0	0.3
x_5	0	0.3	0	0	0	0	0
x_6	0.6	0	0	0	0	0	0

There are two methods that help to eliminate false initial suspicions without lowering the membership degrees to the correct families. The first one is the use of excluding features and the second one is that the compositions based on generalized quantifiers. In this context, we consider fuzzy set modeling the meaning of the linguistic expression `Roughly Big` (abbr. `RoBi`) which enables us to construct a generalized quantifier Q = **"Majority"**. In a standard context, this fuzzy set takes values $\texttt{RoBi}(1/9) = 0, \texttt{RoBi}(2/9) = 0, \texttt{RoBi}(3/9) = 0, \texttt{RoBi}(4/9) = 0, \texttt{RoBi}(5/9) = 0, \texttt{RoBi}(6/9) = 0.178, \texttt{RoBi}(7/9) = 0.861$ and $\texttt{RoBi}(8/9) = 1, \texttt{RoBi}(1) = 1$. We obtain the results:

$R \circ S^{\backprime}E$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0.2	0	0	0	0	0.9	0
x_2	1	0	0	0	0	0	0
x_3	0	0	1	0	0	0	0.5
x_4	0	0	0	0	0	0	1
x_5	0	1	0	0	0	0	0
x_6	1	0	0	0	0	0	0

$R \square^Q S$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0	0	0.6	0	0	0.6	0
x_2	0.8	0	0	0	0	0.3	0.1
x_3	0	0	0.9	0.7	0	0.3	0.5
x_4	0	0.4	0.7	0	0	0	0.7
x_5	0	0.7	0	0	0	0.1	0.5
x_6	0.7	0	0	0	0	0	0

As we can see from the first result, each animal is classified to one correct family excepting x_3 - Hairless dog, this animal is suspicious of belonging to two families, z_3 (Dog) and z_7 (Reptile). For the second one, there are still a number of suspicions of animals and the families. In some problems like medical diagnosis problems, it may happen that each patient has many diseases. However, for the problem of classification of animals in biology, we expect that each animal is classified to only one family. Thus, from the result of using $R \circ S^{\backprime}E$, it will be better if we can eliminate one of two families, z_3, z_7 from the suspicion of belonging of x_3 - Hairless dog.

Now let us apply the suggested fuzzy relational composition $R \square^Q S^{\backprime}E$. As we can see below, the composition helped to eliminate the false suspicion provided by $R \circ S^{\backprime}E$ and $R \square^Q S$ without endangering the correct classification. In particular, the membership degree of x_3 (Hairless dog) into the family z_7 (Reptile), that had been determined to 0.5 by the both above mentioned compositions, has been decreased to 0.3 by their combination.

$R \square^Q S^{\backslash} E$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0	0	0	0	0	0.6	0
x_2	0.8	0	0	0	0	0	0
x_3	0	0	0.9	0	0	0	0.3
x_4	0	0	0	0	0	0	0.7
x_5	0	0.7	0	0	0	0	0
x_6	0.7	0	0	0	0	0	0

6 Discussion

Of course, the results are fully dependent on the choice of the fuzzy relation E with excluding features, which is a task based on expert knowledge that is at disposal. The practical example above may seem not enough convincing as the composition $R \circ S^{\backslash} E$ already provided very good results. On the other hand, it has to be taken into account, that the example is only illustrative and very simple. In practice, the number of objects, features and classes is much higher and then, the determination of the fuzzy relation E is often a difficult step that does not lead to so idealistic results as in the case of the above introduced example.

We recall the application of excluding features in fuzzy relational compositions into the expert system for classification of Odonata (dragonfly) [8]. That application dealt with 140 families of dragonflies, nearly 106 thousand objects and 60 features. In such case, the excluding features matrix E determined by an expert may significantly help to improve the results, as demonstrated in the recalled work [8], on the other hand, hardly may lead to a unique classification family. The expert system suggested a sort of set of "guessed classes" with 20 classes on average. In other words, the expert system with nearly perfect accuracy (98.9%) decreased the uncertainty of the correct class from 140 classes to only 20 classes. Although such results were not obtained by any other tested approach, one could hardly say that there would be no room for any improvement. Combination with generalized quantifiers provides a potential to obtain such an improvement but, has to be carefully tested first.

So, let the above introduced example is viewed only as for the sake of illustration of the behavior of such a combination, not as a real practical example to convince readers about superiority of the combination compared to the other approaches.

7 Conclusion

We have recalled the two methods extending the standard fuzzy relational compositions and we have proposed the new concept of excluding features in fuzzy relational compositions based on generalized quantifiers that can be considered as a combination of the two methods. The strength of the proposed approach is that it can help to solve two problems which are existing in compositions: filling in a huge gap between existential and universal quantifiers and solving the problems of existence of excluding features for some particular objects. Apart from providing the definitions and showing the corresponding properties, we have presented that a list of valid properties holding for the two mentioned approaches are still preserved for the proposed compositions. Moreover, the use of excluding features in fuzzy relational compositions based on generalized quantifiers was considered on an illustrative example which was stemming from a purpose: to see how it can be more effective than the use of the single approach.

These results have significantly contributed for extending the standard fuzzy relational compositions that is widely used in many areas of application, such as inference systems, flexible query answering systems in database applications.

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A New Optimization Metaheuristic Based on the Self-defense Techniques of Natural Plants Applied to the CEC 2015 Benchmark Functions

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Abstract. A new optimization metaheuristic algorithm based on the mechanisms of self-defense of plants in nature in this work is presented. The proposed optimization algorithm is applied to optimize mathematical functions of CEC 2015, this suite of functions are proposed as a challenge for the area of algorithm bio-inspired, with the purpose of creating a competition of performance and stability between algorithms of search and optimization. We propose a new meta-heuristic inspired in the coping techniques of plants in nature, as there techniques are developed by plants as a defense from predators. The proposed algorithm is based on the Lotka and Volterra model better known as the prey predator model, this model consists of two non-linear equations and is used to model the growth of two populations that competing with each other.

Keywords: Aggressor · Lotka and Volterra model · Mechanism · Plants · Self-defense · Lévy flights

1 Introduction

In the literature there are many optimization algorithms that have been applied to multiple problems and in some cases are successful and in others not. Each algorithm is selected depending on the problem to be solved. In the area of engineering sciences and computation there have been different meta-heuristics of optimization that have ben proposed, such as Particle swarm optimization (PSO) [15], Genetic algorithm (GA) [12], Flower pollination algorithm (FPA), Gravitational Search Algorithm (SGA), Ant colony optimization (ACO), Bee colony optimization (BCO) [1, 8].

The aforementioned algorithms have been applied to various problems such as optimization of neurons, in a neural network to improve the level of recognition of people's faces, others for optimizing fuzzy controllers, other authors apply it to optimize mathematical functions in some cases normal functions and also in hybrid or composite functions [14, 15, 17].

All over the planet, all living organisms are exposed to a large number of threats that inhabit the environment. Therefore they force us to be in constant fight and adaptation to be immune to this type of threats [3, 4, 6]. The meta-heuristic proposed in

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this paper uses as a basis the Lotka and Volterra model predator-prey, which is a system formed by two nonlinear equations, are used to model the behavior of two interacting populations [7, 10].

2 Self-defense Techniques of Plants in the Nature

In nature, all living organisms on the planet are constantly fighting against different predators (fungi, bacteria, for mention some), which cause extinction or death of species [8, 13, 14, 17, 18].

Self-defense techniques are natural or developed processes that protect every living organism against different threats. The Plants are also sensitive to different stimuli. In [3–5, 9] the authors define the mechanisms of defense of plants in nature.

In Fig. 1 we can observe a general scheme of the behavior of the plants when they detect the attack of a predatory organism [3, 4, 9].

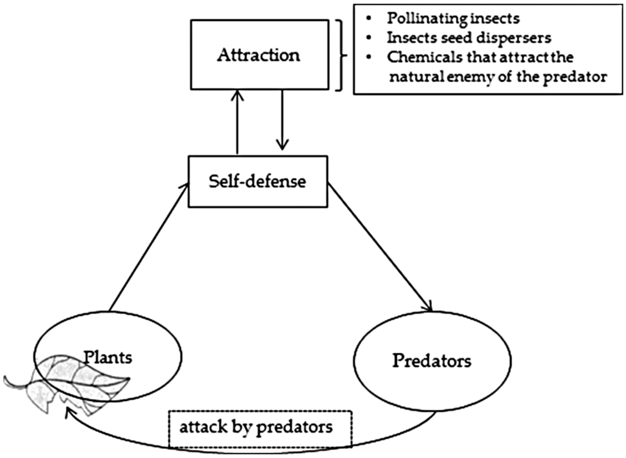

Fig. 1. Illustration of self-defense techniques

All plants have different strategies for example, defense techniques against predators, also techniques for adaptation to different climates such as humid, cold, sunny areas, these strategies prevent the extension of the species. However at the same time the predatory species also develop other adaptation and coping techniques, therefore both species prey and predator are always in constant fight for the survival of the best. In this work we only consider the self-defense strategies of the plants in nature.

3 Predator-Prey Model

The Lotka-Volterra equations are a biomathematical model that represents the growth of two populations interacting with each other, and the model is formed by the following Eqs. (1) and (2) [1, 3, 10–12]:

$$\frac{dx}{dt} = \alpha x - \beta xy \quad (1)$$

$$\frac{dy}{dt} = -\delta xy + \lambda y \quad (2)$$

The definition of the parameters are observed below.
Where:

$\boldsymbol{x}$: Represent the number of prey
$\boldsymbol{y}$: Represent the number of predators
$\frac{dx}{dt}$ Represent the growth of the population of prey time t
$\frac{dy}{dt}$ Represent the growth of the population of predator at time t
$\boldsymbol{\alpha}$: It represents the birth rate of prey in the absence of predator
$\boldsymbol{\beta}$: It represents the death rate of predators in the absence of prey.
$\boldsymbol{\delta}$: Measures the susceptibility of prey.
λ: Measures the ability of predation.

4 Case Study

In this work we propose a new application of the algorithm of plant defense, in the previous works the algorithm was used to optimize traditional mathematical benchmark functions with different methods of biological reproduction [5, 6].

In this work the proposed metaheuristic was used to optimize the functions of CEC 2015 competition, in this set of functions there are some that are composed and others that are hybrid [12, 15, 18]. Figure 2 shows a graphical representation of the proposed algorithm and the prey predator model.

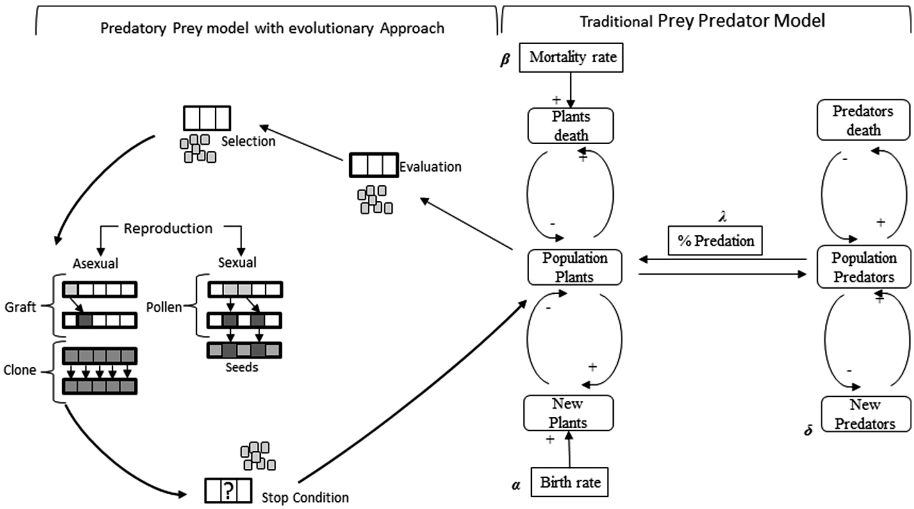

Fig. 2. General representation of metaheuristics

In Fig. 2 we can observe both methods, the predatory prey and our proposal, where we can observe that we are applying evolutionary processes in the plant population, however the population of predators are also affected since its size is dependent on the size of the population of prey in time (t).

Plants and any other living thing in nature have different methods of biological reproduction, and in this work we are only considering the most common. For example: pollen, clone and graft. In [5, 6, 9], the authors define the different reproduction operators.

Prey and predator populations are created using the Lotka and Volterra equations, where equation one is used to generate the population of the plants and Eq. (2) is used to generate the population of the predators. In Fig. 3 shows a diagram that describes the steps of the optimization metaheuristics.

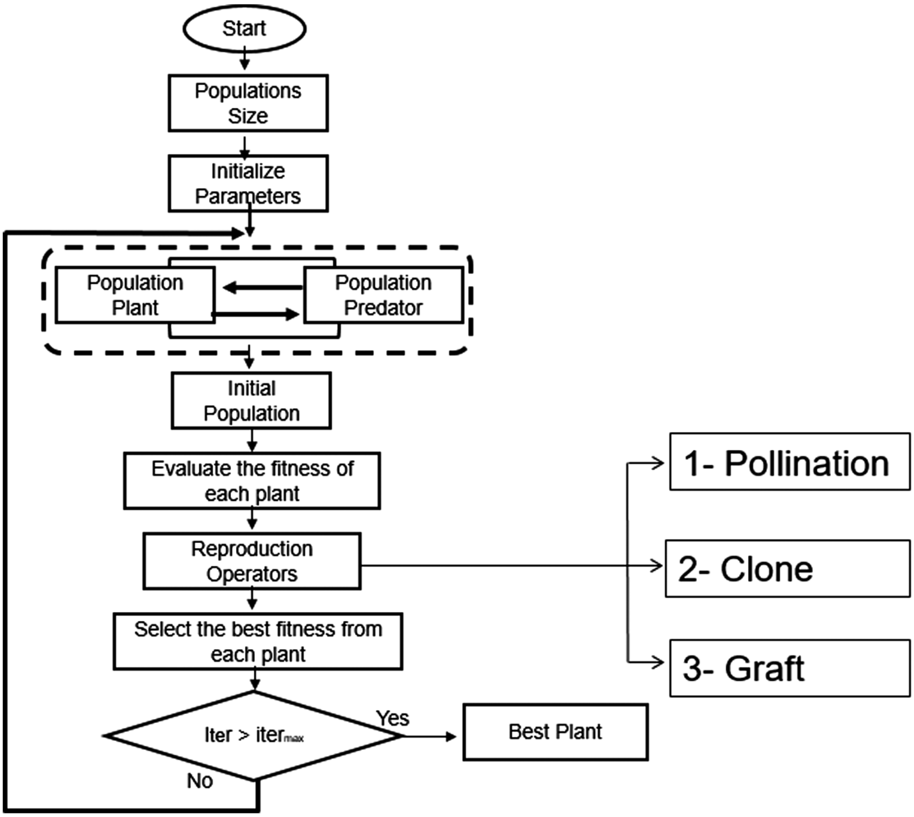

Fig. 3. Flowchart illustrating the proposed algorithm

Population sizes (prey, predators) and the values of the variables (α, β, δ, λ) In [4, 6] we explain the recommended values for the variables used in the equations of the model in our proposal and also the values recommended by the creators of the prey and predator model. In [3–5] the authors publish results with other variants of the algorithm and we can also consult the definition of the proposed biological operators for this algorithm.

5 Simulation Results

This section shows the results obtained from the experiments performed using the optimization algorithm bioinspired on the self-defense mechanisms of the plants to the set of eight functions of the CEC-2015 [15, 18]. Based on previous publications the authors recommend using the method of pollination as reproduction operator, because it has a higher performance. 30 experiments were performed for the following mathematical functions and the evaluation is for 10, 30 Variables, Some data of the functions used can be find in Table 1, for more information of the functions please review [15, 18].

Table 1. Mathematical functions

Type	No.	Function
Unimodal functions	1	Rotated high conditioned elliptic function
	2	Rotated cigar function
Simple multimodal functions	3	Shifted and rotated Ackley's function
	4	Shifted and rotated Rastrigin's function
	5	Shifted and rotated Schwefel's function
Hybrid functions	6	Hybrid function 1 (N = 3)
	7	Hybrid function 2 (N = 4)
	8	Hybrid function 3 (N = 5)

6 Parameters for the Algorithm

For this work the parameters for the variables of ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$), were moved in a specific range, as mentioned before some publicaciones of the algorithm where the authors recommend a range of optimum values to improve the performance of the meta-heuristic [5, 6]. And also the configuration of other parameters such as the size of populations of prey (plants), predators (herbivores).

For the CEC 2015 function problem, they recommend a range of values to be able to compete against the results found by other algorithms, in this work we only want to show that the proposed algorithm can also be used to optimize complex problems. The configuration parameters are defined below: we use plants = 400, Herbivores = 350, and the ranges for the iterations were 1000–900 to observe the behavior.

In Table 2 we can find the results obtained for the case study used in this paper. In the table we observed the results of 30 experiments for each function, using 10 and 30 dimensions, we consider important to the reader the following data the worse, best, average, and standard deviation [4, 6].

Table 2. Results for 10 dimensions

Function	Important results of the algorithm			
	Best	Worse	σ	Average
F1	4.95E+04	4.38E+06	1.06E+06	1.03E+06
F2	1.40E+05	2.87E+06	7.45E+05	1.15E+06
F3	2.00E+01	2.04E+01	1.08E-01	2.03E+01
F4	8.08E+00	6.67E+01	1.67E+01	2.69E+01
F5	2.45E+02	1.08E+03	2.07E+02	6.24E+02
F6	3.55E+02	4.75E+04	8.66E+03	5.93E+03
F7	1.42E+00	1.23E+01	1.96E+00	2.89E+00
F8	9.41E+02	6.75E+03	1.34E+03	2.30E+03

Table 3. Results for 30 dimensions

Function	Important results of the algorithm			
	Best	Worse	σ	Average
F1	2.80E+06	2.94E+07	6.77E+06	1.199E+07
F2	1.74E+07	7.07E+09	1.34E+09	4.275E+08
F3	2.02E+01	2.10E+01	1.58E-01	2.09E+01
F4	1.62E+02	2.99E+02	3.90E+01	2.132E+02
F5	2.67E+03	5.54E+03	7.77E+02	3.91E+03
F6	3.57E+02	4.86E+04	8.82E+03	5.14E+03
F8	2.67E+04	1.18E+06	2.29E+05	2.22E+05

In Tables 2 and 3, we show the results obtained from 30 experiments performed for 10 and 30 variables, we can observe that in the experiments it was very difficult to approximate the value of the function to zero. The mathematical functions used are very complex, some are hybrid, multimodal and composite, and this increases the complexity therefore the algorithms need to be more efficient or use the help of other intelligent techniques such as fuzzy logic or the hybridization with another optimization algorithm. It is important to mention that some of the functions do not have their objetive value as zero.

However, in some functions, for example: f2, f4, f7, the algorithm was able to find a very good near to zero, in comparison to the others, this was for 10 variables see Table 2. Also in Table 3 we can observe the performance of the algorithm for 30 variables, where we only succeeded in the following functions: f3, f4, f6.

To conclude this work it is necessary to make a statistical comparison against other published results, the test used is z-test, in Table 4 we can observe the parameters used in this test, the results obtained with the algorithm of the mechanisms of the plants (MSPA) are compared with the Dynamic Search Fireworks Algorithm (dynFWA) [20].

Table 4. Parameters for statistical comparison.

Parameters	Values
Level of significance	0.05%
Ha	$\mu_1 < \mu_2$
H0	$\mu_1 \geq \mu_2$
p value	1.6715

In applying the statistic Z-test, with a significance level of 0.05, and the alternative hypothesis says that the average of the proposed method is lower than the average of dynFWA [20], and of course the null hypothesis tells us that the average of the proposed method is greater than or equal to the average of dynFWA [20], with a rejection region for all values fall below of **−1.6715**. In the Table 5 we can observe the results of the statistical comparison.

Table 5. Results of applying the statistical z-test for 10D.

Case study	Our method	DynFWA	Z-value	Evidence
F1	MSPA	dynFWA	4.7282	Not significant
F2	MSPA	dynFWA	8.3890	Not significant
F3	MSPA	dynFWA	15.2145	Not significant
F4	MSPA	dynFWA	3.1080	Not significant
F5	MSPA	dynFWA	1.8319	Not significant
F6	MSPA	dynFWA	2.5840	Not significant
F7	MSPA	dynFWA	3.9964	Not significant

The authors of this work can observe that the comparison is not fair, the algorithm of fireworks uses fuzzy logic to adjust the parameters and our proposal is simple algorithm, however in some functions the results are very similar.

7 Conclusions

To conclude this work we consider important to mention that the main contribution of this work is to demonstrate that the proposed algorithm can also be applied to more complex problems and in this case we decided to use it for mathematical functions of the CEC-2015 benchmark, however we managed to find some values close to zero in some functions. We consider it important to mention that in this work we only use the method of reproduction by polinization, and this is recommended by the authors based on the previous experiments using this algorithm in other simpler problems for example using traditional benchmark functions. Based on the results obtained we can conclude that it is necessary to consider some improvements to the algorithm, we observe problems of local minima, and it is important to investigate other biological processes of the plants to apply them to the algorithm to solve the problem of local minima.

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A Multi-objective Evolutionary Algorithm for Tuning Type-2 Fuzzy Sets with Rule and Condition Selection on Fuzzy Rule-Based Classification System

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Abstract. This paper presents a Multi-Objective Evolutionary Algorithm (MOEA) for tuning type-2 fuzzy sets and selecting rules and conditions on Fuzzy Rule-Based Classification Systems (FRBCS). Before the tuning and selection process, the Rule Base is learned by means of a modified Wang-Mendel algorithm that considers type-2 fuzzy sets in the rules antecedents and in the inference mechanism. The Multi-Objective Evolutionary Algorithm used in the tuning process has three objectives. The first objective reflects the accuracy where the correct classification rate of the FRBCS is optimized. The second objective reflects the interpretability of the system regarding complexity, by means of the quantity of rules and is to be minimized through selecting rules from the initial rule base. The third objective also reflects the interpretability as a matter of complexity and models the quantity of conditions in the Rule Base. Finally, we show how the FRBCS tuned by our proposed algorithm can achieve a considerably better classification accuracy and complexity, expressed by the quantity of fuzzy rules and conditions in the RB compared with the FRBCS before the tuning process.

Keywords: Fuzzy Rule-Based Classification System · Type-2 fuzzy sets · Multi-Objective Evolutionary Algorithm

1 Introduction

Type-2 fuzzy sets were introduced by Zadeh in 1975 [1] as a generalization of the type-1 fuzzy sets that offer a more powerful representation of imprecision by allowing the membership values to be represented as fuzzy sets. Following this concept, a new class of fuzzy systems was presented in 1999 [2] where the antecedent or consequent membership functions were type-2 fuzzy sets. Due to their representational power, type-2 fuzzy sets have been used in several applications [14]. Additional discussion on the advantages of using type-2 fuzzy sets can be found in [15,16].

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According to [3], the use of type-2 fuzzy sets is still an open problem in the context of Multi-Objective Evolutionary Fuzzy Systems (MOEFS). MOEFS aim at using the learning capabilities of Multi-Objective Evolutionary Algorithms (MOEA) to generate fuzzy systems favoring a balance among conflicting objectives that represent accuracy and interpretability. The problem can be approached from different modeling schemes that are dedicated to generate or optimize one or more components of a fuzzy system [3,17]. Those work that focus on tuning the fuzzy sets parameters, the use of type-2 fuzzy sets poses an increase in complexity.

In an attempt to investigate this open and promising research field, this work studies the tuning of type-2 fuzzy sets with simultaneous selection of rules and conditions, in the helm of Fuzzy Rule Based Classification Systems (FRBCS). Before the multi-objective evolutionary process, the type-2 fuzzy sets are generated and an initial Rule Base (RB) is learned using a modified version WM algorithm [4]. We use the well known, fast sorting and elite MOEA called Non-dominated Sorting Genetic Algorithm-II (NSGA-II) [5], with three objectives. The first objective optimizes the accuracy of the classification system varying the parameters of type-2 fuzzy sets. The second objective optimizes the interpretability by means of the selection of fuzzy rules in the RB. The third objective optimizes the interpretability too by means of the selection of conditions in fuzzy rules selected in the second objective.

This paper is organized as follows. In Sect. 2, we present the basic concepts of type-2 fuzzy sets. In Sect. 3, we explain the fuzzy rule learning process used in this paper. In Sect. 4, we introduce the multi-objective evolutionary algorithm for tuning type-2 fuzzy sets with rule and condition selection on FRBCS used. In Sect. 5, we report the experiments and result of our study. Finally, the conclusions and future works are presented in Sect. 6.

2 Type-2 Fuzzy Sets

The type-2 fuzzy sets theory was proposed by Lotfi Zadeh in 1975 for modelling the uncertainties inherent to the definition of the membership functions of antecedents and consequents in a fuzzy inference system. The main concerns about the use of type-2 fuzzy sets were related to the cost of inference. To surpass this problem, the interval fuzzy sets were proposed, which are simpler type-2 fuzzy sets where the value of the secondary membership is always one.

An interval type-2 fuzzy set, $\tilde{A}$ on X, is defined by a type-2 membership function, $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$, where $x \in X$ and $J_x \subseteq [0, 1]$, i.e. [16]:

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \quad (1)$$

A type-2 fuzzy set can be represented by means a geometric figure, for example Fig. 1. This triangular geometric type-2 fuzzy set is used in our proposed algorithm.

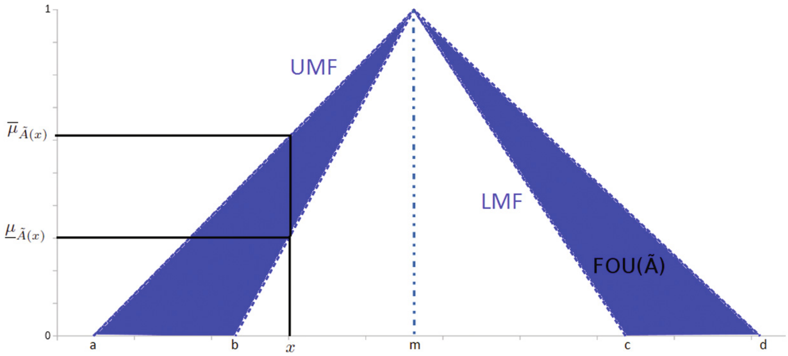

Fig. 1. Interval type-2 fuzzy set

In Fig. 1 the value $\underline{\mu}_{\tilde{A}(x)}$ for x is defined by LMF(Lower Membership Function) for x and the value $\overline{\mu}_{\tilde{A}(x)}$ for x is defined by UMF (Upper membership function). The uncertainty of $\tilde{A}$ is represented by the Footprint of Uncertainty (FOU(Ã)).

3 Fuzzy Rule Learning Process

The fuzzy rule learning process used to generate the initial RB based on the WM algorithm. First, we predefined the Data Base (DB) with type-2 membership functions uniformly distributed adopting a process similar to the one used in [6] and considering the minimum and maximum input values (*min* and *max*). Figure 2 shows an example of a linguistic variable with five type-2 fuzzy sets where each one have five parameters $\{a_i, b_i, m_i c_i, d_i\}$ where $b_i - a_i = m_i - b_i = c_i - m_i = d_i - c_i$.

Second, for example pattern in $E = \{e_1, e_2, ..., e_p\}$ labelled with a class from the set of classes $C = \{C_1, C_2, ..., C_m\}$, where each $e_q \in E$ is defined by a set of k features $e_q = \{a_{q1}, a_{q2}, ..., a_{qk}\}$, the values of $\underline{\mu}_{\tilde{A}(a_{i_j})}$ and $\overline{\mu}_{\tilde{A}(a_{i_j})}$ are calculated for each type-2 fuzzy set.

After that, the linguistic term with maximum value for $\underline{\mu}_{\tilde{A}(a_{i_j})} + \overline{\mu}_{\tilde{A}(a_{i_j})}$ is included as a condition in the fuzzy rule. For each $e_q \in E$ a fuzzy rule is learned in the form:

$$R_i : \textbf{ IF } V_1 \textbf{ IS } T_{1l_1} \textbf{ AND } V_2 \textbf{ IS } T_{2l_2} \textbf{ AND } ... \textbf{ AND } T_k \textbf{ IS } T_{kl_k} \textbf{ THEN } \text{Class } C_j$$

where:

R_i	: Index of the fuzzy rule i.
$V_1, V_2, ..., V_n$	: Linguistic variables or features of each example e_q.
$T_{1l_1}, T_{2l_2}, ..., T_{nl_n}$	: Linguistic terms or type-2 fuzzy sets for each V_r.
C_j	: Class of the fuzzy rule R_i.

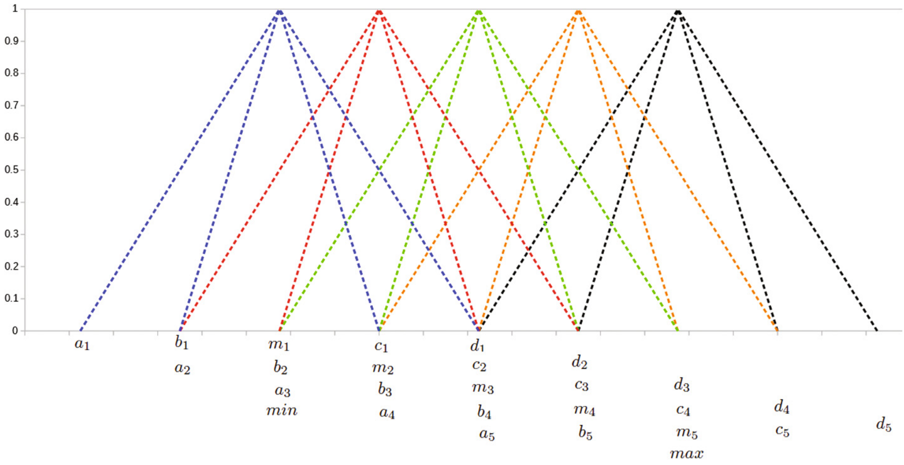

Fig. 2. DB uniformly distributed using type-2 fuzzy sets

Third, for each fuzzy rule in the RB with the same antecedent (conflicting and redundant) the fuzzy rule with the highest degree is selected to remain in the RB and the other ones are eliminated. The degree of fuzzy rule i (D_{R_i}) is defined by means of the fuzzification and inference illustrated in Fig. 3 for the particular case of rules with two antecedents. That output-processing blocks, fuzzification and inference, are similar to blocks used in [7].

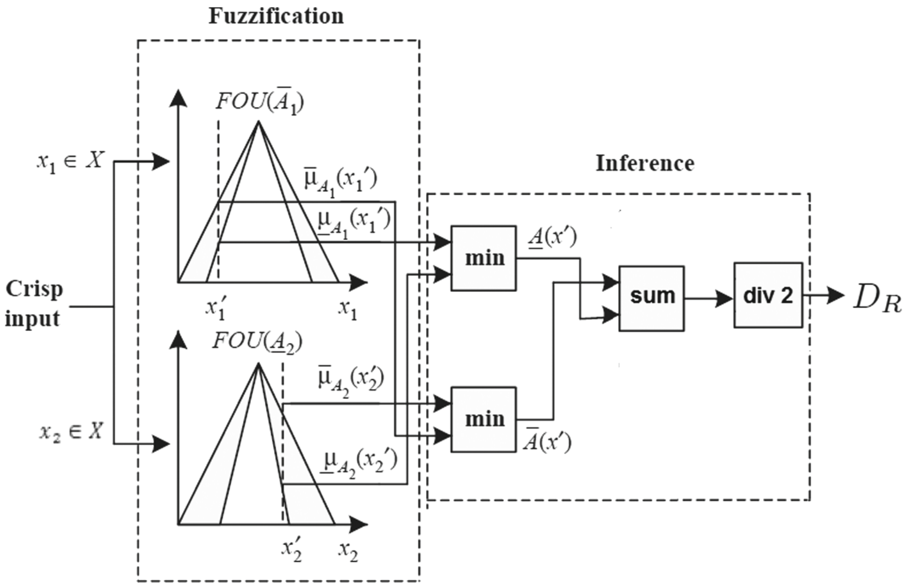

Fig. 3. Fuzzification and inference using type-2 fuzzy sets

4 Multi-objective Evolutionary Algorithm for Tuning Type-2 Fuzzy Sets

In this section we present our proposed algorithm for tuning type-2 fuzzy sets in a FRBCS with selection of rules and conditions using NSGA-II algorithm. This algorithm follows the steps of NSGA-II algorithm shown in Fig. 4.

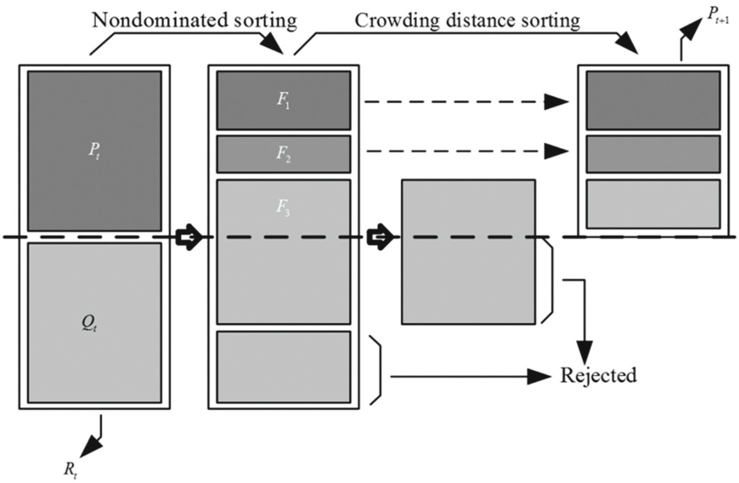

Fig. 4. NSGA–II algorithm

First, we define the initial population P_t $(t = 0)$ with size N. Each chromosome (CR_i), similar to the structure proposed in [8], is represented as:

$$CR_i = CR_{M_i} + CR_{R_i} + CR_{Co_i} \tag{2}$$

The first part of the chromosome CR_{M_i} encodes the parameters of the type-2 fuzzy sets for each linguistic variable in the in DB. Each gene in the chromosome encodes a parameter of a linguistic term in a linguistic variable. Figure 5 shows an example of CR_{M_i} considering five linguistic terms for each linguistic variable, where v represents the quantity of linguistic variables.

$a1_1$	$b1_2$	$m1_3$	$c1_4$	$d1_5$	$a2_1$	$b2_2$	$m2_3$	$c2_4$	$d2_5$	...	a_{v_1}	b_{v_2}	m_{v3}	c_{v_4}	d_{v_5}

Fig. 5. Chromosome encoding for the MF parameters

In order to preserve the semantics of fuzzy sets, we define the limits of each parameter as shown in Fig. 6 and defined according to the following equations:

$$Diff = (b - a)/2 = (m - b)/2 = (c - m)/2 = (d - c)/2 \tag{3}$$

$$a_{lower} = a - Diff;\ a_{upper} = a + Diff \tag{4}$$

$$b_{lower} = a_{upper};\ b_{upper} = b + Diff \tag{5}$$

$$m_{lower} = b_{upper};\ m_{upper} = m + Diff \tag{6}$$

$$c_{lower} = m_{upper};\ c_{upper} = c + Diff \tag{7}$$

$$d_{lower} = c_{upper};\ d_{upper} = d + Diff \tag{8}$$

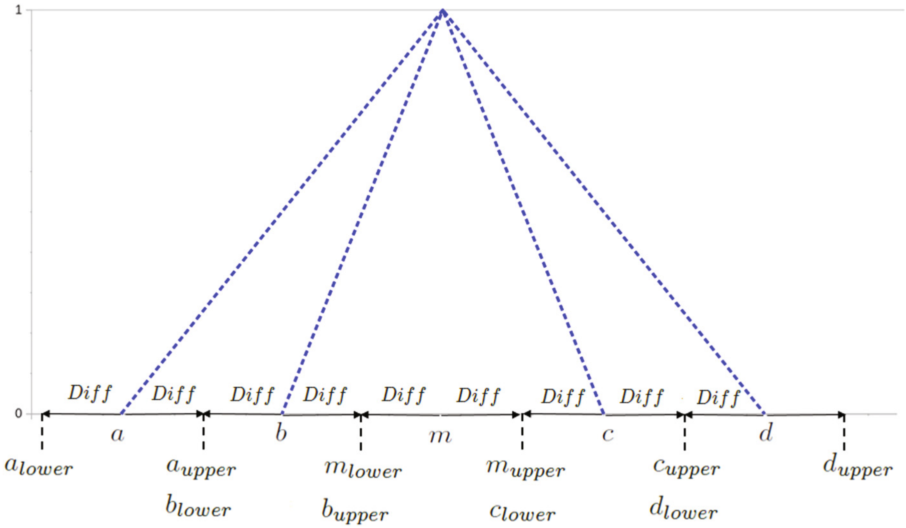

Fig. 6. Type-2 MF parameters

The fuzzy rules used in the fuzzy inference are encoded in the second part of the chromosome, CR_R. The upper and lower limit for each gene in CR_R are 1.0 and 0.0 respectively. A fuzzy rule is considered as part of the RB and used in the fuzzy inference if the gene value is greater than 0.5. Figure 7 shows an example of CR_R considering r fuzzy rules in the RB obtained by learning process described in the previous section.

$0.0 \leq R_1 \leq 1.0$	$0.0 \leq R_2 \leq 1.0$	$0.0 \leq R_3 \leq 1.0$	...	$0.0 \leq R_r \leq 1.0$

Fig. 7. Chromosome encoding for CR_R

The conditions in the RB used in the fuzzy inference are encoded in the thrid part of the chromosome, CR_{Co}. As in CR_R, the upper and lower limit for each gene in CR_{Co} are 1.0 and 0.0 respectively. A condition is considered as part of the fuzzy rule and used in the fuzzy inference if the gene value is greater than 0.5. Obviously, the condition is considered if it is in a valid fuzzy rule. Figure 8 shows an example of CR_{Co} considering a linguistic term for v linguistic variables in r fuzzy rules in RB.

$0.0 \leq Co_{1_1} \leq 1.0$	$0.0 \leq Co_{2_1} \leq 1.0$	...	$0.0 \leq Co_{v_1} \leq 1.0$	...	$0.0 \leq Co_{v_r} \leq 1.0$

Fig. 8. Chromosome encoding for CR_{Co}

In the initial population P_t, the first chromosome encodes the uniformly distributed DB, all fuzzy rules and conditions obtained in the WM based learning process. The other chromosomes are encoded randomly considering the upper and lower limits of each gene.

Each chromosome is evaluated by means of the calculation of the three objectives, three objectives are calculated. The first objective is defined by the error rate of the FRBCS and it is based in single-winner inference. The winner fuzzy rule is the fuzzy rule i with gene value greater than or equal 0.5 in CR_M and D_{R_i} is the maximum value compared with the other fuzzy rules in RB.

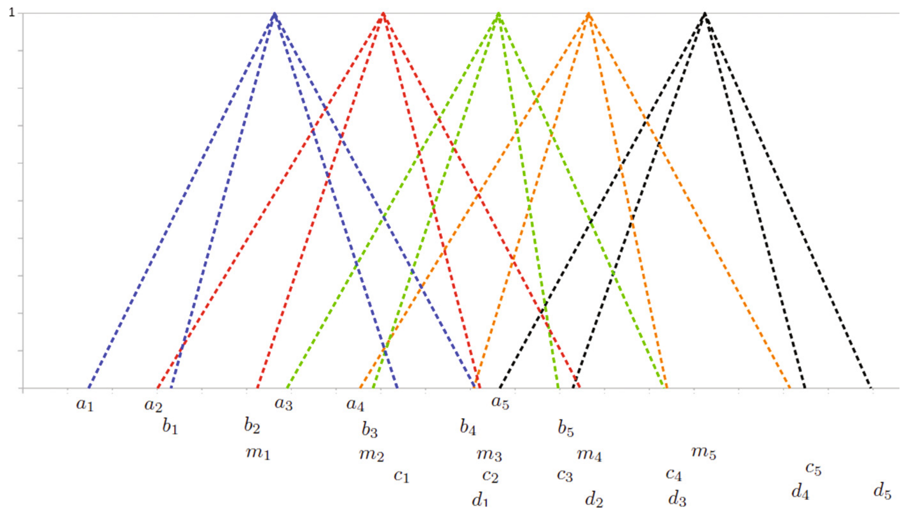

Fig. 9. Tuned type-2 fuzzy sets in a linguistic variable

The value of the second objective is calculated by counting genes with values greater than 0.5 in CR_M. These genes represent the indexes of the fuzzy rules considered in the inference system.

Similar to the to second objective, the third objective is calculated by counting genes with values greater than 0.5 in CR_{Co} taking into consideration that this condition is in a fuzzy rule considered in the inference system.

New populations are generated using the genetic operators of selection, crossover and mutation. Tournament selection, based on the dominance of the solutions and crowding distances, is the method used for selecting a chromosome for crossover operator. Simulated Binary Crossover (SBX) [9] and polinomial mutation [9] are the crossover and mutation operator used in our proposal respectively.

An example of tuned type-2 fuzzy sets in a linguistic variable is shown in Fig. 9.

5 Experiments and Results

The multi-objective evolutionary algorithm for tuning type-2 fuzzy sets with rule and condition selection to optimized FRBCS described in the last section was applied on ten well-known data sets, extracted from KEEL repository [10] and UCI repository [11], as shown in Table 1.

Table 1. Data sets used in the experiments

Data set	Pattern number	Atributes	Class number
Appendicitis	106	7	2
Bloodtransfusion	748	5	2
Bupa	345	6	2
Haberman	306	3	2
Hayes-roth	160	4	3
Hepatitis	80	19	2
Iris	150	4	3
Newthyroid	215	5	3
Tae	151	5	3
Zoo	87	8	3

All experiments were run using 10-fold cross validation. Results of the initial learning process before the tuning of fuzzy sets and selection of rules and conditions are shown in Table 2. First and second columns show the mean error rate in training (Tr_{er}) and test (Te_{er}) dataset respectively. Third and fourth columns show the quantity of fuzzy rules and conditions in the RB respectively. Standard deviation is shown in parenthesis for each result.

Table 2. Results obtained by the initial learning process

Data set	Trer	Teer	Rules number	Conditions number
Appendicitis	0.1384 (0.0551)	0.3173 (0.1506)	61.8000 (1.4697)	432.6000 (10.2879)
Bloodtransfusion	0.4923 (0.0197)	0.4987 (0.0321)	29.4000 (0.4899)	117.6000 (1.9596)
Bupa	0.3317 (0.0176)	0.4937 (0.0868)	120.8000 (1.7205)	724.8000 (10.3228)
Haberman	0.4122 (0.0325)	0.4514 (0.0608)	51.7000 (1.2689)	155.1000 (3.8066)
Hayes-roth	0.1049 (0.0085)	0.3625 (0.0729)	78.7000 (1.2689)	314.8000 (5.0754)
Hepatitis	0.0000 (0.0000)	0.7826 (0.1653)	71.2000 (2.1354)	1352.8000 (40.5729)
Iris	0.0674 (0.0420)	0.0467 (0.0427)	40.8000 (1.5362)	163.2000 (6.1449)
Newthyroid	0.0595 (0.0253)	0.0742 (0.0421)	48.0000 (1.0000)	240.0000 (5.0000)
Tae	0.2641 (0.0243)	0.4496 (0.1254)	64.4000 (1.6852)	322.0000 (8.4261)
Zoo	0.0000 (0.0000)	0.3528 (0.1260)	54.2000 (1.4000)	867.2000 (22.4000)

Table 3. Parameters of NSGA-II algorithm

Parameter	Value
Size of the population	100.0
Crossover probability	1.0
Mutation probability	0.1
Number of generations	2000.0

Table 4. Results after tuning process

Data set	Tr_{er}	Te_{er}	Rules number	Conditions number
Appendicitis	0.0933 (0.0275)	0.0655 (0.0592)	12.9000 (14.6318)	56.4000 (80.0165)
Bloodtransfusion	0.2148 (0.0094)	0.2033 (0.0291)	5.3000 (1.9000)	10.7000 (5.1778)
Bupa	0.3317 (0.0243)	0.2866 (0.0557)	45.0000 (16.2358)	158.3000 (89.5735)
Haberman	0.2375 (0.0084)	0.2649 (0.0532)	4.7000 (1.6155)	7.5000 (3.4713)
Hayes-roth	0.1090 (0.0250)	0.2938 (0.0793)	74.7000 (4.3370)	284.1000 (26.9757)
Hepatitis	0.0948 (0.0911)	0.2568 (0.1603)	30.8000 (18.5462)	322.4000 (298.7869)
Iris	0.0385 (0.0162)	0.0200 (0.0306)	4.0000 (0.8944)	5.6000 (1.9079)
Newthyroid	0.0295 (0.0111)	0.0182 (0.0302)	23.8000 (10.7219)	89.0000 (47.3793)
Tae	0.2583 (0.0262)	0.3508 (0.0887)	56.5000 (6.2330)	264.3000 (39.3066)
Zoo	0.0000 (0.0000)	0.0650 (0.0554)	31.8000 (5.7931)	326.6000 (68.9321)

Table 3 shows the parameters used in NSGA-II algorithm.

The results after the evolutionary process considering the a chromosome in the middle of first frontier F_1 are shown in Table 4. The content of columns of Table 4 are the same as Table 2.

Finally, Table 5 shows the difference (diff) between results obtained after and before the evolutionary process for tuning the type-2 fuzzy sets and selecting rules and conditions in the RB. Based on Table 5, one can conclude that the reduction in the compared values are considerable, mainly in the number of rules (R. number) and number of conditions (C. number).

Table 5. Reduction in accuracy, rules number and conditions number after the evolutionary process. Tuning process (TP). Learning process (LP)

Data set	diff(Tr_er) TP–LP	diff(Te_er) TP–LP	diff(R. number) TP–LP	diff(C. number) TP–LP
Appendicitis	−0.0451	−0.2518	−48.9000	−376.2000
Bloodtransfusion	−0.2775	−0.2954	−24.1000	−106.9000
Bupa	0.0000	−0.2071	−75.8000	−566.5000
Haberman	−0.1747	−0.1865	−47.0000	−147.6000
Hayes-roth	0.0041	−0.0687	−4.0000	−30.7000
Hepatitis	0.0948	−0.5258	−40.4000	−1030.4000
Iris	−0.0289	−0.0267	−36.8000	−157.6000
Newthyroid	−0.0300	−0.0560	−24.2000	−151.0000
Tae	−0.0058	−0.0988	−7.9000	−57.7000
Zoo	0.0000	0.0000	−22.4000	−540.6000
Mean	**−0.0463**	**−0.1717**	**−33.1500**	**−316.5200**

6 Conclusions

This paper describes a multi-objective evolutionary algorithm with three objectives for tuning the parameters of type-2 fuzzy sets and selection of rules and conditions. The first objective is to minimize the error rate on FRBCS. The second and third objectives are to minimize the quantity of fuzzy rules and conditions in RB respectively. Before the tuning process we use a modified version of the WM algorithm considering type-2 fuzzy sets uniformly distributed for leaning the initial set of fuzzy rules. The results obtained show a considerable difference between the values before and after the optimization process with reduction in the error rate both for training and test datasets, number of rules and number of conditions.

As future work, we intend to explore in the following directions. 1) Apply the multi-objective evolutionary algorithm for tuning type-2 fuzzy sets after another fuzzy rule learning process for compare the proposed algorithm with the most recent similar works that include a tuning process. 2) Apply others multi-objective evolutionary algorithms in the tuning process, for example, MOEA/D [12] or SPEA2 [13], with others geometric type-2 fuzzy sets. 3) To increase the quantity of interpretability objectives in the tuning process, for example, the quantity of fuzzy rule throw in the inference system to evaluate the benefits of adopting type-2 fuzzy sets, comparing the algorithm described here with the ones using type-1 fuzzy sets.

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Sugeno Integral on Property-Based Preference Domains

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Abstract. We consider decision problems in which we have to compare and rank a set of alternatives and each alternative is defined by its attributes or properties. We introduce and characterize property-based preference domains. This paper proposes also a characterization and a generalization of Sugeno integral in our framework.

Keywords: Poset · Topological space · Congruence · Aggregation functional · Sugeno integral

1 Introduction

Decision problems are characterized by a plurality of points of view and there are different dimensions from which the alternatives can be viewed. In order to solve a decision problem we have to compare and rank a set of alternatives and each alternative is often defined by its attributes or properties.

We consider the model of abstract aggregation model introduced in [18] and more recently studied in [9], that represents a decision problem in terms of a set of Boolean properties specifying for every alternative a list of properties that are satisfied.

A *property-based domains* is a pair $(X, \mathscr{H})$ where X is a non-empty set and $\mathscr{H}$ is a collection of non-empty subsets of X that separates points (i.e. if $x, y \in X$ and $x \neq y$ there exists $H \in \mathscr{H}$ such that $x \in H$ and $y \notin H$). The elements of $\mathscr{H}$ are referred to as properties and if $x \in H$ we say that x has property associated to the subset H. The "property space" model has received attention in the literature on judgement aggregation for studying the problem of aggregating sets of logically interconnected propositions. Moreover, it provides a general framework for representing preferences and then aggregation of preferences (see [18]).

Our principal goal is to introduce a general framework to study property-based preference domains, in particular we do not consider only finite spaces as in [9,18].

The paper is organized as follows. In the next section some basic information is given. In Sect. 3 property-based preference domains are introduced and then we study the categorical equivalence between the description of a partially ordered

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set by means of objects and properties and the representation of the corresponding topological space. In Sect. 4 we consider congruence in property-based domains while in Sect. 5 we focus on aggregation operators over property-based domains and we introduce Sugeno integrals in our framework as aggregation operators that are compatible with congruences (see [12–14]).

2 Basic Notions and Terminology

The aim of this section is to introduce some basic definitions, terminology and notation. More detailed introduction to the subject of order can be found in e.g., Caspard, Leclerc and Monjardet [2], Davey and Priestley or Grätzer [11].

A *partially ordered set* (*poset* as a shorthand) $(P, \leq)$ is a set P with a reflexive, antisymmetric and transitive binary relation $\leq$. We will write $(x, y) \in R$ as $x \leq y$ (or equivalently, $y \geq x$) and we will use $x > y$ to mean that $x \geq y$ and $x \neq y$.

If for a poset $(P, \leq)$ we have that all elements can be compared to each other i.e. for all $x, y \in P$, at least one of $x \leq y$ and $x \geq y$ holds we call $(P, \leq)$ a *chain*.

A relation that is reflexive and transitive is said to be a *preorder*. This is a rather general concept, as every partial order and every equivalence order is a preorder.

If P, Q are posets, the function $f\colon P \to Q$ is called *order-preserving* if for all $x, y \in P$ with $x \leq y$ we have $f(x) \leq f(y)$.

Given a poset P and a set $S \subseteq P$, then $y \in P$ is called an *upper bound* of S if $x \leq y$ for all $x \in P$. A lower bound is defined dually. The set of all upper bound is denoted by $U_P(S)$, and the set of all lower bounds by $L_P(S)$. When the set $U_P(S)$ has a least element p with respect to $\leq$ we say that p is the *join* of S and write $\bigvee S = \bigvee_{x \in S} x = p$. Similarly when $L_P(S)$ has a greatest element q with respect to $\leq$ and we say q is the *meet* of S and write $\bigwedge S = \bigwedge_{x \in S} x = q$.

A lattice is a poset in which every pair of elements (and thus every finite subset) has both a meet and a join. Every lattice L constitutes a partially ordered set endowed with the partial order $\leq$ such that for every $x, y \in L$, write $x \leqslant y$ if $x \wedge y = x$ or, equivalently, if $x \vee y = y$. A lattice L is said to be *bounded* if it has a least and a greatest element, denoted by 0 and 1, respectively.

A lattice L is said to be *distributive*, if for every $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \text{or, equivalently,} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice L is said to be *complete* if $\bigwedge I = \bigwedge_{x \in I} x$ and $\bigvee I = \bigvee_{x \in I} x$ exist for every $I \subseteq L$. Clearly, every complete lattice is also bounded.

If L, M are lattices, the function $f\colon L \to M$ is called a *lattice homomorphism* if for all $x, y \in P$ we have $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$.

A subset $S \subseteq P$ of a poset P is said to be an *upper set* (*down set*) if when $x \in S$ and $x \leq y$ $(x \geq y)$ then $y \in S$.

A *filter* of a poset P is a subset F of P such that

(i) if $x \in F$ and $x \leq y$ then $y \in F$,
(ii) if $x, y \in F$ there is $z \in F$ such that $z \leq x$ and $z \leq y$.

The dual notation is that of an *ideal*. If $a \in P$ we define the *principal filter* generated by x as $\uparrow x = \{y \in L : y \geq x\}$. It is easy to prove that $\uparrow x$ is a filter for every $x \in P$. It can be proved that in a finite lattice each filter and each ideal are principal.

In a lattice L a filter F is an upset such that if $x, y \in F$ then $x \wedge y \in F$.

A *proper filter* is a filter that is neither empty nor the whole lattice while a *prime filter* is a proper filter F such that whenever $\bigvee_{i \in I} x_i$ is defined in P for a finite set I we have $x_i \in F$ for some $i \in I$.

If $(X, \mathscr{T})$ be a topological space where the members of $\mathscr{T}$ are the *open* subsets of X then a family of open sets $\mathscr{B}$ is a *base* (*subbase*) if every open set in $\mathscr{T}$ is union of elements in $\mathscr{B}$ (union of finite intersections of elements in $\mathscr{B}$).

A topological space $(X, \mathscr{T})$ satisfies property T_0 if for all $x, y \in X$ there exists an open set A such that $x \in A$ and $y \notin A$ or such that $y \in A$ and $x \notin A$.

There is a natural order defined on the set of the points of every topological space $(X, \mathscr{T})$ that is the *specialization preorder* defined by

$$x \leq y \quad \text{if and only if when} \quad A \in \mathscr{T} \;\; \text{and} \;\; x \in A \;\; \text{then} \quad y \in A$$

The open sets are upper sets while the closed sets are down sets with respect to the specialization preorder. we can also note that the specialization order is a partial order if the topological space satisfies property T_0.

3 Property-Based Domains

In this section, we consider the framework of abstract aggregation introduced in [9] and in [18] that represents a decision problem in terms of set of Boolean properties specifying for every alternative a list of properties that are satisfied.

A *property-based domain* is a pair $(X, \mathscr{H})$ where X is a non-empty set and $\mathscr{H}$ is a collection of non-empty subsets of X and if $x, y \in X$ and $x \neq y$ there exists $H \in \mathscr{H}$ such that $x \in H$ and $y \notin H$. The elements of $\mathscr{H}$ are referred to as properties and if $x \in H$ we say that x has property represented by the subset H. Our definition is slightly more general than that of [9] and of [18], in fact we do not assume that the set X is finite and we do not consider that the set H^c is a property if H is a property.

The "property space" model provides a very general framework for representing preferences and then aggregation of preferences. Here are some important examples.

Example 1. Any chain X is a property-based domain with respect to the family of subsets $\{H_a\}$ where $H_a = \{x \in X : x \geq a\}$.

Example 2. It is well known that if x, y are two elements of a distributive lattice L and $x \ngeq y$ there exists a prime filter F with $y \in F$ and $x \notin F$ and so if X is a lattice the family $\mathscr{H}$ of prime filters of X defines a property-based domain.

Example 3. The problem of preference aggregation can be viewed as a property-based domain. If we consider a finite set of alternatives A and a a set $\mathscr{R}$ of binary relations in A. We can consider different requirements on the set $\mathscr{R}$ and so $\mathscr{R}$ can be the set of preorders or the set of linear orders in A.

If we define for each pair $a, b \in A$ the set

$$H_{a,b} = \{R \in \mathscr{R} : aRb\}$$

the family $\mathscr{H} = \{H_{a,b} : a, b \in A\}$ defines a property-based domain structure on the set $\mathscr{R}$. See [18] for more details on Arrowian framework.

We can define a natural preorder in any property-based domain. In fact if $(X, \mathscr{H})$ is a property-based domain we can define the relation $\leq$ by

$$x \leq y \quad \text{for every} \quad H \in \mathscr{H} \quad \text{if} \quad x \in H \quad \text{then} \ \ y \in H.$$

We can also note that if $(X, \mathscr{T})$ is a topological space that satisfies property T_0, X is a property-based domain with respect to the family $\mathscr{T}$ and the preorder associated with the property-based domain is the specialization preorder of the topological structure.

We can also prove that there is a correspondence between topological spaces and property-based domains and that we can extend this correspondence to morphism of the two structures. If $(X, \mathscr{H})$ is a property-based domain we consider the topological space $(X, \mathscr{T}(\mathscr{H}))$ that is generated by the subbase $\mathscr{H}$. So $(X, \mathscr{H})$ and $(X, \mathscr{T}(\mathscr{H}))$ are two property-based domains on the same set of alternatives X but we can prove that the two structures are associated to the same preorder on the set X.

Proposition 1. *The property-based domains* $(X, \mathscr{H})$ *and* $(X, \mathscr{T}(\mathscr{H}))$ *define the same preorder on the set* X.

Proof. Let $\leq_1$ and $\leq_2$ the preorders associated respectively to $\mathscr{T}(\mathscr{H})$ and to $\mathscr{H}$.

It is straightforward to prove that if $x, y \in X$ are such that $x \leq_1 y$ then $x \leq_2 y$. Then we consider two elements $x, y \in X$ with $x \leq_2 y$. If $x \in \bigcup H_i$ where $H_i \in \mathscr{H}$ for every i then there exists a set H_i such that $x \in H_i$ and so we have that $y \in H_i$ and then $y \in \bigcup H_i$. We can also prove that if $x \in \bigcap H_i$ then $y \in \bigcap H_i$ and so we can conclude that $x \leq_1 y$.

We introduce the definition of morphism between property-based domains as continuous functions between the associated topological spaces.

If $(X, \mathscr{H})$ and $(Y, \mathscr{K})$ are a property-based domains a function $f : X \to Y$ is a *morphism* if for every $K \in \mathscr{K}$, $f^{-1}(K)$ is an element of $\mathscr{T}(\mathscr{H})$. It is important to note that a morphism between property-based domains is a continuous function between the topological spaces associated and an isotone function between the posets associated.

4 Congruences on Property-Based Domains

Congruence relations are studied in lattices (see [1,8]) and there are various definitions of a congruence relation in a poset (for example in [12]).

In this section a definition of a congruence on a property-based domain is proposed. Congruence relations are equivalence relations that are compatible with the structure defined on the space and an equivalence relation in a set is a reflexive, symmetric and transitive relation. If $(X, \mathscr{H})$ is a property-based domain for every $\mathscr{H}' \subseteq \mathscr{T}(\mathscr{H})$ we define a *congruence* $\sim$ in $(X, \mathscr{H})$ by

$$x \sim y \text{ when } x \in H \text{ if and only if } y \in H, \text{ for every } H \in \mathscr{H}'.$$

It is straightforward to prove that the relation $\sim$ is an equivalence relation in X, and we are going to prove that there are nice characterizations of the proposed definition.

If $x \in X$, $[x]$ is the equivalence class of the element x with respect to the equivalence relation $\sim$.

Proposition 2. *If $\sim$ is a congruence in the property-based domain $(X, \mathscr{H})$ defined by the set $\mathscr{H}' \subseteq \mathscr{H}$ then the family $\mathscr{K}$ of subsets of $X/\sim$ where $K \in \mathscr{K}$ if $K = \{[x] : x \in H\}$ for some $H \in \mathscr{H}'$ defines a property-based structure on the set $X/\sim$.*

Proof. It is almost evident that if x, y are elements of X such that $x \sim y$ then $x \in K$ if and only if $y \in K$ for every $K \in \mathscr{K}$ and the set $K \in \mathscr{K}$ is well defined. Moreover if $[x] \neq [y]$ there exists an element $H \in \mathscr{H}'$ such that $x \in H$ and $x \notin H$ and we can prove by the definition of the set $\mathscr{K}$ that for a set $K \in \mathscr{K}$ we have that $[x] \in K$ and $[y] \notin K$.

There is a natural morphism between a property-based domain $(X, \mathscr{H})$ and the space $(X/\sim, \mathscr{K})$ defined in the above proposition.

Proposition 3. *If $\sim$ is a congruence on the property-based domain $(X, \mathscr{H})$ there is a surjective morphism $f \colon X \to X/\sim$.*

If $f \colon X \to X$ is a surjective morphism from $(X, \mathscr{H})$ to itself then the relation defined in X by

$$x \sim y \quad \textit{if and only if} \quad f(x) = f(y)$$

is a congruence in X.

Proof. It is not restrictive to suppose that $\mathscr{H} = \mathscr{T}(\mathscr{H})$. Then if we define a function $i \colon X \to X/\sim$ such that $i(x) = [x]$ i is a surjective function. If $K \in \mathscr{K}$ is such that $K = \{[x] : x \in H\}$ for some $H \in \mathscr{H}'$ we have that $f^{-1}(K) = H \cup \{H' \in \mathscr{H} : H' \notin \mathscr{H}'\}$ hence the function i is a morphism between property-based spaces.

Conversely if $f \colon X \to X$ is a surjective morphism from $(X, \mathscr{H})$ to itself the relation $\sim$ defined by $x \sim y$ if and only if $f(x) = f(y)$ is the congruence relation defined with respect to the family $\mathscr{H}' = \{f^{-1}(H) : H\mathscr{H}\}$ that is a family of subset of $\mathscr{H}$ since f is a morphism of $(X, \mathscr{H})$ onto itself.

An equivalence relation $\sim$ defined on a lattice L is a *congruence* that is compatible with the two lattice operations i.e.

(i) if $x \sim y$ then $x \wedge z \sim y \wedge z$ for each $z \in L$;
(ii) if $x \sim y$ then $x \vee z \sim y \vee z$ for each $z \in L$.

If $(X, \mathscr{H})$ is a property-based domain and $Y \subseteq X$ we consider the set

$$L_X(Y) = \{x \in X : \ x \leq y \ \text{for every} \ \ y \in Y\},$$
$$U_X(Y) = \{x \in X : \ x \geq y \ \text{for every} \ \ y \in Y\}$$

where $\leq$ is the preorder relation associated with the property-based domain structure.

In a property-based domain we can prove the following proposition on the ordered structure of the space.

If $Y = \{x_1, \ldots, x_n\}$ we write $L_X(Y) = L_X(x_1, \ldots, x_n)$ and $U_X(Y) = U_X(x_1, \ldots, x_n)$.

Proposition 4. *Let $\sim$ is a congruence on the property-based domain $(X, \mathscr{H})$. If $x, y \in X$ we have*

$$\{[z] : z \in L_X(x, y)\} = L_{X/\sim}([x], [y]).$$

Proof. The proof follows directly if we note that if $x, y \in X$ then $x \leq y$ if and only if for every $H \in \mathscr{H}$ if $x \in H$ then $y \in H$ and that $[x] \leq [y]$) if and only if for every $H \in \mathscr{H}'$ if $x \in H$ then $y \in H$.

Then we can conclude that congruences are morphism preserving upper and lower bounds.

5 Compatible Aggregation Functional on Property-Based Domains

The process of merging or combining sets of values (numerical or qualitative) into a single one is usually achieved by the so-called aggregation functionals; see [10] for a comprehensive overview on aggregation theory. The importance of aggregation functionals is made apparent by their wide use in several fields such as decision sciences, computer and information sciences, economics, and social sciences. There are a large number of different aggregation operators that differ on the assumptions on the inputs and on the information that you want to consider in the model.

One of the most important aggregation functional making sense in a qualitative framework is Sugeno integral that is a very useful non-linear functional in several applications in mathematics, economics and decision making (see [3–5]). The definition of Sugeno integral primarily introduced on real intervals can be extended to bounded distributive lattices (see [3–5]).

The aim of this section is to introduce a Sugeno-type integral representation for aggregation operators defined on property-based domains. We follow the approach of [13,14] where are characterized congruence preserving aggregation functionals acting on a bounded distributive lattice as discrete Sugeno integrals.

Let N be a non empty set which can be either finite or infinite and X a property-based poset. We define an *aggregation functional* as a map $F\colon X^N \to \mathscr{P}(X)$. Then we consider the case in which there are more than one equivalent solutions and also the case in which the only solution is the element with no properties.

We consider now some of the properties that an aggregation functional $F\colon X^n \to \mathscr{P}(X)$ may or may not satisfy.

Let F be an aggregation functional $F\colon X^N \to \mathscr{P}(X)$ acting on a property-based domain $(X, \mathscr{H})$ F is *monotone* if $F(x_1, \dots x_i \dots,) \in H$ for $H \in \mathscr{H}$ and $y_i \in H$ then $F(x_1, \dots y_i \dots,) \in H$. F is said to be *compatible with the congruence* $\sim$ if when $\mathbf{x}, \mathbf{y} \in X^N$ and for every i, $1 \leq i \leq n$, $x_i \sim y_i$ then $F(\mathbf{x}) \sim F(\mathbf{y})$.

F is *idempotent* if and only if for every $x \in X$ $F(x, x, \dots, x \dots) = x$.

We want to characterize the class of multivariate functionals compatible with every congruence of a property-based domain. Thus we propose a definition of Sugeno integral which generalizes well known definitions to the more general setting of property-based domains.

Proposition 5. *Let $(X, \mathscr{H})$ be a property-based domain and $F\colon X^n \to \mathscr{P}(X)$ an idempotent aggregation functional that is compatible with every congruence in X. Then F is monotone and there exists for every $H \in \mathscr{H}$ a non empty family $\mathscr{F}_H$ of subsets of N such that*

$$F(\mathbf{x}) = \bigcap \{H : N(\mathbf{x}, H) \in \mathscr{F}_H\}$$

where $N(\mathbf{x}, H) = \{i \in N : x_i \in H\}$.

Proof. If H is an element of $\mathscr{H}$ we can consider the congruence $\sim_H$ in X defined by

$$x \sim_H y \ \text{ when } \ x, y \in H \ \text{ or } \ x, y \notin H.$$

F is compatible with the congruence $\sim_H$ so if we consider two elements $\mathbf{x}, \mathbf{y} \in X^N$ such that for every $i \in N$, $x_i \in H$ if and only if $y_i \in H$ then $F(\mathbf{x}) \in H$ if and only if $F(\mathbf{y}) \in H$.

We say that a set A is H-decisive if there exists $\mathbf{x} \in X^N$ such that $N(\mathbf{x}, a) = A$ and $F(\mathbf{x}) \in H$. Being F compatible with the congruence $\sim_H$ a set is H-decisive if and only if for every $\mathbf{x} \in X^n$ such that $N(\mathbf{x}, H) = A$, $F(\mathbf{x}) \in H$.

For every $H \in \mathscr{H}$ let $\mathscr{F}_H$ the family of H-decisive subsets of N. Hence for every $\mathbf{x} \in X^N$, $F(\mathbf{x}) \in H$ if and only if $N(\mathbf{x}, H) \in \mathscr{F}_H$. Note that the family $\mathscr{F}_H$ is non empty since F is idempotent. Moreover the characterization of the functional F implies that F is a monotone functional.

Note that our framework is very general, we do not assume neither that X or N are finite sets. Moreover, we consider the case in which there are more than one equivalent solutions and also the case in which there are no solutions.

6 Concluding Remarks

In this paper we have introduced a general framework for studying preferences representation. Our framework is abstract and the crucial operations are the joining and meet of two properties that are subsets of the considered space. It appears that there are many connections between the work presented here with the results of [9,15,17,18]. Applications of these types of results can be found in [6,18]. There are however many opportunities for much more detailed research in this area from the point of view of aggregation theory.

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Integrating a Tourism Service Quality Evaluation Linguistic Multi-criteria Decision Making Model into a Relational Database Management System

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Abstract. In this paper we present an implementation, using a conventional relational database management system, of a linguistic *multi-criteria decision making* model to integrate the hotel guests' opinions included in the WWW and expressed on several dimensions (or attributes) in order to obtain a SERVQUAL scale evaluation value of service quality. SERVQUAL scale is commonly used in tourism to standardize the service quality evaluation and is a five-item scale consisting of: tangibles, reliability, responsiveness, assurance and empathy. As a particular case study, we show an application example of the implemented model using TripAdvisor website.

1 Introduction

In [1, 2] we have presented a model to integrate the hotel guests' opinions included in several websites (and expressed in several dimension or attributes) in order to obtain a standard overall evaluation value of service quality by means of *linguistic multi-criteria decision making* (LMCDM) processes based on the 2-tuple fuzzy linguistic approach [3].

The fuzzy linguistic approach is a tool intended for modeling qualitative information in a problem. It is based on the concept of linguistic variable and has been satisfactorily used in *multi-criteria decision making* (MCDM) problems [4]. The 2-tuple fuzzy linguistic approach [3] is a model of information representation that carries out processes of "computing with words" without the loss of information.

The SERVQUAL scale [6] is the standard used in [1, 2] to obtain the evaluation value of tourism service quality. It is a survey instrument which claims to measure the service quality in any type of service organization.

In this paper we present a general implementation of this LMCDM model integrating it in a conventional relational database management system (RDBMS). Specifically, we have chosen the well-known Oracle© RDBMS and therefore in the

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implementation we will use its corresponding PL/SQL language [7]. This makes able to be widely applied at a practical level and not only at a theoretical one.

The rest of the paper is organized as follows: Sect. 2 revises the preliminary concepts, i.e. the 2-tuple linguistic modeling and the SERVQUAL scale. Section 3 presents the implementation of the LMCDM model using an Oracle© environment. Section 4 shows an application example using the implemented model to integrate customers' opinions collected from TripAdvisor [8] website. Finally, we point out some concluding remarks and future work.

2 Preliminaries

2.1 The 2-Tuple Fuzzy Linguistic Approach

Let $S = \{s_0,\ldots,s_T\}$ be a linguistic term set with odd cardinality, where the mid-term represents a indifference value and the rest of terms are symmetric with respect to it. We assume that the semantics of labels is given by means of triangular membership functions and consider all terms distributed on a scale on which a total order is defined, i.e. $s_i \leq s_j \Longleftrightarrow i < j$. In this fuzzy linguistic context, if a symbolic method aggregating linguistic information obtains a value $b \in [0, T]$, and $b \notin \{0,\ldots,T\}$, then an approximation function is used to express the result in S.

Definition 1 [3]. *Let b be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set S, i.e. the result of a symbolic aggregation operation,* $b \in [0, T]$. *Let* $i = round(b)$ *and* $\alpha = b - i$ *be two values, such that* $i \in [0, T]$ *and* $\alpha \in [-0.5, 0.5)$, *then* α *is called a Symbolic Translation.*

The 2-tuple fuzzy linguistic approach [3] is developed from the concept of symbolic translation by representing the linguistic information by means of 2-tuple (s_i, α_i), $s_i \in S$ and $\alpha_i \in [-0.5, 0.5)$, where s_i represents the information linguistic label, and α_i is a numerical value expressing the value of the translation from the original result b to the closest index label, i, in the linguistic term set S. This model defines a set of transformation functions between numeric values and 2-tuple:

Definition 2 [3]. *Let* $S = \{s_1,\ldots, s_T\}$ *be a linguistic term set and* $b \in [0, T]$ *a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to b is obtained with the following function:*

$$\begin{aligned} &\Delta : [0, T] \rightarrow S \times [-0.5, 0.5) \\ &\Delta(b) = (s_i, \alpha), with\ s_i,\ i = round(b) \text{ and } \alpha = b - i,\ \alpha \in [-0.5, 0.5) \end{aligned} \tag{1}$$

where $round(\cdot)$ *is the usual round operation,* s_i *has the closest index label to b and* α *is the value of the symbolic translation.*

$$\text{For all } \Delta, \text{there exists } \Delta^{-1},\ \text{defined as } \Delta^{-1}(s_i, \alpha) = i + \alpha. \tag{2}$$

Below, we describe the aggregation operators which we use in our model:

Definition 3 [5]. *Let* $A = \{(l_1, \alpha_1), \ldots, (l_n, \alpha_n)\}$ *be a set of linguistic 2-tuple and* $W = \{(w_1, \alpha_1^w), \ldots, (w_n, \alpha_n^w)\}$ *be their linguistic 2-tuple associated weights. The 2-tuple linguistic weighted average* $\bar{A}^w$ *is:*

$$\bar{A}^w\left[\left((l_1,\alpha_1),(w_1,\alpha_1^w)\right),\ldots,\left((l_n,\alpha_n),(w_n,\alpha_n^w)\right)\right] = \Delta\left(\frac{\sum_{i=1}^{n}\beta_i \cdot \beta_{wi}}{\sum_{i=1}^{n}\beta_{wi}}\right), \quad (3)$$

with $\beta_i = \Delta^{-1}(l_i, \alpha_i)$ and $\beta_{wi} = \Delta^{-1}(w_i, \alpha_i^w)$.

2.2 The SERVQUAL Scale

Below, we will explain the five resultant scales proposed for SERVQUAL [6] and their adaptation to hotel guests' perceptions [1, 2]:

- *Tangibles*: It makes reference to the appearance of the physical facilities, equipment, personnel, and communication materials.
- *Reliability*: This is the ability to perform the promised service dependably and accurately. Customers generally place heavy emphasis on the image, sanitary condition, safety and privacy of the hotel.
- *Responsiveness*: Willingness to help customers and provide prompt service. A courteous and friendly attitude by the service personnel makes the consumer feel respected, and definitely enhances the customer's appraisal of the hotel.
- *Assurance*: Knowledge and courtesy of employees and their ability to inspire trust and confidence. The price level is usually one of the most important factors that will influence the evaluation result by customers.
- *Empathy*: Caring and individualized attention that the firm provides its customers. If the hotel is located in a remote district, whether the hotel provides a tourist route suggestion, convenient traffic routes, or a shuttle bus to pick up customers will influence customers' desire to go to the hotel.

3 Implementing a LMCDM into a RDBMS

In a LMCDM [4] process, the goal consists in searching the best alternatives of the set $id\_alternative = \{id\_alternative_1, \ldots, id\_alternative_n\}$ according to the linguistic assessments provided by a group of experts, $id\_criterion = \{id\_criterion_1, \ldots, id\_criterion_m\}$ with respect to a set of evaluation criteria. In our model, we assume that these assessments are weighted by the self-rated expertise level set $weight = \{weight_1, \ldots, weight_m\}$. We also assume that we have p decision problems, one for each dimension (attribute or set of attributes of the problem domain) value. Therefore, we define the assessments set as $assessments = \{assessments_{dij}\}\ \forall i, j, d,\ i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}, d \in \{1, \ldots, p\}$.

In order to obtain an easy linguistic interpretability and the high precision of the model results, we assume that all the information provided by the experts is in the 2-tuple form [3]. In our system we represent the 2-tuple values with a single attribute (string data type), we denote the pair (s_i, α), $s_i \in S$, with "s_i $sign(\alpha)$ $abs(\alpha)$", e.g. $(s_0, -0.1)$ is denoted by the string "s_0, −0.1" and $(s_0, 0)$ by "s_0".

Table 1. Database table *T_S*

ID_LABEL	RANKING_ LABEL	ALPHA	BETA	GAMMA
SD	0	0.0	0.0	0.25
D	1	0.0	0.25	0.5
N	2	0.25	0.5	0.75
A	3	0.5	0.75	1.0
SA	4	0.75	1.0	1.0

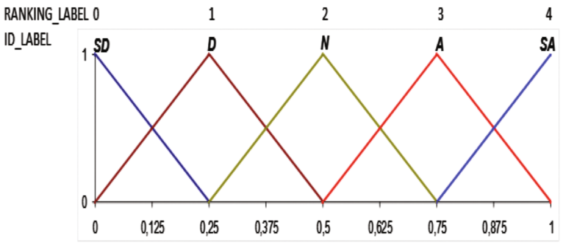

Fig. 1. Representation of the Table 1.

Our goal is to design some PL/SQL functions [7] that solve all the problems that are under the previous approach.

First, we need to store information about the symmetric and uniformly distributed domain $S = \{s_0,\ldots, s_T\}$, $T = 4$: $s_0 = Strongly\ Disagree = SD$, $s_1 = Disagree = D$, $s_2 = Neutral = N$, $s_3 = Agree = A$, and $s_4 = Strongly\ Agree = SA$. For this purpose we use a database table called *T_S* (see Table 1 and Fig. 1).

Below we explain the function implementations made to solve this problem:

`AVG_2T UDA Function.` In a conventional RDBMS to create a user-defined aggregate (UDA) functions the user must implement the following basic routines [9, 10]:

(1) *Init*: This function is used to initialize any variables needed for the computation later on. Intuitively, it is similar to a constructor. In PL/SQL this function is called *ODCIAggregateInitialize*.

(2) *Terminate*: This function is used to end the calculation and return the final value of the aggregate function. It may involve some calculations on variables which were defined to use them with the aggregate function. In PL/SQL this function is *ODCIAggregateTerminate*.

(3) *Accumulate*: This function is called once for each aggregated value. Generally, this function will "add" the value to the running total computed so far. In PL/SQL this function is called *ODCIAggregateIterate*.

An UDA function called *AVG_2T* based on an Oracle© object type (class) has been used to implement the 2-tuple linguistic weighted average $\bar{A}^w$. In this function the inputs are the 2-tuples values (l_i, α_i) and (w_i, α_i^w), $i = 1,\ldots, n$, showed in Definition 3. These two 2-tuple values (labels to be aggregated and their weights) have the string format mentioned above and are included in a single string variable separated with the "*" character. This object is composed of:

- Three attributes related to the aggregator to be implemented:
  - *n*: number of rows that are added at a given time. In the end (*ODCIAggregateTerminate*) it will match the value *n* of Eq. 3.
  - *sum_num*: it contains the expression $\sum_{i=1}^{n} \beta i \cdot \beta wi$ of mentioned equation.
  - *sum_den*: it contains the expression $\sum_{i=1}^{n} \beta wi$ of the same equation.
- The PL/SQL UDA routines shown in Fig. 2.

```
STATIC FUNCTION ODCIAggregateInitialize (
                        sctx IN OUT typ_obj_AVG_2T
                        ) RETURN NUMBER IS
    BEGIN
      sctx := typ_obj_AVG_2T(0,0,0);
      RETURN ODCIConst.Success;
    END;
```

```
MEMBER FUNCTION ODCIAggregateIterate (
                        self  IN OUT typ_obj_AVG_2T,
                        value IN     VARCHAR2
                        ) RETURN NUMBER IS
    valueSI VARCHAR2(20);
    valueWI VARCHAR2(20);
    pos NUMBER(3);
    BEGIN
      pos := INSTR(value,'*');
      IF pos <> 0 THEN
          valueSI := TRIM(SUBSTR(value,1,pos-1));
          valueWI := TRIM(SUBSTR(value,pos+1,20));
          self.sum_num := self.sum_num + REVERSE_2T(valueSI)*REVERSE_2T(valueWI);
          self.sum_den := self.sum_den + REVERSE_2T(valueWI);
      ELSE
          self.sum_num := self.sum_num + REVERSE_2T(value);
          self.sum_den := self.sum_den + 1;
      END IF;
      self.n := self.n + 1;
      RETURN ODCIConst.Success;
    END;
MEMBER FUNCTION ODCIAggregateTerminate (
                        self   IN  typ_obj_AVG_2T,
                        returnValue OUT VARCHAR2,
                        flags  IN  NUMBER
                        ) RETURN NUMBER IS
    v_avg   NUMBER;
    BEGIN
      returnValue := NULL;
      IF self.n = 0 THEN
         returnValue := 'NULL';
      ELSE
         IF self.sum_den = 0
         THEN v_avg := 0;
         ELSE v_avg := self.sum_num/self.sum_den;
         END IF;
         returnValue := DIRECT_2T (v_avg);
      END IF;
      RETURN ODCIConst.Success;
    END;
```

Fig. 2. *AVG_2T* PL/SQL UDA routines for $\bar{A}^w$ calculation

The implementation of this function has used the functions *REVERSE_2T* and *DIRECT_2T* explained below:

REVERSE_2T Function. This PL/SQL function implements the Δ^{-1} function (Eq. 2). For this purpose the function uses the table *T_S* showed in Table 1. The code is shown in Fig. 3.

```
FUNCTION REVERSE_2T (value_input IN VARCHAR2)
  RETURN NUMBER IS
  margen FLOAT;  pos NUMBER;  value_w1 VARCHAR2(30);  value_label VARCHAR2(30);
  rec_T_S T_S%ROWTYPE;
BEGIN
        value_w1 := TRIM(UPPER(value_input));
        value_w1 := REPLACE(value_w1,')','');

        CASE
            WHEN INSTR(value_w1,'-') <> 0 THEN -- left
                pos := INSTR(REPLACE(value_w1,'.',','),'-');
                value_label := TRIM(SUBSTR(value_w1,1,pos-1));
                margen := CAST(SUBSTR(REPLACE(value_w1,'.',','),pos+1,10) AS FLOAT)*-1;
            WHEN INSTR(value_w1,'+') <> 0 THEN -- right
                pos := INSTR(REPLACE(value_w1,'.',','),'+');
                value_label := TRIM(SUBSTR(value_w1,1,pos-1));
                margen := CAST(SUBSTR(REPLACE(value_w1,'.',','),pos+1,10) AS FLOAT);
            ELSE
                value_label := TRIM(value_w1);
                margen := 0;
        END CASE;

        SELECT * into rec_T_S FROM T_S WHERE S=value_label;
        SELECT MAX(RANKING_LABEL) into T FROM T_S;

        RETURN rec_T_S.RANKING/T+margen;
END REVERSE_2T;
```

Fig. 3. *REVERSE_2T* PL/SQL function for Δ^{-1} calculation

DIRECT_2T Function. This PL/SQL function implements the Δ function (Eq. 1) using the table *T_S* (see Table 1). The code is shown in Fig. 4.

```
FUNCTION DIRECT_2T (value_input IN NUMBER)
  RETURN VARCHAR2 IS
  margen FLOAT;
  value_label VARCHAR2(30);
  rec_T_S T_S%ROWTYPE;
  s_label VARCHAR2(30);
  ranking_label NUMBER(1);
BEGIN
        SELECT * into rec_T_S FROM T_S WHERE RANKING=ROUND(value_input*4,0);
        SELECT MAX(RANKING_LABEL) into T FROM T_S;
        ranking_label:=ROUND(value_input*T,0);
        margen := value_input-ranking_label/T;
        CASE
            WHEN margen=0 THEN value_label:=s_label;
            ELSE value_label:=s_label||TO_CHAR(ROUND(margen,3),'S0.000');
        END CASE;
        RETURN value_label;
END DIRECT_2T;
```

Fig. 4. *DIRECT_2T* PL/SQL function for Δ calculation

Once the *AVG_2T* UDA function is defined, we are already in a position to define the following function:

`2T_MCDM Function.` This PL/SQL function implements the LMCDM process:

`Inputs:` $\forall d \in \{1,\ldots, p\}$

$$\begin{array}{c} \\ \\ id\_alternative_1 \\ \ldots \\ id\_alternative_n \end{array}\begin{array}{ccc} id\_criterion_1 & \ldots & id\_criterion_m \\ weight_1 & \ldots & weight_m \\ \left[\begin{array}{c} assessment_{d11} \\ \vdots \\ assessment_{d1n} \end{array}\right. & \begin{array}{c} \cdots \\ \ddots \\ \cdots \end{array} & \left.\begin{array}{c} assessment_{dm1} \\ \vdots \\ assessment_{dmn} \end{array}\right] \end{array}$$

`Outputs:` $\forall d \in \{1,\ldots, p\}$

$$\begin{array}{c} \\ id\_alternative_1 \\ \ldots \\ id\_alternative_n \end{array}\begin{array}{ccc} id\_dimension_1 & \ldots & id\_dimension_p \\ \left[\begin{array}{c} performance\_value\_2t_{11} \\ \vdots \\ performance\_value\_2t_{1n} \end{array}\right. & \begin{array}{c} \cdots \\ \ddots \\ \cdots \end{array} & \left.\begin{array}{c} performance\_value\_2t_{p1} \\ \vdots \\ performance\_value\_2t_{pn} \end{array}\right] \end{array}$$

In Fig. 5 a scheme of the implemented function is shown. Inputs are included in database table format:

- *2T_CRITERIA*: it stores the $id\_criterion_j$ their $weight_j$ and an additional description of each criterion j (*DES_CRITERION*), $j \in \{1,\ldots, m\}$.
- *2T_ALTERNATIVES*: it stores the $id\_alternative_i$ and an additional description of each alternative i (*DES_ALTERNATIVE*), $i \in \{1,\ldots, n\}$.
- *2T_DIMENSIONS*: it stores the $id\_dimension_d$ and an additional description of each dimension d (*DES_DIMENSION*), $d \in \{1,\ldots, p\}$.

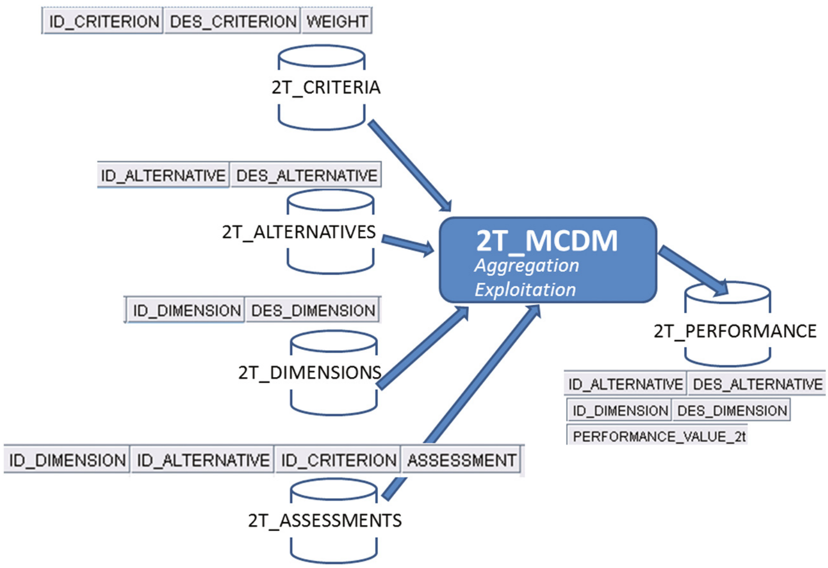

Fig. 5. *2T_MCDM* PL/SQL function for LMCDM process implementation

- *2T_ASSESSMENTS*: it stores the $assessments_{dij}$ for each $id\_criterion_j$, $id\_alternative_i$ and $id\_dimension_d$.

And also the corresponding output:

- *2T_PERFORMANCE*: it stores the $performance\_value_{di}$ for each $id\_alternative_i$ and $id\_dimension_d$ which allows to rank these alternatives.

Our function implements the two typical phases of the decision processes [4]:

(1) *Aggregation* that combines the expert preferences. The 2-tuple performance values are obtained using the *AVG_2T* UDA function using the corresponding clause *GROUP BY* (Fig. 6).
(2) *Exploitation* that obtains a solution set of alternatives for the decision problem sorted descending by the 2-tuple performance values. For which these values are previously converted to crisp numbers by the function *REVERSE_2T* and used in the corresponding clause *ORDER BY* (Fig. 6).

```
SELECT *
    (SELECT ID_DIMENSION,DES_DIMENSION,
            ID_ALTERNATIVE,DES_ALTERNATIVE,
            AVG_2T(ASSESSMENT||'*'||WEIGHT) PERFORMANCE_VALUE_2T
    FROM 2T_ASSESSMENTS ASS
    JOIN 2T_CRITERIA CR ON ASS.ID_CRITERION=CR.ID_CRITERION
    JOIN 2T_ALTERNATIVE AL ON ASS.ID_ALTERNATIVE=AL.ID_ALTERNATIVE
    JOIN 2T_DIMENSION DI ON ASS.ID_DIMENSION=DI.ID_DIMENSION
    GROUP BY ID_DIMENSION,DES_DIMENSION,ID_ALTERNATIVE,DES_ALTERNATIVE) AGGREGATION_PHASE
ORDER BY REVERSE_2T(PERFORMANCE_VALUE_2T) DESC
```

Fig. 6. Extract of the *2T_MCDM* PL/SQL function to implement the LMCDM processes

4 Application Example

In [1, 2] we have presented a LMCDM model to integrate the hotel guests' opinions included in several websites in order to get a SERVQUAL evaluation value of service quality. In this section, we present a RDBMS Oracle© implementation of this model, using the function *2T_MCDM* presented in the previous section, with an application example using the TripAdvisor website [8]. In this website customers write reviews on the following dimensions: *sleep quality*, *location*, *rooms*, *service*, *value* and *cleanliness* (see Fig. 6) using a linguistic five scale which can be modeled with $S = \{s_0, \ldots, s_T\}$, $T = 4$: $s_0 = Terrible = SD$, $s_1 = Poor = D$, $s_2 = Average = N$, $s_3 = Very\ Good = A$, and $s_4 = Excellent = SA$ (see Table 1 and Fig. 1).

In the first step of this model the objective is to obtain the linguistic importance of the dimensions (Fig. 7) for each SERVQUAL scale. In this step, we have counted on the collaboration of five experts. Using the implementation explained in Sect. 3, the process to solve this phase is very simple: we collect the input information provided by experts (Table 2) and then we execute the function *2T_MCDM* (Fig. 5) obtaining the output data (Table 3), i.e., the linguistic importance of each dimensions for each SERVQUAL scale.

Table 2. Input data: (a) *2T_CRITERIA*: Expert's criteria and self-rated weight. (b) *2T_ALTERNATIVES*: SERVQUAL scales. (c) *2T_DIMENSIONS*: Attributes on which the TripAdvisor users express their opinion. (d) *2T_ASSESSMENTS*: Provided by experts (we only show for the alternative PZB1 -Tangibles-).

ID_CRITERION	DES_CRITERION	WEIGHT
E1	EXPERT 1	SA
E2	EXPERT 2	SA
E3	EXPERT 3	SA
E4	EXPERT 4	SA
E5	EXPERT 5	A

(a)

ID_ALTERNATIVE	DES_ALTERNATIVE
PZB1	TANGIBLES
PZB2	RELIABILITY
PZB3	RESPONSIVENESS
PZB4	ASSURANCE
PZB5	EMPATHY

(b)

ID_DIMENSION	DES_DIMENSION
D1	Sleep quality
D2	Location
D3	Rooms
D4	Service
D5	Value
D6	Cleanliness

(c)

ID_DIMENSION	ID_ALTERNATIVE	ID_CRITERION	ASSESSMENT
D1	PZB1	E1	SA
D1	PZB1	E2	SA
D1	PZB1	E3	SA
D1	PZB1	E4	SA
D1	PZB1	E5	A
D2	PZB1	E1	SA
D2	PZB1	E2	SA
D2	PZB1	E3	SA
D2	PZB1	E4	SA
D2	PZB1	E5	A
D3	PZB1	E1	SA
D3	PZB1	E2	A
D3	PZB1	E3	SA
D3	PZB1	E4	SA
D3	PZB1	E5	SA
D6	PZB1	E1	SA
D6	PZB1	E2	A
D6	PZB1	E3	SA
D6	PZB1	E4	SA
D6	PZB1	E5	A

(d)

Fig. 7. Evaluation form of a hotel in TripAdvisor website

Table 3. Output data *2T_LMCDM_1* (*2T_PERFORMANCE*): Importance of the dimensions (Fig. 7) for each scale (alternative)

ID_ALTERNATIVE	DES_ALTERNATIVE	ID_DIMENSION	DES_DIMENSION	PERFORMANCE_VALUE_2t
PZB1	TANGIBLES	D2	Location	SA-0.039474
PZB1	TANGIBLES	D1	Sleep quality	SA-0.039474
PZB1	TANGIBLES	D3	Rooms	SA-0.052632
PZB2	RELIABILITY	D4	Service	SA-0.092105
PZB1	TANGIBLES	D6	Cleanliness	SA-0.092105
PZB4	ASSURANCE	D5	Value	SA-0.105263
PZB4	ASSURANCE	D4	Service	A+0.052632
PZB4	ASSURANCE	D1	Sleep quality	A+0.052632
PZB3	RESPONSIVENESS	D4	Service	A+0.052632
PZB2	RELIABILITY	D6	Cleanliness	A+0.052632
PZB5	EMPATHY	D4	Service	A+0.013158
PZB2	RELIABILITY	D3	Rooms	A-0.052632
PZB4	ASSURANCE	D6	Cleanliness	N+0.105263
PZB3	RESPONSIVENESS	D1	Sleep quality	N+0.052632

5 Concluding Remarks and Future Work

We have presented a RDBMS Oracle© implementation of a 2-tuple LMCDM model using UDA functions. Thus, we have implemented the model [1, 2] for integrating the opinions expressed by hotel guests in the TripAdvisor website [8] in order to obtain the overall value of service quality under the SERVQUAL instrument perspective. This makes able to be widely applied at a practical level on several types of problems. The implementation proposed here can also be applied to other types of problems solved with 2-tuple LMCDM [11–14]. Therefore, as future work we will implement these models with the proposed scheme

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Fuzzy Fingerprints for Item-Based Collaborative Filtering

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Abstract. Memory-based Collaborative filtering solutions are dominant in the Recommender Systems domain, due to its low implementation effort and service maintenance when compared with Model-based approaches. Memory-based systems often rely on similarity metrics to compute similarities between items (or users). Such metrics can be improved either by improving comparison quality or minimizing computational complexity. There is, however, an important trade-off—in general, models with high complexity, which significantly improve recommendations, are computationally unfeasible for real-world applications. In this work, we approach this issue, by applying Fuzzy Fingerprints to create a novel similarity metric for Collaborative Filtering. Fuzzy Fingerprints provide a concise representation of items, by selecting a relatively small number of user ratings and using their order to describe them. This metric requires from 23% through 95% less iterations to compute the similarities required for a rating prediction, depending on the density of the dataset. Despite this reduction, experiments performed in three datasets show that our metric is still able to have comparable recommendation results, in relation to state-of-art similarity metrics.

1 Introduction

Users of the digital world are overloaded with information [13]. Recommender Systems (RSs) allow us to cope with this, by cataloging a vast list of items, that later can be recommended. Due to their success, RSs can be found in a number of services, providing recommendations for movies, music, news, products, events, services, among others [1].

However, turning state of the art solutions into real-world scenarios is still challenging, mainly due to a large amount of data available and the scalability issues that ensue. For this reason, more traditional approaches, such as item-based Collaborative Filtering (CF) are still the most widely used [16]. Despite its simplicity, item-based CF can provide quite accurate results, thus yielding an advantageous trade-off between engineering effort and user satisfaction.

In CF systems, the issue of scalability is closely related to the need to compute similarities between a high number of items in the database. To solve this, two complementary types of solution are usually proposed. One is to provide scalability by distributing the storage and computational cost over several

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machines [10,15]. The other is to devise computationally efficient similarity metrics [3,8,14,18]. Our work focuses on the latter.

Our main contribution is a novel similarity metric for RSs, using the concept of Fuzzy Fingerprints (FFPs) [7]. More specifically, we propose to represent items by their low-dimensional Fingerprints, which can then be directly used to determine similarities between them. A similar idea has been previously applied to text authorship identification [7] with success. Our goal is to apply the same principle to RSs. This solution has three major advantages: (1) it has a smaller computational cost than traditional similarity metrics; (2) it requires a minimal implementation effort; and (3) the proposed representation of the items is also easily maintainable.

To demonstrate our claims, experiments were performed on three datasets. Results show that FFPs are a promising route to be applied for recommendations, requiring from 23% through 95% less iterations to compute the similarities for a rating prediction, depending on the density of the dataset. This improvement is achieved while maintaining a comparable quality of results.

The remainder of this paper is organized as follows. Section 2 contains literature review on similarity metrics for CF. Section 3 presents how FFPs can be applied to RSs. Section 4 presents an experimental evaluation. Finally, in Sect. 5 some conclusions are drawn from the results and directions for future work are proposed.

2 Related Work

Even thought Fuzzy systems have been previously applied to RSs, they have never been specifically used to improve the RS similarity metric [12,17]. Our proposal applies concepts of Fuzzy Systems to the problem of item-based Collaborative Filtering. More specifically, we use the Fuzzy Fingerprints to represent items in a CF system.

CF systems usually rely on the ratings given to items by users to determine similarities between items (or users), through the use of a similarity metric. This allows the creation of neighborhoods of similar items, to predict new ratings. Traditionally, the similarity is measured using metrics such as Pearson Correlation (PC) or the Cosine similarity (COS) [2]. Nevertheless, many other ways of measuring similarity have been proposed, ranging from simple variations of PC and COS, through the design of more complex functions.

An example is the work of [5], where ratings are combined with a measure of *trust* between users, which is inferred from social information. The authors show that such combination does improve the overall rating prediction. On a different approach, in [4], the authors propose a combination of the mean squared difference between the user's ratings with the Jaccard coefficient. Through experiments, they demonstrate that results are improved, when compared to traditional CF.

Liu et al. [9] also propose a new similarity metric, which attributes penalties to *bad* similarities, while rewarding *good* similarities. Defining a similarity as good or bad depends on several factors, such as the popularity of the rated items or the similarity of the rating to the other user's ratings. Finally, in [18], authors propose an alternative called M-distance-based recommendation (MBR). They leverage the average rating of each item and use the difference of such averages as the distance between items. Authors also have shown that it is possible to pre-compute such averages and compute the similar items neighborhood in linear time. Since this is the most time-efficient work that we know of, we use it as a baseline in our experiments.

The above works show that improving the similarity measures has a beneficial impact on the overall RS results. Nevertheless, this is often done at the expense of an increase in the computational complexity. In this work, we introduce a similarity metric based on FFPs, adapted for item-based CF, that aims to improve time-efficiency while maintaining a low implementation effort and a comparable, or even better, recommendation accuracy.

3 Fuzzy Fingerprints for Collaborative Filtering

We now explain how the Fingerprint of a given item is created and, following, how a Fuzzifying Function can be applied to obtain the corresponding FFP.

Let r_i be the set of ratings that a given set of users $u_1 \cdots, u_N$ has provided for item i be: $r_i = \{(u_1, r_{1i}), (u_2, r_{2i}), \cdots, (u_N, r_{Ni})\}$. To build the Fingerprint ϕ_i, we start by choosing a subset of k ratings in r_i, where k is the parameter that controls the size of the Fingerprint. The idea is that the selected ratings should be those that best represent item i. To this effect, we select the k highest ratings.

However, since users usually provide ratings on a small discrete scale (e.g. $1, 2, 3, 4$, or 5 stars), we still need to give a different importance to the possibly many ratings with the same value. Thus, when two ratings are equal, we use what we call a sorting scheme (SS) to decide which one will have a higher rank. In our experiments, we evaluated three different SSs. Let $\#u_j$ be the total number of items that user u_j has rated: (1) Random: equal ratings are ordered randomly (to be used as a baseline); (2) Higher to Lower (HL): equal ratings are sorted in descending order according to $\#u_j$; (2) Lower to Higher (LH): equal ratings are sorted in ascending according to $\#u_j$.

To illustrate, let $r_i = \{(a, 5), (b, 2), (d, 5), (e, 4), (f, 2), (h, 1), (i, 2)\}$, assume that $k = 4$, and $\#a > \#b > \cdots > \#i$. The resulting Fingerprints ϕ_i, using each of the above sorting schemes would be $\phi_i^{(\text{random})} = \{(d, 5), (a, 5), (e, 4), (f, 2)\}$, $\phi_i^{(\text{HL})} = \{(a, 5), (d, 5), (e, 4), (b, 2)\}$, and $\phi_i^{(\text{LH})} = \{(d, 5), (a, 5), (e, 4), (i, 2)\}$.

The Fingerprint ϕ_i is, in fact, an *ordered set* of ratings. This order, determined by a SS and, reflects the importance of each rating to represent items. It is by leveraging on this importance that we determine the Fuzzy Fingerprint of item i, Φ_i.

A Fuzzyfying Function (FF) $\mu(idx)$ assigns a weight to each position in a Fingerprint. In this case, the FF is used to assign a weight to each user in ϕ_i.

There are many alternatives to define the FF [6]. Here, we have tested three possibilities, shown in Eq. 1, where p_{u_j} is the position of user u_j within ϕ_i.

$$\mu_{one}(\mathrm{p}_{u_j}) = 1 \qquad \mu_{linear}(\mathrm{p}_{u_j}) = \frac{k - \mathrm{p}_{u_j}}{k} \qquad \mu_{erfc}(\mathrm{p}_{u_j}) = 1 - \mathrm{erfc}(\frac{2 \times \mathrm{p}_{u_j}}{k}) \quad (1)$$

The function μ_{one} assigns an equal weight to all users ratings. Using function μ_{linear}, the weight of a user decreases linearly, according to its position p_{u_j}. Finally, function μ_{erfc}, uses a variation of the *complementary error function* to yield a faster decrease in weights. It is important to note that these functions are not the only available options [7]. However, preliminary experiments have indicated that using other variations does not significantly improve the quality of the results. For this reason, we have not tested further alternatives.

Using one of the above fuzzifying functions, we can now define the FFP Φ_i as: $\Phi_i = \{(u_j, \mu(\mathrm{p}_{u_j})), \forall u_j \in \phi_i\}$. The FFP is, therefore, the set of users in the original Fingerprint, each with an associated weight, given by the Fuzzyfying Function. It is, in effect, a *fuzzy set of users* that rated item i.

Once the FFP for each item is determined, it is possible to compute similarities between items.

Consider Φ_i and Φ_j the FFPs of items i and j, respectively. Let U_i be the set of users in Φ_i and U_j be the set of users in Φ_j. The FFP similarity between items i and j is defined as:

$$\mathrm{sim}(\Phi_i, \Phi_j) = \sum_{u_v \in U_i \cap U_j} \frac{\min(\Phi_i(u_v), \Phi_j(u_v))}{k} \quad (2)$$

where $\Phi_x(u_v)$ denotes the the value associated to user u_v in Φ_x. This similarity, in fact, corresponds to a *minimum t-norm* between the two fuzzy sets represented by the FFPs.

Rating predictions can now be obtained in a process similar to traditional Collaborative Filtering. More specifically, let $\widehat{r}_{vi}$ be the *predicted* rating that a given user u_v would assign to item i. We start by computing the *neighborhood* of item i, i.e. the set of n items in the database that are more similar to i, $N_i(v)$, using the similarity function defined in Eq. (2). The value of $\widehat{r}_{vi}$ is defined as:

$$\widehat{r}_{vi} = \bar{r}_i + \frac{\sum_{j \in N_i(v)} \mathrm{sim}(\Phi_i, \Phi_j) \times (r_{vj} - \bar{r}_j)}{\sum_{j \in N_i(v)} sim(\Phi_i, \Phi_j)} \quad (3)$$

where r_{vj} is the rating assigned by user u to item j, $\bar{r}_x$ is the average of all ratings assigned to item x. A RS will usually perform these predictions for a large set of items and return those with the highest rating predictions, thus creating recommendations for a user.

4 Evaluation

4.1 Experimental Setup

To assert the effectiveness of FFPs, experiments were performed using four baseline similarity metrics and three distinct datasets. The similarity metrics used as a baseline for comparison are the traditional Pearson Correlation (PC) and Cosine similarity (COS). In addition, we also include the Jaccard Mean Squared Difference (JMSD) [4], an improvement on previous metrics that offers a high rating prediction accuracy, while using a lower number of neighbors. Finally, an alternative similarity metric, named MBR [18], is also compared since its authors have the exact same goal as ours.

The Pearson Correlation coefficient has been widely used since it is simple to implement, intuitive, and provides good quality results [4]. PC is defined in Eq. 4, where U is the set users that rated both items i and j.

$$sim_{PC}(i,j) = \frac{\sum_{u \in U}(r_{u,i} - \bar{r}_i) \times (r_{u,j} - \bar{r}_j)}{\sqrt{\sum_{u \in U}(r_{u,i} - \bar{r}_i)^2} \times \sqrt{\sum_{u \in U}(r_{u,j} - \bar{r}_j)^2}} \tag{4}$$

The resulting similarity will be in within the interval $[-1, 1]$, where -1 corresponds to an inverse correlation, $+1$ to a positive correlation, and values near zero show that no linear correlation exists between the two items.

Another often used similarity measure is the Cosine similarity, as defined in Eq. 5. COS will yield a value between 0 and 1, where 0 corresponds to no similarity between i and j and 1 to exactly proportional ratings between both users.

$$sim_{COS}(i,j) = \frac{\sum_{u \in U} r_{u,i} \times r_{u,j}}{\sqrt{\sum_{u \in U} r_{u,i}^2} \times \sqrt{\sum_{u \in U} r_{u,j}^2}} \tag{5}$$

The idea behind Jaccard Mean Squared Difference (JMSD) it to combine the Jaccard coefficient, which captures the number of ratings in common between items, with the Mean Square Difference (MSD) of those ratings, resulting in Eq. 6:

$$sim_{JMSD}(i,j) = Jaccard(i,j) \times \big(1 - MSD(i,j)\big); \tag{6}$$

where $Jaccard$ and MSD are defined as:

$$Jaccard(i,j) = \frac{|U_i \cap U_j|}{|U_i \cup U_j|} \qquad MSD(i,j) = \frac{\sum_{u \in U}(r_{u,i} - r_{u,j})^2}{|U|} \tag{7}$$

where U_s is the set of items ranked by user s.

The MBR metric uses a different principle, as shown in Eq. 8. It starts by computing the average rating $\bar{r}_j$ of each item j. The absolute value of the difference between these average ratings (called MBR) determines the similarity between the items. The set of neighbors H_i of item i is defined as all items $j \neq i$ such that $MBR(i,j) \leq T$, where T is a predefined threshold. Rating predictions $\hat{r}_{u,i}$ can then be determined such that the rating predicted for user u and item i is the average of all ratings given by u to items in H_i.

$$MBR(i,j) = |\bar{r}_i - \bar{r}_j| \qquad \hat{r}_{u,i} = \frac{\sum_{j \in H_i \cap U_u} r_{u,j}}{|H_i \cap U_u|} \tag{8}$$

Evaluation was conducted using three standard datasets: (1) MovieLens-1M (ML-1M), a dataset from the movie domain; (2) Netflix, a large dataset, also from the movie domain with a very sparse user-items ratings matrix; and (3) Jester, a dataset for recommending jokes, with a high number of ratings per item. Table 1 shows some statistics regarding their contents.

Table 1. Statistics for the experimental datasets. Column *sparsity* shows the percentage of not rated items in the rating matrix and column $\#\bar{r_i}$ shows the average number of ratings per item.

Dataset	Ratings	Users	Items	Sparsity	$\#\bar{r_i}$
ML-1M	1 000 209	6 040	3 706	95.53%	217
Jester	1 728 785	79 681	150	75.64%	12348
Netflix	100 000 000	480 189	17 770	98.82%	5576

All experiments were conducted using the RIVAL framework [11]. All measurements result from a 5 fold cross-validation, where the ratings are split on a user basis. The exception is the Netflix dataset, where we used the provided *probe* test set, to make our results comparable to those found in most literature.

4.2 Results

Recommendation Effectiveness. We start by evaluating the results yielded by different sorting schemes. Figure 1 presents the RMSE for the three proposed sorting schemes, while varying the value of k, using μ_{linear} as the Fuzzyfying Function. We note that the different scale for the Jester dataset is required since its ratings vary between 1 and 10, whereas the ratings for the ML-1M and Netflix datasets vary between 1 and 5.

We can see that, in all datasets, the three sorting schemes show a similar behavior. The best results are achieved, in general, by the LH sorting scheme, while the worst are achieved by the HL sorting scheme. This indicates that ratings given by the least active users are better sources of information. A possible explanation for this phenomena is that very active users tend to give the same rating to a large number of items that, in practice, vary widely in quality (as perceived by the user). On the other hand, less active users are more discriminatory when evaluating the items.

We now evaluate the impact on RMSE for different Fuzzyfying Functions. Figure 2 presents the values obtained, while varying the value of k, using the LH sorting scheme.

As for the sorting schemes, all FFs show a similar behavior on all the datasets. In addition, all show a similar performance, with only very small differences in RMSE. The difference is slightly more evident in the ML-1M dataset, although still lower than 1% between the best (μ_{erfc}) and worst (μ_{One}) FFs. This small difference is, in fact, coherent with other results found in the literature [7].

It is also interesting to show a comparison between the results achieved by our proposed FFP similarity and the baselines described in Sect. 4.1. Figure 3 shows

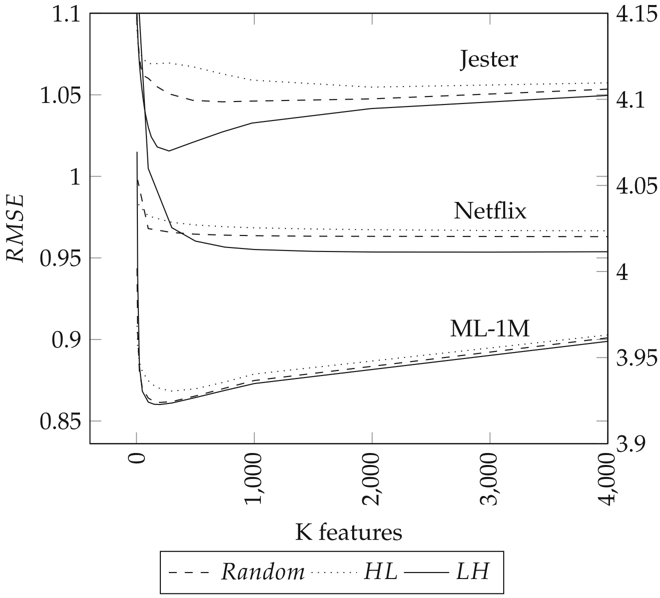

Fig. 1. Impact on RMSE of different sorting schemes. We use 100 neighbors for ML-1M, 200 for Netflix and 50 for Jester. The scale on the left y axis is for the ML-1M and Netflix datasets. The scale on the right y axis is for the Jester dataset.

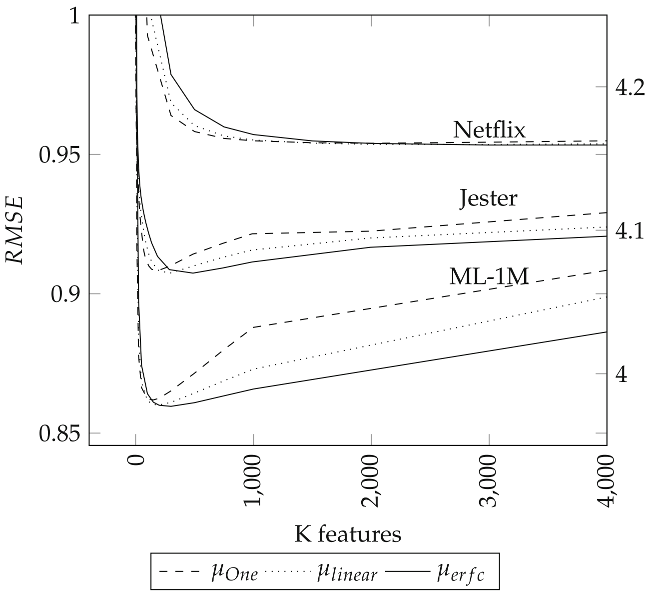

Fig. 2. Impact on RMSE of different Fuzzyfying Functions. We use 100 neighbors for ML-1M, 200 for Netflix and 50 for Jester. The scale on the left y axis is for the ML-1M and Netflix datasets. The scale on the right y axis is for the Jester dataset.

the results obtained, while varying the number of neighbors while computing rating predictions. We note that the number of neighbors is not applicable to the MBR similarity metric (see Sect. 4.1). Thus, results for MBR are shown as an horizontal line across the plot.

The Figure shows that results are clearly comparable, independently of the number of neighbors used. Furthermore, the lower RMSE was yielded, in general, by the FFP similarity metrics. We also note that FFP metrics seem to be more resilient to variations in the number of neighbors used, with results remaining almost constant as this number increases.

Computational Efficiency. To measure computational efficiency, we count the number of iterations required to make a single rating prediction. Since computing the actual prediction, using Eq. (3) is independent of the similarity metric used (i.e. in all cases, the same items will be compared to the item whose rating is being predicted) we are only interested in the iterations required to compute the similarity between any two items.

For the purpose of this work, when computing the similarity between any pair of items i and j, we define an iteration as: (1) a comparison between a value in the FFP of item i and a value in the FFP of item j, as in Eq. (2); (2) a multiplication of a rating of item i by a rating of item j, as required for the Pearson correlation or Cosine similarity; or (3) a subtraction of a rating of item i from a rating of item j, as in Eq. (7). In practice, for the baselines, this will be the number of ratings in common between the two items being compared. For the FFP, this will be the highest value between k and the number of ratings in common between the two items. We expect the gain in our proposal to come from the fact that k will be lower.

Figure 4 shows a plot of the average number of iterations performed per similarity computed, on each dataset. The number of operations for the FFP metric is shown as a function of k. A vertical line is drawn where the best results were achieved. It should be noted that MBR only requires one iteration, since it compares items by simply computing the difference between their ratings average.

In Fig. 4a, we observe that, for the ML-1M dataset, the FFP similarity requires on average 100 iterations when $k = 200$. This corresponds to a reduction of about 23% per similarity, since the baselines require 130 iterations. In the Netflix dataset (Fig. 4b), the gain is even more evident, with the FFP metric requiring about 1009 iterations, when $k = 1500$, whereas the baselines use 3791 iterations—a reduction of 73%. Finally, for the Jester dataset (Fig. 4c), the FFP similarity requires, on average, 281 iterations when $k = 300$, compared to the baseline, requiring 5822 iterations. The gain is, therefore, of 95%.

In conclusion, FFP has shown gains in all cases. This is, of course, dependent on the data. However, it is natural to expect that, the bigger the dataset, the most likely it is that items have a high number of ratings in common and, thus, the more gains can be achieved by our proposal.

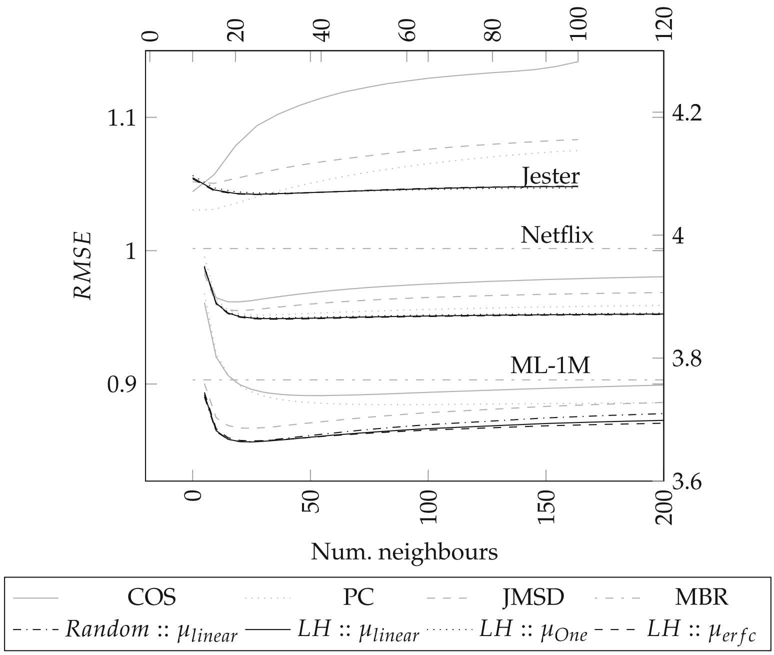

Fig. 3. Comparison of FFPs with the baseline similarity metrics. Baseline metrics are represented in light grey, while the FFP metrics are represented in black. The scales on the left y axis and bottom x axis are for the ML-1M and Netflix datasets. The scales on the right y axis and top y axis are for the Jester dataset.

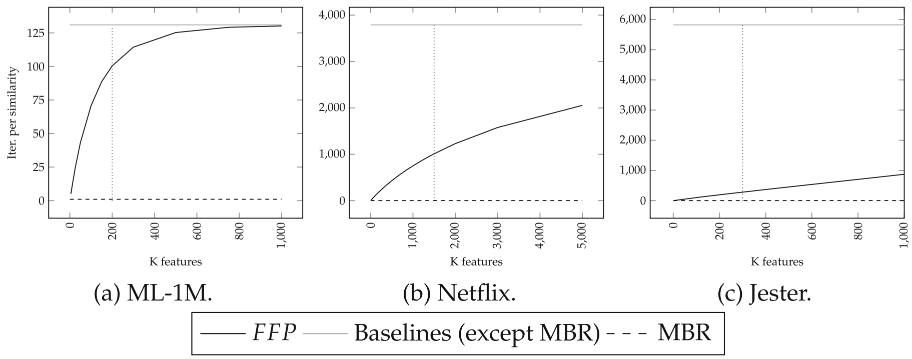

Fig. 4. Average number of iterations performed per similarity computed, on the three experimental datasets. Vertical lines show the value of k for which the best results were achieved.

Table 2. Best results for FFPs and baselines, on ML-1M. Column N contains the number of nearest-neighbour items used to compute the predicted rating.

Sim.	ML-1M					Netflix					Jester				
	SS	μ	k	N	$RMSE$	SS	μ	k	N	$RMSE$	SS	μ	k	N	$RMSE$
FFP	LH	μ_{linear}	200	20	**0.8565**	LH	μ_{One}	1500	35	0.9497	LH	μ_{linear}	200	25	4.0664
	LH	μ_{erfc}	200	25	0.8577	LH	μ_{One}	1500	35	0.9497	LH	μ_{erfc}	300	25	4.0660
	Rand	μ_{linear}	200	20	0.8568	LH	μ_{erfc}	3000	35	**0.9486**	LH	μ_{One}	200	25	4.0685
COS	-	-	-	50	0.8914	-	-	-	15	0.9616	-	-	-	15	4.0983
PC	-	-	-	75	0.8847	-	-	-	30	0.9517	-	-	-	15	**4.0419**
JMSD	-	-	-	20	0.8670	-	-	-	20	0.9549	-	-	-	15	4.0842
MBR	-	-	-	-	0.9031	-	-	-	-	1.0016	-	-	-	-	4.4063

Summary of Results. To summarize our experiments, we now present the best results achieved by each tested similarity metric. Results for the ML-1M, Netflix, and Jester datasets are shown in Table 2. The lowest values for RMSE are highlighted using bold.

The best results for the ML-1M dataset were obtained with FFPs, which outperforms all four baselines. This was achieved using at most 200 ratings to describe the items and 20 neighbors to compute rating predictions. JMSD, the best performing baseline, uses the same number of neighbors, as the best FFP similarity, but still requires using all available ratings to compute the similarities.

Similarly, on the Netflix dataset, the best results were also achieved by the FFPs. However, the lowest value in RMSE was obtained using the μ_{erfc} Fuzzyfying Function and $k = 3000$.

On the Jester dataset, the best results were obtained using Pearson Correlation. Nevertheless, the results for our proposal are still highly relevant, for several reasons. First, Jester is a somewhat unusual dataset, with a highly number of ratings per item (see Table 1) thus, we could expect the similarity metrics to behave differently. Second, the difference in RMSE to the best FFP similarity is small (0.02). Finally, as shown in Fig. 4c, the gain in efficiency obtained by the FFP is clearly significant, since on average we need 95% less iterations to compute a similarity than PC.

In conclusion, the use of FFPs allows the reduction of the similarity computational complexity, while improving, or at least maintaining, the quality of recommendations, in comparison with the baselines COS, PC, JMSD, and MBR. The improvements become more noticeable in larger datasets, which translates to a better solution in real world RSs, where we can expect very sparse data and a higher number of users and items.

5 Conclusion

In this work, we have applied the concept of Fuzzy Fingerprints to item-based Collaborative Filtering. FFPs are used to create a new concise item representation and an efficient and effective similarity metric. They have a smaller computational cost than traditional similarity metrics while requiring a low engineering effort to implement.

We have experimentally compared our proposal to two traditional similarity measures, Pearson Correlation and Cosine similarity, and two state of the art similarity metrics, Jaccard Mean Squared Difference and MBR. Results show that FFPs are a promising approach since they can be applied with success in recommendation tasks. In fact, using FFPs we were able to obtain a reduction of the number of operations needed per similarity computation between 23% and 95%, depending on the density of the rating matrix. This was achieved with an overall improvement in $RMSE$.

Future work will be conducted with the goal of exploring further configuration options for the FFPs, such as new sorting schemes and Fuzzyfying Functions. Also, we will study the application of FFPs to content-based RS.

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A Survey on Nullnorms on Bounded Lattices

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Abstract. Nullnorms are generalizations of triangular norms (t-norms) and triangular conorms (t-conorms) with a zero element to be an arbitrary point from an arbitrary bounded lattice. In this paper, we study nullnorms on bounded lattices. We examine some properties of nullnorms considering the concepts of idempotency, local internality, conjunctivity and disjunctivity on bounded lattices. We investigate relationships between such concepts for nullnorms on bounded lattices and some illustrative examples are added to clearly show connections between these. Moreover, we give two methods to obtain nullnorms on bounded lattices with a zero element by using the given nullnorm and t-norm (t-conorm) with some constraints.

1 Introduction

Aggregation functions play an important role [15,22] in the fuzzy set theory and its applications. Recently, nullnorms were introduced in [6], which are a generalization of triangular norms and triangular conorms with the zero element a anywhere in the unit interval and have to satisfy some additional conditions. In case of $a = 1$, we obtain t-conorms and in case of $a = 0$, we obtain t-norms. In addition that t-operators were introduced in [23], which are also the generalizations of the concepts of triangular norms and triangular conorms. And then in [24], it was demonstrated that nullnorms and t-operators are equivalent since they have the same block structures in $[0,1]^2$. Namely, if a binary operation F is a nullnorm then it is also a t-operator and vice versa. A characterization of such binary operations is interesting not only from a theoretical point of view but also for their applications, since they have been proved to be useful in several fields like fuzzy logic frame-work, expert system, neural networks and fuzzy quantifiers [21,25]. Idempotent nullnorms were introduced [17] as the standard median. The certain characterization and various properties of nullnorms (particularly idempotent nullnorms) were also studied in the papers [5,11,12,14,18,27].

Karaçal, İnce and Mesiar [20] have studied nullnorms on bounded lattices. They have showed the presence of nullnorms with the zero element $a \in L\backslash\{0,1\}$ by using t-norms and t-conorms on an arbitrary bounded lattice L. And the existence of the smallest nullnorm and the greatest nullnorm on L has been observed. Furthermore, it has been proved the existence of idempotent nullnorms on a distributive bounded lattice L for any element $a \in L\backslash\{0,1\}$ playing the role of a zero element in [19].

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In this paper, we study nullnorms on bounded lattices and theirs properties by using especially the concepts of idempotency, local internality, conjunctivity and disjunctivity on bounded lattices. The paper is organized as follows. We shortly recall some basic notions in Sect. 2. In Sect. 3, it is proved that considering an arbitrary bounded lattice L, if the nullnorm V is locally internal on L, V is either conjunctive or disjunctive. But we exemplify that it may not true vice versa. And it is showed that if the nullnorm V is locally internal on L, it is idempotent. We give an example every idempotent nullnorm on bounded lattices need not be locally internal. Moreover, in Theorem 1 (Theorem 2), we show that the structure of the bounded lattice L such that every nullnorm on L with the zero element $a \in L$ is idempotent (locally internal). In Sect. 4, we give two methods to obtain nullnorms on an arbitrary bounded lattice L with the zero element $s \in L$ with underlying the given nullnorm on sublattice $[0, a]$ of L (nullnorm on sublattice $[a, 1]$ of L) and t-norm on $[a, 1]$ (t-conorm on $[0, a]$) for arbitrary element $a \in L$ under some additional assumptions. And some illustrative examples are added to clarity. Finally, some concluding remarks are given.

2 Preliminaries

In this section, some preliminaries concerning bounded lattices, t-norms, t-conorms and nullnorms on them are recalled.

Definition 1 ([4]). A lattice $(L, \leq)$ is bounded if L has top and bottom elements, which are denoted as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Let L be a bounded lattice. An upper bound of the elements $x, y \in L$ is an element $a \in L$ containing the elements both x and y. The least upper bound of the elements $x, y \in L$ is an upper bound contained by every other upper bound, it is denoted $\sup\{x, y\}$ or $x \vee y$. The notations of lower bound and the gratest lower ($\inf\{x, y\}$ or $x \wedge y$) of the elements $x, y \in L$ are defined dually.

Definition 2 ([4]). Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So, $I_a = \{x \in L \mid x \parallel a\}$.

If a and b are comparable, then we use the notation $a \nparallel b$.

Definition 3 ([4]). Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, the subinterval $[a, b]$ of L is defined as $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

Similarly, we define $(a, b] = \{x \in L \mid a < x \leq b\}$, $[a, b) = \{x \in L \mid a \leq x < b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 4 ([1,2]). An operation $T : L^2 \rightarrow L$ is called a t-norm if it is commutative, associative, increasing with respect to both variables and has as neutral element 1.

Definition 5 ([7,9,10]). An operation $S : L^2 \to L$ is called a t-conorm if it is commutative, associative, increasing with respect to both variables and has as neutral element 0.

Definition 6 ([3,20]). Let $(L, \leq, 0, 1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $V : L^2 \to L$ is called a nullnorm if there is an element $a \in L$ such that $V(x, 0) = x$ for all $x \leq a$ and $V(x, 1) = x$ for all $x \geq a$.

It can be easily obtained that $V(x, a) = a$ for all $x \in L$. So $a \in L$ is the zero element for V.

Consider the set $\mathscr{V}$ of all nullnorms on L with the following order: For $V_1, V_2 \in \mathscr{V}$,

$V_1 \leq V_2 \Leftrightarrow V_1(x, y) \leq V_2(x, y)$ for all $(x, y) \in L^2$.

It can be easily shown that $\mathscr{V}$ is a partially ordered set. If we denote the set of all nullnorms on L with the zero element $a \in L$ by $\mathscr{V}(a)$, then each $\mathscr{V}(a)$ is also a partially ordered set.

We use D_a to represent the following set:

$D_a = [0, a] \times [a, 1] \cup [a, 1] \times [0, a]$ for $a \in L \setminus \{0, 1\}$.

Definition 7 ([19]). Let $(L, \leq, 0, 1)$ be a bounded lattice. An element $x \in L$ is called an idempotent element of a function $V : L^2 \to L$ if $V(x, x) = x$. The function V is called idempotent on L if all elements of L are idempotent.

Definition 8 ([26]). Let $(L, \leq, 0, 1)$ be a bounded lattice and $F : L^2 \to L$ be a function on L. Then

(i) F is called conjunctive on L if $F(x, y) \leq x \wedge y$ for all $x, y \in L$.
(ii) F is called disjunctive on L if $F(x, y) \geq x \vee y$ for all $x, y \in L$.

Proposition 1 ([13,20]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0, 1\}$ and V be a nullnorm on L with the zero element a. Then*

(i) $V|_{[0,a]^2} : [0, a]^2 \to [0, a]$ *is a t-conorm on* $[0, a]$.
(ii) $V|_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ *is a t-norm on* $[a, 1]$.

The next results, characterizing general properties of nullnorms on a bounded lattice L, are immediate from the definition of nullnorms.

Proposition 2 ([13,20]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0, 1\}$ and V be a nullnorm on L with the zero element a. The following properties hold:*

(i) $V(x, y) = a$ *for all* $(x, y) \in D_a$.
(ii) $a \leq V(x, y)$ *for all* $(x, y) \in [a, 1]^2 \cup [a, 1] \times I_a \cup I_a \times [a, 1]$.
(iii) $V(x, y) \leq a$ *for all* $(x, y) \in [0, a]^2 \cup [0, a] \times I_a \cup I_a \times [0, a]$.
(iv) $V(x, y) \leq y$ *for all* $(x, y) \in L \times [a, 1]$.
(v) $V(x, y) \leq x$ *for all* $(x, y) \in [a, 1] \times L$.
(vi) $x \leq V(x, y)$ *for all* $(x, y) \in [0, a] \times L$.
(vii) $y \leq V(x, y)$ *for all* $(x, y) \in L \times [0, a]$.

(viii) $x \vee y \leq V(x,y)$ *for all* $(x,y) \in [0,a]^2$.
(ix) $V(x,y) \leq x \wedge y$ *for all* $(x,y) \in [a,1]^2$.
(x) $(x \wedge a) \vee (y \wedge a) \leq V(x,y)$ *for all* $(x,y) \in [0,a] \times I_a \cup I_a \times [0,a] \cup I_a \times I_a$.
(xi) $V(x,y) \leq (x \vee a) \wedge (y \vee a)$ *for all* $(x,y) \in [a,1] \times I_a \cup I_a \times [a,1] \cup I_a \times I_a$.

3 Nullnorms on Bounded Lattices

Proposition 3. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $a \in L \backslash \{0,1\}$ *and* V *be a nullnorm on* L *with the zero element* a*. In this case the nullnorm* V *is neither conjunctive nor disjunctive on* L*.*

Proposition 4. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $a \in L$ *and* V *be a nullnorm on* L *with the zero element* a*.*

(i) $a = 0$ *if and only if the nullnorm* V *is conjunctive on* L*.*
(ii) $a = 1$ *if and only if the nullnorm* V *is disjunctive on* L*.*

Definition 9 ([8]). Let $(L, \leq, 0, 1)$ be a bounded lattice. The function $F : L^2 \to L$ is called locally internal on L if it satisfies $F(x,y) \in \{x, x \wedge y, x \vee y, y\}$ for all $x, y \in L$.

Proposition 5. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $a \in L \backslash \{0,1\}$ *and* V *be a nullnorm on* L *with the zero element* a*. In this case the nullnorm* V *is not locally internal on* L*.*

Proposition 6. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $a \in L$ *and* V *be a nullnorm on* L *with the zero element* a*. If the nullnorm* V *is locally internal on* L*, then* $a = 0$ *or* $a = 1$*.*

Corollary 1. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $a \in L$ *and* V *be a nullnorm on* L *with the zero element* a*. If the nullnorm* V *is locally internal on* L*, then it is either conjunctive or disjunctive on* L*.*

Remark 1. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L$ and V be a nullnorm on L with the zero element a. By Proposition 6, we know that if V is locally internal on L, then $a = 0$ or $a = 1$. It occurs a natural question: if $a = 0$ or $a = 1$, does the nullnorm V always need to be locally internal on L? The following example illustrates the fact that this hypothesis is false.

Example 1. Given the bounded lattice $L = \{0, b, c, d, 1\}$ with the order given in Fig. 1 and define the mapping $V : L^2 \to L$ by Table 1. However the mapping V is a nullnorm on L with the zero element 0, V is not locally internal on L since $V(c,d) = 0$ not $V(c,d) \in \{c,\ d,\ c \wedge d = b,\ c \vee d = 1\}$.

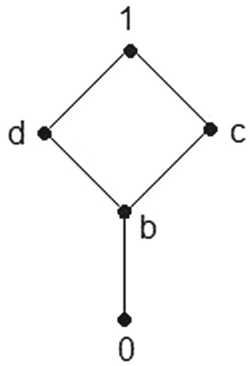

Fig. 1. The lattice L

Table 1. The nullnorm V on L

V	0	b	c	d	1
0	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	0	c
d	0	0	0	0	d
1	0	b	c	d	1

Proposition 7. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L$ and V be a nullnorm on L with the zero element a. If the nullnorm V is locally internal on L, then it is idempotent on L.*

Remark 2. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L$ and V be a nullnorm on L with the zero element a. If the nullnorm V is locally internal on L, we know that either $a = 0$ or $a = 1$ from Proposition 6. We obtain t-norm while $a = 0$ and we obtain t-conorm while $a = 1$. So, if the nullnorm V is locally internal on L, there are the only two idempotent nullnorms on L based on the fact that the only idempotent t-conorm (sup) $S_\vee : L^2 \to L$, $S_\vee(x, y) = x \vee y$ and the only idempotent t-norm (inf) $T^\wedge : L^2 \to L$, $T^\wedge(x, y) = x \wedge y$.

Remark 3. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L$ and V be a nullnorm on L with the zero element a. In that case, the natural question arises: if the nullnorm V is idempotent on L, is the nullnorm V locally internal on L? By Proposition 5, we know that the nullnorm V can not be locally internal on L while $a \in L \backslash \{0, 1\}$. The another natural question arises: if the nullnorm V is idempotent on L with the zero element a such that $a = 0$ or $a = 1$, is the nullnorm V locally internal on L? In the following proposition, we give positive answer to this question.

Proposition 8. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L$ and V be a nullnorm on L with the zero element a such that $a = 0$ or $a = 1$. If the nullnorm V is idempotent on L, the nullnorm V is locally internal on L.*

Theorem 1. *The structure of the bounded lattice $(L, \leq, 0, 1)$ such that every nullnorm defined on L with the zero element $a \in L \mid \{0, 1\}$ is idempotent is as shown in Fig. 2.*

Fig. 2. The lattice L

Theorem 2. *The structure of the bounded lattice $(L, \leq, 0, 1)$ such that every nullnorm defined on L with the zero element $a \in L$ is locally internal is as shown in Figs. 3 or 4.*

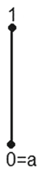

Fig. 3. The lattice L

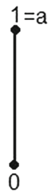

Fig. 4. The lattice L

4 Some Methods to Obtain Nulnorms on Bounded Lattices

In this section, we introduce two construction methods to obtain nullnorms on a bounded lattice $(L, \leq, 0, 1)$, for the indicated element $a \in L$, by using the existence of the nullnorm given on sublattice $[0, a]$ of L (the nullnorm given on sublattice $[a, 1]$ of L) and the t-norm on $[a, 1]$ (the t-conorm on $[0, a]$) with some constraints. The zero element $s \in L$ of the nullnorms obtained by these construction methods allows the freedom to be an arbitrary element from the sublattice $[0, a]$ of L, see Theorem 3 (the sublattice $[a, 1]$ of L, see Theorem 4).

Theorem 3. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $s, a \in L \backslash \{0, 1\}$, $[0, a]$ be a sublattice of L and $s \in [0, a]$ such that $x \nparallel s$ for all $x \in [0, a]$. If $x \geq a$ for all $x \in L \setminus [0, a]$, V^* is a nullnorm on $[0, a]$ with the zero element s and T is a t-norm on $[a, 1]$, then the following operation $V_1 : L^2 \rightarrow L$ is a nullnorm with the zero element s, where*

$$V_1(x, y) = \begin{cases} V^*(x, y) \; if \;\; (x, y) \in [0, a]^2, \\ s \qquad\qquad if \;\; (x, y) \in [0, s] \times [a, 1] \cup [a, 1] \times [0, s], \\ T(x, y) \;\;\; if \;\; (x, y) \in [a, 1]^2, \\ x \wedge y \qquad otherwise. \end{cases} \tag{1}$$

Example 2. Given the bounded lattice $L = \{0, k, s, a, p, q, 1\}$, see Fig. 5. If the nullnorm V^* on $[0, a]$ is defined by Table 2 and we take the t-norm T on $[a, 1]$ as $T(x, y) = \begin{cases} y \; if \; x = 1, \\ x \; if \; y = 1, \\ a \; otherwise. \end{cases}$ then the operation $V : L^2 \rightarrow L$ defined in Table 3 using the formula (1) is a nullnorm on L with the zero element s from Theorem 3.

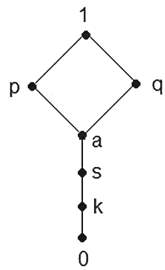

Fig. 5. The lattice L

Table 2. The nullnorm V^* on $[0, a]$

V^*	0	k	s	a
0	0	k	s	s
k	k	k	s	s
s	s	s	s	s
a	s	s	s	a

Table 3. The nullnorm V on L

V	0	k	s	a	p	q	1
0	0	k	s	s	s	s	s
k	k	k	s	s	s	s	s
s	s	s	s	s	s	s	s
a	s	s	s	a	a	a	a
p	s	s	s	a	a	a	p
q	s	s	s	a	a	a	q
1	s	s	s	a	p	q	1

The following theorem is given as dual of Theorem 3.

Theorem 4. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $s, a \in L \backslash \{0, 1\}$*,* $[a, 1]$ *be a sublattice of* L *and* $s \in [a, 1]$ *such that* $x \nparallel s$ *for all* $x \in [a, 1]$*. If* $x \leq a$ *for all* $x \in L \setminus [a, 1]$*,* V_* *is a nullnorm on* $[a, 1]$ *with the zero element* s *and* S *is a t-conorm on* $[0, a]$*, then the following operation* $V_2 : L^2 \to L$ *is a nullnorm with the zero element* s*, where*

$$V_2(x, y) = \begin{cases} V_*(x, y) & if \ \ (x, y) \in [a, 1]^2, \\ S(x, y) & if \ \ (x, y) \in [0, a]^2, \\ s & if \ \ (x, y) \in [s, 1] \times [0, a] \cup [0, a] \times [s, 1], \\ x \vee y & otherwise. \end{cases} \tag{2}$$

Example 3. Given the bounded lattice $L = \{0, p, q, a, s, k, t, 1\}$, see Fig. 6. If the nullnorm V_* on $[a, 1]$ is defined by Table 4 and we take the t-conorm S on $[0, a]$ as $S(x, y) = \begin{cases} y \ if \ x = 0, \\ x \ if \ y = 0, \\ a \ otherwise. \end{cases}$ then the operation $V : L^2 \to L$ defined in Table 5 using the formula (2) is a nullnorm on L with the zero element s from Theorem 4.

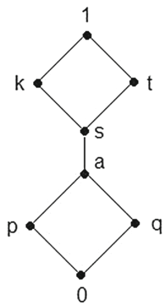

Fig. 6. The lattice L

Table 4. The nullnorm V_* on $[a, 1]$

V_*	a	s	k	t	1
a	a	s	s	s	s
s	s	s	s	s	s
k	s	s	k	s	s
t	s	s	s	t	s
1	s	s	s	s	1

Table 5. The nullnorm V on L

V	0	p	q	a	s	k	t	1
0	0	p	q	a	s	s	s	s
p	p	a	a	a	s	s	s	s
q	q	a	a	a	s	s	s	s
a	a	a	a	a	s	s	s	s
s	s	s	s	s	s	s	s	s
k	s	s	s	s	s	k	s	s
t	s	s	s	s	s	s	t	s
1	s	s	s	s	s	s	s	1

Remark 4. Let $(L, \leq, 0, 1)$ be a bounded lattice, $s, a \in L \backslash \{0, 1\}$, $[0, a]$ be a sublattice of L, $s \in [0, a]$ such that $x \nparallel s$ for all $x \in [0, a]$ and $x \geq a$ for all $x \in L \setminus [0, a]$. In Theorem 3, if we take t-norm T on $[a, 1]$ as the only idempotent t-norm (inf) $T^{\wedge}$, then we obtain the following nullnorm on L, where

$$V_1(x, y) = \begin{cases} V^*(x, y) & if \ \ (x, y) \in [0, a]^2, \\ s & if \ \ (x, y) \in [0, s] \times [a, 1] \cup [a, 1] \times [0, s], \\ x \wedge y & otherwise. \end{cases} \quad (3)$$

(i) If the nullnorm V^* is idempotent on $[0, a]$ with the zero element s, then the nullnorm V_1 given by the formula (3) is idempotent on L with the zero element s.
(ii) If the nullnorm V^* is locally internal on $[0, a]$ with the zero element s, then $s = a$ since $s \in L \backslash \{0, 1\}$. So, even if the nullnorm V^* is locally internal on $[0, a]$ with the zero element s, then the nullnorm V_1 given by the formula (3) can not be locally internal on L from Proposition 5.

Remark 5. Let $(L, \leq, 0, 1)$ be a bounded lattice, $s, a \in L \backslash \{0, 1\}$, $[a, 1]$ be a sublattice of L, $s \in [a, 1]$ such that $x \nparallel s$ for all $x \in [a, 1]$ and $x \leq a$ for all $x \in L \setminus [a, 1]$. In Theorem 4, if we take t-conorm S on $[0, a]$ as the only idempotent t-conorm (sup) $S_{\vee}$, then we obtain the following nullnorm on L, where

$$V_2(x,y) = \begin{cases} V_*(x,y) & if \quad (x,y) \in [a,1]^2, \\ s & if \quad (x,y) \in [s,1] \times [0,a] \cup [0,a] \times [s,1], \\ x \vee y & otherwise. \end{cases} \qquad (4)$$

(i) If the nullnorm V_* is idempotent on $[a,1]$ with the zero element s, then the nullnorm V_2 given by the formula (4) is idempotent on L with the zero element $s \in L \backslash \{0,1\}$.
(ii) If the nullnorm V_* is locally internal on $[a,1]$ with the zero element s, then $s = a$ since $s \in L \backslash \{0,1\}$. So, even if the nullnorm V_* is locally internal on $[a,1]$ with the zero element s, then the nullnorm V_2 given by the formula (4) can not be locally internal on L from Proposition 5.

Remark 6. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, s \in L \backslash \{0,1\}$, $[0,a]$ be a sublattice of L and $s \in [0,a]$ such that $x \nparallel s$ for all $x \in [0,a]$. Turning back to Theorem 3, observe that the constraint $x \geq a$ for all $x \in L \setminus [0,a]$ can not be omitted, in general. The next example illustrates the fact that the associativity of V can be violated.

Example 4. Given the bounded lattice $L = \{0, s, a, n, m, 1\}$ by Fig. 7 and the nullnorm V^* on $[0,a]$ by Table 6. And define the mapping $V : L^2 \to L$ as Table 7 using the formula (1) in Theorem 3.

Since $V(V(1,n), m) = V(n,m) = 0$ and $V(1, V(n,m)) = V(1,0) = s$ for the elements $1, n, m \in L$, the mapping V does not satisfy associativity. So, we obtain that the mapping V is not a nullnorm on L, however the mapping V is defined by using the formula (1) in Theorem 3.

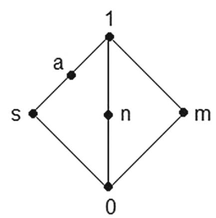

Fig. 7. The lattice L

Table 6. The nullnorm V^* on $[0,a]$

V^*	0	s	a
0	0	s	s
s	s	s	s
a	s	s	a

Table 7. The mapping V on L

V	0	s	a	n	m	1
0	0	s	s	0	0	s
s	s	s	s	s	s	s
a	s	s	a	0	0	a
n	0	s	0	n	0	n
m	0	s	0	0	m	m
1	s	s	a	n	m	1

5 Conclusion Remarks

In this paper, some properties of nullnorms on an arbitrary bounded lattice L with a zero element are investigated considering the notations of idempotency, local internality, conjunctivity and disjunctivity. Some results on the relationship between such notions for nullnorms on L are given. In addition that the structure of the bounded lattice L such that every nullnorm on L with the given zero element is idempotent (locally internal) is researched. Moreover, we give two methods yielding nullnorms on bounded lattices with the indicated zero element under some constraints.

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Characterizing Ordinal Sum for t-norms and t-conorms on Bounded Lattices

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Abstract. The ordinal sum of triangular norms on the unit interval has been proposed to construct new triangular norms. However, considering general bounded lattices, the ordinal sum of triangular norms and conorms may not generate triangular norms and conorms. In this paper, we study and propose some new construction methods yielding triangular norms and conorms on general bounded lattices. Moreover, we generalize these construction methods by induction to a ordinal sum construction for triangular norms and conorms, applicable on any bounded lattice. And some illustrative examples are added for clarity.

1 Introduction

The triangular norms (t-norms for short) and the triangular conorms (t-conorms for short) firstly appeared in mathematical literature in the study "Statistical Metrics" of Menger [19]. The main aim of introducing these concepts was that triangular inequalities were extended from classical metric spaces to probabilistic metric spaces using the theory of t-norm and t-conorm. The t-norms with 1 as neutral element and t-conorms with 0 as neutral element which are considered as special semigroup on unit interval and play a key role in the theory of fuzzy metric spaces were given as equivalent by Schweizer and Sklar [25,26]. T-norms and t-conorms also play an important role in decision making, statistics as well as in the theories of non-additive measures and cooperative games [5,16,22,23]. Therefore, the knowledge of the structure of the class of t-norms and t-conorms is very important. These operators originally were defined on the unit interval, although it is crucial and fundamental to work on general structure of them on bounded lattices. Additionally, the concepts of t-norms and t-conorms were also studied theoretically by many authors in other papers [1,2,7,17,20].

Constructions for t-norms and t-conorms on $[0,1]$ by using ordinal sums in the sense of lattices of Birkhoff [4] and ordinal sums of semigroups [9] are the ordinal sum construction. Although, the ordinal sum of t-norms on $[0,1]$ has been used to construct other t-norms, on a bounded lattice, an ordinal sum of t-norms may not be a t-norm. Considering an arbitrary bounded lattice L, in order to construct these operators on L as ordinal sums of t-norms and t-conorms, Goguen has proposed to study on fuzzy sets with membership values from L [15]. Saminger

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has studied on ordinal sums of t-norms acting on some bounded lattice which is not necessarily a chain or an ordinal sum of posets [24]. Furthermore, Medina has given some methods that ordinal sum of arbitrary t-norms on a bounded lattice is a t-norm under some additional assumptions [18]. Drygaś has investigated some properties of uninorm-like operations generalizating of t-norms and t-conorms based on operations are given by ordinal sums [12]. For more details on uninorms, we refer to [6,11,13,21].

Ertuğrul, Karaçal and Mesiar have proposed a modification of ordinal sums of t-norms and t-conorms underlying with t-norms and t-conorms valid on general bounded lattices [14]. In this paper, we propose new construction ways to generate t-norms and t-conorms on an arbitrary bounded lattice different from given in [14]. If x or y are from $[a,1]$ in triangular norm case, then both constructions coincide and our new construction puts on the remainder 0. Similarly if x or y are from $[0,a]$ in triangular conorm case, then both constructions coincide and our new construction puts on the remainder 1. As by-product, our constructions generate the smallest t-norm T on L such that on $[a,1]^2$ t-norm T coincides with an a priori fixed t-norm V on $[a,1]$ and the greatest t-conorm S on L such that on $[0,a]^2$ t-conorm S coincides with an a priori fixed t-norm W on $[0,a]$. Furthermore, we generalize these construction ways by induction to a ordinal sum construction for t-norms and t-conorms, applicable on any bounded lattice. And we give some examples to clearly understand these construction ways for t-norms and t-conorms on general bounded lattices.

Definition 1. [4,10] A bounded lattice $(L,\leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1,0\in L$ such that $0\leq x\leq 1$, for all $x\in L$.

Definition 2. [4] Given a bounded lattice $(L,\leq,0,1)$ and $a,b\in L$, if a and b are incomparable, in this case, we use the notation $a\parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a=\{x\in L \mid x\parallel a\}$.

Definition 3. [3,8] Let $(L,\leq,0,1)$ be a bounded lattice. Operation $T: L^2\to L$ is called a triangular norm (t-norm) if it is commutative, associative, increasing with respect to both variables and it satisfies $T(x,1)=x$ for all $x\in L$.

Definition 4. [8] Let $(L,\leq,0,1)$ be a bounded lattice. Operation $S: L^2\to L$ is called a triangular conorm (t-conorm) if it is commutative, associative, increasing with respect to both variables and it satisfies $S(x,0)=x$ for all $x\in L$.

2 Construction of t-norms and t-conorms on Bounded Lattices

Consider a bounded lattice $(L,\leq,0,1)$, an element $a\in L\backslash\{0,1\}$, a t-norm $V: [a,1]^2\to[a,1]$ and a t-conorm $W:[0,a]^2\to[0,a]$. An ordinal sum extension T of V to L and S of W to L is given by (see [24])

$$T(x,y)=\begin{cases} V(x,y) \; if \;\; (x,y)\in[a,1]^2, \\ x\wedge y \quad otherwise \end{cases} \tag{1}$$

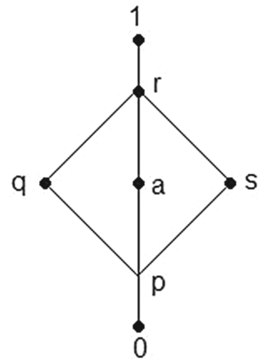

Fig. 1. The lattice L

and

$$S\left(x,y\right)=\begin{cases} W\left(x,y\right) \; if \; \left(x,y\right)\in\left[0,a\right]^2, \\ x\vee y \qquad otherwise \end{cases} \tag{2}$$

However, the above-defined mapping T need not be a t-norm, in general. Similarly, S need not be a t-conorm, in general.

Example 1. Given the lattice $L=\{0,p,q,s,a,r,1\}$ with the order given in Fig. 1.

(i) Consider the t-norm $V:[a,1]^2\to[a,1]\,,\, V(x,y)=\begin{cases} x\wedge y \; if \; 1\in\{x,y\}, \\ a \qquad otherwise \end{cases}$ for all $x,y\in[a,1]$. Then the operation T is constructed as Table 1 by using the formula (1), but T is not a t-norm on L.

Table 1. The operation T on L

T	0	p	q	s	a	r	1
0	0	0	0	0	0	0	0
p	0	p	p	p	p	p	p
q	0	p	q	p	p	q	q
s	0	p	p	s	p	s	s
a	0	p	p	p	a	a	a
r	0	p	q	s	a	a	r
1	0	p	q	s	a	r	1

If we take the elements $s,r\in L$, then $s\leq r$. But we have that $T(s,r)=s\parallel a=T(r,r)$. Hence, T does not satisfy monotonicity. Moreover, $T\left(T\left(r,r\right),q\right)=T\left(a,q\right)=p$ and $T\left(r,T\left(r,q\right)\right)=T\left(r,q\right)=q$ for the elements $r,q\in L$. Hence, T does not satisfy associativity. So, we obtain that T is not a t-norm on L.

(ii) Consider the t-conorm $W : [0,a]^2 \to [0,a]$, $W(x,y) = \begin{cases} x \vee y \; if \, 0 \in \{x,y\}, \\ a \quad\quad otherwise \end{cases}$ for all $x, y \in [0,a]$. Then the operation S is constructed as Table 2 by using the formula (2), but S is not a t-conorm on L.

Table 2. The operation S on L

S	0	p	a	q	s	r	1
0	0	p	a	q	s	r	1
p	p	a	a	q	s	r	1
a	a	a	a	r	r	r	1
q	q	q	r	q	r	r	1
s	s	s	r	r	s	r	1
r	r	r	r	r	r	r	1
1	1	1	1	1	1	1	1

we take the elements $p, q \in L$, then $p \leq q$. But we have that $S(p,p) = a \parallel q = S(q,p)$. Hence, S does not satisfy monotonicity. Moreover, $S(S(p,p), s) = S(a,s) = r$ and $S(p, S(p,s)) = S(p,s) = s$ for the elements $p, s \in L$. Hence, S does not satisfy associativity. So, we obtain that S is not a t-conorm on L.

Theorem 1. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \backslash \{0,1\}$. If V is a t-norm on $[a,1]$ and W is a t-conorm on $[0,a]$, then the functions $T : L^2 \to L$ and $S : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where*

$$T(x,y) = \begin{cases} V(x,y) \; if \; (x,y) \in [a,1[^2, \\ x \wedge y \quad\;\; if \, 1 \in \{x,y\}, \\ 0 \quad\quad\quad\; otherwise \end{cases} \tag{3}$$

and

$$S(x,y) = \begin{cases} W(x,y) \; if \; (x,y) \in]0,a]^2, \\ x \vee y \quad\;\; if \, 0 \in \{x,y\}, \\ 1 \quad\quad\quad\; otherwise \end{cases} \tag{4}$$

Corollary 1. *Let $a \in L \backslash \{0,1\}$. If we put $V(x,y) = \begin{cases} x \wedge y \; if \, 1 \in \{x,y\}, \\ a \quad\quad otherwise \end{cases}$ on $[a,1]$ in the formula (3) in Theorem 1, the following t-norm T is the smallest t-norm on L that extends V.*

$$T(x,y) = \begin{cases} a \quad\quad if \; (x,y) \in [a,1[^2, \\ x \wedge y \; if \, 1 \in \{x,y\}, \\ 0 \quad\quad otherwise \end{cases}$$

Corollary 2. *Let* $a \in L\backslash\{0,1\}$. *If we put* $W(x,y) = \begin{cases} x \vee y \; if \, 0 \in \{x,y\}, \\ a \quad\quad otherwise \end{cases}$ *on* $[0,a]$ *in the formula (4) in Theorem 1, the following t-conorm* S *is the greatest t-conorm on* L *that extends* W.

$$S(x,y) = \begin{cases} a & if \; (x,y) \in \left]0,a\right]^2, \\ x \vee y & if \, 0 \in \{x,y\}, \\ 1 & otherwise \end{cases}$$

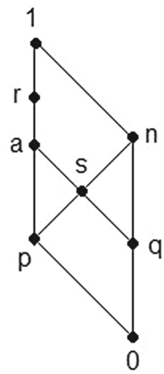

Fig. 2. The lattice L in Example 2

Example 2. Given a bounded lattice $L = \{0,p,q,s,a,n,r,1\}$, with the order given in Fig. 2. Consider t-norm $V : [a,1]^2 \to [a,1]$, $V(x,y) = x \wedge y$ for all $x,y \in [a,1]$. By using Theorem 1, the corresponding t-norm $T : L^2 \to L$ is given as Table 3.

Table 3. The t-norm T on L

T	0	p	q	s	n	a	r	1
0	0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	0	p
q	0	0	0	0	0	a	0	q
s	0	0	0	0	0	0	0	s
n	0	0	0	0	0	0	0	n
a	0	0	a	0	0	a	r	a
r	0	0	0	0	0	r	r	r
1	0	p	q	s	n	a	r	1

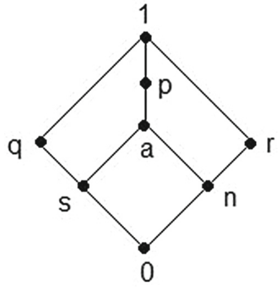

Fig. 3. The lattice L in Example 3

Example 3. Given a bounded lattice $L = \{0, s, n, a, q, r, p, 1\}$, with order given in Fig. 3. Consider t-conorm $W : [0, a]^2 \to [0, a]\,, W(x, y) = x \vee y$ for all $x, y \in [0, a]$. By using Theorem 1, the corresponding t-conorm $S : L^2 \to L$ is given as Table 4.

Table 4. The t-conorm S on L

S	0	s	n	a	q	r	p	1
0	0	s	n	a	q	r	p	1
s	s	s	a	a	1	1	1	1
n	n	a	n	a	1	1	1	1
a	a	a	a	a	1	1	1	1
q	q	1	1	1	1	1	1	1
r	r	1	1	1	1	1	1	1
p	p	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

3 Ordinal Sum Characterization for t-norms and t-conorms on Bounded Lattices

Theorem 2. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $\{a_0, a_1, a_2, ..., a_n\}$ *be a finite chain in* L *such that* $a_0 = 1 > a_1 > a_2 > ... > a_n = 0$. *Let* $V : [a_1, 1]^2 \to [a_1, 1]$ *be a t-norm on the sublattice* $[a_1, 1]$. *Then the operation* $T = T_n : L^2 \to L$ *defined recursively as follows is a t-norm, where* $V = T_1$ *and for* $i \in \{2, 3, ..., n\}$ *the operation* $T_i : [a_i, 1]^2 \to [a_i, 1]$ *is given by*

$$T_i\,(x, y) = \begin{cases} T_{i-1}\,(x, y) & if\ (x, y) \in [a_{i-1}, 1[^2\,, \\ x \wedge y & if\, 1 \in \{x, y\}\,, \\ a_i & otherwise \end{cases} \qquad (5)$$

The proof follows easily from Theorem 1 by induction and therefore it is omitted. The construction described inductively by formula (5) can be considered as a ordinal sum construction for t-norms. Obviously, if L in Theorem 2 is a chain then the formula (5) reduces to

$$T_i(x,y)=\begin{cases} T_{i-1}(x,y) & if\ (x,y)\in [a_{i-1},1[^2, \\ x\wedge y & if\, 1\in\{x,y\}, \\ a_i & if\ (x,y)\in [a_i,a_{i-1}[^2 \cup [a_i,a_{i-1}[\times[a_{i-1},1[\cup[a_{i-1},1[\times[a_i,a_{i-1}[. \end{cases}$$

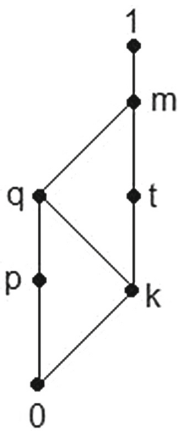

Fig. 4. The lattice L in Example 4

Example 4. Consider the lattice $L = \{0,p,q,k,t,m,1\}$ with the order given in Fig. 4 and in L the finite chain $\{0,k,t,m,1\}$ such that $V(x,y) = x \wedge y$ *for all* $x,y \,\epsilon\, [m,1]$. Let $V : [m,1]^2 \to [m,1]$ be the t-norms on the sublattice $[m,1]$ such that $\vee(x,y) = x \wedge y$ for all $x,y \in [m,1]$. By using Theorem 2, where $V = T_1$, the t-norms $T_2 : [t,1]^2 \to [t,1]$, $T_3 : [k,1]^2 \to [k,1]$ and $T = T_4 : L^2 \to L$ are defined as follows (Tables 5, 6 and 7)

By using the formula (3) in Theorem 1, on the bounded lattice L which is depicted by Fig. 4 the corresponding t-norm $T : L^2 \to L$ is defined as Table 8 for the given t-norm $V = T_1 : [m,1]^2 \to [m,1]$ on the sublattice $[m,1]$.

Table 5. The t-norm T_2 on L

T_2	t	m	1
t	t	t	t
m	t	m	m
1	t	m	1

Table 6. The t-norm T_3 on L

T_3	k	q	t	m	1
k	k	k	k	k	k
q	k	k	k	k	q
t	k	k	t	t	t
m	k	k	t	m	m
1	k	q	t	m	1

Table 7. The t-norm T on L

$T = T_4$	0	p	k	q	t	m	1
0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	p
k	0	0	k	k	k	k	k
q	0	0	k	k	k	k	q
t	0	0	k	k	t	t	t
m	0	0	k	k	t	m	m
1	0	p	k	q	t	m	1

Table 8. The t-norm T on L

T	0	p	k	q	t	m	1
0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	p
k	0	0	0	0	0	0	k
q	0	0	0	0	0	0	q
t	0	0	0	0	0	0	t
m	0	0	0	0	0	m	m
1	0	p	k	q	t	m	1

Theorem 3. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $\{b_0, b_1, b_2, ..., b_n\}$ *be a finite chain in* L *such that* $b_0 = 0 < b_1 < b_2 < ... < b_n = 1$*. Let* $W : [0, b_1]^2 \to [0, b_1]$ *be a t-conorm on the sublattice* $[0, b_1]$*. Then the operation* $S = S_n : L^2 \to L$ *defined recursively as follows is a t-conorm, where* $W = S_1$ *and for* $i \in \{2, 3, ..., n\}$ *the operation* $S_i : [0, b_i]^2 \to [0, b_i]$ *is given by*

$$S_i(x, y) = \begin{cases} S_{i-1}(x, y) & if \ (x, y) \in]0, b_{i-1}]^2, \\ x \vee y & if \ 0 \in \{x, y\}, \\ b_i & otherwise \end{cases} \quad (6)$$

The proof follows easily from Theorem 1 by induction and therefore it is omitted. The construction described inductively by formula (6) can be considered as a ordinal sum construction for t-conorms. Obviously, if L in Theorem 3 is a chain then the formula (6) reduces to

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & if\ (x,y) \in]0,b_{i-1}]^2, \\ x \vee y & if\ 0 \in \{x,y\}, \\ b_i & if\ (x,y) \in]b_{i-1},b_i]^2 \cup]0,b_{i-1}] \times]b_{i-1},b_i] \cup]b_{i-1},b_i] \times]0,b_{i-1}]. \end{cases}$$

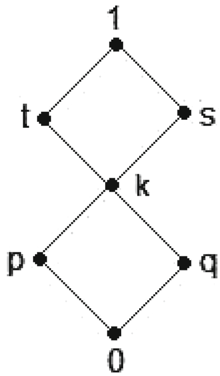

Fig. 5. The lattice L in Example 5

Example 5. Consider the lattice $L = \{0,p,q,k,t,s,1\}$ with the order given in Fig. 5 and in L the finite chain $\{0,p,k,t,1\}$ such that $0 < p < k < t < 1$. Let $W : [0,p]^2 \to [0,p]$ be a t-conorm on the sublattice $[0,p]$ such that $W(x,y) = x \vee y$ for all $x,y \in [0,p]$. By using Theorem 3, where $W = S_1$, the t-conorms $S_2 : [0,k]^2 \to [0,k]$, $S_3 : [0,t]^2 \to [0,t]$ and $S = S_4 : L^2 \to L$ are defined as follows (Tables 9, 10 and 11)

By using the formula (4) in Theorem 1, on the bounded lattice L which is depicted by Fig. 5, the corresponding $P : L^2 \to L$ is defined as Table 12 for the given t-norm $W = S_1 : [0,p]^2 \to [0,p]$ on the sublattice $[0,p]$.

Table 9. The t-conorm S_2 on L

S_2	0	p	q	k
0	0	p	q	k
p	p	p	k	k
q	q	k	k	k
k	k	k	k	k

Table 10. The t-conorm S_3 on L

S_3	0	p	q	k	t
0	0	p	q	k	t
p	p	p	k	k	t
q	q	k	k	k	t
k	k	k	k	k	t
t	t	t	t	t	t

Table 11. The t-conorm S on L

$S = S_4$	0	p	q	k	t	s	1
0	0	p	q	k	t	s	1
p	p	p	k	k	t	1	1
q	q	k	k	k	t	1	1
k	k	k	k	k	t	1	1
t	t	t	t	t	t	1	1
s	s	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Table 12. The t-conorm S on L

S	0	p	q	k	t	s	1
0	0	p	q	k	t	s	1
p	p	p	1	1	1	1	1
q	q	1	1	1	1	1	1
k	k	1	k	1	1	1	1
t	t	1	1	1	1	1	1
s	s	1	1	1	1	1	1
1	1	1	1	1	1	1	1

4 Concluding Remarks

Considering an arbitrary bounded lattice L, we have researched and exemplified the new ways to characterization of t-norms and t-conorms on L by using the existence of t-norms on the sublattice $[a, 1]$ and t-conorms on the sublattice $[0, a]$ for the indicated element $a \in L \setminus \{0, 1\}$. Furthermore, we have proposed that the construction methods considered as a ordinal sum of t-norms and t-conorms on L. The construction methods as observed in [18,24] be in need of several constraint conditions ensuring an ordinal sum on L of arbitrary t-norms and t-conorms. But our ordinal sum construction methods be valid on general bounded lattices without any additional assumption. And these methods can be applied to define connectives for fuzzy sets type 2, interval-valued fuzzy sets, intuitionistic fuzzy sets, etc.

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Crisp vs. Fuzzy Data in Multicriteria Decision Making: The Case of the VIKOR Method

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Abstract. In this contribution we want to shed light onto the following research question: in the context of multicriteria decision making problem, does the nature of the information available (either crisp or fuzzy) has any impact in the ranking of the alternatives? We explore this situation using randomly generated decision problems and the VIKOR method as an example.

Keywords: MCDM · VIKOR · Fuzzy data · Crisp data

1 Introduction

A multicriteria (or multiattribute) decision making (MCDM) problem can be represented using a decision matrix as the one shown in Table 1. There are m rows, each one associated with an alternative $\{A_1, A_2, \ldots, A_m\}$. Every column (out of n) is associated with a set of criteria $\{C_1, C_2, \ldots, C_n\}$. Finally, it is assumed that a decision maker is able to reflect the importance of the criteria using a set of n weights $\{w_1, w_2, \ldots, w_n\}$. The value of the alternative A_i under criterion C_j is denoted as x_{ij}.

Then, a MCDM method takes as input a decision matrix and gives, as an output, a ranking of the alternatives. Usually, the ranking is derived from a rating or scoring of the alternatives.

There are many popular MCDM methods in the literature, as those based on pairwise comparison of alternatives, Analytic Hierarchy Process (AHP), methods that consider the distance to the ideal solution as the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) or methods that works with preferences, as the Preference Ranking Organisation Methods for Enrichment Evaluations (PROMETHEE). The interested reader can check recent books like [6] for further information.

It is well known that given the same MCDM problem, different methods may lead to different results, as it have been shown in [1,2], but here we focus on the *VIseKriterijumska Optimizacija I Kompromisno Resenje* (VIKOR) [9,11]. VIKOR has been consistently used in a wide range of areas in the last years.

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Table 1. Decision matrix of a MCDM problem.

MCDM	w_1 C_1	w_2 C_2	... ...	w_n C_n
A_1	x_{11}	x_{12}	...	x_{1n}
A_2	x_{21}	x_{22}	...	x_{2n}
...	...	...	x_{ij}	...
A_m	x_{m1}	x_{m2}	...	x_{mn}

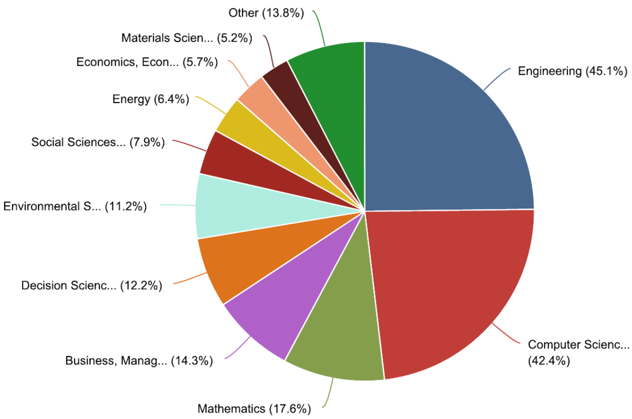

Fig. 1. Distribution of VIKOR related publications (698 since 1990) by research areas.

A search with the query *TITLE-ABS-KEY (vikor)* in www.scopus.com, and considering the results since 1990, retrieved 698 documents. But what is more interesting is the research areas where these publications appeared, which reveals the wide range of potential applications of the method. This distribution is shown in Fig. 1.

In many real life situations, the values x_{ij} are not known precisely. In other words, the imprecision, vagueness, uncertainty, etc. of the data should be taken into account. In a situation where instead of saying the value of A_i under C_j is *exactly* x_{ij}, we are allowed to say *around* x_{ij}, then fuzzy numbers [4] are suitable tools to model such kind of imprecision leading to fuzzy multicriteria decision making problems [7,12]. In this way, we can obtain a proper modeling of the imprecision in the nature of the data.

The research question we pose here is: *when using the VIKOR method, does it make any impact to consider or not this imprecision (crisp vs. fuzzy) in the problem from the point of view of the obtained rankings?* In order to shed light into this topic, we design and perform a simulation based experiment to assess such impact.

This paper is organized as follows. Section 2 introduces the VIKOR method and its fuzzy version, Fuzzy VIKOR. In Sect. 4 we describe the computational experiments and the results obtained. Finally, Sect. 5 summarizes the conclusions of our research.

2 Description of the VIKOR Method

Here we outline the basic aspects of the VIKOR method following the description presented in [11].

VIKOR Method for Precise Information

Considering the notation of the Table 1, the method consists of the following steps:

Step 1: Normalization procedure:

$$n_{ij} = \frac{(f_j^+ - x_{ij})}{(f_j^+ - f_j^-)} \tag{1}$$

where f_j^+ and f_j^- are defined as follow:

$$\text{If } C_j \text{ is a benefit criterion } \begin{cases} f_j^+ = \max_i f_{ij} \\ f_j^- = \min_i f_{ij} \end{cases} \tag{2}$$

$$\text{If } C_j \text{ is a cost criterion } \begin{cases} f_j^+ = \min_i f_{ij} \\ f_j^- = \max_i f_{ij} \end{cases} \tag{3}$$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2: Compute the values S_i and R_i

$$S_i = \sum_{j=1}^{m} w_j * n_{ij} \tag{4}$$

$$R_i = \max_j \left[w_j * n_{ij} \right] \tag{5}$$

where w_j is the weight associated to the criteria C_j, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$.

Step 3: Compute the values Q_i, $i = 1, 2, \ldots, m$ as:

$$Q_i = v \frac{(S_i - S^+)}{(S^- - S^+)} + (1 - v) \frac{(R_i - R^+)}{(R^- - R^+)} \tag{6}$$

where

$$S^+ = \min_i \{S_i\} \qquad S^- = \max_i \{S_i\}$$
$$R^- = \min_i \{R_i\} \qquad R^- = \max_i \{R_i\}$$

Parameter $v \in [0,1]$ plays the following role. When $v > 0.5$, this represents a decision-making process that could use the strategy of maximum group utility (i.e., if v is big, group utility is emphasized), or by consensus when $v \approx 0.5$, or with veto when $v < 0.5$.

Step 4: Rank the alternatives in terms of their Q_i values. The lower the Q_i value, the higher position in the ranking.

VIKOR Method for Imprecise Information

VIKOR was modified to deal with imprecise (fuzzy) information in the decision matrix. This modification led to Fuzzy VIKOR [10]. Now, the input information considers that the values x_{ij} and w_j are triangular fuzzy numbers denoted as $\tilde{x}_{ij}$ and $\tilde{w}_j$, respectively. The fuzzy decision matrix is shown in Table 2.

Table 2. Fuzzy decision matrix of a MCDM problem.

MCDM	$\tilde{w}_1$ C_1	$\tilde{w}_2$ C_2	... ...	$\tilde{w}_n$ C_n
A_1	$\tilde{x}_{11}$	$\tilde{x}_{12}$	...	$\tilde{x}_{1n}$
A_2	$\tilde{x}_{21}$	$\tilde{x}_{22}$	...	$\tilde{x}_{2n}$
...	...	...	$\tilde{x}_{ij}$	...
A_m	$\tilde{x}_{m1}$	$\tilde{x}_{m2}$	...	$\tilde{x}_{mn}$

From an algorithmic point of view, Fuzzy VIKOR replaces the "classic arithmetic" with the triangular fuzzy numbers' arithmetic. For the sake of completeness, the corresponding operations are described below. Being $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ two triangular fuzzy numbers, the required operations are:

$$\begin{aligned}
Addition : \tilde{A} \oplus \tilde{B} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \\
Subtraction : \tilde{A} \ominus \tilde{B} &= (a_1 - b_3, a_2 - b_2, a_3 - b_1) \\
Multiplication : \tilde{A} \otimes \tilde{B} &= (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3) \\
Division : \tilde{A} \oslash \tilde{B} &= (a_1/b_3, a_2/b_2, a_3/b_1) \\
ScalarDivision : \tilde{A}/k &= (a_1/k, a_2/k, a_3/k) \\
Maximum : MAX(\tilde{A}, \tilde{B}) &= (\max(a_1, b_1), \max(a_2, b_2), \max(a_3, b_3)) \\
Minimum : MIN(\tilde{A}, \tilde{B}) &= (\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3))
\end{aligned}$$

At the end of step 3, Fuzzy VIKOR has a fuzzy value $\tilde{Q}_i = \{Q_{i1}, Q_{i2}, Q_{i3}\}$ for every alternative. Instead of sorting them, a defuzzification process is made as follows:

$$Q_i = \frac{Q_{i1} + 2Q_{i2} + Q_{i3}}{4}$$

where now, the best alternative is again the one with the lowest value of Q_i.

3 Illustrative Examples

In this section, we will show using three examples, that the nature of the data (fuzzy or crisp) may or may not have an impact in the ranking of the alternatives.

Example 1: the nature of the data has no impact.

Table 3 shows (on top) a fuzzy decision matrix. A corresponding crisp decision matrix (in the bottom) is derived from the fuzzy one, taking the central values of each fuzzy number.

The last two columns on each matrix show the scores (Q_i values) for the alternatives together with their corresponding rankings. Please note that the Q_i values in the 7^{th} column of the fuzzy matrix correspond to the defuzzified value of $\tilde{Q}_i$.

For these matrices, the ranking of the alternatives is the same. As a consequence, we can say that in this example, the nature of data is irrelevant or has no impact.

Table 3. Example 1: fuzzy (top) and crisp (bottom) decision problems. The ranking of the alternatives is also shown.

Alt.	C_1	C_2	C_3	C_4	C_5	Q_i	Rank
A_1	(47,51,54)	(11,11,12)	(50,53,56)	(79,88,93)	(43,43,45)	0.38	3
A_2	(29,31,33)	(17,19,20)	(9,9,9)	(1,1,1)	(79,86,93)	0.55	5
A_3	(24,26,27)	(57,60,63)	(83,90,91)	(44,49,51)	(72,77,82)	0.00	1
A_4	(38,40,42)	(29,31,34)	(54,56,60)	(85,92,99)	(38,41,45)	0.30	2
A_5	(8,9,10)	(70,72,78)	(15,16,17)	(29,31,33)	(51,52,57)	0.50	4

Alt.	C_1	C_2	C_3	C_4	C_5	Q_i	Rank
A_1	51	11	53	88	43	0.76	3
A_2	31	19	9	1	86	1.00	5
A_3	26	60	90	49	77	0.00	1
A_4	40	31	56	92	41	0.73	2
A_5	9	72	16	31	52	0.99	4

Example 2: different rankings but the same top alternative.

Table 4 shows the decision matrices and the corresponding results. For these matrices, the ranking of the alternatives are almost completely different, but the top alternative (A_2) is the same.

Two reversals occurred when going from the fuzzy to the crisp data: A_1 with A_5 and A_3 with A_4. If we analize the Q_i values of these pairs of alternatives, we can observe that they are very similar. Thus, a minor change in the decision matrix data, may lead to a change in their rank position. In other words, we can say that the alternatives (A_1, A_5) and (A_3, A_4) have the "same quality".

Table 4. Example 2: fuzzy (top) and crisp (bottom) decision problems. The ranking of the alternatives is also shown.

Alt.	C_1	C_2	C_3	C_4	C_5	Q_i	Rank
A_1	(32,34,37)	(83,92,97)	(20,22,24)	(62,67,70)	(52,55,58)	0.06	3
A_2	(89,99,105)	(75,77,78)	(13,13,14)	(24,24,25)	(42,47,51)	0.04	1
A_3	(73,79,84)	(4,4,4)	(5,5,5)	(84,91,100)	(36,36,39)	0.33	5
A_4	(59,62,67)	(73,76,79)	(22,22,24)	(19,21,23)	(31,32,33)	0.22	4
A_5	(36,38,43)	(86,87,95)	(43,45,46)	(86,95,104)	(22,23,25)	0.05	2

Alt.	C_1	C_2	C_3	C_4	C_5	Q_i	Rank
A_1	34	92	22	67	55	0.50	2
A_2	99	77	13	24	47	0.10	1
A_3	79	4	5	91	36	0.95	4
A_4	62	76	22	21	32	1.00	5
A_5	38	87	45	95	23	0.51	3

Example 3: different rankings and the top alternatives changed.

Table 5 shows the problems information and the results.

In this example we observe that the top 3 alternatives changed when the problem is considered as crisp. In the fuzzy problem, the top alternatives were A_1, A_5, A_2, while in the crisp case, the order was A_2, A_1, A_5.

Moreover, in the fuzzy case, the difference in the Q_i values is smaller between A_1 (0.06) and A_5 (0.07) than the one between A_5 (0.07) and A_2 (0.13), so a minor change in the data may alter which is the top alternative. However in the crisp

Table 5. Example 3: fuzzy (top) and crisp (bottom) decision problems. The ranking of the alternatives is also shown.

Alt.	C_1	C_2	C_3	C_4	C_5	Q_i	Rank
A_1	(80,86,93)	(81,88,92)	(3,3,3)	(72,76,83)	(50,53,58)	0.06	1
A_2	(34,37,39)	(81,90,91)	(8,9,9)	(68,73,80)	(33,34,35)	0.13	3
A_3	(27,29,30)	(16,17,17)	(75,78,85)	(18,20,21)	(20,20,20)	0.35	5
A_4	(58,59,63)	(85,90,97)	(54,55,57)	(32,34,35)	(9,10,10)	0.22	4
A_5	(76,84,85)	(59,60,61)	(91,99,108)	(4,5,5)	(70,74,78)	0.07	2

Alt.	C_1	C_2	C_3	C_4	C_5	Q_i	Rank
A_1	86	88	3	76	53	0.50	2
A_2	37	90	9	73	34	0.22	1
A_3	29	17	78	20	20	1.00	5
A_4	59	90	55	34	10	0.73	4
A_5	84	60	99	5	74	0.51	3

case, the top alternative is clearly the best one: $Q_2 = 0.22$ while the second one has $Q_1 = 0.5$. Here, a minor change in the data may cause a reversal between the second and third alternatives (as $Q_5 = 0.51$).

4 Experiments and Results

Using a simulation based approach, we want assess the nature of the data' impact. We will firstly solve a set of randomly generated fuzzy decision making problems and their derived crisp versions and secondly, we will compare the corresponding rankings.

More specifically we consider 100 decision matrices with $n = 5$ criteria and $m = 5$ alternatives. The data generation procedure is outlined below.

We first define a fuzzy decision making problems dataset (FDP), where each element $\tilde{dp}_k \in FDP, k = 1, \ldots, 100$ is a decision matrix with fuzzy data. The corresponding values $\tilde{x}_{ij}$ in the matrix are randomly generated as follows:

$$\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3}) = \begin{cases} x_{ij1} \in x_{ij2} - \cup(1, 10) * x_{ij2}/100 \\ x_{ij2} \in \cup(1, 100) \\ x_{ij3} \in x_{ij2} + \cup(1, 10) * x_{ij2}/100 \end{cases}$$

Regarding the criteria, and for the sake of simplicity, we assumed that all of them are equally important and should be maximized (benefit criteria). They are defined as follows:

$$\tilde{w}_j = (w_{j1}, w_{j2}, w_{j3}) = \begin{cases} w_{j1} = 1/n - (0.1 * 1/n) \\ w_{j2} = 1/n \\ w_{j3} = 1/n + (0.1 * 1/n)) \end{cases}$$

Departing from the set FDP we construct a crisp dataset CDP, where each $dp_k \in CDP, k = 1, \ldots, 100$, is the defuzzified version of the corresponding $\tilde{dp}_k$. Recalling that $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})$, we define the crisp value $x_{ij} = x_{ij2}$. In these crisp problems, the weights are defined as $w_j = 1/n$.

Then, for each pair $(\tilde{dp}_k, dp_k), k = 1, \ldots, 100$ we apply Fuzzy VIKOR and VIKOR to obtain two rankings: r_k^f, r_k^c, respectively. Finally we count the number of times where $r_k^f = r_k^c$ and those cases where $r_k^f \neq r_k^c$. In the later case, we count if the top alternative is the same or not in both rankings.

The whole process is repeated for three different values of the VIKOR's v parameter: $v = \{0.0, 0.5, 1.0\}$.

The computational experiments have been done in `R-Project` work environment [13], using the VIKOR and Fuzzy VIKOR implementation provided in the `MCDM` [8] and `FuzzyMCMD` [5] packages respectively. Both packages are available on the `CRAN` repository [3].

4.1 Results

In this section we analyze the results obtained.

Solved Problems. It is well known that VIKOR may fail to solve a problem (unable to return a ranking), when $v < 1$.

The reason for the failure lies on the R_i values, which can be the same for all the alternatives in some problems. In such situation, $\min_i\{R_i\} = \max_i\{R_i\}$, thus $R^- = R^+$ and then, in the following calculation

$$Q_i = v\frac{(S_i - S^+)}{(S^- - S^+)} + (1-v)\frac{(R_i - R^+)}{(R^- - R^+)}$$

a division by zero occurs.

Table 6 shows an example from our experiment, where VIKOR failed to provide a solution when $v = 0.5$. As the R_i, $i = 1, \ldots, m$ values are the same, then the Q_i, $i = 1, \ldots, m$ can not be calculated.

Table 6. Unsolved problems using VIKOR with $v = 0.5$.

Alt.	C_1	C_2	C_3	C_4	C_5	S_i	R_i	Q_i	Rank
A_1	24	38	100	38	66	0.55	0.20	-	-
A_2	85	7	41	39	86	0.57	0.20	-	-
A_3	5	69	59	88	95	0.35	0.20	-	-
A_4	19	91	21	47	70	0.59	0.20	-	-
A_5	12	20	69	88	23	0.63	0.20	-	-
f^+	85	91	100	88	95	$S^+ = 0.35$	$R^+ = 0.20$		
f^-	5	7	21	38	23	$S^- = 0.63$	$R^- = 0.20$		

This situation also happened in our experiments when considering crisp data and $v = 0$ and $v = 0.5$. However when the data have a fuzzy nature, Fuzzy VIKOR did not fail. The number of solved cases for both data sets are showed in Table 7.

The reason behind this "good" behaviour of Fuzzy VIKOR lies on the fact that it works with fuzzy numbers. Then the chance of obtaining "exactly" the same fuzzy numbers after the calculations is definitely low.

Of course, we can not ensure that Fuzzy VIKOR is free of the indeterminacies of its crisp counterpart, but in our experiments, all the problems have been solved.

Comparison of the Rankings. We compared if the ranking of the VIKOR method and the ranking of the Fuzzy VIKOR method are the same. In those cases where the rankings were different, we checked if the top alternative was

Table 7. Number of solved problems considering crisp and fuzzy data.

v	Solved Problems	
	Crisp	Fuzzy
0.0	95	100
0.5	95	100
1.0	100	100

the same or not. The 5 unsolved problems when VIKOR was run with $v = 0$ and $v = 0.5$ in the crisp case were not taken into account in the following analysis.

The results are in Table 8, where for each v value we show the number of cases where both rankings were the same and, for those cases where the rankings were different, we also indicate the cases where the top alternative was the same or not.

Table 8. Number of cases where the rankings were equal. When the rankings were different, we show the number of cases where the top alternative is the same or not.

Value of v	Different Rank					
	Same Rank		Same 1^{st}		Different 1^{st}	
0.0	9/95	(9%)	82/86	(95%)	4/86	(5%)
0.5	59/95	(62%)	29/36	(80%)	7/36	(20%)
1.0	86/100	(86%)	11/14	(78%)	3/14	(22%)

Several aspects should be highlighted. The first one is the influence of the v parameter. As v increased, the number of cases with the same ranking also increased. The second one is that when the rankings are different, the top alternative is almost always the same. However, as v increased, the number of cases with the same top alternative decreased. In other words, as v increased, the cases where the top alternative changed also increased: 5% cases when $v = 0$, 20% when $v = 0.5$ and 22% when $v = 1.0$.

5 Conclusions

In this work we wanted to shed light into the following question: does the nature of the information available (either crisp or fuzzy) has any impact in the ranking of the alternatives produced by the VIKOR method?

Using three examples, with fuzzy and crisp data, we showed that three situations may occurr with the corresponding rankings: they could be equal, they could be different but the top alternative is maintained or the rankings could be different and the top alternative changed.

Under this situation, we conducted a simulation based experiment where we randomly generated 100 fuzzy multicriteria decision making problems and their corresponding crisp variant that were then solved using VIKOR and Fuzzy VIKOR.

Two conclusions can be derived from the experiments. The first one is that when dealing with fuzzy data, VIKOR is able to solve all the problems in the dataset. This is not happened for the crisp dataset, where a reduced number of problems were not solved due to some indeterminacies occurring in the inner VIKOR calculations.

The second one regards with the question under study: our experiment revealed that considering fuzzy information of the model may lead to different results in terms of the rankings of the alternatives. Moreover these differences depended on the VIKOR's v parameter. When $v = 0.0$, in just 9% of the solved cases the rankings were equal. In most of the cases where the rankings were different, the top alternative remained unchanged (95% of the cases). As v increased, the cases with the same ranking also increased (up to 86%). The same occurred with the number of cases where the top alternative changed.

From this initial study, several lines of research emerge. The first one is to extend the datasets to consider different numbers of criteria and alternatives. The second one is related with the methods: the same kind of experiment should be reproduced using other MCMD methods. Both lines are under study.

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Facility Location Selection Employing Fuzzy DEA and Fuzzy Goal Programming Techniques

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Abstract. Facility location selection have strategic importance for companies because it influences not only manufacturing and transportation costs but also productivity and lead times to a great extend. Additionally, it is considered as hard and complicated tasks with respect to its multi-objective nature and difficulties resulted from collecting necessary data. Therefore this problem has always been an important subject of engineering literature. The aim of this study is to solve a facility location selection problem in a manufacturing company that locates in Tekirdağ/Turkey. This company has six different factories in the same facility and is considering about establishing a plastic injection factory in the future for producing some of the important plastic components in order to gain cost advantage and also to increase know-how. For this purpose, facility location selection problem is aimed to be solved by applying fuzzy data envelopment analysis (Fuzzy DEA) and fuzzy goal programming (Fuzzy GP) methods.

Keywords: Facility location selection · Fuzzy DEA · Fuzzy goal programming · MCDM

1 Introduction

The most common corporate growth strategies are mainly related with growing in global markets, like entrance to new markets or attempting start new businesses to get the benefit of economies of scale [1]. Many researchers emphasized the significance of facility location selection problem under the existence of unsteady and versatile environment of global economy [2]. This kind of location problems that consist of global development are mostly connected with social, economic, legal, cultural factors and moreover they require considerable capital investment which will affect the limitation of manufacturing and logistics in long term. Like any other real-life problems, facility location problems are mostly complicated in nature and their dependence to other processes change from situation to situation. The basic reasons of difficulty to solve these problems are determining necessary considerations that will have further direct effect on selection procedure and fulfilling necessary adjustments between these considerations [3].

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For manufacturing companies, selecting the most optimal location has gained significance since minimizing costs and maximizing the use of resources is one of the most important objectives to achieve. While selecting a location, there are several criteria to pay attention like human resources, climate conditions, availability of raw material etc. Depending on this reason, plant selection can be considered as a multiple criteria decision making problem [4]. According to the literature review, facility location selection is a group decision making problem which is a non-repetitive process, requires the contribution of different departments of the organization and usually cleared up through a procedure that is not supported by a well-defined framework. Within the overall operation management, the methodology of facility location is too broad including product/service design, planning of the capacity and facility layout design issues. Depending on the fact that decisions related with the design of the facility location influence each part of the organization, they cannot be made by only operational managers [5]. Solely top managers can be responsible of these decisions or the company can also outsource necessary support to as well [6].

Over the last five years, scholars have contributed to facility location selection problem by applying a few number of multi-criteria decision making approaches. [7] ranked the nuclear power plant sites in Turkey by integrating fuzzy ENTROPY and fuzzy compromise programming. [8] solved the carbon dioxide geological storage location decision problem in Turkey using fuzzy TOPSIS, fuzzy ELECTRE, and fuzzy VIKOR methods. [9] identified the most appropriate location for a textile manufacturing facility in Istanbul by combining Geographic Information Systems and fuzzy AHP (analytic hierarchy process). [10] selected the most suitable location for the production of nuclear power by employing interval type-2 fuzzy TOPSIS methodology, and also conducted a case study in Turkey.

In this study, in order to find the best location for a new factory, four location alternatives are evaluated by fuzzy DEA ranking methodology, then a fuzzy goal programming approach is proposed, for selecting the most suitable alternative, according to the different achievement degrees of goals.

The rest of the paper is organized as follows: Facility location selection problem is detailed in Sect. 2. Fuzzy DEA and Fuzzy GP methods are explained in Sect. 3. Numerical application is made in Sect. 4 for a manufacturing company that locates in Tekirdağ/Turkey and the paper is concluded in Sect. 5.

2 Facility Location Selection Problem

Facility location selection problems are basically dealt with identifying the most suitable site for a firm for conducting operations. Not only locating but also relocating or expansion is considered as facility location decisions. Determination, examination, assessment and choosing between options are the steps of facility location decision processes [11]. As they are long term, high-investment required and irreversible decisions, location selection problems have strategic importance. Moreover selected facility has an observable influence on costs and revenues. A careless location decision might be the most important reason of excessive transportation costs, lack of raw material or labor, loss of adequate logistics network or any other kind of problems those will badly affect operations [12].

Facility location selection can be considered not only as a decision making problem at the strategic management level but also a partly constructed process of supply chain management. Looking at the significance of facility location selection decisions, current situation and possible future trends have to be elaborately examined by decision-maker to determine all possible aspects before location selection process begins. Therefore it is so obvious that many factors have to be considered during the selection procedure. Although many criterions exist for facility site selection, some of them those may have a possible impact on decisions are more crucial. As an example of the possible facility location selection criterions, [13] conducted their study on five criteria which are: favorable labor climate, proximity to markets, community considerations, quality of life, proximity to suppliers and resources.

Facility location selection problem is a multi-criteria decision making problem by nature. In order to evaluate location alternatives, in the beginning managers must define the affecting criteria which are important for the company. Set of alternatives in the case of selecting probable alternatives of location are generally determined by top managers in connection with their business environment, published reports, individual processes and etc. Information which is necessary to select facility location is provided externally and based on the human judgments. It is a non-repetitive decision and therefore has to be examined from many aspects. [14] indicated that most of the criterions, weights and the rules of decisions are assessed by human perceptions which are impossible to express exactly with numerical values. This is also a natural result of ambiguity typically involved in location selection processes which is resulted from linguistic evaluation and multiple attribute decision making in these problems [15]. Fuzzy sets are valid instruments to deal with such ambiguities in the literature. Fuzzy numbers have a certain success in representing linguistic data in decision making models. Fuzzy based methods have been recently emerged in such fields that verbal statements can be used as translators between verbal statements and quantitative estimates when it is a necessity to deal with the ambiguity [16].

In this study, we consider the problem of selecting a location for a new facility of a manufacturing company that locates in Tekirdağ/Turkey. Decision makers are four managing engineers of the company. The problem is observed in two parts; in the first part Fuzzy DEA is applied in order to find the technical efficiency rankings of the alternatives and in the second part, as a further step, to meet the targets of the company given as intervals, we employed a fuzzy goal programming model.

3 Fuzzy MCDM Techniques

3.1 Fuzzy Data Envelopment Analysis

Data envelopment analysis (DEA) model, pioneered by [17], calculates the relative efficiency of a decision making unit (DMU) by maximizing the ratio of its total weighted outputs to its total weighted inputs with a constraint that the output to input ratio of every decision making unit (DMU) is less than or equal to unity. The standard

CCR model is as follows; where E_{j_0} is the efficiency score of the evaluated DMU, u_r is the weight assigned to output r, v_i is the weight assigned to input i, y_{rj} is the quantity of output r generated and x_{rj} is the amount of input i consumed by DMU j, respectively, and ε is a small positive scalar.

$$Max\,E_{j_0} = \frac{\sum_{r=1}^{s} u_r y_{rj_0}}{\sum_{i=1}^{m} v_i x_{ij_0}} \tag{1}$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad \forall j$$
$$u_r, v_i \geq \varepsilon \;\; \forall r, i$$

The classical model has several drawbacks, namely poor weight dispersion and poor discriminating power among the DMUs. In the literature, several approaches such as weight restriction and cross-efficiency analysis have been developed in order to handle the unrealistic weight distribution and improve the discriminating power of DEA [18, 19]. Moreover, some proposed models like minsum and minimax efficiency models restrict favorable consideration to the DMU under evaluation unlike the traditional DEA model. Minimax efficiency is to minimize maximum deviation from efficiency whereas minsum efficiency minimizes the total deviation from efficiency [19, 20].

The conventional DEA models are limited for dealing technology, supplier or health-care service systems evaluation and selection problems, where the observed data set may provide vague and imprecise knowledge about the generating process [21]. Then use of the fuzzy measures and fuzzy mathematical programs in the DEA models is unavoidable, which are obtained from the experts, generally by linguistic terms, and then denoted as fuzzy numbers [22, 23]. Fuzzy DEA, is an extension of DEA which incorporates imprecision in DEA. There are several approaches in fuzzy DEA literature, such as tolerance approach [21], the α-cut approach [24, 25], fuzzy ranking approach [26, 27] and the possibility approach. [28]. In this study, the model proposed by [25] is employed to deal with the facility location selection problem. The selected framework, deal the problem for different α-cuts in order to rank the alternatives.

$$Min\,z = \theta \tag{2}$$

subject to

$$\theta\left(\alpha x_{ip}^{m} + (1-\propto)x_{ip}^{l}\right) \geq \sum_{j=1}^{n} \lambda_j\left(\propto x_{ij}^{m} + (1-\propto)x_{ij}^{u}\right) \quad \forall i,$$
$$\propto y_{rp}^{m} + (1-\propto)y_{rp}^{u} \leq \sum_{j=1}^{n} \lambda_j\left(\propto y_{rj}^{m} + (1-\propto)y_{rj}^{l}\right) \quad \forall r,$$
$$\lambda_j \geq 0 \quad \forall j,$$

In this formulation; $\propto \in (0,\ 1]$ is a parameter. $\tilde{x}_{ij} = (x^l, x^m, x^u)$ and $\tilde{y}_{ij} = (y^l, y^m, y^u)$ are fuzzy triangular numbers.

3.2 Fuzzy Goal Programming

Fuzzy goal programming (FGP) is to determine imprecise aspiration levels of the goals in a fuzzy decision environment by using membership functions. In a FGP model, the goals are considered fuzzy and their priorities can be expressed by both using linguistic variables such as "important", "medium", "low", and ordinal numbers namely "first", "second", etc. A membership function of each goal can be provided by making interviews with the decision-makers for determining the achievement degrees that are to be satisfied for each objective function.

In this study, the linear membership function of Zimmerman [29] is adapted, the membership function is as follows:

$$\mu_i = \begin{cases} 1 & if \ \ G_i(x) \geq g_i \\ \frac{G_i(x) - L_i}{g_i - L_i} & if \ \ L_i \leq G_i(\mathrm{x}) \leq g_i \\ 0 & if \ \ G_i(x) \leq L_i \end{cases} \tag{3}$$

or

$$\mu_i = \begin{cases} 1, & if \ \ G_i(x) \leq g_i \\ \frac{U_i(x) - G_i(x)}{U_i - g_i}, & if \ \ g_i \leq G_i(x) \leq U_i \\ 0, & if \ \ G_i(x) \leq L_i \end{cases} \tag{4}$$

Where L_i is the lower tolerance limit, U_i is the upper tolerance limit for the ith fuzzy goal and g_i is the aspiration level. The simple additive fuzzy goal programming model [30] is as follows:

$$Maximize f(\mu) = \sum\nolimits_{k=1}^{n} \mu_k \tag{5}$$

subject to

$$\begin{aligned} &\mu_i = \frac{G_i(x) - L_i}{g_i - L_i} \quad \text{for some i,} \\ &\mu_j = \frac{U_i(x) - G_i(x)}{U_i - g_i} \quad \text{for some j, } \ \mathrm{j} \neq \mathrm{i}, \\ &Ax \leq b, \\ &\mu_i, \mu_j \leq 1, \\ &x, \mu_i, \mu_j \geq 0; \quad \mathrm{i}, \mathrm{j} \in \{1, \ldots, \mathrm{n}\} \end{aligned}$$

Where $Ax \leq b$ are the crisp system constraints and μ_i, μ_j are the goal's achievement degree.

4 Case Study: New Plastic Injection Factory

4.1 Problem Definition

The company in which this study is conducted has five other factories in one facility. Production in all of these six factories basically only consists of assembly lines in which parts are supplied from selected vendors and assembled inside the factory. There is a possibility of establishing a new plastic injection factory to manufacture some of the selected plastic parts as in-house production especially the ones that are expensive and the others those have frequent quality problems. It has to be emphasized that various plastic parts with many different sizes are required to produce any kind of white goods. Establishing a plastic factory regarded as a necessary decision not only for financial point of view but also for increasing know-how and not to be behind competitors technically.

For the application, first of all a committee of four decision makers (D1, D2, D3, D4) are selected. These four decision makers are all managing engineers of the company from different departments. Objective is defined as finding optimal location for the new plastic factory among four alternatives which are Tekirdağ, Hadımköy, Gebze and Yalova (A1, A2, A3, A4). These alternatives are especially selected depending on the proximity to location of the facility that it is going to serve and closeness to related industries depending on the meeting of qualified labor force requirement. Evaluation criteria (C1, C2, C3, C4, C5) have been determined by the decision makers, as shown explicitly in Fig. 1, in the hierarchical structure of the problem.

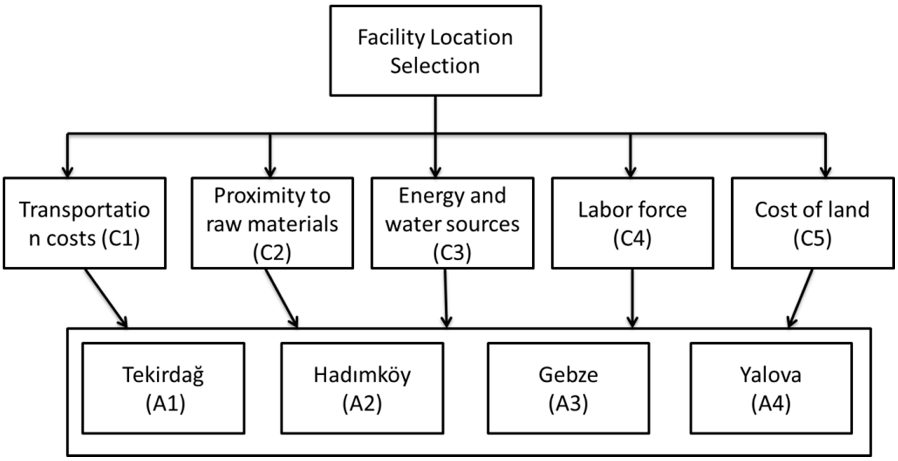

Fig. 1. Hierarchical structure of facility location selection methodology

- ***Transportation costs (C1)***

Transportation is a significant criteria to consider as it has a certain impact on costs and therefore on facility location selection decisions. With the logic of “Just In Time”, in the future the aim is to deliver materials to concerned factories as much frequent as possible. Therefore how close the selected location is, and how good transportation sources of selected location are, have a certain impact on costs and efficiency of production system. It is important to provide that the new established factory has to be located close to the facility of other six factories in order to assure the easiness of transport conditions.

- ***Proximity to raw materials (C2)***

Transportation costs of raw material supply to the factory also consist of a great portion among the expenditure item. New plastic factory will serve to 6 factories as much as its total capacity, it can be estimated how frequent raw materials will be transported to factory. Therefore, selected alternatives have to be analyzed with respect to the sources of raw materials and the existence of potential suppliers.

- ***Energy and water sources (C3)***

It is a fact that the facility requirement for energy and water will affect the costs in a large portion. Therefore it is important to establish related factory in a location that the energy and water sources are cheaper comparing to other alternatives.

- ***Labor force (C4)***

Plastic injection production requires not only experienced and qualified blue collar but also white collar employees. It has to be analyzed in details if the labor force is both quantitatively and qualitatively adequate. Selected location should also provide necessary conditions that employees and their families can live during their carrier.

- ***Cost of land (C5)***

The first and the most important cost item at the beginning of the establishment process is the investment on the land. Selected location also should not be a constriction for further expanding strategies of the factory.

4.2 Fuzzy DEA Application

In order to find the technical efficiency rankings of the alternatives, Fuzzy DEA is implemented. First, decision makers are asked to evaluate location alternatives with respect to the pre-defined criteria using linguistic terms set given below in Fig. 2. The data set of the problem is given in Table 1.

Then, evaluations of four decision makers are aggregated with equal weights for each decision maker. Using Eq. (2), the efficiency ranking results are obtained and given in Table 2.

As observed from the table given above, the first and second alternatives have the best results according to the different possibility levels (alfa-cuts). At $\alpha = 1$, which is the pessimistic scenario, alternative 4 is the only non-efficient alternative among the four alternatives.

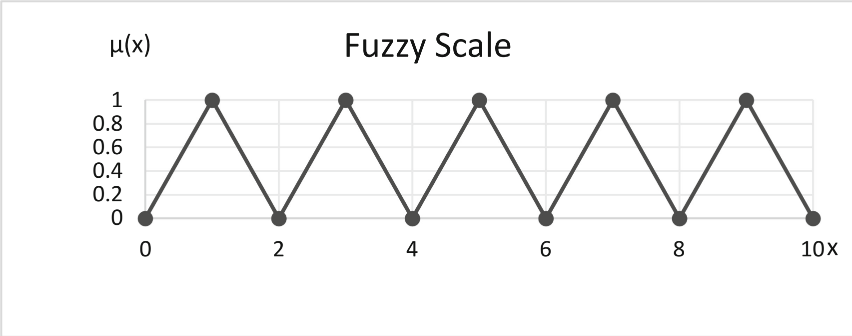

Fig. 2. A linguistic term set where VL: (0, 1, 2), L: (2, 3, 4), M: (4, 5, 6), H: (6, 7, 8), VH: (8, 9, 10)

Table 1. Four DM's evaluation of four alternatives with respect to five criteria

Criteria	Alternatives	Decision-makers			
		D1	D2	D3	D4
C1	A1	VG	VG	VG	VG
	A2	G	G	G	G
	A3	MG	G	G	MG
	A4	G	MG	MG	G
C2	A1	VG	G	VG	VG
	A2	VG	VG	G	VG
	A3	F	MG	F	MG
	A4	MG	F	MG	F
C3	A1	VG	VG	G	VG
	A2	G	G	VG	G
	A3	G	MG	MG	G
	A4	MG	G	MG	F
C4	A1	VG	G	VG	VG
	A2	VG	G	G	G
	A3	MG	G	F	G
	A4	F	MG	G	F
C5	A1	F	MG	MP	F
	A	MG	F	P	MG
	A3	VG	G	MG	VG
	A4	MG	F	G	G

Table 2. Fuzzy DEA results

	α = 0	α = 0.2	α = 0.4	α = 0.6	α = 0.8	α = 1
A1	2.097	1.796	1.545	1.333	1.154	1
A2	2.097	1.796	1.545	1.333	1.154	1
A3	1.771	1.571	1.399	1.248	1.116	1
A4	1.615	1.429	1.267	1.127	1.004	0.897

4.3 Fuzzy Goal Programming Model

Ranking of the alternatives is obtained at the first step with the Fuzzy DEA model, however, as a further step, a fuzzy goal programming model is constructed according to the company's goals, in order to find the best alternative which minimizes the total deviation from company's goals. The goals were pre-defined by the company in accordance with its financial resources, technical analyses, market research etc. In this study, interactions between goals are not observed because the company claimed that there is not any interaction between their goals.

The company has four major goals in this new factory project: Material (G1), labor force (G2), Capital Investment (G3) and Energy (G4).

Hence, the fuzzy goal programming model is obtained:

$$
\begin{aligned}
&G1: 49000x_1 + 42000x_2 + 47000x_3 + 44000x_4 \cong 45000 \pm 5000\\
&G2: 130000x_1 + 170000x_2 + 120000x_3 + 180000x_4 \cong 150000 \pm 50000\\
&G3: 200000x_1 + 400000x_2 + 320000x_3 + 270000x_4 \cong 350000 \pm 150000\\
&G4: 75000x_1 + 120000x_2 + 110000x_3 + 90000x_4 \cong 100000 \pm 50000
\end{aligned}
$$

$Max\ \mu_1 + \mu_2 + \mu_3 + \mu_4$
subject to

$$
\begin{aligned}
&\mu_1 \le (-49000x_1 - 42000x_2 - 47000x_3 - 44000x_4 + 50000)/(50000 - 45000)\\
&\mu_1 \le (49000x_1 + 42000x_2 + 47000x_3 + 44000x_4 - 40000)/(45000 - 40000)\\
&\mu_2 \le (-130000x_1 - 170000x_2 - 120000x_3 - 180000x_4 + 200000)/(200000 - 150000)\\
&\mu_2 \le (130000x_1 + 170000x_2 + 120000x_3 + 180000x_4 - 100000)/(150000 - 100000)\\
&\mu_3 \le (-200000x_1 - 400000x_2 - 320000x_3 - 270000x_4 + 500000)/(500000 - 350000)\\
&\mu_3 \le (200000x_1 + 400000x_2 + 320000x_3 + 270000x_4 - 200000)/(350000 - 200000)\\
&\mu_4 \le (-75000x_1 - 120000x_2 - 110000x_3 - 90000x_4 + 150000)/(150000 - 100000)\\
&\mu_4 \le (75000x_1 + 120000x_2 + 110000x_3 + 90000x_4 - 50000)/(100000 - 50000)
\end{aligned}
$$

$$
\begin{aligned}
&\mu_i \ge 0; i = 1, 2, 3, 4\\
&x_j \in \{0, 1\}; j = 1, 2, 3, 4
\end{aligned}
$$

The problem is solved using GAMS program, and the solution is $(x_1, x_2, x_3, x_4) = (0, 0, 1, 0)$ with the respective achievement degree of $(\mu_1, \mu_2, \mu_3, \mu_4) = (0.6, 0.4, 0.8, 0.8)$. The best location alternative is the third alternative-Gebze. Therefore, the maximum achieved goals are capital investment and energy with 0.8. However, selecting the third alternative yields to a poor achievement degree of 0.4 for the second goal which is the labor force, and a moderate achievement degree of 0.6 for the first goal.

5 Conclusions

Facility location selection problem, being a strategic level decision, is highly examined problem in the literature. Due to its ambiguous nature where crisp data does not exist, fuzzy set theory and its extensions are often employed in this problem type. In this study, we employed a fuzzy DEA methodology for ranking the location alternatives, based on linguistic assessment of decision makers. Although, fuzzy DEA is a powerful tool to derive the technical efficiency score of alternatives, it does not reflect totally the goals of the company. Therefore, as a further step, a fuzzy goal programming methodology is used considering four goals of the company with respect to four location alternatives. As is mentioned in section before, while the first and second alternatives are the most efficient location alternatives, when the goals are incorporated problem, it is found that the third alternative is the selected alternative, for maximum achievement degree of the mentioned fuzzy goals. Further research, may focus on different scenario on the location selection problem, as differentiating the weights of the goals or prioritizing some of them.

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Finding the Optimal Number of Features Based on Mutual Information

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Abstract. For high dimensional data analytics, feature selection is an indispensable preprocessing step to reduce dimensionality and keep the simplicity and interpretability of models. This is particularly important for fuzzy modeling since fuzzy models are widely recognized for their transparency and interpretability. Despite the substantial work on feature selection, there is little research on determining the optimal number of features for a task. In this paper, we propose a method to help find the optimal number of feature effectively based on mutual information.

Keywords: Feature selection · Mutual information · Number of features · Fuzzy models

1 Introduction

In recent decades, fuzzy models have been widely recognized as valuable tools for data-driven modeling because of their interpretability and transparency (e.g. [21,22]). Fuzzy models are based on IF-THEN fuzzy rules, in which the partitions of variables are formed by fuzzy sets. Since these fuzzy sets can be used to represent linguistic concepts, fuzzy models describe the relationships between the input variables and output variables in a way similar to the natural language [1].

However, the number of fuzzy rules of a fuzzy model grows exponentially with the increasing number of input variables, which undermines the understandability of the model. With the increasing dimensionality of the data collected in various fields, this problem becomes more and more prominent. Therefore, to avoid "the curse of dimensionality" and to lay the foundation for the interpretability of fuzzy models, dimensionality reduction is an indispensable preprocessing step in fuzzy modeling applications.

Dimensionality reduction techniques have been extensively investigated by the machine learning and data mining community. Those techniques can be

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divided into two categories: feature extraction and feature selection. In feature extraction, the input dimensionality can be reduced through linear or non-linear combination of features, e.g. principle components analysis (PCA) [2,10]. Because of the combination of features, it is difficult to interpret the model when using feature extraction techniques. Different from feature extraction, in feature selection, only the most relevant feature subset is selected. Feature selection algorithms, therefore, are widely applied in fuzzy modeling applications.

The objective of feature selection is to select the smallest feature subset from the full feature space that yields the best performance. Various feature selection methods haven been developed in the literature [4,5,9]. In general, feature selection methods are grouped into three categories: filter methods, wrapper methods and embedded methods. There is not a “best method” and each type of feature selection methods presents advantages and disadvantages. Filters are independent of the classifier, have good generalization ability and are computationally cheaper than wrappers, but may fail to select the best feature subset for the classifier. Wrappers interact with the classifier and can capture the dependency among features, but are computationally expensive and may have the risk of overfitting. Embedded methods are computationally less expensive than wrappers but are limited to certain learning machines, e.g. SVM [11]. Therefore, new feature selection methods are constantly appearing [8,18,23].

Deciding the optimal number of features is still an open question, especially for filter feature selection methods. Typically, the number of features is selected in an heuristic fashion which may sometimes lead to suboptimal performance. In fact, in many situations, the chosen feature number maybe too large and a much smaller feature subset can achieve similar or even better performances. In other cases, the chosen feature number maybe a bit small and a small increase of feature number can create big increase of performance. In this paper, we propose a structured way to determine the optimal number of features based on filter feature selection, specially based on mutual information feature ranking. We apply the proposed method to cardiac resynchronization therapy (CRT) data set and the results are promising.

The remainder of the paper is organized as follows. Section 2 introduces the background of filter feature selection methods and mutual information. In Sect. 3, the proposed method to find the optimal number of features is described. Section 4 describes the experiments and discusses the results. Finally, Sect. 5 concludes the paper.

2 Background

Filter feature selection methods are popular because of their simplicity and generality. Filters select feature subsets by ranking the features' relevance based on some criteria. The relevance of features can be viewed as a measurement of the ability of features to predict the target features. Various feature relevance ranking methods have been proposed by the researchers [7,12,13,19]. One of the simplest criteria to rank features is the correlation coefficient, e.g. Pearson correlation. However, Pearson correlation can only measure the linear relation between input features and targets.

Compared with the correlation coefficient, mutual information (MI) can measure any kind of relationship between random variables, including nonlinear relationships [3,6]. Therefore, mutual information attract more and more attention in the feature selection field. MI quantifies the "amount of information" (in units such as bits) obtained about one random variable, through an other random variable. Formally, given two random discrete variables X and Y, their mutual information is defined as follows in terms of their probabilistic density functions $p(x)$, $p(y)$ and $p(x, y)$:

$$I(X;Y) = \sum_{x \in X} \sum_{y \in Y} p(x,y) log \left(\frac{p(x,y)}{p(x)p(y)} \right). \tag{1}$$

In the case of continuous random variables, the summation is replaced by a definite double integral:

$$I(X;Y) = \int_Y \int_X p(x,y) log \left(\frac{p(x,y)}{p(x)p(y)} \right) dxdy. \tag{2}$$

Extensive work has been done on mutual information based feature selection. For example, the "minimum-redundancy-maximum-relevance" (mRMR) method proposed in [19] selects features that have the highest relevance with the target classes and are also minimally redundant. Both optimization criteria of maximum relevance and minimum redundancy are based on mutual information. Max-relevance is to search a feature subset S satisfying (3), which approximates $D(S, c)$: the mean value of all mutual information values between individual feature f_i and class c.

$$\max D(S,c),\ D = \frac{1}{|S|} \sum_{f_i \in S} I(f_i; c). \tag{3}$$

The feature subset selected according to max-relevance could have rich redundancy because of the dependency among features. Therefore, min-redundancy criterion is added to exclude dependent features:

$$\min R(S),\ R = \frac{1}{|S|^2} \sum_{f_i, f_j \in S} I(f_i; f_j). \tag{4}$$

Finally, mRMR method combines the above two criterions to optimize D and R simultaneously:

$$\max \Phi(D,R),\ \Phi = D - R. \tag{5}$$

After getting the ranked feature list based on incremental mRMR selection, the author proposed to determine the optimal number of features through more sophisticated schemes (wrappers). However, the computational cost of wrappers is expensive.

While there is substantial research on filter feature selection, there is much less work on how to determine the optimal number of features. Usually, filters rank the features based on the score of relevance and a threshold is used to remove the variables below the threshold. However, the threshold is generally chosen heuristically. Sometimes, the number of features h is decided arbitrarily by users to select the top h features. In most of these situations, optimal performance will not be achieved.

Typically (not always), for a finite data set, the quality of a model increases at first with the number of features increasing. After a certain point, the quality becomes stable or even decreasing. This phenomenon was described for discrete classification in [15]. Exploiting this phenomenon, we propose a method to find the optimal number of features efficiently using mutual information.

3 Method

The method proposed in this paper aims to provide a structured way to decide the optimal number of features for filter feature selection. It is based on the observation that the performance of a model increases as more and more relevant features are added to the model. As more irrelevant features are added, the model performance will decrease. Then the optimal number of features for the model should be at the turning-point (peak). Our method to find the peak is described below.

To select a feature subset S from the full feature space F with N features, the optimal number h of S could be determined through the following steps using cross validation.

1. Rank the features in F using the mutual information between individual feature and target class according to Eq. (1). The sequenced feature list is denoted as F_1.
2. Build models with the first k features of F_1 and test the quality (e.g. AUC, accuracy, AUK, see [14,17]) of models. k is from 1 to N.
3. Plot the figure of quality of models vs. number of features. According to the figure, the corresponding feature number of one of the peaks will be the optimal number of features h.

In the first step, note that except for ranking the features using mutual information, this method can be applied to feature rankings by any other methods, e.g. correlation based filters. We use mutual information based filters in this work because mutual information can be used to measure any kind of relationship between random variables [6]. In the second step, different classifiers can be used to build models, e.g. C4.5 decision tree, k-nearest neighbor classification, logistic regression and fuzzy models. In the third step, the peak need to be selected according to the real applications. For example, as is shown in Fig. 1, there are two obvious peaks, and the performance is similar. Therefore, the peak corresponding to the fewer number of features is preferred. However, if the performance of the other peak is much better than the first peak, then the second peak could be selected to get better performance.

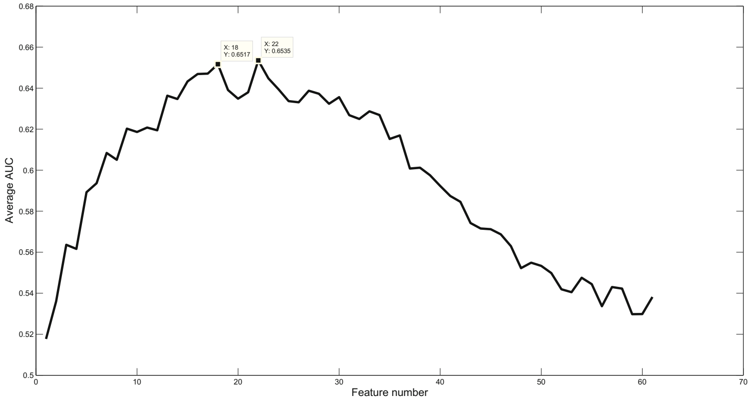

Fig. 1. Example of performance vs. feature number.

4 Experiments

We test our method on cardiac resynchronization therapy (CRT) data set from Catharina Hospital in Eindhoven, the Netherlands. The data set contains 187 patients who underwent CRT in the period between January 1, 2008 and July 1, 2015. Each patient has 137 input variables: gender, age, surgery type, 11 lab variables and 123 ultra sound variables. There are two output classes, which are responsive and non-responsive. The data set is imbalanced because 22 percent of patients are in responsive class and the remaining patients are non-responsive. Because accuracy can be misleading with imbalanced data sets, we use AUC to evaluate the performance of models. The features within this data set have different characteristics, being binary, discrete or continuous. The continuous features are discretized using a quantized feature space [20] to calculate the mutual information.

Two classifiers are used to evaluate the quality of the selected feature subsets: Multinomial Logistic Regression and first order Takagi-Sugeno fuzzy model (also known as TSK fuzzy model). We choose multinomial logistic regression in our experiments for its speed and simplicity. We choose TSK fuzzy model because it is one of the most commonly used fuzzy systems. We build multinomial logistic regression models with 1 to 137 features and fuzzy models with 1 to 30 features. We did not build fuzzy models with more than 30 features for keeping the computational cost small. Experiments are repeated 5 times using ten-fold cross validation. We calculate the average AUC to evaluate each model.

In the experiments, the features are ranked in three different ways:

1. Mutual information ranking according to the mutual information between individual variables and class labels.
2. Reverse ranking of mutual information ranking.
3. Random ranking.

The reverse ranking of mutual information ranking and random ranking are used to test whether the mutual information ranking provide added value, i.e.

the reverse mutual information ranking and the random ranking are not expected to outperform the mutual information ranking. The experimental framework is shown in Fig. 2.

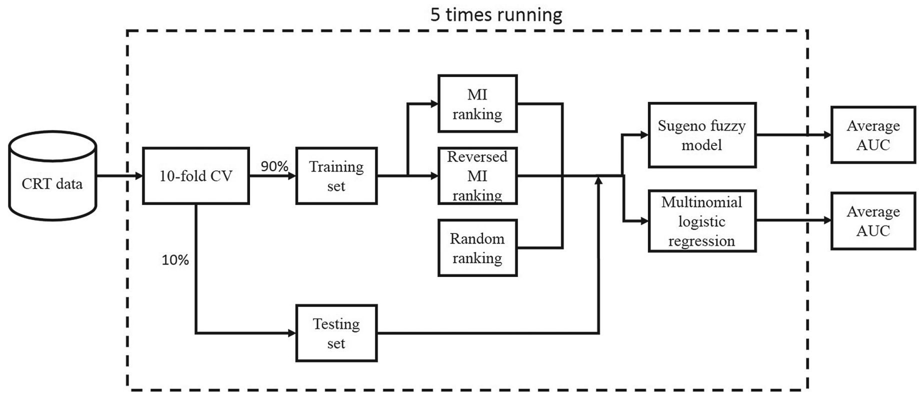

Fig. 2. Experimental framework.

4.1 Results of Multinomial Logistic Regression

Figure 3 shows the average AUC of the multinomial logistic regression model over the whole size of the input variables, from 1 feature up to 137 features. In this figure, one can clearly see that the highest average AUC (0.6535) is achieved by mutual information ranking with 22 features in the second peak. The model with 18 features achieve the first peak with average AUC 0.6517, which is similar to the second peak. Since the model with fewer features are more understandable,

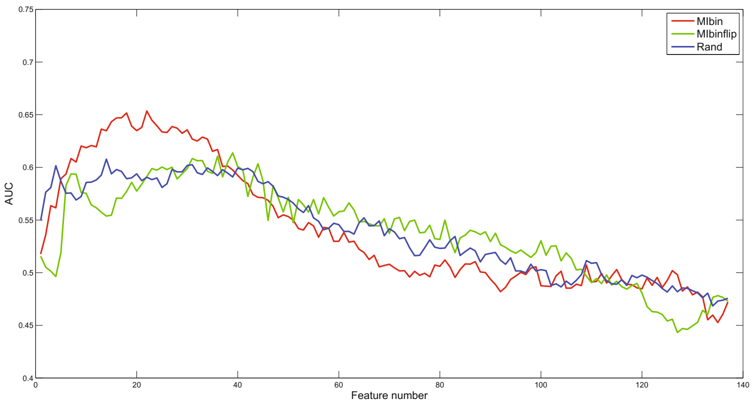

Fig. 3. Average AUC vs. number of features using multinomial logistic regression.

the optimal number of features for multinomial logistic regression model on CRT data set will be 18.

In comparison, the reverse mutual information ranking and random ranking reach significantly lower AUC, i.e. the models based on random ranking keep relatively stable performance (around 0.60) from 4 to 40 features and the models based on reverse mutual information ranking are reaching stable AUC (around 0.61) from 23 to 44 features.

4.2 Results of Fuzzy Inference Systems

The second experiment builds TSK fuzzy model using fuzzy c-means (FCM) clustering with 1 to 30 features [16]. Fuzzy models with clusters from two to 20 are built. As Fig. 4 shows, in general, fuzzy models with three clusters have better performance than other models and the best performance is achieved by fuzzy models with three clusters. Besides, with the number of clusters increasing, the performance fluctuates from 1 to 30 features. For visual convenience, Fig. 4 only displays the average performance of models with 7 to 20 clusters to prevent clutter.

In Fig. 5, the performance of fuzzy models with 3 clusters based on three feature rankings is shown. Similar to the results of multinomial logistic regression model, the fittest fuzzy model is obtained with mutual information ranking with 14 features. Besides, there is another peak point with 9 features. Compared with multinomial logistic regression, fuzzy models reach the peak with fewer features.

In addition, from the figure we can see that the performance of reverse mutual information ranking is also contrary to the performance of mutual information ranking. Therefore, mutual information provide import information regarding the importance of ranking criteria.

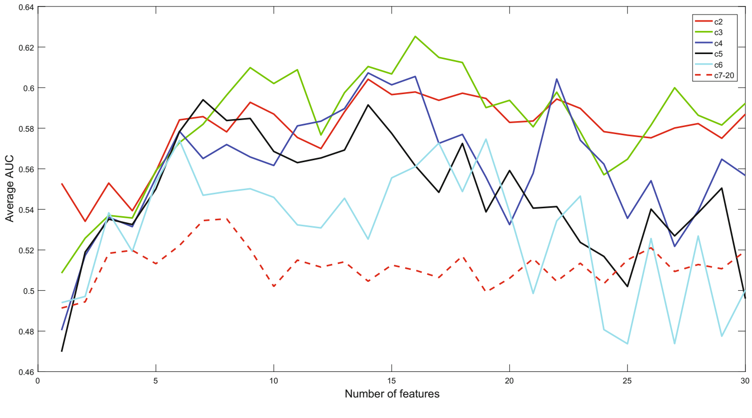

Fig. 4. Average AUC of fuzzy models with two to 20 clusters.

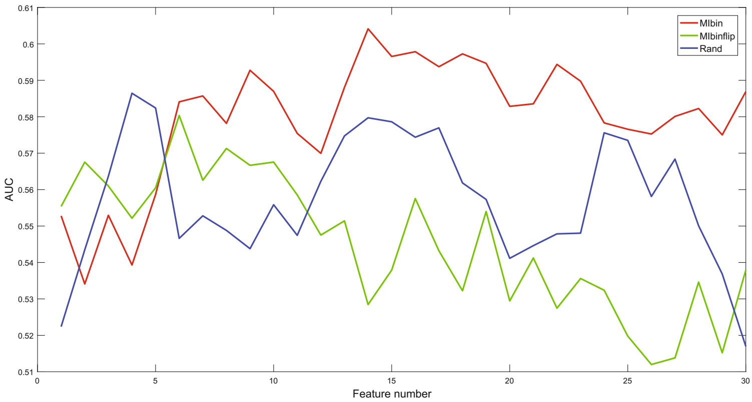

Fig. 5. Average AUC vs. number of features using fuzzy models with 3 clusters.

5 Conclusion

This paper presents a method to find the optimal number of features based on mutual information. This method is designed to resolve the issue of deciding the optimal number of features for the final model, particularly for filter feature selection.

The method has been evaluated on a clinical data set of CRT patients. The model performance using the ranking of feature based on mutual information has been compared with reverse ranking of mutual information ranking and random ranking. Two classifiers, multinomial logistic regression and fuzzy models, are used to evaluate the performance of the selected features. Overall, the performance of models using mutual information ranking is significantly better than the reverse mutual ranking and random ranking. Furthermore, from the figure of performance vs. number of features, one can easily find the optimal number of features. Comparing the optimal number of features for multinomial logistic regression and fuzzy model, we can conclude that, for different modeling methods, the optimal number of features could be different and fuzzy model reaches the peak with less features than multinomial logistic regression.

In the future, we will test our proposed method on more data sets and with more classifiers. We will also compare this method with other alternative approaches to decide the optimal number of features.

In this paper, the simplest feature ranking criterion for mutual information is used. Further improvements of the proposed method could be made by implementing better methods to rank features, e.g. mRMR. In addition, further work will be done to decide which features should be included.

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Selection Among Solar Power Plants Using Fuzzy Economics

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Abstract. Alternative energy sources are gaining popularity against the world's fossil energy sources. There is an increasing energy demand due to the growing population. Renewable energy sources are used as alter-natives for fulfilling this demand. Since these sources are non-exhaustible and can renew themselves, they are considered as primary energy sources for the future. Although solar energy has the highest capacity among renewable energy sources, currently it has the disadvantage of high equipment and installation costs. Therefore, the economic analysis of investments in solar energy systems should be accurate and realistic but the uncertainty and ambiguity inherit in the parameters make this analysis complex and inaccurate. In this work, the solar economic model containing economic and technical uncertainties has been evaluated by using fuzzy logic. Realistic solutions from the developed solar fuzzy economic model direct investors to more accurate solar power plant investments.

1 Introduction

The energy that is used at every stage of our life has an important role in sustaining life. With increasing population and industrialization, the demand for energy is increasing rapidly [1]. Fossil fuels are the most important sources of energy as a result of existing technologies and their price advantages. Despite these advantages, carbon dioxide and other greenhouse gases are emitted when fossil fuels burn down, that is, when they enter the energy conversion process. These gases causing the ozone layer to be thinned cause harmful radiation that endangers the food chain and ecological order [2]. The ecological problems that have emerged at the end of this process have led the world to turn to alternative energy sources.

Non-fossil energy sources such as hydraulics, wind, solar, geothermal, and biomass are defined as renewable energy sources [3]. Renewable energy emerges as the most important alternative energy source for fossil fuels with their environmental and social benefits [4]. The sun which is the most important renewable energy source is the source of all renewable and conventional sources (except nuclear and tidal energies) [5]. In addition to its indirect use, solar energy is directly utilized with newly developed technologies. These special advantages distinguish the sun from the renewable energy sources and make the sun a promising resource.

Determining the most suitable locations and conditions to promote the use of solar energy systems, which require a high initial cost, provides price advantage against

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competitors. The economic analyzes and evaluations to be made are of critical importance at this stage. Taking into account the uncertainties in the expenditure and earnings contributes to the development of realistic solutions. For this purpose, solar economics calculation models should be developed based on fuzzy logic. The following sections refer to solar energy, fuzzy logic, and fuzzy economic parameters. The developed models are tested with a sample application in Sect. 4. General evaluation and future studies are concluded in the conclusion section.

2 Solar Energy and Economic Bases

The Sun is a star with a blackbody character with a surface temperature of 5777 K degrees. The mass that disappears in the fusion reaction is released as energy and is transferred to the solar surface and this energy spreads to the space [2, 6]. This energy spreading in space reaches as far as our Earth and is transformed into direct and indirect usable forms on earth. Solar energy can be used directly with simple (cooking, hot water) or modern technology (PV, CSP). These special advantages distinguish the sun between renewable energy sources and strengthen the belief that future use of solar energy systems will increase. It is seen that solar energy does not have enough interest, which meets only 1% (2016) of global electricity generation [4]. Nevertheless, the presence of the potential of solar energy is obvious and it is necessary to increase the efforts to help the decision makers to evaluate this potential [7].

The major disadvantage for the solar energy industry is the high initial costs required for installation. The costs of solar energy system components and the installation of these components are important factors for the solar economy. These factors include hardware equipment such as collectors, pumps, storage unit and connection equipment, and software infrastructure and their installations. It is of great importance to ensure the correct economic equilibrium between the high installation costs and the gain from the solar energy system. Therefore, it is crucial for investors to make accurate economic analyzes of solar energy systems operated with high installation costs and low operating costs. The economic analysis is based on the comparison between the present total cost of the solar energy system and the present value of fuel economy to be realized in the future [6]. The economic analyzes provide to determine the lowest cost method to meet the energy need by comparing solar energy and non-solar energy alternatives and determine the optimal solar energy system size that provides cost and gain balance. The life cycle saving method is commonly used in the analysis by taking into account the lifetime defined for the energy system.

3 Fuzzy Economic Analysis of Solar Energy Plants

Economic analyzes are influenced by many vague factors such as national and international social, political, economic and so on. Economic assessments based on long-term temporal calculations cannot be instantly measured and analyzed within these uncertainties. There are similar uncertainties in the economic analysis of solar energy systems as in all other economic analysis methods. The energy sector, which has a complex market on a global scale, has more complex economic and political uncertainties [8]. Fuzzy logic based calculations should be done to incorporate these uncertainties into economic evaluations and to produce realistic solutions.

3.1 Fuzzy Logic

In 1965, L.A. Zadeh introduced fuzzy clusters [9] to the science literature in order to illuminate the classes of unexplained events and objects such as 0–1. Zadeh's proposal explains that objects and events have properties defined in a class between 0 and 1. The fact that real world situations and events are ambiguous, hesitant, vague and unknown in a complex and dynamic structure supports this proposition. Fuzzy logic generates more accurate solutions for real life by taking intermediate alternatives between events and situations. The fuzzy sets (X) and subsets (e.g. $\tilde{A}$) defined in the range [0,1] are expressed by the membership function $\left(\tilde{A}(x)\right)$. For example, $\overline{X}(0.5)$ represents that x has membership value 0.5 in $\tilde{X}$. Basic fuzzy set operations can be expressed as follows [10];

$$\text{Union of fuzzy sets}: \mu_{\tilde{\tilde{X}} \cup \tilde{\tilde{Y}}} = max[\mu_{\tilde{X}}(x) \cdot \mu_{\tilde{Y}}(x)] \quad (1)$$

$$\text{Intersection of fuzzy sets}: \mu_{\tilde{\tilde{X}} \bigcap \tilde{\tilde{Y}}} = min[\mu_{\tilde{X}}(x) \cdot \mu_{\tilde{Y}}(x)] \quad (2)$$

$$\text{Complement of a fuzzy set}: \mu_{\tilde{Z}^c} = 1 - \mu_{\tilde{Z}} \quad (3)$$

$$\text{Addition}: \tilde{X}(+)\tilde{Y} = [x_1, x_3](+)[\mathrm{y}_1, \mathrm{y}_3] = [\mathrm{x}_1 + \mathrm{y}_1, \mathrm{x}_3 + \mathrm{y}_3] \quad (4)$$

$$\text{Subtraction}: \tilde{X}(-)\tilde{Y} = [x_1, x_3](-)[\mathrm{y}_1, \mathrm{y}_3] = [\mathrm{x}_1 - \mathrm{y}_3, \mathrm{x}_3 - \mathrm{y}_1] \quad (5)$$

where $\tilde{X}$ and $\tilde{Y}$ are assumed as numbers in the interval as $\tilde{X} = [x_1, x_3]$, $\tilde{Y} = [y_1, y_3]$.

In defining fuzzy membership functions, fuzzy sets are divided into discrete and continuous fuzzy sets. Fuzzy numbers have an important place in fuzzy operations, and triangular (Fig. 1), trapezoidal, and Gaussian numbers are commonly used in real world problems.

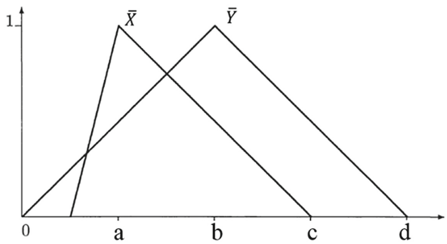

Fig. 1. Continuous triangular fuzzy sets.

Triangular fuzzy numbers are expressed by three points of the function they describe such as $\tilde{Y} = (0, b, d)$ (Fig. 1). Fuzzy membership function of the triangular fuzzy set $\tilde{Y}$ can be defined as;

$$\mu_{\tilde{Y}}(y) = \begin{cases} 0. & y < 0 \\ \frac{y}{b}. & 0 \leq y \leq b \\ \frac{b-y}{d-b}. & b \leq y \leq d \\ 0. & y > d \end{cases} \quad (6)$$

Where $\tilde{X} = (x_1, x_2,\ x_3)$ and $\tilde{Y} = (y_1, y_2, y_3)$, fuzzy arithmetic operations for triangular fuzzy numbers can be defined as:

$$\tilde{X}(+)\tilde{Y} = (x_1.x_2.\ x_3)(+)(y_1.y_2.y_3) = (x_1 + y_1.x_2 + y_2.x_3 + y_3) \quad (7)$$

$$\tilde{X}(-)\tilde{Y} = (x_1.x_2.\ x_3)(-)(y_1.y_2.y_3) = (x_1 - y_3.x_2 - y_2.x_3 - y_1) \quad (8)$$

$$-\left(\tilde{X}\right) = (-x_3.\ -x_2.\ -x_1) \quad (9)$$

$$\tilde{X}(\cdot)\tilde{Y} \cong (x_1 * y_1.x_2 * y_2.\ x_3 * y_3) \quad (10)$$

$$\tilde{X}(/)\tilde{Y} \cong (\frac{x_1}{y_3}.\frac{x_2}{y_2}.\frac{x_3}{y_1}) \quad (11)$$

Although triangular fuzzy numbers are obtained at the end of the addition and subtraction of two triangular fuzzy numbers, triangular fuzzy numbers cannot be obtained at the end of the multiplication and division of two triangular fuzzy numbers. However, multiplication and division operations are performed on three points to obtain approximate triangles.

In this study, the uncertainty parameters included in the installation and operation costs of the solar energy system are defined by the triangular fuzzy membership functions and the economic analysis calculations are made on a fuzzy logic basis. The inflation rate, the market discount rate, the discount rate, the effective income tax rate, the solar fraction, the fuel cost increase rate and the variable costs are the main uncertainties, and these headings are included in the economic analysis calculations by defining them with fuzzy numbers. Fuzzy numbers obtained at the end of fuzzy calculations are converted to crisp values by defuzzification method for use in comparative evaluations. In this study, it is appropriate to use the weighted average method in Eq. (12) [11] for defuzzification in accordance with our method of defining the uncertainties with fuzzy numbers.

$$z^* = \frac{\sum \mu_A(\overline{Z})\overline{Z}}{\sum \mu_A(\overline{Z})} \quad (12)$$

where z is the centroid of each membership function.

3.2 Fuzzy Economic for Solar Systems

This part deals with methods of fuzzy economic analysis based on life-cycle saving method. In this method, the costs and benefits over the time value of the money are examined in detail [2, 6]. The process initiated by defining the installation and operating costs is followed by the incorporation of the periodical gains to be obtained into the life cycle analysis [12]. The installation cost of the solar energy system is calculated by summing the two costs, which are dependent and independent of the collector field in Eq. (13) [6].

$$C_S = C_A A_C + C_E \tag{13}$$

where C_S is the total installation cost of solar energy equipment, C_A is the sum of the costs dependent to collector area, A_C is the collector area and C_E is the total cost of equipment independent for collector area.

Operating costs associated with the operation of the solar system occur periodically. These costs consist of fuel cost, mortgage payment, maintenance and insurance costs, parasite energy cost and property tax. Revenue generating equipment may be subject to depreciation and income tax may also decrease. In the process of using the system, fuel is saved and fuel cost is reduced. The lifetime of the equipment can eventually be salvage or resale, which creates a return on capital. There may be income tax effects in the purchase of solar equipment. Revenue generating assets and equipment may be subject to depreciation, in which case the taxable income is reduced. The use of solar energy system reduces fuel costs by saving fuel. The annual costs of energy systems can be expressed as [6, 8, 13, 14]:

$$\begin{aligned} Annual\ cost\ &=\ Fuel\ cost\ +\ mortgage\ payment +\ maintenance\ and\ insurance\ + \\ &cost\ of\ interference\ energy\ +\ property\ taxes\ -\ income\ tax\ savings \end{aligned} \tag{14}$$

Income tax savings for a non-income system can be expressed as:

$$Income\ tax\ saving =\ effective\ tax\ rate * \ (interest\ payment\ +\ property\ tax) \tag{15}$$

Income tax savings for income generating systems are as follows;

$$\begin{aligned} &Income\ tax\ saving =\ effective\ tax\ rate\ x\ (interest\ payment\ + property\ tax\ + \\ &fuel\ cost\ +\ maintenance\ and\ insurance\ +\ parasitic\ energy\ costs\ -\ depreciation) \end{aligned} \tag{16}$$

The concept of solar savings describes the system requirement and is expressed in terms of the difference between traditional system cost and solar system cost. Negative savings indicate value losses.

$$Solar\ energy\ savings\ =\ traditional\ energy\ costs\ -\ solar\ energy\ costs \tag{17}$$

In this study, the life-cycle cost (LCC) economic criteria are used to evaluate and optimize solar energy systems. Life-cycle analysis, which adds the time value of money to the account, is the sum of the current worth of all the costs associated with the system over the lifetime defined for the energy system or the selected analysis period. The life cycle cost method compares future costs with current costs. All expected costs are reduced to the present value and the best alternative investment rate is determined to cover all expected expenses. The best approach for a solar energy economy is to use a life cycle costing method that takes into account all future costs. Thus, it is determined that the best alternative investment rate must be invested in order to be able to use funds in the future to cover all expected expenses.

The reason for the reduction of the cash flow is the time value of money. In order to determine the present worth of a money amount, the discount rate (d) in the future market and N period (usually years) must be known. The discount rate implies uncertainty due to future vagueness and must be defined as a fuzzy number ($\tilde{d}$) [6, 12, 13]. Periodic costs are assumed to be inflated or deflated in a given period of time. Inflation rate including future ambiguity as in the discount rate also includes uncertainty and is defined as fuzzy number ($\tilde{i}$). The current value of the N. period is generated at the end of the first time period. These present fuzzy worth calculations can be shown as [6, 12, 13]:

$$\widetilde{PW} = \frac{F}{\left(1+\tilde{d}\right)^N} \tag{18}$$

The present worth factor (PWF) is used to determine the periodic and fixed payment amount of the loan used for system installation. The present value factor of a series of N periodic payments of a repetitive payment that is subjected to head inflation every period is defined as follows [6, 12, 13].

$$\widetilde{PWF}\left(N,\tilde{i},\tilde{d}\right) = \sum_{j=1}^{N} \frac{\left(1+\tilde{i}\right)^{j-1}}{\left(1+\tilde{d}\right)^{j}} = \begin{cases} \frac{1}{\tilde{d}-\tilde{i}}\left[1-\left(\frac{1+\tilde{i}}{1+\tilde{d}}\right)^N\right], & if\ \tilde{i}\neq\tilde{d} \\ \frac{N}{\tilde{i}+1}, & if\ \tilde{i}=\tilde{d} \end{cases} \tag{19}$$

where equity between $\tilde{i}$ and $\tilde{d}$ is concluded according to user preferences. Because all mortgage payments are equal, a series of payments with an inflation rate of zero occurs. In this case, the discount rate in the $\widetilde{PWF}$ equation becomes the mortgage interest rate, and the periodic loan payment (PLP) is defined as [12, 13];

$$\widetilde{PLP} = \frac{M}{\widetilde{PWF}\ (N_L,\ 0,\ \tilde{m})} \tag{20}$$

where M is the mortgage principal, N_L is the mortage period, and m is the mortage interest rate. The savings achieved in the jth year of fossil fuels not used in the solar energy system are calculated as follows [12, 13]:

$$\widetilde{FS} = \tilde{F} * L * C_{F1}\left(1+\tilde{i}_F\right)^{j-1} \tag{21}$$

The fuel cost increase ($\tilde{i_F}$) is defined by the fuzzy number because it is uncertain depending on national and international economic indicators. The solar fraction ($\tilde{F}$) that determines the amount of fuel savings is defined in the fuzzy logic because it is variable and uncertain depending on equipment and atmospheric conditions. The C_{F1} parameter in the equation represents the unit energy cost of the fuel saved in the first period.

Life cycle costs of insurance, maintenance, parasite energy, property taxes and mortgage payments and life cycle fuel saving can be determined using appropriate present worth factors ($\widetilde{PWF}$). The present fuzzy values of fuel saving ($\widetilde{PW}_{FS}$), mortgage payments series ($\widetilde{PW}_M$), other expenses ($\widetilde{PW}_O$), property tax ($\widetilde{PW}_{PT}$), income tax saving on payment of mortgage interest ($\widetilde{PW}_{ITS}$), mortgage interest payments ($\widetilde{PW}_{int}$) and income tax savings ($\widetilde{PW}_{ITS}$) are calculated as following methods [6, 12, 13]:

$$\widetilde{PW}_{FS} = \tilde{F} * L * C_{F1} * \widetilde{PWF}\left(N_e,\ \tilde{i_F}, \tilde{d}\right) \tag{22}$$

$$\widetilde{PW}_M = -M * \widetilde{PWF}(N_L, 0, \tilde{d}) \tag{23}$$

$$\widetilde{PW}_O = -MS_1 * \widetilde{PWF}(N_e, \tilde{i}, \tilde{d}) \tag{24}$$

$$\widetilde{PW}_{PT} = -PT_1 * PWF(N_e,\ \tilde{i},\ \tilde{d}) \tag{25}$$

$$\widetilde{PW}_{ITS} = \tilde{t} * \widetilde{PW}_{int} \tag{26}$$

$$\widetilde{PW}_{int} = M[\frac{\widetilde{PWF}\left(N_{min}, 0, \tilde{d}\right)}{PWF(N_L,\ 0, \tilde{m})} + \widetilde{PWF}(N_{min}, \tilde{m}, \tilde{d})(\tilde{m} - \frac{1}{\widetilde{PWF}(N_L, 0, \tilde{m})})] \tag{27}$$

The present value of income tax savings [6, 12, 13];

$$\widetilde{PW}_{ITS} = \tilde{t} * PT_1 * \widetilde{PWF}(N_e,\ \tilde{i}, \tilde{d}) \tag{28}$$

where $\tilde{t}$ is the effective income tax, MS_1 is the various expenses to be paid at the end of the first period (maintenance, insurance, parasitic power) and PT_1 is the property tax to be paid at the end of the first period.

4 Application

The high initial equipment and installation costs are the most critical disadvantage of solar energy systems. Therefore, solar energy investment should be done carefully at the right time and in the right place. The energy sector, which is affected by many national and international factors, includes uncertainties due to political and economic dynamism and complexity. These uncertainties that are taken into account in the economic analysis phase of the solar energy system enable more realistic solutions. In this section, fuzzy-based solutions are introduced to the uncertainties in the solar economic analysis through the example application.

4.1 Case Study

Managers of Turkey's largest white goods factory wants decrease energy costs in a long-term by using renewable energy. The company also has a sustainable environment policy. At the end of the feasibility studies made for this purpose, factory roofs were found to be the most suitable place with their wide use area. The factory splits have been found to be very suitable for photovoltaic solar energy panels and it has been decided that the solar energy system is the most suitable choice. The economic data obtained at the end of the market researches are explained below. In this study, we assess the suitability of the company's decision with the solar fuzzy economic analysis.

The energy system is supported by the solar energy system and it is planned to meet the energy demand with a hybrid system. The cost of the solar system, which is financed at €97,000 and 80% with an interest rate of 7% over 20 years, is expected to reduce fuel consumption by 63%. The cost of energy consumption in the first year of a fuel-only (non-solar) energy system is €12,000. The equipment is expected to have a resale value of 30% of its original cost after 20 years. The market discount rate is %8 per year and the fuel cost inflation rate is 9% per year. In the first year, insurance, maintenance and interference energy costs are estimated at $500 and the property tax at €720. These expenses are expected to increase by a general inflation rate of 5% per year. Property taxes and mortgage interest can be deducted from tax purposes. During the analysis period, the effective income tax rate is expected to be 40%.

As a results, fuzzy present values are obtained by fuzzy operations for each fuzzy calculation method. These fuzzy values are converted into crisp values by using the weighted average method as shown below (Table 1);

Table 1. Defuzzified results of solar fuzzy economics.

	Present value (€)
Fuel saving	153026,71
Mortgage payments	−71916,86
Other expenses	−7179,00
Property tax	−10337,75
Income saving on mortgage interest	15912,79
Income tax savings	4135,10
Resale value	6243,35
Advance payment	−19400,00
Total present value of solar savings	70484,35

Calculations made by using economic analysis methods for solar energy systems based on fuzzy logic refer to profit at the present value. The results show that the high initial investment made can be economically offset by solar savings in the long run. Incorporating economic and technical uncertainties into economic analysis with fuzzy logic enables more realistic results to be obtained.

5 Conclusion

The most important problem encountered in the selection of solar energy systems is high equipment and installation costs. High initial investment in solar energy systems must be planned accurately and realistically. Therefore, uncertainties involving economic analyzes must be included in the calculations. In this study, economic and technical uncertainty parameters in solar economic calculation methods are defined on the basis of fuzzy logic. In this way, the problems of deciding to install the solar energy plant are decided by more realistic solutions. The uncertainties in solar economic calculations have been attributed to economic and technical reasons such as interest rate, discount rate, fuel cost inflation rate, other expense increases by inflation, and solar fraction. These parameters are fuzzified by using triangular fuzzy numbers which are most suitable for economic analysis methods. At the end of calculations made with triangular fuzzy operators, fuzzy present worth value and present worth factors are obtained. The defuzzified method is used to convert these fuzzy values to crisp form and more meaningful values are obtained. The validity of the solar fuzzy economic calculation methods is verified by the case study in the last section.

The experience gained from this work carried out for a hybrid model can only develop fuzzy economic models for energy systems consisting solely of solar energy as a future study. Fuzzy economic models to be obtained from these studies can be combined with fuzzy technical calculation methods to develop more general models for the installation of solar energy systems.

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Co-words Analysis of the Last Ten Years of the Fuzzy Decision Making Research Area

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Abstract. The main aim of this contribution is to develop a co-words analysis of the Fuzzy Decision Making research area in the last ten years (2007–2016). The software tool SciMAT is employed using an approach that allows us to uncover the main research themes and analyze them according to their performance measures (qualitative and quantitative). Using an advance query, an amount of 1,465 documents were retrieved from the ISI Web of Science. The corpus was divided into two consecutive periods (2007–2011 and 2012–2016). Our key findings are that the most important research themes in the first and second period was *Consensus* and *Aggregation-Operators*, respectively.

Keywords: Fuzzy group decision making · Bibliometric analysis · Science mapping analysis · Co-Words analysis

1 Introduction

The Fuzzy Decision Making [3,14] research field born from the synergy of the Decision Making and Fuzzy Sets research fields. Decision Making is a common task carried out by humans each day. Its goal is to find a best decision from among some possible options [14]. A lot of real world decision making processes take place in an environment in which the aims, the constraints and the consequences of possible actions are not precisely known. Thus, Fuzzy Sets theory [22,23] is a common tool used to deal with imprecision and vagueness problem, and also to represent the concept in a natural way through linguistic terms. In this sense, to deal with imprecision in the Decision Making research field, Fuzzy Set theory is employed.

Fuzzy Decision making is a growing research area, publishing a high amount of research documents each years. Although an expert on the field can discover

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and analyze the different subtopics of the research field, it is obvious that the high volume of research documents that are available makes this a difficult and daunting task to carry out. Therefore, scientific support tools to uncover the conceptual structure of a research area of interest are worth and necessary. In that sense, Science Mapping Analysis is a powerful bibliometric technique to study the conceptual structure of a particular research field [9,18].

So, the main aim of this contribution is to carry out a conceptual science mapping analysis [4,6,9] of the research conducted by the Fuzzy Decision Making research area from 2007 to 2016 (the last ten years). The analysis is developed using SciMAT [10] software tool and partially based in the approach presented in [8].

This article is organized as follows: Sect. 2 introduces the methodology employed in the analysis. In Sect. 3, the dataset is described. In Sect. 4, the science mapping analysis of the Fuzzy Decision Making research area is presented. Finally, some conclusions are drawn in Sect. 5.

2 Methodology

Science mapping or bibliometric mapping is a spatial representation of how disciplines, fields, specialities, and documents or authors are related to one another [21]. It has been widely used to show and uncover the hidden key elements (documents, authors, institutions, topics, etc.) in different research fields [7,11,16,19,20].

Science mapping analysis can be carried out with different software tools [9]. Particularly, SciMAT was presented in [10] as a powerful tool that integrates the majority of the advantages of available science mapping software tools [9]. SciMAT was designed according to the science mapping analysis approach presented in [8], combining both performance analysis tools and science mapping tools to analyze a research field and detect and visualize its conceptual subdomains (particular topics/themes or general thematic areas) and its thematic evolution.

Therefore, in this contribution, SciMAT was employed to develop a longitudinal conceptual science mapping analysis [4,9] based on co-words bibliographic networks [2,6]. Thus, the analysis was carried out in three stages:

1. *Detection of the research themes.* In each period of time studied the corresponding research themes are detected by applying a co-word analysis [6] to raw data for all the published documents in the research field, followed by a clustering of keywords to topics/themes [12], which locates keyword networks that are strongly linked to each other and that correspond to centres of interest or to research problems that are the subject of significant interest among researchers. The similarity between the keywords is assessed using the equivalence index [5].
2. *Visualizing research themes and thematic network.* In this phase, the detected themes are visualized by means of two different visualization instruments:

strategic diagram [13] and thematic network [8]. Each theme can be characterized by two measures [5]: *centrality* and *density*. Centrality measures the degree of interaction of a network with other networks. On the other hand, density measures the internal strength of the network. Given both measures, a research field can be visualized as a set of research themes, mapped in a two-dimensional strategic diagram (Fig. 1) and classified into four groups:

(a) Themes in the upper-right quadrant are both well developed and important for the structure of the research field. They are known as the *motor-themes* of the specialty, given that they present strong centrality and high density.
(b) Themes in the upper-left quadrant have well-developed internal ties but unimportant external ties and so, they are of only marginal importance for the field. These themes are very *specialized and peripheral*.
(c) Themes in the lower-left quadrant are both weakly developed and marginal. The themes in this quadrant have low density and low centrality and mainly represent either *emerging or disappearing* themes.
(d) Themes in the lower-right quadrant are important for a research field but are not developed. This quadrant contains *transversal and general*, basic themes.

3. *Performance analysis*. In this phase, the relative contribution of the research themes to the whole research field is measured (quantitatively and qualitatively) and used to establish the most prominent, most productive and highest-impact subfields. Some of the bibliometric indicators to use are: number of published documents, number of citations, and different types of h-index [1,15,17].

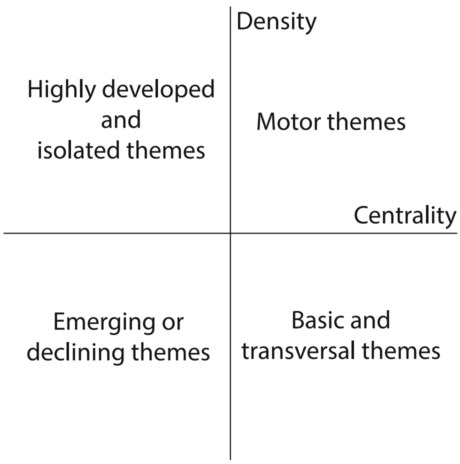


Fig. 1. The strategic diagram.

3 Dataset

In order to carry out the performance and science mapping analysis, the research documents published in the Fuzzy Decision Making research area during the last ten years must be collected and also, preprocessed.

Since ISI Web of Science (ISIWoS) is the most important bibliometric database, the research documents published by Fuzzy Decision Making research area were downloaded from it using the following advance query: *SO=("FUZZY SETS AND SYSTEMS" OR "IEEE TRANSACTIONS ON FUZZY SYSTEMS" OR "INTERNATIONAL JOURNAL OF UNCERTAINTY FUZZINESS AND KNOWLEDGE BASED SYSTEMS" OR "JOURNAL OF INTELLIGENT FUZZY SYSTEMS" OR "INTERNATIONAL JOURNAL OF FUZZY SYSTEMS" OR "IRANIAN JOURNAL OF FUZZY SYSTEMS" OR "FUZZY OPTIMIZATION AND DECISION MAKING" OR "FUZZY LOGIC AND APPLICATIONS" OR "ROUGH SETS FUZZY SETS DATA MINING AND GRANULAR COMPUTING" OR "INFORMATION FUSION" OR "INFORMATION SCIENCE" OR "INTERNATIONAL JOURNAL OF INFORMATION TECHNOLOGY & DECISION MAKING" OR "IEEE TRANSACTIONS ON SYSTEMS MAN AND CYBERNETICS PART A-SYSTEMS AND HUMANS" OR "IEEE TRANSACTIONS ON SYSTEMS MAN AND CYBERNETICS PART B-CYBERNETICS" OR "INTERNATIONAL JOURNAL OF GENERAL SYSTEMS" OR "APPLIED SOFT COMPUTING" OR "SOFT COMPUTING" OR "KNOWLEDGE-BASED SYSTEMS" OR "CONTROL AND CYBERNETICS" OR "COMPUTERS & MATHEMATICS WITH APPLICATIONS" OR "EUROPEAN JOURNAL OF OPERATIONAL RESEARCH" OR "EXPERT SYSTEMS WITH APPLICATIONS" OR "INTERNATIONAL JOURNAL OF APPROXIMATE REASONING" OR "INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS") AND TS=("fuzzy decision making" OR "fuzzy group decision making" OR "fuzzy preference*" OR "aggregation operator*" OR "fuzzy AHP*" OR "fuzzy analytic hierarchy process" OR "fuzzy majority" OR "fuzzy quantifier*") AND PY=2007-2016 NOT TS="FUZZY QUERYING".*

This query retrieved a total of 1,465 documents from 2007 to 2016 (Fig. 2). The corpus was further restricted to articles and reviews. Citations of these documents are also used in this study; they were counted up to 17th April 2017.

The raw data was downloaded from ISIWoS as plain text and entered into SciMAT to build the knowledge base for the science mapping analysis. Thus, it contains the bibliographic information stored by ISIWoS for each research document. To improve the data quality, a de-duplicating process was applied (the author's keywords and the ISI keywords plus were used as unit of analysis). Words representing the same concepts were grouped. Furthermore, some meaningless keywords in this context, such as stop-words or words with a very broad and general meaning, e.g. "SYSTEMS", were removed.

Next, using the SciMAT period manager, the corpus was divided in different slices in order to analyze the evolution. To avoid data smoothness, the best option would have been to choose one-year periods. However, it was found that

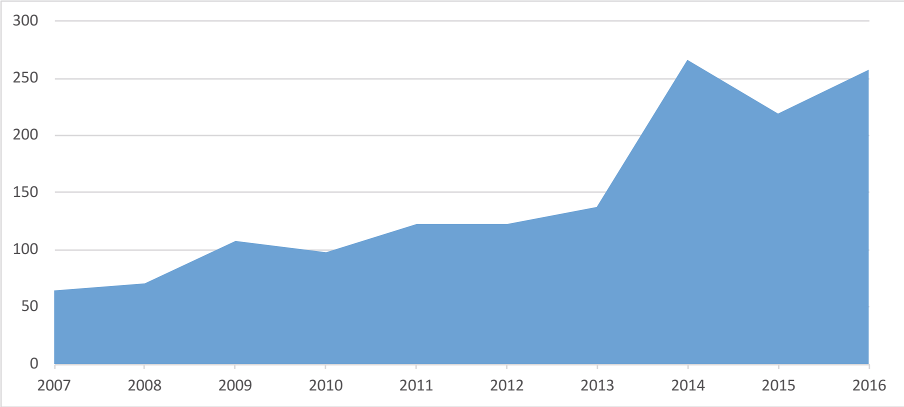


Fig. 2. Distribution of documents retrieved by years.

not enough data were generated in the span of a single year to obtain good results from science mapping analysis. For this reason, two consecutive periods of five years were established (Fig. 3): 2007–2011 and 2012–2016, with 1,606 and 2,898 keywords, respectively.

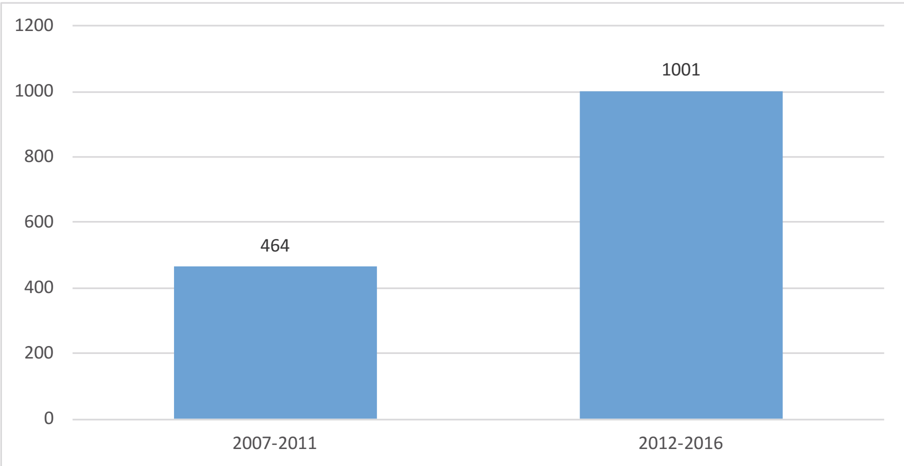


Fig. 3. Distribution of documents retrieved by years.

4 Conceptual Analysis

In order to analyze the most highlighted themes of the Fuzzy Decision Making research area, a strategic diagram is shown for each period. In addition, the spheres size is proportional to the number of published documents associated with each research theme.

First Period (2007–2011). According to the strategic diagram shown in Fig. 4, during this period the research area of Fuzzy Decision Making pivoted on nineteen themes, with the following ten major themes (motor themes plus basic themes): *Consensus*, *Computing-with-Words*, *Fuzzy-Analytic-Hierarchy-Process*, *Intuitionistic-Fuzzy-Sets*, *Transitivity*, *Prioritize-Design-Requirements*, *T-Norm*, *Priority-Vector*, *Uncertainty* and *Fuzzy-Sets*.

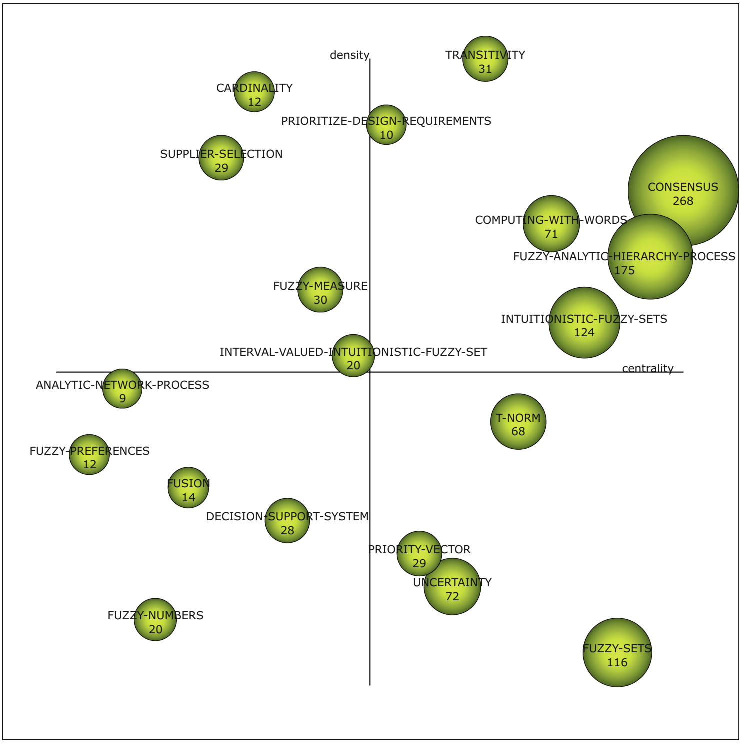


Fig. 4. Strategic diagram for the 2007–2011 period.

The performance measures of the themes are given in Table 1, showing the number of documents, numbers of citations and h–index per theme. According to this performance measures, the following six themes stand out: *Consensus*, *Fuzzy-Analytic-Hierarchy-Process*, *Intuitionistic-Fuzzy-Sets*, *Fuzzy-Sets*, *Uncertainty* and *Computing-with-Words*. These research themes get important impact rates, achieving more than one-thousand citations and getting higher h-index in comparison with the other themes.

The motor theme *Consensus* (Fig. 5a) gets the highest citation count and h-index in that period. It plays a central role in this period and it associated

Table 1. Performance of the themes in the 2007–2011 period

Name	Number of documents	Number of citations	h-index
CONSENSUS	268	13,346	67
FUZZY-ANALYTIC-HIERARCHY-PROCESS	175	7,465	49
INTUITIONISTIC-FUZZY-SETS	124	7,757	50
FUZZY-SETS	116	4,727	40
UNCERTAINTY	72	3,512	32
COMPUTING-WITH-WORDS	71	4,382	33
T-NORM	68	1,580	24
TRANSITIVITY	31	835	16
FUZZY-MEASURE	30	1,088	18
PRIORITY-VECTOR	29	1,760	20
SUPPLIER-SELECTION	29	1,861	19
DECISION-SUPPORT-SYSTEM	28	1,709	20
INTERVAL-VALUED-INTUITIONISTIC-FUZZY-SET	20	1,527	17
FUZZY-NUMBERS	20	1,052	15
FUSION	14	562	10
CARDINALITY	12	412	8
FUZZY-PREFERENCES	12	455	7
PRIORITIZE-DESIGN-REQUIREMENTS	10	608	8
ANALYTIC-NETWORK-PROCESS	9	263	9

with topics such as, fuzzy preference relations, linguistic variables, majority and operators.

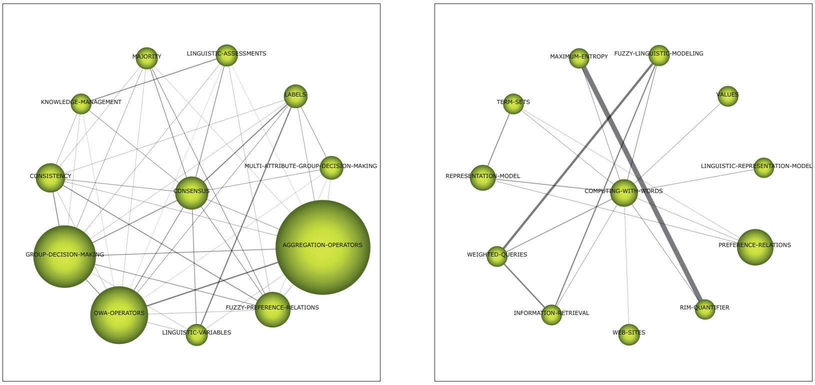
The motor theme *Fuzzy-Analytic-Hierarchy-Process*, achieves important citations score (over seven-thousand citations). It is mainly focused on fuzzy TOPSIS, multicriteria decision making, balanced scorecards, among others.

The motor theme *Intuitionistic-Fuzzy-Sets*, gets the second highest impact rate, both in citations and h-index. It specializes in aspect related with vague set theory, operators, distance measures, similarities measures, etc.

The motor theme *Computing-with-Words* (Fig. 5b) collects the research conducted on linguistic modeling, linguistic representation and information retrieval.

Second Period (2012–2016). The research conducted in this period pivots in twenty-two themes. According to the strategic diagram shown in Fig. 6, during this period eleven themes could be highlighted (motor themes plus basic themes): *Uninorms*, *Priority-Weights*, *Aggregation-Operators*, *Analytic-Hierarchy-Process*, *Entropy*, *Multi-Attribute-Group-Decision-Making*, *T-Norm*, *Decision-Analysis*, *Ranking, Consensus-Model*, *Choquet-Integral.*

According to the performance measures shown in Table 2 four themes stand out since they have the highest citations count (more than two-thousand): *Aggregation-Operators*, *Analytic-Hierarchy-Process*, *Multi-Attribute-Group-Decision-Making* and *Hesitant-Fuzzy-Set*. Moreover, *Ranking*, *Priority-Weights*,



(a) Theme Consensus. (b) Theme Computing with words

Fig. 5. Thematic networks for the period 2007–2011.

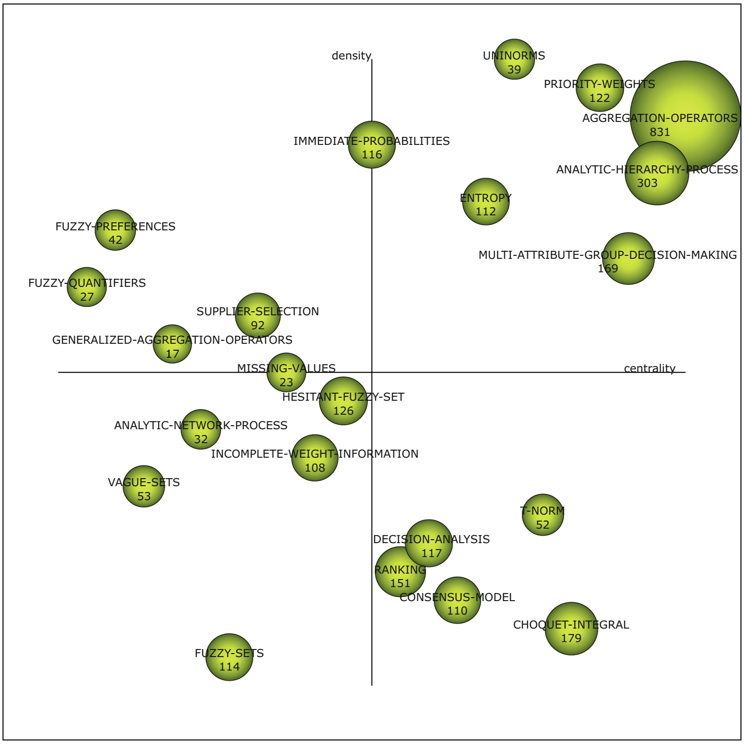


Fig. 6. Strategic diagram for the 2012–2016 period.

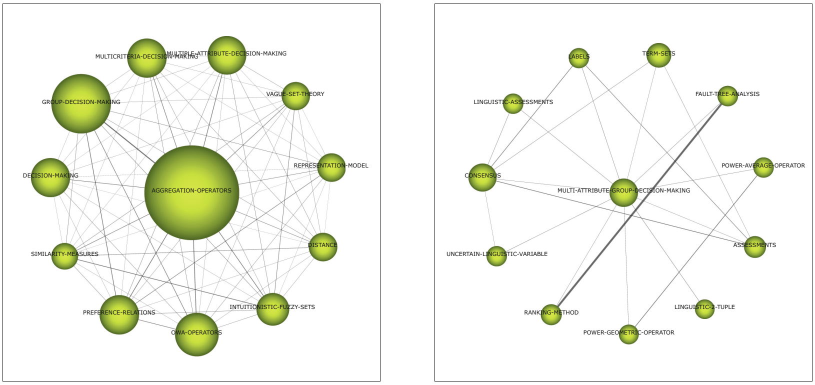
Decision-Analysis, *Fuzzy-Sets*, *Entropy*, *Consensus-Model*, *Incomplete-Weight-Information* and *Supplier-Selection* are also remarkable since they get more than one-thousand citations.

Table 2. Performance of the themes in the 2012–2016 period

Name	Number of documents	Number of citations	h-index
AGGREGATION-OPERATORS	831	9,747	46
ANALYTIC-HIERARCHY-PROCESS	303	4,163	32
CHOQUET-INTEGRAL	179	1,476	20
MULTI-ATTRIBUTE-GROUP-DECISION-MAKING	169	2,441	28
RANKING	151	1,821	25
HESITANT-FUZZY-SET	126	2,018	25
PRIORITY-WEIGHTS	122	1,962	26
DECISION-ANALYSIS	117	1,241	18
IMMEDIATE-PROBABILITIES	116	965	18
FUZZY-SETS	114	1,130	20
ENTROPY	112	1,515	20
CONSENSUS-MODEL	110	1,470	23
INCOMPLETE-WEIGHT-INFORMATION	108	1,155	16
SUPPLIER-SELECTION	92	1,644	21
VAGUE-SETS	53	946	16
T-NORM	52	341	10
FUZZY-PREFERENCES	42	316	11
UNINORMS	39	207	9
ANALYTIC-NETWORK-PROCESS	32	546	14
FUZZY-QUANTIFIERS	27	329	9
MISSING-VALUES	23	401	10
GENERALIZED-AGGREGATION-OPERATORS	17	183	6

The motor theme *Aggregation-Operators* (Fig. 7a) is the theme with the highest impact rate. In fact, it doubles the citations achieved by the second ranked theme, *Analytic-Hierarchy-Process*. *Aggregation-Operators* could be seen as one of the central topics studies in the Fuzzy Decision Making Area. Particularly, it collects the research related with the different kind of operators (i.e. OWA), multi-criteria and multi attribute aspects. It could be seen as an evolution of the *Consensus* theme appeared in the previous period.

The motor theme *Analytic-Hierarchy-Process*, evolves from the previous period. It gets the second best impact score, doubling the third one. It encompasses topics related with TOPSIS, pairwise comparisons, preference relation among others.



(a) Theme Aggregation Operators.

(b) Theme Multi-Attribute-Decision-Making.

Fig. 7. Thematic networks for the period 2012–2016.

The motor theme *Multi-Attribute-Group-Decision-Making* is related with topics such as, linguistic 2-tuple, power operators, etc.

5 Conclusions

In this contribution a conceptual science mapping analysis of the research conducted in the field of Fuzzy Decision Making in the last ten years (2007–2016) was carried out. The analysis, was performed using SciMAT.

An amount of 1,465 documents (articles and reviews) were retrieved. The whole corpus was divided in two consecutive period of five years length: 2007–2011 and 2012–2016.

In the first period, the themes *Consensus*, *Fuzzy-Analytic-Hierarchy-Process*, *Intuitionistic-Fuzzy-Sets*, *Fuzzy-Sets*, *Uncertainty* and *Computing-with-Words* stand out due to their highest impact rates. We should point out that *Consensus* doubles the citations achieved by the second one. Similarly, in the second period four themes must be highlighted according to their impact scores: *Aggregation-Operators*, *Analytic-Hierarchy-Process*, *Multi-Attribute-Group- Decision-Making* and *Hesitant-Fuzzy-Set*. Also, *Aggregation- Operators* gets two time more citations that the remaining themes.

Finally, we would like to address some future works. First, a global analysis could be carried out taking into account a wider time span. Second, the evolution of the research themes could be studied across the consecutive time periods.

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Real Option Analysis with Interval-Valued Fuzzy Numbers and the Fuzzy Pay-Off Method

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Abstract. This paper presents an extension of the fuzzy pay-off method for real option valuation using interval-valued fuzzy numbers. To account for a higher level of imprecision that can be present in many applications, we propose to use triangular upper and lower membership functions as the basis of real option analysis. In the paper, analytical formulas are derived for the triangular case by calculating the possibilistic mean of truncated interval-valued triangular fuzzy numbers. A numerical example of a cash-flow analysis is presented to illustrate the use of the proposed approach.

Keywords: Real option valuation · Fuzzy pay-off · Interval-valued fuzzy numbers · Possibilistic mean

1 Introduction

As one of the most prominent tools to model imprecise information, fuzzy set theory has been introduced in [15] as an approach to deal with uncertainty different from randomness. In many real life situations, the available data cannot be specified either precisely, or by relying on the tools of traditional probability theory. A typical example of this is, when one needs to construct models and then operate them based on estimates given by experts. This situation frequently occurs in the context of real option valuation, as in many situations there is no sufficient historical data available for constructing data-based estimates of the value of the underlying asset.

From a practical point of view, real options resemble financial options, but they give their holder the possibilities that are connected to tangible real-world investments. Due to this resemblance, until recently, the traditional financial option valuation tools were used almost exclusively in real option analysis (ROA), and still dominate the ROA literature. One of the most promising alternative approaches that utilizes methodological basis different from probability theory is the fuzzy pay-off method, introduced in [3]. In order to account for the imprecision and inaccuracy present in the real investment analysis context, the authors proposed to use fuzzy set and possibility theory as the basis for ROA, resulting in an easy-to-use and intuitively understandable model. There

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are also other fuzzy ROA approaches that include binomial option valuation based methods in [6,9], and hybrid stochastic-fuzzy Black-Scholes pricing model based models [18].

In this article, we propose an extension to the fuzzy pay-off method that replaces the underlying fuzzy net present value estimations with interval-valued fuzzy numbers (IVFNs) to account for a higher level of uncertainty. IVFNs (and in general type-2 fuzzy sets) are extensively used in different problems of decision-making [10], particularly in financial modelling and option pricing [9]. In addition to the derivation of analytical formulas for the possibilistic mean of truncated IVFNs used in the extended fuzzy pay-off method, a numerical case is presented to illustrate the difference in results, when compared to the original method.

The rest of the paper is structured as follows. In the following section, we will shortly present the necessary definitions and notations. In Sect. 3, the extension of the original fuzzy pay-off method with interval-valued fuzzy numbers is presented. An application of the approach to evaluating cash-flow estimations is discussed in Sect. 4. Finally, the conclusions are presented in Sect. 5.

2 Preliminaries

In this section we present the definitions and concepts used in the discussion of the model, including fuzzy sets, ordinary and interval-valued fuzzy numbers, and the definition and main properties of the possibilistic mean value.

2.1 Fuzzy Numbers and Possibilistic Mean

A fuzzy subset A of a non-empty set X can be defined as a mapping,

$$\mu_A \colon X \to [0,1].$$

μ_A is termed as the membership function specifying the degree to which elements from X belong to A. In the following, as it is a common tradition, we will write simply $A(x)$ instead of $\mu_A(x)$ to denote the membership function. A typical way to characterize fuzzy sets is through the γ-cuts denoted by $[A]^\gamma$ defined as

$$[A]^\gamma = \{x \in X | A(x) \geq \gamma\},$$

if $\gamma > 0$ and cl(supp$A$) if $\gamma = 0$, where cl(supp$A$) denotes the closure of the support of A. A fuzzy set A of X is called *convex* if $[A]^\gamma$ is a convex subset of X for all $\gamma \in [0,1]$. A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support [1]. An important reason for utilizing fuzzy numbers in many applications is that they constitute a family of possibility distributions. Formally, possibility [17] can be defined as a maxitive normalized monotone measure (in contrast to probability, which is an additive measure). In case of fuzzy numbers, the possibility measure of a set B can be specified as follows:

$$\mathrm{Pos}(B \subset \mathbb{R}) = \sup_{x \in B} A(x),$$

where $A(x)$ is the membership function of the fuzzy number A. This interpretation offers an important motivation in utilizing fuzzy numbers instead of probability distributions in (real) option analysis.

Fuzzy numbers can be uniquely characterized by their γ-cuts that constitute an interval for every γ.

Definition 1. *Let A be a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\gamma \in [0,1]$. Let us introduce the notations*

$$a_1(\gamma) = \min[A]^\gamma, \quad a_2(\gamma) = \max[A]^\gamma$$

where $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut, $\gamma \in [0,1]$.

In most applications, a special class of fuzzy numbers is utilized, namely the class of triangular fuzzy numbers [11].

Definition 2. *A fuzzy set A is called a triangular fuzzy number with center a, left width $\alpha > 0$ and right width $\beta > 0$, if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \le t \le a \\ 1 - \dfrac{t-a}{\beta} & \text{if } a \le t \le a + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha, \beta)$.

A typical, non-trivial problem frequently occurring in various applications of fuzzy numbers is the process of ranking [13]. One often used approach is simply to map fuzzy numbers into the real line by using any one of the many applicable methods available, and then ranking the resulting values. One of the most widely used ranking approaches, the possibilistic mean, is studied in [1], and can be defined as follows.

Definition 3. *The possibilistic mean value of fuzzy number A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ is defined as*

$$E(A) = \int_0^1 (a_1(\gamma) + a_2(\gamma))\gamma \, \mathrm{d}\gamma.$$

2.2 Interval-Valued Fuzzy Numbers

An important aspect of fuzzy set theory-based modelling in recent decades has been that of incorporating various types and levels of imprecision in the different approaches. In addition to the contributions relying on traditional fuzzy sets, the number of contributions that are based on, for example, interval-valued, intuitionistic, and hesitant fuzzy sets or vague sets, increases continuously. The extensions that aim to more completely model imprecision involved in defining the limits of fuzzy sets include the class of type-2 fuzzy sets and as their special case interval-valued fuzzy sets (IVFS) [16]. In the following, we will only consider IVFS's, and introduce the most important definitions used.

An IVFS is a mapping A from the universe X to the set of closed intervals of $[0, 1]$ [5]. By using the notations $A(u) = [A^L(u), A^U(u)]$, the traditional terminology is to call A^L and A^U as lower fuzzy set and upper fuzzy set of A, respectively. An interval-valued fuzzy set is said to be an interval-valued fuzzy number (IVFN), if A^U and A^L are fuzzy numbers [12]. IVFN's have been extensively used in different problems of decision-making [7,14].

The γ-cuts of A^L and A^U are denoted as

$$[A^L]^\gamma = [a_1(\gamma), a_2(\gamma)],$$
$$[A^U]^\gamma = [A_1(\gamma), A_2(\gamma)]$$

with

$$[A]^\gamma = ([A^L]^\gamma, [A^U]^\gamma).$$

The extension of the possibilistic mean value for interval-valued fuzzy numbers can be defined as follows.

Definition 4 [2]. *The possibilistic mean value of $A \in IVFN$ is defined as*

$$E_I(A) = \int_0^1 \gamma(M(U_\gamma) + M(L_\gamma))\mathrm{d}\gamma, \tag{1}$$

where U_γ and L_γ are uniform probability distributions defined on $[A^U]^\gamma$ and $[A^L]^\gamma$, respectively, and M stands for the probabilistic mean operator.

Intuitively, the possibilistic mean of an interval-valued fuzzy number is the arithmetic mean of the mean values of its upper and lower fuzzy numbers. If $A = A^U = A^L$ is an ordinary fuzzy number, this definition collapses into the possibilistic mean value.

It is additionally proven in [2] that this definitions satisfies numerous reasonable properties required from a ranking method on fuzzy sets [13], and that the operator is linear in the sense of the max-min extended operations addition and multiplication by a scalar on IVFN.

Lemma 1. *If $A, B \in IVFN$ and $c \in \mathbb{R}$, then*

1. $E_I(cA) = cE_I(A)$,
2. $E_I(A + B) = E_I(A) + E_I(B)$.

As we will use this property extensively, it is important to highlight that the possibilistic mean of an IVFN can be calculated as the arithmetic mean of the possibilistic mean of the upper and lower fuzzy numbers:

$$E_I(A) = \frac{E(A^U) + E(A^L)}{2}.$$

3 Fuzzy Pay-Off with IVFNs

The fuzzy pay-off method introduced in [3] is based on the same option valuation logic underlying simulation-based real option methods see, e.g., [4,8], where the main idea is that of by simulation creating probability distributions for the net present value (NPV), from which a pay-off distribution for the real option is created. In the real option pay-off distribution the possible negative values are accounted for as zero and consequently the real option value can be calculated as the probability weighted average of the real option pay-off distribution. In the fuzzy pay-off method, fuzzy numbers (possibility distributions) are used to represent the NPV and the real option pay-off distributions. A single real option value is reached by calculating an area weighted possibilistic mean of the positive NPV outcomes.

Definition 5. *We calculate the real option value from the fuzzy NPV as follows*

$$\mathrm{ROV} = \frac{\int_0^\infty A(x)\mathrm{d}x}{\int_{-\infty}^\infty A(x)\mathrm{d}x} \times E(A_+) \tag{2}$$

where A stands for the fuzzy NPV, $E(A_+)$ denotes the possibilistic mean value of the positive side of the NPV.

3.1 An Extension of the Pay-Off Method

As we pointed out above, interval-valued fuzzy sets extend the traditional fuzzy sets by introducing a second level of imprecision. In the traditional case, while the exact value of an object is not precisely identified, the membership function estimates the degree to which a specific values belong to the underlying object. In the interval-valued case, we assume that the membership function is imprecisely known, an assumption that is in line with the reality facing the context of this paper, real investments. Compared to using general type-2 fuzzy sets, IVFSs are more intuitive from a practical point of view as they only require the specification of an interval for membership values, and not second level fuzzy sets. Moreover, as we consider in connection with the application, triangular IVFN's can be specified by five unique real numbers, while allowing a more complete representation.

Motivated by these observations, we propose an extension of the fuzzy pay-off method that uses IVFN's as follows.

Definition 6. *The real option value from an interval-valued fuzzy NPV is calculated as follows*

$$\mathrm{IV-ROV} = \frac{\int_0^\infty (A^U(x) + A^L(x))\mathrm{d}x}{\int_{-\infty}^\infty (A^U(x) + A^L(x))\mathrm{d}x} \times E_I(A_+) \tag{3}$$

where A stands for the interval-valued fuzzy NPV, $E_I(A_+)$ denotes the possibilistic mean value of the positive side of the NPV. $\int_{-\infty}^\infty 0.5(A^U(x) + A^L(x))\mathrm{d}x$

computes the average of the area below the upper and lower membership function of A, while $\int_0^\infty 0.5(A^U(x) + A^L(x))\mathrm{d}x$ is the average of the area below the positive part of A^U and A^L.

In the following, we will calculate the possibilistic mean value for the positive part of a triangular interval-valued fuzzy number A, with upper triangular fuzzy number $A^U = (a, \alpha_1, \beta_1)$ and lower triangular fuzzy number $A^L = (a, \alpha_2, \beta_2)$.

Specifically, if $0 = a - \alpha_1 + z$, $E_I(A|z)$ denotes the possibilistic mean of the truncated interval-valued fuzzy number A. In the following, we will look at the different cases depending on the location of the value of 0. It is important to mention that the possibilistic mean value of a triangular interval-valued fuzzy number can be calculated as

$$E_I(A) = a + \frac{\beta_2 - \alpha_2}{12} + \frac{\beta_1 - \alpha_1}{12}.$$

This implies in our specific application that, if the fuzzy number A is greater than 0, then the possibilistic mean of the truncated fuzzy number is the same as the mean of the original fuzzy number specified in the above formula. In case $a+\beta_1 < 0$, the support of the fuzzy number does not contain any positive values; this situation implies that the possibilistic mean of the truncated interval-valued fuzzy number (and the real option value) is 0.

Additionally to these trivial cases, depending on the location of 0, there are four situations to be considered:

- $a - \alpha_1 < 0 < a - \alpha_2$;
- $a - \alpha_2 < 0 < a$;
- $a < 0 < a + \beta_2$;
- $a + \beta_2 < 0 < a + \beta_1$.

In the first case, we need to consider the situation when

$$a - \alpha_1 < 0 < a - \alpha_2.$$

In this case, we only truncate the upper membership function, the lower will remain the same. Before presenting the calculations, we introduce the following notation for the upper membership value of 0:

$$z_1 = A^U(0) = A^U(a - \alpha_1 + z) = 1 - \frac{\alpha_1 - z}{\alpha_1} = \frac{z}{\alpha_1}.$$

According to this, for the lower membership function, the γ-cuts remain the same,

$$[A^L]^\gamma = [a_1(\gamma), a_2(\gamma)]$$

while for the upper membership function, they remain the same for $\gamma > z_1$, and for $\gamma < z_1$

$$[A^U]^\gamma = [a - \alpha_1 + z, A_2(\gamma)].$$

We need to calculate the following possibilistic mean value:

$$\begin{aligned}E_I(A|z) &= \frac{1}{2}\,(I_1 + I_2 + I_3)\\ &= 0.5\int_0^{z_1} \gamma(a - \alpha_1 + z + a + (1-\gamma)\beta_1)\mathrm{d}\gamma\\ &+ 0.5\int_{z_1}^{1} \gamma(a - (1-\gamma)\alpha_1 + a + (1-\gamma)\beta_1)\mathrm{d}\gamma\\ &+ 0.5\int_0^{1} \gamma(a - (1-\gamma)\alpha_2 + a + (1-\gamma)\beta_2)\mathrm{d}\gamma\end{aligned} \tag{4}$$

The integrals can be computed as follows.

$$I_1 = \int_0^{z_1} [(2a - \alpha_1 + z)\gamma + \gamma(1-\gamma)\beta_1]\mathrm{d}\gamma$$

$$= (2a - \alpha_1 + z + \beta_1)\frac{z^2}{2\alpha_1^2} - \beta_1\frac{z^3}{3\alpha_1^3}$$

$$= (2a - \alpha_1 + \beta_1)\frac{z^2}{2\alpha_1^2} - \beta_1\frac{z^3}{3\alpha_1^3} + \frac{z^3}{2\alpha_1^2}$$

$$I_2 = \int_{z_1}^{1} [2a\gamma + \gamma(1-\gamma)(\beta_1 - \alpha_1)]\mathrm{d}\gamma$$

$$= a - \frac{az^2}{\alpha_1^2} + \frac{\beta_1 - \alpha_1}{6} - (\beta_1 - \alpha_1)\frac{z^2}{\alpha_1^2} + (\beta_1 - \alpha_1)\frac{z^3}{3\alpha_1^3}$$

$$I_3 = a + \frac{\beta_2 - \alpha_2}{6}$$

Using these values, one can obtain that

$$E_I(A|z) = a + \frac{\beta_2 - \alpha_2}{12} + \frac{\beta_1 - \alpha_1}{12} + \frac{z^3}{12\alpha_1^3}$$

In the second case, we need to calculate the truncated mean when

$$a - \alpha_2 < 0 < a.$$

In this case, additionally to z_1, we also need to calculate the membership value of 0 in the lower membership function:

$$z_2 = A^L(0) = A^L(a - \alpha_1 + z) = 1 - \frac{\alpha_1 - z}{\alpha_2} = \frac{\alpha_2 - \alpha_1 + z}{\alpha_2}.$$

The possibilistic mean of the positive part can be calculated as:

$$\begin{aligned} E_I(A|z) &= \frac{1}{2}\left(I_1 + I_2 + I_3 + I_4\right) \\ &= 0.5\int_0^{z_1} \gamma(a-\alpha_1+z+a+(1-\gamma)\beta_1)\mathrm{d}\gamma \\ &+ 0.5\int_{z_1}^{1} \gamma(a-(1-\gamma)\alpha_1+a+(1-\gamma)\beta_1)\mathrm{d}\gamma \\ &+ 0.5\int_0^{z_2} \gamma(a-\alpha_1+z+a+(1-\gamma)\beta_2)\mathrm{d}\gamma \\ &+ 0.5\int_{z_2}^{1} \gamma(a-(1-\gamma)\alpha_2+a+(1-\gamma)\beta_2)\mathrm{d}\gamma \end{aligned} \tag{5}$$

The integrals can be computed as follows.

$$I_1 = (2a-\alpha_1+\beta_1)\frac{z^2}{2\alpha_1^2} - \beta_1\frac{z^3}{3\alpha_1^3} + \frac{z^3}{2\alpha_1^2}$$

$$I_2 = a - \frac{az^2}{\alpha_1^2} + \frac{\beta_1-\alpha_1}{6} - (\beta_1-\alpha_1)\frac{az^2}{\alpha_1^2} + (\beta_1-\alpha_1)\frac{z^3}{3\alpha_1^3}$$

$$\begin{aligned} I_3 &= \int_0^{z_2} [(2a-\alpha_1+z)\gamma + \gamma(1-\gamma)\beta_2]\mathrm{d}\gamma \\ &= (2a-\alpha_1+\beta_2+z)\frac{z_2^2}{2} - \beta_2\frac{z_2^2}{3} + \frac{z_2^3}{2} \end{aligned}$$

$$\begin{aligned} I_4 &= \int_{z_2}^{1} [2a\gamma + \gamma(1-\gamma)(\beta_2-\alpha_2)]\mathrm{d}\gamma \\ &= a - az_2^2 + \frac{\beta_2-\alpha_2}{6} - (\beta_2-\alpha_2)\frac{z_2^2}{2} + (\beta_2-\alpha_2)\frac{z_2^3}{3} \end{aligned}$$

Using these values, one can obtain that

$$E_I(A|z) = a + \frac{\beta_2-\alpha_2}{12} + \frac{\beta_1-\alpha_1}{12} + \frac{z^3}{12\alpha_1^3} + \frac{(z-\alpha_1+\alpha_2)^3}{12\alpha_2^2}$$

In the third case, we need to calculate the truncated mean when

$$a < 0 < a + \beta_2.$$

In this case, we have the following expression specifying the membership value of 0:

$$z_1 = A^U(0) = A^U(a-\alpha_1+z) = 1 - \frac{z-\alpha_1}{\beta_1} = \frac{\beta_1+z-\alpha_1}{\beta_1}.$$

Additionally, we also need the membership in the lower membership function:

$$z_2 = A^L(0) = A^L(a-\alpha_1+z) = 1 - \frac{z-\alpha_1}{\beta_2} = \frac{\beta_2+z-\alpha_1}{\beta_2}.$$

The possibilistic mean of the positive part can be calculated as:

$$\begin{aligned} E_I(A|z) &= \frac{1}{2}(I_1 + I_2) \\ = 0.5 \int_0^{z_1} &\gamma(a - \alpha_1 + z + a + (1-\gamma)\beta_1)\mathrm{d}\gamma \\ +0.5 \int_0^{z_2} &\gamma(a - \alpha_1 + z + a + (1-\gamma)\beta_2)\mathrm{d}\gamma \end{aligned} \tag{6}$$

The integrals can be computed as follows.

$$I_1 = (2a - \alpha_1 + \beta_1 + z)\frac{z_1^2}{2} - \beta_1 \frac{z_1^3}{3}$$

$$I_2 = (2a - \alpha_1 + \beta_2 + z)\frac{z_2^2}{2} - \beta_2 \frac{z_2^3}{3}$$

Using these values, one can obtain that

$$E_I(A|z) = a\left(\frac{\beta_1 - \alpha_1 + z}{\beta_1^2} + \frac{(\beta_2 - \alpha_1 + z)^2}{\beta_2^2}\right)$$

$$+ \frac{\beta_2 - \alpha_1 + z)^3}{6\beta_1^2} + \frac{\beta_2 - \alpha_1 + z)^3}{6\beta_2^2}$$

Finally, in the last case, we need to calculate the truncated mean when

$$a + \beta_1 < 0 < a + \beta_2.$$

In this case we only consider the upper membership, as the support of the lower triangular fuzzy number does not contain positive values.

The possibilistic mean of the positive part can be calculated as:

$$E_I(A|z) = 0.5 I_1 = 0.5 \int_0^{z_1} \gamma(a - \alpha_1 + z + a + (1-\gamma)\beta_1)\mathrm{d}\gamma \tag{7}$$

The integral can be computed as

$$I_1 = (2a - \alpha_1 + \beta_1 + z)\frac{z_1^2}{2} - \beta_1 \frac{z_1^3}{3}$$

Finally,

$$E_I(A|z) = a\left(\frac{\beta_1 - \alpha_1 + z}{\beta_1^2}\right) + \frac{\beta_2 - \alpha_1 + z)^3}{6\beta_1^2}$$

In all the four cases, to compute the real option value, according to the definition, the ratio between the average of the positive area of the upper and lower triangular fuzzy number and the mean of the total area under the upper and lower triangular fuzzy numbers has to be calculated as the weight for the above calculated possibilistic mean of the truncated interval-valued fuzzy number.

4 Numerical Example

In this section we present a numerical example to illustrate the application of the proposed approach. In particular, we will discuss how to assess uncertain cash-flows. The data was used in [3] to illustrate the benefits of the original fuzzy pay-off method. The original data is presented in the form of three points corresponding to triangular fuzzy numbers that in turn characterize future cash-flow scenarios. In Fig. 1, the data presented in [3] and used in the following example is shown.

The scenario values can be specified by managers as crisp values, or generated from any preliminary analysis. Costs and revenues are specified separately, with the cost cash-flows discounted at the risk-free rate and the revenue discount rate is selected according to the risk adjusted discount rate. The final fuzzy NPV is the fuzzy pay-off distribution for the investment.

	Investment / cost cash-flows (PV@"risk free" level)						Rf= 0,05									
	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	6	7	8	9	10
CF Maximum	15,00	0,00	0,00	0,00	325,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
CF Base	15,00	0,00	0,00	0,00	325,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
CF Minimum	15,00	0,00	0,00	0,00	325,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
PV Maximum	15,00	0,00	0,00	0,00	294,78	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
PV Base	15,00	0,00	0,00	0,00	294,78	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
PV Minimum	15,00	0,00	0,00	0,00	294,78	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7	0-8	0-9	0-10	0-11
ΣPV Maximum	15,00	15,00	15,00	15,00	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78
ΣPV Base	15,00	15,00	15,00	15,00	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78
ΣPV Minimum	15,00	15,00	15,00	15,00	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78	309,78
	Revenue cash-flows (PV @ "risk adjusted" level)						Ra = 0,15									
	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	6	7	8	9	10
CF Maximum	0,00	0,00	0,00	0,00	0,00	0,00	80,00	0,00	116,00	0,00	153,00	177,00	223,00	268,00	314,00	0,00
CF Base	0,00	0,00	0,00	0,00	0,00	0,00	52,00	0,00	62,00	0,00	74,00	77,00	89,00	104,00	122,00	0,00
CF Minimum	0,00	0,00	0,00	0,00	0,00	0,00	20,00	0,00	23,00	0,00	24,00	18,00	20,00	20,00	22,00	0,00
PV Maximum	0,00	0,00	0,00	0,00	0,00	0,00	52,60	0,00	66,32	0,00	76,07	76,52	83,83	87,61	89,26	0,00
PV Base	0,00	0,00	0,00	0,00	0,00	0,00	34,19	0,00	35,45	0,00	36,79	33,29	33,46	34,00	34,68	0,00
PV Minimum	0,00	0,00	0,00	0,00	0,00	0,00	13,15	0,00	13,15	0,00	11,93	7,78	7,52	6,54	6,25	0,00
	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7	0-8	0-9	0-10	0-11
ΣPV Maximum	0,00	0,00	0,00	0,00	0,00	0,00	52,60	52,60	118,92	118,92	194,99	271,51	355,35	442,96	532,22	532,22
ΣPV Base	0,00	0,00	0,00	0,00	0,00	0,00	34,19	34,19	69,64	69,64	106,43	139,72	173,18	207,18	241,86	241,86
ΣPV Minimum	0,00	0,00	0,00	0,00	0,00	0,00	13,15	13,15	26,30	26,30	38,23	46,01	53,53	60,07	66,33	66,33
	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7	0-8	0-9	0-10	0-11
FNPV Maximum	-15,00	-15,00	-15,00	-15,00	-309,78	-309,78	-257,18	-257,18	-190,86	-190,86	-114,79	-38,27	45,56	133,17	222,43	222,43
FNPV Base	-15,00	-15,00	-15,00	-15,00	-309,78	-309,78	-275,59	-275,59	-240,15	-240,15	-203,35	-170,06	-136,61	-102,61	-67,93	-67,93
FNPV Minimum	-15,00	-15,00	-15,00	-15,00	-309,78	-309,78	-296,63	-296,63	-283,48	-283,48	-271,55	-263,77	-256,25	-249,71	-243,46	-243,46

Fig. 1. Data for the numerical example

To extend this process of acquiring data in the form of interval-valued fuzzy costs and revenues, from a practical point of view, the experts specifying the values can be asked to specify, instead of three crisp values, the following:

– a crisp value corresponding to the most likely scenario: the center of both the upper and lower fuzzy numbers;
– a real interval that can potentially contain the minimum possible outcome: the interval $[a - \alpha_1, a - \alpha_2]$
– a real interval that can potentially contain the maximum possible outcome: the interval $[a + \beta_2, a + \beta_1]$

This description illustrates the main benefit of the proposed approach: we are not restricted any more to a crisp minimum and maximum possible scenarios, as they can be difficult to estimate precisely in many applications, but we can estimate two intervals that contain these two scenarios, respectively.

For illustrative purposes we have used the estimated values presented in Fig. 1 as a starting point and artificially generated intervals around. The interval-valued triangular fuzzy numbers use the base scenarios from Fig. 1 as the center, while the minimum and maximum possible intervals around the center are specified with endpoints plus and minus $t\%$ of the center value, for $t \in \{1, 5, 10\}$. The results from the original analysis presented in [3] and for the three considered t values can be seen in Table 1.

Table 1. Results from the numerical example

	Pay-off bad	Pay-off base	Pay-off good	ROV
Original model	−243.46	−67.93	222.43	13.56
$t = 1\%$	$[-247.22, -238.28]$	−67.93	$[218.74, 225.42]$	13.69
$t = 5\%$	$[-255.66, -234.02]$	−67.93	$[215.63, 231.04]$	13.89
$t = 10\%$	$[-270.45, -222.23]$	−67.93	$[205.02, 241.12]$	14.11

5 Conclusions

In this paper, we have proposed a new extension that uses interval-valued fuzzy sets in the fuzzy pay-off method [3]. The proposed method allows for a more complete representation of the imprecision connected to cash-flow estimation underlying real option valuation. While the structure and the complexity of the method remains the same, with an extra effort one can potentially improve the quality of the final real option value prediction. We specifically considered the case of triangular interval-valued fuzzy numbers and an extension of the possibilistic mean in this paper, and derived analytical formulae for the possibilistic mean values of truncated numbers.

This paper is one of the first contributions in the literature on real options to account for higher level of imprecision in terms of the underlying mathematical representation by utilizing interval-valued fuzzy numbers. While the case of trapezoidal interval-valued fuzzy numbers is not presented, the derivation of the formulae can be done straight forward based on our results, while accounting for the presence of the central interval. From a practical point of view, the input data requires the experts to specify an interval for the outcome of some basic scenarios, which on one hand does not complicate their task excessively, but allows for a more complete picture as the basis of real option value to be obtained.

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Measuring the Incoherent Information in Multi-adjoint Normal Logic Programs

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Abstract. Databases usually contain incoherent information due to, for instance, the presence of noise in the data. The detection of the incoherent information is an important challenge in different topics. In this paper, we will consider a formal notion for this kind of information and we will study different measures in order to detect incoherent information in a general fuzzy logic programming framework. As a consequence, we can highlight some irregular data in a multi-adjoint normal logic program and so, in other useful and more particular frameworks.

Keywords: Multi-adjoint normal logic program · Coherence interpretation · Incoherence measure

1 Introduction

Fuzzy logic programming is the computational branch of fuzzy logic, which has widely been studied [1,10,15,16,29]. One of the most general frameworks is multi-adjoint logic programming (MALP) [13,14,25,27,30], which was introduced by Medina, Ojeda-Aciego and Vojtáš in [26] as a general framework in which the minimal mathematical requirements are only considered in order to ensure the main properties given in the diverse usual logic programming frameworks. For instance, this theory generalizes the annotated logic programming [17], possibilistic logic programming [7], monotonic and residuated logic programming [5,6], fuzzy logic programming [32], etc. All these frameworks are part of the set of "positive" logic frameworks which only consider monotonic operators.

The use of negation operators enriches the flexibility of the logic language considered in order to model a particular knowledge database. However, this consideration hinders the computational operability of the obtained framework. Recently, multi-adjoint normal logic programming (MANLP) arises as a generalization of MALP in which negation operators can be considered [3,4] and different studies on the syntax and semantics of this general framework have been presented.

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Recently, in [3], the suitable notion of coherence and incoherence in a MANLP has been studied and a final definition has been proposed. Based on this definition, we will study in this paper different measures in order to detect incoherent information in a general fuzzy logic programming framework. As a consequence, we can highlight some irregularity in the data in a multi-adjoint normal logic program and so, in other useful and more particular frameworks, previously enumerated. This fact will provide a procedure in order to assess the quality of the data of the considered knowledge system and so, of the obtained results from it.

The structure of the paper is the following. Section 2 recalls the basic definitions in multi-adjoint normal logic programming and graph theory needed throughout of the paper. Section 3 set the notion of coherent interpretation we will consider, introduce different properties and diverse incoherent measures. The paper finishes with the conclusions and several aims for future work.

2 Preliminaries

This section will provide an overview with some necessary definitions and results of the multi-adjoint normal logic programming framework and graph theory, which will contribute to a better understanding of the carried out study.

2.1 Multi-adjoint Normal Logic Programming

The first definitions we need to recall in this framework are the notions of multi-adjoint normal lattice and multi-adjoint normal logic program.

Definition 1. *The tuple* $(L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ *is a* multi-adjoint normal lattice *if the following properties are verified:*

(1) $(L, \preceq)$ *is bounded lattice, i.e. it has a bottom* $(\bot)$ *and a top* $(\top)$ *element;*
(2) $(\&_i, \leftarrow_i)$ *is an adjoint pair in* $(L, \preceq)$*, for* $i \in \{1, \dots, n\}$*;*
(3) $\top \&_i \vartheta = \vartheta \&_i \top = \vartheta$*, for all* $\vartheta \in L$ *and* $i \in \{1, \dots, n\}$*;*
(4) $\neg$ *is a negation operator on* $(L, \preceq)$*, that is,* $\neg$ *is a decreasing mapping satisfying that* $\neg(\bot) = \top$ *and* $\neg(\top) = \bot$*.*

From a multi-adjoint normal lattice together with an additional (symbol of) negation $\sim$, we can define a multi-adjoint normal logic program $\mathbb{P}$ as a set of weighted rules in which different implications are used. The elements appearing in the rules of $\mathbb{P}$ can be either (positive) propositional symbols or negated propositional symbols by $\sim$. All these elements are called literals and they are collected in a set denoted by $Lit_{\mathbb{P}}$. The set composed only by (positive) propositional symbols appearing in $\mathbb{P}$ is denoted by $\Pi_{\mathbb{P}}$.

Definition 2. *Let* $(L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ *be a multi-adjoint normal lattice and* $\sim$ *be a strong negation. A* multi-adjoint normal logic program (MANLP) $\mathbb{P}$ *is a finite set of weighted rules of the form:*

$$\langle l \leftarrow_i @[l_1, \dots, l_m, \neg l_{m+1}, \dots, \neg l_n]; \vartheta \rangle$$

where $i \in \{1, \dots, n\}$*,* $@$ *is an aggregator operator,* ϑ *is an element of* L *and* $l, l_1, \dots, l_n$ *literals such that* $l_j \neq l_k$*, for all* $j, k \in \{1, \dots, n\}$*, with* $j \neq k$*.*

Notice that the negation operators $\neg$ and $\sim$ play different roles, that is, the truth value of $\sim\phi$ can straightforwardly be inferred from the program whereas the value of $\neg\phi$ is obtained from the truth value of ϕ. As a consequence, we will call "default negation" to $\neg$ and "strong negation" to $\sim$, as it is usual. It is important to recall that the strong negation operator should not be confused in this paper with the well-known notion of involutive operator.

With respect to the semantics of multi-adjoint logic programming framework, it is based on the notion of stable model [11] which is closely related to the notion of minimal model. For that reason, the existence of minimal models in multi-adjoint normal logic programs was studied in [4] following the philosophy considered in [23]. Now, we will introduce the definition of interpretation and an interesting property corresponding to the whole set of interpretations which will play a crucial role throughout the paper.

Definition 3. *Given a complete lattice $(L, \preceq)$, a mapping $I\colon Lit_{\mathbb{P}} \to L$, which assigns to every literal appearing in $Lit_{\mathbb{P}}$ an element of L, is called L-*interpretation*. The set of all L-interpretations is denoted by $\mathcal{I}_{\mathfrak{L}}$.*

Proposition 1. *If $(L, \preceq)$ is a complete lattice, then $(\mathcal{I}_{\mathfrak{L}}, \sqsubseteq)$ is a complete lattice where the ordering relation $\sqsubseteq$ is given in the following way:*

$$I_1 \sqsubseteq I_2 \; if \text{ and only } if \; I_1(l) \preceq I_2(l), \; for \text{ all } l \in Lit_{\mathbb{P}} \text{ and } I_1, I_2 \in \mathcal{I}_{\mathfrak{L}}.$$

Satisfaction and model are other fundamental notions for multi-adjoint normal logic programming semantics. In order to present these definitions, we need to consider some previous notational conventions:

(a) The interpretation of a operator symbol ω under a multi-adjoint normal lattice will be denoted by $\dot{\omega}$.
(b) The evaluation of a formula $\mathcal{A}$ under an interpretation I will be denoted as $\hat{I}(\mathcal{A})$. It will be proceeded inductively as usual, until all propositional symbols in $\mathcal{A}$ are reached and evaluated under I.

Definition 4. *Given an interpretation $I \in \mathcal{I}_{\mathfrak{L}}$, we say that:*

(1) A weighted rule $\langle l \leftarrow_i @[l_1, \ldots, l_m, \neg l_{m+1}, \ldots, \neg l_n]; \vartheta\rangle$ is satisfied *by I if and only if $\vartheta \preceq \hat{I}\left(\langle l \leftarrow_i @[l_1, \ldots, l_m, \neg l_{m+1}, \ldots, \neg l_n]\right)$.*
(2) An L-interpretation $I \in \mathcal{I}_{\mathfrak{L}}$ is a model *of a MANLP $\mathbb{P}$ if and only if all weighted rules in $\mathbb{P}$ are satisfied by I.*

The first two challenges in this research topic were: (1) obtaining an existence theorem for stable models in MANLPs, and (2) choosing a suitable notion in order to handle inconsistent information given by the stable models of a multi-adjoint normal logic program. Both goals were achieved in [4] and in [3], respectively. Now, we are focused on the fundamental problem of providing an incoherence measure for MANLPs defined on a finite lattice. To carry out this task, we will need some notions related to graph theory which will be recalled in the following section.

2.2 Some Useful Notions of Graph Theory

Graph theory has become an indispensable mathematical tool in diverse fields such as industrial engineering [8], medical diagnosis [12] and decision making [33], among others. In this paper, the next basic notions will lead us to obtain another application of graph theory helpful for our logic programming framework.

Definition 5. *A* graph $G = (V, E)$ *is an ordered pair composed by a non-empty set whose elements are called* vertices*, denoted as* V*, and a set whose elements are called* edges*, denoted as* E*, together with an* incidence mapping $\gamma_G \colon E \to \{\{u, v\} \mid u, v \in V\}$ *that associates each edge in* E *with two vertices in* V*.*

Definition 6. *Let* G *be a graph. A* path of length n *in* G *is a sequence of edges* $e_1 e_2 \dots e_n$ *together with a sequence of vertices* $v_1 v_2 \dots v_{n+1}$ *satisfying that* $\gamma_G(e_i) = \{v_i, v_{i+1}\}$*, for all* $i \in \{1, \dots, n\}$*. In this case, we say that the path* $e_1 e_2 \dots e_n$ *is a path between the vertex* v_1 *and the vertex* v_{n+1}*.*

Considering the above definitions, we can interpret the Hasse diagram of a finite lattice $(L, \preceq)$ as a graph $G = (V, E)$ in which $V = L$ and each edge in E correspond to an edge in the diagram. Taking into account this consideration, we can define the distance between two elements in a finite lattice $(L, \preceq)$ as follows.

Definition 7 ([2]). *Let* $G = (V, E)$ *be the graph associated with the Hasse diagram of a finite lattice* $(L, \preceq)$*,* P_G *be the set of all paths in* G *and* $l_G \colon P_G \to \mathbb{R}$ *be a mapping which assigns to each path in* G *its length. The mapping* $d \colon V \times V \to \mathbb{R}$ *defined as:*

$$d(x, y) = \min\{l_G(p) \mid p \, is \, a \, path \, between \, the \, vertices \; x \; and \; y\}$$

for all $x, y \in V$*, is called the* geodesic distance *between the vertices* x *and* y *belonging to the graph* G*.*

The relevance of the geodesic distance in this paper will be shown in Sect. 3.2, where it will be used to give original incoherence measures for multi-adjoint normal logic programs defined on a finite lattice. It can be straightforwardly proved that d is in fact a distance.

3 Coherence Interpretations and Incoherence Measures

According to the syntactic structure of MANLPs, we can ensure that the inconsistency causes - instability and incoherence - explained by Madrid and Ojeda-Aciego for residuated logic programs [18–22], can be also given in this framework. The instability is characterized by the absence of stable models in a MANLP, therefore it is easy to classify a MANLP in this case as unstable. By contrast, the incoherence is given by stable models in a MANLP which assign contradictory values to a propositional symbol p and to its corresponding negation $\sim p$. In this case, the determination of incoherent programs is not clear. From this fact, the necessity to establish definitions that allows us both to decide when a program is incoherent and what is a good measure in order to assess such incoherent information.

A detailed survey on different notions related to the concept of coherence was carried out in [3], in order to select an appropriate coherence notion capable of handling inconsistent information contained in MANLPs. In this section, we will include the coherence notion chosen for multi-adjoint normal logic programming theory [3], and we will introduce some interesting properties related to this coherence notion as well as different incoherence measures for MANLPs defined on a finite lattice.

3.1 Coherence: Definition and Properties

The concept of coherence has been an interesting object of study in recent years [9,24,28,31]. In [3], we justified the selection of the coherence interpretation notion considered in [22,24] as the most suitable definition for dealing with the inconsistent information included in MANLPs. Madrid and Ojeda-Aciego provided more reasons in order to justify why the definition of coherent interpretation is a good generalization of the consistent interpretation in a fuzzy environment. The flexibility of the coherent interpretation definition comes from we can accept an interpretation contradicting the next inference rule: *"If the truth value of a propositional symbol* p *is* ϑ *then the truth value of* $\neg p$ *is* $n(\vartheta)$", where n is a negation operator. Formally:

Definition 8 ([22,24]). *Let* $\mathcal{L} = (L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ *be a multi-adjoint normal lattice,* $\sim$ *be a strong negation and* I *an interpretation. We say that:*

(a) $p \in \Pi_{\mathbb{P}}$ *is a* coherent propositional symbol *with respect to* I *if and only if the inequality* $I(\sim p) \preceq \dot{\sim} I(p)$ *holds. Otherwise the propositional symbol* p *is called* incoherent.
(b) I *is a* coherent interpretation *if and only if the inequality* $I(\sim p) \preceq \dot{\sim} I(p)$ *holds, for every* $p \in \Pi_{\mathbb{P}}$.

Notice that the definition above allows a possible lack of information but not an excess of information. For instance, the interpretation $I \in \mathcal{I}_{\mathfrak{L}}$ satisfying $I(p) = I(\sim p) = 0$ for all $p \in \Pi_{\mathbb{P}}$, which provides no information, is a coherent interpretation. The following definition is closely related to the coherence interpretation and model notions previously presented. Now, we will introduce what is a coherent multi-adjoint normal program.

Definition 9. *Let* $\mathcal{L} = (L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ *be a multi-adjoint normal lattice,* $\sim$ *be a strong negation and* $\mathbb{P}$ *the multi-adjoint normal logic program defined on* $\mathcal{L}$*. We say that* $\mathbb{P}$ *is a* coherent program *if there exists at least a coherent model.*

An interesting property, corresponding to coherent interpretations, ensures that if there exists a coherente interpretation then all interpretations less or equal to it are also coherent.

Proposition 2 ([24]). *Let I and J be two L-interpretations satisfying $I \sqsubseteq J$. If J is coherent, then I is coherent as well.*

Observe that, by Proposition 2, we can deduce that if a MANLP $\mathbb{P}$ has a least model, then $\mathbb{P}$ is coherent program if and only if its least model is coherent. This statement will be useful for defining incoherence measures in Sect. 3.2.

Moreover, from the proposition above, we obtain a clear idea of how coherent and incoherent interpretations are distributed in the lattice of interpretations. Before presenting the algebraic structure obtained from these interpretations, we need to define the following sets:

$$C_I = \{I \in \mathcal{I}_{\mathfrak{L}} \mid I(\sim p) \preceq \dot{\sim} I(p) \text{ for each } p \in \Pi_{\mathbb{P}}\}$$
$$\bar{C}_I = \{I \in \mathcal{I}_{\mathfrak{L}} \mid \text{there exists } p \in \Pi_{\mathbb{P}} \text{ such that } I(\sim p) \not\preceq \dot{\sim} I(p)\}$$

The previous sets C_I and $\bar{C}_I$ denote the set of all coherent interpretations and the set of all incoherent interpretations belonging to $\mathcal{I}_{\mathfrak{L}}$, respectively. The ordering relation $\sqsubseteq$ introduced in Proposition 1 provides to C_I and $\bar{C}_I$ with the structure of semilattice.

Proposition 3. *The following statements are satisfied:*

(1) $(C_I, \sqsubseteq)$ *is a lower semilattice.*
(2) $(\bar{C}_I, \sqsubseteq)$ *is an upper semilattice.*

It is worth mentioning that, knowing the algebraic structure formed by the set of coherent interpretations and the set of incoherent interpretations is a great achievement, since it can reduce the computation time and improve the effectiveness of an incoherence measure. In the next section, we present incoherence measures for multi-adjoint normal logic programs defined on a finite lattice.

3.2 Incoherence Measures for MANLPs defined on a Finite Lattice

An interesting survey on incoherence measures in multi-adjoint normal logic programs has been given by Madrid and Ojeda-Aciego in [24]. Specifically, the authors measure the degree of incoherence generated by one interpretation on each negated literal considering only its negation and for that, they make use of an information measure [24, Definition 7] in order to assign a positive real value to each element in the general lattice considered in the program. In this paper, we are interested in defining incoherence measures directly from the given lattice, without considering any extra mapping. From now on, we will consider a finite multi-adjoint normal lattice $\mathcal{L} = (L, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ together with a fixed strong negation $\sim$.

We will present our first proposal of incoherence measure taking into account the meaning of a coherent/incoherent propositional symbol. This proposal is a simple definition in which the ratio of incoherent propositional symbols under an L-interpretation is considered with respect to the number of propositional symbols appearing in a MANLP $\mathbb{P}$. The idea under the following definition consists in estimating the average number of incoherent propositional symbols [24].

Definition 10. *Let I be an L-interpretation. We define the measure of incoherence as:*

$$\mathcal{M}_1(I) = \frac{\mathcal{NI}(I)}{|\Pi_{\mathbb{P}}|}$$

where $\mathcal{NI}(I)$ denotes the number of incoherent propositional symbols in $\mathbb{P}$.

This notion is similar to the one introduced by Madrid and Ojeda-Aciego in [24]. The main drawback of this definition is that it does not take into account how incoherent a propositional symbol is. For example, given two L-interpretations $I, J \in \mathcal{I}_{\mathfrak{L}}$ and a propositional symbol $p \in \Pi_{\mathbb{P}}$, the definition above does not indicate if p is more incoherente with respect to I than with respect to J. In order to introduce incoherence measures capable of dealing with such degree of incoherence, we will define the set of coherent pairs with respect to $\mathcal{L}$ as follows:

$$\Delta_{\mathcal{L}} = \{(x, y) \in L \times L \mid y \preceq \dot{\sim} x\}$$

The set $\Delta_{\mathcal{L}}$ contains all the values that a coherent interpretation can consider. Furthermore, we can characterize coherent interpretations by means of $\Delta_{\mathcal{L}}$, as follows:

Proposition 4 ([24]). *Let I be an L-interpretation. Then, I is coherent if and only if the pair $(I(p), I(\sim p)) \in \Delta_{\mathcal{L}}$, for all $p \in \Pi_{\mathbb{P}}$.*

Now, we are ready to present different alternatives to measure the distance from the elements in $L \times L$ to the set of coherent pairs. Considering the geodesic distance d associated with the Hasse diagram of the multi-adjoint normal lattice $\mathcal{L}$, we introduce our first definition of incoherence measure in a multi-adjoint approach taking into account the degree of incoherence of a pair of elements $(a, b) \in L \times L$ given by:

$$d_1((a, b), \Delta_{\mathcal{L}}) = d(b, \dot{\sim}(a)) \tag{1}$$

The definition of d_1 makes sense since the pair $(a, \dot{\sim}(a))$ belongs to $\Delta_{\mathcal{L}}$, for each $a \in L$. Notice that $\dot{\sim}(a)$ is the greatest value for b satisfying that $(a, b) \in \Delta_{\mathcal{L}}$. Therefore, if there exists $c \in L$ such that $c \prec \dot{\sim}(a)$, that is $(a, c) \in \Delta_{\mathcal{L}}$, we obtain $d_1((a, c), \Delta_{\mathcal{L}}) > 0$ because d is a distance and $c \neq \dot{\sim}(a)$. Hence, this first measure must be improved in order to avoid this fact, since the distance from a pair (a, b) in $\Delta_{\mathcal{L}}$ to the set $\Delta_{\mathcal{L}}$ must be 0. Therefore, we can conclude that d_1 is not the most suitable definition of incoherence measure in this framework.

A similar result is obtained if we define the degree of incoherence of a pair $(a, b) \in L \times L$ as follows:

$$d_2((a, b), \Delta_{\mathcal{L}}) = d(z_b, a) \tag{2}$$

where $z_b = \inf\{z \in L \mid \dot{\sim} z \preceq b\}$ (see Fig. 1). The definition of z_b is needed because $\dot{\sim}$ is not necessarily an involutive negation.

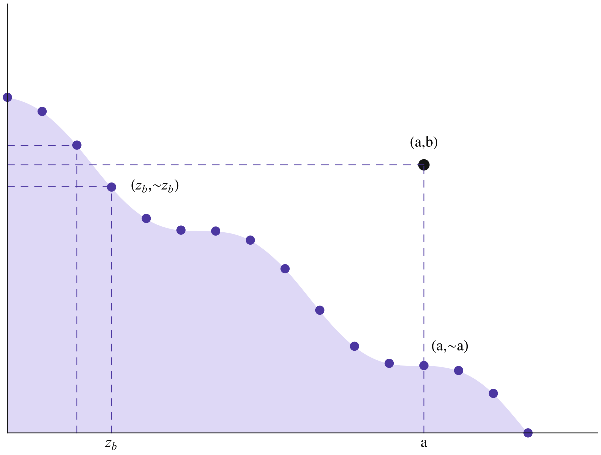

Fig. 1. Definition of z_b in distance d_2.

It is important to observe that somehow d_1 measures how much one has to be removed from b to obtain that (a, b) is a coherent pair, while d_2 measures how much one has to be removed from a to ensure that (a, b) is a coherent pair.

According to the previous considerations, we propose to consider a more general definition for the degree of incoherence of a pair $(a, b) \in L \times L$ which combines d_1 and d_2. We will consider the geodesic distance of the graph associated with the Hasse diagram of the lattice $(L \times L, \preceq)$, where $(a_1, b_1) \preceq (a_2, b_2)$ if and only if $a_1 \preceq a_2$ and $b_1 \preceq b_2$, for each $a_1, a_2, b_1, b_2 \in L$. Formally, we define the degree of incoherence of a pair $(a, b) \in L \times L$ as:

$$d_G((a, b), \varDelta_{\mathcal{L}}) = \inf\{d((a, b), (c, d)) \mid z_b \preceq c \preceq a, d \preceq \dot{\sim} c\} \tag{3}$$

It is easy to see that if $(a, b) \in \varDelta_{\mathcal{L}}$ then $d_G((a, b), \varDelta_{\mathcal{L}}) = 0$. As we noted above, this fact is not satisfied with the distances d_1 and d_2.

From the previous notion corresponding to the degree of incoherence given by d_G, we have many different ways to define a measure of incoherence for an interpretation and a given propositional symbol. We have chosen the next two incoherence measures due to their simplicity and utility:

$$\mathcal{M}_2(I) = \max\{d_G((I(p), I(\sim p)), \varDelta_{\mathcal{L}}) \mid p \in \varPi_{\mathbb{P}}\}$$

$$\mathcal{M}_3(I) = \frac{\sum_{p \in \varPi_{\mathbb{P}}} d_G((I(p), I(\sim p)), \varDelta_{\mathcal{L}})}{|\varPi_{\mathbb{P}}|}$$

Note that, given an interpretation I, $\mathcal{M}_2(I)$ measures the maximum size of incoherence while $\mathcal{M}_3(I)$ indicates the average size of incoherence. These previous incoherence measures will be helpful for reaching the main goal in this paper, that is, to decide how much incoherent a program is.

Now, we want to extend the incoherence measures given above for an interpretation and a given propositional symbol to an arbitrary MANLP $\mathbb{P}$. In order to achieve this aim, we will define the incoherence measure, for all $i \in \{1, 2, 3\}$, as:

$$\mathbb{M}_i(\mathbb{P}) = \inf\{\mathcal{M}_i(I) \mid I \text{ is a model of } \mathbb{P}\}$$

It is fundamental to observe that if $\mathbb{P}$ has a least model, then Proposition 2 lead us to ensure that $\mathbb{P}$ is coherent if and only if its least model is coherent. As a consequence, the previous incoherence measure can be written in the following way $\mathbb{M}_i(\mathbb{P}) = \mathcal{M}_i(M_{\mathbb{P}})$, for all $i \in \{1, 2, 3\}$, where $M_{\mathbb{P}}$ represents the least model of the MANLP $\mathbb{P}$.

Below, we will include an illustrative example useful for clarifying the presented incoherence measures. First of all, we show a coherent program such that the incoherence measures with respect to it are equal to zero, as one can expect. After that, we complete the program adding one rule more and we prove that the obtained program is incoherent.

Example 1. Consider the multi-adjoint normal lattice $\mathcal{L} = (L, \preceq, \leftarrow, \&, \neg)$ composed by the complete lattice $(L, \preceq)$ displayed in Fig. 2, the adjoint pair $(\&, \leftarrow)$ with respect to L defined as $x \,\&\, y = \min\{x, y\}$, $z \leftarrow x = \top$ if $x \preceq z$ and $z \leftarrow x = z$ otherwise, for all $x, y, z \in L$, and the negation operator $\neg\colon L \to L$ shown in Fig. 2. Moreover, we will consider a strong negation $\sim$ associated with a mapping $\dot{\sim}\colon L \to L$ which is also defined in Fig. 2.

$$\neg(x) = \begin{cases} \bot \text{ if } x \not\preceq d \\ \top \text{ if } x \preceq d \end{cases} \qquad \dot{\sim}(x) = \begin{cases} \bot \text{ if } & x = \top \\ g \text{ if } & d \preceq x \preceq a \\ b \text{ if } & x = e \\ c \text{ if } & x = f \\ \top \text{ if } & x \preceq g \end{cases}$$

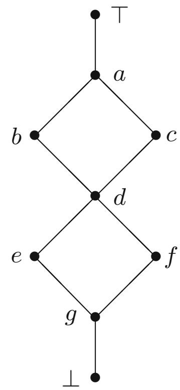

Fig. 2. Definition of negation operators appearing in $\mathbb{P}$ and the Hasse diagram of $(L, \preceq)$.

From this multi-adjoint normal lattice, the following MANLP $\mathbb{P}$ is defined:

$$\begin{array}{lll} r_1: & \langle p \leftarrow q \,\&\, \neg(\sim r) \;; & b\rangle \\ r_2: & \langle \sim p \leftarrow r \;; & d\rangle \\ r_3: & \langle q \leftarrow s \;; & e\rangle \\ r_4: & \langle r \leftarrow \top \;; & f\rangle \\ r_5: & \langle s \leftarrow \top \;; & d\rangle \end{array}$$

whose least model is given by the interpretation $M_{\mathbb{P}}$:

$$\begin{array}{ll} M_{\mathbb{P}}(p) = e & M_{\mathbb{P}}(\sim p) = f \\ M_{\mathbb{P}}(q) = e & M_{\mathbb{P}}(\sim q) = \bot \\ M_{\mathbb{P}}(r) = f & M_{\mathbb{P}}(\sim r) = \bot \\ M_{\mathbb{P}}(s) = d & M_{\mathbb{P}}(\sim s) = \bot \end{array}$$

Clearly, the propositional symbols p, q, r and s are coherent since they satisfy the inequality $M_{\mathbb{P}}(\sim x) \preceq \dot{\sim} M_{\mathbb{P}}(x)$ for all $x \in \{p, q, r, s\}$. Therefore, we can conclude that $M_{\mathbb{P}}$ is a coherent interpretation by Definition 8. Hence, the least model of the program $\mathbb{P}$ is coherent which let us to ensure that $\mathbb{P}$ is coherent. Indeed, $\mathbb{M}_1(\mathbb{P}) = \mathbb{M}_2(\mathbb{P}) = \mathbb{M}_3(\mathbb{P}) = 0$.

Now, we will consider a MANLP $\mathbb{P}^*$ which is formed by the rules appearing in $\mathbb{P}$ together with the following rule $r_2^* : \langle p \leftarrow r;\ f\rangle$. In this case, the least model $M_{\mathbb{P}^*}$ of the program $\mathbb{P}^*$ is defined as follows:

$$\begin{array}{ll} M_{\mathbb{P}^*}(p) = d & M_{\mathbb{P}^*}(\sim p) = f \\ M_{\mathbb{P}^*}(q) = e & M_{\mathbb{P}^*}(\sim q) = \bot \\ M_{\mathbb{P}^*}(r) = f & M_{\mathbb{P}^*}(\sim r) = \bot \\ M_{\mathbb{P}^*}(s) = d & M_{\mathbb{P}^*}(\sim s) = \bot \end{array}$$

which is not a coherent interpretation due to Definition 8 is not satisfied. In particular, we obtain that $M_{\mathbb{P}^*}(\sim p) = f \not\preceq g = \dot{\sim}(d) = \dot{\sim} M_{\mathbb{P}^*}(p)$.

Note that, $(M_{\mathbb{P}^*}(p), M_{\mathbb{P}^*}(\sim p)) = (d, f)$ is the unique pair which does not belongs to $\Delta_{\mathcal{L}}$. Therefore, to compute the incoherence measures $\mathbb{M}_2(\mathbb{P}^*)$ and $\mathbb{M}_3(\mathbb{P}^*)$, we will only need to make the computation for the value $d_G((d, f), \Delta_{\mathcal{L}})$. Since $z_{M_{\mathbb{P}^*}(\sim p)} = z_f = \inf\{z \in L \mid \dot{\sim} z \preceq f\} = d$, applying the definition of distance d_G given by Eq. (3), we obtain that

$$\begin{aligned} d_G((d, f), \Delta_{\mathcal{L}}) &= \inf\{d((d, f), (d, x)) \mid x \preceq \dot{\sim}(d)\} \\ &= \inf\{d((d, f), (d, g)), d((d, f), (d, \bot))\} = \inf\{1, 2\} = 1 \end{aligned}$$

Finally, considering the definitions for the incoherence measures with respect to a program, we can conclude that:

$$\begin{aligned} \mathbb{M}_1(\mathbb{P}^*) &= \mathcal{M}_1(M_{\mathbb{P}^*}) = \frac{1}{4} = 0.25 \\ \mathbb{M}_2(\mathbb{P}^*) &= \mathcal{M}_2(M_{\mathbb{P}^*}) = \max\{1, 0, 0, 0\} = 1 \\ \mathbb{M}_3(\mathbb{P}^*) &= \mathcal{M}_3(M_{\mathbb{P}^*}) = \frac{1}{4} = 0.25 \end{aligned}$$

The first incoherence measure shows that the ratio of incoherent propositional symbols in the program $\mathbb{P}^*$ is 25%. The second and third ones provide information on the degree of incoherence of the propositional symbols appearing in $\mathbb{P}^*$. Specifically, the equality $\mathbb{M}_2(\mathbb{P}^*) = 1$ means that any propositional symbol $p \in \mathbb{P}^*$ is either coherent or $(M_{\mathbb{P}^*}(p), M_{\mathbb{P}^*}(\sim p))$ is neighbor of a pair of elements $(x, y) \in L \times L$ satisfying $(x, y) \in \Delta_{\mathcal{L}}$. $\mathbb{M}_3(\mathbb{P}^*) = 0.25$ shows that the average of incoherence in the program from all the propositional symbols.

4 Conclusions and Future Work

Our contribution has been focused on introducing incoherence measures in order to ensure how much coherent a multi-adjoint normal logic program is. First of all, we have presented different distances to measure the degree of incoherence of a pair of elements belonging to a finite multi-adjoint normal lattice. We have presented different ways to obtain an incoherence measure for an interpretation and a given propositional symbol. Finally, these measures have been extended to a multi-adjoint normal logic program whose computations are different depending on whether the program has a least model or does not have it.

In the future, we will apply these measure to real examples, we will compare them with other measures and we will study new measures, if we detect they are needed from the practical examples.

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Enhancing the Expressive Power of Sugeno Integrals for Qualitative Data Analysis

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Abstract. Sugeno integrals are useful for describing families of multiple criteria aggregation functions qualitatively. It is known that Sugeno integrals, as aggregation functions, can be represented by a set of rules. Each rule refers to the same threshold in the conditions about the values of the criteria and in the conclusion pertaining to the value of the integral. However, in the general case we expect rules where several thresholds appear. Some of these rules involving different thresholds can be represented by Sugeno utility functionals where criteria values are rescaled by means of utility functions associated with each criterion. But as shown in this paper, their representation power is quite restrictive. In contrast, we provide evidence to conjecture that the use of disjunctions or conjunctions of Sugeno integrals with utility functions drastically improves the expressive power and that they can capture any aggregation function on a finite scale, understood as piecewise unary aggregation functions.

Keywords: Sugeno integrals · Piecewise unary functions · Rule-based representation

1 Introduction

Sugeno integrals are aggregation functions that return a global evaluation in-between the minimum and the maximum of the combined partial evaluations. They are used in multiple criteria decision making and in decision under uncertainty [5,8,9,15]. They are qualitative aggregation functions because they can be defined on any completely ordered scale. The idea is to use a lattice polynomial (using min and max operations) whereby the importance of each subset of criteria is assessed by means of a monotonic set-function called a capacity. Sugeno integrals include weighted minimum and weighted maximum as particular cases.

The problem of representing a function of several variables with a Sugeno integral is discussed in [15]. More precisely, for a given piece of data composed of a vector of partial evaluations and a global evaluation, the set of Sugeno

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integrals that agree with this piece of data is determined. Moreover, necessary and sufficient conditions are presented.

The problem of eliciting Sugeno integrals agreeing with a set of data has received some attention both from a theoretical and a practical point of view [13,14]. The idea is to define a pair of best upper and lower capacities with importance weights bearing on the same subsets of criteria, corresponding to a pair of Sugeno integrals that enclose the dataset. For each piece of data, this approach computes tightest constraints from above and constraints from below on the capacity needed for representing the dataset. In [13], a general approach to the elicitation of several such families of Sugeno integrals is proposed in cases in which the data are not altogether compatible with a unique family of capacities.

In [7,10] a Sugeno integral S is shown to represent a set of single-threshold if-then rules of the form $x \geq \alpha$ and $y \geq \alpha$ and $\dots z \geq \alpha \Rightarrow S \geq \alpha$, or yet, $x \leq \alpha$ and $y \leq \alpha$ and $\dots z \leq \alpha \Rightarrow S \leq \alpha$. These representations are used to select or reject some alternatives, respectively.

Recently, Sugeno integral has been generalized into Sugeno utility functionals [2] that introduce a utility function for each criterion. In the domain of multiple criteria decision making, this aggregation function can be viewed as the combination of the Sugeno integral and order preserving one-argument maps on each criterion. In [3] the Sugeno utility functional is extended to distributive lattices with more general maps.

In this paper, we take a step beyond the above results by considering disjunctions or conjunctions of Sugeno utility functionals. We claim that this class covers all monotonic piecewise unary functions on finite scales, and can represent multi-threshold rules of the form

$$x \geq \alpha \text{ and } y \geq \beta \text{ and } \dots z \geq \gamma \Rightarrow S \geq \delta \text{ (selection rules);}$$

or yet

$$x \leq \alpha \text{ and } y \leq \beta \text{ and } \dots z \leq \gamma \Rightarrow S \leq \delta \text{ (deletion rules);}$$

The paper is organized as follows: The next section is devoted to the background on Sugeno integrals and the kind of if-then rules they can represent. Section 3 presents necessary and sufficient conditions for a set of rules to be represented with a Sugeno utility functional, that is Sugeno integral on utility functions that modify the value scale of each criterion. The main purpose of Sect. 4 is the extension of Sugeno utility functionals to conjunctive and disjunctive combinations thereof, that capture the class of non-decreasing piecewise unary functions. This class of functions is shown to be very expressive and can capture any aggregation function on a finite scale.

2 Sugeno Integrals and Qualitative Datasets

We use the terminology of multiple criteria decision-making where some objects are evaluated according to criteria. We denote by $C = \{1, \cdots, n\}$ the set of criteria, 2^C the power set and L a totally ordered scale with top 1, bottom 0, and the order-reversing operation denoted by ν (ν is involutive and such that $\nu(1) = 0$ and $\nu(0) = 1$). An object is represented by a vector $x = (x_1, \ldots, x_n)$ where x_i is the evaluation of x according to criterion i.

Sugeno integral. In the definition of Sugeno integral the relative weights of the set of criteria are represented by a capacity (or fuzzy measure) which is a set function $\mu : 2^C \to L$ that satisfies $\mu(\emptyset) = 0$, $\mu(C) = 1$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. The conjugate capacity of μ is defined by $\mu^c(A) = \nu(\mu(A^c))$ where A^c is the complement of A. Sugeno integral was originally defined in [16,17]. The most common definition is as follows:

Definition 1. *The Sugeno integral of a function $x : i \in C \mapsto x_i \in L$ with respect to a capacity $\mu : 2^C \to L$ is defined by:*
$S_\mu(x) = \max_{\alpha \in L} \min(\alpha, \mu(x \geq \alpha))$, *where* $\mu(x \geq \alpha) = \mu(\{i \in C | x_i \geq \alpha\})$.

It can be equivalently written under various forms [6,11,12,16], especially:

$$S_\mu(x) = \max_{A \subseteq \mathcal{C}} \min(\mu(A), \min_{i \in A} x_i) = \min_{A \subseteq \mathcal{C}} \max(\mu(A^c), \max_{i \in A} x_i) \tag{1}$$

Sugeno integrals compatible with a dataset. Let us recall how to elicit a family of Sugeno integrals that are compatible with a given dataset that is a collection of pairs $(x^k, \alpha_k), k = 1, \ldots, N$ where each x^k is a tuple $(x_1^k, \ldots, x_n^k)$ of local evaluations of object k with respect to criteria $i \in C$ and α_k is the global evaluation of object k.

In [15] it is proved that for a given piece of data (x, α) the set of capacities μ such that $S_\mu(x) = \alpha$ is such that $\forall A \subseteq C, \check{\mu}_{x,\alpha}(A) \leq \mu(A) \leq \hat{\mu}_{x,\alpha}(A)$, where $\check{\mu}_{x,\alpha}$ and $\hat{\mu}_{x,\alpha}$ are capacities defined by

$$\check{\mu}_{x,\alpha}(A) = \begin{cases} \alpha \text{ if } \{i | x_i \geq \alpha\} \subseteq A \\ 0 \text{ otherwise} \end{cases} \text{ and } \hat{\mu}_{x,\alpha}(A) = \begin{cases} \alpha \text{ if } A \subseteq \{i | x_i > \alpha\} \\ 1 \text{ otherwise.} \end{cases}$$

Note that $\check{\mu}_{x,\alpha}$ is a necessity measure with respect to the possibility distribution

$$\check{\pi}_{x,\alpha}(i) = \begin{cases} 1 \text{ if } x_i \geq \alpha \\ \nu(\alpha) \text{ otherwise} \end{cases},$$

and $\hat{\mu}_{x,\alpha}(A)$ is a possibility measure with respect to the possibility distribution

$$\hat{\pi}_{x,\alpha}(i) = \begin{cases} 1 \text{ if } x_i \leq \alpha \\ \alpha \text{ otherwise} \end{cases}.$$

It is worth noticing that a capacity μ is compatible with the piece of data (x, α) in the above sense if and only if $\mu(x > \alpha) \leq \alpha$ and $\mu(x \geq \alpha) \geq \alpha$. Note that for the set of compatible μ's to be not empty we need that $\min_{i=1}^n x_i \leq \alpha \leq \max_{i=1}^n x_i$, due to idempotence.

The set of capacities compatible with the dataset $(x^k, \alpha_k)_k$ is the set of capacities μ satisfying $\max_k \check{\mu}_{x^k,\alpha_k} \leq \mu \leq \min_k \hat{\mu}_{x^k,\alpha_k}$. This set of solutions can be empty, even if the set of compatible μ's is not empty for each piece of data. In order to compare $\max_k \check{\mu}_{x^k,\alpha_k}$ and $\min_k \hat{\mu}_{x^k,\alpha_k}$ it is not necessary to calculate their values and to compare them on each subset of criteria. It is proved in [15] that the set of compatible capacities is not empty if and only if for all $\alpha_k < \alpha_l$ we have $\{i|x_i^l \geq \alpha_l\} \not\subseteq \{i|x_i^k > \alpha_k\}$.

Sugeno integral as a set of if-then rules. In [6] it is described how to express if-then rules associated to Sugeno integrals. We have two sorts of rules: selection rules and deletion rules. Their construction is based on the inner qualitative Moebius transform of a capacity μ which is a mapping $\mu_\# : 2^C \to L$ defined by

$$\mu_\#(E) = \mu(E) \text{ if } \mu(E) > \max_{B \subset E} \mu(B) \text{ and 0 otherwise.}$$

A set E such that $\mu_\#(E) > 0$ is called a focal set. The set of focal sets of μ is denoted by $\mathcal{F}(\mu)$. Sugeno integral can be expressed in terms of $\mu_\#$ using Eq. (1) as follows [7]:

$$S_\mu(x) = \max_{E \in \mathcal{F}(\mu)} \min(\mu_\#(E), \min_{i \in E} x_i) = \min_{T \in \mathcal{F}(\mu^c)} \max(\nu(\mu^c_\#(T)), \max_{i \in T} x_i).$$

A selection rule is a rule whose conclusion is of the form $S \geq \alpha$. A deletion rule is a rule whose conclusion is of the form $S \leq \alpha$. A Sugeno integral corresponds to the following rules:

- *Selection rules associated to* S_μ. Each focal set E of μ corresponds to the selection rule:

$$R_E^s : \text{ If } x_i \geq \mu_\#(E) \text{ for all } i \in E \text{ then } S_\mu(x) \geq \mu_\#(E).$$

- *Deletion rules associated to* S_μ. Each focal set T of the conjugate μ^c corresponds to the deletion rule:

$$R_T^e : \text{ If } x_i \leq \nu(\mu^c_\#(T)) \text{ for all } i \in T \text{ then } S_\mu(x) \leq \nu(\mu^c_\#(T)).$$

Note that a Sugeno integral is equivalent to a set of single-thresholded rules. In the following, single-thresholded selection rules will be denoted by $(\bigwedge_{i \in E_j} x_i \geq \delta_j) \Rightarrow S(x) \geq \delta_j$ and single-thresholded deletion rules will be denoted by $(\bigwedge_{i \in T_j} x_i \leq \delta_j) \Rightarrow S(x) \leq \delta_j$.

As Sugeno integrals are idempotent, the set of selection rules of the form $(\bigwedge_{i \in C} x_i \geq \delta_j) \Rightarrow S(x) \geq \delta_j$ or deletion rules of the form $(\bigwedge_{i \in C} x_i \leq \delta_j) \Rightarrow S(x) \leq \delta_j$, is always valid.

Let us denote by $r_i, i \in I$ the rules in a single-thresholded rule set R, and A^{r_i} $i \in I$ the set of criteria involved in the rule r_i and δ_i $i \in I$ the associated threshold. In some cases, we can define a capacity μ with focal sets A^{r_i} such that $\mu_\#(A^{r_i}) = \delta_i$ such that the corresponding Sugeno integral induces R.

Proposition 1. *Any set of single-thresholded selection rules R is representable by a Sugeno integral.*

Proof: A single-thresholded rule set is equivalent to a set of pairs $\{(A^{r_i}, \delta_i), i = 1, \dots, N\}$. Consider the set of rules $R(A) = \{r_i : A^{r_i} \subseteq A\}$. Then define a set-function $\mu : 2^C \to L$, by $\mu(A) = \max\{\delta_i : r_i \in R(A)\}$ and $\mu(C) = 1$. This set-function induced by R is clearly a capacity. Consider a rule r_i, the min-term associated to it is $\min(\mu(A^{r_i}), \min_{i \in A^{r_i}} x_i) \geq \delta_i$. So $\forall r_i \in R, S_\mu(x) \geq \delta_i$ whenever $x \geq \delta_i, i \in A^{r_i}$. Moreover μ is the smallest capacity ensuring these inequalities. □

Note that the set of rules associated to focal sets of μ will provide a minimal representation of the set of selection rules R, deleting the redundant ones. For instance, if R consists of $x_1 \geq \alpha \wedge x_2 \geq \alpha \Rightarrow S \geq \alpha$ and $x_1 \geq \beta \wedge x_2 \geq \beta \Rightarrow S \geq \beta$, where $\{1, 2\} \subset C$ (there are more than two criteria), and $\alpha > \beta$, the second rule is redundant, if we assume they are represented by a Sugeno integral with respect to $\mu : \mu(\{1, 2\}) = \alpha$ (and 0 otherwise), since if $x_1 \geq \beta \wedge x_2 \geq \beta$ then $S_\mu(x) = \min(\alpha, \beta)$.

A similar proposition holds for deletion rules. However, in general rules are multi-thresholded. So, we need to go beyond pure Sugeno integrals to represent them.

3 Generalizing Sugeno Integrals with Utility Functions

In this paper we are going to consider multi-thresholded rules. It is then clear we need to go beyond the mere use of Sugeno integrals. A first generalization is the following:

Definition 2: *The Sugeno utility functional with respect to a capacity μ is $S_{\mu,\varphi}(x) = S_\mu(\varphi(x))$ where $\varphi(x) = (\varphi_1(x_1), \cdots, \varphi_n(x_n))$ and each mapping $\varphi_i : L \to L$ is an increasing function in the wide sense, with limit conditions $\varphi_i(0) = 0$ and $\varphi_i(1) = 1$.*

Note that $S_{\mu,\varphi}(x) = \max\limits_{E \in \mathcal{F}(\mu)} \min(\mu_\#(E), \min\limits_{i \in E} \varphi_i(x_i))$. It is also worth noticing that $S_{\mu,\varphi}$ is not always an idempotent aggregation function. Note that when the value scale L is finite, the effect of function φ_i is essentially one of shrinking the value scale since when φ_i is not the identity, $\varphi_i(L) \subset L$. Despite this remark, Sugeno utility functionals are strictly more expressive than Sugeno integrals, as shown in [1], for instance.

It is easy to figure out that $S_{\mu,\varphi}$ expresses the rules:

$$(\bigwedge\nolimits_{i \in E_j} \varphi_i(x_i) \geq \delta_j) \Rightarrow S_{\mu,\varphi}(x) \geq \delta_j \text{ and } (\bigwedge\nolimits_{i \in T_j} \varphi_i(x_i) \leq \delta_j) \Rightarrow S_{\mu,\varphi}(x) \leq \delta_j.$$

Let α_i be such that $\varphi_i(\alpha_i) = \delta_j$. Then the above single-thresholded rules express multi-thresholded selection rules of the form $(\bigwedge_{i \in E_j} x_i \geq \alpha_i) \Rightarrow S_{\mu,\varphi}(x) \geq \delta_j$. Let us show that any multi-thresholded selection rule can be represented by a Sugeno utility functional.

Example 1: Consider the selection rule $x_1 \geq \alpha$ and $x_2 \geq \beta \Rightarrow S \geq \delta$ with $1 \geq \alpha \geq \beta > \delta \geq 0$, where C contains at least 3 criteria. Define utility functions φ_1, φ_2 such that $\varphi_1(x_1) \geq \delta$ if $x_1 \geq \alpha$ and $\varphi_1(x_1) < \delta$ otherwise; $\varphi_2(x_2) \geq \delta$ if $x_2 \geq \beta$ and $\varphi_2(x_2) < \delta$ otherwise. Then we do have that the single-thresholded rule $\varphi_1(x_1) \geq \delta$ and $\varphi_2(x_2) \geq \delta \Rightarrow S \geq \delta$ is equivalent to the previous multi-thresholded rule. This is because, by construction, $x_1 \geq \alpha$ is equivalent to $\varphi_1(x_1) \geq \delta$, and likewise for x_2. Then we can use a capacity with weight δ assigned to focal set $\{1, 2\}$ and weight 1 assigned to C; the Sugeno utility functional $\max(\min(\varphi_1(x_1), \varphi_2(x_2), \delta), \min_{i \in C} \varphi_i(x_i))$ induces the original multi-thresholded selection rule, provided that, for $i > 2$ we let $\varphi_i(1) = 1$ and $\varphi_i(x_i) = 0$ otherwise.

More generally, consider the selection rule $x_1 \geq \alpha_1$ and $x_2 \geq \alpha_2$ and $\ldots x_\ell \geq \alpha_\ell \Rightarrow S \geq \lambda_j$, with weights on a finite scale $L = \{0, \lambda_1 < \cdots < \lambda_k = 1\}$. We can represent its effect by means of utility functions φ_i such that $\varphi_i(x_i) \in [\lambda_j, 1]$ if $x_i \geq \alpha_i$ and $\varphi_i(x_i) \in [0, \lambda_{j-1}]$ otherwise. The weight λ_j assigned to focal set $\{1, 2, \ldots, \ell\}$ and weight 1 assigned to C; the Sugeno utility functional representing the rule is $\max(\min(\min_{i=1}^{\ell} \varphi_i(x_i), \lambda_j), \min_{i \in C} \varphi_i(x_i))$, provided that, for $i > \ell$ we let $\varphi_i(1) = 1$ and $\varphi_i(x_i) = 0$ otherwise.

Likewise, for the deletion rule, $x_1 \leq \alpha_1$ and $x_2 \leq \alpha_2$ and $\ldots x_\ell \leq \alpha_\ell \Rightarrow S \leq \lambda_j$, we can represent its effect by means of utility functions ψ_i such that $\psi_i(x_i) \in [0, \lambda_j]$ if $x_i \leq \alpha_i$ and $\psi_i(x_i) \in [\lambda_{j+1}, 1]$ otherwise. The weight λ_j assigned to focal set $\{1, 2, \ldots, \ell\}$ and weight 1 assigned to C; the Sugeno utility functional representing the rule is $S_{\mu,\psi}(x) = \min(\max(\max_{i=1}^{\ell} \psi_i(x_i), \lambda_j), \max_{i \in C} \psi_i(x_i))$, provided that, for $i > \ell$ we let $\psi_i(0) = 0$ and $\psi_i(x_i) = 1$ otherwise.

However, it is sometimes impossible to represent the behavior of several rules by a single Sugeno utility functional, because the constraints on the utility functions induced by the rules may be in conflict.

Proposition 2: *Consider two selection rules r^1 and r^2 sharing one criterion x, and of the form "if ... and $x \geq \alpha_i$ and ... then $S \geq \delta_i$ such that $\alpha_1 > \alpha_2$ but $\delta_1 \leq \delta_2$. There is no Sugeno integral with utility functions that can represent both of them.*

Proof: Indeed the utility function φ for criterion x is submitted to the following constraints: for rule 1: $\varphi(x) \geq \delta_1$ if $x \geq \alpha_1$, and $\varphi(x) < \delta_1$ otherwise. For rule 2: $\varphi(x) \geq \delta_2$ if $x \geq \alpha_2$, and $\varphi(x) < \delta_2$ otherwise. But since $\alpha_1 > \alpha_2$, suppose $\alpha_1 > x \geq \alpha_2$. Then the conditions enforce $\varphi(x) < \delta_1$ and $\varphi(x) \geq \delta_2$, which is impossible.

Example 2. Let us consider the rules: $\left\{ \begin{array}{l} \text{if } x_1 \geq \lambda_2 \text{ and } x_2 \geq \lambda_3 \text{ then } S \geq \lambda_3 \\ \text{if } x_2 \geq \lambda_2 \text{ and } x_3 \geq \lambda_2 \text{ then } S \geq \lambda_3 \end{array} \right\}$ where $\lambda_2 < \lambda_3$. Variable x_2 is common to both rules. Due to the first rule, we must add a utility function φ_2 such that $\varphi_2(\lambda_3) \geq \lambda_3$ and $\varphi_2(x_2) < \lambda_3$ if $x_2 < \lambda_3$ (for instance $\varphi_2(x_2) = x_2$). In particular, $\varphi_2(\lambda_2) < \lambda_3$. But according to the other rule, one must have that $\varphi_2(\lambda_2) \geq \lambda_3$, which creates a contradiction.

Note that in the above proposition, if the two rules involve the same criteria ($A^{r_1} = A^{r_2}$) and the thresholds in r^1 for criteria other than x are not less than the thresholds in r^2 for these criteria, then rule r^1 is just a consequence of rule r^2 and can be dropped.

A proposition similar to Proposition 2 would hold for deletion rules.

If the condition in the above proposition is not encountered in a set of selection rules R, that is, $\forall r_1, r_2 \in R, \forall i \in A^{r_1} \cap A^{r_2}$, if $\alpha_i^{r_1} > \alpha_i^{r_2}$ implies $\delta_1 > \delta_2$, then the set of rules can be accounted for by a Sugeno integral based on a capacity μ such that $\mu(A^{r_j}) = \delta_j, r^j \in R$, provided that we delete redundant rules from R.

Example 3. Let us consider the rules: $\left\{\begin{array}{l}\text{if } x_1 \geq \lambda_3 \text{ and } x_2 \geq \lambda_5 \text{ then } S \geq \lambda_4\\ \text{if } x_1 \geq \lambda_2 \text{ and } x_2 \geq \lambda_3 \text{ then } S \geq \lambda_3\end{array}\right\}$ where $\lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$. Both rules involve the same criteria. Observe that the impossibility condition of Proposition 2 is not met. Due to the first rule, we must add a utility function φ_1 such that $\varphi_1(\lambda_3) \geq \lambda_4$ and $\varphi_1(x_1) < \lambda_4$ if $x_1 < \lambda_3$. Due to the second rule, φ_1 must also satisfy $\varphi_1(\lambda_2) \geq \lambda_3$ and $\varphi_1(x_1) < \lambda_3$ if $x_1 < \lambda_2$. For instance, one may choose $\varphi_1(\lambda_1) = \lambda_2; \varphi_1(\lambda_2) = \lambda_3; \varphi_1(\lambda_3) = \lambda_4$. Likewise for attribute 2, φ_2 must also satisfy $\varphi_2(\lambda_5) \geq \lambda_4$ and $\varphi_2(x_2) < \lambda_4$ if $x_2 < \lambda_5$, and $\varphi_2(\lambda_3) \geq \lambda_3$ and $\varphi_2(x_2) < \lambda_3$ if $x_1 < \lambda_3$, for instance $\varphi_2(\lambda_2) = \lambda_2$, $\varphi_2(\lambda_3) = \lambda_3$, $\varphi_2(\lambda_4) = \lambda_3$, $\varphi_2(\lambda_5) = \lambda_4$. Using utility functions we get single-thresholded rules

$$\left\{\begin{array}{l}\text{if } \varphi_1(x_1) \geq \lambda_4 \text{ and } \varphi_2(x_2) \geq \lambda_4 \text{ then } S \geq \lambda_4\\ \text{if } \varphi_1(x_1) \geq \lambda_3 \text{ and } \varphi_2(x_2) \geq \lambda_3 \text{ then } S \geq \lambda_3.\end{array}\right.$$

which can be represented by the single expression $S = \min(\varphi_1(x_1), \varphi_2(x_2), \lambda_4)$.

Here a question arises: what sort of rule sets can be represented with a Sugeno utility functional $S_{\mu,\varphi}$?

The above results suggest that the set of rules must have a locally strict monotonic behavior, in the following sense: Let $R(j)$ be the set of rules where attribute x_j appears. Let Θ_j be the set of thresholds α_i appearing in the rules r^i of $R(j)$ in the form $x_j \geq \alpha_i$, and let $\Gamma(\alpha)$ be the set of conclusion thresholds δ_i for rules r^i such that $\alpha_i = \alpha \in \Theta_j$. Then the multifunction Γ must be strictly monotonic in the sense that $\forall \alpha, \alpha' \in \Theta_j, \alpha > \alpha'$ implies $\min \Gamma(\alpha) > \max \Gamma(\alpha')$. Indeed, note that if there are several conclusion thresholds $\delta \in \Gamma(\alpha)$, corresponding to several rules having the same condition threshold $x_j \geq \alpha$, the utility function for x_j will have to satisfy $\varphi_j(\alpha) \geq \max \Gamma(\alpha)$ and $\varphi_j(x_j) < \min \Gamma(\alpha)$ if $x_j < \alpha$.

4 Combination of Sugeno Utility Functionals

In order to find an aggregation operation that can represent any set of multi-thresholded selection rules, we consider non decreasing functions $f : L^n \to L$ of the form

$$f(x_1, \ldots x_n) = \max_{i \in I} \min(\min_{j \in A_i} \varphi_{ij}(x_j), \delta_i)$$

where L is a finite chain and each mapping $\varphi_{i,j} : L \to L$ is an increasing function such that $\varphi_{i,j}(0) = 0$ and $\varphi_{i,j}(1) = 1$. We call such functions f *piecewise unary functions in disjunctive form* (df-PUF), in the sense that the domain L^n, can be partitioned into subsets where $f(x_1, \dots x_n) = \varphi_{ij}(x_j)$ or is a constant δ_i for some $i \in C$.

The main purpose of this part is to study whether there exists a family of K Sugeno utility functionals S_k such that $f = \vee_{i=1}^{K} S_k$, and to show that any aggregation function g (non-decreasing and such that $g(1, 1, \dots, 1) = 1$ and $g(0, 0, \dots, 0) = 0$) can be expressed in this way. If this is so, we can then find a disjunction of Sugeno utility functionals that accounts for a set of selection rules, and more generally we can hope to learn such aggregation from qualitative data.

On the first issue we can prove the following result:

Proposition 3. *Any df-PUF on a finite domain such that $f(1, 1, \dots, 1) = 1$ and $f(0, 0, \dots, 0) = 0$ is a disjunction of Sugeno utility functionals.*

Proof: First notice that as $f(1, 1, \dots, 1) = 1$, there exists $i \in I$ such that $\delta_i = 1$, and $\forall j \in A_i, \varphi_{ij}(1) = 1$; moreover $\forall j \in A_i, \varphi_{ij}(0) = 0$. It is clear that we can rewrite f as $f(x_1, \dots, x_n) = \max_{i \in I} \max(\min(\min_{j \in A_i} \varphi_{ij}(x_j)), \delta_i), \min_{k \in C} \varphi_{ik}(x_k))$, provided that $\forall k \notin A_i, \varphi_{ik}(1) = 1$ and $\varphi_{ik}(x_k) = 0$ if $x_k < 1$. The inner expression

$$\max(\min(\min_{j \in A_i} \varphi_{ij}(x_j)), \delta_i), \min_{k \in C} \varphi_{ik}(x_k))$$

is a Sugeno utility functional with respect to the capacity μ_i such that A_i is a focal set with $\mu_i(A_i) = \delta_i$ and $\mu_i(C) = 1$ in the case $A_i \neq C$. □

The decomposition of f as $\bigvee_{i \in I} S_{\mu_i, \varphi_i}(x)$ in Proposition 3 is not parsimonious. Some terms inside $\bigvee_{i \in I}$ can be grouped into a single Sugeno utility functional with respect to a more complex capacity by unifying the utility functions for each attribute into a single one. To do so, the idea is that we extract a maximal number of subsets $A_i \subset C$, such that

- whenever $A_i \cap A_{i'} \neq \emptyset$, the utility functions φ_{ij} and $\varphi_{i'j}$ for all $j \in A_i \cap A_{i'}$ must be equal.
- whenever $A_i \subset A_{i'}$, we have that $\delta_i < \delta_{i'}$.

Let I_1 be the maximal subset of indices of terms that can form a Sugeno utility functional using the corresponding subsets of A_i's as described above. The idea is then to apply the same procedure to the remaining $\{A_i : i \in I \setminus I_1\}$, until no index remains in I.

Likewise we can consider *piecewise unary functions in conjunctive form* (cf-PUF), namely expressions such as:

$$f(x_1, \dots x_n) = \min_{i \in I} \max(\max_{j \in A_i} \psi_{ij}(x_j), \gamma_i).$$

Proposition 4: *Any cf-PUF on a finite domain such that $f(1,1,\ldots,1)=1$ and $f(0,0,\ldots,0)=0$ is a conjunction of Sugeno utility functionals.*

Proof: First notice that as $f(0,0,\ldots,0)=0$, there exists $i\in I$ such that $\gamma_i=0$, and $\forall j\in A_i, \psi_{ij}(0)=0$; moreover $\forall j\in A_i, \psi_{ij}(1)=1$. It is clear that we can rewrite f as $f(x_1,\ldots,x_n)=\min_{i\in I}\min(\max(\max_{j\in A_i}\psi_{ij}(x_j), \gamma_i), \max_{k\in C}\psi_{ik}(x_k))$, provided that $\forall k\notin A_i, \psi_{ik}(0)=0$ and $\psi_{ik}(x_k)=1$ if $x_k>0$. The inner expression

$$\min(\max(\max_{j\in A_i}\psi_{ij}(x_j),\gamma_i),\max_{k\in C}\psi_{ik}(x_k))$$

is a Sugeno utility functional (in conjunctive form) with respect to the capacity μ_i such that A_i^c is a focal set with $\mu_i(A_i^c)=\gamma_i$ (and $\mu_i(\emptyset)=0$). □

Based on these results, we can try to model any aggregation function g completely defined by a n-dimensional table, by means of a disjunction or a conjunction of Sugeno utility functionals. Indeed, we can represent such a table by means of a set of multi-thresholded selection or deletion rules.

Example 4. Let us consider the function $f(x_1,x_2)$ in the table below where the scale is $0<\lambda<1$

$x_2\uparrow x_1\rightarrow$	0	λ	1
1	1	1	1
λ	0	λ	1
0	0	0	λ

We can describe the positive values in position (α_1,α_2) in the table by means of selection rules of the form "if $x_1\geq\alpha_1$ and $x_2\geq\alpha_2$ alors $S\geq\delta$". In our example the following rules are enough:

For output value 1

r^1 $x_2=1\Rightarrow S=1$; (upper line)
r^2 $x_1=1$ and $x_2\geq\lambda\Rightarrow S=1$ (value 1 in line 2).

For output value λ

r^3 $x_1\geq\lambda$ and $x_2\geq\lambda\Rightarrow S\geq\lambda$ (λ in line 2);
r^4 $x_1=1\Rightarrow S\geq\lambda$ (λ in bottom line).

This set of rules can be expressed by means of a df-PUF formed by the maximum of the following terms

- r^1: $\varphi_{12}(x_2)$, with $\varphi_{12}(1)=1$ and $\varphi_{12}(\lambda)<1(\delta_1=1)$.
- r^2: $\min(\varphi_{21}(x_1),\varphi_{22}(x_2))$ with $\varphi_{21}(1)=1$ and $\varphi_{21}(\lambda)<1$; $\varphi_{22}(\lambda)=1$ and $\varphi_{22}(0)=0(\delta_2=1)$.
- r^3: $\min(\varphi_{31}(x_1),\varphi_{32}(x_2),\lambda)$ with $\varphi_{31}(\lambda)\geq\lambda$; $\varphi_{32}(\lambda)\geq\lambda(\delta_3=\lambda)$.
- r^4: $\min(\varphi_{41}(x_1),\lambda)$, with $\varphi_{41}(1)=1$ and $\varphi_{41}(\lambda)=0(\delta_4=\lambda)$.

By construction, the df-PUF that corresponds to the superposition of tables in Fig. 1,

$$\max(\varphi_{12}(x_2), \min(\varphi_{21}(x_1), \varphi_{22}(x_2)), \min(\varphi_{31}(x_1), \varphi_{32}(x_2), \lambda), \min(\varphi_{41}(x_1), \lambda)) \le f(x_1, x_2),$$

i.e., is a lower bound of function f.

1	1	1
≥ 0	≥ 0	≥ 0
0	0	0

0	≥ 0	1
0	≥ 0	1
0	0	0

0	λ	λ
0	λ	λ
0	0	0

0	0	λ
0	0	λ
0	0	λ

Fig. 1. r^1: $\varphi_{12}(x_2)$ r^2: $\min(\varphi_{21}(x_1), \varphi_{22}(x_2))$ r^3: $\min(\varphi_{31}(x_1), \varphi_{32}(x_2), \lambda)$ r^4: $\min(\varphi_{41}(x_1), \lambda)$

It can be checked that the df-PUF acting as a lower bound of f can be made equal to f provided that $\varphi_{12}(\lambda) = 0$.

Let us represent f by the maximum of Sugeno utility functionals. First, rules r^1 and r^4 together correspond to the Sugeno utility functional $S_{\mu,\varphi}(x)$ where $\mu_\#(1) = \lambda$, $\mu_\#(2) = 1$, where $\varphi_1(\lambda) = 0 = \varphi_2(\lambda)$, i.e., we can choose $S_{\mu,\varphi}(x) = (\lambda \wedge \varphi_1(x_1)) \vee \varphi_2(x_2)$. Rule r^2 alone corresponds to the Sugeno utility functional $S_{\mu',\varphi'}(x)$ where $\mu_\#(\{1, 2\}) = 1$, and $\varphi'_2(\lambda) = 1$, that is $S_{\mu',\varphi'}(x) = x_1 \wedge \varphi'_2(x_2)$. Note that the utility functions for rule r^3 can be chosen as being the same as those for rule r^2, in which case the term $\min(\varphi_{31}(x_1), \varphi_{32}(x_2), \lambda)$ is subsumed by $S_{\mu',\varphi'}(x)$. It yields the following expression of f:

$$f(x_1, x_2) = S_{\mu,\varphi}(x) \vee S_{\mu',\varphi'}(x) = (\lambda \wedge \varphi_1(x_1)) \vee \varphi_2(x_2) \vee (x_1 \wedge \varphi'_2(x_2)).$$

The values other than 1 in position (α_1, α_2) in the table can also be represented by means of deletion rules of the form "if $x_1 \le \alpha_1$ and $x_2 \le \alpha_2$ then $S \le \delta$". In our example the following rules are enough:

For output value λ

r'^1 $x_2 = 0 \Rightarrow S \le \lambda$; (value λ in line 1)
r'^2 $x_1 \le \lambda$ and $x_2 \le \lambda \Rightarrow S \le \lambda$ (value in line 2).

For output value 0

r'^3 $x_1 = 0$ and $x_2 \le \lambda \Rightarrow S = 0$ (value 0 in column 1);
r'^4 $x_1 \le \lambda$ and $x_2 = 0 \Rightarrow S = 0$ (value 0 in line 1).

This set of rules can be expressed by means of a piecewise unary function formed by the minimum of the following max-terms

- r'^1: $\max(\psi_{12}(x_2), \lambda)$;
- r'^2: $\max(\psi_{21}(x_1), \psi_{22}(x_2), \lambda)$ with $\psi_{21}(\lambda) \le \lambda; \psi_{22}(\lambda) \le \lambda$;
- r'^3: $\max(\psi_{31}(x_1), \psi_{32}(x_2))$ with $\psi_{32}(\lambda) = 0$;
- r'^4: $\max(\psi_{41}(x_1), \psi_{42}(x_2))$ with $\psi_{41}(\lambda) = 0$;

The piecewise unary function that corresponds to the superposition of tables in Fig. 2 verifies

$$\min(\max(\psi_{12}(x_2), \lambda), \max(\psi_{21}(x_1), \psi_{22}(x_2), \lambda),$$

$$\max(\psi_{31}(x_1), \psi_{32}(x_2)), \max(\psi_{41}(x_1), \psi_{42}(x_2))) \geq f(x_1, x_2).$$

It can be checked that this piecewise unary function acting as an upper bound of f becomes equal to f, provided that $\varphi_{12}(\lambda) = 1$, which ensures value 1 on entry $(1, \lambda)$.

1	1	1
≤ 1	≤ 1	≤ 1
λ	λ	λ

1	1	1
λ	λ	1
λ	λ	1

1	1	1
0	≤ 1	1
0	≤ 1	1

1	1	1
≤ 1	≤ 1	1
0	0	1

Fig. 2. r'^1 r'^2 r'^3 r'^4

We can find utility functions in such a way that the upper bound coincides with the function f and can be expressed as a minimum of Sugeno integrals. Namely rules r'^1 and r'^4 can be put together and yield a capacity μ such that $\mu(2) = \lambda$ and $\mu(\{1, 2\}) = 1$. We can unify the utility functions appearing in the max-terms for these rules: $\psi_1(\lambda) = 0$, $\psi_2(\lambda) = 1$; rule r'^3 yields a capacity ν such that $\nu(\{1, 2\}) = 1$. We need utility function $\psi'_2(\lambda) = 0$, while ψ'_1 can be the identity. Rule $r'2$ holds if we use the same utility functions as for rule r'^3. We get:

$$f(x_1, x_2) = \min(S_{\mu,\psi}(x), S_{\nu,\psi'}(x)) = \min(\max(\lambda, \psi_2(x_2)),$$

$$\max(\psi_1(x_1), \psi_2(x_2)), \max(x_1, \psi'_2(x_2)))$$

□

To generalize the approach outlined in the above example we can consider the following steps.

1. Transforming the function into a set of selection (resp. deletion) rules.
2. Expressing each rule as a weighted min-term (resp. max-term) involving unary functions, and building the corresponding df-PUF (resp. cf-PUF).
3. Grouping min-terms (resp. max-terms) into Sugeno utility functionals, by unifying the utility functions for each involved variable.

In order to find the minimal set of selection rules that can represent (a lower bound of) an aggregation function f, we can proceed as follows. Consider $\delta = f(\lambda_1, \ldots, \lambda_n)$ and the Cartesian products of the form $\Lambda_\delta = \times_{i=1}^{n}[\lambda_i, 1]$. We can restrict to the maximal sets of this form in the sense of inclusion, i.e.,

$$\mathcal{K}_\delta = \max_{\subseteq}\{\Lambda_\delta : \delta = f(\lambda_1, \ldots, \lambda_n), (\lambda_1, \ldots, \lambda_n) \in L^n\}.$$

For each such maximal hypercube $\Lambda_\delta \in \mathcal{K}_\delta$ and each $\delta > 0$, we can write the selection rule

$$\bigwedge_{i:\lambda_i>0} x_i \geq \lambda_i \Rightarrow f \geq \delta$$

and construct a max-term following the procedure described earlier in this paper.

For deletion rules, the procedure is similar, but we consider maximal sets of the form $M_\delta = \times_{i=1}^{n}[0, \lambda_i], \delta < 1$ and for each of them, define the deletion rule

$$\bigwedge_{i:\lambda_i<1} x_i \leq \lambda_i \Rightarrow f \leq \delta,$$

and construct a min-term.

While Step 2 of the above procedure is obvious to get a df-PUF (resp. cf-PUF), we again get a non-parsimonious representation. Moreover in order to have an exact representation of the aggregation function f using selection rules only (or deletion rules only), we may need to enforce additional constraints on the utility functions as patent in Example 4. Finally, Step 3 should be more formally defined, as the choice of the groupings of max-terms (resp. min-terms) of the df-PUF (resp.cf-PUF), and the alignment of utility functions so as to form several Sugeno utility functionals to be combined does not seem to be unique. The question of finding a minimal representation of any aggregation function on a finite scale by means of conjunction or a disjunction of Sugeno utility functionals is a matter of further research.

It is interesting to measure the improved expressive power, when going from Sugeno integrals to monotonic aggregation functions on L. For instance in the case when $|L| = 3$ as in Example 4, it is easy to check from Fig. 3 that there are $49 = 7 \times 7$ idempotent aggregation functions, only 9 of which are Sugeno integrals, all of the form $\max(\min(x_1, \mu(1)), \min(x_2, \mu(2)), \min(x_1, x_2))$, with $\mu(1), \mu(2) \in L$.

$x_2 \uparrow x_1 \rightarrow$	0	λ	1
1	$d \geq c$	$\geq \max(\lambda, c)$	1
λ	$c \leq \lambda$	λ	$\geq \max(\lambda, b)$
0	0	$a \leq \lambda$	$b \geq a$

Fig. 3. Aggregation functions on the three-valued scale

5 Conclusion

The main result of this paper is to show that any set of rules involving thresholds acting as lower bounds (resp. upper bounds) on attribute values or global evaluation can be represented by piecewise unary functions that in turn can be expressed in the form of fuzzy conjunctions or disjunctions of Sugeno integrals on suitable transformations of the common attribute scale. We have shown that this family of functions corresponds to monotonic aggregation functions on a finite scale. We have shown how to express such an aggregation function by means of a set of multi-thresholded rules, that in turn can be captured by combination of Sugeno utility functionals.

These results could be applied to learning aggregation operations (hence rules with thresholds) from qualitative data, viewing the latter as a partially defined aggregation table. There is another approach to this problem, based on Sugeno integrals and single-thresholded rules [4,6]. In these papers, the idea is to approach a set of data from above and from below by two standard Sugeno integrals with respect to an upper and a lower capacity, which is not always possible. In this method, there is no utility function. In contrast, an approach based on our result seems to lead to the conjunction or the disjunction of several Sugeno integrals (hence several capacities) and several unary functions acting as utility functions, which may require the tuning of many parameters. However, the latter drawback can be alleviated by searching for a minimal representation, which is a topic for further research.

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The Novel Shape Normalization Operator for Fuzzy Numbers in OFN Notation

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Abstract. In the article, the authors undertook to resolve a burdensome problem in the OFN calculus concerning so-called improper shapes. Although the calculations are possible for all shapes of the numbers in the OFN notation but the interpretation of the numbers of improper shapes has been rather poor and little intuitive. Moreover, indeed, an effective fast calculus in OFN arithmetic has lost some of its reliability due to those shapes, which was indicated by the critics. First the authors presented the definition for the adoption of an order by the created OFN number, which has a significant impact on the results of the calculations. Then they defined a new, unprecedented normalization operator - ShapeNO. For given four coordinates of an OFN number the authors presented all 256 variants of its theoretically possible shapes and normalized all of them, which resulted in only a dozen or so repetitions. This article is another element of the series of related basic studies on the artificial intelligence inspired by nature, where new methods in the OFN area allow creation and development of new meta-heuristic methods of swarm intelligence. Thanks to that operator, arithmetic operations have been simplified, fuzzy input and output data do not cause consternation during the interpretation and the time needed to perform the calculation itself has been shortened.

Keywords: Fuzzy numbers · OFN · Normalization operator

1 Introduction

One of important areas of the computational intelligence is the fuzzy logic. It allows to define and to infer under conditions of naturally imprecise concepts (terms) and literals. The origins of that theory go back to 1960's. The professor of the Columbia University in New York and of Berkeley University in California Lotfii Askar Zadeh is regarded as the author of that theory. He published the paper entitled Fuzzy sets in the journal Information and Control in 1965. He defined the "fuzzy set" term. This allowed to record the uncertainty and perform the operations on imprecise terms [42].

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By generalizing, one can say that fuzzy numbers enable presentation of possible data values as well as abstract values either fully or partially [10,35]. Diverse approaches to that issue can be noticed in the definition of the fuzzy number [21,22,28]. According to some authors, the definition includes both single kernel sets, which are referred to as actual fuzzy numbers, as well as sets with a kernel in form of an interval, which is called a fuzzy interval. In accordance with [3,7,22] the above definitions come down to the following Definition (1). Whereas J. Kacprzyk in his paper [24,40] defined a fuzzy number as a fuzzy set on R with a continuous membership function, and named such a number a normal number [28,30,39].

Definition 1. A fuzzy number is defined as a fuzzy set A specified on a set of real numbers, $A \subseteq R$, for which the membership function:

$$\mu_A : \mathbb{R} \to [0,1] \tag{1}$$

- a fuzzy set A is normal when there is an argument for which it satisfies the following conditions:

$$supp\ \ \mu_A(X) = 1, x \in R \tag{2}$$

- a fuzzy set A is convex when its membership function is convex:

$$\forall_{x,y\in\mathbb{R}}\ \ \forall_{\lambda\in[0,1]}\ \ \mu_A[\lambda x + (1-\lambda)y] \geq min\ \ \mu_A\{\mu_A(x), \mu_A(y)\} \tag{3}$$

- a membership function $\mu_A(x)$ is constant at intervals, has a finite number of discontinuity numbers, i.e. such that the left-hand limit is different from the right-hand limit.

Figure 1 is an example of the interpretation of a fuzzy number. The number A is a negative number, while the numbers C and D are called positive fuzzy numbers, accordingly. The number B is neither positive nor negative, i.e. its support contains zero.

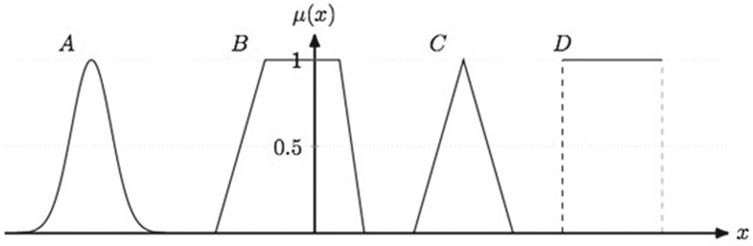
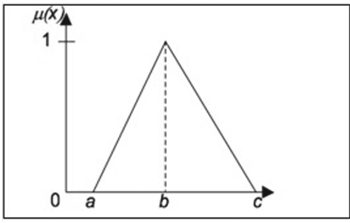


Fig. 1. The example of fuzzy numbers

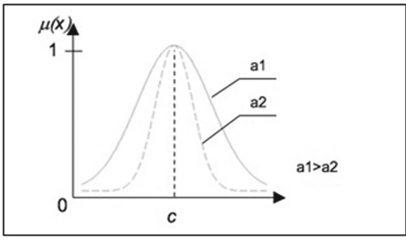
Definition 2. $\mu_A : R[0,1]$ (1) a fuzzy set A is normal when there is an argument for which it satisfies the following conditions:
The fuzzy number $A \subseteq R$ is positive, if:
$\forall x < 0 \ \mu_A(x) = 0$
The fuzzy number $A \subseteq R$ is negative, if:
$\forall x > 0 \ \mu_A(x) = 0$

As various shapes of the membership function are possible, as specified by the authors in their numerous papers [5,12,16,17], the development of matching fuzzy logic arithmetic has become a non-trivial problem. a triangular membership function:



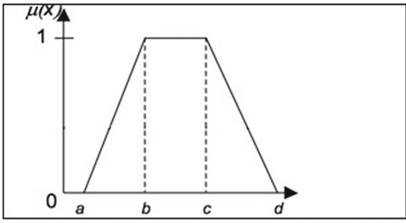
$$\mu_A(x;s,b,c) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ \dfrac{c-x}{c-b}, & b < x \le c \\ 0, & c < x \end{cases}$$

a Gaussian membership function:



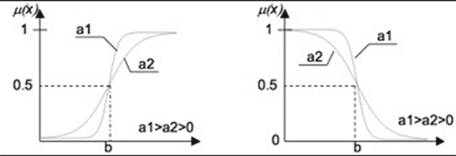
$$\mu_A(x;c,a) = \exp\left(-\frac{1}{2}\left(\frac{x-c}{a}\right)^2\right)$$

a trapezoid membership function:



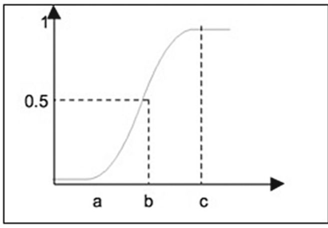
$$\mu_A(x;a,b,c,d) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ 1, & b < x \le c \\ \dfrac{d-x}{d-c}, & c < x \le d \\ 0, & d < x \end{cases}$$

a sigmoidal membership function:



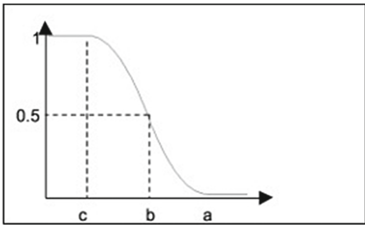
$$\mu_A(x;a,b)=\frac{1}{1+\exp(-a(x-b))}$$

an S class membership function:



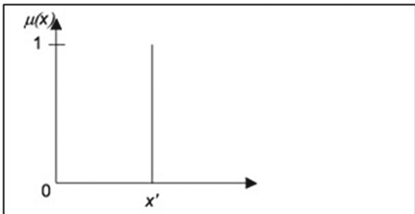
$$\mu_A(x;a,b,c)=\begin{cases}0, & x\ge c\\ 1-2\left(\frac{x-c}{c-a}\right)^2, & c>x\ge b\\ 2\left(\frac{x-a}{c-a}\right)^2, & b>x\ge a\\ 1, & x<a\end{cases};\quad gdzie\quad b=\frac{a+c}{2}$$

a Z class membership function:



$$\mu_A(x;a,b,c)=\begin{cases}0, & x\ge a\\ 2\left(\frac{x-a}{c-a}\right)^2, & a>x\ge b\\ 1-2\left(\frac{x-c}{c-a}\right)^2, & b>x\ge c\\ 1, & x<c\end{cases};\quad gdzie\quad b=\frac{a+c}{2}$$

Singleton (defuzzyfied value):



$$\mu_A(x;x')=\begin{cases}1, & x=x'\\ 0, & x\ne x'\end{cases}$$

Many researchers took up the issue of building an Zadeh's arithmetic enabling to perform calculations for selected shapes of the membership function [41]. One of the most popular arithmetics was the LR one proposed by Dubois and Prade [42]. An interesting approach, i.e. constrained fuzzy arithmetic (CFA) can be found in articles of Klir [13,14] and in the article of Chang and Hung (see [6]). Another approach has been proposed by Piegat and Plucincki in RMD arithmetic [32,33]. However, for the needs of this article, the most important is the OFN arithmetic introduced by Kosinski, Slezak and Prokopowicz [7,25,36], which, in accordance with the intentions of the authors, enables calculations on fuzzy numbers of various membership function shapes [17–20].

2 The OFN Numbers Normalization Operator

However, an ordered fuzzy number is, in accordance with the intention of the creators [25,36], the generalization of shapes and takes the trapezium form (Fig. 2).

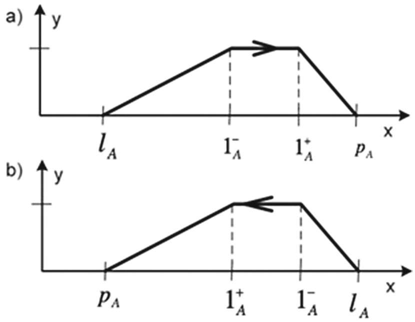


Fig. 2. An ordered fuzzy number with (a) positive order (b) negative order

Definition 3. A membership function of an ordered fuzzy number A is the function μ_A: $R \to [0,1]$ if defined for $x \in R$ as follows:

$$\begin{aligned} x \notin supp_A &\Rightarrow \mu_A(x) = 0 \\ x \in (1_A^-, 1_A^+) &\Rightarrow \mu_A(x) = 1 \\ x \in supp_A \wedge x \notin (1_A^-, 1_A^+) &\Rightarrow \mu_A(x) = max(f_A^{-1}(x), g_A^{-1}(x)) \end{aligned} \quad (4)$$

The above membership function can be used in the control rules similarly to the way in which the membership of classic fuzzy numbers is used [1,2,4,8,11,15,16,29,34,37,38]. All quantities that can be found in fuzzy control describe a selected element of reality [26,27]. The process of determining this value is called a fuzzy observation. Unfortunately, as repeatedly mentioned by the authors $[YXR)$ the arithmetic operations, although they are always possible in the OFN domain, encounter difficulties in interpretation among the representations of numbers which have traditionally been called improper numbers [9,10,23,31]. Improper numbers are for example numbers for which $core(A) > sup(A)$. This means that the support of the number is smaller than the core of the number or sometimes equal to one. The interpretation of this type of numbers is dubious but a number of such shape may be the result of a calculation or an argument for the calculation. To solve problems of that type, the authors of this article proposed the Shape Normalization Operator (SNO).

3 Mathematical Grounds of SNO Operator

Let $\mathcal{A}$ be the space of fuzzy numbers. A fuzzy number $A \in \mathcal{A}$ is described by the four ordered real numbers: a, b, c, d, i.e. $A = (a, b, c, d)$. These numbers are called coordinates of A. For any $a \in R$ the fuzzy number $A = (a, a, a, a)$ is called a singleton. The set of singletons is denoted by S. We distinguish the following subsets P, Q, T of $\mathcal{A}$, defined by

$P = \{A \in \mathcal{A} : \quad a \neq b\}$,
$Q = \{A \in \mathcal{A} : \quad a = b \ \wedge \ a \neq c\}$,
$T = \{A \in \mathcal{A} : \quad a = b = c \ \wedge \ a \neq d\}$.

Of course, the sets P, Q, T and S are pairwise disjoint and moreover family $\mathcal{F} = \{P, Q, T, S\}$ is a partition of $\mathcal{A}$, i.e. $\mathcal{A} = P \cup Q \cup T \cup S$. We define order of fuzzy number in the following way:

an arbitrary $A \in P$ has positive order if $a < b$, otherwise order of A is negative,

an arbitrary $A \in Q$ has positive order if $a < c$, otherwise order of A is negative,

an arbitrary $A \in T$ has positive order if $a < d$, otherwise order of A is negative.

Now, we define the operator $\Psi : \mathcal{A} \longrightarrow \mathcal{A}$ of the form

$$\Psi(A) = \Psi(a, b, c, d) = \begin{cases} (\alpha_1, \alpha_2, \alpha_3, \alpha_4) & \text{if order of A is positive,} \\ (\beta_1, \beta_2, \beta_3, \beta_4) & \text{if order of A is negative.} \end{cases}$$

where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a permutation of (a, b, c, d) such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$,(in other words $\alpha_1 = \min\{a, b, c, d\}$, $\alpha_2 = \min\{\{a, b, c, d\} \setminus \{\alpha_1\}\}$, $\alpha_3 = \min\{\{a, b, c, d\} \setminus \{\alpha_1, \alpha_2\}\}$, $\alpha_4 = \max\{a, b, c, d\}$.) and $(\beta_1, \beta_2, \beta_3, \beta_4)$ is a permutation of (a, b, c, d) such that $\beta_1 \geq \beta_2 \geq \beta_3 \geq \beta_4$, (in other words $\beta_1 = \max\{a, b, c, d\}$, $\beta_2 = \max\{\{a, b, c, d\} \setminus \{\beta_1\}\}$, $\beta_3 = \max\{\{a, b, c, d\} \setminus \{\beta_1, \beta_2\}\}$, $\beta_4 = \min\{a, b, c, d\}$.) The operator Ψ on the set S is the identity i.e. $\forall\ A \in S,\ \ \Psi(A) = A$.

4 The Catalog of OFN Shapes in Their Notation

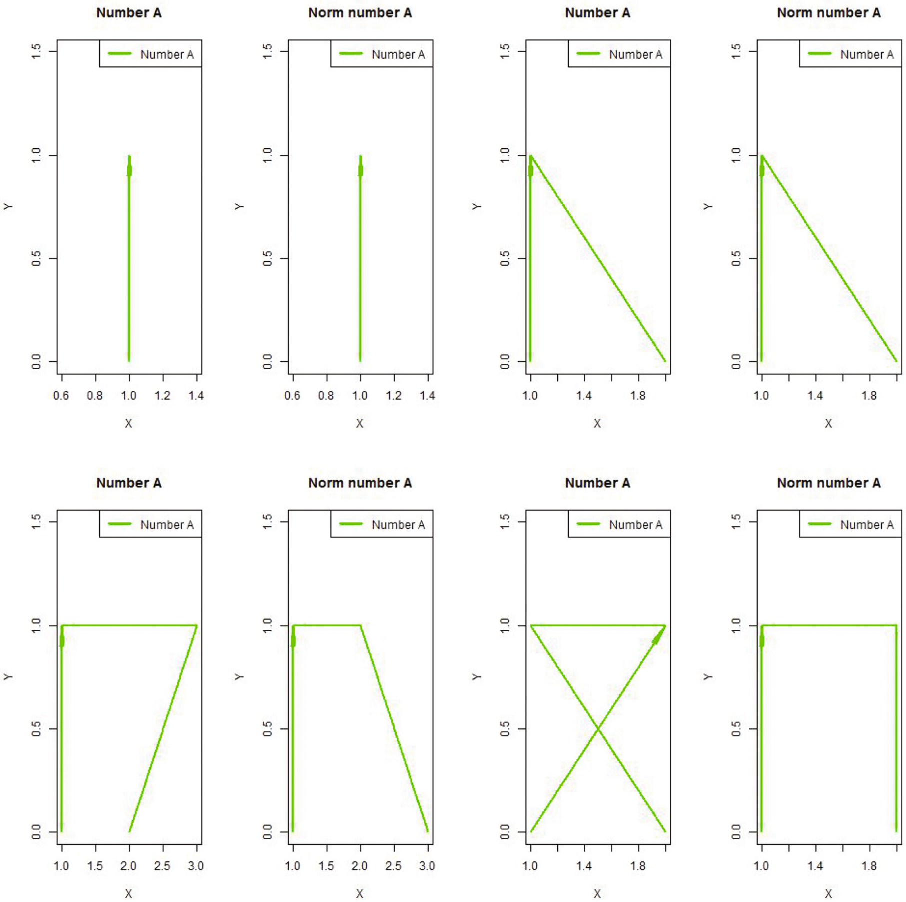
Below is a catalog of all the 256 possible shapes which may be adopted by A(a, b, c, d) number in OFN notation. However, as stated in the paragraph concerning the mathematical background, each of a, b, c, d coordinates belongs to R set of real numbers and their sequence is relevant.

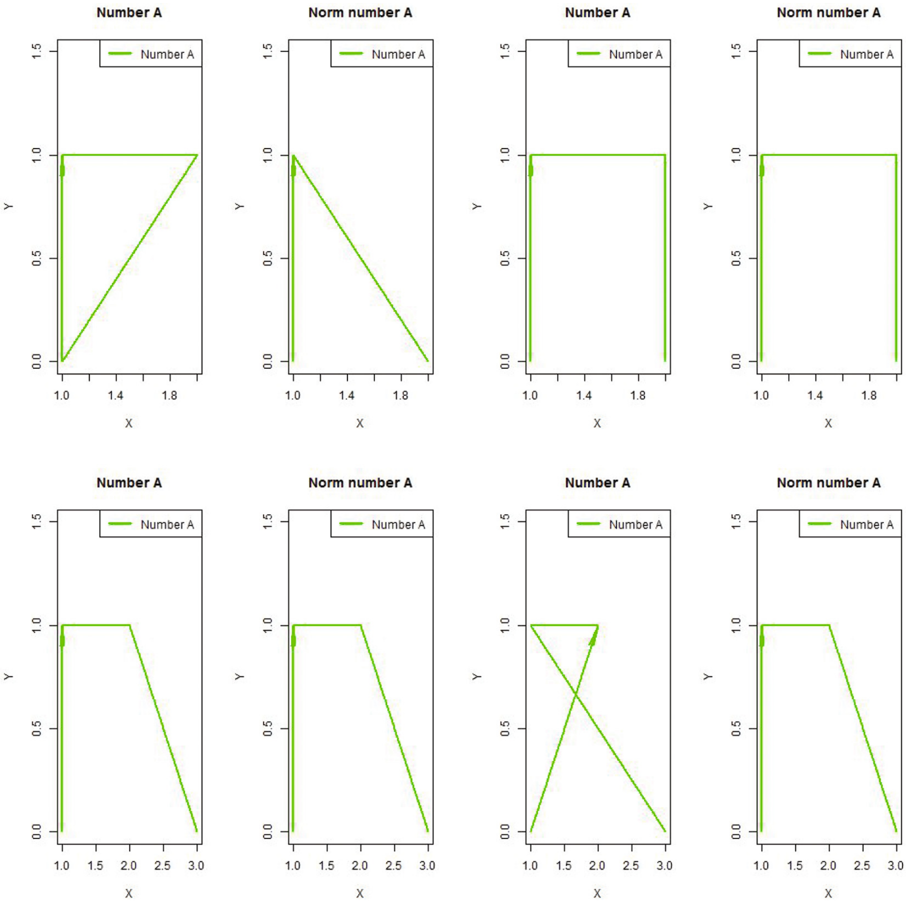
$A_1 = [1,1,1,1]$	$A_2 = [1,1,1,2]$	$A_3 = [1,1,1,3]$	$A_4 = [1,1,1,4]$
$A_5 = [1,1,2,1]$	$A_6 = [1,1,2,2]$	$A_7 = [1,1,2,3]$	$A_8 = [1,1,2,4]$
$A_9 = [1,1,3,1]$	$A_{10} = [1,1,3,2]$	$A_{11} = [1,1,3,3]$	$A_{12} = [1,1,3,4]$
$A_{13} = [1,1,4,1]$	$A_{14} = [1,1,4,2]$	$A_{15} = [1,1,4,3]$	$A_{16} = [1,1,4,4]$
$A_{17} = [1,2,1,1]$	$A_{18} = [1,2,1,2]$	$A_{19} = [1,2,1,3]$	$A_{20} = [1,2,1,4]$
$A_{21} = [1,2,2,1]$	$A_{22} = [1,2,2,2]$	$A_{23} = [1,2,2,3]$	$A_{24} = [1,2,2,4]$
$A_{25} = [1,2,3,1]$	$A_{26} = [1,2,3,2]$	$A_{27} = [1,2,3,3]$	$A_{28} = [1,2,3,4]$
$A_{29} = [1,2,4,1]$	$A_{30} = [1,2,4,2]$	$A_{31} = [1,2,4,3]$	$A_{32} = [1,2,4,4]$
$A_{33} = [1,3,1,1]$	$A_{34} = [1,3,1,2]$	$A_{35} = [1,3,1,3]$	$A_{36} = [1,3,1,4]$
$A_{37} = [1,3,2,1]$	$A_{38} = [1,3,2,2]$	$A_{39} = [1,3,2,3]$	$A_{40} = [1,3,2,4]$
$A_{41} = [1,3,3,1]$	$A_{42} = [1,3,3,2]$	$A_{43} = [1,3,3,3]$	$A_{44} = [1,3,3,4]$
$A_{45} = [1,3,4,1]$	$A_{46} = [1,3,4,2]$	$A_{47} = [1,3,4,3]$	$A_{48} = [1,3,4,4]$
$A_{49} = [1,4,1,1]$	$A_{50} = [1,4,1,2]$	$A_{51} = [1,4,1,3]$	$A_{52} = [1,4,1,4]$
$A_{53} = [1,4,2,1]$	$A_{54} = [1,4,2,2]$	$A_{55} = [1,4,2,3]$	$A_{56} = [1,4,2,4]$
$A_{57} = [1,4,3,1]$	$A_{58} = [1,4,3,2]$	$A_{59} = [1,4,3,3]$	$A_{60} = [1,4,3,4]$
$A_{61} = [1,4,4,1]$	$A_{62} = [1,4,4,2]$	$A_{63} = [1,4,4,3]$	$A_{64} = [1,4,4,4]$
$A_{65} = [2,1,1,1]$	$A_{66} = [2,1,1,2]$	$A_{67} = [2,1,1,3]$	$A_{68} = [2,1,1,4]$
$A_{69} = [2,1,2,1]$	$A_{70} = [2,1,2,2]$	$A_{71} = [2,1,2,3]$	$A_{72} = [2,1,2,4]$
$A_{73} = [2,1,3,1]$	$A_{74} = [2,1,3,2]$	$A_{75} = [2,1,3,3]$	$A_{76} = [2,1,3,4]$
$A_{77} = [2,1,4,1]$	$A_{78} = [2,1,4,2]$	$A_{79} = [2,1,4,3]$	$A_{80} = [2,1,4,4]$
$A_{81} = [2,2,1,1]$	$A_{82} = [2,2,1,2]$	$A_{83} = [2,2,1,3]$	$A_{84} = [2,2,1,4]$
$A_{85} = [2,2,2,1]$	$A_{86} = [2,2,2,2]$	$A_{87} = [2,2,2,3]$	$A_{88} = [2,2,2,4]$
$A_{89} = [2,2,3,1]$	$A_{90} = [2,2,3,2]$	$A_{91} = [2,2,3,3]$	$A_{92} = [2,2,3,4]$
$A_{93} = [2,2,4,1]$	$A_{94} = [2,2,4,2]$	$A_{95} = [2,2,4,3]$	$A_{96} = [2,2,4,4]$
$A_{97} = [2,3,1,1]$	$A_{98} = [2,3,1,2]$	$A_{99} = [2,3,1,3]$	$A_{100} = [2,3,1,4]$
$A_{101} = [2,3,2,1]$	$A_{102} = [2,3,2,2]$	$A_{103} = [2,3,2,3]$	$A_{104} = [2,3,2,4]$
$A_{105} = [2,3,3,1]$	$A_{106} = [2,3,3,2]$	$A_{107} = [2,3,3,3]$	$A_{108} = [2,3,3,4]$
$A_{109} = [2,3,4,1]$	$A_{110} = [2,3,4,2]$	$A_{111} = [2,3,4,3]$	$A_{112} = [2,3,4,4]$
$A_{113} = [2,4,1,1]$	$A_{114} = [2,4,1,2]$	$A_{115} = [2,4,1,3]$	$A_{116} = [2,4,1,4]$
$A_{117} = [2,4,2,1]$	$A_{118} = [2,4,2,2]$	$A_{119} = [2,4,2,3]$	$A_{120} = [2,4,2,4]$
$A_{121} = [2,4,3,1]$	$A_{122} = [2,4,3,2]$	$A_{123} = [2,4,3,3]$	$A_{124} = [2,4,3,4]$
$A_{125} = [2,4,4,1]$	$A_{126} = [2,4,4,2]$	$A_{127} = [2,4,4,3]$	$A_{128} = [2,4,4,4]$
$A_{129} = [3,1,1,1]$	$A_{130} = [3,1,1,2]$	$A_{131} = [3,1,1,3]$	$A_{132} = [3,1,1,4]$
$A_{133} = [3,1,2,1]$	$A_{134} = [3,1,2,2]$	$A_{135} = [3,1,2,3]$	$A_{136} = [3,1,2,4]$
$A_{137} = [3,1,3,1]$	$A_{138} = [3,1,3,2]$	$A_{139} = [3,1,3,3]$	$A_{140} = [3,1,3,4]$
$A_{141} = [3,1,4,1]$	$A_{142} = [3,1,4,2]$	$A_{143} = [3,1,4,3]$	$A_{144} = [3,1,4,4]$
$A_{145} = [3,2,1,1]$	$A_{146} = [3,2,1,2]$	$A_{147} = [3,2,1,3]$	$A_{148} = [3,2,1,4]$
$A_{149} = [3,2,2,1]$	$A_{150} = [3,2,2,2]$	$A_{151} = [3,2,2,3]$	$A_{152} = [3,2,2,4]$
$A_{153} = [3,2,3,1]$	$A_{154} = [3,2,3,2]$	$A_{155} = [3,2,3,3]$	$A_{156} = [3,2,3,4]$
$A_{157} = [3,2,4,1]$	$A_{158} = [3,2,4,2]$	$A_{159} = [3,2,4,3]$	$A_{160} = [3,2,4,4]$
$A_{161} = [3,3,1,1]$	$A_{162} = [3,3,1,2]$	$A_{163} = [3,3,1,3]$	$A_{164} = [3,3,1,4]$
$A_{165} = [3,3,2,1]$	$A_{166} = [3,3,2,2]$	$A_{167} = [3,3,2,3]$	$A_{168} = [3,3,2,4]$
$A_{169} = [3,3,3,1]$	$A_{170} = [3,3,3,2]$	$A_{171} = [3,3,3,3]$	$A_{172} = [3,3,3,4]$
$A_{173} = [3,3,4,1]$	$A_{174} = [3,3,4,2]$	$A_{175} = [3,3,4,3]$	$A_{176} = [3,3,4,4]$

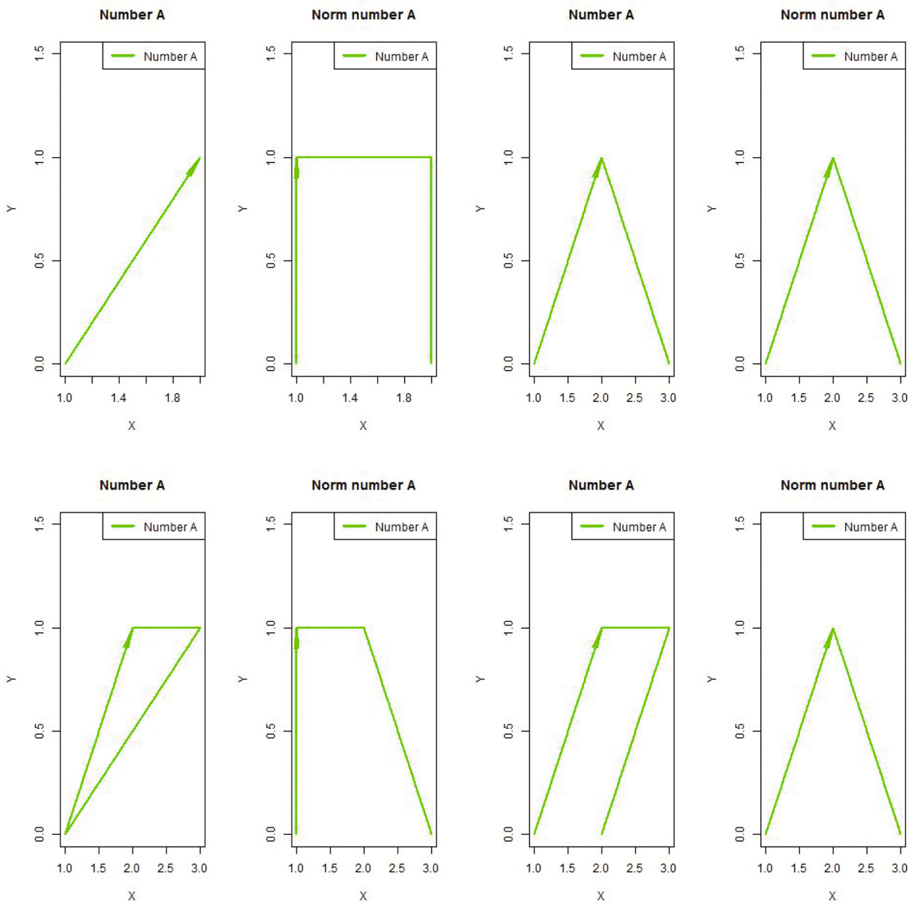
$A_{177} = [3, 4, 1, 1]$ $A_{178} = [3, 4, 1, 2]$ $A_{179} = [3, 4, 1, 3]$ $A_{180} = [3, 4, 1, 4]$
$A_{181} = [3, 4, 2, 1]$ $A_{182} = [3, 4, 2, 2]$ $A_{183} = [3, 4, 2, 3]$ $A_{184} = [3, 4, 2, 4]$
$A_{185} = [3, 4, 3, 1]$ $A_{186} = [3, 4, 3, 2]$ $A_{187} = [3, 4, 3, 3]$ $A_{188} = [3, 4, 3, 4]$
$A_{189} = [3, 4, 4, 1]$ $A_{190} = [3, 4, 4, 2]$ $A_{191} = [3, 4, 4, 3]$ $A_{192} = [3, 4, 4, 4]$
$A_{193} = [4, 1, 1, 1]$ $A_{194} = [4, 1, 1, 2]$ $A_{195} = [4, 1, 1, 3]$ $A_{196} = [4, 1, 1, 4]$
$A_{197} = [4, 1, 2, 1]$ $A_{198} = [4, 1, 2, 2]$ $A_{199} = [4, 1, 2, 3]$ $A_{200} = [4, 1, 2, 4]$
$A_{201} = [4, 1, 3, 1]$ $A_{202} = [4, 1, 3, 2]$ $A_{203} = [4, 1, 3, 3]$ $A_{204} = [4, 1, 3, 4]$
$A_{205} = [4, 1, 4, 1]$ $A_{206} = [4, 1, 4, 2]$ $A_{207} = [4, 1, 4, 3]$ $A_{208} = [4, 1, 4, 4]$
$A_{209} = [4, 2, 1, 1]$ $A_{210} = [4, 2, 1, 2]$ $A_{211} = [4, 2, 1, 3]$ $A_{212} = [4, 2, 1, 4]$
$A_{213} = [4, 2, 2, 1]$ $A_{214} = [4, 2, 2, 2]$ $A_{215} = [4, 2, 2, 3]$ $A_{216} = [4, 2, 2, 4]$
$A_{217} = [4, 2, 3, 1]$ $A_{218} = [4, 2, 3, 2]$ $A_{219} = [4, 2, 3, 3]$ $A_{220} = [4, 2, 3, 4]$
$A_{221} = [4, 2, 4, 1]$ $A_{222} = [4, 2, 4, 2]$ $A_{223} = [4, 2, 4, 3]$ $A_{224} = [4, 2, 4, 4]$
$A_{225} = [4, 3, 1, 1]$ $A_{226} = [4, 3, 1, 2]$ $A_{227} = [4, 3, 1, 3]$ $A_{228} = [4, 3, 1, 4]$
$A_{229} = [4, 3, 2, 1]$ $A_{230} = [4, 3, 2, 2]$ $A_{231} = [4, 3, 2, 3]$ $A_{232} = [4, 3, 2, 4]$
$A_{233} = [4, 3, 3, 1]$ $A_{234} = [4, 3, 3, 2]$ $A_{235} = [4, 3, 3, 3]$ $A_{236} = [4, 3, 3, 4]$
$A_{237} = [4, 3, 4, 1]$ $A_{238} = [4, 3, 4, 2]$ $A_{239} = [4, 3, 4, 3]$ $A_{240} = [4, 3, 4, 4]$
$A_{241} = [4, 4, 1, 1]$ $A_{242} = [4, 4, 1, 2]$ $A_{243} = [4, 4, 1, 3]$ $A_{244} = [4, 4, 1, 4]$
$A_{245} = [4, 4, 2, 1]$ $A_{246} = [4, 4, 2, 2]$ $A_{247} = [4, 4, 2, 3]$ $A_{248} = [4, 4, 2, 4]$
$A_{249} = [4, 4, 3, 1]$ $A_{250} = [4, 4, 3, 2]$ $A_{251} = [4, 4, 3, 3]$ $A_{252} = [4, 4, 3, 4]$
$A_{253} = [4, 4, 4, 1]$ $A_{254} = [4, 4, 4, 2]$ $A_{255} = [4, 4, 4, 3]$ $A_{256} = [4, 4, 4, 4]$

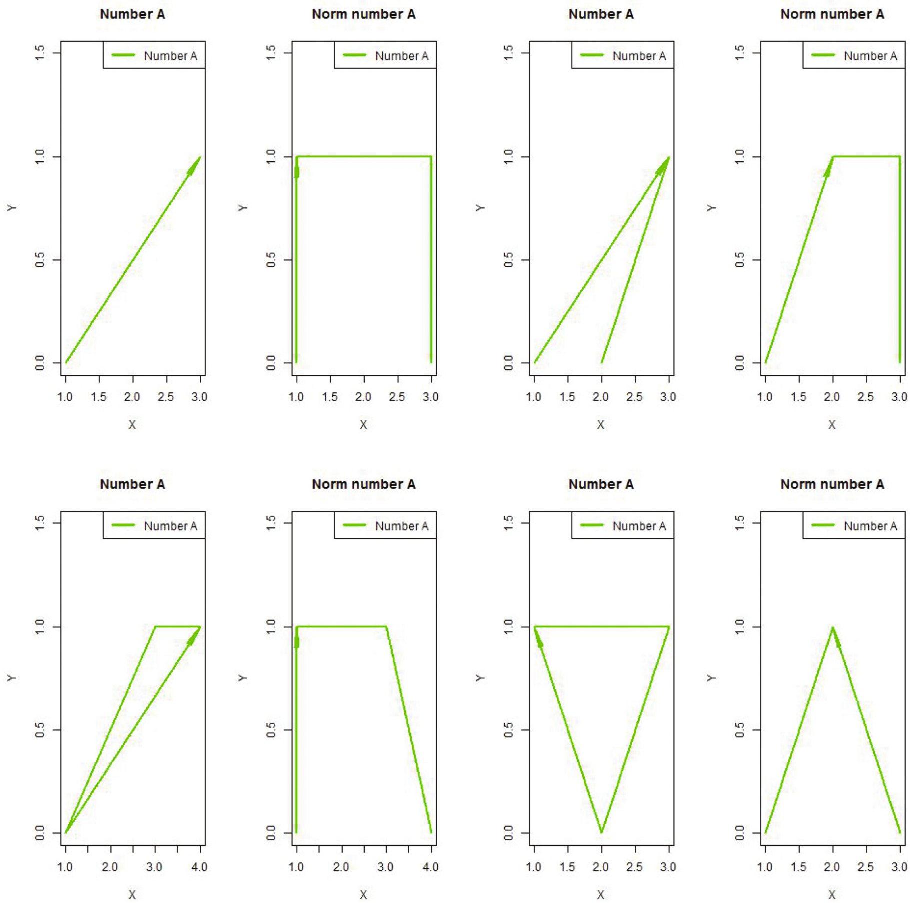
5 The Set of Normalizations for OFN Numbers of Unique Shapes

Below is the list of unique shapes of OFN numbers. Each of them was subject to SNO(A) normalization. As a result, the set of fuzzy numbers in OFN notation was obtained which does not include any improper numbers. Each number is clear and its interpretation is unambiguous.


Number A
Norm number A
Number A
Norm number A
Number A
Norm number A
Number A
Norm number A
Number A
X
Y


Number A
Norm number A
Number A
Norm number A
Number A
Norm number A
Number A
Norm number A
Number A
Y
X


Number A
Norm number A
Number A
Norm number A
Number A
Norm number A
Number A
Norm number A
Number A
X
Y
0.0
0.5
1.0
1.5
1.0
1.4
1.8
1.5
2.0
2.5
3.0



6 Conclusions

As can be seen in the previous paragraph, 256 possible shapes of the number in OFN notation have been analyzed. Such value of combinations of the coordinates is the result of variations with repetitions of a vector with four fields. Half of the shapes of the analyzed numbers were characterized by a positive orientation, while the other half by a negative one. Prior to normalization of the shape of the number, its current order is checked using the formula specified in paragraph 3. Then the normalization itself is performed. The normalization process must lead to achievement of so-called proper shapes from all the shapes of the numbers. As a result of this operation only shapes of numbers for which the core(A) value was not higher than the supp(A) value were obtained. Numbers of that type are suitable for interpretation in all conditions and their accompanying orders in OFN notation allow to carry more information than in other notations. The results of the conducted experiments show that the normalization has influence on 12, 7% of possible numbers. Proper numbers do not change, the same as the singleton. As a rule, the observation of the normalization experiment indicates that numbers are normalized identically irrespective of the order. However, the original order is recorded and assigned to the number after normalization because it carries information on a trend which is desirable and critical for OFN.

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Fuzzy Relations and Fuzzy Functions in Partial Fuzzy Set Theory

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Abstract. In this paper, we will study partially-defined fuzzy relations i.e., fuzzy relations with membership functions that are not necessarily defined everywhere. We will handle them in a suitable framework that is a partial fuzzy set theory. It provides tools for dealing with undefined values of the membership by means of special operations based on the connectives and quantifiers of a background fuzzy logic. We analyze a suitability of operations of a partial fuzzy set theory for a meaningful definition of the functionality property. This property determines fuzzy functions and we will be concerned with their properties.

Keywords: Fuzzy function · Partial function · Undefined values · Functional relation

1 Introduction

Partial fuzzy set theory (PFST) that has been introduced in [1], formalizes fuzzy sets that can have undefined membership degrees and offers basic fuzzy set operations and properties for their handling. Consequently, it provides tools for dealing with undefined values of the membership that may be useful in may practical applications, where the undefinedness is present. The background logic for PFST is the fuzzy partial propositional logic that has been proposed in [2], there, the motivations and explanations for the choice of a logical formalism can be found.

We will touch only a few aspects of the theory. Our purpose is to present first steps towards analysis of fuzzy functions in PFST. In our approach, fuzzy function is a fuzzy relation fulfilling the functionality property—a direct generalization of the classical property that specifies functions out of relations. In the fuzzy community the functionality property has been studied by many authors [3–6] and it is also known as the unique mapping [7]. We will propose the notion of functionality for fuzzy relations with variable domains in agreement with our intuitive expectations. Further, we will study properties of fuzzy functions w.r.t. fuzzy set operations and a relational composition.

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We stem from [8] where the functionality has been explored in the framework of Fuzzy Class Theory [9]. We define the semantics of a simple first-order extension of fuzzy partial propositional logic and a simple theory of partial fuzzy sets of the first order. We next introduce a selection of basic partial fuzzy set—theoretic notions and present a few results about these notions. Because of space limitation, we omit all proofs; they will be given in the upcoming full paper.

2 Partial Fuzzy Sets

We call fuzzy sets that can have undefined membership degrees—*partial fuzzy sets*. We identify partial fuzzy sets with pairs $A = (X_A, \mu_A)$, where $X_A \subseteq U$ is a crisp domain of A, U is a universe of discourse, and $\mu_A\colon X_A \to L$ is a membership function from X_A to a suitable structure L of membership degrees. We denote this fact by $A \subsetneqq\!\!\sim X_A$.

Whenever it is clear from the context, we refer to partial fuzzy sets simply as fuzzy sets. But a reader should keep in mind that from the point of view of the universe U a fuzzy set $A \subsetneqq\!\!\sim X_A$ is partial.

2.1 The Representation of Partial Fuzzy Sets

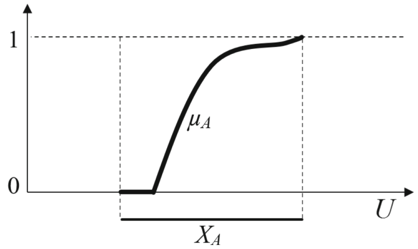
The main idea of the representation of partial fuzzy sets is to replace undefined membership values by a dummy element $\bullet$ that stands outside the scale for truth values L and is incomparable with any $a \in L$. Consequently, original partial membership functions to L of fuzzy sets (with undefined membership values) are replaced by total functions to the extended scale $L \cup \{\bullet\}$ that represent partial fuzzy sets.

Let $A = (X_A, \mu_A)$ be a fuzzy set, where X_A is a crisp domain and $\mu_A\colon X_A \to L$ is a membership function from X_A to a suitable structure L of membership degrees. We introduce a new dummy index $\bullet \notin L$ as a *new membership degree,* designed for undefined membership values. And we set $\mu_A(x) = \bullet$ for all $x \notin X_A$.

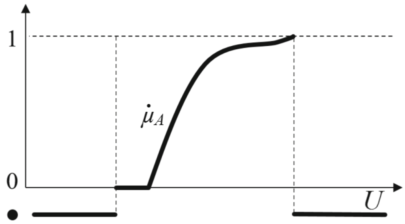
Definition 1. *Let $L \neq \emptyset$ and $L^\bullet = L \cup \{\bullet\}$. We shall say that a fuzzy set $A = (X_A, \mu_A)$ in a universe $U \supseteq X_A$ is* represented *by a $L^\bullet$-valued membership function $\dot{\mu}_A$ on X, defined for each $x \in U$ as:*

$$\dot{\mu}_A(x) = \begin{cases} \mu_A(x) & \text{if } x \in X_A \\ \bullet & \text{if } x \in U \setminus X_A. \end{cases} \tag{1}$$

This representation is clearly one to one correspondence between partial fuzzy sets on the universe U and $L^\bullet$-valued functions on U. Note that the original fuzzy set A (Fig. 1(a)) can be recovered from $\dot{\mu}_A$ (Fig. 1(b)) by setting $X_A = \{x \in U \mid \dot{\mu}_A(x) \neq \bullet\}$ and defining μ_A as the restriction of $\dot{\mu}_A$ to X_A. Therefore we may handle partial fuzzy sets by means of total $L^\bullet$-valued functions on a common universe U.



(a) The fuzzy set $A = (X_A, \mu_A)$.



(b) The representation $\dot{\mu}_A$ for A.

Fig. 1. The representation of the fuzzy set $A = (X_A, \mu_A)$ on a universe $U \supseteq X_A$.

2.2 Operations with Undefined Degrees

Then, a natural question arise: how do we define usual fuzzy set-theoretical operations such as the unions or intersections of fuzzy sets for fuzzy sets on different domains? There are several meaningful options that follows from suitable extensions of connectives used in definitions of the intended operations. In this paper, we will recall only two families of extended connectives descended from classical 3-valued connectives [10–12]. For more explanation and other extensions see [2].

In the sequel, let us assume $\mathcal{L}$ be an MTL-algebra of the form

$$\mathcal{L} = \langle L, \vee, \wedge, *, \Rightarrow, 0, 1 \rangle. \tag{2}$$

We will call the operation $*$ *product* and $\Rightarrow$ *residuum*.

Convention: To reduce the number of parenthesis used in mathematical expressions we set that $*$ has the highest priority and $\Rightarrow$ the lowest priority out of all operations that are at the disposal.

Definition 2. *Let $\mathcal{L}$ be an* MTL-*algebra of the form (2).*

- *The* Bochvar *operation* $c_{\mathrm{B}} \in \{\wedge_{\mathrm{B}}, \vee_{\mathrm{B}}, *_{\mathrm{B}}, \Rightarrow_{\mathrm{B}}\}$, $c_{\mathrm{B}} : L^\bullet \times L^\bullet \to L^\bullet$ *is interpreted by the following truth table for all binary operations of $\mathcal{L}$ (and similarly for higher and lower arities):*

c_{B}	β	$\bullet$
α	$\alpha\, c\, \beta$	$\bullet$
$\bullet$	$\bullet$	$\bullet$

(3)

- *The* Sobociński *operation* $c_{\mathrm{S}} \in \{\wedge_{\mathrm{S}}, \vee_{\mathrm{S}}, *_{\mathrm{S}}\}$, $c_{\mathrm{S}} : L^\bullet \times L^\bullet \to L^\bullet$, *which treat $\bullet$ as the neutral element; and the Sobociński-style residuum $\Rightarrow_{\mathrm{S}}$ residuated with $*_{\mathrm{S}}$:*

c_{S}	β	$\bullet$
α	$\alpha\, c\, \beta$	α
$\bullet$	β	$\bullet$

$\Rightarrow_{\mathrm{S}}$	β	$\bullet$
α	$\alpha \Rightarrow \beta$	$\neg\alpha$
$\bullet$	β	$\bullet$

(4)

The names Bochvar and Sobociński in the above definitions of operations have been chosen according to the interpretation of three-valued connectives (namely, Bochvar and Sobociński conjunction and disjunction, see, e.g., [12]) with which they coincide on the three values $\{0, 1, \bullet\}$.

Definition 3. *Let $\alpha_i \in L^\bullet$ for each $i \in I$ (where I is an arbitrary index set). Then we define:*

– *The* Bochvar infimum $\bigwedge_{i\in I}^{\mathrm{B}} \alpha_i = \begin{cases} \inf_{i\in I} \alpha_i & \text{if } \alpha_i \neq \bullet \text{ for each } i \in I \\ \bullet & \text{otherwise} \end{cases}$

– *The* Bochvar supremum $\bigvee_{i\in I}^{\mathrm{B}} \alpha_i = \begin{cases} \sup_{i\in I} \alpha_i & \text{if } \alpha_i \neq \bullet \text{ for each } i \in I \\ \bullet & \text{otherwise} \end{cases}$

– *The* Sobociński infimum $\bigwedge_{i\in I}^{\mathrm{S}} \alpha_i = \begin{cases} \inf_{\substack{i\in I \\ \alpha_i\neq\bullet}} \alpha_i & \text{if } \alpha_i \neq \bullet \text{ for some } i \in I \\ \bullet & \text{otherwise} \end{cases}$

– *The* Sobociński supremum $\bigvee_{i\in I}^{\mathrm{S}} \alpha_i = \begin{cases} \sup_{\substack{i\in I \\ \alpha_i\neq\bullet}} \alpha_i & \text{if } \alpha_i \neq \bullet \text{ for some } i \in I \\ \bullet & \text{otherwise.} \end{cases}$

Observe that all four operators coincide with the usual supremum and infimum on L if all of their operands are defined. Otherwise, the Bochvar operators $\bigwedge_{\mathrm{B}}, \bigvee_{\mathrm{B}}$ yield the undefined value $\bullet$ as soon as any operands are undefined, while the Sobociński operators $\bigwedge_{\mathrm{S}}, \bigvee_{\mathrm{S}}$ *ignore* the undefined values and only yield $\bullet$ if all of their operands are undefined.

Remark 1. The Bochvar (Sobociński) infimum and supremum correspond with the Bochvar (Sobociński) universal and existential quantifiers, respectively, see [1].

2.3 Unions and Intersections of Fuzzy Partial Sets

Definition 4. *Let us consider two fuzzy sets $A = (X_A, \mu_A)$ and $B = (X_B, \mu_B)$:*

– *We define*

$$A \cup_{\mathrm{B}} B = (X_A \cap X_B, \mu_{A\cup B}) \quad \textit{Bochvar union of } A \textit{ and } B$$
$$A \cap_{\mathrm{B}} B = (X_A \cap X_B, \mu_{A\cap B}) \quad \textit{Bochvar intersection of } A \textit{ and } B$$
$$A \Cap_{\mathrm{B}} B = (X_A \cap X_B, \mu_{A\Cap B}) \quad \textit{Bochvar strong intersection of } A \textit{ and } B$$

where

$$\mu_{A\cup_{\mathrm{B}}B}(x) = \mu_A(x) \vee \mu_B(x), \text{ for all } x \in X_A \cap X_B$$
$$\mu_{A\cap_{\mathrm{B}}B}(x) = \mu_A(x) \wedge \mu_B(x), \text{ for all } x \in X_A \cap X_B$$
$$\mu_{A\Cap_{\mathrm{B}}B}(x) = \mu_A(x) * \mu_B(x), \text{ for all } x \in X_A \cap X_B$$

Bochvar union *is denoted by* $\cup_{\mathrm{B}}$, Bochvar intersection *is denoted by* $\cap_{\mathrm{B}}$ *and* Bochvar strong intersection *is denoted by* $\Cap_{\mathrm{B}}$.

– *We define* Sobociński union *of* A *and* B *is defined by*

$$\begin{aligned} A \cup_{\mathrm{S}} B &= (X_A \cup X_B, \mu_{A \cup_{\mathrm{S}} B}) && \text{Sobociński union } \textit{of } A \textit{ and } B \\ A \cap_{\mathrm{S}} B &= (X_A \cup X_B, \mu_{A \cap_{\mathrm{S}} B}) && \text{Sobociński intersection } \textit{of } A \textit{ and } B \\ A \Cap_{\mathrm{S}} B &= (X_A \cup X_B, \mu_{A \Cap_{\mathrm{S}} B}) && \text{Sobociński strong intersection } \textit{of } A \textit{ and } B \end{aligned}$$

where

$$\begin{aligned} \mu_{A \cup_{\mathrm{S}} B}(x) &= \mu_A(x) \vee \mu_B(x), && \textit{for } x \in X_A \cap X_B, \\ \mu_{A \cap_{\mathrm{S}} B}(x) &= \mu_A(x) \wedge \mu_B(x), && \textit{for } x \in X_A \cap X_B, \\ \mu_{A \Cap_{\mathrm{S}} B}(x) &= \mu_A(x) * \mu_B(x), && \textit{for } x \in X_A \cap X_B, \\ \mu_{A \cup_{\mathrm{S}} B}(x) = \mu_{A \cap_{\mathrm{S}} B}(x) = \mu_{A \Cap_{\mathrm{S}} B}(x) &= \mu_A(x), && \textit{for } x \in X_A \setminus X_B, \\ \mu_{A \cup_{\mathrm{S}} B}(x) = \mu_{A \cap_{\mathrm{S}} B}(x) = \mu_{A \Cap_{\mathrm{S}} B}(x) &= \mu_B(x), && \textit{for } x \in X_B \setminus X_A. \end{aligned}$$

That is, the membership degree of x in $A \cup_{\mathrm{B}} B$ is considered to be defined only if the membership degrees of x in both A and B are defined.

By means of operations given in Definition 2, the representation of Bochvar and Sobociński unions and intersections can be expressed in a uniform way:

$$\dot\mu_{A \cap_{\mathrm{B}} B}(x) = \dot\mu_A(x) \wedge_{\mathrm{B}} \dot\mu_B(x) \tag{5}$$
$$\dot\mu_{A \Cap_{\mathrm{B}} B}(x) = \dot\mu_A(x) *_{\mathrm{B}} \dot\mu_B(x) \tag{6}$$
$$\dot\mu_{A \cup_{\mathrm{B}} B}(x) = \dot\mu_A(x) \vee_{\mathrm{B}} \dot\mu_B(x) \tag{7}$$
$$\dot\mu_{A \cap_{\mathrm{S}} B}(x) = \dot\mu_A(x) \wedge_{\mathrm{S}} \dot\mu_B(x) \tag{8}$$
$$\dot\mu_{A \Cap_{\mathrm{S}} B}(x) = \dot\mu_A(x) *_{\mathrm{S}} \dot\mu_B(x) \tag{9}$$
$$\dot\mu_{A \cup_{\mathrm{S}} B}(x) = \dot\mu_A(x) \vee_{\mathrm{S}} \dot\mu_B(x), \tag{10}$$

for $x \in U$.

Thus, Bochvar and Sobociński operations with fuzzy sets can be defined straightforwardly by means of Bochvar and Sobociński operations with their representations.

2.4 Characteristics of Fuzzy Partial Sets

Characteristics that have been introduced for fixed-domain fuzzy sets can be modified for fuzzy partial sets in a straightforward manner. Since, we have two types of operations at a disposal, consequently, more than one meaningful modification is available. It appears, there are also several characteristics which are meaningful for fuzzy partial sets.

Definition 5. *Let* $A = (X_A, \mu_A)$ *be a fuzzy set,* $X_A \subseteq U$*, and* $\dot\mu_A$ *the representation of* A *due to Definition 1.*

– *We say that* A *is* total *on* U *and write* $\mathrm{Tot}_X(A)$ *(or simply* $\mathrm{Tot}(A)$ *if the universe* U *is fixed) if* $\mathrm{dom}\, A = U$*, i.e., if* $\dot\mu_A(x) \neq \bullet$ *for all* $x \in U$*.*
– *We say that* A *is* crisp *and write* $\mathrm{Crisp}(A)$ *if* $\mu_A(x) \in \{0, 1\}$ *for all* $x \in X_A$*, i.e., if* $\dot\mu_A(x) \in \{0, 1, \bullet\}$ *for all* $x \in U$*.*

Alongside, we will use binary relations between partial fuzzy sets, such as equality and inclusion.

Definition 6. *Let $A = (X_A, \mu_A)$ and $B = (X_B, \mu_B)$ be fuzzy sets, where $\mu_A \colon X_A \to L$ and $\mu_B \colon X_B \to L$, and let $U \supseteq X_A \cup X_B$. Then we say that:*

1. *A and B are* strongly equal, *written $A = B$, if $X_A = X_B$ and $\mu_A(x) = \mu_B(x)$ for all $x \in X_A$.*
2. *A is a* subfunction *of B, written $A =_{sub} B$, if $X_A \subseteq X_B$ and $\mu_A(x) = \mu_B(x)$ for all $x \in X_A$.*
3. *A is* strongly included *in B, written $A \subseteq B$, if $X_A = X_B$ and $\mu_A(x) \le \mu_B(x)$ for all $x \in X_A$.*
4. *A is* subincluded *in B, written $A \subseteq_{sub} B$, if $X_A \subseteq X_B$ and $\mu_A(x) \le \mu_B(x)$ for all $x \in X_A$.*

Remark 2. The relations introduced in Definition 6 are bivalent (yes/no). *Graded* notions of inclusion and equality (cf. [13], [14, Sect. 18.2.2]) can be defined too, e.g., by means of the operators $\bigwedge_{\mathrm{B}}, \bigwedge_{\mathrm{S}}$ and $\Rightarrow_{\mathrm{B}}, \Rightarrow_{\mathrm{S}}$ of Sect. 2.2. For instance, the degree of *Bochvar–Sobociński inclusion* might be defined as:

$$(A \subseteq_{\mathrm{BS}} B) = \bigwedge_{x \in U}{}_{\mathrm{B}} \big(\dot{\mu}_A(x) \Rightarrow_{\mathrm{S}} \dot{\mu}_B(x)\big).$$

In this paper, though, we shall leave graded relations between partial fuzzy sets aside.

3 Fuzzy Relations in PFST

Let us start with the Cartesian product of (two) fuzzy sets. In our apparatus, we set $\dot{\mu}_{A\times B}(x, y) = \bullet$ for (x, y) outside the domain of $A \times B$. As can be easily observed, it is the *Bochvar* extension $*_{\mathrm{B}}$ of the product $*$, which should then be used to make the domain of $A \times B$ equal to $X_A \times X_B$. Therefore we define:

Definition 7. *Let $A = (X_A, \mu_A)$ and $B = (Y_B, \mu_B)$ be fuzzy sets, where $\mu_A \colon X_A \to L$ and $\mu_B \colon Y_B \to L$. Let $U \supseteq X_A \cup Y_B$ and let $\dot{\mu}_A, \dot{\mu}_B \colon U \to L^\bullet$ be the representations of A, B. The* Bochvar Cartesian product *$A \times_{\mathrm{B}} B$ is defined by the representation $\dot{\mu}_{A\times_{\mathrm{B}} B} \colon U \times U \to L^\bullet$ as follows:*

$$\dot{\mu}_{A\times_{\mathrm{B}} B}(x, y) = \dot{\mu}_A(x) *_{\mathrm{B}} \dot{\mu}_B(y) \quad \textit{for each}\,(x, y) \in U \times U. \tag{11}$$

Remark 3. Expanding the definition of $*_{\mathrm{B}}$, we obtain:

$$\dot{\mu}_{A\times_{\mathrm{B}} B}(x, y) = \begin{cases} \mu_A(x) * \mu_B(y) & \text{if}\, x \in X_A \,\text{and}\, y \in Y_B \\ \bullet & \text{otherwise.} \end{cases}$$

Thus, the Bochvar Cartesian product thus captures the usual definition of Cartesian product of fuzzy sets.

Besides the Bochvar product $*_{\mathrm{B}}$, we could also use another extension of $*$ to $L^\bullet$ in (11), for instance the Sobociński product $*_{\mathrm{S}}$.

We can define the notion of binary fuzzy relation between two fuzzy sets as follows:

Definition 8. *Let $A = (X_A, \mu_A)$ and $B = (Y_B, \mu_B)$ be fuzzy sets, and $U \supseteq X_A, Y_B$ a common universe. Let $R = (X_R, \mu_R)$, where $X_R \subseteq U \times U$ and $\mu_R \colon X_R \to L$.*

We say that R is a fuzzy relation between A and B *if $R \subseteq_{sub} A \times_{\mathrm{B}} B$.*

If $A = B$, we speak of fuzzy relations on A.

The notion of binary fuzzy relations between two crisp sets is a special case of the one given in the above definition.

Proposition 1. *Let R, R_1, R_2 be fuzzy relations between fuzzy sets A and B.*

1. *If $A \subseteq_{sub} A'$ and $B \subseteq_{sub} B'$, then R is also a fuzzy relation between A' and B'.*
2. *The fuzzy relations $R_1 \cap_{\mathrm{B}} R_2$, $R_1 \cap_{\mathrm{S}} R_2$, $R_1 \Cap_{\mathrm{B}} R_2$, $R_1 \Cap_{\mathrm{S}} R_2$, are fuzzy relations between A and B as well.*
3. *If A and B are crisp, then analogous claims hold also for $\cup_{\mathrm{B}}, \cup_{\mathrm{S}}$ and $\Cup_{\mathrm{B}}, \Cup_{\mathrm{S}}$.*
4. *$\lambda = (\emptyset, \emptyset)$ is the smallest and $A \times_{\mathrm{B}} B$ the largest fuzzy relation between A and B with respect to subinclusion $\subseteq_{sub}$.*

4 Relational Composition of Fuzzy Relations

Relational composition sup-T of fuzzy relations is intended to have the following values:

$$\dot{\mu}_{R \circ_{\mathrm{SB}} S}(x, y) = \begin{cases} \mu_{R \circ S}(x, y) & \text{for } (x, y) \in A \times C \\ \bullet & \text{for } (x, y) \in (U \times U) \setminus (A \times C) \end{cases} \tag{12}$$

It leads to the following setting of operations: a domain of composition is obtained when using the *Bochvar* operation $*_{\mathrm{B}}$ for combining the degrees of both relations and the *Sobociński* supremum for aggregating them. Hence, we obtain the following generalization of sub-T composition of fuzzy relations:

Definition 9. *Let $R = (A \times B, \mu_R)$ and $S = (B \times C, \mu_S)$, where $\mu_R \colon A \times B \to L$ and $\mu_S \colon B \times C \to L$ be fuzzy relations and $U \supseteq A \cup B \cup C$ a common universe. Let $\dot{\mu}_R, \dot{\mu}_S \colon U \times U \to L^{\bullet}$ be the representation of R, S as in Definition 1. Then, we define the* Sobociński–Bochvar sup-T composition $R \circ_{\mathrm{SB}} S$ *as the fuzzy relation on U such that for all $x, y \in U$:*

$$\dot{\mu}_{R \circ_{\mathrm{SB}} S}(x, y) = \bigvee_{z \in U}^{\mathrm{S}} \big(\dot{\mu}_R(x, z) *_{\mathrm{B}} \dot{\mu}_S(z, y)\big).$$

5 Fuzzy Functions in PFST

In the sequel, let A and B be crisp sets, $U \supseteq A, B$ be a common universe, and $\approx_i = (X_{\approx_i}, \mu_{\approx_i})$, for $i = 1, 2$, be fuzzy relations on A, B, respectively. Moreover, let $R = (X_R, \mu_R)$ be a fuzzy relation between A and B.

We require from the definition of fuzzy function the following:

$$\mu_{\approx_1}(x, x') * \mu_R(x, y) * \mu_R(x', y') \leq \mu_{\approx_2}(y, y'), \tag{13}$$

for all $(x, x') \in X_{\approx_1}$, $(y, y') \in X_{\approx_2}$ and $(x, y), (x', y') \in X_R$.

This specification leads to the definition using the representation of fuzzy sets which combines Sobociński infimum and Bochvar operations.

Definition 10. *Let $\dot{\mu}_R, \dot{\mu}_{\approx_1}, \dot{\mu}_{\approx_2} : U \times U \to L^\bullet$ be the representation of $R, \approx_1, \approx_2$ as in Definition 1.*

– *We say that R is a* fuzzy function *between A and B w.r.t. $\approx_1, \approx_2$ if*

$$\bigwedge\nolimits^{\mathrm{S}}_{x,x',y,y' \in U} (\dot{\mu}_{\approx_1}(x, x') *_{\mathrm{B}} \dot{\mu}_R(x, y) *_{\mathrm{B}} \dot{\mu}_R(x', y') \Rightarrow_{\mathrm{B}} \dot{\mu}_{\approx_2}(y, y')) = 1. \tag{14}$$

– *We say that R is a* function *between A and B if R is a fuzzy function between A and B w.r.t. $=, =$, and moreover,* $\mathrm{Crisp}(R)$.

5.1 Set Operations and the Relational Composition with Fuzzy Functions

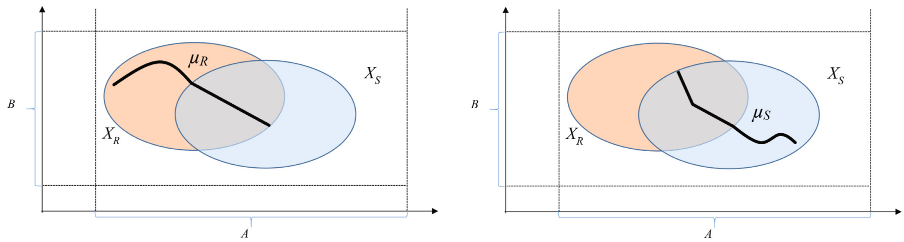
Let us summarize properties of fuzzy functions.

Theorem 1. *Let $R = (X_R, \mu_R)$ and $S = (X_S, \mu_S)$ be fuzzy functions between A and B, respectively, w.r.t. $\approx_1, \approx_2$, moreover, let $T = (X_T, \mu_T)$ be a fuzzy function between B and C w.r.t. $\approx_2, \approx_3$ Then*

– *$R \cap_{\mathrm{B}} S = (X_R \cap X_S, \mu_{R \cap_{\mathrm{B}} S}), R \cap_{\mathrm{S}} S = (X_R \cup X_S, \mu_{R \cap_{\mathrm{S}} S})$ are fuzzy functions between A and B, respectively, w.r.t. $\approx_1, \approx_2$,*
– *$R \Cap_{\mathrm{B}} S = (X_R \cap X_S, \mu_{R \Cap_{\mathrm{B}} S})$ is a fuzzy function between A and B, respectively, w.r.t. $\approx_1 \Cap_{\mathrm{B}} \approx_1, \approx_2 \Cap_{\mathrm{B}} \approx_2$.*

Figure 3 demonstrates Bochvar and Sobociński intersections of crisp functions. Note that $R \Cap_{\mathrm{S}} S = (X_R \cup X_S, \mu_{R \Cap_{\mathrm{S}} S})$ is not a fuzzy function between A and B, respectively, w.r.t. $\approx_1 \Cap_{\mathrm{B}} \approx_1, \approx_2 \Cap_{\mathrm{B}} \approx_2$, because R and S are not fuzzy functions w.r.t. $\approx_1 \Cap_{\mathrm{B}} \approx_1, \approx_2 \Cap_{\mathrm{B}} \approx_2$ on $X_R \setminus X_S$ and $X_S \setminus X_R$, respectively.

We will demonstrate it on a simple example over the standard Łukasiewicz algebra. Put $R = ([0, 0.5]^2, \mu_R)$, $S = ([0, 1]^2, \mu_S)$ and $\approx_1 = \approx_2 = S$, where $\mu_S(x, y) =_{\mathrm{df}} (1 - |x - y|) \vee 0$ and $\mu_R(x, y) = \mu_S(x, y)$ for $x, y \in [0, 0.5]^2$ and $\mu_{\approx_1} = \mu_{\approx_2} = \mu_S$. Then, $R \Cap_{\mathrm{B}} R$ is a fuzzy function between $[0, 1]$ and $[0, 1]$ w.r.t.



(a) A function $R = (X_R, \mu_R)$ (b) A function $S = (X_S, \mu_S)$

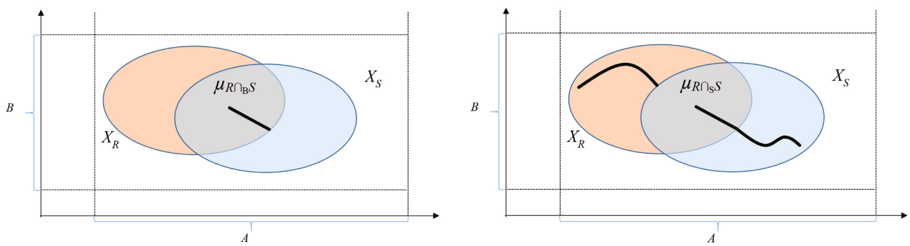
Fig. 2. Functions with different domains.

$\approx_1 \Cap_{\mathrm{B}} \approx_1, \approx_2 \Cap_{\mathrm{B}} \approx_2$. But $R \Cap_{\mathrm{S}} R$ is not a fuzzy function w.r.t. $\approx_1 \Cap_{\mathrm{B}} \approx_1, \approx_2 \Cap_{\mathrm{B}} \approx_2$, take e.g. $x, x' = 0.6, y = 0.6, y' = 0.7$ then

$$\dot{\mu}_{\approx_1}(x, x') *_{\mathrm{B}} \dot{\mu}_{R \Cap_{\mathrm{S}} R}(x, y) *_{\mathrm{B}} \dot{\mu}_{R \Cap_{\mathrm{S}} R}(x', y') \Rightarrow_{\mathrm{B}} \dot{\mu}_{\approx_2}(y, y')$$
$$= 1 *_{\mathrm{B}} 1 *_{\mathrm{B}} 0.9 *_{\mathrm{B}} \bullet \Rightarrow \underbrace{0.9 *_{\mathrm{B}} 0.9}_{0.8} < 1.$$

Theorem 2. *Let the assumptions of Theorem 1 hold. Moreover, let $(y, y') \in \approx_2$ for all x, x', z, z' such that $(x, y), (x', y') \in R$ and $(y, z), (y', z') \in T$.*

Then $R \circ_{\mathrm{SB}} T$ is a fuzzy function between A and C w.r.t. $\approx_1, \approx_3$,



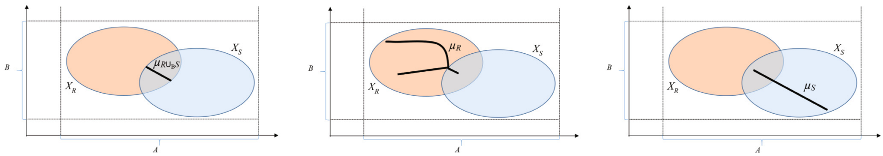
(a) The function $R \cap_{\mathrm{B}} S$ (b) The function $R \cap_{\mathrm{S}} S$

Fig. 3. Intersections of the functions from Fig. 2.

Theorem 3. *Let $R = (X_R, \mu_R)$ and $S = (X_S, \mu_S)$ be fuzzy functions between A and B w.r.t. $\approx_1, \approx_2$.*

If $R \cup_{\mathrm{S}} S$ is a fuzzy function between A and B w.r.t. $\approx_1, \approx_2$ then R and S are fuzzy functions between A and B w.r.t. $\approx_1, \approx_2$.

If $R \cup_{\mathrm{B}} S$ is a fuzzy function between A and B w.r.t. $\approx_1, \approx_2$ then R and S need not be fuzzy functions between A and B w.r.t. $\approx_1, \approx_2$. It is easy to find a crisp counterexample see Fig. 4, where the Bochvar-union $R \cup_{\mathrm{B}} S$ is crisp function but R is not.



(a) The function $R \cup_{\mathrm{B}} S$ (b) A relation R (c) A function S

Fig. 4. Bochvar-union of relations.

6 Conclusions

We have introduced basic notions of the fuzzy set and fuzzy relational calculus in PFST. The main focus was put to the section devoted to fuzzy functions for which we have provided some basic properties and illustrative examples. Let us emphasize the fact that results from the classical fuzzy set theory are not directly transferable to PFST. As an example we recall Theorem 2, where we have to add non-trivial requirements. Moreover, we have shown that there are properties of fuzzy functions that hold in the classical fuzzy set theory but not in PFST for all types of partial fuzzy set operations.

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Medical Fuzzy Control Systems with Fuzzy Arden Syntax

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Abstract. Arden Syntax is a formal language for representing and processing medical knowledge that is employed by knowledge-based medical systems. In HL7 International's Arden Syntax version 2.9 (Fuzzy Arden Syntax), the syntax was extended by formal constructs based on fuzzy set theory and fuzzy logic, including fuzzy control. These concepts are used to model linguistic and propositional uncertainty – which is inherent to medical knowledge – in a variety of clinical situations. Using these fuzzy methods, we can create medical fuzzy control systems (MFCSs), in which linguistic control rules are used and evaluated in parallel. Their results are aggregated so that gradual transitions between otherwise discrete control states are enabled. In this paper, we discuss the implementation of MFCSs in Fuzzy Arden Syntax. Through code examples from FuzzyArdenKBWean, an MFCS for weaning support in mechanically ventilated patients after cardiac surgery, we illustrate the implementation of fuzzy control.

Keywords: Arden Syntax · Fuzzy logic · Fuzzy control · Clinical decision support systems · Weaning from ventilation

1 Introduction

Arden Syntax is a widely known international standard for computerized knowledge representation and processing that supports the collection, description, and processing of medical knowledge in a machine-executable format. With Arden Syntax, medical rules and procedures can be expressed using algorithmic expressions and conditional statements. The rule sets are known as medical logic modules (MLMs) and usually contain sufficient logic to make at least a single medical decision [1]. Due to the fact that Arden Syntax MLMs can be interconnected and invoke each other, modularized packages for certain clinical decision support tasks can be established [2].

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A drawback of earlier versions of Arden Syntax is that the modeling of fuzziness of linguistic clinical terms and uncertainty with respect to clinical conclusions were not intrinsically supported. Such linguistic and propositional uncertainties are, however, inherent to medical knowledge. For example, clinical guidelines are sometimes expressed using linguistic constructs such as “usually” or “often”, which are subject to interpretation and lead to inter-rater variability. The same is true of clinical concepts such as “fever”, “increased glucose levels”, “leukopenia”, and many others. As of version 2.9, Arden Syntax supports formal operators for fuzzy sets and fuzzy logic. Hence we will refer to this version as Fuzzy Arden Syntax [3]. Fuzzy sets can be employed to formally model the unsharpness of linguistic clinical concepts in relation to underlying medical data [4]; fuzzy logic can then be used to evaluate logical combinations of declared clinical concepts in order to draw conclusions about more abstract higher-level clinical concepts, and propagate the results through an inference network.

A number of medical applications have been based on fuzzy sets and fuzzy logic [5, 6]. *Fuzzy control* is of special interest in this paper. Medical fuzzy control systems (MFCSs) are based on linguistic control rules. The rules are evaluated in parallel, and their outcome is aggregated such that small transitions between “on” and “off” are possible. Examples of MFCSs include the control of drug dosages for human immunodeficiency virus and acquired immune deficiency syndrome (HIV/AIDS)-infected patients [7], the control of limb prostheses [8], and the regulation of mechanical ventilators in intensive care units [9].

In the present report, we discuss how fuzzy control is intrinsically supported in Fuzzy Arden Syntax. Using examples from FuzzyArdenKBWean, a system for weaning support in mechanically ventilated patients after cardiac surgery [10], we show how fuzzy sets and fuzzy logic control rules can be implemented in Fuzzy Arden Syntax. These MLMs were implemented and executed using the ArdenSuite framework for medical knowledge representation and rule-based inference, which supports Arden Syntax to version 2.10 [11] (including Fuzzy Arden Syntax).

2 Methods

2.1 Arden Syntax

Arden Syntax is a medical knowledge representation and processing standard with properties that make it well suited for the computerized representation of medical knowledge [12]. In Arden Syntax, the program code resembles natural language, thus healthcare professionals can understand the code more easily. Furthermore, it supports data types tailored to the needs of medical documentation, such as data concerning time and duration. Finally, medical knowledge is separated from technical code, which improves code transparency. We will describe Arden Syntax version 2.9 to the extent that the reader will be able to understand the present report and the examples mentioned therein. For a complete description of the syntax we refer to the Arden Syntax version 2.9 specification [3].

In Arden Syntax, knowledge bases are segmented into MLMs. Each MLM is constructed hierarchically. At the top level, an MLM is divided into four categories:

maintenance, library, knowledge, and resources. These categories, in turn, are divided into slots. The maintenance category contains metadata on the MLM; it includes self-explanatory slots that describe various aspects of the MLM, such as title, author, or version. The library category is meant to provide contextual and background information about the MLM. Using slots such as purpose, explanation, keywords, citations, and links, the MLM author(s) can describe why the MLM was created, what it does, and refer to external sources. The actual implementation of algorithms and rules takes place in the knowledge category. MLM parameters can be declared in the data slot. Apart from input parameters, MLMs can also acquire data from external sources through curly braces expressions, which allow for dynamic interaction between an MLM and the data-providing host system. The MLM algorithms, rules, or program logic expressions are implemented in the logic slot. Other MLMs can also be invoked. Execution in the logic slot is finished with a concluding statement. If the statement proves to be true, the content of the action slot is executed, such as sending data to an external data source or returning a value. Finally, the conditional resources category allows for the construction of localized messages.

2.2 Fuzzy Arden Syntax

In this section, we present a selection of fuzzy extensions implemented in Fuzzy Arden Syntax. This is not a complete overview; for an extended survey of fuzzy methods in Fuzzy Arden Syntax, we refer to previously published work on the subject [13].

An underlying principle of fuzzy methods in Fuzzy Arden Syntax is the extension of the syntax's truth value model. In Fuzzy Arden Syntax, a truth value is defined over a continuous spectrum in a range [0,1] rather than a dichotomous "true/false" model. In this range 0 stands for false, 1 for true, and an intermediate value indicates a degree of truth (or compatibility). Based on this extension, Fuzzy Arden Syntax intrinsically supports fuzziness with data types, built-in propositional fuzzy logic operators, and handling of fuzzy conditions in conditional branches.

With the fuzzy set data type, the unsharpness of boundaries in definitions of linguistic concepts can be conveniently modeled and explicitly calculated. A fuzzy set declaration requires that the boundaries of the fuzzy region be specified. Based on these boundaries, a linear membership function is associated to the variable, which is then able to calculate the truth value of measured data with respect to the clinical linguistic concept under consideration.

Three basic propositional fuzzy logic operations are implemented in Fuzzy Arden Syntax: conjunction, disjunction, and negation. These operations are equipped to handle all truth values in the specified range [0,1]. In Fuzzy Arden Syntax, these operators were implemented by the standard intersection, union, and complement operators *min*, *max*, and *1-x*, respectively [14].

Because of the extended truth value model, it is possible that conditions in conditional branches are neither true nor false. When this happens, the affected conditional branches are executed in parallel; they are also assigned a *degree of applicability* (DoA), which refers to the degree to which it is reasonable to use the value of a variable or set of variables modified or assigned in the respective branch [3]. By default, the DoA equals 1, and is reduced automatically to a weighted average when a program branches on a fuzzy condition; after all, since the condition enabling this program branch was neither true nor false, any values resulting from its execution are relativized accordingly.

In Fuzzy Arden Syntax, the DoA was implemented as follows: When *n* conditional statements are grouped in if-then-elseif statements with fuzzy values as conditions, the execution of the MLM is split into *n* branches, which are executed in parallel. In this process, each branch is provided with its own set of duplicated variables. Furthermore, each branch is assigned a DoA, which is the truth value of its condition divided by the sum of all truth values in the if-then-elseif block.

If, after execution of all conditional branches, the if-then-elseif block is not subsumed using the "*aggregate*" keyword, the different sets of duplicated variables will remain separate and the MLM will conclude with multiple return values, each with a DoA equal to the DoA of its respective program branch. However, if the branches are aggregated, the values of the variables are joined using a weighted average, and only one value for each variable is returned, here with a DoA equal to 1.

As a practical example of using fuzzy constructs in Fuzzy Arden Syntax, consider the MLM code below. Note that we have limited the example to the knowledge category.

```
knowledge:
  type: data_driven;;
  data: (lcnt) := argument;; // Laboratory result
  priority: ;;
  evoke: ;;
  logic:
    // Fuzzy set definitions
    fs_leukopenia := fuzzy set (4000,1), (5000,0);
    fs_leukocytosis := fuzzy set (11000,0), (12000,1);

    // Leukocyte count analysis
    if (lcnt is in fs_leukopenia) or
      (lcnt is in fs_leukocytosis) then
    msg := "Leukocyte count is in pathological range";
    else
      msg := "Leukocyte count is in normal range";
    endif;
    conclude true;;
  action: return msg;;
  urgency: ;;
end:
```

The logic in this MLM is based on infection surveillance criteria defined by the European Centre for Disease Prevention and Control [15]. Surveillance definitions for leukopenia (*4,000 white blood cells (WBC) per mm*^{3} *blood* or less) and leukocytosis (*12,000 WBC/mm*^{3} *blood* or more) are included in these criteria. However, one might argue that patients with measured values close to these thresholds are also of interest. As such, we created fuzzy sets for both clinical concepts that extend beyond the defined thresholds (Fig. 1).

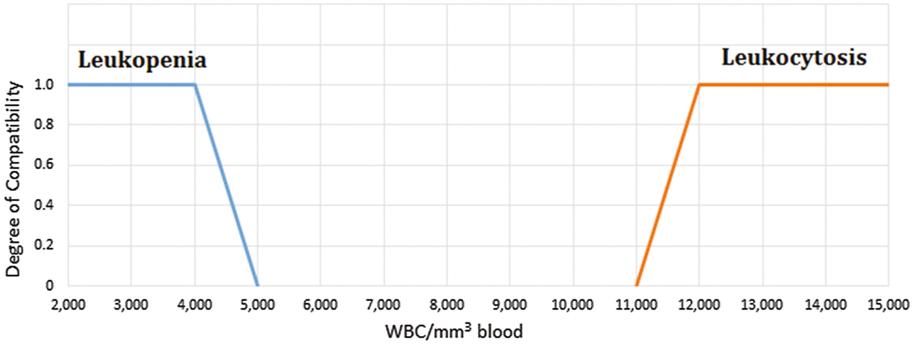

Fig. 1. Graphical depiction of leukopenia and leukocytosis fuzzy sets. Note: WBC, white blood cells.

Truth values are determined during leukocyte count analysis with both fuzzy sets. The truth values are then combined using a logical fuzzy disjunction. In case the outcome is neither true nor false, both conditional branches are executed; as the branches are not aggregated, this would cause the MLM to return two copies of *msg*, each with its own DoA. For example, if the laboratory result were to be *4,400 WBC/mm*^{3}, the outcome of the conditional statement would be *0.6*, due to the evaluation with *fs_leukopenia*. Consequentially, the DoA of that conditional branch would be *0.6,* and for the *else* branch it is automatically *0.4*. As no *aggregate* keyword was provided at the *endif* branch closure, two copies of the *msg* variable are returned: one with a DoA of *0.6* that specifies that "*leukocyte count is in pathological range*", and another with a DoA of *0.4* that states that "*leukocyte count is in normal range*".

2.3 Fuzzy Control and FuzzyArdenKBWean

In fuzzy control, the control strategy is written in "if-then-else" statements similar to natural language rather than using abstract mathematical equations. In these statements, the unsharpness of linguistic terms is represented by fuzzy sets. As a result, transition between control states in fuzzy control systems is more gradual than in traditional control systems.

In general, a fuzzy controller performs the following actions: First, (discrete) system inputs are acquired from the device to be controlled and possibly from additional data sources. The inputs are then interpreted with fuzzy sets (fuzzification) to produce truth values for linguistic concepts. Using the resulting truth values as conditions, all linguistic control rules in the knowledge base are evaluated in parallel, yielding a set of values for each output parameter. In the last processing step, values in each set are aggregated and made discrete again (defuzzification). These discrete results are then either interpreted by the user to manually adjust the system (open-loop system), or fed back into the system itself (closed-loop system).

FuzzyArdenKBWean is an open-loop MFCS that works as described above. This system was developed to improve weaning support in mechanically ventilated patients after cardiac surgery in intensive care units. More specifically, the system optimizes the weaning process (i.e., the transition from full to no ventilation support) by trying to achieve optimal values for arterial oxygen partial pressure (P_aO_2), arterial carbon dioxide partial pressure (P_aCO_2), and the fraction of inspired oxygen (F_iO_2). It is an open-loop, knowledge-based control system that proposes changes to peak inspiratory pressure (PIP), positive end expiratory pressure (PEEP), and F_iO_2, which patient caregivers then implement or, due to reasons unknown to the MFCS, deviate from these proposals.

On average, measured in our tests, FuzzyArdenKBWean reacted 131 min earlier than the attending physicians, with a standard error of mean (SEM) of 47 min. The mean delay in case of hyperventilation was 127 min, (SEM 34); the corresponding value for hypoventilation was 50 min (SEM 21).

2.4 ArdenSuite

In order to obtain examples for the present report, MLMs in FuzzyArdenKBWean were created with the ArdenSuite software [11, 16]. ArdenSuite is a framework for medical knowledge representation, rule-based inference, and extended integration in health IT landscapes. It comprises an integrated development and test environment (ArdenSuite IDE) and the ArdenSuite Server, including software to interconnect with data sources (Fig. 2).

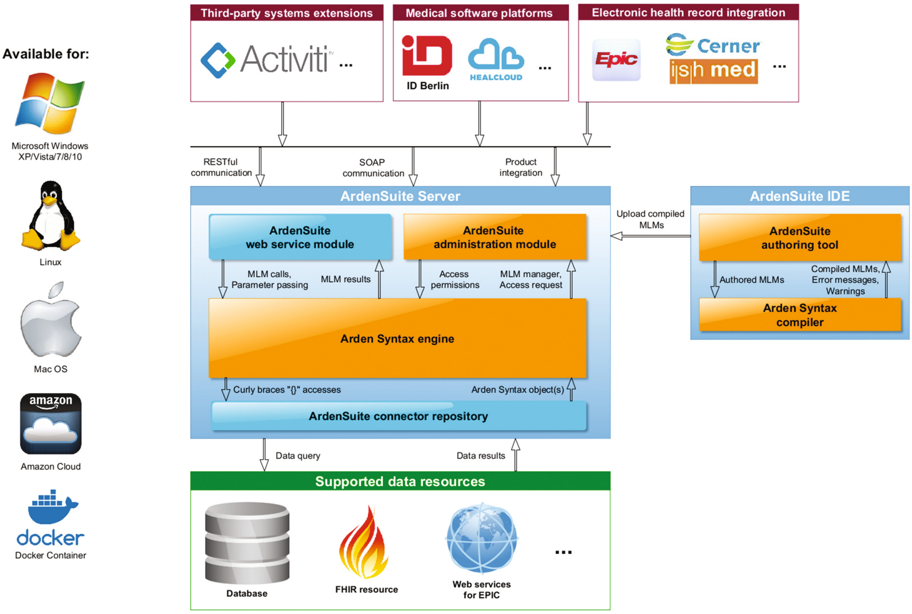

Fig. 2. The ArdenSuite framework for medical knowledge representation, rule-based inference, and health IT integration.

The ArdenSuite IDE includes an authoring component, which allows users to write and compile MLMs. Given the appropriate data, the IDE also allows users to immediately test the written MLMs. After MLMs have been compiled, they are uploaded to the ArdenSuite server. The central element of the ArdenSuite server is the Arden Syntax engine, which executes the compiled MLMs. On top of the engine, an administration module is provided. Functionality reaches from being a repository for compiled Arden Syntax projects to allowing for the management of those projects (such as MLM version management, activation or deactivation of MLMs in an application, or implementing temporal restrictions). Furthermore, the server hosts a web service component that enables service-oriented access to the server by arbitrary clients.

To promote interoperability between the ArdenSuite server and host systems, such as electronic health records (EHRs), the system is provided with several forms of server and data access as well as multiple communication standards. For data exchange and MLM execution, the ArdenSuite server supports various web services. MLM and event calls are realized by Simple Object Access Protocol (SOAP) or Representational State Transfer (REST); the data required for MLM processing can also be provided in this call. Alternatively, data can also be acquired from external databases, Fast Healthcare Interoperability Resources (FHIR) resources, and through web services for EPIC using the ArdenSuite connector repository.

3 Results

The first step in implementing FuzzyArdenKBWean was the construction of fuzzy sets for classifying the inputs in linguistic terms. Based on medical experience, as well as statistical data from prospective randomized trials and archived data, fuzzy sets were

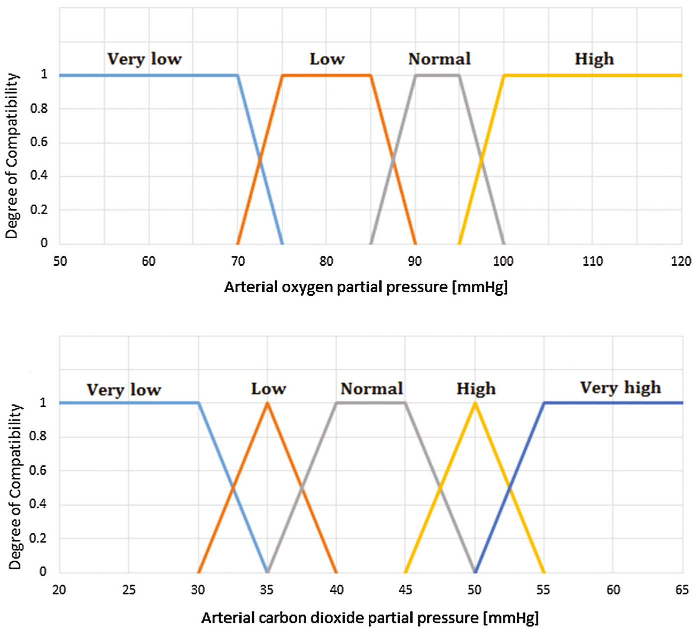

Fig. 3. Graphical depiction of fuzzy sets defined in FuzzyArdenKBWean.

created for linguistic classifications (Very low, Low, Normal, High, and Very high, respectively) for both inputs (Fig. 3).

The fuzzy sets in Fig. 3 were used as conditions of fuzzy control rules in the knowledge base. The following rules were selected for this paper:

R1: IF O_2 IS NORMAL AND CO_2 IS VERY HIGH THEN PIP = +5;
R2: IF O_2 IS LOW AND CO_2 IS VERY HIGH THEN PIP = +5;
R3: IF O_2 IS LOW AND CO_2 IS HIGH THEN PIP = +0;
R4: IF O_2 IS NORMAL AND CO_2 IS HIGH THEN PIP = +0;

Note that despite rules R3 and R4 seem to have no impact, their presence is vital in the control mechanism, as they influence the final PIP increase, depending on the truth value of their condition.

Finally, for defuzzification we used the aggregate command. The resulting (partial) MLM code is shown below. Due to space constraints, we only defined a subset of the fuzzy sets used in the rule selection mentioned earlier, namely those that were used in conditions in this MLM.

```
knowledge:
  type: data-driven;;
  data: (O2, CO2) := argument;; // PaO2 and PaCO2
  priority: ;;
  evoke: ;;
  logic:
    // Fuzzy set definitions
    O2_low := FUZZY SET (70,0), (75,1), (85,1), (90,0);
    O2_normal := FUZZY SET (85,0), (90,1), (95,1), (100,0);
    CO2_high := FUZZY SET (45,0), (50,1), (55,0);
    CO2_very_high := FUZZY SET (50,0), (55,1);

    // Rule analysis
    if (O2 is in O2_normal) and (CO2 is in CO2_very_high) then
      PIP_inc := 5; // R1
    elseif (O2 is in O2_low) and (CO2 is in CO2_very_high) then
      PIP_inc := 5; // R2
    elseif (O2 is in O2_low) and (CO2 is in CO2_high) then
      PIP_inc := 0; // R3
    elseif (O2 is in O2_normal) and (CO2 is in CO2_high) then
      PIP_inc := 0; // R4
    endif aggregate;
    conclude true;
  ;;
  action:
    return PIP_inc;;
  urgency: ;;
end:
```

To clarify the workings of this MLM, let us consider an example. Assume that PaO2 equals *89* and PaCO2 equals *52*. This yields the following truth values for the defined fuzzy sets: (O2_normal, *0.8*), (O2_low, *0.2*), (CO2_high, *0.6*), and (CO2_very_high, *0.4*). Using the standard intersection operator, the truth values for the rule conditions evaluate to: (R1, *min(0.8, 0.4) = 0.4*), (R2, *min(0.2, 0.4) = 0.2*), (R3, *min (0.2, 0.6) = 0.2*), and (R4, *min(0.8, 0.6) = 0.6).* Given that the total sum of truth values for these conditions equals (*0.4 + 0.2 + 0.2 + 0.6 = 1.4*), the weighted average of individual condition truth values, thus the DoA for the conditional branches, is: (R1, *(0.4 /1.4) = 0.285*), (R2, *(0.2 /1.4) = 0.143*), (R3, *(0.2 /1.4) = 0.143*), (R4, *(0.6 / 1.4) = 0.429*).

Finally, the *aggregate* keyword causes a defuzzification of all the program branches and different *PIP_inc* copies, resulting in a single *PIP_inc* value:

$$PIP_{inc} = (0.285 * 5) + (0.143 * 5) + (0.143 * 0) + (0.429 * 0) = 2.14$$

Thus, the program will return the suggestion that the ventilator's PIP should be increased by 2.14 percentage points.

4 Discussion

In the present report, we showed how MFCSs can be implemented using fuzzy methods supported by Arden Syntax version 2.9, an international HL7 standard for computerized knowledge representation and processing that incorporates fuzzy methods. This is important because fuzzy control is used increasingly often in medical devices and systems and has yielded encouraging results [6]. As logical rules are implemented in natural language, clinical experts together with clinical knowledge engineers can implement their expertise quite easily without having to learn complex syntaxes of current programming languages. Uncertainty and incompleteness of knowledge can be modeled by introducing fuzzy sets for linguistic concepts that are part of these rules. Through the ArdenSuite server, with its standardized communication as well as information exchange capabilities, MFCSs can be more easily integrated into small and large healthcare institutions or single medical applications.

In our experience, the implementation of FuzzyArdenKBWean in Fuzzy Arden Syntax yields the MLMs to be clearer and easier to understand in comparison to implementations done with functional programming languages (Delphi or Java). Although clinicians indicated that they were not able to rapidly produce MLMs by themselves, they did find it straightforward and easy to validate written MLMs and identify logical flaws. As such, the cooperation between knowledge engineer and clinician becomes more productive, resulting in high quality medical software.

The limitations of the present report are worthy of note. The study is limited to a single application, FuzzyArdenKBWean. Other forms of fuzzy control in medicine, e.g., those mentioned in [5, 6], need to be studied to see whether those can be implemented in Fuzzy Arden Syntax as well. For example, for defuzzification we used a weighted average based on truth values and the DoA. Other ways, such as centroid methods or mean–max methods need to be studied too. Furthermore, we have not yet fully tested how the program behaves in real time. Such performance is crucial, especially in intensive care units.

We performed the first steps in implementing fuzzy control with Fuzzy Arden Syntax. In the future, we plan to study and address aforementioned limitations, and also apply Fuzzy Arden Syntax for other medical areas and tasks, such as fuzzy automata for real-time monitoring purpose.

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Convolution on Bounded Lattices

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Abstract. The union and intersection of two membership degrees of type-2 fuzzy sets are defined using a generalization of the mathematical operation of convolution. In the literature, it has been deeply studied when these convolution operations constitute a bounded distributive lattice. In this paper, we generalize the union and intersection convolution operations by replacing the functions from $[0,1]$ to itself with functions from a bounded lattice $\mathbb{L}_1$ to a frame $\mathbb{L}_2$, a particular type of bounded lattice. Similarly to some previous studies in the literature, we analyze when these new convolution operations constitute a bounded distributive lattice.

Keywords: Convolution operation · Bounded lattice · Fuzzy logic

1 Introduction

The mathematical operation of convolution is at the basis of mathematical morphology, both for binary [7] and gray-scale images [3]. Specifically, for gray-scale images the dilation and erosion of an image f by a structuring function b are defined as follows:

$$(f \oplus b)(x) = \bigvee_{y \in E} (f(y) + b(x-y))$$

and

$$(f \ominus b)(x) = \bigwedge_{y \in E} (f(y) - b(y-x)),$$

where $\bigvee$ represents supremum and $\bigwedge$ infimum, and E is the set of pixels of f.

Similar convolution operations are at the basis of Zadeh's extension principle. For a function on a composite universe $X = X_1 \times X_2$, the extension principle is defined as

$$f(A_1, A_2)(z) = \bigvee_{f(x_1,x_2)=z} \min(A_1(x_1), A_2(x_2)).$$

Zadeh's extension principle has been successfully used to define union and intersection operations on type-2 fuzzy sets that generalize Zadeh's definition of union and intersection on fuzzy sets [11].

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Let X be the universe of discourse of a type-2 fuzzy set A. The membership degree $A(x)$ (with $x \in X$) is a function from $[0,1]$ to itself, i.e., a function f belonging to the set $\mathcal{F}([0,1],[0,1])$. Hence, the union and intersection convolution operations of two membership degrees are defined as:

$$(f \sqcup g)(x) = \bigvee_{u \vee v = x} f(u) \wedge g(v) := \sup\{f(u) \wedge g(v) \mid u \vee v = x\};$$

and

$$(f \sqcap g)(x) = \bigvee_{u \wedge v = x} f(u) \wedge g(v) := \sup\{f(u) \wedge g(v) \mid u \wedge v = x\},$$

for any $f, g \in \mathcal{F}([0,1],[0,1])$.

Moreover, some relevant studies on the algebra $(\mathcal{F}([0,1],[0,1]), \sqcup, \sqcap)$ show that by considering suitable restrictions on the functions, the union and intersection constitute a bounded distributive lattice [5,6,8]. Some other works study the algebraic/lattice-theoretical properties of similar convolution operations when the functions from $[0,1]$ to itself are replaced by functions from a finite chain to another one [9,10]. However, as far as we know, there is no study about these convolution operations in more general lattice frameworks.

The main purpose of this paper is to study the algebraic laws of the union and intersection convolution operations when we replace the functions of $\mathcal{F}([0,1],[0,1])$ with functions from a bounded lattice to a frame (a particular type of bounded lattice). Specifically, we focus on the algebraic laws of a bounded distributive lattice. We also analyze two different lines of research. On the one hand, the algebraic laws can be ensured by restricting to particular instances of bounded lattices. On the other hand, they can also be ensured by restricting to appropriate subsets of functions that satisfy additional properties.

The structure of the paper is as follows. In Sect. 2 we recall the notion of bounded distributive lattice and introduce the convolution operations. In Sect. 3, we analyze whether or not there exist particular classes of lattices where the union and intersection convolution operations constitute a bounded distributive lattice. Similarly, in Sect. 4 we study which additional properties should be imposed on the functions in order to constitute a bounded lattice. In Sect. 5 we study which subsets of functions are closed under the convolution operations in order to ensure that the operations are well-defined. We finish with some conclusions and future research.

2 Convolution on Bounded Lattices

Let L be a set equipped with two binary operations, in our context usually referred to as *join* ($\vee$) and *meet* $\wedge$, and two elements $0_L \in L$ and $1_L \in L$. Table 1 summarizes some potential algebraic properties of binary operations.

The universal algebra constituted by the set L equipped with join and meet operations that satisfy commutativity, associativity and absorptions laws is

Table 1. Some algebraic properties of binary operations.

The idempotency laws	$a \vee a = a$ for any $a \in L$ $a \wedge a = a$ for any $a \in L$
The commutativity laws	$a \vee b = b \vee a$ for any $a, b \in L$ $a \wedge b = b \wedge a$ for any $a \in L$
The associativity laws	$a \vee (b \vee c) = (a \vee b) \vee c$ for any $a, b, c \in L$ $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ for any $a, b, c \in L$
The absorption laws[a]	$a \vee (a \wedge b) = a$ for any $a, b \in L$ $a \wedge (a \vee b) = a$ for any $a, b \in L$
The identity laws[a]	$a \vee 0_L = a$ for any $a \in L$ $a \wedge 1_L = a$ for any $a \in L$
The distributivity laws[a]	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for any $a, b, c \in L$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$

[a]In order not to overload the notation we only define the algebraic properties for commutative operations.

called a lattice [1,2]. It worth mentioning that a lattice can be also defined as a partially ordered set where each pair of elements has a supremum and an infimum. If it is a linearly ordered set, the lattice is called a chain.

The join and meet operations of a lattice satisfy the idempotency laws as consequence of the two absorption laws. Additionally, if the identity laws are satisfied, the lattice is said to be bounded. Similarly, if the distributivity laws are satisfied the lattice is said to be distributive. A bounded lattice $\mathbb{L} = (L, \vee, \wedge, 0_L, 1_L)$ is said to be complete, if the supremum and infimum of each subset of elements exist. A frame [4] is a complete lattice that satisfies the meet continuity property: for any $a \in L$ and any $\emptyset \subset B \subseteq L$, it holds that

$$a \wedge \left(\bigvee_{b \in B} b \right) = \bigvee_{b \in B} (a \wedge b) \,. \tag{1}$$

It is important to mention that any frame is a distributive bounded lattice. In this paper, we consider a bounded lattice $\mathbb{L}_1 = (L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and a frame $\mathbb{L}_2 = (L_2, \vee_2, \wedge_2, 0_2, 1_2)$, and the corresponding set of functions between them $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2) = \{f \mid f : L_1 \to L_2\}$, called lattice functions further on. Note that $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$ depends on lattices $\mathbb{L}_1$ and $\mathbb{L}_2$, but we will refer to $\mathcal{F}$ without explicitly indicating the lattices.

Let $\mathbb{L}_1$ be a finite lattice. We visualize a function $f : L_1 \to L_2$ by replacing the elements of L_1 in the Hasse diagram of $\mathbb{L}_1$ by their corresponding function values in L_2. For example, let $\mathbb{L}_1 = \mathbb{M}_2$ be the lattice depicted in Fig. 1(a) and

(a) $\mathbb{L}_1$ (b) Function f_1

Fig. 1. (a) Hasse diagram of the lattice $\mathbb{L}_1 = \mathbb{M}_2$ and (b) graphical representation of the function f_1.

let f_1 from $\mathbb{L}_1$ to $\mathbb{L}_2 = ([0,1], \max, \min, 0_2, 1_2)$ be defined as

$$f_1(x) = \begin{cases} 0.2, & \text{if } x = 0_1\,, \\ 0_2, & \text{if } x = x_1\,, \\ 0.5, & \text{if } x = y_1\,, \\ 0.7, & \text{if } x = 1_1\,. \end{cases}$$

The function f_1 is depicted in Fig. 1(b).

The following operations are of major importance in our work.

Definition 1. *The join- and meet-convolution on $\mathcal{F}$ are defined, for any $f, g \in \mathcal{F}$, as:*

$$(f \sqcup g)(x) = \bigvee_{u \vee_1 v = x} f(u) \wedge_2 g(v) := \sup\{f(u) \wedge_2 g(v) \mid u \vee_1 v = x\}\,;$$

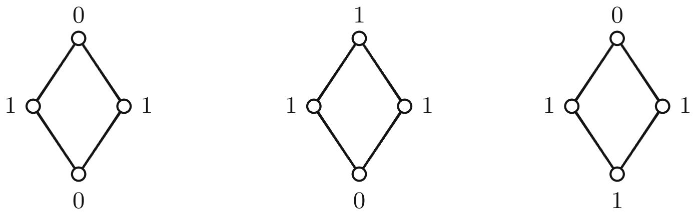
and

$$(f \sqcap g)(x) = \bigvee_{u \wedge_1 v = x} f(u) \wedge_2 g(v) := \sup\{f(u) \wedge_2 g(v) \mid u \wedge_1 v = x\}\,.$$

Note the we are studying the union and intersection convolution operations of two membership degrees of type-2 fuzzy sets [6,8] when the functions from $[0,1]$ to itself are replaced by functions belonging to $\mathcal{F}$. The purpose of this paper is to study when these operations constitute a bounded distributive lattice. In order not to overload the notations and since no confusion can occur, we will drop the subindices 1 and 2 from here on.

Theorem 1. *Let $\mathcal{F}$ be the set of lattice functions. The convolution operations $\sqcup$ and $\sqcap$ satisfy the commutativity and associativity laws.*

In general, the convolution operations on the set of functions $\mathcal{F}([0,1],[0,1])$ satisfy the idempotency laws. However, this no longer holds when $\mathbb{L}_1$ is a bounded lattice.



(a) Function f (b) Function $f \sqcup f$ (c) Function $f \sqcap f$

Fig. 2. Graphical representations of the functions in Example 1: (a) the function f, (b) the join-convolution $f \sqcup f$, and (c) the meet-convolution $f \sqcap f$.

Example 1. Let $\mathbb{L}_1 = \mathbb{M}_2$ and $\mathbb{L}_2 = (\{0, 1\}, \max, \min, 0, 1)$. Consider the function $f \in \mathcal{F}$ depicted in Fig. 2(a). The join- and meet-convolution $f \sqcup f$ and $f \sqcap f$ are depicted in Figs. 2(b)–(c). One easily verifies that $f \neq f \sqcup f$ and $f \neq f \sqcap f$.

Remark 1. To study the remaining of the algebraic laws of a lattice, there are two different lines of research. On the one hand, we can restrict the classes of lattices $\mathbb{L}_1$ and $\mathbb{L}_2$. In this manner, the more restrictive the lattices are, the more algebraic properties the convolution operations satisfy. On the other hand, we can restrict the set of functions $\mathcal{G} \subseteq \mathcal{F}$, only considering functions $g \in \mathcal{G}$ that satisfy some additional properties. These two lines of research are analyzed in Sects. 3 and 4, respectively.

3 Restricting the Lattices

As a first step, we study in which classes of lattices the idempotency laws are satisfied.

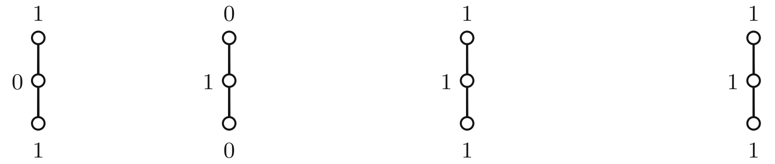
Theorem 2. *The following statements hold:*

(i) the operation $\sqcup$ satisfies the idempotency law on $\mathcal{F}$ if and only if $\mathbb{L}_1$ is chain;
(ii) the operation $\sqcap$ satisfies the idempotency law on $\mathcal{F}$ if and only if $\mathbb{L}_1$ is chain.

Chains are a very special class of lattices, i.e., the restriction in the preceding theorem is a very restrictive condition. Moreover, even if we restrict to chains, the absorption laws are not satisfied as we show in the following example.

Example 2. Let $\mathbb{L}_1 = (\{0, \frac{1}{2}, 1\}, \max, \min, 0, 1)$ and $\mathbb{L}_2 = (\{0, 1\}, \max, \min, 0, 1)$. Consider the functions $f, g \in \mathcal{F}$ depicted in Figs. 3(a)–(b). The corresponding functions $f \sqcup (f \sqcap g)$ and $f \sqcap (f \sqcup g)$ are depicted in Figs. 3(c)–(d). One easily verifies that $f \neq f \sqcup (f \sqcap g)$ and $f \neq f \sqcap (f \sqcup g)$.

Note that the preceding counterexample shows that when $\mathbb{L}_1$ is a chain with at least three elements ($\mathbb{L}_1 = \{0, \frac{1}{2}, 1\}$) and $\mathbb{L}_2$ has at least two elements ($\mathbb{L}_2 = \{0, 1\}$) the absorption laws fail. Hence, the possibility of restricting the lattice such that the absorption laws hold does not seem a suitable option. From now on, we will study appropriate restrictions on the set of functions.



(a) Function f (b) Function g (c) Function $f \sqcup (f \sqcap g)$ (d) Function $f \sqcap (f \sqcup g)$

Fig. 3. Graphical representation of the functions in Example 2: (a) the function f, (b) the function g, (c) the corresponding function $f \sqcup (f \sqcap g)$, and (d) the corresponding function $f \sqcap (f \sqcup g)$.

4 Restricting the Set of Functions

Let $s_f \in L_2$ denote supremum of the function f, defined as $s_f := \bigvee_{x \in L_1} f(x)$ and let $\mathbf{0}_a$ and $\mathbf{1}_a$ (with $a \in L_2$) denote the functions

$$\mathbf{0}_a(x) = \begin{cases} a, & \text{if } x = 0\,, \\ 0, & \text{otherwise}\,; \end{cases} \qquad \mathbf{1}_a(x) = \begin{cases} a & , \text{ if } x = 1\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

In order to study all the algebraic laws of a lattice, we consider the following sets of functions:

$$\mathcal{N}_a = \{f \in \mathcal{F} \mid s_f = a\} \text{ for some } a \in L_2\,,$$
$$\mathcal{I}_\sqcup = \left\{f \in \mathcal{F} \mid (\forall (x,y) \in {L_1}^2)(f(x) \wedge f(y) \leq f(x \vee y))\right\}\,,$$
$$\mathcal{I}_\sqcap = \left\{f \in \mathcal{F} \mid (\forall (x,y) \in {L_1}^2)(f(x) \wedge f(y) \leq f(x \wedge y))\right\}\,,$$
$$\mathcal{C} = \{f \in \mathcal{F} \mid (\forall (x_1,x_2,x_3) \in {L_1}^3)(x_1 \leq x_2 \leq x_3 \Rightarrow f(x_1) \wedge f(x_3) \leq f(x_2))\}\,.$$

Theorem 3. *Let $f, g, h \in \mathcal{F}$. The following statements hold:*

(i) $f \sqcup \mathbf{0}_a = f$ *if and only if* $s_f \leq a$;
(ii) $f \sqcap \mathbf{1}_a = f$ *if and only if* $s_f \leq a$;
(iii) $f \sqcup f = f$ *if and only if* $f \in \mathcal{I}_\sqcup$;
(iv) $f \sqcap f = f$ *if and only if* $f \in \mathcal{I}_\sqcap$;
(v) $f \sqcup (f \sqcap g) = f$, *for any* $g \in \mathcal{N}_{s_f}$, *if and only if* $f \in \mathcal{I}_\sqcup \cap \mathcal{C}$;
(vi) $f \sqcap (f \sqcup g) = f$, *for any* $g \in \mathcal{N}_{s_f}$, *if and only* $f \in \mathcal{I}_\sqcap \cap \mathcal{C}$;

The only algebraic properties we have not studied yet are the distributivity laws. For their fulfillment we impose $\mathbb{L}_1$ to be distributive.

Theorem 4. *Let $\mathbb{L}_1$ be a distributive lattice and $f, g, h \in \mathcal{F}$. The following statements hold:*

(i) If $f \in \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$, *then* $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$ *holds, for any* $g, h \in \mathcal{F}$;
(ii) If $f \in \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$, *then* $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$ *holds, for any* $g, h \in \mathcal{F}$.

Corollary 1. *Let $\mathbb{L}_1$ be a distributive lattice. The convolution operations satisfy all the algebraic laws of a bounded distributive lattice if we restrict to the set $\mathcal{N}_a \cap \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$, for some $a \in L_2$.*

Remark 2. Note that all the algebraic laws of a bounded distributive lattice are satisfied on $\mathcal{N}_a \cap \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$, for some $a \in L_2$. Moreover, if $\mathbb{L}_1$ is distributive, then the distributivity laws are also satisfied. However, we have not studied whether or not the convolution operations are internal on these subsets, i.e., we have not studied if the convolution operations are well-defined on these subsets.

5 Closedness

We devote this section to study whether or not the subsets of functions that have appeared in Sect. 4 are closed under the convolution operations. Specifically, for any subset of functions $\mathcal{G}$, and for any $f, g \in \mathcal{G}$, we should study if $f \sqcup g$ and $f \sqcap g$ belong to $\mathcal{G}$.

Theorem 5. *The following statements hold:*

(i) The sets $\mathcal{N}_a$ (with $a \in L_2$) are closed under join- and meet-convolution.
(ii) The set $\mathcal{I}_\sqcup$ is closed under join-convolution.
(iii) The set $\mathcal{I}_\sqcap$ is closed under meet-convolution.

In the following example we show that neither $\mathcal{I}_\sqcap$ nor $\mathcal{C}$ is closed under join-convolution. Similarly, neither $\mathcal{I}_\sqcup$ nor $\mathcal{C}$ is closed under meet-convolution.

Example 3. Let $\mathbb{L}_1$ be the distributive lattice with Hasse diagram depicted in Fig. 4(a) and $\mathbb{L}_2 = (\{0,1\}, \max, \min, 0, 1)$.

(i) Consider the functions $f_1, g_1 \in \mathcal{I}_\sqcap$ depicted in Figs. 4(b)–(c). The join-convolution $f_1 \sqcup g_1$ is depicted in Fig. 4(d). One easily verifies that $x_2 \wedge x_4 = x_3$, while $(f_1 \sqcup g_1)(x_2) \wedge (f_1 \sqcup g_1)(x_4) = 1 > (f_1 \sqcup g_1)(x_3) = 0$. Hence, $f_1 \sqcup g_1 \notin \mathcal{I}_\sqcap$.
(ii) Consider the functions $f_2, g_2 \in \mathcal{C}$ depicted in Figs. 4(e)–(f). The join-convolu-
tion $f_2 \sqcup g_2$ is depicted in Fig. 4(g). One easily verifies $x_1 \leq x_2 \leq 1$, while $(f_2 \sqcup g_2)(x_1) \wedge (f_2 \sqcup g_2)(1) = 1 > 0 = (f_2 \sqcup g_2)(x_2)$. Hence, $f_2 \sqcup g_2 \notin \mathcal{C}$.

Due to Theorem 3, in order to constitute a bounded lattice the restriction to the set $\mathcal{N}_a \cap \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$, for some $a \in L_2$, must be considered. Hence, the non-closedness of the sets $\mathcal{I}_\sqcup$, $\mathcal{I}_\sqcap$ and $\mathcal{C}$ is of major importance. Fortunately, the intersection of the sets $\mathcal{I}_\sqcup$, $\mathcal{I}_\sqcap$ and $\mathcal{C}$ is closed under the convolution operations with the additional assumption of $\mathbb{L}_1$ being distributive. Moreover, the latter will turn out to be a necessary and sufficient condition.

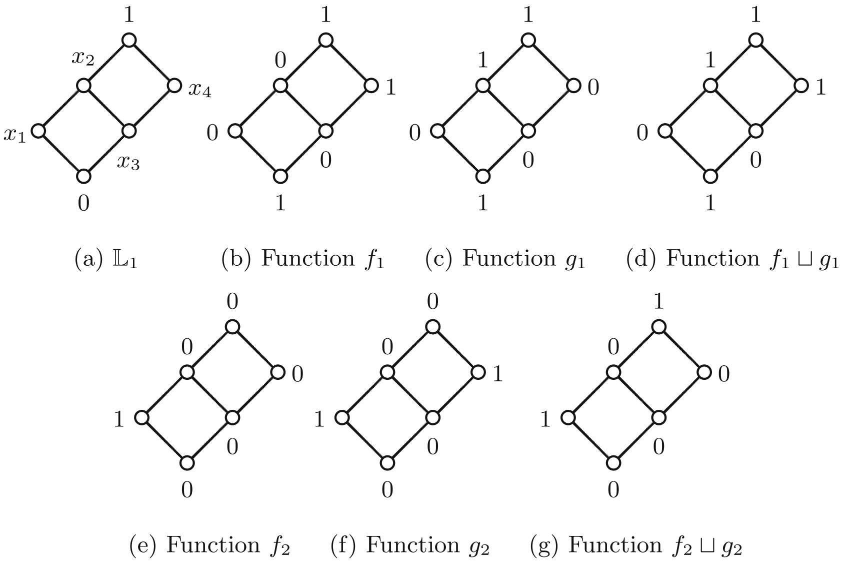


Fig. 4. Graphical representation of the functions in Example 3: (a) the Hasse diagram of the lattice $\mathbb{N}_5$, (b) the function f_1, (c) the function g_1, (d) the join-convolution $f_1 \sqcup g_1$, (e) Hasse diagram of the lattice $\mathbb{L}_1$, (f) the function f_2, (g) the function g_2, and (h) the join-convolution $f_2 \sqcup g_2$.

Theorem 6.

(i) The set $\mathcal{I}_{\sqcup} \cap \mathcal{I}_{\sqcap} \cap \mathcal{C}$ *is closed under join-convolution if and only if* $\mathbb{L}_1$ *is a distributive lattice.*
(ii) The set $\mathcal{I}_{\sqcup} \cap \mathcal{I}_{\sqcap} \cap \mathcal{C}$ *is closed under meet-convolution if and only if* $\mathbb{L}_1$ *is a distributive lattice.*

Proof. Note that we have omitted all the preceding proofs due to space constraints. Since we want to illustrate the general idea of the proofs, we include here the proof of statement (i).

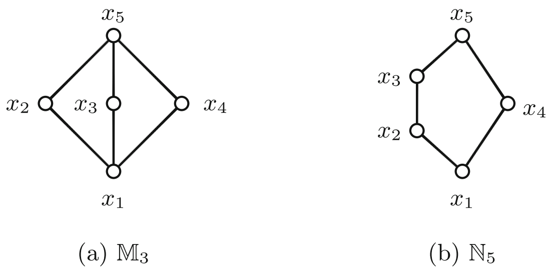


Fig. 5. Hasse diagram of: (a) the sublattice $\mathbb{M}_3$, and (b) the sublattice $\mathbb{N}_5$.

⇒ Suppose that $\mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$ is closed under join-convolution, while $\mathbb{L}_1$ is not distributive. Due to the well-known $\mathbb{M}_3$–$\mathbb{N}_5$ theorem (see [1] or [2]), $\mathbb{L}_1$ has a sublattice that is isomorphic to $\mathbb{M}_3$ or to $\mathbb{N}_5$ (depicted in Fig. 5(a)–(b)). We distinguish two cases.

(a) The case that $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{M}_3$. We refer to the elements of this sublattice as in Fig. 5(a). We consider the functions $f, g \in \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1, & \text{if } x \in \{x_1, x_2\}\,, \\ 0, & \text{otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1 & , \text{ if } x \in \{x_1, x_3\}\,, \\ 0 & , \text{ otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_1, x_2, x_3, x_5\}$, where $f \sqcup g$ takes the value 1. Since $x_1 \leq x_4 \leq x_5$ and $(f \sqcup g)(x_4) = 0 < 1 = (f \sqcup g)(x_1) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

(b) The case that $\mathbb{L}_1$ has a sublattice isomorphic to $\mathbb{N}_5$. We refer to the elements of this sublattice as in Fig. 5(b). We consider the functions $f, g \in \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$ defined as:

$$f(x) = \begin{cases} 1, & \text{if } x = x_2\,, \\ 0, & \text{otherwise}\,; \end{cases} \qquad g(x) = \begin{cases} 1, & \text{if } x \in \{x_1, x_4\}\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

It holds that $(f \sqcup g)(x) = 0$ for any $x \in L_1$ unless $x \in \{x_2, x_5\}$, where $f \sqcup g$ takes the value 1. Since $x_2 \leq x_3 \leq x_5$ and $(f \sqcup g)(x_3) = 0 < 1 = (f \sqcup g)(x_2) \wedge (f \sqcup g)(x_5)$, we conclude that $f \sqcup g \notin \mathcal{C}$, a contradiction.

⇐ Let $\mathbb{L}_1$ be a distributive lattice and $f, g \in \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$. Since $\mathcal{I}_\sqcup$ is closed under join-convolution, it holds that $f \sqcup g \in \mathcal{I}_\sqcup$ and we only need to show that $f \sqcup g \in \mathcal{I}_\sqcap \cap \mathcal{C}$.

Firstly, we prove that $f \sqcup g \in \mathcal{I}_\sqcap$. For any $x_1, x_2 \in L_1$, it holds that

$$\begin{aligned} (f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) &= \left(\bigvee_{u_1 \vee v_1 = x_1} f(u_1) \wedge g(v_1) \right) \wedge \left(\bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) \right) \\ &= \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1) \wedge f(u_2) \wedge g(v_1) \wedge g(v_2)\,. \end{aligned}$$

Since $f \in \mathcal{I}$, it holds that $f(u_1) \wedge f(u_2) \leq f(u_1 \vee u_2)$ and $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2)$, and, hence, $f(u_1) \wedge f(u_2) \leq f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2)$. Similarly, since $g \in \mathcal{I}$, it holds that $g(v_1) \wedge g(v_2) \leq g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)$. This leads to

$$\begin{aligned} &(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) \\ &\qquad \leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_2 \vee v_2 = x_2}} f(u_1 \wedge u_2) \wedge f(u_1 \vee u_2) \wedge g(v_1 \wedge v_2) \wedge g(v_1 \vee v_2)\,. \end{aligned}$$

Taking into account that $u_1 \leq u_1 \vee v_1 = x_1$ and $u_2 \leq u_2 \vee v_2 = x_2$, it holds that $u_1 \wedge u_2 \leq x_1 \wedge x_2$. Moreover, since $u_1 \wedge u_2 \leq u_1 \vee u_2$, we find that

$$u_1 \wedge u_2 \leq (x_1 \wedge x_2) \wedge (u_1 \vee u_2) \leq (u_1 \vee u_2)\,.$$

Analogously, it follows that

$$v_1 \wedge v_2 \leq (x_1 \wedge x_2) \wedge (v_1 \vee v_2) \leq (v_1 \vee v_2)\,.$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}&(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2)\\ &\qquad \leq \bigvee_{\substack{u_1 \vee v_1 = x_1\\ u_2 \vee v_2 = x_2}} f((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \wedge g((x_1 \wedge x_2) \wedge (v_1 \vee v_2))\,.\end{aligned}$$

Finally, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}&((x_1 \wedge x_2) \wedge (u_1 \vee u_2)) \vee ((x_1 \wedge x_2) \wedge (v_1 \vee v_2))\\ &\qquad = (x_1 \wedge x_2) \wedge ((u_1 \vee u_2) \vee (v_1 \vee v_2))\\ &\qquad = (x_1 \wedge x_2) \wedge ((u_1 \vee v_1) \vee (u_2 \vee v_2))\\ &\qquad = (x_1 \wedge x_2) \wedge (x_1 \vee x_2) = x_1 \wedge x_2\,.\end{aligned}$$

By denoting $u = (x_1 \wedge x_2) \wedge (u_1 \vee u_2)$ and $v = (x_1 \wedge x_2) \wedge (v_1 \vee v_2)$, it holds that

$$(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_2) \leq \bigvee_{u \vee v = x_1 \wedge x_2} f(u) \wedge g(v) = (f \sqcup g)(x_1 \wedge x_2)\,.$$

Consequently, $f \sqcup g \in \mathcal{I}_\sqcap$.

Secondly, we prove that $f \sqcup g \in \mathcal{C}$. For any $x_1, x_2, x_3 \in L_1$ such that $x_1 \leq x_2 \leq x_3$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) &= \left(\bigvee_{u_1 \vee v_1 = x_1} f(u_1) \wedge g(v_1)\right) \wedge \left(\bigvee_{u_3 \vee v_3 = x_3} f(u_3) \wedge g(v_3)\right)\\ &= \bigvee_{\substack{u_1 \vee v_1 = x_1\\ u_3 \vee v_3 = x_3}} f(u_1) \wedge f(u_3) \wedge g(v_1) \wedge g(v_3)\,.\end{aligned}$$

Analogously to the case $\mathcal{I}_\sqcap$, since $f, g \in \mathcal{I}$, it holds that

$$f(u_1) \wedge f(u_3) \leq f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3)$$

and

$$g(v_1) \wedge g(v_3) \leq g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3)\,.$$

This leads to

$$\begin{aligned}&(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3)\\ &\qquad \leq \bigvee_{\substack{u_1 \vee v_1 = x_1\\ u_3 \vee v_3 = x_3}} (f(u_1 \wedge u_3) \wedge f(u_1 \vee u_3)) \wedge (g(v_1 \wedge v_3) \wedge g(v_1 \vee v_3))\,.\end{aligned}$$

Taking into account that $u_1 \wedge u_3 \leq u_1 \leq u_1 \vee v_1 = x_1 \leq x_2$ and $u_1 \wedge u_3 \leq u_1 \vee u_3$, it holds that

$$u_1 \wedge u_3 \leq x_2 \wedge (u_1 \vee u_3) \leq u_1 \vee u_3\,.$$

Analogously, it follows that

$$v_1 \wedge v_3 \leq x_2 \wedge (v_1 \vee v_3) \leq v_1 \vee v_3 .$$

Since $f, g \in \mathcal{C}$, it holds that

$$\begin{aligned}(f \sqcup g)(x_1) &\wedge (f \sqcup g)(x_3) \\ &\leq \bigvee_{\substack{u_1 \vee v_1 = x_1 \\ u_3 \vee v_3 = x_3}} (f(x_2 \wedge (u_1 \vee u_3))) \wedge (g(x_2 \wedge (v_1 \vee v_3))) .\end{aligned}$$

Finally, since $\mathbb{L}_1$ is a distributive lattice, it holds that

$$\begin{aligned}(x_2 \wedge (u_1 \vee u_3)) \vee (x_2 \wedge (v_1 \vee v_3)) &= x_2 \wedge ((u_1 \vee u_3) \vee (v_1 \vee v_3)) \\ &= x_2 \wedge ((u_1 \vee v_1) \vee (u_3 \vee v_3)) \\ &= x_2 \wedge (x_1 \vee x_3) = x_2 \wedge x_3 = x_2 .\end{aligned}$$

By denoting $u_2 = x_2 \wedge (u_1 \vee u_3)$ and $v_2 = x_2 \wedge (v_1 \vee v_3)$, it holds that

$$(f \sqcup g)(x_1) \wedge (f \sqcup g)(x_3) \leq \bigvee_{u_2 \vee v_2 = x_2} f(u_2) \wedge g(v_2) = (f \sqcup g)(x_2) .$$

Consequently, $f \sqcup g \in \mathcal{C}$.

Taking into account all the preceding results, we can conclude the following.

Theorem 7. *The algebraic structure* $\mathbb{F} = (\mathcal{N}_a \cap \mathcal{I} \cap \mathcal{C}, \sqcup, \sqcap, \mathbf{0}_a, \mathbf{1}_a)$ *(with* $a \in L_2$*) is a bounded distributive lattice if and only if* $\mathbb{L}_1$ *is a distributive lattice.*

6 Conclusions

This paper studies some convolution operations generated by replacing the functions of the set $\mathcal{F}([0,1],[0,1])$ with functions of the set $\mathcal{F}(\mathbb{L}_1, \mathbb{L}_2)$. Firstly, it is shown that the idempotency laws do not hold unless $\mathbb{L}_1$ is a chain. But even in this case, the absorption laws fail. The second part of the paper focuses on the restriction to a suitable subset of functions where the convolution operations fulfill the algebraic properties of a bounded distributive lattice. Moreover, the question which of these subset are closed under the convolution operations is also answered. The most important result of the paper is that the convolution operations constitute a bounded distributive lattice on the set $\mathcal{N}_a \cap \mathcal{I}_\sqcup \cap \mathcal{I}_\sqcap \cap \mathcal{C}$ (with $a \in L_2$) if and only if $\mathbb{L}_1$ is a distributive lattice.

We expect to expand the present work in the future studying the subsets on which the convolution operations constitute a bounded lattice when $\mathbb{L}_1$ is not distributive.

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Representing Uncertainty Regarding Satisfaction Degrees Using Possibility Distributions

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Abstract. Evaluating flexible criteria on data leads to degrees of satisfaction. If a datum is uncertain, it can be uncertain to which degree it satisfies the criterion. This uncertainty can be modelled using a possibility distribution over the domain of possible degrees of satisfaction. In this work, we discuss the meaningfulness thereof by looking at the semantics of such a representation of the uncertainty. More specifically, it is shown that defuzzification of such a representation, towards usability in (multi-criteria) decision support systems, corresponds to expressing a clear attitude towards uncertainty (optimistic, pessimistic, cautious, etc.)

1 Introduction

Consider for the remainder of this paper that a data set consists of *objects* which represent real-world entities as collections of *attribute* values. An example data set might store information on people by tracking attributes such as "age", "name", "sex", "weight", "height", and so on. As such, each actual person corresponds to a collection of values for these attributes taken from their respective domains.

Often, our knowledge of entities is limited to information stored in a database. In the best case scenario, *all* attributes of an entity are *precisely* known. In practice however, attribute values are often missing (not yet measured or inapplicable), outdated, incorrect or vague. In all cases where the real, exactly correct value of an attribute is not known, it is said that (the value of) each such an attribute is *uncertain*. This lack of knowledge can be represented by a possibility distribution over the domain of the attribute, whereby each value in the domain is associated with a degrees of possibility that it is the real value of the attribute. To deal with inapplicability, the domain of each attribute might have to be extended with a special symbol to denote this [6].

Data are commonly subjected to criteria in order to test their suitability for a specific purpose. Examples hereof are querying systems, (multi-criteria) decision support systems, recommender tools, and so on. Mathematically, criteria can be seen as functions which are used to test attribute values. Evaluating a criterion on an attribute then corresponds to testing the function in the attribute value. The resulting score can be interpreted as a score indicating how acceptable the value is.

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Evaluating an uncertain attribute is not straightforward because if the exact value is not known and it can not be tested. It follows that the resulting degree to which an uncertain attribute satisfies a criterion is also uncertain. As such, it can be treated as an uncertain 'attribute' and the uncertainty thus modelled by a possibility distribution. This has already been suggested by Dubois and Prade in [10] but was not investigated further. Instead, they proposed using possibility and necessity degrees to represent respectively the possibility and the necessity that the uncertain attribute satisfies the criterion. To that end, the formulae to compute them are generalized. The advantage hereof is that these degrees are well known and compatible with (and comparable to) the case when evaluating non-flexible criteria on uncertain data. However, as mentioned by Dubois and Prade themselves, the result of the generalization comes at the cost of some properties which have a non-negligable impact on their interpretability, which can lead to counter-intuitive results. Indeed, when evaluating different entities in order to compare them using their approach, it can occur that an entity which certainly satisfies a criterion to a certain degree (and possibly fully satisfies it) is ranked worse than an entity which is not in the least guaranteed to satisfy the criterion.

In this work, we further explore the feasability of using possibility distributions to model the uncertainty over the degree to which an uncertain attribute satisfies a criterion. Towards multi-criteria decision support, we discuss how defuzzification can be applied to reduce these distributions to numbers so they may be used for further calculations. We will show that different ways of performing defuzzification correspond to different attitudes a decision maker can have towards uncertainty.

The remainder of this paper is structured as follows. In Sect. 2 some preliminaries and relevant research are mentioned. Afterwards, mathematical notations used throughout the remainder of the paper are given in Sect. 3 to define possibility distributions over degrees of satisfaction. Then, the usefulness of such distributions is discussed in Sect. 4. Towards usability in existing tools, defuzzification is carefully studied. The findings are illustrated in Sect. 5 by means of an example. To conclude, Sect. 6 summarizes the feasibility and usefulness of using possibility distributions.

2 Preliminaries

2.1 Flexible Criteria

Traditional criteria evaluation is Boolean: either the criteria are satisfied or they are not. Flexible criteria, in contrast, compute a degree (typically from the unit interval) for each entity denoting how well it satisfies the criterion. This has many advantages, the most obvious one being the fact that flexible criteria do not filter but rather sort the objects in a data set. It also allows agents to model *vague* preferences. For example, one may be interested in identifying which people are "old".

Flexible criteria are typically modelled by fuzzy sets [21]. A fuzzy set F over a domain X is a generalization of a regular set, characterized by a membership function μ_F which associates each element from X with a real number in a partially ordered set (usually the unit interval), with the value $\mu_F(x)$ of element x representing the "grade of membership" of x in F. It can easily be seen that a regular set is a special case of a fuzzy set where the membership can only take values 0 and 1, respectively denoting non-membership and membership.

Treating a criterion as a function that is used to test attribute values, it can easily be seen that, mathematically, the relation between fuzzy sets and regular sets is identical to the relation between regular (crisp) criteria and flexible criteria. In a sense, a regular criterion can be seen as a function which partitions the data set in regular sets (satisfied and not satisfied). In contrast, a flexible criterion can be seen as a function which describes the membership of all objects in the data set in the fuzzy set of satisfaction.

Fuzzy sets are at the base of many flexible systems [4,5,12,15,22,23]. Note that flexible criteria are in no way related to uncertainty [9]. To reflect uncertainty in criteria, extended fuzzy sets such as type-2 fuzzy sets, interval-valued fuzzy sets, Atanassov's intuitionistic fuzzy sets [2] and, more recently, hesitant fuzzy sets [17] have been proposed.

2.2 Representation of Uncertainty

There has been a lot of research towards representing uncertain data. For a singular attribute (i.e. it can only take one value), the underlying idea is that we are incapable of storing the correct value because it is not known. Instead, in accordance to the information that we do have, we must store each possible value that the property might take, associated with a degree of belief that it is the correct value. Mathematically, this can be represented by a function u, associating each value x from the attribute's domain X with a real number in a partially ordered set (usually the unit interval), where $u(x)$ denotes the "degree of belief" that x is the correct value of the attribute.

A classical example of modeling stochastic uncertainty is by means of a probability distribution. Alternatively, uncertainty due to a lack of knowledge is generally modelled using a possibility distribution [10,14,18,20].

3 Uncertainty Regarding Satisfaction Degrees

In this section we introduce a formal notation of the representation of uncertainty regarding the extent to which an uncertain attribute satisfies a criterion. Considering the precise value of an uncertain attribute is not known, it is not possible to say to which degree it satisfies a given criterion. As such, the degree of satisfaction of the uncertain attribute is inherently uncertain and can be modelled using a possibility distribution.

With the understanding that A is an attribute with domain X, and that E is an entity for which the value of A is not exactly known, let $\pi_A : X \to [0, 1]$:

$x \mapsto \pi_A(x)$ be a possibility distribution over X where $\pi_A(x)$ denotes the possibility that E takes value x for A. Let further $E[A]$ denote the value that E takes for A, whatever it may be. Let then $\pi_\sigma : [0,1] \rightarrow [0,1]$ be a possibility distribution expressing the uncertainty regarding the degree σ to which $E[A]$ satisfies a flexible criterion defined on A.

Using a possibility distribution to represent the uncertainty over the degree of satisfaction of an uncertain attribute regarding a preference brings all the advantages of working with possibility distributions. As such, it is immediately apparent that such distributions can represent full certainty, full uncertainty and any degree in between. Further more, factoring in the semantics of the domain of these possibility distribution as being a degree of satisfaction, such a model can be used to represent statements like "possibly fully satisfied", "certainly at least x satisfied", "at most x satisfied", "certainly not satisfied", "could be anything" and many more, which are semantically rich and intuitive.

4 Attitudes Towards Uncertainty

Assume there is a way to construct such possibility distributions. Given that the evaluation of a flexible criterion on a data set containing uncertain data yields, for each object, a possibility distribution regarding the degree of satisfaction, we can immediately ask ourselves the question how the results may be ranked. After all, this is one of the key advantages of flexible criteria evaluation which is no longer straightforward. Indeed, ranking objects by a degree of satisfaction comes down to sorting a list of numbers in descending (or ascending) order, but how should possibility distributions be compared?

There have been many studies devoted to the comparison of fuzzy sets (also known as comparing fuzzy numbers) [1,3,7]. There is no real consensus in this area of research, which testifies to the flexibility of fuzzy sets. It also proves that there is no one-size-fits-all approach because the semantics of fuzzy sets play a fundamental role when it comes to how they should be compared [8].

We can ask ourselves the question: how would *we* compare two uncertain attributes? We argue the only correct answer is *it depends*. It depends on the application in which we are considering the uncertainty and on the impact of making a mistake. For example, when it comes to comparing travel options we might be inclined to choose for a suboptimal route which will certainly get us there in time rather than a different route that might get us there faster at the risk of running late, especially if arriving on time is of critical importance.

Another field of application which is built on flexible criteria evaluation is multi-criteria decision support. After evaluating all the criteria, a multi-criteria decision support system aggregates all elementary satisfaction degrees into a single, global score. There exist techniques for aggregating membership functions [19,20]. However, these techniques are tightly coupled to the context in which they are used. As such, the conjunction of two such functions is different for possibility distributions on one hand and characteristic functions on the other. Furthermore, these functions are mostly limited to either simple aggregators or force the user to use a specific technique. To be able to tap into the power of any proven aggregation technique (including but not limited to: OWA, WOWA, LSP, Choquet, Sugeno) it is logical to reduce uncertain satisfaction degrees represented by possibility distributions to numerical values. This comes down to the defuzzification of a possibility distribution. There are different ways of performing defuzzification. We will discuss and compare three defuzzification strategies, considering especially the semantics of each. If a decision maker purposefully chooses for a specific defuzzification strategy, aware of the semantics, this can be viewed as explicitly expressing an attitude towards uncertainty.

4.1 Maximal Possible Satisfaction

Let us first look at defuzzification by taking the maximal possible degree of satisfaction that is associated with a degree of belief larger than 0. In this case, however unlikely, the result of the defuzzification corresponds to the event that the attribute takes the best possible value regarding the preferences of the decision maker. Obviously, this might not be the reality, but assuming this degree of satisfaction clearly indicates an optimistic attitude by believing in the best possible case. Alternatively, it could be seen as a greedy attitude, aggressively assuming the best possible case, neglecting the fact that reality might be less optimal. This kind of attitude is typical for prediction systems such as GPS-based routing software, which assume you can drive at the highest speed for each road and that no sudden accidents happen which could influence travel time.

One could also choose for the maximal fully possible value, e.g. the maximal value which has a degree of belief equal to 1. Such cases can be seen as a greedy attitude assuming "normal circumstances", yielding a natural trade off between what is desired and what can be expected if nothing unforseeable happens.

4.2 Minimal Possible Satisfaction

Let us now look at defuzzification by taking the minimal possible value that is associated with a degree of belief larger than 0. Instead of the best possible case, the worst possible case is assumed. Consequently, this defuzzification strategy denotes a pessimistic attitude. This might be valuable when the outcome of the decision is of critical importance and there is no room for error. As such, it can be seen as an attitude of safeness, avoiding risk.

However, one might also choose a slightly less pessimistic attitude by choosing the minimal fully possible value, e.g. the worst value with degree of belief equal to 1. As such, possibly disastrous but very unlikely outcomes are purposefully ignored, again assuming "normal circumstances".

4.3 Center of Mass

A common approach for defuzzification is to compute the abscissa of the center of mass of the area under the possibility distribution [16]. This can be viewed as a sort of weighted average of all possible outcomes, where the degrees of belief are used as weights. Semantically, the center of mass can be viewed as an indicator of the "expected" degree of satisfaction, taking into account all possible outcomes. As such, it portrays an intermediary attitude which is generally more robust to outliers with low possibility. However, it might produce unexpected results in case of a non-convex possibility distribution by producing an average degree of satisfaction that corresponds to a value that the uncertain attribute might not even be able to take. Consider for example that we know a certain bottle is either completely empty or completely full. Defuzzification through center of mass might lead to an "expected" satisfaction of 0.5, though it is not possible the bottle is half-full. However, if interpreted as a real average, the center of mass strategy can still be useful. One can see that aggregated degrees of satisfaction near 0.5 indicate uncertainty or otherwise mediocre objects, values near 1 denote rather certainly good objects and values near 0 denote rather certainly poor objects. Then one can use the center of mass approach to represent a cautious attitude to reliably identify the good and bad objects, leaving the uncertain and mediocre objects in the middle. For attributes with a defuzzified satisfaction degree below 0.5 it can be said that there is more reason to believe that it will take an unsatisfactory value than that it will take a satisfactory value. However, it should be kept in mind that this degree should not be reverse engineered to an attribute value to guesstimate which value the uncertain attribute might take.

5 Example

Suppose we are trying to evaluate if an area of the subsurface is suitable for extracting a specific lithological resource. To that end, a model of the subsurface is queried. This model is a collection of discrete 3D cuboids (voxels) representing minimal extractable volumes. For each voxel, the lithological classification of the extractable resource (medium sand, fine sand, clay...) and a degree of impurities (percentage rocks, shells...) are stored. In practice, such subsurface models are largely generated through statistical interpolations from only a very small amount of soil samples that typically cover less than 1% of the actual model area. As a result, practically all voxels denote interpolated data and are as such inherently uncertain.

Let us say we are looking for fine sand without impurities for some industrial purpose. Because we can not say with certainty whether or not a voxel contains fine sand nor if it is pure (both are possibly uncertain), we must express an attitude towards the possibility that the voxel contains a different resource or is impure. Assume our application absolutely requires fine sand. Any other resource is unusable. Assuming our preference reflects that only fine sand results in the maximal satisfaction degree and that other lithological classes result in a low(er) degree of satisfaction, we use the minimal possible satisfaction strategy to ensure that only voxels that only contain fine sand (i.e. have a high satisfaction degree) are defuzzified to a high satisfaction degree. Voxels that possibly contain other resources (i.e. have a low possible satisfaction degree) will be defuzzified to a low satisfaction degree. Towards impurities we are more forgiving, as we can filter these out after extraction. However, purer is still better, and we want to exclude voxels that are so impure that the remaining usable volume after extraction would be too low to make the extraction cost worthwhile. Here we choose the center of mass strategy for defuzzification of uncertainty regarding impurities to reliably reject those voxels that contain many impurities but including the voxels that are both likely and possibly pure.

6 Conclusions

In this work, we have briefly examined the feasibility of using possibility distributions to represent uncertainty regarding the degree to which an uncertain attribute satisfies a flexible preference. Further, we have discussed how different strategies for defuzzifying these distributions correspond to specific attitudes regarding uncertainty. When dealing with uncertainty in decision support, we argue the decision maker must necessarily express his or her attitude towards uncertainty. A small example illustrates the intuitiveness of the approach and highlights how a decision maker may express different attitudes towards different properties in multi-criteria decision problems. There is still a lot to be done, such as defining how possibility distributions representing uncertainty regarding the satisfaction degree of a flexible criterion on uncertain data should be derived, and how such a representation can be further expanded to be capable of dealing with bipolar queries [11,13,24], interval-valued fuzzy sets, hesitant fuzzy sets [17], two-fold fuzzy sets, Atanassov's intuitionistic fuzzy sets [2] and type-2 fuzzy sets.

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Triangular Expanding, A New Defuzzification Method on Ordered Fuzzy Numbers

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Abstract. At the beginning of the article the authors describe a new trend in the artificial intelligence, associated with fuzzy sets and the accompanying derivative solutions which may include L-R numbers by Dubois and Prade. On their basis the redefined theory has started a new trend in the form of ordered fuzzy numbers (OFN). Main features of ordered fuzzy numbers further in this article. Due to the nature of this article, which is related to the proposed defuzzyfication method, the authors mentions a fuzzy controller model and in particular the defuzzyfication process. Criteria used for conventional solutions of fuzzy numbers and ordered fuzzy numbers were also presented. In the further part of the article the defuzzyfication method called Triangular Expanding was presented. The author compared it to the Geometrical Mean method introduced earlier, which inspired his solution. Results of comparison with other methods such as FOM, LOM, COG were presented in the paper as well. The summary including conclusions and directions of further research were provided at the end.

Keywords: Fuzzy logic · Ordered fuzzy numbers

1 Introduction

Although fuzzy control and fuzzy logic control were introduced over 50 years ago, no doubt they can still be regarded as modern technology. That statement is confirmed by three items of the references [6], describing new concepts in the field of research or applications. One of those novelties are undoubtedly Ordered Fuzzy Numbers introduced by professor Witold Kosiński and his research team including P. Prokopowicz and D. Ślęzak, which, since 2016, should be called Kosinski Fuzzy Numbers. It can easily be noticed that the introduction of the

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fuzzy logic theory was a turning point in the manner of perceiving the traditional logic, which assumes only 0 or 1 value. The same turning point was the appearance of the ordered fuzzy numbers theory. One of the main reasons for which the classic theory of fuzzy sets has been redefined is its inconsistency in the execution of arithmetic operations on fuzzy sets that is observed in the case of operations on real numbers.

2 Reasons Behind Development of OFN

The theory of ordered fuzzy numbers is not an isolated being, but it is a postulate that extends previous achievements of the fuzzy logic. Contrary to some opinions, OFN is rather an abstraction or a generalization of fuzzy numbers rather than just their notation. To explain this position, one has to refer to the Zadeh's definition of a fuzzy set. However, this definition provides too narrow concept of a fuzzy set. In his paper [50] the author on the one hand treats a fuzzy set as classes of a membership function and on the other hand the definition has a specific form, i.e. it applies to the membership function. In their considerations on the ordered fuzzy numbers, their precursors perceived the possibility of introducing novelties in the formulation of the function class. The current form of the ordered fuzzy number results from a transitory, which is related to the use of the so-called quasi-convex membership functions.

A comprehensive polemic on quasi convexity of functions and related concepts can be found in the articles [22,27,35].

The basic form of a fuzzy number according to the definition together with the individual conditions has become the foundation for new insights on fuzzy numbers. Since the support of the fuzzy number is an interval then the convexity condition, can be replaced by quasi-convexity condition of the membership function, as is the case in the paper [23]. The authors of the ordered fuzzy number solution, [29,31], take things step further by thus modifying the convexity condition of the fuzzy set A into strictly quasi-convexity condition of a function over its support. Since the membership function takes the above form, then on the basis of the theory of convex functions [35] the representations are reversible intervals. One can state that, respectively:

Theorem 1. The scalar function $\varphi(x)$ *is strictly quasi-convex in the convex set* X *only and exclusively when any segment* $\left[x^1, x^2\right] \subset X$ *can be divided into three such sections (each cutting point belongs at least to one of the sections and closes it) that* $\varphi(x)$ *is decreasing in the first, constant in the second and growing in the third section. Any one or two of these sections may be empty or degenerated to points* x^1 *and (or)* x^2*.*

On the contrary, while observing strictly quasi-concave relationships one can notice the change in the order of individual sections i.e. monotonicity properties of the interval $\left[x^1, x^2\right] \subset X$. In this case, the order shall be from the increasing to the decreasing part. Thus, it is concluded that for the fuzzy number $A = \{R, \mu_A\}$. As regards strictly quasi-convex relationship, the membership function is strictly

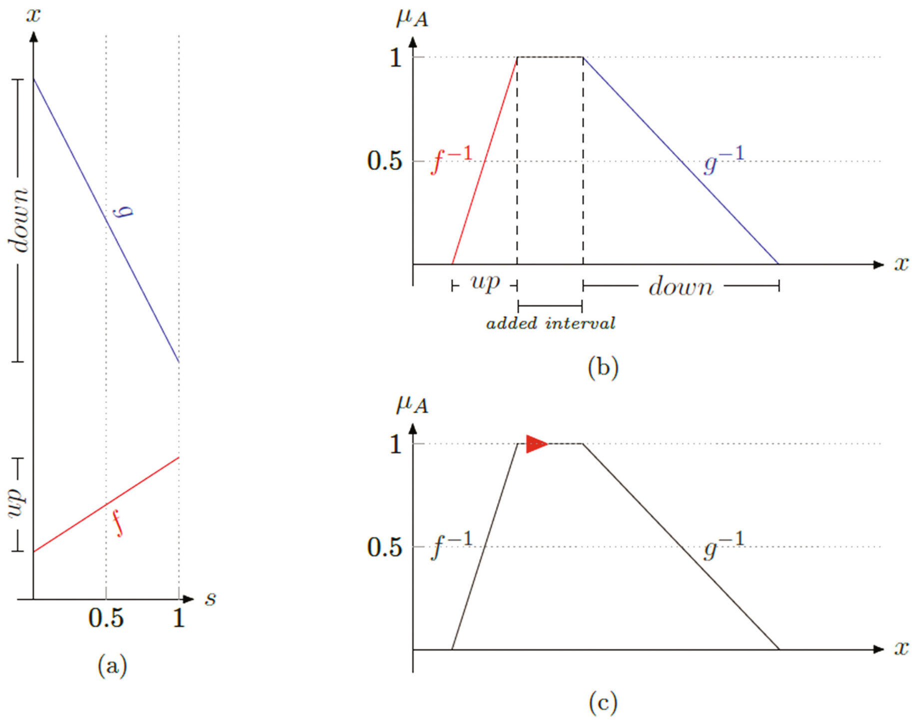

Fig. 1. Example (a) of an ordered fuzzy number and interpretation (b) and (c) referring to a convex fuzzy number

quasi convex, and, consequently, the support of the fuzzy number is treated as an interval (Fig. 1).

3 Classic Defuzzyfication Methods

The popularity of classic defuzzyfication methods which are used in the reduction process of the resulting membership function, results mainly from the domain of application in a given environment. Basically, there are two approaches among the available mechanisms: the first one includes the methods of maxima. These are solutions characterized by simplicity either as regards interpretation or implementation. The second group of widely used algorithms are so called center of gravity methods together with derivatives. These methods are derived from the geometry and are particularly important for solving problems in the field of physics or mechanics. Thus it can be stated that the new defuzzyfication methods should either be based on their simplicity or concern a generally known property taken from nature. Those two main approaches, in a sense, form a lodestar when developing the methods described below. It follows from the assumptions adopted for the purpose of this article that the proposed defuzzyfication methods are to be order sensitive. This means that on the basis of the theory of OFN numbers we will be able to describe physical phenomena over time, although the time will not be formally given. Similarly to time series fully reflecting the time, in case of OFN numbers we use only so called discrete (left, right) form of a trend. For example, there is: for positively ordered OFN numbers - growing trend, for negatively ordered OFN numbers - declining trend to the expected value.

3.1 Method 1 - The Triangular Expanding Method

The assumption to the Triangular Expanding method in the defuzzyfication process for ordered fuzzy numbers is that the method shall be order sensitive. The possibility to apply the concept of ordered fuzzy numbers allows to capture the trend in a given phenomenon. The order - as the approaching some status of a value extends the existing concept of conventional fuzzy numbers. The achieved expansion introduces an additional dimension into the control process. The target in many cases may be more important than the initial status. Referring to the defuzzyfication process itself, where in the conventional approach we obtain a real (defuzzyfied) value from the resulting membership function, we underline order in the result for the ordered system using OFN numbers. The wording "in the result" is meant as the obtained defuzzyfication value, which shall additionally bear the trend index. Graphic interpretation of the proposed method is shown in Fig. 2, where for the ordered fuzzy number $H = (f_H, g_H)$ the defuzzyfication is executed. The figure shown herein, intuitively reflects the character of the method in the further analysis. In a simplified case, when both functions in the OFN representation are affine, i.e. continuous, the direction of the order coming from the UP to DOWN arm defines the place where the extension is created. The triangle generated in this place from the DOWN part (as one of the triangle sides) extends the possible values of defuzzyfication. For the number $H = (f_H, g_H)$, where both functions are continuous and $f_H, g_H : [0, 1] \rightarrow \mathbb{R}$, we take the DOWN part of the OFN number expressed as g_H, hereinafter called the falling slope. The DOWN arm is selected due to the emphasis on the order as the trend to be achieved. The defuzzyfication process of the number proceeds as follows. Two points $A = (0, g(0))$ and $B = (1, g(1))$ are centers of circuits $K_1(A, r)$ and $K_2(B, r)$ respectively. The radius r is the same for K_1 circle and for K_2 circle. The radius r is a function g_H, previously named the DOWN part. The selection of this part is related to the order which approaches that arm. In the example shown in Fig. 2, two circles K_1 and K_2 intersect each other thus creating two intersection points. Points C_1 and C_2 are candidates for vertices C of the triangle, whose aim will be to extend the order. The selected vertex is the one located in the positive part of s axis. The so created common part of three planes limited by vertice ABC forms a triangle. Defuzzyfication occurs at the intersection of the straight lines l_1 oraz l_2. The straight line l_1 is created by points C and $D = (0, f(0))$ while straight line l_2 is determined by points $A = (0, g(0))$ and $E = (1, f(1))$. The presented intuitive operation aspect of the Triangular Expanding defuzzyfier method is intended to outline the essence of

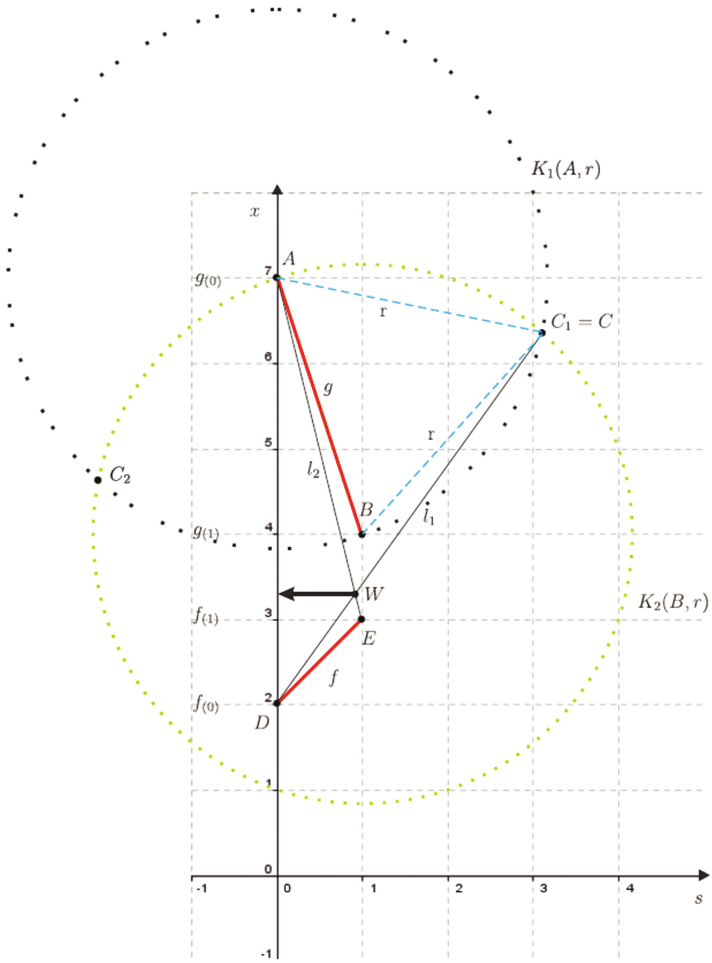

Fig. 2. Graphic interpretation of the Triangular Expanding method

that proposal. Analytical form of the defuzzyfier shall be discussed further in this article.

3.2 Analytical Formula of TR (Triangular Expanding) Method

The above considerations include formal description of the proposed method in the OFN number defuzzyfication process that comes down to determining the equations for the intersection of two circles and the intersection of two linear functions.

Following those assumptions, for the circle K_1 we obtain the Eq. (1) expressed with the formula:

$$K_1(A, r) : \{(s, x) : (s - 0)^2 + (x - g(0))^2 = r^2\} \quad (1)$$

while the circle K_2 is described in form of the Eq. (2)

$$K_2(B, r) : \{(s, x) : (s - 1)^2 + (x - g(1))^2 = r^2\} \quad (2)$$

whereas $x \in K_1 \cap K_2$.

The determination of radius r which is equal for both circles K_1 and K_2, results from the formula (3) presented below:

$$r = \left|\overrightarrow{AB}\right| = \sqrt{((g(0) - g(1))^2 + 1^2)} \quad (3)$$

In the analyzed method, the radii of the circle K_1 and K_2 are identical, as they lie at the ends of the common straight line. The only assumption for those radii is that they must meet the following inequality:

$$0 < r < 2r \quad (4)$$

for which there are two points of intersection of the circles.

When considering the Eqs. (1) and (2), with the assumption that the radii of the two equations are the same, we receive the Eq. (5) of the following form:

$$(s - 0)^2 + (x - g(0))^2 = (s - 1)^2 + (x - g(1))^2 \quad (5)$$

On the basis of the above formula, x of the s function is determined, and as a result we obtain:

$$x(s) = -\frac{2s - g(1)^2 + g(0)^2 - 1}{2g(1) - 2g(0)} \quad (6)$$

by substituting the formula (6) to the Eq. (1) the following equation is obtained:

$$s^2 + \left(\frac{g(1)^2 - g(0)^2 - 2s + 1}{2g(1) - 2g(0)} - g(0)\right)^2 = r^2 \quad (7)$$

The above square Eq. (7) is solved against s, with the assumption that $\Delta > 0$, and we obtain respectively:

$$s_1 = -\frac{g(1) - g(0)}{2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2} * \frac{\sqrt{Z_1}}{2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2} - \frac{g(1)^2 - 2g(0)g(1) + g(0)^2 + 1}{2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2} \quad (8)$$

the denominator is such that $2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2 \neq 0$

$$s_2 = \frac{g(1) - g(0)}{2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2} * \frac{\sqrt{Z_2}}{2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2} - \frac{g(1)^2 - 2g(0)g(1) + g(0)^2 + 1}{2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2} \quad (9)$$

the denominator is such that $2g(1)^2 - 4g(0)g(1) + 2g(0))^2 + 2 \neq 0$
where:

$$Z_1 = \left(g(1)^2 - 8g(0)g(1) + 4g(0)^2 + 4\right) r^2 - g(1)^4 + 4g(0)g(1)^3 + \\ \left(-6g(0)^2 - 2\right) g(1)^2 + \left(4g(0)^3 + 4g(0)\right) g(1) - g(0)^4 - 2g(0)^2 - 1 \quad (10)$$

$$Z_2 = \left(g(1)^2 - 8g(0)g(1) + 4g(0)^2 + 4\right) r^2 - g(1)^4 + 4g(0)g(1)^3 + \\ \left(-6g(0)^2 - 2\right) g(1)^2 + \left(4g(0)^3 + 4g(0)\right) g(1) - g(0)^4 - 2g(0)^2 - 1 \quad (11)$$

The variables s_1, s_2 described by formulas (8, 9) specify x_1 and x_2, respectively, then we obtain:

$$x_1 = -\frac{2s_1 - g(1)^2 + g(0)^2 - 1}{2g(1) - 2g(0)} \quad (12)$$

for $2g(1) - 2g(0) \neq 0$

$$x_2 = -\frac{2s_2 - g(1)^2 + g(0)^2 - 1}{2g(1) - 2g(0)} \quad (13)$$

for $2g(1) - 2g(0) \neq 0$

The calculated intersection points of the circuit K_1, K_2 as $C_1(s_1, x_1)$, $C_2(s_2, x_2)$, make it possible to determine the point C, which is a vertex of the triangle expanding the OFN number. The vertex selection criterion has been determined as the following condition:

$$\text{if} \quad s_1 > 0 \wedge s_2 < 0 \quad \text{then} \quad C = C_1 \quad (14)$$

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$$\text{if} \quad s_1 < 0 \wedge s_2 > 0 \quad \text{then} \quad C = C_2 \quad (15)$$

For the OFN number shown in figure (2) the selected point of the expanding triangle shall be the point C_1, where according to the selection criterion, the condition (14) is met.

A further stage of the method consists in defining the equations of straight lines passing through the points, for the straight line l_1 determined by the points $C = (s_1, x_1)$ and $D = (s_2, x_2)$ respectively. We get the equality:

$$s(x) = l_1(x) = \frac{s_1 x_2 - s_2 x_1 + (s_2 - s_1)\, x}{x_2 - x_1} \quad (16)$$

assuming that $s_2 = 0$, $x_2 = f(0)$, the equation of a straight line l_1 as:

$$s(x) = l_1(x) = \frac{s_1(x - f(0))}{x_1 - f(0)} \quad (17)$$

For straight line l_2 passing through points $A = (s_3, x_3)$ and $E = (s_4, x_4)$ the equation of the straight line assumes the form:

$$s(x) = l_2(x) = \frac{s_3 x_4 - s_4 x_3 + (s_4 - s_3)\, x}{x_4 - x_3} \tag{18}$$

taking into consideration the characteristic values for the ordered OFN number, such that $x_3 = g(0)$, $x_4 = f(1)$ and $s_3 = 0$ and $s_4 = 1$ the Eq. (18) assumes the form:

$$s(x) = l_2(x) = \frac{x - g(0)}{f(1) - f(0)} \tag{19}$$

Considering the Eqs. (17) and (19) for straight lines l_1 and l_2, the searched intersection point is expressed as:

$$\begin{cases} s(x) = l_1(x) = \frac{s_1(x - f(0))}{x_1 - f(0)} \\ s(x) = l_2(x) = \frac{x - g(0)}{f(1) - f(0)} \end{cases} \tag{20}$$

The solution of the set of Eq. (20) is the point $W = (s_w, x_w)$, of the following coordinates:

$$x_w = \frac{f(0)(f(1)s_1 - g(0)s_1 + g(0)) - g(0)x_1}{f(0) + f(1)s_1 - g(0)s_1 - x_1} \tag{21}$$

where: x_1, s_1 are the determined coordinates of the point C as per formulas (12) and (8), considering the vertex selection criterion (14).

Determining a real (defuzzyfied) value in the defuzzyfication process consists in calculation of the argument x_w as the Eq. (21).

3.3 Method 2 - Extended TR Method

In the extended Triangular Expanding method the second intersection point is taken into account. The point of intersection that was rejected, i.e. $C2$ with the point $C1$ applied in the original Triangular Expanding method will be used to build a straight line passing through those points. The straight line created using those two points together with the function g of the ordered fuzzy number shall determine the defuzzyfication value at the point of intersection of these straight lines. The defuzzyfication value will be mainly determined in the area of the function g, i.e. the DOWN part of the OFN number. Thanks to that property the method focuses solely to the goal underlined in the order of the OFN number. Explanation of the proposed solution is shown in Fig. 3, where the straight line lc intersecting the falling edge of the OFN number is presented. Point W indicates the defuzzyfication value in x axis. The analytical formula of the extended Triangular Expanding method is determined similarly to the original Triangular Expanding method. That concept includes similar phases related to the determination of the intersection points with the circles. The calculated intersection points of the circuits K_1, K_2 as $C_1(s_1, x_1)$, $C_2(s_2, x_2)$,

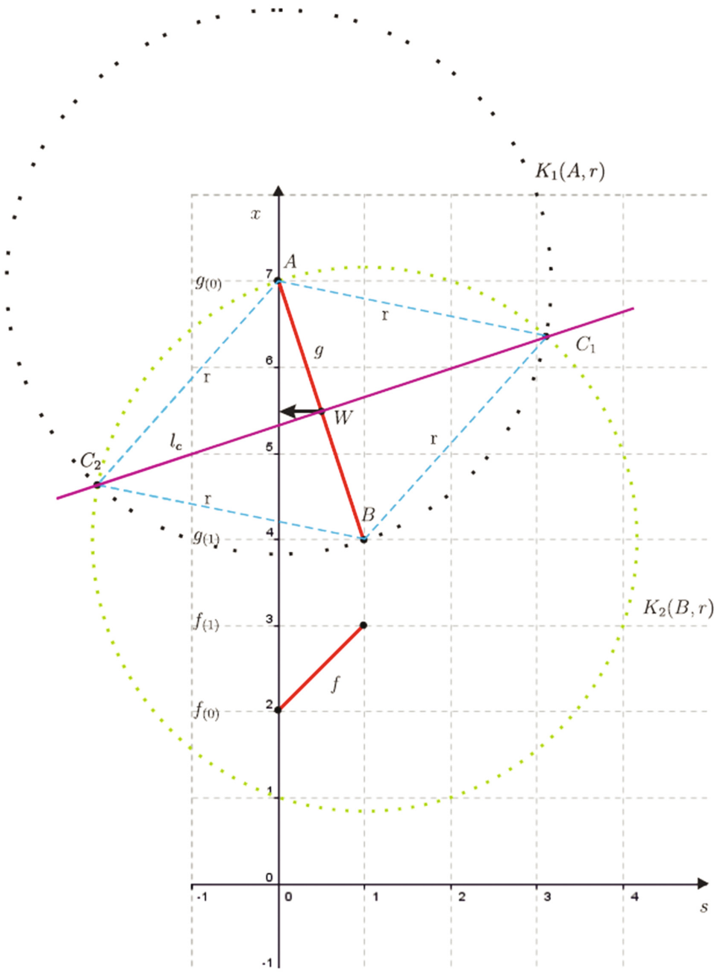

Fig. 3. Graphic interpretation of the extended Triangular Expanding method

make up two triangles extending the OFN number. A straight line l_c drawn through the designated points C_1 and C_2, is described as the function:

$$l_c(x) = \frac{s_1 x_2 - s_2 x_1 + (s_2 - s_1)\, x}{x_2 - x_1} \tag{22}$$

where: s_1 and s_2 and x_1 and x_2 are determined based on the following Eqs. (8, 9, 12, 13).

Together with the function g, of the formula:

$$g(x) = \frac{s_3 x_4 - s_4 x_3 + (s_4 - s_3)\, x}{x_4 - x_3} \tag{23}$$

where: s_3 and s_4 as well as x_3 and x_4 are values $0, 1$ of the function g and arguments $g(0)$, $g(1)$, respectively.

The intersection point of the functions (22, 23) is calculated by comparison of the above functions:

$$\begin{cases} l_c(x) = \frac{s_1x_2 - s_2x_1 + (s_2 - s_1)x}{x_2 - x_1} \\ g(x) = \frac{s_3x_4 - s_4x_3 + (s_4 - s_3)x}{x_4 - x_3} \end{cases} \tag{24}$$

The result is the point $W = (s_w, x_w)$, the value x_w of which defines the defuzzyfication:

$$x_w = \frac{((s_3 - s_1)\ x_2 + (s_2 - s_3)\ x_1)\ x_4 + ((s_1 - s_4)\ x_2 + (s_4 - s_2)\ x_1)\ x_3}{(s_2 - s_1)\ x_4 + (s_1 - s_2)\ x_3 + (s_3 - s_4)\ x_2 + (s_4 - s_3)\ x_1} \tag{25}$$

and

$$s_w = \frac{(s_2 - s_1)\ s_3\, x_4 + (s_1 - s_2)\ s_4\, x_3 + (s_1\, s_3 - s_1\, s_4)\ x_2 + (s_2\, s_4 - s_2\, s_3)\ x_1}{(s_2 - s_1)\ x_4 + (s_1 - s_2)\ x_3 + (s_3 - s_4)\ x_2 + (s_4 - s_3)\ x_1} \tag{26}$$

Taking into account the points of the OFN number, the formulas (25, 26) assume the following form:

$$x_w = \frac{((g(1) - g(0))\ s_1 - g(0))\ x_2 + ((g(0) - g(1))\ s_2 + g(0))\ x_1}{x_2 - x_1 + (g(0) - g(1))\ s_2 + (g(1) - g(0))\ s_1} \tag{27}$$

and

$$s_w = \frac{s_1\, x_2 - s_2\, x_1 - g(0)\, s_2 + g(0)\, s_1}{x_2 - x_1 + (g(0) - g(1))\ s_2 + (g(1) - g(0))\ s_1} \tag{28}$$

4 Comparison of Methods

The proposed defuzzyfication method as the Triangular Expanding presented in the third chapter of this paper was intended to be sensitive to the order of the ordered fuzzy number present in the argument. It is going to be demonstrated further in the article. In this section, the proposed method will be compared with the methods FOM, LOM, GM and COG. The first two methods belong to the group of maxima methods. In the FOM (First Of Maxima) method, the defuzzyfication value is determined for the first element of the kernel, where the kernel means the section of the domain in which the membership function achieves the maximum value equal one. In the case when the defuzzyfication value is associated with the last element of the kernel, the LOM (Last Of Maxma) method is achieved. The Geometrical Mean (GM) method by D. Wilczyńska is described in detail in the article [49]. It is based on determination of the intersection of two lines located at the poles of the OFN number. The last method, which was selected for comparison, is the Center Of Gravity method. This method yields good results both in case of ordered fuzzy numbers and convex fuzzy numbers. The calculations for that method were carried out by means of the software developed by L. Lewiński and M. Szymański. For the other methods, the defuzzyfication value was determined geometrically using the tool: GeoGebra - Dynamic Mathematics for Everyone. The following two numbers were used for the needs of the test: $H = [2, 3, 4, 7]$ and $Z = [7, 4, 3, 2]$. Thire graphic interpretation is shown in Fig. 4, where: (a) H number and (b) the reversely ordered Z number. The presented numbers are characterized by the same shape but differ in the opposite order. Those numbers were subject to defuzzyfication using individual methods. The calculation results are presented in Table 1. For the GM and COG methods, the order does not affect the defuzzyfication value. In the case when the maxima method is used, either FOM or LOM, the defuzzyfication value depends on the order of the number, thus yielding extreme values of defuzzyfication in the kernel area. The results of the Triangular Expanding method indicate that the method is order sensitive. For the H number, the defuzzyfication value is 3.3 and

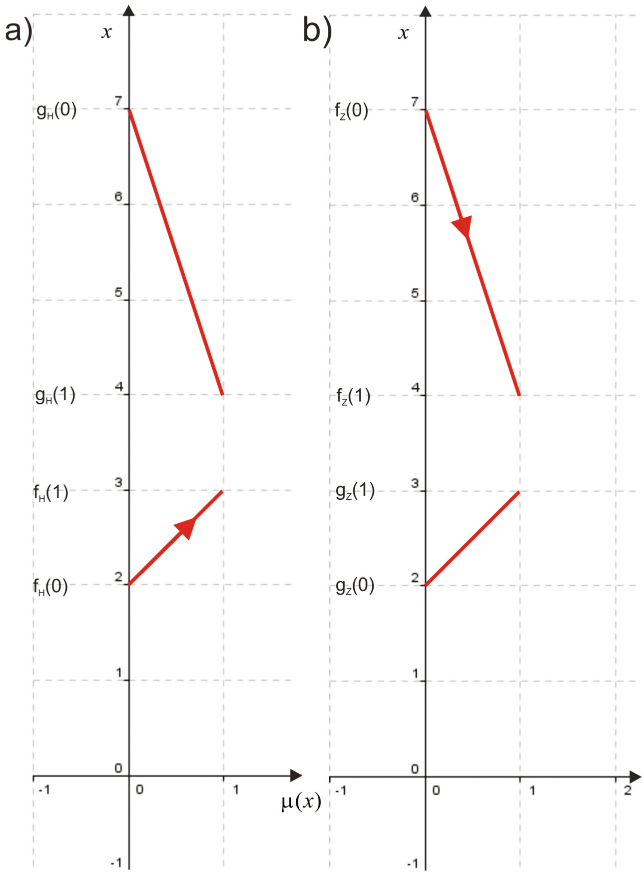

Fig. 4. Two OFN numbers, (a) H number and (b) the reversely ordered Z number

Table 1. Defuzzification measurements for two OFN numbers

Ordered Fuzzy Number	Defuzzyfication method				
	TE	FOL	LOM	GM	COG
H = [2,3,4,7]	3.3	3	4	3.67	4.11
Z = [7,4,3,2]	3.67	4	3	3.67	4.11

for Z number it amounts to 3.67. It can also be noticed that for Z number, the TE method yields the same defuzzyfication as GM method for both numbers: H and Z. Thus it proves that TE method is order sensitive.

5 Conclusions

Ordered fuzzy numbers, as the mathematical method providing broad possibilities for the information describing and processing, has become a new solution in the construction of controller models used as a tool for inference or control. Main advantages of this approach include smooth application of real numbers algebra. The proposed defuzzyfication method as a Triangular Expanding for ordered fuzzy numbers, can be an alternative to existing solutions that do not take into account an order. It has been proved in this article that the method is order sensitive. The results presented clearly show the course of the defuzzyfication value change. Further research, the authors intends to take, will be related to determination of the efficiency and its comparison to widely used methods such as COA. In his article, the authors intended to use ordered fuzzy numbers to create the OFCL (Ordered Fuzzy Control Language) for the fuzzy controller. Although the specification of FCE (Fuzzy Control Language) is available for the conventional fuzzy logic, which is used by manufacturers of re-programmable controllers, the ability to compare the method with other ones and to disseminate the technology is a sufficient motivation to continue research in that scope.

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Construction of Intuitionistic Fuzzy Cognitive Maps for Target Marketing Strategy Decisions

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Abstract. Shopping malls today, are the centers of social life. There exists an increasing demand of shopping malls. The market's profit margin is promising however from the investor's perspective, shopping malls require big investments and the risk level is high. For the success of a shopping mall, its target market needs to be chosen carefully in the beginning and its marketing mix needs to be in the same direction. There exist many factors influencing the target market strategy selection of a shopping mall; this process can be observed as an MCDM problem. The purpose of this study is to determine the interrelations between the criteria affecting a shopping mall's target market differentiation degree, to select a target marketing strategy while considering the hesitations of decision makers and to represent this complex decision making system with intuitionistic fuzzy cognitive maps. Numerical application is made for investment decision making process of a new shopping mall that will locate in Istanbul, Turkey.

Keywords: Shopping mall · Target market strategy · Intuitionistic fuzzy cognitive map · MCDM

1 Introduction

Since consumption habits have increased worldwide, there is a growing interest in shopping malls. This interest in Turkey has also increased in proportion to the world. In Turkey, where the first shopping mall was opened in 1988, there are currently more than 400 active shopping malls. As the green areas gradually diminished, shopping malls became the center of social life. Especially in cities with parking problems, shopping malls with parking facilities have become common meeting points. By the early 1990s, a growing population and a strong economy encouraged a demand for better shopping facilities and a recognition that there was too much of an emphasis on the central area [1]. The first shopping mall in Turkey was founded in 1988 followed by an average of one per year between 1988–1997 [2]. By the end of 2016, there are around 100 shopping malls in İstanbul, followed by 35 in Ankara, 19 in İzmir and 16 in Antalya and overall 400 across country, visited by over a billion visitors per year. Due to the intense competition, there is an increasing pressure on shopping malls to clearly differentiate themselves more distinctively [3]. Therefore the target market of a shopping mall should be chosen carefully in the strategic marketing planning.

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Strategic planning process starts with thorough internal and external analysis. Then, business-units strategies are formed accordingly. Once the strategies are formed, it needs to be implemented and constant surveillance and feedbacks are crucial in order to keep the strategies up to date and prevent them to become obsolete. This surveillance must be made with concrete performance measures. In this paper, we used Intuitionistic Fuzzy Sets (IFS) Theory, which incorporates the hesitation of decision makers; combined with Fuzzy Cognitive Maps (FCM), which provides a representation and quantitative solution of the causal relationships among the decision criteria given by the decision makers.

Strategy prioritization in shopping malls is a multi-criteria decision making problem with many conflicting tangible/intangible and independent/dependent criteria. Scholars contributed to literature with MCDM approaches to shopping mall decisions. [4] used Fuzzy ANP method for target market strategy selection, [5] used Stepwise Weight Assessment Ratio Analysis (SWARA) and Weighted Aggregated Sum Product Assessment (WASPAS) for location selection, [6, 7] used ANP for retail tenant mix planning and location selection, respectively.

Many decision-making and problem-solving tasks are too complex to be understood quantitatively; however, people succeed by using knowledge that is imprecise rather than precise. Fuzzy set theory resembles human reasoning in its use of approximate information and uncertainty to generate decisions. It was specifically designed to mathematically represent uncertainty and vagueness and provide formalized tools for dealing with the imprecision intrinsic to many problems. By contrast, traditional computing demands precision down to each bit. Since knowledge can be expressed in a more natural by using fuzzy sets, many engineering and decision problems can be greatly simplified. Fuzzy set theory implements classes or groupings of data with boundaries that are not sharply defined (i.e., fuzzy). There are few studies on shopping mall using fuzzy logic [8–10]. As an extension of fuzzy sets, intuitionistic fuzzy sets provide the decision makers the flexibility of expressing their hesitations on the information that they give. In this paper IFSs are used to represent hesitations of the decision makers mathematically

After a thorough literature survey, one can see that, in such shopping mall case, IFCM and/or FCM have not been used as a decision tool and furthermore, although FCM has been used, albeit once, in marketing strategy selection [11] but IFCM has never been used. Therefore, the originality of our study comes from the novel approach to marketing strategy selection and a novel decision making application area.

The rest of the paper is organized as follows: In Sect. 2, target marketing strategy is described. In Sect. 3, IFCM method is explained in detail. In Sect. 4, numerical application is given with a case study and the paper is concluded in Sect. 5.

2 Target Marketing Strategy

Marketing strategy of an organization consists of determining a target market or target markets and afterwards determining a marketing mix (4P of product, price, place and promotion). The strategy has to come up with the right combination of target market(s) and marketing mix(es) to ensure competitive advantage on the rivals.

There are two main options in order to select the target market:

– Undifferentiated Strategy: This is also called mass marketing and refers to considering the market as one homogeneous group of customers whose requirements are somewhat same.
– Multi-segment Marketing: In this approach, the market is considered to be formed by clusters/segments of customers where each segment has its own requirements/ needs. Therefore, in this approach, each segment needs its own strategy.

Under multi-segment marketing, one can specifically focus on one or few segments in a whole. This is called Concentrated (Niche) Marketing. And if the focus is on specific individuals, then the concept is called Micromarketing.

Companies need to consider many factors when choosing a market-targeting strategy. Which strategy is best depends on the company's resources. When the firm's resources are limited, concentrated marketing makes the most sense. The best strategy also depends on the degree of product variability. Undifferentiated marketing is more suited for uniform products, such as grapefruit or steel. Products that can vary in design, such as cameras and cars, are more suited to differentiation or concentration. The product's life-cycle stage also must be considered. When a firm introduces a new product, it may be practical to launch one version only, and undifferentiated marketing or concentrated marketing may make the most sense. In the mature stage of the product life cycle, however, differentiated marketing often makes more sense [12].

For our application we adopted the model presented in [4].

Four main criteria of the decision model and their brief explanation including their sub-criteria are as follows:

– Brand Mix: Refers to the brands which will be included in the shopping mall. This will determine the mall's success from the perspective of sales, tenants etc. The brands included in a mall will form a perception of attractiveness in consumers' minds and this will affect the time consumed in that mall. The sub-criteria are: Product type, Profit, Brand Awareness, Brand Loyalty, Life Cycle, Market Share and Economy of Scale.
– Company: Refers to the owner and manager of the shopping mall in question. A successful shopping mall is a result of an effective management. A review of the retailing literature reveals six dominant attributes in shopping center's company concept: company resources, risk level, accessibility, capacity, price determinant, innovation and adaptation to technological developments [4].
– Market: Refers to the understanding of the market in general in order to form an appropriate strategy. Now with the technological advances and ease of use that come with it in online shopping, the fierce competition among the shopping malls is even fiercer. Mall market indicator was formed by the following factors: market structure, segment size, market growth, and number of competitors [4].
– Consumer: Marketing Strategy is built around an understanding of the consumer. A shopping mall will stand a chance to satisfy its customers once the consumers' needs and requirements are analyzed. Shopping mall consumer criteria were formed by the following factors: Heterogeneity of Consumer Shopping Behaviour, Ease of Consumer Profiling, Consumer Sensitivity to Point of Purchase Promotions, and Consumer Sensitivity to Prices [4].

3 Intuitionistic Fuzzy Cognitive Maps

3.1 Fuzzy Sets and Intuitionistic Fuzzy Sets

A conventional fuzzy set is a set containing elements that have varying degrees of membership in the set. This idea is in contrast with classical or crisp sets because members of a crisp set would not be members unless their membership is full, or complete, in that set (i.e., their membership is assigned a value of 1). Elements in a fuzzy set, because their membership need not be complete, can also be members of other fuzzy sets on the same universe [13]. If an element in the universe U, say x, is a member of fuzzy set $\tilde{A}$ then this mapping is given by the membership function; $\mu_{\tilde{A}}(x) \in [0, 1]$. The membership value is the belongingness of the element $x \in U$ to the set $\tilde{A}$. Fuzzy set $\tilde{A}$ has a form as shown in Eq. (1).

$$\tilde{A} = \left(\langle x, \mu_{\tilde{A}}(x)\rangle | x \in U\right) \tag{1}$$

Intuitionistic Fuzzy Sets (IFSs), which are introduced by Atanassov in 1986 [14, 15], represent the generalization of conventional fuzzy sets. IFSs have an additional value which is the non-membership degree $\eta_A(x) \in [0, 1]$. It is the degree of "not being" in the set. For every $x \in U$, $0 \leq \mu_A(x) \leq 1$ and $0 \leq \eta_A(x) \leq 1$. Also, $0 \leq \mu_A(x) + \eta_A(x) \leq 1$. IFS A has a form as shown in Eq. (2).

$$\mathrm{A} = (\langle x, \mu_{\mathrm{A}}(x), \eta_A(x)\rangle | x \in U) \tag{2}$$

In application, IFSs are abbreviated and represented as $\mu_{\mathrm{A}}(x), \eta_A(x)$. If $\mu_{\mathrm{A}}(x) + \eta_A(x) = 1$, then x is a fuzzy set. If $\mu_A(x) + \eta_A(x) < 1$, then x is an intuitionistic fuzzy set with the degree of hesitancy $\pi_A(x) \in [0, 1]$. For every $x \in U$,

$$\mu_{\mathrm{A}}(x) + \eta_A(x) + \pi_A(x) = 1 \tag{3}$$

The degree of hesitancy represents the lack of information and the hesitations of the decision makers in the problem.

3.2 Fuzzy Cognitive Map

Cognitive Maps are a type of directed graph that offers a means to model interrelationships or causalities among concepts; there are various forms of cognitive maps, such as signed digraphs, weighted graphs, and functional graphs. The use of simple binary relationships (i.e., increase and decrease) is done in a conventional (crisp) cognitive map. Cognitive maps have a clear way to visually represent causal relationships, they expand the range of complexity that can be managed, they allow users to rapidly compare their mental models with reality, they make evaluations easier, and they promote new ways of thinking about the issue being evaluated [13]. Fuzzy Cognitive Map (FCM) is an extension of conventional cognitive map that includes various degrees of increase or decrease (small increase, large decrease, almost no decrease, etc.). FCMs include concept nodes and weighted arcs that are graphically showed as a signed weighted graph with feedback. Signed weighted arcs, connecting the concept nodes, display the causal relationship that exists among concepts [16]. Graphical representation of FCM is shown in Fig. 1. The weight value w represents the strength of relationship between two concepts.

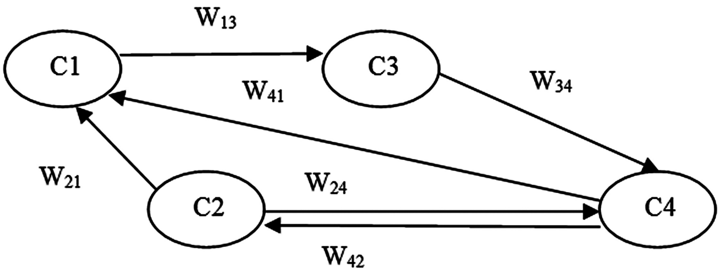

Fig. 1. FCM representation

In conventional cognitive maps, the weight w can only have three values; −1, 0 and 1. In FCMs, w values can be fuzzy numbers and this gives an infinite number of choices to decision makers to speak their mind.

The value of each concept is calculated, computing the influence of other concepts to the specific concept, by applying the following calculation rule:

$$A_i^{(k+1)} = f\left(A_i^k + \sum\nolimits_{j=1}^{N} A_j^{(k)} w_{ji}\right) \tag{4}$$

where $A_i^{(k)}$ is the value of concept C_i at iteration step k, $A_j^{(k+1)}$ is the value of the concept C_j at iteration k + 1, w_{ji} is the weight of the connection from C_j to C_i and f is a threshold function [16].

3.3 Intuitionistic Fuzzy Cognitive Maps

Intuitionistic Fuzzy Cognitive Map (IFCM) is introduced in 2009 [17] and applied in medical decision making. Then it has been developed and discussed [18–20] in recent years. Its applications proved its usefulness in medical decision making when the decision makers have hesitations. In marketing field, decision makers usually have hesitations because it is very expensive and time-consuming to gather the customer data. IFCM is a powerful tool to cope with hesitations.

IFCM has the same iteration-based system with FCM however its equation is slightly different:

$$A_i^{(k+1)} = f\left(A_i^k + \sum\nolimits_{j=1}^{N} A_j^{(k)} w_{ji}^{\mu} - A_j^{(k)} w_{ji}^{\pi}\right) \tag{5}$$

In Eq. (5), $w_{ji}^{\mu} \in [-1, 1]$ represents the influence weight and $w_{ji}^{\pi} \in [-1, 1]$ represents the hesitancy weight. Hesitancy has a negative effect on the interrelations but the weight factor $(w_{ji}^{\mu} - w_{ji}^{\pi})$ may not always conserve the direction of the influence. Hence the linguistic scale must be chosen carefully which all have $\left|w_{ji}^{\mu}\right| > \left|w_{ji}^{\pi}\right|$.

As a decision making tool, the steps of IFCM method is as follows:

Step 1: The key-factors of the system, namely the concepts, are defined by the experts (decision makers). Input concepts and output concepts are determined.
Step 2: The interactions between these concepts are first specified by the experts as "positive relation" or "negative relation" and then the strengths of these relations are determined by the experts using linguistic terms.
Step 3: According to a pre-defined scale, linguistic terms are signed to IFSs and all the weight values are obtained.
Step 4: Using Eq. (5), the final value of each concept is calculated.

4 Case Study: New Shopping Mall in Istanbul

Numerical application of IFCM method is the case study of a new shopping mall in Istanbul. The exact location and the capacity of the mall have been already decided however, investors and managers need to determine a target marketing strategy as soon as possible because all the brands that will serve in the mall, must be chosen considering this strategy. The investors of this mall are experienced in the market and the managers are specialized in shopping mall initialization and management. The mall will be installed in a region where high income segment (A++) is located. However in that region there exist other malls that people got used to. Therefore the competition will be challenging.

Decision makers of this application are one investor and two managers of the mall. They prepared the data with consensus hence no aggregation method is used. Numerical application steps are as follows:

Step 1: The concepts of IFCM are the criteria that have influence on target market differentiation degree. There exist 21 criteria which are given in Sect. 2:

C1: Product type, C2: Profit, C3: Brand Awareness, C4: Brand Loyalty, C5: Life Cycle, C6: Market Share, C7: The Economy of Scale, C8: Company Resources, C9: Risk Level, C10: Accessibility, C11: Capacity, C12: Price Determinant, C13: Innovation and Adaptation to Technological Developments, C14: Segment Size, C15: Number of Competitors, C16: Market Growth, C17: Market Structure, C18: Heterogeneity of Consumer, C19: Ease of Consumer Profiling, C20: Consumer Sensitivity to Point of Purchase Promotions, C21: Consumer Sensitivity to Prices.

These criteria are the input concepts of the model. There is only one output concept, C22: Target Market Differentiation Degree. Target marketing strategy alternatives are explained in Sect. 2. The scale of the output concept, that relates final values to the alternatives, is pre-defined according to expert decisions and marketing indicators as shown in Fig. 2. The thermal scale is used to represent the relations between output values and alternatives, since there are not strict boundaries and there exist intersections. For example, if the output value is calculated as 0.35, that means mass marketing or differentiated marketing can be used or a mixed strategy of these two can be employed.

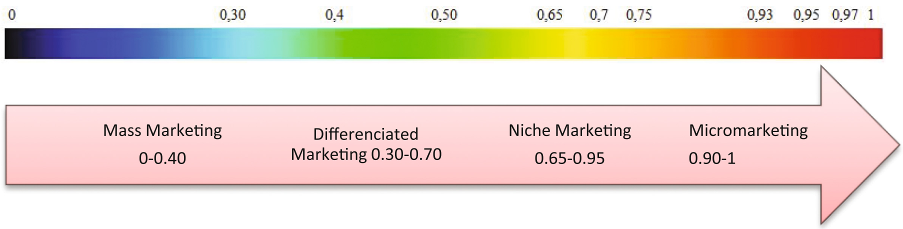

Fig. 2. Scale of the output concept

Step 2: Directions and strengths of relations between concepts are determined by the experts using linguistic terms: "Positive-Negative" for directions and "Very Low-Low-Medium-High-Very High" for strengths. All the relations are listed in Table 2. The IFCM with the concepts and relations are shown in Fig. 3. In the graph, blue and red arrows represent positive and negative relations, respectively. Strengths of the relations are represented with the width of arrows; maximum width represents "Very High" (Table 1).

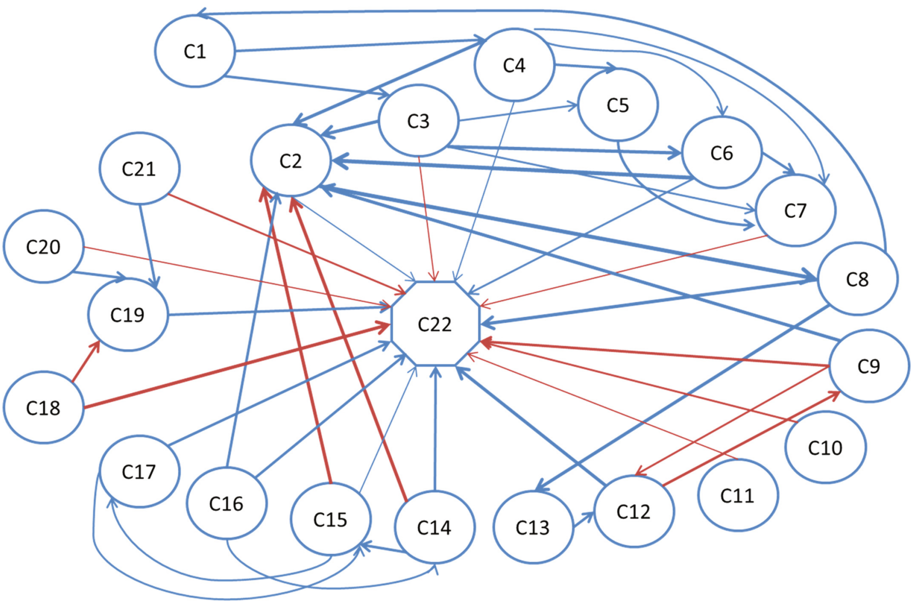

Fig. 3. The IFCM

Step 3: The scale, which assigns IFSs to linguistic terms, is chosen from the literature of multi-attribute group decision making with IFSs. The scale used in the numerical application by Li and Huang [21] is suitable for this study. The scale is given in Table 2 using the representation given in Eq. (2). This scale assumes that boundary

Table 1. Relations between concepts

Relation	Direction	Strength	Relation	Direction	Strength
C1–C3	Positive	Medium	C1–C4	Positive	Medium
C2–C8	Positive	Very High	C2–C22	Positive	Very Low
C3–C2	Positive	High	C3–C5	Positive	Low
C3–C6	Positive	High	C3–C7	Positive	Low
C3–C22	Negative	Very Low	C4–C2	Positive	High
C4–C5	Positive	Medium	C4–C6	Positive	Low
C4–C7	Positive	Low	C4–C22	Positive	Very Low
C5–C7	Positive	Medium	C6–C2	Positive	Very High
C6—C7	Positive	Medium	C6–C22	Positive	Low
C7–C22	Negative	Very Low	C8–C1	Positive	Medium
C8–C13	Positive	High	C8–C22	Positive	High
C9–C2	Positive	High	C9–C12	Negative	Low
C9–C22	Negative	Medium	C10–C22	Negative	Low
C11–C22	Negative	Very Low	C12–C9	Negative	Medium
C12–C22	Positive	High	C13–C12	Positive	Medium
C14–C2	Negative	High	C14–C15	Positive	Medium
C14–C22	Positive	Medium	C15–C2	Negative	High
C15–C17	Positive	Low	C15–C22	Positive	Very Low
C16–C2	Positive	Medium	C16–C14	Positive	Low
C16–C22	Positive	Medium	C17–C15	Positive	Low
C17–C22	Positive	Medium	C18–C19	Negative	Medium
C18–C22	Negative	High	C19–C22	Positive	Medium
C20–C19	Positive	Medium	C20–C22	Negative	Very Low
C21–C19	Positive	Medium	C21–C19	Positive	Medium
C21–C22	Negative	Low			

Table 2. The scale of linguistic terms and IFSs

Linguistic terms	IF sets
Very Low	$\langle 0.05, 0.95 \rangle$
Low	$\langle 0.25, 0.7 \rangle$
Medium	$\langle 0.5, 0.4 \rangle$
High	$\langle 0.7, 0.25 \rangle$
Very High	$\langle 0.95, 0.05 \rangle$

values do not include hesitations; if a decision maker says “Very Low” instead of “Low” or “Very High” instead of “High”, it means he/she is certain about this information, therefore the sum of membership and non-membership degrees equals to 1 as opposed to what appears for the other linguistic values for which a degree of hesitation exists.

Step 4: According to Eq. (5), iterations are coded in MATLAB. Initial values of concepts are taken as 1; $A_i^0 = 1$, for all i = 1, …, 22. The sigmoid function is used as

Table 3. Final values of concepts

C1	Product Type	0.8898
C2	Profit	0.9124
C3	Brand Awareness	0.7517
C4	Brand Loyalty	0.7517
C5	Life Cycle	0.7727
C6	Market Share	0.8098
C7	Eco. of Scale	0.8570
C8	Company Resources	0.8474
C9	Risk Level	0.5707
C10	Accessibility	0.6590
C11	Capacity	0.6590
C12	Price Determinant	0.7146
C13	Innovation	0.7931
C14	Segment Size	0.6959
C15	No of Competitors	0.7657
C16	Market Growth	0.6590
C17	Market Structure	0.7016
C18	Heterogeneity	0.6590
C19	Ease of Profiling	0.7298
C20	Sensit. to Promo's	0.6590
C21	Sensit. to Prices	0.6590
C22	Differentiation Degree	0.9048

threshold function since the value of the output concept must be between 0 and 1. The values are converged in 8 iterations and the final values are given in Table 3.

The final value of the output concept "Target Market Differentiation Degree" is 0.9048. According to the scale given in Fig. 2, this value is in the intersection of two alternatives "Niche Marketing" and "Micromarketing". The IFCM model claims that the managers of this shopping mall should use niche marketing or micromarketing strategy, or a combination of them as the target marketing strategy of this new shopping mall.

5 Conclusions

Target market strategy selection of shopping malls plays a major role in their future success and maintainability. It is a one-time decision that involves great risks because of huge investments. In this study, we used IFCM method, which employs cognitive mapping to model the complex decision making system while considering hesitations of the decision makers using IFSs, in order to determine the best target marketing strategy of a new shopping mall. 21 input concepts and one output concept are implicated in the decision making model and the result is calculated as "Niche Marketing-Micromarketing". According to the experts, a mix strategy of niche marketing and micromarketing is also the best target marketing strategy for this shopping mall. In addition, when the experts interpret the rest of the final values, some remarks emerged:

The final values of the input concepts can be interpreted as; the influence of the factor over the target marketing strategy being "Niche Marketing-Micromarketing". The greatest final value belongs to "Profit" criterion (0.9124) which is significant for these strategies. Amongst these criteria, "Profit" is the biggest motivation of launching a niche shopping mall because niche markets have the greatest profit margins.

"Product type" is the second with the value (0.8898) which is revealing because the type of the product is one of the essential factors while selecting its target marketing strategy. Type of the products that the shopping mall offers is mostly decisive for its target marketing strategy. If they sell mostly convenience products in that mall, the target marketing strategy cannot be niche.

"Risk Level" has the smallest final value which is also significant; "Niche Marketing-Micromarketing" strategy involves major risks. If these strategies are chosen, it shows the risk seeking behavior and it means that a high level of risk has been already envisaged. Hence the risk has a very minor effect on these strategies.

Other factors also have influences close to 1, therefore no concepts should be excluded from the model. The model is well-constructed.

In this target marketing strategy selection model, effects of the concepts varies according to industry's structure (service or production), periods of economic prosperity and depression, economic conjuncture of the country etc. As a future research, the model can be applied in different industries and the effect of economic changes can be observed.

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Intercriteria Analysis of EU Competitiveness Using the Level Operator N_γ

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Abstract. A recent leg of research on a new level operator over intuitionistic fuzzy sets, $N_{\gamma,}$ inspired the development of a new approach to establishing the thresholds for evaluation of the results of application of the InterCriteria Analysis (ICA) over multiobject multicriteria problems. ICA is a novel method of detecting the levels of pairwise correlations within the set of criteria (termed here positive consonance, negative consonance and dissonance), which uses as input the dataset of measurements or evaluations of a set of objects against these criteria. The output of ICA, being a matrix of intuitionistic fuzzy pairs, gives all possible consonances and dissonances between the pairs of criteria, and it is a matter of either expert decision or algorithmic solution what thresholds of precision will be implemented to outline the top correlating pairs of criteria and yield certain domain-specific conclusions from the data. The present paper discusses practical aspects of selecting these top performing pairs of criteria with the use of the newly proposed intuitionistic fuzzy level operator N_γ. For illustrative purposes, we analyze the dataset of 28 EU member states' performance from the Global Competitiveness Report of the World Economic Forum for the year 2016–2017. Further, we comment on the interval in which parameter γ reasonably varies, making use of the intuitionistic fuzzy interpretational triangle and the topological operators *Interior* and *Closure*.

Keywords: Intercriteria analysis · Intuitionistic fuzzy sets · Multiobject multicriteria decision making · Competitiveness · Intuitionistic fuzzy topological operators · Closure · Interior

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1 Introduction

In [12], a new operator of level type was defined over intuitionistic fuzzy sets (IFSs), expanding the available theoretical knowledge and instrumentation for handing IFSs (see [1, 3, 5, 7]). The new operator N_γ elaborates the idea of the existing operator $N_{\alpha,\beta}$, which aims to produce a subset of an IFS, whose elements have their degrees of membership above a given level (threshold) α and have their degrees of non-membership below a given level β. The formal notation of the operator producing the subset of A, $N_{\alpha,\beta}(A)$, is the following:

$$N_{\alpha,\beta}(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E \& \mu_A(x) \geq a \& \nu_A(x) \leq \beta\},$$

where A is an IFS within a universe E, and α, $\beta \in [0, 1]$, $\alpha + \beta \leq 1$, are fixed numbers [5]. The definition of the new level operator, as given in [12], is as follows.

Definition: Let us call an IFS A *ν-positive*, if for each A we have $(\forall x \in E)(\nu_A(x) > 0)$. Let us define for each ν-positive IFS A the operator

$$N_\gamma(A) = \left\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E \& \frac{\mu_A(x)}{\nu_A(x)} \geq \gamma\right\},$$

where γ is an arbitrary non-negative real number.

Several basic properties of the new level operator N_γ were formulated and proved in [12], showing its application over the union and intersection of two IFSs, as well as the relations between N_γ and the modal operators *Necessity* and *Possibility*, and between N_γ and the topological operators *Interior* and *Closure.*

The so formulated new operator is graphically visualized onto the IF interpretational triangle as presented in the following Fig. 1.

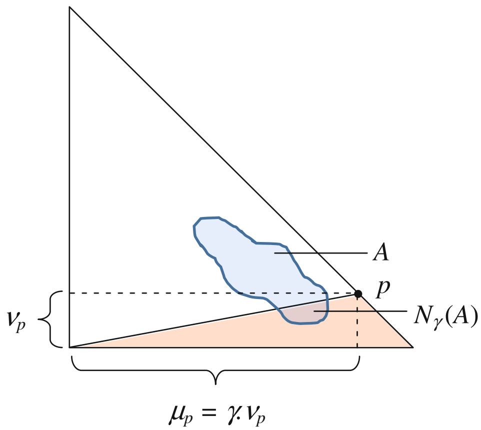

Fig. 1. Graphical visualization of the operator $N\gamma$ onto the IF triangle

The new operator inspires a new leg of research related to the thresholds for evaluation of the results of application of the InterCriteria Analysis over multiobject multicriteria problems (see a detailed overview in [20]).

InterCriteria Analysis (ICA, originally proposed in [9]) is an IFS-based novel method for detecting the levels of pairwise correlations within the set of criteria, based on the measurements or evaluations of the set of objects against these criteria. The ultimate goal of the method is to detect data-supported hypotheses if some of the criteria exhibit high enough correlations with others, so that skipping them from the further decision making process would not affect the whole process. The motivation behind this method is that in certain domains the need has been defined to eliminate some of the criteria, when measurement against these comes at a higher cost, consumes more time or other resources, or is in any other way considered undesirable. Selecting these high enough correlations (termed in ICA as: positive consonance, negative consonance or dissonance) requires either an expert decision or an algorithm for the precise establishment of the thresholds, beyond which the top-correlating criteria are selected in order to yield certain problem-specific conclusions [9].

On the input, ICA requires a table of numerical values of the measuring of m objects against n criteria. It returns as output a matrix of intuitionistic fuzzy pairs (IFPs) with the relations between all the $n(n - 1)/2$ pairs of criteria. Since ICA is based on the concept of IFS and index matrices [8], its algorithm is constructed in such a way to handle the inherent uncertainty, as represented by the use of IFPs, which consist of a pair of numbers in the [0, 1]-interval, staying for the membership and the non-membership, whose sum is also a number in this interval, [10]. These $n(n - 1)/2$ intercriteria pairs, represented as IFPs, can be plotted as points onto the IF interpretational triangle, where the membership and non-membership parts stay respectively for the abscissa and ordinate (see [4, 5]).

2 Discussion of the Parameter γ

In the previous work [12] where the operator N_γ was defined for the first time, we formulated and proved the validity of several statements. At the subsequent step when we aim to apply the operator to a particular dataset, however, we face the question about the practical considerations of the range in which parameter γ changes, so that it cuts a nonempty set as a subset of the IFS. Obviously, if we are interested in the subset of the IFS, which is closer to the *Truth*, we should define γ as $\gamma \in (1, \infty)$, since $\gamma = 1$ gives the angle bisector of the right angle of the IF interpretational triangle, and if we are interested in the subset closer to the *Falsity*, we should define γ as $\gamma \in (0, 1)$.

More precisely, here we can benefit from our knowledge of the topological operators *Interior* and *Closure*, which are defined using the following formulas, and illustrated in Fig. 2. (see [2, 5–7]).

$$I(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}$$

$$C(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\}$$

These two operators help finding the range, from which parameter γ shall reasonably be selected in a particular decision making case, i.e.:

$$\gamma \in \left[\frac{\inf_{y \in E} \mu_A(y)}{\sup_{y \in E} \nu_A(y)}; \frac{\sup_{y \in E} \mu_A(y)}{\inf_{y \in E} \nu_A(y)} \right].$$

For instance, in the illustrative example in Fig. 2, we have some IFS of elements, visually interpreted in the white points in the IF triangle. With the help of the operators *Interior* and *Closure*, expressed with the points with coordinates *C*(*A*) (0.40, 0.49) and *I*(*A*) (0.78, 0.14), parameter γ should be selected from the interval [0.82; 5.57]. For this particular case, if selected outside of this interval, γ will produce an empty subset of the IFS.

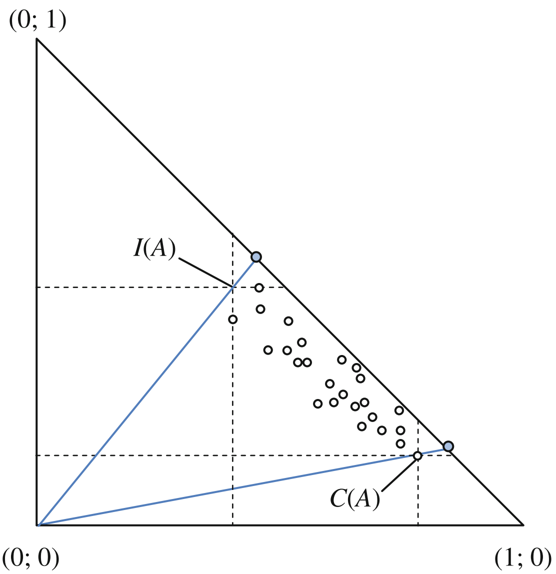

Fig. 2. An illustrative IFS, plotted onto the IF Triangle, with the indicated places of the topological operators *Interior* and *Closure*.

On a side note, the illustrative example in Fig. 2 shows that even finer correction of the boundaries of the range of parameter γ is possible, depending on the actual location of the elements of the set (e.g., see the elements, defining *I*(*A*)).

As it was noted in [12], the idea about the new operator N_γ has been inspired by the theory of the American psychologist John M. Gottman that marital relationships are likely to be stable if they exhibit the "magic ratio" of 5:1 of positive to negative interactions between the partners (see [21]). This idea is transferred here in the ratio μ/ν of the membership–to–non-memberships calculated for the elements of an IFS, which is another way to shortlist some of these elements, along with those already proposed in [11, 14, 16–18] (see also [20] for an overview).

3 Application of the Operator N_γ Over Real Data

We approbate the use of the newly proposed level operator with the dataset of 28 European Union member states' performance from the Global Competitiveness Report (GCR) of the World Economic Forum for the year 2016–2017 [25], taking as a motivation the WEF's general address to policy makers to '*identify and strengthen the transformative forces that will drive future economic growth*' [24]. So far, there has been a long-term research of the application of various threshold of the ICA method with data from the GCRs over the years [13–17], which allows not only comparison of the same threshold techniques over the datasets from consequent years, but also different threshold techniques over the same datasets. Another research of applying ICA too various EU enterprises was performed in [19].

The input data for the ICA are collected in the Table 1, populated with the data of the evaluations of the 28 EU Member States (in ICA: *objects*) against the 12 pillars of competitiveness (in ICA: *criteria*), being '1. Institutions'; '2. Infrastructure'; '3. Macroeconomic stability'; '4. Health and primary education'; '5. Higher education and training'; '6. Goods market efficiency'; '7. Labor market efficiency'; '8. Financial market sophistication'; '9. Technological readiness'; '10. Market size'; '11. Business sophistication'; '12. Innovation'. These evaluations are input in the form of an index matrix with dimensions 28 × 12. According to the adopted methodology of WEF, the set of possible scores for each of the 12 criteria comprises the numbers 1.0, 1.1, …, 6.9, 7.0.

As an output of the software application implementing the algorithm of ICA [22, 23], we obtain two 12 × 12 tables (Tables 2 and 3), giving respectively the membership and the non-membership parts of the IFPs forming the degrees of relation between each pair of criteria (intercriteria pairs). The two matrices are symmetrical according to their main diagonal, along which all the IFPs are all identical to the perfect *Truth* plotted in the interpretational triangle in (1, 0), since every criterion would perfectly correlate with itself.

Taking the results produced by the ICA, with the IF pairs distributed in two index matrices M^μ and M^ν, collected respectively in Tables 2 and 3, we plot them onto the IF

Table 1. Evaluations of the 28 EU Member States against the 12 WEF GCR indicators, 2016–2017 [24]

	1. Institutions	2. Infrastructure	3. Macroeconomic environment	4. Health and primary education	5. Higher education and training	6. Goods market efficiency	7. Labor market efficiency	8. Financial market development	9. Technological readiness	10. Market size	11. Business sophistication	12. Innovation
AT	5.2	5.8	5.5	6.4	5.8	4.9	4.5	4.5	5.7	4.5	5.5	5
BE	5.2	5.5	4.8	6.7	6	5.2	4.5	4.7	6	4.7	5.4	5
BG	3.5	4	5.2	5.9	4.6	4.4	4.4	4.1	5.1	3.9	3.8	3.4
HR	3.6	4.6	4.4	5.8	4.7	4.1	3.9	3.6	4.7	3.5	3.8	3.1
CY	4	4.4	3.8	6.2	4.6	4.7	4.4	3.2	4.6	2.8	4	3.2
CZ	4.2	4.7	5.9	6.3	5.2	4.7	4.5	4.7	5.5	4.4	4.5	3.8
DK	5.5	5.6	5.9	6.4	5.9	5.1	5.1	4.8	6.1	4.2	5.4	5.1
EE	5.1	5	6.1	6.5	5.5	5.1	5	4.8	5.4	3	4.3	4.1
FI	6.1	5.3	5.1	6.9	6.2	5.1	4.8	5.5	6	4.1	5.3	5.7
FR	4.9	6.1	4.7	6.4	5.5	4.7	4.4	4.6	5.9	5.7	5.2	4.9
DE	5.2	6.1	6	6.5	5.6	5	4.8	4.9	6.1	6	5.6	5.6
GR	3.8	4.8	2.9	6.1	4.9	4.2	3.8	2.5	5	4.2	3.9	3.3
HU	3.3	4.2	5.1	5.6	4.4	4.4	4.1	4	4.5	4.3	3.5	3.2
IE	5.6	5.2	5.2	6.5	5.7	5.4	5.1	4	6.1	4.3	5.2	4.8
IT	3.5	5.4	4.2	6.4	4.9	4.3	3.6	3.1	5	5.6	4.8	3.9
LV	4	4.4	5.6	6.2	5	4.5	4.6	4.2	5.2	3.2	4.1	3.4
LT	4.2	4.7	5.4	6.3	5.3	4.6	4.4	4.1	5.6	3.5	4.3	3.7
LU	5.8	5.7	6.2	6.2	4.8	5.5	5	5	6.4	3.2	5.2	4.9
MT	4.5	4.7	5.8	6.4	5	4.7	4.5	4.4	5.8	2.5	4.3	3.7
NL	5.7	6.4	5.7	6.7	6.1	5.4	5.1	4.5	6.2	5.1	5.6	5.4
PL	4	4.3	5.1	6.2	5	4.6	4.1	4.2	4.8	5.1	4.1	3.4
PT	4.3	5.5	3.7	6.4	5	4.7	4.3	3.3	5.6	4.3	4.2	3.9
RO	3.6	3.6	5.5	5.5	4.4	4.2	4	3.7	4.7	4.5	3.6	3.1
SK	3.5	4.2	5.3	6	4.5	4.5	4	4.6	4.8	4	4.1	3.3
SI	4.1	4.8	4.9	6.5	5.4	4.6	4.1	3.2	5.2	3.3	4.2	3.9
ES	4.1	5.9	4.3	6.3	5.1	4.5	4.2	4	5.6	5.4	4.5	3.8
SE	5.9	5.6	6.3	6.4	5.6	5.3	4.9	5.2	6.3	4.6	5.6	5.5
GB	5.5	6	4.4	6.5	5.5	5.3	5.5	4.9	6.3	5.7	5.6	5

Table 2. Membership parts of the intercriteria IFPs calculated from the input in Table 1.

μ	1	2	3	4	5	6	7	8	9	10	11	12
1	1.00	0.74	0.63	0.75	0.81	0.83	0.77	0.74	0.85	0.51	0.80	0.83
2	0.74	1.00	0.50	0.70	0.74	0.70	0.65	0.61	0.77	0.66	0.81	0.79
3	0.63	0.50	1.00	0.48	0.56	0.62	0.67	0.74	0.63	0.42	0.59	0.60
4	0.75	0.70	0.48	1.00	0.79	0.70	0.63	0.59	0.70	0.53	0.74	0.77
5	0.81	0.74	0.56	0.79	1.00	0.73	0.69	0.66	0.76	0.57	0.78	0.82
6	0.83	0.70	0.62	0.70	0.73	1.00	0.81	0.72	0.80	0.50	0.75	0.75
7	0.77	0.65	0.67	0.63	0.69	0.81	1.00	0.74	0.77	0.47	0.71	0.71
8	0.74	0.61	0.74	0.59	0.66	0.72	0.74	1.00	0.72	0.53	0.70	0.72
9	0.85	0.77	0.63	0.70	0.76	0.80	0.77	0.72	1.00	0.57	0.83	0.81
10	0.51	0.66	0.42	0.53	0.57	0.50	0.47	0.53	0.57	1.00	0.63	0.60
11	0.80	0.81	0.59	0.74	0.78	0.75	0.71	0.70	0.83	0.63	1.00	0.87
12	0.83	0.79	0.60	0.77	0.82	0.75	0.71	0.72	0.81	0.60	0.87	1.00

Table 3. Non-membership parts of the intercriteria IFPs calculated from the input in Table 1.

ν	1	2	3	4	5	6	7	8	9	10	11	12
1	0.00	0.21	0.32	0.13	0.13	0.08	0.14	0.20	0.09	0.43	0.12	0.12
2	0.21	0.00	0.46	0.19	0.20	0.22	0.28	0.34	0.17	0.29	0.12	0.15
3	0.32	0.46	0.00	0.39	0.39	0.30	0.26	0.21	0.32	0.54	0.34	0.35
4	0.13	0.19	0.39	0.00	0.08	0.14	0.21	0.28	0.17	0.35	0.12	0.10
5	0.13	0.20	0.39	0.08	0.00	0.18	0.22	0.28	0.18	0.38	0.15	0.11
6	0.08	0.22	0.30	0.14	0.18	0.00	0.09	0.19	0.11	0.42	0.15	0.15
7	0.14	0.28	0.26	0.21	0.22	0.09	0.00	0.19	0.15	0.45	0.19	0.20
8	0.20	0.34	0.21	0.28	0.28	0.19	0.19	0.00	0.21	0.42	0.23	0.22
9	0.09	0.17	0.32	0.17	0.18	0.11	0.15	0.21	0.00	0.38	0.10	0.13
10	0.43	0.29	0.54	0.35	0.38	0.42	0.45	0.42	0.38	0.00	0.30	0.34
11	0.12	0.12	0.34	0.12	0.15	0.15	0.19	0.23	0.10	0.30	0.00	0.07
12	0.12	0.15	0.35	0.10	0.11	0.15	0.20	0.22	0.13	0.34	0.07	0.00

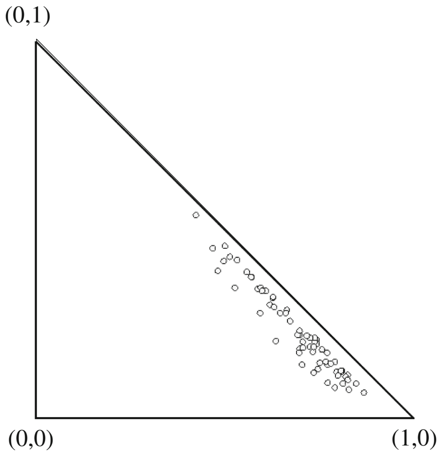

Fig. 3. Intercriteria IFPs calculated from the input in Table 1, plotted on the intuitionistic fuzzy interpretational triangle.

interpretational triangle in Fig. 3. We further render this data in one more layout (Table 4), showing the intercriteria pairs, sorted by the calculated μ/ν ratio, which relates with the determination of the threshold parameter γ. In addition, we also give the distance to *Truth*. Each of the columns in Table 4 is conditionally formatted with a 3-color scale based on their values, where the minimum is the lowest value per column and the maximum is the highest value per column (justified by the different ranges in which these values belong to).

From Table 4 we notice the obvious similarity between the results as sorted by μ/ν ratio, and by distance to the *Truth*. We reason that this is a particular result from the form of this IFS. Let us compare which are the best correlating pairs of criteria, as seen from the point of view of membership, nonmembership, μ/ν ratio, and distance to the *Truth*, and let for example take in each case the top 20% and the bottom 20% of the pairs of criteria, as sorted in these four ways (e.g. in this case of 66 different pairs, the highest 14 and lowest 14 pairs). Table 5 gives these results when the intercriteria pairs are sorted by membership (descending); Table 6 – when the pairs are sorted by non-membership (ascending); Table 7 – when the pairs are sorted by μ/ν ratio (descending); Table 8 – when the pairs are sorted by distance from *Truth* (ascending). Such detailed comparison of the ranking of the top correlating pairs of criteria has not been made so far in the research of intercriteria analysis, with any of the case studies, where ICA has been applied.

Table 4. Intercriteria IFPs calculated from the input in Table 1, sorted by μ/ν ratio.

C_i	C_j	μ	ν	μ/ν ratio	distance to T
11. Business sophistication	12. Innovation	0.868	0.069	12.615	0.149
1. Institutions	6. Goods market efficiency	0.828	0.077	10.793	0.188
4. Health and primary education	5. Higher education and training	0.791	0.082	9.645	0.225
1. Institutions	9. Technological readiness	0.847	0.093	9.143	0.179
6. Goods market efficiency	7. Labor market efficiency	0.812	0.093	8.771	0.209
4. Health and primary education	12. Innovation	0.772	0.095	8.111	0.247
9. Technological readiness	11. Business sophistication	0.825	0.103	8.000	0.203
5. Higher education and training	12. Innovation	0.820	0.111	7.381	0.211
1. Institutions	12. Innovation	0.825	0.116	7.091	0.210
6. Goods market efficiency	9. Technological readiness	0.799	0.114	7.023	0.231
2. Infrastructure	11. Business sophistication	0.807	0.124	6.489	0.230
1. Institutions	11. Business sophistication	0.796	0.124	6.404	0.239
1. Institutions	5. Higher education and training	0.810	0.127	6.375	0.229
9. Technological readiness	12. Innovation	0.807	0.127	6.354	0.231
4. Health and primary education	11. Business sophistication	0.735	0.122	6.043	0.291
1. Institutions	4. Health and primary education	0.746	0.132	5.640	0.286
1. Institutions	7. Labor market efficiency	0.772	0.143	5.407	0.269
5. Higher education and training	11. Business sophistication	0.780	0.146	5.364	0.263
2. Infrastructure	12. Innovation	0.791	0.151	5.246	0.258
7. Labor market efficiency	9. Technological readiness	0.767	0.151	5.088	0.277
6. Goods market efficiency	11. Business sophistication	0.751	0.148	5.071	0.289
6. Goods market efficiency	12. Innovation	0.751	0.148	5.071	0.289
4. Health and primary education	6. Goods market efficiency	0.704	0.143	4.926	0.329
2. Infrastructure	9. Technological readiness	0.770	0.175	4.409	0.289

(*continued*)

Table 4. (*continued*)

C_i	C_j	μ	ν	μ/ν ratio	distance to T
5. Higher education and training	9. Technological readiness	0.757	0.183	4.145	0.304
5. Higher education and training	6. Goods market efficiency	0.733	0.177	4.134	0.321
4. Health and primary education	9. Technological readiness	0.696	0.175	3.985	0.351
7. Labor market efficiency	8. Financial market development	0.735	0.190	3.861	0.326
6. Goods market efficiency	8. Financial market development	0.725	0.190	3.806	0.335
2. Infrastructure	4. Health and primary education	0.698	0.185	3.771	0.354
7. Labor market efficiency	11. Business sophistication	0.706	0.188	3.761	0.349
1. Institutions	8. Financial market development	0.743	0.198	3.747	0.324
2. Infrastructure	5. Higher education and training	0.738	0.204	3.623	0.332
1. Institutions	2. Infrastructure	0.743	0.209	3.557	0.331
7. Labor market efficiency	12. Innovation	0.706	0.204	3.468	0.357
3. Macroeconomic environment	8. Financial market development	0.738	0.214	3.444	0.338
8. Financial market development	9. Technological readiness	0.725	0.214	3.383	0.349
8. Financial market development	12. Innovation	0.717	0.220	3.265	0.358
2. Infrastructure	6. Goods market efficiency	0.698	0.222	3.143	0.375
5. Higher education and training	7. Labor market efficiency	0.693	0.222	3.119	0.379
4. Health and primary education	7. Labor market efficiency	0.635	0.206	3.077	0.419
8. Financial market development	11. Business sophistication	0.698	0.233	3.000	0.381
3. Macroeconomic environment	7. Labor market efficiency	0.672	0.259	2.592	0.418
5. Higher education and training	8. Financial market development	0.661	0.280	2.358	0.440
2. Infrastructure	10. Market size	0.664	0.288	2.303	0.443
2. Infrastructure	7. Labor market efficiency	0.646	0.280	2.302	0.452
10. Market size	11. Business sophistication	0.630	0.296	2.125	0.474
4. Health and primary education	8. Financial market development	0.593	0.280	2.113	0.495
3. Macroeconomic environment	6. Goods market efficiency	0.619	0.302	2.053	0.486
1. Institutions	3. Macroeconomic environment	0.627	0.320	1.959	0.492
3. Macroeconomic environment	9. Technological readiness	0.627	0.323	1.943	0.493
2. Infrastructure	8. Financial market development	0.608	0.339	1.797	0.518
10. Market size	12. Innovation	0.598	0.339	1.766	0.526
3. Macroeconomic environment	12. Innovation	0.595	0.347	1.718	0.533
3. Macroeconomic environment	11. Business sophistication	0.587	0.344	1.708	0.537
5. Higher education and training	10. Market size	0.571	0.376	1.521	0.570
4. Health and primary education	10. Market size	0.526	0.347	1.519	0.587
9. Technological readiness	10. Market size	0.569	0.376	1.514	0.572
3. Macroeconomic environment	5. Higher education and training	0.558	0.389	1.435	0.589
8. Financial market development	10. Market size	0.532	0.421	1.264	0.629
3. Macroeconomic environment	4. Health and primary education	0.481	0.392	1.230	0.650
1. Institutions	10. Market size	0.513	0.429	1.198	0.649
6. Goods market efficiency	10. Market size	0.497	0.418	1.190	0.654
2. Infrastructure	3. Macroeconomic environment	0.500	0.458	1.092	0.678
7. Labor market efficiency	10. Market size	0.468	0.452	1.035	0.698
3. Macroeconomic environment	10. Market size	0.423	0.540	0.784	0.790

Table 5. Top 20% and Bottom 20% intercriteria IFPs from Table 4, sorted by membership (descending)

C_i	C_j	μ
11. Business sophistication	12. Innovation	0.868
1. Institutions	9. Technological readiness	0.847
1. Institutions	6. Goods market efficiency	0.828
9. Technological readiness	11. Business sophistication	0.825
1. Institutions	12. Innovation	0.825
5. Higher education and training	12. Innovation	0.820
6. Goods market efficiency	7. Labor market efficiency	0.812
1. Institutions	5. Higher education and training	0.810
2. Infrastructure	11. Business sophistication	0.807
9. Technological readiness	12. Innovation	0.807
6. Goods market efficiency	9. Technological readiness	0.799
1. Institutions	11. Business sophistication	0.796
4. Health and primary education	5. Higher education and training	0.791
2. Infrastructure	12. Innovation	0.791
3. Macroeconomic environment	12. Innovation	0.595
4. Health and primary education	8. Financial market development	0.593
3. Macroeconomic environment	11. Business sophistication	0.587
5. Higher education and training	10. Market size	0.571
9. Technological readiness	10. Market size	0.569
3. Macroeconomic environment	5. Higher education and training	0.558
8. Financial market development	10. Market size	0.532
4. Health and primary education	10. Market size	0.526
1. Institutions	10. Market size	0.513
2. Infrastructure	3. Macroeconomic environment	0.500
6. Goods market efficiency	10. Market size	0.497
3. Macroeconomic environment	4. Health and primary education	0.481
7. Labor market efficiency	10. Market size	0.468
3. Macroeconomic environment	10. Market size	0.423

Table 6. Top 20% and Bottom 20% intercriteria IFPs from Table 4, sorted by non-membership (ascending)

C_i	C_j	ν
11. Business sophistication	12. Innovation	0.069
1. Institutions	6. Goods market efficiency	0.077
4. Health and primary education	5. Higher education and training	0.082
1. Institutions	9. Technological readiness	0.093
6. Goods market efficiency	7. Labor market efficiency	0.093
4. Health and primary education	12. Innovation	0.095
9. Technological readiness	11. Business sophistication	0.103
5. Higher education and training	12. Innovation	0.111
6. Goods market efficiency	9. Technological readiness	0.114
1. Institutions	12. Innovation	0.116
4. Health and primary education	11. Business sophistication	0.122
2. Infrastructure	11. Business sophistication	0.124
1. Institutions	11. Business sophistication	0.124
1. Institutions	5. Higher education and training	0.127
10. Market size	12. Innovation	0.339
3. Macroeconomic environment	11. Business sophistication	0.344
3. Macroeconomic environment	12. Innovation	0.347
4. Health and primary education	10. Market size	0.347
5. Higher education and training	10. Market size	0.376
9. Technological readiness	10. Market size	0.376
3. Macroeconomic environment	5. Higher education and training	0.389
3. Macroeconomic environment	4. Health and primary education	0.392
6. Goods market efficiency	10. Market size	0.418
8. Financial market development	10. Market size	0.421
1. Institutions	10. Market size	0.429
7. Labor market efficiency	10. Market size	0.452
2. Infrastructure	3. Macroeconomic environment	0.458
3. Macroeconomic environment	10. Market size	0.540

Table 7. Top 20% and Bottom 20% intercriteria IFPs from Table 4, sorted by μ/ν ratio (descending)

C_i	C_j	μ/ν ratio
11. Business sophistication	12. Innovation	12.615
1. Institutions	6. Goods market efficiency	10.793
4. Health and primary education	5. Higher education and training	9.645
1. Institutions	9. Technological readiness	9.143
6. Goods market efficiency	7. Labor market efficiency	8.771
4. Health and primary education	12. Innovation	8.111
9. Technological readiness	11. Business sophistication	8.000
5. Higher education and training	12. Innovation	7.381
1. Institutions	12. Innovation	7.091
6. Goods market efficiency	9. Technological readiness	7.023
2. Infrastructure	11. Business sophistication	6.489
1. Institutions	11. Business sophistication	6.404
1. Institutions	5. Higher education and training	6.375
9. Technological readiness	12. Innovation	6.354
10. Market size	12. Innovation	1.766
3. Macroeconomic environment	12. Innovation	1.718
3. Macroeconomic environment	11. Business sophistication	1.708
5. Higher education and training	10. Market size	1.521
4. Health and primary education	10. Market size	1.519
9. Technological readiness	10. Market size	1.514
3. Macroeconomic environment	5. Higher education and training	1.435
8. Financial market development	10. Market size	1.264
3. Macroeconomic environment	4. Health and primary education	1.230
1. Institutions	10. Market size	1.198
6. Goods market efficiency	10. Market size	1.190
2. Infrastructure	3. Macroeconomic environment	1.092
7. Labor market efficiency	10. Market size	1.035
3. Macroeconomic environment	10. Market size	0.784

Table 8. Top 20% and Bottom 20% intercriteria IFPs from Table 4, sorted by distance to *Truth* (ascending)

C_i	C_j	Distance to T
11. Business sophistication	12. Innovation	0.149
1. Institutions	9. Technological readiness	0.179
1. Institutions	6. Goods market efficiency	0.188
9. Technological readiness	11. Business sophistication	0.203
6. Goods market efficiency	7. Labor market efficiency	0.209
1. Institutions	12. Innovation	0.210
5. Higher education and training	12. Innovation	0.211
4. Health and primary education	5. Higher education and training	0.225
1. Institutions	5. Higher education and training	0.229
2. Infrastructure	11. Business sophistication	0.230
6. Goods market efficiency	9. Technological readiness	0.231
9. Technological readiness	12. Innovation	0.231
1. Institutions	11. Business sophistication	0.239
4. Health and primary education	12. Innovation	0.247
10. Market size	12. Innovation	0.526
3. Macroeconomic environment	12. Innovation	0.533
3. Macroeconomic environment	11. Business sophistication	0.537
5. Higher education and training	10. Market size	0.570
9. Technological readiness	10. Market size	0.572
4. Health and primary education	10. Market size	0.587
3. Macroeconomic environment	5. Higher education and training	0.589
8. Financial market development	10. Market size	0.629
1. Institutions	10. Market size	0.649
3. Macroeconomic environment	4. Health and primary education	0.650
6. Goods market efficiency	10. Market size	0.654
2. Infrastructure	3. Macroeconomic environment	0.678
7. Labor market efficiency	10. Market size	0.698
3. Macroeconomic environment	10. Market size	0.790

4 Discussion and Conclusion

From the conducted analysis of the discussed case study, we observe that the following pairs of criteria exhibit similar high occurrence in the Top 20% (positive consonance) regardless of the approach of ranking them: with four occurrences of the pairs 11–12, 1–9, 1–6, 9–11, 1–12, 5–12, 1–5, 2–11, 6–9, 1–11, 4–5, with three occurrences of the pairs 9–12, 4–12 and one occurrence of the pairs 2–12, 4–11. The following pairs of criteria exhibit similar high occurrence in the Bottom 20% (negative consonance) regardless of the approach of ranking them: with four occurrences of the pairs 3–10, 7–10, 3–4, 6–10, 2–3, 1–10, 4–10, 8–10, 3–5, 9–10, 5–10, 3–11, 3–12, with three occurrences of the pairs 10–12, and one occurrence of the pair 4–8. Practically, with one exception, all of these intercriteria pairs contain either criterion 3 or criterion 10. These results outline the high relevance of institutions, goods market efficiency technological readiness, higher education and training, business sophistication and innovation on the competitiveness of EU Member States, and demonstrates the limited relation of the macroeconomic environment and market size on the EU competitiveness.

The findings in the case study are arguably the result of the location of the IFS onto the IF interpretational triangle and more specifically due to the low levels of uncertainty (proximity of the IFS to the hypotenuse of the IF interpretational triangle). We reason that in cases of IFSs having elements exhibiting higher levels of uncertainty, the different approaches of cutting a subset of the IFS and ranking them will demonstrate more substantial differences in the selection and ranking of the highest intercriteria correlations in ICA, as suggested in Fig. 4.

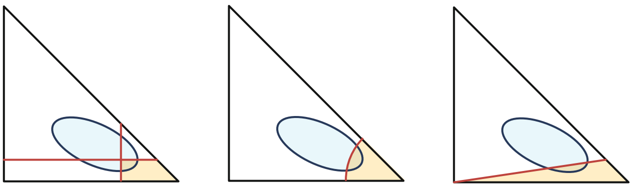

Fig. 4. Different scenarios to cut a subset of an IFS, plotted on the IF interpretational triangle.

We will specifically discuss on the operator N_γ over ICA case studies. The use and applicability of the operator would be more visible in cases where the IFS elements exhibit both high degrees of uncertainty and high membership-to-non-membership ratios of the elements. The use of thresholds α and β for the membership and non-membership, respectively, to outline the positive correlations, which was part of the original formulation of the ICA approach, tends to isolate such elements due to the higher degree uncertainty. In the same manner, these elements would remain undiscussed if the IFS elements are ranked with respect to their distance from the *Truth*. Even with higher degree of uncertainty, however, the high μ/ν ratio can prove to be indicative and helpful for the decision maker. Studying more cases can make this open question clearer and can contribute to the wider adoption and applicability of this newly proposed operator over IFSs.

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Some Remarks About Idempotent Uninorms on Complete Lattice

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Abstract. In this paper we study the properties of idempotent uninorms on the lattice, that are one of the binary operations. It is shown that in any lattice idempotent uninorms need not be internal (with the extended definition of the term "internal"). But with additional assumptions, we get that the uninorm is locally internal. With this assumption, we present the theorem of Czogała and Drewniak for a complete lattice. Moreover, many properties of idempotent uninorm in this case is shown.

Keywords: Uninorms · Aggregation operators · Complete lattice · Idempotent operations

1 Introduction

Uninorms were introduced by Yager and Rybalov [19] as a generalization of triangular norms and triangular conorms [18,20] with more applications, allowing the freedom for the neutral element e to be an arbitrary element from unit interval $[0, 1]$, which is 1 for t-norms and 0 for t-conorms. Fodor et al. and De Baets [4,5,12] is among others that have studied the structure of these operators in considerable detail. If the neutral element is not 0 or 1, the construction of uninorm aggregation operators is an important work. These operators play an important role both in theoretical investigations and in practical applications such as the expert systems, neural networks, fuzzy logics, etc.

Martin, Mayor and Torrens [16] characterize idempotent uninorms on unit interval $[0, 1]$ as improvement of a well-known theorem of Czogała and Drewniak [3] on idempotent, associative and increasing operations with a neutral element. Associative, monotonic, idempotent operations with a neutral element are special combinations of minimum and maximum and, consequently, locally internal. The classes $U_{\min}$ and $U_{\max}$ are well-known examples of idempotent uninorms, which are the smallest and largest idempotent uninorms with the neutral element e. The characterization of idempotent uninorms given in [4] are also locally internal.

De Baets et al. [5] characterize all idempotent uninorms defined on a finite ordinal scale, similarly as on the unit interval the characterization of idempotent uninorms [3,4,16]. They have proved that any idempotent uninorm is uniquely determined by a decreasing function from the set of scale elements not greater

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than the neutral element to the set of scale elements not smaller than the neutral element. Some other studies related to uninorms can be found also in [7–11, 13, 17].

In this paper, after some preliminaries concerning locally internal operation on unit interval $[0, 1]$ we present the properties of idempotent uninorms on the lattice. It is shown that in any lattice idempotent uninorms need not be internal (with the extended definition of the term "internal"). We remaind a sufficient condition for the lattice to any idempotent uninorm be internal. In addition, many properties of idempotent uninorm in this case is shown and at the end there is given a description of idempotent uninorms on such lattices.

2 Preliminaries

We recall here some definitions and results about binary operations on [0, 1] that are monotonic and satisfy the locally internal property, i.e. the value of such operation at any point (x, y) is always one of its arguments. Next we consider uninorms on bounded lattice and its basic properties.

Definition 1 ([16]). *We say that a binary operation $F : [0, 1]^2 \to [0, 1]$ is locally internal if it satisfies the following condition:*

$$F(x, y) \in \{x, y\}$$

for all $x, y \in [0, 1]$.

The following results shows the relationship between commutativity and associativity for locally internal, monotonic operations.

Proposition 1 ([16]). *If a locally internal, monotonic operation is commutative, then it is associative.*

Proposition 2 ([3]). *Idempotent, associative, monotonic operation with neutral element is locally internal.*

Definition 2 ([2]). *A bounded lattice we denote by $(L, \leq, 0, 1)$ where the top and bottom elements are written as 1 and 0, respectively. Moreover, for $a, b \in L$, if a and b are* incomparable, *we use the notation $a \parallel b$.*

Definition 3 ([2]). *Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a* subinterval $[a, b]$ *of L is defined as*

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, we define $(a, b] = \{x \in L \mid a < x \leq b\}, [a, b) = \{x \in L \mid a \leq x < b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L$. Let $A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$ and $I_e = \{x \in L \mid x \parallel e\}$.

Definition 4 ([14]). *Let $(L, \leq, 0, 1)$ be a bounded lattice. Operation $U: L^2 \to L$ is called a uninorm on L (shortly a uninorm, if L is fixed) if it is commutative, associative, increasing with respect to both variables and there exist an element $e \in L$ such that $U(x, e) = x$, for all $x \in L$. The element e is called the neutral element of U.*

We denote by $\mathcal{U}(e)$ the set of all uninorms on L with the neutral element $e \in L$.

Definition 5 ([14]). *Operation $T : L^2 \to L$ $\big(S : L^2 \to L\big)$ is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1$ $(e = 0)$.*

Proposition 3 ([14]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U a uninorm on L with the neutral element e. Then*

(i) $T = U|[0, e]^2 : [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]$.
(ii) $S = U|[e, 1]^2 : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$.

Proposition 4 ([14]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U a uninorm on L with the neutral element e. The following properties hold:*

(i) $x \wedge y \leq U(x, y) \leq x \vee y$ *for all* $(x, y) \in A(e)$,
(ii) $U(x, y) \leq x$ *for* $(x, y) \in L \times [0, e]$,
(iii) $U(x, y) \leq y$ *for* $(x, y) \in [0, e] \times L$,
(iv) $x \leq U(x, y)$ *for* $(x, y) \in L \times [e, 1]$,
(v) $y \leq U(x, y)$ *for* $(x, y) \in [e, 1] \times L$.

3 Idempotent Uninorms

Definition 6 ([6]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U a uninorm on L with the neutral element e. U is called an idempotent uninorm if $U(x, x) = x$ for all $x \in L$.*

Proposition 5 ([6]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U an idempotent uninorm on L with the neutral element e. Then it holds:*
(i) $U(x, y) = x \wedge y$ *for all* $(x, y) \in [0, e]^2$,
(ii) $U(x, y) = x \vee y$ *for all* $(x, y) \in [e, 1]^2$.

Proposition 6 (cf. [7]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U an idempotent uninorm on L with the neutral element e. Then it holds:*

(i) $U(x, y) \in \{x, y\}$ *or* $U(x, y) \in I_e$ *for all* $(x, y) \in A(e)$,
(ii) $U(x, y) = x \wedge y$ *or* $U(x, y) \in I_e$ *for all* $x \in [0, e]$ *and* $y \in I_e$,
(iii) $U(x, y) = x \vee y$ *or* $U(x, y) \in I_e$ *for all* $x \in [e, 1]$ *and* $y \in I_e$,
(iv) $U(x, y) = x \wedge y$ *or* $U(x, y) \in I_e$ *for all* $y \in [0, e]$ *and* $x \in I_e$,
(v) $U(x, y) = x \vee y$ *or* $U(x, y) \in I_e$ *for all* $y \in [e, 1]$ *and* $x \in I_e$,
(vi) $U(x, y) \in \{x \wedge y, x \vee y\}$ *or* $U(x, y) \in I_e$ *for all* $x \in I_e$ *and* $y \in I_e$.

Proposition 7 (cf. [7]). *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* U *an idempotent uninorm on* L *with the neutral element* $e \in L \backslash \{0, 1\}$ *such that all* $x \in L$ *comparable with* e*. Then,* $U(x, y) \in \{x \wedge y, x \vee y\}$ *for all* $(x, y) \in L^2$*.*

Directly from Proposition 5, we see that an idempotent uninorm can not be locally internal in the sense of the Definition 1. In accordance with the above proposition, we see that under the appropriate assumptions an idempotent uninorm is locally internal in the broader sense. A natural question arises: If there are some $x \in L$ incomparable with e, it must always be the fact that $U(x, y) \in \{x, y, x \wedge y, x \vee y\}$ for $U \in \mathcal{U}(e)$? In the following example, we give a negative answer regarding the above hypothesis.

Example 1. Given a bounded lattice $L = \{0, x, y, e, z, t, 1\}$ with order in the Fig. 1, define a mapping $U : L^2 \to L$ by Table 1. Then U is an idempotent uninorm on L with a neutral element e and $U(z, x) = t$.

Table 1. The idempotent uninorm given in Example 1

U	0	a	b	e	c	d	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	d	d
b	0	a	b	b	d	d	d
e	0	a	b	e	c	d	1
c	0	a	d	c	c	d	1
d	0	d	d	d	d	d	d
1	0	d	d	1	1	d	1

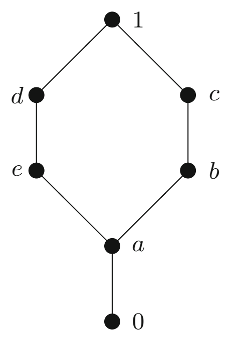

Fig. 1. The lattice given in Example 1

Proposition 8. (cf. [7]). *Let $(L, \leq, 0, 1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e \in L \backslash \{0, 1\}$ such that all $x \in L$ comparable with e. For $x, y, z \in L$ such that $x, y \geq e$, $x \parallel y$ and $z \leq e$, it may be possible only one of the following conditions:*

(i) If $U(x, z) = z$, then $U(y, z) = z$, $U(x \vee y, z) = z$ and $U(x \wedge y, z) = z$.
(ii) If $U(x, z) = x$, then $U(y, z) = y$ and $U(x \vee y, z) = x \vee y$.

Unfortunately, value of $U(x \wedge y, z)$ is not always equal to $x \wedge y$ while $U(x, z) = x$ for $x, y \geq e$, $x \parallel y$ and $z \leq e$.

Example 2. Given a lattice $L = \{0, a, e, b, c, d, 1\}$ with order given on Fig. 2, define a mapping $U : L^2 \to L$ by Table 2. Then, U is an idempotent uninorm on L with the neutral element e. But $U(c \wedge d, a) = a$ while $U(c, a) = c$ for $c, d \geq e$, $c \parallel d$ and $a \leq e$.

Proposition 9. (cf. [7]). *Let $(L, \leq, 0, 1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e \in L \backslash \{0, 1\}$ such that all $x \in L$ comparable with*

Table 2. The idempotent uninorm given in Example 2

U	0	a	e	b	c	d	1
0	0	0	0	0	c	d	1
a	0	a	a	a	c	d	1
e	0	a	e	b	c	d	1
b	0	a	b	b	c	d	1
c	c	c	c	c	c	1	1
d	d	d	d	d	1	d	1
1	1	1	1	1	1	1	1

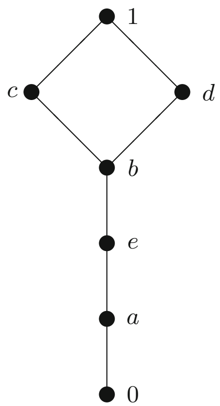

Fig. 2. The lattice given in Example 2

e. For $x, y, z \in L$, such that $x, y \leq e$, $x \parallel y$ and $z \geq e$, it may be possible only one of the following conditions:

(i) If $U(x, z) = z$, then $U(y, z) = z$, $U(x \vee y, z) = z$ and $U(x \wedge y, z) = z$.
(ii) If $U(x, z) = x$, then $U(y, z) = y$ and $U(x \wedge y, z) = x \wedge y$.

4 Separating Function of Idempotent Uninorms on Some Types of Lattices

In this section we will consider the structure of uninorms under some assumptions, that L is a complete lattice, $e \in L \setminus \{0, 1\}$ and all elements are comparable with e.

Definition 7. *Let $(L, \leq, 0, 1)$ be a complete lattice, U be an idempotent uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$ such that all $x \in L$ comparable with e. Then the function $g : L \to L$ defined by*

$$g(x) = \begin{cases} \sup\{z \in L \mid U(x, z) = \min(x, z)\} \; if \; x \leq e, \\ \inf\{z \in L \mid U(x, z) = \max(x, z)\} \; otherwise. \end{cases} \tag{1}$$

is called a separating function of idempotent uninorm U.

Lemma 1. *Let $(L, \leq, 0, 1)$ be a complete lattice, U be an idempotent uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$ such that all $x \in L$ comparable with e. Then the function $g : L \to L$ defined by (1) has the following properties:*

(i) $g(e) = e$.
(ii) For all $x < e$, it holds that $g(x) \geq e$ and $U(x, y) = x \wedge y$ whenever $y < g(x)$.
(iii) For all $x > e$, it holds that $g(x) \leq e$ and $U(x, y) = x \vee y$ whenever $y > g(x)$.

Proof. Let $x \in L$, $x < e$. Then $U(x, e) = x$ and consequently the set $B_x = \{z \in L \mid U(x, z) = \min(x, z)\}$ is nonempty. Moreover, by Propositions 8 and 9 we obtain, that $\sup B_x$ exist. Similarly we have for $x \geq e$. It means, that the function $g : L \to L$ defined by (1) is well defined.

To prove (i) we have

$$\begin{aligned} g(e) &= \sup\{z \in L \mid U(e, z) = \min(e, z)\} \\ &= \sup\{z \in L \mid z = \min(e, z)\} \\ &= \sup\{z \in L \mid z \leq e\} = e \end{aligned}$$

(ii) For all $x < e$, we have that $U(x, e) = x = \min(x, e)$ and

$$g(x) = \sup\{z \in L \mid U(x, z) = \min(x, z)\} \geq e$$

Moreover, because of Proposition 5 (i) it is enough to consider the condition $x < e \leq y < g(x)$. By monotonicity of U and definition of g, we obtain that $U(x, y) = \min(x, y) = x$. So, $U(x, y) = x = x \wedge y$.

(iii) It can be proved as (ii).

Corollary 1. *Let $(L, \leq, 0, 1)$ be a complete lattice, U be an idempotent uninorm on L with the neutral element $e \in L \backslash \{0, 1\}$ such that all $x \in L$ comparable with e. Then the function $g : L \to L$ defined by (1) separate the values* min *and* max *in $A(e)$*

Lemma 2. *Let $(L, \leq, 0, 1)$ be a complete lattice, U be an idempotent uninorm on L with the neutral element $e \in L \backslash \{0, 1\}$ such that all $x \in L$ comparable with e and $g : L \to L$ be defined by (1). Then*

(i) There does not exist $y \in L$ such that $y \parallel g(x)$ for all $x < e$.
(ii) There does not exist $y \in L$ such that $y \parallel g(x)$ for all $x > e$.

Proof. (i) Suppose that there exists $y \in L$ such that $y \parallel g(x)$ for some $x < e$. By Proposition 7, $U(x, y) \in \{x, y\}$. If $U(x, y) = x$, then by Proposition 8, $U(x, y \vee g(x)) = x$. So, we obtain that

$$g(x) = \sup\{z \in L | U(x, z) = \min(x, z)\} \geq y \vee g(x) > g(x).$$

This is a contradiction. If $U(x, y) = y$, by Proposition 8, $U(x, g(x)) = g(x)$. Taking $t = y \wedge g(x)$ we have two possibilities:
(a) $U(x, t) = t$, then we have a contradiction with Lemma 1.
(b) $U(x, t) = x$, then for all $z > t$ we have $U(x, z) = z$. So, t is is the supremum of the set $\{z \in L | U(x, z) = \min(x, z)\}$ and $t < g(x)$. This is contradiction.

(ii) It can be proved as (i).

Theorem 1. *Let $(L, \leq, 0, 1)$ be a complete lattice, U be an idempotent uninorm on L with neutral element $e \in L \backslash \{0, 1\}$ such that all $x \in L$ are comparable with e. Then there exists a decreasing function $g : L \to L$ with $g(e) = e$ such that*

$$U(x, y) = \begin{cases} x \wedge y & \textit{if } y < g(x) \\ x \vee y & \textit{if } y > g(x) \\ x \textit{ or } y & \textit{otherwise.} \end{cases} \tag{2}$$

Proof. Consider the function $g : L \to L$ defined by (1). By using Proposition 7 and Lemma 1, we obtain that U is given by (2). The monotonicity of U immediately implies that g is decreasing.

Example 3. Let $L = \{0, a, b, c, d, h, e, x, y, z, u, v, w, 1\}$ ba a lattice with order given on Fig. 3 and a mapping $U : L^2 \to L$ be given by Table 3. Then, U is an

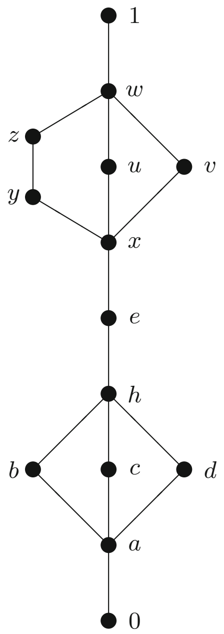

Fig. 3. The lattice given in Example 3

Table 3. The idempotent uninorm given in Example 3

U	0	a	b	c	d	h	e	x	y	z	u	v	w	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a	a	a	a	a	a	1
b	0	a	b	a	a	b	b	b	b	b	b	b	b	1
c	0	a	a	c	a	c	c	c	c	c	c	c	c	1
d	0	a	a	a	d	d	d	d	d	d	d	d	d	1
h	0	a	b	c	d	h	h	x	y	z	u	v	w	1
e	0	a	b	c	d	h	e	x	y	z	u	v	w	1
x	0	a	b	c	d	h	x	x	y	z	u	v	w	1
y	0	a	b	c	d	y	y	y	y	z	w	w	w	1
z	0	a	b	c	d	z	z	z	z	z	w	w	w	1
u	0	a	b	c	d	u	u	u	w	w	u	w	w	1
v	0	a	b	c	d	v	v	v	w	w	w	v	w	1
w	0	a	b	c	d	w	w	w	w	w	w	w	w	1
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1

idempotent uninorm on L with the neutral element e with a separating function given by

$$g(s) = \begin{cases} 1, & s = 0 \\ v, & s \in \{a, b, c, d\} \\ x, & s = h \\ e, & s = e \\ h, & x \in \{x, y, z, u, v, w\} \\ a, & x = 1. \end{cases}$$

5 Conclusion

In this paper we studied the properties of idempotent uninorms on the lattice. It is shown that under additional assumption about the lattice idempotent uninorms need to be internal (with the extended definition of the term "internal"). In this case the representation of idempotent uninorm using the separating function is given. Unfortunately, in contrast to the finite lattice considered in [7], in this case we do not get uniqueness, i.e. for one function g we can get several uninorms (as in the case of characterization idempotent uninorms on a unit interval). Moreover, the full characterization of idempotent uninorms on the considered type of latices remains an open problem. In addition, it is an open problem to provide a condition equivalent for a complete lattice so that the idempotent uninorm on the considered lattice is locally internal.

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Ordinal Sum of Fuzzy Implications Fulfilling Left Ordering Property

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Abstract. In the article new methods of constructing of ordinal sum of fuzzy implications are proposed. The concepts are based both on a construction of ordinal sums of overlap functions and residual implication of ordinal sum of triangular norms. Moreover, preservation of some properties of the ordinal sums of fuzzy implications are examined.

Keywords: Fuzzy connectives · Fuzzy implication · Ordinal sum · Overlap functions · Residual implication

1 Introduction

Fuzzy implications find applications in many fields such as approximate reasoning, decision support systems, and fuzzy control. For this reason new families of these connectives are the subject of investigation. One of the directions of such research is considering an ordinal sum of fuzzy implications on the pattern of the ordinal sum of t-norms. Some interesting results connected to representation of the residual implication corresponding to a fuzzy conjunction (for example continuous or at least left-continuous t-norm) given by an ordinal sum were obtained in [5,13,19]. In [21] Su et al. introduced an idea of ordinal sum of fuzzy implications similar to the construction of the ordinal sum of t-norms. In [2,10,11,18] other constructions of ordinal sums of fuzzy implications were proposed.

In this paper, new ways of generating ordinal sums of fuzzy implications are proposed. The ideas are on the basis of a construction of ordinal sums of overlap functions and residual implication of ordinal sum of triangular norms. The proposed constructions generates a fuzzy implication without any additional assumptions on summands.

In Sect. 2 some basic information about fuzzy connectives, in particular triangular norms, overlap functions and fuzzy implications, including their ordinal sums are recalled. In Sect. 3 two new constructions of ordinal sums of fuzzy implications are presented and properties of these operations are examined.

2 Preliminaries

Here we recall the notions of a triangular norm, an overlap function, and a fuzzy implication including residual implication. Moreover, we recall and illustrate some of the constructions of ordinal sums of the fuzzy connectives.

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2.1 Triangular Norms

Firstly, we put definition of a triangular norm and one important class of t-norms with some examples of these operations.

Definition 1 ([16]). *A triangular norm (a t-norm) is an increasing, commutative and associative operation* $T\colon [0,1]^2 \to [0,1]$ *with a neutral element* 1*.*

Example 1 (cf. [14, p. 7], [16, p. 4]). Here, we list well-known t-norms.

$T_M(x,y) = \min(x,y)$, $\quad T_P(x,y) = xy$,

$T_L(x,y) = \max(x+y-1,0)$, $\quad T_D(x,y) = \begin{cases} x, & if\ y=1 \\ y, & if\ x=1 \\ 0, & otherwise \end{cases}$,

$$T_{nM}(x,y) = \begin{cases} 0, & if x+y \le 1 \\ \min(x,y), & otherwise \end{cases}.$$

Next, let us recall the generalized Ordinal Sum Theorem for triangular norms [15, Corollary 2].

Theorem 1 (cf. [9,15,16]). *Let* $([a_k,b_k])_{k\in\mathcal{A}}$ *be a countable family of nonoverlapping, closed, proper subintervals of* $[0,1]$*, where* $\mathcal{A}$ *is a finite or infinite index set. Let* T *be an operation in* $[0,1]$ *defined by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & if\ (x,y) \in (a_k,b_k]^2, \\ \min(x,y), & otherwise, \end{cases} \tag{1}$$

where for each k *the binary operation* $T_k\colon [0,1]^2 \to [0,1]$ *is associative, commutative increasing such that* $T_k \le \min$*, i.e.,* T_k *is a t-subnorm. Moreover, if* $b_k = a_l$ *for some* l*,* k *and* T_l *is with a zero divisor, then* T_k *has the neutral element* $e = 1$*. We also assume that if* $b_k = 1$ *for some* k*, then the operation* T_k *has the neutral element* $e = 1$*. Then the operation* T *is a t-norm.*

Definition 2. *T-norm* T *defined as in Theorem 1 in Eq. (1) is called an ordinal sum of* $(\langle [a_k,b_k], T_k\rangle)_{k\in\mathcal{A}}$ *and each* T_k *is called a summand.*

For the general structure of such ordinal sum of t-norms see Fig. 1. Please note that in all our figures we indicate summands for subdomains, having in mind that they need to be linearly transformed (as given precisely in related formulas), but because of space problems we use this abbreviated indication.

Ordinal sums of t-norms are important not only because in this way we can construct new t-norms but also because they are important in the characterization of continuous t-norms.

Theorem 2. *For a function* $T\colon [0,1]^2 \to [0,1]$ *the following statements are equivalent:*

(i) T *is a continuous t-norm.*
(ii) T *is uniquely representable as an ordinal sum of continuous Archimedean t-norms.*

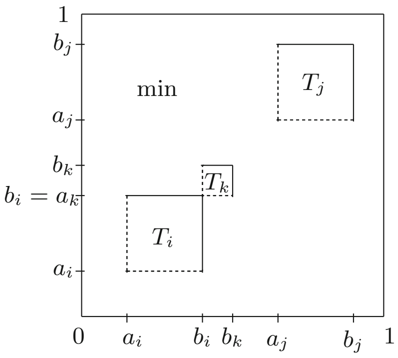

Fig. 1. The structure of an ordinal sum of t-norms given by Eq. (1).

2.2 Overlap Functions

The concept of overlap functions was introduced by Bustince et al. [4]. Some properties of these functions (e.g., migrativity, homogeneity, idempotency, convex combination, additive generators) were studied by Bedregal et al. [3,7].

Definition 3. *A function* $O\colon [0,1]^2 \to [0,1]$ *is called an overlap function if it satisfies the following conditions:*

- **(O1)** *O is commutative,*
- **(O2)** *$O(x,y) = 0$ if and only if $xy = 0$,*
- **(O3)** *$O(x,y) = 1$ if and only if $xy = 1$,*
- **(O4)** *O is non-decreasing,*
- **(O5)** *O is continuous.*

Example 2. As an example of an overlap function, we can take any positive continuous t-norm. Examples of overlap functions that are not t-norms are the following functions:

$$O_p(x,y) = x^p y^p \text{ with } p > 0 \text{ and } p \neq 1.$$

Definition 4 ([7]). *Let $\mathcal{A}$ be a countable set of indexes, $(O_k)_{k\in\mathcal{A}}$ be a family of overlap functions and $(a_k, b_k)_{k\in\mathcal{A}}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. An ordinal sum of $(O_k)_{k\in\mathcal{A}}$ is a binary function $O\colon [0,1]^2 \to [0,1]$ defined by*

$$O(x,y) = \begin{cases} a_k + (b_k - a_k) O_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } (x,y) \in [a_k,b_k]^2, \\ \min(f_A(x), f_A(y)), & \text{otherwise,} \end{cases} \tag{2}$$

where $f_A\colon [0,1] \to [0,1]$ is given by

$$f_A(x) = \begin{cases} a_k + (b_k - a_k) O_k\left(\frac{x-a_k}{b_k-a_k}, 1\right), & \text{if } \exists_{k\in\mathcal{A}} \mid x \in [a_k,b_k], \\ x, & \text{otherwise.} \end{cases}$$

Theorem 3 (cf. [7]). *The ordinal sum of overlap functions given by (2) is an overlap function.*

2.3 Fuzzy Implications

Now, we focus on fuzzy implications.

Definition 5 (cf. [1,14]). *A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:*

(I1) *I is non-increasing with respect to the first variable,*
(I2) *I is non-decreasing with respect to the second variable,*
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

Directly from the definition we obtain as follows.

Corollary 1. *Each fuzzy implication I is constant for $x = 0$ and for $y = 1$, i.e., it satisfies the following properties, called left and right boundary condition, respectively: $I(0,y) = 1$, for all $y \in [0,1]$ and $I(x,1) = 1$ for all $x \in [0,1]$.*

There are other properties the fuzzy implication may also have. Some of them are listed below.

Definition 6 (cf. [1,7,8]). *We say that a fuzzy implication I fulfils:*

- *the (left) neutrality property (NP), if*

$$I(1,y) = y, \quad y \in [0,1], \tag{NP}$$

- *the identity principle (IP), if*

$$I(x,x) = 1, \quad x \in [0,1], \tag{IP}$$

- *the ordering property (OP), if*

$$I(x,y) = 1 \Leftrightarrow x \le y, \quad x,y \in [0,1], \tag{OP}$$

- *the property (CB), if*

$$I(x,y) \ge y, \quad x,y \in [0,1], \tag{CB}$$

- *the left ordering property (LOP), if*

$$x \le y \Rightarrow I(x,y) = 1, \quad x,y \in [0,1], \tag{LOP}$$

- *the right ordering property (ROP), if*

$$I(x,y) = 1 \Rightarrow x \le y, \quad x,y \in [0,1], \tag{ROP}$$

- *the strong boundary condition (SBC), if*

$$x \neq 0 \Rightarrow I(x,0) = 0, \quad x, y \in [0,1], \tag{SBC}$$

- *the strong corner condition for 0 (SCC0), if*

$$I(x,y) = 0 \Rightarrow x = 1 \wedge y = 0, \quad x, y \in [0,1], \tag{SCC0}$$

- *the strong corner condition for 1 (SCC1), if*

$$I(x,y) = 1 \Rightarrow x = 0 \vee y = 1, \quad x, y \in [0,1]. \tag{SCC1}$$

Remark 1. Let us notice that the property (CB) is equivalent to the following one

$$I(1,y) \geq y, \quad x, y \in [0,1]. \qquad \text{(CB')}$$

Moreover, if a fuzzy implication satisfies NP, then it satisfies (CB).

Example 3 ([1, 17]). The following are well-known examples of fuzzy implications.

$$I_{\text{ŁK}}(x,y) = \min(1 - x + y, 1), \quad I_{\text{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if} x > y \end{cases},$$

$$I_{\text{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}, \qquad I_{\text{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases},$$

$$I_{\text{RC}}(x,y) = 1 - x + xy, \qquad I_{\text{YG}}(x,y) = \begin{cases} 1, & \text{if } x, y = 0 \\ y^x, & \text{if else} \end{cases},$$

$$I_{\text{DN}}(x,y) = \max(1 - x, y), \qquad I_{\text{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1 - x, y), & \text{if } x > y \end{cases},$$

$$I_{\text{WB}}(x,y) = \begin{cases} 1, & \text{if } x \leq 1 \\ y, & \text{if } x = 1 \end{cases}, \qquad I_{\text{DP}}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1 - x, & \text{if } y = 0 \\ 1, & \text{if } x < 1,\ y > 0 \end{cases}.$$

Now, let us recall an important class of fuzzy implications which are called residual implications.

Definition 7. *A function $I : [0,1]^2 \to [0,1]$ is called a residual implication (an R-implication) if there exists a t-norm T such that for all $x, y \in [0,1]$*

$$I(x,y) = I_T(x,y) = \sup\{t \in [0,1] : T(x,t) \leq y\}. \tag{3}$$

Example 4. Table 1 shows R-implications obtained by formula (3) from basic t-norms presented in Example 1.

Now, let us recall the structure of residual implications (3) whose corresponding triangular norms are an ordinal sums of triangular norms.

Table 1. Basic R-implications.

t-norm T	R-implication I_T
T_M	I_{GD}
T_P	I_{GG}
T_L	I_{LK}
T_D	I_{WB}
T_nM	I_{FD}

Theorem 4 ([1], p. 83, [19]). *If T is a continuous triangular norm with an ordinal sum structure (see Theorem 2), then the corresponding R-implication I_T is given by the formula*

$$I_T(x,y)=\begin{cases} 1, & \text{if } x \leq y \\ a_k + (b_k - a_k) I_{T_k}\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x,y \in [a_k,b_k], x > y \\ y, & \text{otherwise} \end{cases} . \quad (4)$$

3 Main Results

In this section we propose two methods of generating a new fuzzy implication from given ones based on ordinal sum of overlap functions and residual implication obtained from ordinal sum of t-norms, which take into account differences in monotonicity between overlap functions and fuzzy implications. Let start with the first method.

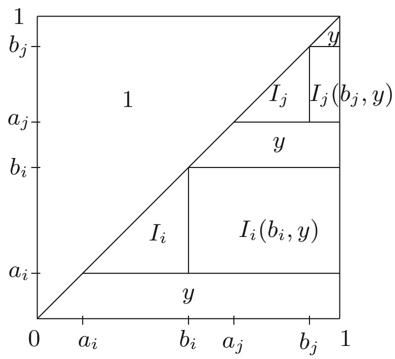

Fig. 2. The structure of ordinal sum of implications given by Eq. (5).

Definition 8. *Let $(I_k)_{k\in\mathcal{A}}$ be a family of fuzzy implications and $(a_k, b_k)_{k\in\mathcal{A}}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ for all $k \in \mathcal{A}$,* *where $\mathcal{A}$ is a finite or countably infinite index set. Let us define an operation $I\colon [0,1]^2 \to [0,1]$ by the following formula:*

$$I(x,y)=\begin{cases}1, & \text{if } x\le y,\\ a_k+(b_k-a_k)I_k\left(\frac{x-a_k}{b_k-a_k},\frac{y-a_k}{b_k-a_k}\right), & \text{if } x,y\in[a_k,b_k],\ x>y\\ a_k+(b_k-a_k)I_k\left(1,\frac{y-a_k}{b_k-a_k}\right), & \text{if } x\in(b_k,1], y\in[a_k,b_k],\\ y, & \text{otherwise,}\end{cases} \tag{5}$$

For the general structure of the above ordinal sum of fuzzy implications see Fig. 2 (the symbols $I_i(b_i,y)$ and $I_j(b_i,y)$ are a shortcut and should be understand as a linear combination of these values).

Our first new result is the following.

Theorem 5. *The operation I given by (5) is a fuzzy implication.*

Proof. Let $x_1,x_2,y\in[0,1]$ and $x_1<x_2$. If $y\in[a_k,b_k]$ for some $k\in\mathcal{A}$, then we consider the following cases:

1. $x_1\le y$, then $I(x_1,y)=1\le I(x_2,y)$;
2. $x_1\in[a_k,b_k]$, $x_1>y$ and
   (a) $x_2\in[a_k,b_k]$, then using the monotonicity of I_k we have

$$\begin{aligned}I(x_1,y)&=a_k+(b_k-a_k)I_k\left(\frac{x_1-a_k}{b_k-a_k},\frac{y-a_k}{b_k-a_k}\right)\\&\ge a_k+(b_k-a_k)I_k\left(\frac{x_2-a_k}{b_k-a_k},\frac{y-a_k}{b_k-a_k}\right)\\&=I(x_2,y).\end{aligned}$$

   (b) $x_2>b_k$, then again using the monotonicity of I_k we have

$$\begin{aligned}I(x_1,y)&=a_k+(b_k-a_k)I_k\left(\frac{x_1-a_k}{b_k-a_k},\frac{y-a_k}{b_k-a_k}\right)\\&\ge a_k+(b_k-a_k)I_k\left(\frac{b_k-a_k}{b_k-a_k},\frac{y-a_k}{b_k-a_k}\right)\\&=a_k+(b_k-a_k)I_k\left(1,\frac{y-a_k}{b_k-a_k}\right)\\&=I(x_2,y).\end{aligned}$$

3. $x_1>b_k$, then using (5) we have $I(x_1,y)=I(b_k,y)=I(x_2,y)$.

If $y\notin[a_k,b_k]$ for all $k\in\mathcal{A}$, then $I(x_1,y)=y=I(x_2,y)$. Thus, I satisfies (I1).

Next, let consider the condition (I2). Let $x,y_1,y_2\in[0,1]$ and $y_1<y_2$. If $x\in[a_k,b_k]$ for some $k\in\mathcal{A}$, then we consider the following cases:

1. $y_2>x$, then $I(x,y_1)\le 1=I(x,y_2)$.

2. $y_1, y_2 \in [a_k, b_k]$, $y_2 < x$ then using the monotonicity of I_k we have

$$I(x, y_1) = a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y_1 - a_k}{b_k - a_k} \right) \\ \leq a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y_2 - a_k}{b_k - a_k} \right) \\ = I(x, y_2).$$

3. $y_1 \in [a_i, b_i]$, $y_2 \in [a_j, b_j]$, $y_2 < x$ and $b_i < a_j$ for some $i, j \in \mathcal{A}$, then $I(x, y_1) \in [a_i, b_i]$, $I(x, y_2) \in [a_j, b_j]$, so $I(x, y_1) \leq I(x, y_2)$.
4. $y_1 \in [a_i, b_i]$ for some $i \in \mathcal{A}$ and $y_2 \notin [a_j, b_j]$ for all $j \in \mathcal{A}$, $y_2 < x$, then $I(x, y_1) \in [a_i, b_i]$, $I(x, y_2) = y_2$, so $I(x, y_1) \leq b_i < y_2 = I(x, y_2)$.
5. $y_1 \notin [a_i, b_i]$ for all $i \in \mathcal{A}$ and $y_2 \in [a_j, b_j]$ for some $j \in \mathcal{A}$, $y_2 < x$, then $I(x, y_1) = y_1$, $I(x, y_2) \in [a_j, b_j]$, so $I(x, y_1) = y_1 \leq a_j \leq I(x, y_2)$.
6. $y_1, y_2 \notin [a_i, b_i]$ for all $i \in \mathcal{A}$, $y_2 < x$, then $I(x, y_1) = y_1 < y_2 = I(x, y_2)$.

If $x \notin [a_k, b_k]$ for all $k \in \mathcal{A}$, $x \neq 0$ then we consider the following cases:

1. $y_2 > x$, then $I(x, y_1) \leq 1 = I(x, y_2)$.
2. $y_1, y_2 \in [a_k, b_k]$ and $b_k < x$, then using the monotonicity of I_k we have

$$I(x, y_1) = a_k + (b_k - a_k) I_k \left(1, \frac{y_1 - a_k}{b_k - a_k} \right) \\ \leq a_k + (b_k - a_k) I_k \left(1, \frac{y_2 - a_k}{b_k - a_k} \right) \\ = I(x, y_2).$$

3. $y_1 \in [a_i, b_i]$, $y_2 \in [a_j, b_j]$, $y_2 < x$ and $b_i < a_j$ for some $i, j \in \mathcal{A}$, then $I(x, y_1) \in [a_i, b_i]$, $I(x, y_2) \in [a_j, b_j]$, so $I(x, y_1) \leq I(x, y_2)$.
4. $y_1 \in [a_i, b_i]$ for some $i \in \mathcal{A}$ and $y_2 \notin [a_j, b_j]$ for all $j \in \mathcal{A}$, $y_2 < x$, then $I(x, y_1) \in [a_i, b_i]$, $I(x, y_2) = y_2$, so $I(x, y_1) \leq b_i < y_2 = I(x, y_2)$.
5. $y_1 \notin [a_i, b_i]$ for all $i \in \mathcal{A}$ and $y_2 \in [a_j, b_j]$ for some $j \in \mathcal{A}$, $y_2 < x$, then $I(x, y_1) = y_1$, $I(x, y_2) \in [a_j, b_j]$, so $I(x, y_1) = y_1 \leq a_j \leq I(x, y_2)$.
6. $y_1, y_2 \notin [a_i, b_i]$ for all $i \in \mathcal{A}$, $y_2 < x$, then $I(x, y_1) = y_1 < y_2 = I(x, y_2)$.

Thus, I satisfies (I2).

Directly from (5) we have $I(0, 0) = 1$ and $I(1, 1) = 1$. If $0 \notin [a_k, b_k]$ for all $k \in \mathcal{A}$, then $I(1, 0) = 0$. If $0 \in [a_k, b_k]$ for some $k \in \mathcal{A}$ i.e. $a_k = 0$, then $I(1, 0) = b_k I_k(1, 0) = 0$. Therefore I fulfils (I3), (I4) and (I5).

Now, let us examine some properties of the operation I given by (5).

Theorem 6. *Let I be given by (5).*

(i) I satisfies (NP) if and only if I_k satisfies (NP) for all $k \in \mathcal{A}$.
(ii) I satisfies (IP).
(iii) I satisfies (LOP).

(iv) I satisfies (ROP) if and only if one of following conditions holds
  (a) $1 \notin [a_k, b_k]$ *for all* $k \in \mathcal{A}$*,*
  (b) there exists $k \in \mathcal{A}$ *such that* $b_k = 1$ *and* I_k *satisfies (ROP).*
(v) I satisfies (OP) if and only if one of following conditions holds
  (a) $1 \notin [a_k, b_k]$ *for all* $k \in \mathcal{A}$*,*
  (b) there exists $k \in \mathcal{A}$ *such that* $b_k = 1$ *and* I_k *satisfies (OP).*
(vi) I satisfies (CB) if and only if I_k *satisfies (CB) for all* $k \in \mathcal{A}$*.*
(vii) I satisfies (SBC) if and only if one of following conditions holds
  (a) $0 \notin [a_k, b_b]$ *for all* $k \in \mathcal{A}$*,*
  (b) there exists $k \in \mathcal{A}$ *such that* $a_k = 0$ *and* I_k *satisfies (SBC).*
(viii) I satisfies (SCC0) if and only if there exists $k \in \mathcal{A}$ *such that* $[a_k, b_k] = [0, 1]$ *and* I_k *satisfies (SCC0).*
(ix) I does not satisfy (SCC1).

Proof. Directly by (5) we obtain that I satisfies (IP) and (LOP).

(iv) Firstly, let us assume that I satisfies (ROP). If there exists $k \in \mathcal{A}$ such that $b_k = 1$ then

$$I_k(x, y) = 1,$$
$$\frac{I\left((1 - a_k)x + a_k, (1 - a_k)y + a_k\right) - a_k}{1 - a_k} = 1,$$
$$I\left((1 - a_k)x + a_k, (1 - a_k)y + a_k\right) = 1.$$

Since I satisfies (ROP) we have

$$(1 - a_k)x + a_k \leq (1 - a_k)y + a_k,$$
$$(1 - a_k)x \leq (1 - a_k)y,$$
$$x \leq y.$$

This means that I_k satisfies (ROP).

Now, let us assume that $x > y$ and consider two cases

1. $1 \notin [a_k, b_b]$ for all $k \in \mathcal{A}$. If $y \in [a_k, b_k]$ for some $k \in \mathcal{A}$, then $I(x, y) \in [a_k, b_k]$ and $I(x, y) < 1$. If $y \notin [a_k, b_k]$ for all $k \in \mathcal{A}$ then $I(x, y) = y < 1$.
2. there exists $k \in \mathcal{A}$ such that $b_k = 1$ and I_k satisfies (ROP). If $y \in [a_k, 1]$ then we have

$$I(x, y) = 1,$$
$$a_k + (1 - a_k)I_k\left(\frac{x - a_k}{1 - a_k}, \frac{y - a_k}{1 - a_k}\right) = 1,$$
$$(1 - a_k)I_k\left(\frac{x - a_k}{1 - a_k}, \frac{y - a_k}{1 - a_k}\right) = 1 - a_k,$$
$$I_k\left(\frac{x - a_k}{1 - a_k}, \frac{y - a_k}{1 - a_k}\right) = 1,$$

Since I_k satisfies (ROP) we have

$$\frac{x-a_k}{1-a_k} \leq \frac{y-a_k}{1-a_k}$$
$$x \leq y.$$

a contradiction.
If $y < a_k$ then the proof is similar as in 1.

(v) By joining (iii) and (iv) we get (OP).
The rest of the properties are simple consequence of Eq. (5).

In this paper we propose yet another method of generating fuzzy implication by the use of an ordinal sum of fuzzy implications.

Definition 9. *Let $(I_k)_{k\in\mathcal{A}}$ be a family of fuzzy implications and $(a_k, b_k)_{k\in\mathcal{A}}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ for all $k \in \mathcal{A}$, where $\mathcal{A}$ is a finite or countably infinite index set. Let us define an operation $I\colon [0,1]^2 \to [0,1]$ by the following formula:*

$$I(x,y) = \begin{cases} 1, & \text{if } x \geq y, \\ a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x, y \in [a_k, b_k], \\ a_k, & \text{if } x \in (b_k, 1], y \in [a_k, b_k], \\ y, & \text{otherwise,} \end{cases} \tag{6}$$

For the general structure of the above ordinal sum of fuzzy implications see Fig. 3.

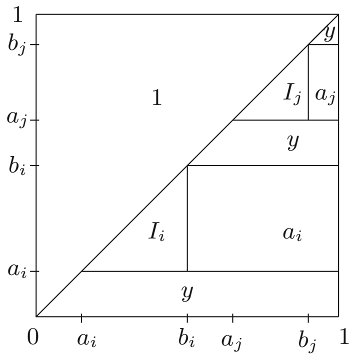

Fig. 3. The structure of ordinal sum of implications given by Eq. (6).

Similarly as in the previous method we obtain a fuzzy implication.

Theorem 7. *The operation I given by (6) is a fuzzy implication.*

Now, let us examine some properties of the operation I given by (6).

Theorem 8. *Let I be given by (6).*

(i) I satisfies (NP) if and only if I has only one summand I_{k_0} which satisfies (NP) with underlying interval $[a_{k_0}, 1]$.
(ii) I satisfies (IP).
(iii) I satisfies (LOP).
(iv) I satisfies (ROP) if and only if one of following conditions holds
  (a) $1 \notin [a_k, b_k]$ for all $k \in \mathcal{A}$,
  (b) there exists $k \in \mathcal{A}$ such that $b_k = 1$ and I_k satisfies (ROP).
(v) I satisfies (OP) if and only if one of following conditions holds
  (a) $1 \notin [a_k, b_k]$ for all $k \in \mathcal{A}$,
  (b) there exists $k \in \mathcal{A}$ such that $b_k = 1$ and I_k satisfies (OP).
(vi) I does not satisfy (CB).
(vii) I satisfies (SBC) if and only if one of following conditions holds
  (a) $0 \notin [a_k, b_b]$ for all $k \in \mathcal{A}$,
  (b) there exists $k \in \mathcal{A}$ such that $a_k = 0$ and I_k satisfies (SBC).
(viii) I satisfies (SCC0) if and only if there exists $k \in \mathcal{A}$ such that $[a_k, b_k] = [0, 1]$ and I_k satisfies (SCC0).
(ix) I does not satisfy (SCC1).

4 Conclusions

In the paper two new methods of constructing ordinal sums of fuzzy implications which lead to fuzzy implications are presented. Basic properties of thus obtained operation have been examined.

It seems useful to examine other properties of the component of introduced ordinal sums which can be preserved by the ordinal sums. It seems also necessary to compare the here obtained methods with other existing methods of constructing ordinal sum of fuzzy implications.

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Relativization of Fuzzy Quantifiers: Initial Investigations

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Abstract. We discuss so-called relativization of various classes of generalized quantifiers (bivalent, fuzzy, semi-fuzzy). We show that for fuzzy quantifiers, relativization cannot be defined in a satisfactory way. Hence we provide a generalization of fuzzy quantifiers consisting of use of a fuzzy set as a universe for quantification. Relativization of these fuzzy quantifiers fulfills intuitive requirements.

Keywords: Fuzzy quantifier · Generalized quantifier · Relativization · Residuated lattice

1 Introduction

In this paper we study the important operation of *relativization* from the perspective of fuzzy quantifiers. In the most common case of quantifiers[1] with one argument (so-called type $\langle 1 \rangle$[2]), relativization of such (bivalent, i.e., two-valued) quantifier Q is a quantifier Q^{rel} with two arguments (type $\langle 1,1 \rangle$). Q^{rel} behaves as Q, but on a universe provided by its first argument. Formally, for all $A, B \subseteq M$,

$$Q^{\mathrm{rel}}_M(A,B) := Q_A(A \cap B), \tag{1}$$

(see Sect. 3.2).[3]

The importance of relativization follows from its ability to link related quantifiers of different types. It also permits to characterize an important class of type $\langle 1,1 \rangle$ quantifiers with semantic properties of extension and conservativity

[1] In this paper, we will use the term *quantifier* instead of more common *generalized quantifier*. Our presentation of (generalized) quantifiers mainly follows a general approach of book [16], see also [10]. Important papers and books on *fuzzy quantifiers* include [3,7,8,12–15,17].

[2] This notation originated in [11], where quantifiers are understood to be classes of relational structures of a certain type (representing a number of arguments and variable binding). It is widely used in the literature on generalized quantifiers [16].

[3] $Q^{\mathrm{rel}}_M(A,B)$ denotes the truth value of quantifier Q^{rel} defined on universe M for arguments A and B.

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as relativizations of type $\langle 1 \rangle$ quantifiers. However, if we translate formula (1) directly into the language of fuzzy set theory (A and B are now *fuzzy* subsets of M), we run into problems, because there is a type mismatch in the expression on the right side of (1): in a position for the universe for Q, a classical set (such as M) is expected, not a fuzzy set A. It is possible to try to amend this by using, for example, the support of A in this place, but results are not satisfactory (see Sect. 4.2). We are even able to proof (Theorem 2) that it is not possible to define relativization of fuzzy quantifiers (in the sense of Definitions 7 and 8) in a satisfactory way.

Our solution lies in a generalization of the definition of fuzzy quantifiers. It turns out that when we define fuzzy quantifiers not on a crisp universal set, but on a fuzzy set (Definitions 12 and 13), things start to run smoothly and important theorems that hold for bivalent quantifiers start to hold for fuzzy quantifiers too.

These new fuzzy quantifiers (we call them C-fuzzy quantifiers)[4] possess interesting properties. In this paper, we, besides some examples showing how models of common quantifiers "for all", "some" etc., can be defined, concentrate only on their behaviour with respect to relativization.

2 Preliminaries

2.1 Structures of Truth Values

The basic structure of truth values in this paper will be a *commutative bounded integral residuated lattice* $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \bot, \top \rangle$ [6],[5] that is, an algebra with four binary operations and two constants such that $\mathbf{L} = \langle L, \wedge, \vee, \bot, \top \rangle$ is a lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, $\mathbf{L} = \langle L, \otimes, \top \rangle$ is a commutative monoid (i.e., $\otimes$ is associative and commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property

$$a \otimes b \leq c \text{ if and only if } a \leq b \rightarrow c$$

is satisfied for all $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. The operations $\otimes$ and $\rightarrow$ are usually called multiplication and residuum, respectively. *Negation* $\neg$ is a defined operation: $\neg a := a \rightarrow \bot$ for all $a \in L$.

Important special cases of residuated lattices are *IMTL-algebras*, *MV-algebras*, the standard *Łukasiewicz algebra* on $[0, 1]$ and the *Boolean algebra* $\mathbf{2}$ on the two-element support $2 := \{0, 1\}$. Details on these algebras can be found, e.g., in [1,2].

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[4] The C in their name refers to the most natural denotation for the fuzzy set acting as a universe of discourse, because letters A and B are usually reserved for arguments of the most common quantifiers of type $\langle 1, 1 \rangle$ and M is always used for a *crisp* universe.

[5] We will call this structure a residuated lattice for short.

2.2 Sets and Fuzzy Sets

Let X be a set. Then 2^X denotes the set of all mappings from X to $\{0,1\}$. These mappings can be understood as *characteristic functions* of subsets of X. Therefore, 2^X denotes also the set of all subsets of X.

Let M be a set and $\mathbf{L}$ be a residuated lattice. A mapping $A\colon M \to L$ is called a *fuzzy set* on M. A value $A(m)$ is called the *membership degree* of m in the fuzzy set A. The set of all fuzzy sets on M is denoted by L^M. A *support* of a fuzzy set A is denoted by Supp(A) and defined as $\mathrm{Supp}(A) = \{m \in M \mid m > \bot\}$. $\emptyset_M$ denotes the empty fuzzy set on M, that is, $\emptyset_M(m) = \bot$ for all $m \in M$. A fuzzy set A is a *subset* of fuzzy set B if for all $m \in M$, $A(m) \leq B(m)$. The set of all subsets of a fuzzy set A is denoted by L^A. A fuzzy set A is called *crisp* if there is a subset $Z \in M$ such that $A = 1_Z$, where 1_Z denotes the characteristic function of Z.

Let $\{A_i \mid i \in I\}$ be a non-empty family of fuzzy sets on M. Then the *union* and *intersection of* A_i are defined as

$$\left(\bigcup_{i\in I} A_i\right)(m) := \bigvee_{i\in I} A_i(m) \quad \text{and} \quad \left(\bigcap_{i\in I} A_i\right)(m) := \bigwedge_{i\in I} A_i(m), \tag{2}$$

respectively, for any $m \in M$. Finally, an extension of the operations $\otimes$ and $\to$ on L to the operations on L^M is given by

$$(A \otimes B)(m) := A(m) \otimes B(m) \quad \text{and} \quad (A \to B)(m) := A(m) \to B(m), \tag{3}$$

respectively, for any $A, B \in L^M$ and $m \in M$.

Let $f\colon M \to M'$ be a mapping. A mapping $f^{\to}\colon L^M \to L^{M'}$ defined by $f^{\to}(A)(m') := \bigvee_{m\in f^{-1}(m')} A(m)$ is called the *fuzzy extension* of the mapping f. Obviously, if f is a one-to-one mapping from M onto M', then $f^{\to}(A)(f(m)) = A(m)$ for any $m \in M$.

2.3 L-fuzzy Equivalence

In Sect. 5 we will need the following operations. Let M be a set and $\mathbf{L}$ be a complete residuated lattice.

(i) The mapping $\cong_M\colon L^M \times L^M \to L$ defined as

$$A \cong_M B := \bigwedge_{m\in M} (A(m) \leftrightarrow B(m)) \tag{4}$$

is called the $\mathbf{L}$-*fuzzy equivalence* on M.

(ii) The mapping $\cong_M\colon L^M \times L^M \to L$ defined as

$$A \subseteq_M B := \bigwedge_{m\in M} (A(m) \to B(m)) \tag{5}$$

is called the $\mathbf{L}$-*fuzzy subsethood* on M.

3 Bivalent Quantifiers

By *NL-quantifiers*, we in this paper understand natural language expressions such as "for all", "many", "several", etc. For our purposes it is not necessary to delineate the class of NL-quantifiers formally. In fact, we are interested in *mathematical models* of these natural language quantifiers. For the sake of comprehensibility, we in the following informal explanation consider NL-quantifiers with two arguments, such as "some" in sentence "Some people are clever."

3.1 Global and Local Quantifiers

Generally [16], a model of the NL-quantifier "some" takes the form of a functional (the so-called *global* quantifier) some that to any universe of discourse M assigns a *local* quantifier some_M. This local quantifier is a mapping that to any two subsets A and B of M assigns a truth value $\mathsf{some}_M(A,B)$. If we consider only (classical) sets A and B and the truth value of $\mathsf{some}_M(A,B)$ can be either *true* or *false* only, we say that this some is a *bivalent quantifier*. If A and B are fuzzy sets and the truth value of $\mathsf{some}_M(A,B)$ is taken from some many-valued structure of truth degrees, we say that this some[6] is a *fuzzy quantifier*.

Definition 1 (Local bivalent quantifier). *Let M is a universe of discourse. A* local bivalent quantifier *Q_M of type $\langle 1^n, 1\rangle$ on M is a function from $(2^M)^n \times 2^M$ to 2 that to any sets $A_1,\ldots,A_n$ and B from 2^M assigns a truth value $Q_M(A_1,\ldots,A_n,B)$ from 2.*

Definition 2 (Global bivalent quantifier). *A* global bivalent quantifier *Q* of type *$\langle 1^n, 1\rangle$ is a functional that to any universe M assigns a local bivalent quantifier $Q_M\colon (2^M)^n \times 2^M \to 2$ of type $\langle 1^n, 1\rangle$.*

Important examples of bivalent quantifiers of type $\langle 1\rangle$ are $\forall$ and $\exists$. They are defined as $\forall_M(B) := B = M$ and $\exists_M(B) := B \neq \emptyset$.[7] Important examples of bivalent quantifiers of type $\langle 1,1\rangle$ are all and some, defined as $\mathsf{all}_M(A,B) := A \subseteq B$ and $\mathsf{some}_M(A,B) := A \cap B \neq \emptyset$.

There are many semantic properties that can be defined for bivalent quantifiers. Here we recall only three properties necessary for the statement of an important theorem in the next subsection. However, these properties are essential from the point of view of the adequacy of our models with respect to natural language semantics. For discussion on motivation of these properties, see [16] and also [4,5,9]. Briefly, isomorphism invariance (ISOM) holds for quantifiers invariant with respect to bijections between various universes of discourse. It means that these quantifiers are not sensitive to individual objects but to numbers of them (cardinalities of respective sets). Quantifiers with the property of extension

[6] It will be always clear from the context whether some denotes (global/local) bivalent quantifier, fuzzy quantifier or C-fuzzy quantifier (see Sect. 5).

[7] For example, the definition of $\forall$ should be read as follows: If B is equal to M, then the truth value of $\forall_M(B)$ is equal to 1, otherwise it is equal to 0.

(EXT) are invariant with respect to possible extensions of the universe of discourse. This is typical for models of NL-quantifiers of type $\langle 1,1\rangle$, because their first argument (also called *restriction*) provides the universe of discourse for the given quantifier application. Hence, the extension of the underlying universe of discourse should not be relevant. The final semantic property is conservativity (CONS). The intuitive meaning of this property can be expressed as follows. If we want to know the truth value of a quantified statement "Q As are B" (e.g., "All streets are wet"), only those B that are also A are relevant. Specifically, if "all" is conservative, the sentence above is logically equivalent to $S =$ "All streets are streets and are wet". Wet things that are not streets are irrelevant.

Definition 3 (Isomorphism invariance). *We say that a bivalent quantifier Q of type $\langle 1^n,1\rangle$ is* isomorphism invariant *if for any universe M and bijection $f : M \to M'$ and all $A_1,\dots,A_n,B \in 2^M$ it holds that*

$$Q_M(A_1,\dots,A_n,B) = Q_{M'}(f(A_1),\dots,f(A_n),f(B)),$$

where $f(A)$ denotes the image of A under f. The set of all isomorphism-invariant bivalent quantifiers is denoted by ISOM.

Definition 4 (Extension). *We say that a bivalent quantifier Q of type $\langle 1^n,1\rangle$ satisfies the property of* extension *if for any M and M' such that $M \subseteq M'$ it holds that, for any $A_1,\dots,A_n,B \in 2^M$,*

$$Q_M(A_1,\dots,A_n,B) = Q_{M'}(A_1,\dots,A_n,B).$$

The set of all bivalent quantifiers satisfying the property of extension is denoted by EXT.

Definition 5 (Conservativity). *Let $n \geq 1$. We say that a bivalent quantifier Q of type $\langle 1^n,1\rangle$ is* conservative *if for any M and any $A_1,\dots,A_n,B,B' \in 2^M$ it holds that if $A_1 \cap B = A_1 \cap B', \cdots, A_n \cap B = A_n \cap B'$, then*

$$Q_M(A_1,\dots,A_n,B) = Q_M(A_1,\dots,A_n,B').$$

The set of all conservative bivalent quantifiers is denoted by CONS. It can be shown that for the most common quantifiers of type $\langle 1,1\rangle$, a quantifier Q is conservative if and only if for any M and any $A,B \in 2^M$,

$$Q_M(A,B) = Q_M(A, A \cap B).$$

3.2 Relativization of Bivalent Quantifiers

In this paper, we consider only the most common case of relativization of quantifiers of type $\langle 1\rangle$.

Definition 6 ([16]). *Let Q be a global bivalent quantifier of type $\langle 1\rangle$. The* relativization *of Q is a global bivalent quantifier Q^{rel} of type $\langle 1,1\rangle$ defined as*

$$(Q^{\mathrm{rel}})_M(A,B) := Q_A(A \cap B) \tag{6}$$

for all $A,B \in 2^M$.

The importance of relativization follows from its ability to link related quantifiers of different types. For example, it is intuitively clear that the bivalent quantifier $\forall$ of type $\langle 1 \rangle$ is related to the bivalent quantifier all of type $\langle 1, 1 \rangle$. This relation is established by the fact that

$$\forall^{\mathrm{rel}} = \mathsf{all},$$

and, similarly,

$$\exists^{\mathrm{rel}} = \mathsf{some}.$$

Relativization is especially interesting from the point of view of modeling of NL-quantifiers. In [16], it is argued that *all* models of NL-quantifiers of the most common type $\langle 1, 1 \rangle$ should be conservative and satisfy the property of extension. The following theorem shows that there is one-to-one correspondence between bivalent quantifiers of type $\langle 1 \rangle$ and bivalent quantifiers of type $\langle 1, 1 \rangle$ satisfying conservativity and extension. This correspondence is provided just by the relativization. Hence, any model of a NL-quantifier of type $\langle 1, 1 \rangle$ can be established as the relativization of a bivalent quantifier of type $\langle 1 \rangle$. Moreover, if a bivalent quantifier of type $\langle 1 \rangle$ is isomorphism invariant, the same holds for its relativization.

Theorem 1 ([16]). *Let Q be a bivalent quantifier of type $\langle 1 \rangle$ and P be a bivalent quantifier of type $\langle 1, 1 \rangle$. Then*

(a) Q^{rel} is EXT and CONS.
(b) If P is EXT and CONS, then there exists a bivalent quantifier Q of type $\langle 1 \rangle$ such that $Q^{\mathrm{rel}} = P$.
(c) Q^{rel} is ISOM if and only if Q is ISOM.

4 Fuzzy Quantifiers

4.1 Definitions

Straightforward generalization [9] of Definitions 1 and 2 consists of replacing classical sets $A_1, \ldots, A_n$ and B by fuzzy subsets of M and of using a residuated lattice $\mathbf{L}$ instead of the Boolean algebra $\mathbf{2}$.

Definition 7 (Local fuzzy quantifier). *Let M is a universe of discourse. A* local fuzzy quantifier *Q_M of type $\langle 1^n, 1 \rangle$ on M is a function from $(L^M)^n \times L^M$ to L that to any fuzzy sets $A_1, \ldots, A_n$ and B from L^M assigns a truth value $Q_M(A_1, \ldots, A_n, B)$ from L.*

Definition 8 (Global fuzzy quantifier). *A* global fuzzy quantifier *Q of type $\langle 1^n, 1 \rangle$ is a functional that to any universe M assigns a local fuzzy quantifier $Q_M\colon (L^M)^n \times L^M \to L$ of type $\langle 1^n, 1 \rangle$. The set of all global fuzzy quantifiers of type $\langle 1^n, 1 \rangle$ will be denoted by* $\mathrm{QUANT}_{\langle 1^n, 1 \rangle}$.

Important examples of fuzzy quantifiers of type $\langle 1 \rangle$ are, again, $\forall$ and $\exists$. They are standardly defined as

$$\forall_M(B) := \bigwedge_{m \in M} B(m)$$

and

$$\exists_M(B) := \bigvee_{m \in M} B(m).$$

Important examples of fuzzy quantifiers of type $\langle 1, 1 \rangle$ are all and some, defined as

$$\mathsf{all}_M(A, B) := \bigwedge_{m \in M} (A \to B)(m) \tag{7}$$

and

$$\mathsf{some}_M(A, B) := \bigvee_{m \in M} (A \cap B)(m). \tag{8}$$

In some situations, it is advantageous to consider fuzzy quantifiers that are sensitive only to supports of some of its arguments. In other words, they ignore vagueness in some of its arguments. We call such fuzzy quantifiers *semi-fuzzy quantifiers* [7]. The formal definition is as follows.

Definition 9 (Semi-fuzzy quantifiers). *A fuzzy quantifier Q of type $\langle 1^n, 1 \rangle$ is* semi-fuzzy in its i-th component*, where $i \in \{1, \dots, n+1\}$, if for an arbitrary universe M and $A_1, \dots, A_{n+1} \in L^M$ it holds that*

$$Q_M(A_1, \dots, A_i, \dots, A_{n+1}) = Q_M(A_1, \dots, \mathrm{Supp}(A_i), \dots, A_{n+1}).$$

A fuzzy quantifier Q of type $\langle 1^n, 1 \rangle$ that is semi-fuzzy in each of its components is called semi-fuzzy quantifier *of type $\langle 1^n, 1 \rangle$.*

As an example, consider the sentence "All pregnant women are happy". Because the predicate "to be a pregnant woman" is bivalent, it is not necessary to model the NL-quantifier "all" by "fully fuzzy" quantifier (7), but by semi-fuzzy in the first argument quantifier $\mathsf{all}^{\mathsf{sf}}$ defined as

$$\mathsf{all}^{\mathsf{sf}}_M(A, B) := \bigwedge_{m \in \mathrm{Supp}(A)} B(m). \tag{9}$$

Similarly,

$$\mathsf{some}^{\mathsf{sf}}_M(A, B) := \bigvee_{m \in \mathrm{Supp}(A)} B(m). \tag{10}$$

It is easy to see that for any M and $A, B \in L^M$ it holds that if A is crisp, then $\mathsf{all}_M(A, B) = \mathsf{all}^{\mathsf{sf}}_M(A, B)$.

4.2 Relativization of Fuzzy Quantifiers

In [9], the second author of the present paper defined *relativization* and *weak relativization* of fuzzy quantifiers of type $\langle 1 \rangle$ defined according to Definitions 7 and 8.

Definition 10. *Let Q is a fuzzy quantifier of type $\langle 1 \rangle$. Then its* relativization Q^{rel} *or* weak relativization $Q^{\text{w-rel}}$ *is the fuzzy quantifier of type $\langle 1, 1 \rangle$ given by*

$$Q_M^{\text{rel}}(A, B) := Q_{\text{Supp}(A)}(A \cap B) \tag{11}$$

or

$$Q_M^{\text{w-rel}}(A, B) := Q_{\text{Supp}(A)}(\text{Supp}(A) \cap B), \tag{12}$$

respectively, for an arbitrary crisp universe M and $A, B \in L^M$.

We see that these definitions are somewhat awkward, because they use the support of A instead of expected A itself on their right sides (compare with (6)). It is caused by the fact that the universe on which fuzzy quantifiers are defined is always *crisp*. Therefore, we are forced to use a crisp set there, and $\text{Supp}(A)$ seems to be a natural choice. It is easy to see that for any fuzzy quantifier Q of type $\langle 1 \rangle$, $Q^{\text{w-rel}}$ is semi-fuzzy in its first component.

If we try to establish a similar correspondence between fuzzy quantifiers of types $\langle 1 \rangle$ and $\langle 1, 1 \rangle$ by means of relativization (11) or weak relativization (12) as for bivalent quantifiers in Sect. 3.2, we obtain the following results.

$$(\forall^{\text{rel}})_M(A, B) = \bigwedge_{m \in \text{Supp}(A)} (A \cap B)(m), \tag{13}$$

$$(\forall^{\text{w-rel}})_M(A, B) = \bigwedge_{m \in \text{Supp}(A)} (\text{Supp}(A) \cap B)(m) = \bigwedge_{m \in \text{Supp}(A)} (B)(m), \tag{14}$$

$$(\exists^{\text{rel}})_M(A, B) = \bigvee_{m \in \text{Supp}(A)} (A \cap B)(m), \tag{15}$$

$$(\exists^{\text{w-rel}})_M(A, B) = \bigvee_{m \in \text{Supp}(A)} (\text{Supp}(A) \cap B)(m) = \bigvee_{m \in \text{Supp}(A)} (B)(m). \tag{16}$$

We can see that $\forall^{\text{w-rel}} = \mathsf{all}^{\mathsf{sf}}$ and $\exists^{\text{w-rel}} = \mathsf{some}^{\mathsf{sf}}$. Therefore, weak relativization links fuzzy quantifiers of type $\langle 1 \rangle$ to semi-fuzzy in the first component quantifiers of type $\langle 1, 1 \rangle$. However, use of relativization (11) does not provide satisfactory results, because although it holds that $\exists^{\text{rel}} = \mathsf{some}$, it also holds that $\forall^{\text{rel}} \neq \mathsf{all}$ (cf. (7)). Moreover, the fuzzy quantifier $\forall^{\text{rel}}$ from (13) does not model the NL-quantifier "all" in any reasonable sense.

Now we formulate the theorem saying that it is not possible to define a relativization operation for fuzzy quantifiers in such a way that the relativization of $\forall$ is all and, at the same time, the relativization of $\exists$ is some. It means that the failure of the relativization (11) and the weak relativization (12) in this respect is inevitable and cannot be avoided in the frame of fuzzy quantifiers from Definitions 7 and 8. For this purpose, we define an operation called *generalized relativization* in the following way.

Definition 11 (Generalized relativization). *Let M be an arbitrary universe. Let $X_M : L^M \times L^M \to 2^M$ be an operation that to any two fuzzy sets $A, B \in L^M$ assigns a crisp subset of M.*[8] *Let $\varphi_M : L^M \times L^M \to L^M$ be an operation that to any two fuzzy sets $A, B \in L^M$ assigns a fuzzy set on M.*[9] *We say that* grel : $\mathrm{QUANT}_{\langle 1\rangle} \to \mathrm{QUANT}_{\langle 1,1\rangle}$, *defined as*

$$(Q^{\mathrm{grel}})_M(A, B) := Q_{X_M(A,B)}(\varphi_M(A, B)) \tag{17}$$

for any $Q \in \mathrm{QUANT}_{\langle 1\rangle}$, any M and $A, B \in L^M$, is a generalized relativization from fuzzy quantifiers of type $\langle 1\rangle$ to fuzzy quantifiers of type $\langle 1, 1\rangle$.

Theorem 2. *Let $\mathbf{L}$ be a residuated lattice such there exists $\alpha \in L$ for which $\bot < \alpha < \top$. Let* grel *be a generalized relativization (17). Then if $\forall^{\mathrm{grel}} = \mathsf{all}$, then $\exists^{\mathrm{grel}} \neq \mathsf{some}$, and if $\exists^{\mathrm{grel}} = \mathsf{some}$, then $\forall^{\mathrm{grel}} \neq \mathsf{all}$.*

Proof sketch. Let M be an arbitrary non-empty universe. Let us define fuzzy sets A_0 and B_0 on M as follows:

$$A_0 = B_0 = \{\alpha/m_0\}$$

for $m_0 \in M$ and $\bot < \alpha < \top$.

We can prove that, in order to secure that $\exists^{\mathrm{grel}}_M(A_0, B_0) = \mathsf{some}_M(A_0, B_0)$ and, in the same time, $\forall^{\mathrm{grel}}_M(A_0, B_0) = \mathsf{all}_M(A_0, B_0)$, it would be necessary that $\varphi_M(A_0, B_0)(m_0) = \alpha$ and $\varphi_M(A_0, B_0)(m_0) = \top$, which is impossible. □

5 C-fuzzy Quantifiers

In the previous section, we saw that it is not possible to define a relativization of fuzzy quantifiers in a way that satisfies intuitive requirements. In this section, we outline a solution consisting in a generalization of a definition of fuzzy quantifiers.

Definition 12 (Local C-fuzzy quantifier). *Let C be a fuzzy set (on some fixed non-empty universe M). A* local *C-fuzzy quantifier Q_C* of type $\langle 1^n, 1\rangle$ on *C is a function from $(L^C)^n \times L^C$ to L that to any fuzzy sets $A_1, \ldots, A_n$ and B from L^C assigns a truth value $Q_C(A_1, \ldots, A_n, B)$.*

In the rest of this paper, we always assume that M is a non-empty (fixed) set.[10] In many cases, when no misunderstanding can occur, we omit in definitions, examples, etc., that C-fuzzy quantifiers are defined "on M".

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[8] An example of X_M: $X_M(A, B) := \mathrm{Supp}(A)$ as in (11) and (12).

[9] Examples of φ_M: $\varphi^1_M(A, B) := A \cap B$ as in (11) or $\varphi^2_M(A, B) := \mathrm{Supp}(A) \cap B$ as in (12).

[10] It has hardly any sense to consider the empty M here. Note that M has a similar rôle in our approach as a universal class V in the set theory – it permits to pick some (fuzzy) sets and work with them.

Definition 13 (Global C-fuzzy quantifier). *A* global *C-fuzzy quantifier Q* of type $\langle 1^n, 1\rangle$ (on M) *is a functional that assigns to any fuzzy set $C \in L^M$ a local C-fuzzy quantifier $Q_C\colon (L^C)^n \times L^C \to L$ of type $\langle 1^n, 1\rangle$.*

If we write "C-fuzzy quantifier", we always understand the quantifier in question as a *global* one. If M is finite or countable, we call Q a finite or countable C-fuzzy quantifier, respectively.

Example 1. Let $\cong_M$ be an **L**-fuzzy equivalence (4) (on M). A C-fuzzy quantifier $\forall$ of type $\langle 1\rangle$ assigns to any $C \in L^M$ a local fuzzy quantifier $\forall_C$ defined as

$$\forall_C(B) := C \cong_M B$$

for any $B \in L^C$. It can be shown that

$$\forall_C(B) = \bigwedge_{m\in M} (C(m) \to B(m)).$$

If C is crisp, then

$$\forall_C(B) = \bigwedge_{m\in M} (C(m) \to B(m)) = \bigwedge_{m\in M} (\top \to B(m)) = \bigwedge_{m\in M} B(m),$$

that is, it coincides with the standard definition of the fuzzy quantifier $\forall$.

Example 2. Let $\cong_M$ be a **L**-fuzzy equivalence (on M). A C-fuzzy quantifier $\exists$ of type $\langle 1\rangle$ assigns to any $C \in L^M$ a local fuzzy quantifier $\exists_C$ defined as

$$\exists_C(B) := (C \cap B) \ncong_M \emptyset_M \tag{18}$$

for any $B \in L^C$.[11] It can be obtained that

$$\exists_C(B) = \neg\neg \bigvee_{m\in M} B(m).$$

If the negation in **L** is involutive (for example, if **L** is an IMTL-algebra, MV-algebra or Łukasiewicz algebra), then

$$\exists_C(B) = \bigvee_{m\in M} B(m), \tag{19}$$

that is, it coincides with the standard definition of the fuzzy quantifier $\exists$.

Example 3. Let $\subseteq_M$ be a **L**-fuzzy subsethood (5) (on M). A C-fuzzy quantifier all of type $\langle 1, 1\rangle$ assigns to any $C \in L^M$ a local fuzzy quantifier all_C defined as

$$\mathsf{all}_C(A, B) := A \subseteq_M B$$

for any $A, B \in L^C$.

[11] $A \ncong_M B$ denotes $\neg(A \cong_M B)$.

Due to the definition of **L**-fuzzy subsethood (5), we can write[12]

$$\mathsf{all}_C(A, B) = A \subseteq_M B = \bigwedge_{m \in M} (A(m) \to B(m)). \tag{20}$$

Let $\cong_M$ be a **L**-fuzzy equivalence (on M). A C-fuzzy quantifier some of type $\langle 1, 1 \rangle$ assigns to any $C \in L^M$ a local fuzzy quantifier some_C defined as

$$\mathsf{some}_C(A, B) := (A \cap B) \ncong_M \emptyset_M$$

for any $A, B \in L^C$. It can be shown that

$$\mathsf{some}_C(A, B) = \neg\neg \bigvee_{m \in M} (A(m) \wedge B(m)).$$

If the negation in **L** is involutive, then

$$\mathsf{some}_C(A, B) = \bigvee_{m \in M} (A(m) \wedge B(m)).$$

6 Relativization of C-fuzzy Quantifiers

Now we define *relativization* for C-fuzzy quantifiers (cf. Definition 6).

Definition 14 (Relativization of C-fuzzy quantifiers). *Let Q be a global C-fuzzy quantifier of type $\langle 1 \rangle$. The* relativization *of Q is a global C-fuzzy quantifier Q^{rel} of type $\langle 1, 1 \rangle$ defined as*

$$(Q^{\mathrm{rel}})_C(A, B) := Q_A(A \cap B) \tag{21}$$

for all $A, B \in L^C$.

Remark 1. Observe that relativization of C-fuzzy quantifiers is structurally identical with relativization of bivalent quantifiers (Definition 6). It means that the only difference between formulas (6) and (21) on the surface level is in the replacement of M on the left side of (6) by C on the left side of (21).

Example 4. Let $\forall$ be the C-fuzzy quantifier (on M) from Example 1. Its relativization is the C-fuzzy quantifier $\forall^{\mathrm{rel}}$ of type $\langle 1, 1 \rangle$. It can be shown that

$$(\forall^{\mathrm{rel}})_C(A, B) = \forall_A(A \cap B) = \bigwedge_{m \in M} (A(m) \to B(m)) = \mathsf{all}_C(A, B).$$

Hence, we see that the relativization of $\forall$ is indeed the C-fuzzy quantifier all from Example 3.

[12] We can see that the resulting expression for $\mathsf{all}_C(A, B)$ does not depend on the fuzzy set C. It is a natural result considering the fact that the first argument of natural language quantifiers of type $\langle 1, 1 \rangle$ serves as a new universe of discourse.

Example 5. Let $\exists$ be the C-fuzzy quantifier (on M) from Example 2. Its relativization is the C-fuzzy quantifier $\exists^{\mathrm{rel}}$ of type $\langle 1,1\rangle$:

$$(\exists^{\mathrm{rel}})_C(A,B) \;=\; \exists_A(A\cap B) \;=\; \neg\neg\bigvee_{m\in M}(A(m)\wedge B(m)) \;=\; \mathsf{some}_C(A,B).$$

Hence, we see that the relativization of $\exists$ is indeed the fuzzy quantifier some from Example 3, as expected.

Due to lack of space, we cannot elaborate more details and examples on relativization of C-fuzzy quantifiers in this paper. However, for stating the main theorem we need to define generalizations of semantic properties ISOM, EXT and CONS from Definitions 3–5.

Definition 15 (Isomorphism invariance). *A global C-fuzzy quantifier Q of type $\langle 1^n,1\rangle$ (on M) is* isomorphism invariant*, if for all $C\in L^M$ and all permutations $f\in \mathrm{Perm}(M)$ it holds that*

$$Q_C(A_1,\dots,A_n,B)=Q_{f^{\to}(C)}(f^{\to}(A_1),\dots,f^{\to}(A_n),f^{\to}(B))$$

for all $A_1,\dots,A_n,B\in L^C$. The class of all C-fuzzy quantifiers with this property is denoted by C-ISOM.

Definition 16 (Property of extension). *A global C-fuzzy quantifier Q of type $\langle 1^n,1\rangle$ (on M) has a* property of extension*, if for all $C,D\in L^M$ such that $C\subseteq D$ it holds that*

$$Q_C(A_1,\dots,A_n,B)=Q_D(A_1,\dots,A_n,B)$$

for all $A_1,\dots,A_n,B\in L^C$. The class of all C-fuzzy quantifiers with this property is denoted by C-EXT.

Definition 17 (Conservativity). *A global C-fuzzy quantifier Q of type $\langle 1,1\rangle$ (on M) is* conservative*, if for all $C\in L^M$ and all $A,B,B'\in L^C$ it holds that if $A\cap B=A\cap B'$, then*

$$Q_C(A,B)=Q_C(A,B').$$

The class of all C-fuzzy quantifiers with this property is denoted by C-CONS.

Then we can prove the final theorem analogous to Theorem 1.

Theorem 3. *Let Q be a C-fuzzy quantifier of type $\langle 1\rangle$ and P be a C-fuzzy quantifier of type $\langle 1,1\rangle$. Then*

(a) Q^{rel} is C-EXT and C-CONS.
(b) If P is C-EXT and C-CONS, then there exists a C-fuzzy quantifier Q of type $\langle 1\rangle$ such that $Q^{\mathrm{rel}}=P$.
(c) Q^{rel} is C-ISOM if and only if Q is C-ISOM.

Proof. We sketch the proof of the part (b) of the theorem.

(b) Let P be a C-fuzzy quantifier of type $\langle 1, 1\rangle$. Define a C-fuzzy quantifier Q of type $\langle 1\rangle$ as

$$Q_C(B) = P_C(C, B) \tag{22}$$

for all $B \in L^C$. We show that if $P \in$ C-EXT $\cap$ C-CONS, then $Q^{\mathrm{rel}} = P$. Indeed, for all $A, B \in L^C$,

$$(Q^{\mathrm{rel}})_C(A, B) = Q_A(A \cap B) = P_A(A, A \cap B) = P_A(A, B) = P_C(A, B),$$

where we use, in turn, the definition of relativization, the definition of Q (22), the conservativity and the property of extension of P. □

7 Conclusion

In this paper we discussed an important operation of relativization of (generalized) quantifiers. We showed that it is not possible to define relativization of standardly defined fuzzy quantifiers in a satisfactory way. As a solution, we provide a new definition of fuzzy quantifiers (C-fuzzy quantifiers). Relativization of C-fuzzy quantifiers fulfills intuitive requirements. In future research, we will study further properties of C-fuzzy quantifiers, for example so-called *freezing* operation [16], which reduces type $\langle 1, 1\rangle$ quantifiers to type $\langle 1\rangle$. We will also investigate semantic properties of these quantifiers, as well as various ways of defining negations and duals of them. Applications can be expected, e.g., in novel definitions of fuzzy relational compositions and data mining.

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Some Remarks About Crucial and Unsolved Problems on Atanassov's Intuitionistic Fuzzy Sets

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Abstract. In the paper *Crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets* D.-F. Li pointed out that some kind of definitions of operations over Atanassov's intuitionistic fuzzy sets (IFSs) are incorrect. In this paper the Li's reasoning is presented and commented.

Keywords: Intuitionistic fuzzy sets · Operations on IFSs

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1 Introduction

Intuitionistic fuzzy sets (IFSs) were first introduced by Krassimir T. Atanassov in 1983. The latest development of the theory are collected in the monograph [2]. Despite fairly well-defined terms, there may be still some misunderstandings regarding the operations on the intuitionistic fuzzy sets. Such misunderstandings can be found in the paper *Crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets*.

The citations of the original Li's paper will be denoted (except for mathematical symbols and formulas) by the Monotype corsiva font.

2 Main Remarks

In the paper [6] - *Crucial and unsolved problems on Atanassov's intuitionistic fuzzy sets* - the Author, Deng-Feng Li, presents the doubts related to some operations on the intuitionistic fuzzy sets. Based on the Atanassov's paper [1], Li gives the definition of the sum $A + B$ and the product $A \cdot B$ in the form

$$A + B = \{<x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)> \ : x \in X\}, \quad (1)$$

$$A \cdot B = \{ <x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)> \ : x \in X\}, \quad (2)$$

Li presented first the doubts related to operations on particular sets. The Li's examples (a–c) are given below (see: [6], p. 59).

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(a) If $B = \{<x, 1, 0> : x \in X\}$, i.e., B is a fuzzy set, which means that every element x completely belongs to B, then according to Eq. (1), we have

$$A + B = \{ <x, 1, 0> : x \in X\} = B,$$

which also means that every element x completely belongs to $A + B$ despite A is any Atanassov's IFS. Similarly, according to Eq. (2), we have

$$AB = A,$$

which means that whether every element x belonging to AB completely depends on A despite B ensures that all elements x completely belong to B.

Li calls the above set B the fuzzy set. Formally, it is correct. However, the set B can be called a classical set, and, moreover $B = X$.

(b) If $B = \{ <x, 0, 1> : x \in X\}$, i.e., B is a fuzzy set, which means that every element x completely does not belong to B, then according to Eq. (1), we have

$$A + B = A,$$

which means that whether every element x belonging to $A + B$ completely depends on A despite B ensures that all elements x completely do not belong to B. Similarly, according to Eq. (2), we have

$$AB = B,$$

which means that every element x completely does not belong to AB despite A is any Atanassov's IFS.

Li calls the above set B the fuzzy set. Formally, it is correct. However, the set B is in fact the classical empty set.

(c) If $B = \{ <x, 0, 0> : x \in X\}$, which means that every element x cannot be completely determined whether it belongs to B or not, then according to Eq. (1), we have

$$A + B = \{ <x, \mu_A(x), 0> : x \in X\},$$

which means that the membership degree of the element x to $A + B$ is the same as that of x to A whereas the non-membership degree of the element x to $A + B$ is 0 despite A ensures that the non-membership degree of the element x to A is $v_A(x)$. Similarly, according to Eq. (2), we have

$$AB = \{ <x, 0, v_A(x)> : x \in X\},$$

which means that the non-membership degree of the element x to $A + B$ is the same as that of x to A whereas the membership degree of the element x to $A + B$ is 0 despite A ensures that the membership degree of the element x to A is $\mu_A(x)$.

After the next numerical example (see: [6], p. 60), Li argues that the Eqs. (1) and (2) define the operations incompatible/inconsistent with the Zadeh's extension principle.

The example, given by Li, is as follows.

Let A and B be intuitionistic fuzzy sets on the universum $X = \{3, 4, 5, 6, 7\}$, which mean "approximately 5", where

$$A = \{ <3, 0.7, 0.2>, <4, 0.8, 0.1>, <5, 1, 0>, <6, 0.8, 0.1> \} \tag{3}$$

and

$$B = \{ <4, 0.7, 0.2>, <5, 1, 0>, <6, 0.9, 0.05>, <7, 0.85, 0.1> \}, \tag{4}$$

respectively. Using Eqs. (1) and (2), we have

$$A + B = \{ <3, 0.7, 0>, <4, 0.94, 0.02>, <5, 1, 0>, <6, 0.98, 0.005>, <7, 0.85, 0> \}, \tag{5}$$

$$A \cdot B = \{ <3, 0, 0.2>, <4, 0.56, 0.28>, <5, 1, 0>, <6, 0.72, 0.145>, <7, 0, 0.1> \}. \tag{6}$$

In a similar way to the extension principle of the fuzzy sets, we have

$$\begin{aligned} A + B = \{ & <3, 0.7, 0.2>, <7, 0.7, 0.2>, <8, 0.7, 0.2>, <9, 0.8, 0.1> \\ & <10, 1, 0> <11, 0.9, 0.05>, <12, 0.85, 0.1>, <13, 0.8, 0.1> \}, \end{aligned} \tag{7}$$

$$\begin{aligned} A \cdot B = \{ & <12, 0.7, 0.2>, <15, 0.7, 0.2>, <16, 0.7, 0.2>, <18, 0.7, 0.2>, \\ & <20, 0.8, 0.1>, <21, 0.7, 0.2>, <24, 0.8, 0.1>, <25, 1, 0>, \\ & <28, 0.8, 0.1>, <30, 0.9, 0.05>, <35, 0.85, 0.1>, <36, 0.8, 0.1>, \\ & <42, 0.8, 0.1> \}. \end{aligned} \tag{8}$$

which are remarkably different from Eqs. (5) and (6) since Eqs. (7) and (8) compute elements in A and B rather than membership and nonmembership degrees.

It is really easy to see that the results (5), (6) and (7), (8) are different. Li does not consider whether the use of the extension principle - known for classical fuzzy sets - is legitimate for the IFSs. He assumes it is so and, because the extension principle is, in this case, not fulfilled, hence the operations of the sum and the product are incorrect.

The above reasoning has been also presented by Li in the monograph [7] (pp. 33–35).

It is understandable that "approximately 5" added to "approximately 5" must be "approximately 10", and "approximately 5" multiplied by „ approximately 5" must be "approximately 25". Here the argumentation of Li is convincing, and the Atanassov's mistake is evident.

However, in fact, it is Li who makes the mistake!

Namely, he does not correctly understand the signs + and · (the sign · is by Li, typically, omitted). These signs denote not the operations of sum and product in terms of arithmetic of fuzzy numbers (or intuitionistic fuzzy numbers) but the sum and the product in the set-theoretically sense.

Here the mistake of Li is obvious and the rest of his reasoning is invalid.

This means, obviously, that the results in the examples (a), (b), and (c) are not surprising. By the way - similar results we obtain based on typical sum and product using the minimum and maximum operators.

It is possible that the use of the symbols + and · by Atanassov, without direct comments, are not fortunate, but such a notation is very often used in the case of classically fuzzy sets also.

Quite weird is that D.-F. Li, otherwise known as creative researcher of intuitionistic fuzzy sets theory, failed to notice the use of sum and product based on other t- and s-norms, over the most popular norms: minimum and maximum.

In general, the union and intersection of the IFSs are defined based on the IF t-norm and IF s-norm. For IFSs these norms are considered explicitly first by Cornelis and Deschrijver [3], Cornelis, Deschrijver and Kerre [4] and Deschrijver and Kerre [5].

Cornelis, Deschrijver and Kerre, (see: [4], p. 1), defined the intuitionistic fuzzy t-norm (IF triangular norm) on the lattice L as any monotonous, commutative, associative mapping from L^2 to L with the neutral element $<1, 0>$. The intuitionistic fuzzy s-norm (IF triangular co-norm) on the lattice L the Authors call any monotonous, commutative, associative mapping from L^2 to L with the neutral element $<0, 1>$.

The Authors formulated also the theorem given below (the idea of the theorem; see: [3], p. 5, and, [5], p. 3).

3 Theorem

Let T_1 and T_2 are t-norms, and S_1 and S_2 are s-norms.

The mappings $\mathcal{T}, \mathcal{S}$: $L^2 \to L$ on the lattice $L = \{<a, b> \in [0, 1]^2 : a + b \leq 1\}$, given in the form:

$$\mathcal{T}(<a,b>, <c,d>) = <T_1(a,c), S_1(b,d)>,$$

and

$$\mathcal{S}(<a,b>, <c,d>) = <S_2(a,c), T_2(b,d)>,$$

fulfilling the conditions $T_1(a, c) + S_1(b, d) \leq 1$ and $S_2(a, c) + T_2(b, d) \leq 1$, are the IF t-norm and IF s-norm, respectively.

The definition of the t-, s-norms and the lattice are widely known.

It is not difficult to see, in the cited formulas (1) and (2), the use of the probabilistic (product) t-norm

$$T(a,b) = a \cdot b,$$

and the s-norm

$$S(a,b) = a + b - a \cdot b,$$

to define the union and intersection of intuitionistic fuzzy sets.

Li is therefore wrong, when he wrote in the end of his paper *the addition and multiplication operations of Atanassov's IFSs are incorrect*.

4 Conclusion

In this paper we can see that, near 30 years after the first Atanassov's papers, there exist some misunderstandings related, for example, on the basic operations on IFSs. Those misunderstandings can be found not only in the papers of young adepts of science, but also in the papers written by prominent Authors cited by hundreds of scientists. It is worth noticing that despite of several years that passed by from the Li's papers [2012, 2014] being published, the misunderstanding described above were not commented on in the known literature.

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Estimating Fuzzy Life Time with a Fuzzy Reliability Function in the Appliance Sector

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Abstract. Reliability is defined as the length of time a component works without failing. The reliability function shows an estimate for the probable amount of time a component should work without experiencing failure. Fuzzy set theory is rather useful for evaluating ambiguity and vagueness that exists within reliability parameters. Fuzzy reliability can model more appropriately a components lifetime if the inputs on the system are fuzzy numbers. Also fuzzy parameters of function can be estimated as fuzzy numbers in a reliability analysis. In this study, the fuzzy lifetime of a component in a refrigerator with censored data is estimated via a fuzzy reliability function for modeling the uncertainty of the process.

1 Introduction

Although there is a consensus that the concept of reliability is an important feature of a product, there is no universally accepted definition of reliability. Some of the definitions used in the past for reliability are:

- "Reduction of things gone wrong" [1].
- "The capability of a product to meet customer expectations of product performance over time" [2].
- "The probability that a device, product, or system will not fail for a given period of time under specified operating conditions" [3].

Since there is always uncertainty about the future performance of a product, a product's future performance may vary randomly and the mathematical theory of probability may be used to characterize the uncertainty about the future performance of a product. When a product is purchased, the product is expected to work for a certain period of time. Generally, a manufacturer provides a guarantee that the product will not fail, and if it fails, the replacement guarantee is given to customer [4].

If there is a general definition of "reliability" in all these lights,

Reliability is the fulfillment of the desired function or task of a product or system during its lifetime under the conditions of life without any deterioration or malfunction.

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It is expected that this distribution will comply with the Weibull, exponential, log-normal distributions etc. because it is concerned with the time it takes for a product to fail and its reliability.

Weibull probability density function is so flexible that it will provide a good fit for a wide variety of data sets with three parameters in reliability engineering. In addition to being the most useful density function for reliability calculations, the Weibull analysis provides the information required for troubleshooting, classifying failure types, scheduling preventive maintenance and scheduling inspections [5].

It is sometimes difficult to record the data obtained as a result of the application of the Weibull analysis because of unexpected situations. Therefore, deficiencies can be observed in the data to be analyzed. This is called censoring for ignoring certain unknown data that can not be observed for any reason. In one study, it is impossible for various reasons to keep each dataset from the beginning to the end of the study, when the event of interest is the life of a dataset. In this case, the data is called censored [6]. In some cases, life testing of some products takes time and it is difficult to apply real-time monitoring on samples to check if the samples are failing. Under these conditions, periodic inspection is always carried out to obtain failure time data in the life test. When a failure is detected during the inspection, the time of the failure is recorded as an estimate. Therefore, the failure time can be at any time during the next inspection interval and can not be detected immediately when the failure occurs. For this reason, this time is not certain. Another uncertainty in this sense is that the failure time can not be determined precisely because the failure of the samples immediately after the completion of an inspection can not be detected until the next inspection, and because the time between the two inspections is long [7]. For all these reasons, fuzzy set theory studies for reliability analysis have been done.

Since the fault time can not be determined precisely during the periodic examinations, uncertainties arise. Xu, Li and Liu [7] proposed a statistical method for accelerated life testing (ALT) with fuzzy theoretic type 2 censored samples to remove this situation, and assumed that the life span of product followed the Weibull distribution. Katithummarugs et al. [8] proposed an estimation method. The aim of this method is to predict a reliability index of power distribution feeders in the central area of the Provincial Electricity Authority (PEA) of Thailand. Using the Weibull distribution, the time to failure data during 2002–2009 was analyzed over moving 4-year periods to determine the failure rates associated with protective devices. Fuzzy sets were introduced to describe uncertainty of failure rates using a fuzzy arithmetic operation. When the distribution of lifetimes is 2-parameter exponential, Balasooriya [9] provided a failure-censored reliability sampling-plan to save test time. Wu et al. [10] extended the Balasooriya sampling plan to the Weibull distribution and provided a limited failure-censored reliability sampling plan (LFCR) to do life testing when test facilities are scarce. In the paper edited by Cheng, a different approach of assessing the total failure time of censored data was investigated and a fuzzy estimator for the total failure time was discussed. Cheng [11] said that the system test with several identical components could be repeated a few times to evaluate the reliability of the system. The fuzzy estimator used in the study provided more information in total failure time than just a point estimate, or just a single confidence estimate. Jamkhaneh [12] indicated that it is difficult to determine the parameters of the probability distributions in many situations where the data contain uncertainties and imprecision. Therefore, it was assumed that these parameters are fuzzy in the article. At the same time, the lifetimes and repair times of components were assumed to have Weibull distribution with fuzzy parameters. Formulas of a fuzzy reliability function, fuzzy hazard function and their α-cut set were presented. Finally, some numerical examples were presented to illustrate how to calculate the fuzzy reliability characteristics and their α-cut set.

2 Conventional Reliability Theory

Reliability is the fulfillment of the desired function or task of a product or system during its lifetime under the conditions of life without any deterioration or malfunction. There is always uncertainty about the future performance of a product. Therefore, this performance is a random variable and the mathematical theory of probability can be used to characterize this uncertainty. Reliability Engineering is the engineering field that deals with the ability of the product or system to fulfill the desired function in its entirety. It aims to reduce the probability of fault by analyzing faults. It aims to improve the use of the products on this. The failure characteristics of products or systems are shown in Fig. 1. It is called a bath tub pattern, which has three distinct phases.

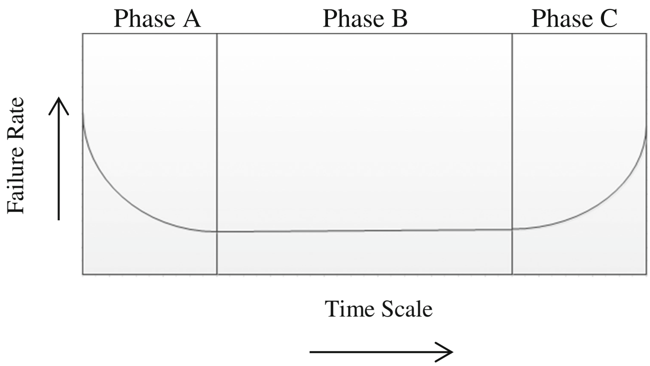

Fig. 1. Failure rate of products or systems [13].

Phase A or the burning in period: The major contributing factor to this failure is poor component quality. When the equipment is given initial trials, there might be many initial failures due to poor design, workmanship, assembly errors, etc. Damaged components and poor joints or connections also contribute to this failure. These are tested and replaced generally at the manufacturer's premises to improve their reliability.

Phase B or the useful life period: Here the failure rate is low, but may occur unexpectedly and at random intervals. They are known as random failures or normal failures. It is during this period, that all our available reliability analysis is based on. The major contributing factor is the stress to which the equipment or products are subjected to and could be due to operating stresses, poor maintenance, operator abuse, and accidents.

Phase C or the wear out period: Beyond the useful period, the wear rate is the major contributing factor because of aging or wear of the components of the system, and could be due to weak design, poor lubrication, wear, fatigue failure, corrosion, and insulation breakdown [13].

The occurrence times of failures occurring at a certain time interval are frequently used as data in reliability analysis. Failure times are continuous random variables that can take any value in a real number interval. For a random variable T representing the data, while the probability density function is f(t), the cumulative distribution function F(t) is calculated as follows.

$$F(t) = P(T \leq t) = \int_0^t f(t)dt \tag{1}$$

In the reliability analysis, the failure of a product until time t is concerned. Thus F(t) is the probability of unit failure here. In this study, two cases "success" or "failure" in the reliability analysis is considered. Because the success and the failure are two mutually independent states of probability, the sum of these probabilities is always equal to 1. The reliability function R(t) is the probability that something will survive past a certain time (t). It is defined as follows.

$$\begin{aligned} &F(t) + R(t) = 1, \text{then} \\ &R(t) = P(T > t) = 1 - F(t) \end{aligned} \tag{2}$$

Hazard rate refers to the rate of death for an item of a given age, and is also known as the failure rate.

$$h(t) = \frac{f(t)}{R(t)} \tag{3}$$

2.1 Weibull Analysis

The Weibull distribution with two parameters have usually been used to model failure times.

The goal is to find a cumulative distribution function that has a wide variety of failure rate shapes, with the constant h(t) = λ as just one possibility. Allowing any polynomial form of the type h(t) = at^b for a failure rate function achieves this objective. In order to derive F(t), it is easier to start with the cumulative hazard function H(t). Because h(t) = $\frac{dH(t)}{dt}$, setting

$$H(t) = (\lambda t)^{\beta} \tag{4}$$

results in

$$h(t) = \beta\lambda(\lambda t)^{\beta-1} \tag{5}$$

This form gives us the exponential constant failure rate when $\beta = 1$ and a polynomial failure rate for other values of β.

Now, we use the basic identity relating F(t) and H(t):

$$F(t) = 1 - e^{-H(t)} = 1 - e^{-(\lambda t)\beta} \tag{6}$$

We obtain the equation for the Weibull cumulative distribution function by making a substitution of $\theta = 1/\lambda$ in the above equation and write

$$F(t) = 1 - e^{-(t/\theta)\beta} \tag{7}$$

The parameter θ is a *scale parameter* that is often called the *characteristic life*. The parameter β is known as the *shape parameter*. θ and β must be greater than zero, and the distribution is a life distribution defined only for positive times $0 \leq t < \infty$.

The Weibull probability density function (PDF) f(t), failure rate h(t) are given by the following:

$$\begin{aligned} f(t) &= \frac{\beta}{t}\left(\frac{t}{\theta}\right)^{\beta} e^{-(t/\theta)^{\beta}} \\ h(t) &= \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} = \frac{\beta}{t}\left(\frac{t}{\theta}\right)^{\beta} \end{aligned} \tag{8}$$

For $\beta > 1$, the failure rate h(t) increases monotonically, with $h(0) = 0$ and $h(\infty) = \infty$. For $\beta < 1$, the failure rate h(t) decreases monotonically, with $h(0) = \infty$ and $h(\infty) = 0$.

The reliability function R(t) is defined by $R(t) = 1 - F(t)$ [14]. Thus,

$$R(t) = e^{-(t/\theta)^{\beta}} \tag{9}$$

2.2 Maximum Likelihood Estimation Method

There are many methods for the parameter estimation of the Weibull distribution. One of the most common of these methods is the Maximum Likelihood Method. The likelihood function for the Weibull distribution is as follows.

$$L = \prod_{i=1}^{n} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^{\beta}} \tag{10}$$

For the unknown θ and β parameters of the Weibull distribution, the estimators of the parameters $\hat{\theta}$ and $\hat{\beta}$ are calculated using the following equations. Firstly, to make easier mathematical operations, the log-likelihood function is obtained by taking the logarithm of the likelihood function to transform the multiplication expression into a sum [15].

$$\ln \mathrm{L} = \mathrm{LL} = \mathrm{n} \ln(\beta) - \mathrm{n}\ \beta \ln(\theta) + (\beta - 1) \sum_{i=1}^{n} \ln(x_i) - \frac{1}{\theta^{\beta}} \sum_{i=1}^{n} (x_i)^{\beta} \quad (11)$$

In order to obtain the θ and β parameters, the first-order partial derivative is obtained according to the θ and β parameters of the log-likelihood function and equalized to zero.

$$\begin{aligned} \frac{\partial LL}{\partial \theta} &= -\mathrm{n} + \frac{1}{\theta^{\beta}} \sum_{i=1}^{n} (x_i)^{\beta} = 0 \text{ then,} \\ \theta &= \left(\frac{1}{n} \sum_{i=1}^{n} (x_i)^{\beta} \right)^{1/\beta} \end{aligned} \quad (12)$$

If the logarithm of θ is taken;

$$\begin{aligned} \ln \theta &= \frac{1}{\beta} \ln \sum_{i=1}^{n} (x_i)^{\beta} - \frac{1}{\beta} \ln(\mathrm{n}) \\ \frac{\partial LL}{\partial \beta} &= \frac{n}{\beta} - \mathrm{n} \ln(\theta) + \sum_{i=1}^{n} \ln(x_i) - \frac{1}{\theta^{\beta}} \sum_{i=1}^{n} (x_i)^{\beta} \ln(x_i) + \frac{1}{\theta^{\beta}} \ln(\theta) \sum_{i=1}^{n} (x_i)^{\beta} = 0 \\ \beta &= \left[\frac{\sum_{i=1}^{n} (x_i)^{\beta} \ln(x_i)}{\sum_{i=1}^{n} (x_i)^{\beta}} - \frac{1}{\mathrm{n}} \ln(x_i) \right]^{-1} \end{aligned} \quad (13)$$

3 Fuzzy Theory

A fuzzy number $\tilde{x}$ is expressed by a characterizing function. This function is called membership function μ(x) which means the degree of membership of element $\tilde{x}$ of the universe X [7].

$$\begin{aligned} &\tilde{x} \in \mathrm{X}, \\ &\mu_{\tilde{x}}(x) \in [0, 1] \end{aligned} \quad (14)$$

An α-cut of $\tilde{x}$, written as $\tilde{x}_{\alpha}$, is described as below:

$$\begin{aligned} &\tilde{x}_{\alpha} = \{x | \mu_{\tilde{x}}(\mathrm{x}) \geq \alpha\} \\ &0 \leq \alpha \leq 1 \end{aligned} \quad (15)$$

A fuzzy number $\tilde{x}$ can be turned into a finite closed interval as below when the value of α is settled.

$$\tilde{x}_{\alpha} \in \left[\tilde{x}_{\alpha}^{L}, \tilde{x}_{\alpha}^{U} \right] \quad (16)$$

4 Fuzzy Reliability

Jamkhaneh suggests a general procedure to construct the reliability characteristics and its α-cut set in his paper, when the parameters are fuzzy. The parameter of the system is represented by a trapezoidal fuzzy number [12]. With the difference of Jamkhaneh's paper, in this study the parameter of the system is considered by a triangular fuzzy number.

4.1 Fuzzy Weibull Distribution

The Cumulative Weibull distribution function with fuzzy scale parameter $\tilde{\theta}$ and α – cut can be defined the following equations [12].

$$\begin{aligned} &F(x,\tilde{\theta}) = \left\{F(x)[\alpha], \mu_{F(x)} | F(x)[\alpha] = F_{\min}(x)[\alpha], F_{\max}(x)[\alpha]], \mu_{F(x)} = \alpha\right\}, \\ &F_{\min}(x)[\alpha] = \inf\left\{F(x,\theta) | \theta \in \tilde{\theta}[\alpha]\right\}, \\ &F_{\max}(x)[\alpha] = \sup\left\{F(x,\theta) | \theta \in \tilde{\theta}[\alpha]\right\}. \end{aligned} \tag{17}$$

When the fuzzy probability of event X ∈ [c, d], the fuzzy probability of X ∈ (c ≤ X ≤ d), c ≥ 0 and its α-cut intervals with crisp shape parameter β can be calculated the following equations:

$$\tilde{P}(c \le X \le d)[\alpha] = \int_c^d \frac{\beta}{x}\left(\frac{x}{\theta}\right)^{\beta} e^{-(x/\theta)^{\beta}} dx | \theta \in \tilde{\theta}[\alpha] = \left[P^L[\alpha], P^U[\alpha]\right] \tag{18}$$

for all α, where

$$\begin{aligned} P^L[\alpha] &= \min\left\{\int_c^d \frac{\beta}{x}\left(\frac{x}{\theta}\right)^{\beta} e^{-(x/\theta)^{\beta}} dx | \theta \in \tilde{\theta}[\alpha]\right\}, \\ P^U[\alpha] &= \max\left\{\int_c^d \frac{\beta}{x}\left(\frac{x}{\theta}\right)^{\beta} e^{-(x/\theta)^{\beta}} dx | \theta \in \tilde{\theta}[\alpha]\right\}, \end{aligned} \tag{19}$$

4.2 Fuzzy Reliability Function

Fuzzy reliability function ($\tilde{R}(t)$) defines the fuzzy probability for the period of time that components do not break down time t. Let the random variable X represents component lifetime, and its fuzzy density function $f\left(x, \tilde{\theta}\right)$ and fuzzy cumulative distribution function $\tilde{F}_x(t) = \tilde{P}(X \le t)$ where parameter, $\tilde{\theta}$ is a fuzzy triangular number. The fuzzy reliability function at time *t* is defined as:

$$\tilde{R}(t) = \tilde{P}(X > t) = 1 - \tilde{F}_x(t) = \left\{1 - F_{max}(x)[\alpha],\ 1 - F_{min}(x)[\alpha],\ \mu_{F(x)} = \alpha\right\},\ t > 0. \quad (20)$$

Because of X has fuzzy Weibull distribution, parameter $\tilde{\theta}$ can be shown with a triangular fuzzy number as $\tilde{\theta} = (a_1, a_2, a_3)$ such that a membership function $\mu_{\tilde{\theta}}(x)$ in the following manner:

$$\mu_{\tilde{\theta}}(x) = \begin{cases} 0, & x \le a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & x \le a_3 \end{cases} \quad (21)$$

The α – cut value of $\tilde{\theta}$ is calculated as follows:

$$\tilde{\theta}[\alpha] = [(a_2 - a_1)\,\alpha + a_1, -(a_3 - a_2)\,\alpha + a_3] \quad (22)$$

The fuzzy reliability function of a component is as follows:

$$\tilde{R}(t)[\alpha] = \{\textstyle\int_t^\infty \frac{\beta}{x}\left(\frac{x}{\theta}\right)^\beta e^{-(x/\theta)^\beta}dx \big| \theta \in \tilde{\theta}[\alpha]\} = \{e^{-(t/\theta)^\beta}\big|\theta \in \tilde{\theta}[\alpha]\Big\}. \quad (23)$$

Then the α – cuts of fuzzy reliability function is as:

$$\tilde{R}(t)[\alpha] = [e\{^{-(t/((a_2-a_1)\alpha+a_1))^\beta}\}, e\{^{-(t/(-(a_3-a_2)\alpha+a_3))^\beta}\}], \quad (24)$$

The triangular membership function of fuzzy reliability function ($\tilde{R}(t_0)$ [α]) is given in the following equation.

$$\mu_{\tilde{R}(t_0)}(x) = \begin{cases} 0, & x < e^{-(t_0/a_1)^\beta} \\ \frac{x-e^{-(t_0/a_1)^\beta}}{e^{-(t_0/a_2)^\beta}-e^{-(t_0/a_1)^\beta}}, & e^{-(t_0/a_1)^\beta} \le x < e^{-(t_0/a_2)^\beta} \\ \frac{e^{-(t_0/a_3)^\beta}-x}{e^{-(t_0/a_3)^\beta}-e^{-(t_0/a_2)^\beta}}, & e^{-(t_0/a_2)^\beta} \le x \le e^{-(t_0/a_3)^\beta} \\ 0, & x > e^{-(t_0/a_3)^\beta} \end{cases} \quad (25)$$

Fuzzy mean time to failure (FMTTF) is presented following equation by using Buckley's definition [16].

$$M\tilde{T}TF[\alpha] = \{\textstyle\int_0^\infty xf(x)dx\Big|\theta \in \tilde{\theta}[\alpha]\} = \{\int_0^\infty R(t)dt|\theta \in \tilde{\theta}[\alpha]\} = [P^L[\alpha], P^U[\alpha]].$$

$$\begin{aligned} P^L[\alpha] &= \min\{\textstyle\int_0^\infty R(t)dt\big|\theta \in \tilde{\theta}[\alpha]\}, \\ P^U[\alpha] &= \max\{\textstyle\int_0^\infty R(t)dt\big|\theta \in \tilde{\theta}[\alpha]\}. \end{aligned} \quad (26)$$

$$\begin{aligned} M\tilde{T}TF\,[\alpha] &= \{\theta\Gamma\left(1+\beta^{-1}\right)\big|\theta \in \tilde{\theta}[\alpha]\} \\ &= [((a_2 - a_1)\alpha + a_1)\Gamma\left(1+\beta^{-1}\right), (-(a_3 - a_2)\alpha + a_3)\Gamma\left(1+\beta^{-1}\right) \end{aligned} \tag{27}$$

The triangular membership function of fuzzy mean time to failure is given as follows.

$$\mu_{M\tilde{T}TF}(x) = \begin{cases} 0, & x < a_1\Gamma(1+\beta^{-1}) \\ \frac{x - a_1\Gamma(1+\beta^{-1})}{(a_2 - a_1)\Gamma(1+\beta^{-1})}, & a_1\Gamma(1+\beta^{-1}) \le x < a_2\Gamma(1+\beta^{-1}) \\ \frac{a_3\Gamma(1+\beta^{-1}) - x}{(a_3 - a_2)\Gamma(1+\beta^{-1})}, & a_2\Gamma(1+\beta^{-1}) \le x < a_3\Gamma(1+\beta^{-1}) \\ 0, & x > a_3\Gamma(1+\beta^{-1}) \end{cases} \tag{28}$$

5 Application

In this study, a product reliability analysis study was carried out in a white goods operation in Turkey. It is sometimes difficult to record the data obtained as a result of the application of the Weibull analysis because of unexpected situations. Therefore, deficiencies can be observed in the data to be analyzed. In some cases, life testing of some products takes time and it is difficult to apply real-time monitoring on samples to check if samples are failing. Therefore, the failure time cannot be detected immediately when the failure occurs. For all these reasons, fuzzy set theory is used in this study.

Parameter values were obtained with the aid of the Minitab v.16 package program and the fuzzy scale parameter values is found like $\tilde{\theta} = (3580, 3661, 3742)$, the shape parameter value is found 5.87. Here the parameter values are in days.

The α – cut of fuzzy scale parameter and fuzzy reliability function are given as follows. The α – cut value is taken 0.35 in study.

$$\tilde{\theta}[\alpha] = [81\alpha + 3580, -81\alpha + 3742] = [3608.35, 3713.65] \tag{29}$$

$$\widetilde{\mathrm{R}}(\mathrm{t})[0.35] = [e\{^{-(t/(3608.35)^{5.87}}\}, e\{^{-(t/(3713.65)^{5.87}}\}] \tag{30}$$

According to these parameter values, the membership function of scale parameter, $\mu_{\tilde{\theta}}(\mathrm{x})$ is given in the Eq. 31. Also, the α – cut of fuzzy survival plot is shown in Fig. 2.

$$\mu_{\tilde{\theta}}(x) = \begin{cases} 0, & x \le 3580 \\ \frac{x - 3580}{81}, & 3580 \le x \le 3661 \\ \frac{3742 - x}{81}, & 3661 \le x \le 3742 \\ 0, & x \ge 3742 \end{cases} \tag{31}$$

When the shape parameter values takes 2, 4, 6 values, the changes in the survival plot are shown in Fig. 3. According to Fig. 3, the increasing value of shape parameter results the decreasing the life time.

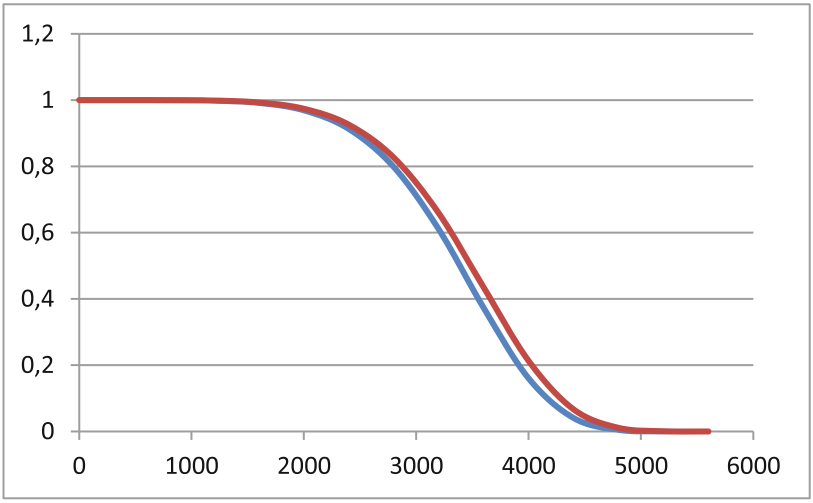

Fig. 2. The α – cut of fuzzy survival plot

Fuzzy mean time to failure is given as follows for β = 5.87

$$M\tilde{T}TF[\alpha] = [36.08.35\,\Gamma\left(1 + 5.87^{-1}\right), 3713.65\,\Gamma\left(1 + 5.87^{-1}\right)] \tag{32}$$

Here, $\Gamma(1 + 5.87^{-1}). \approx 1$, so;

$$M\tilde{T}TF[\alpha] = [3608.35,\ 3713.65]$$

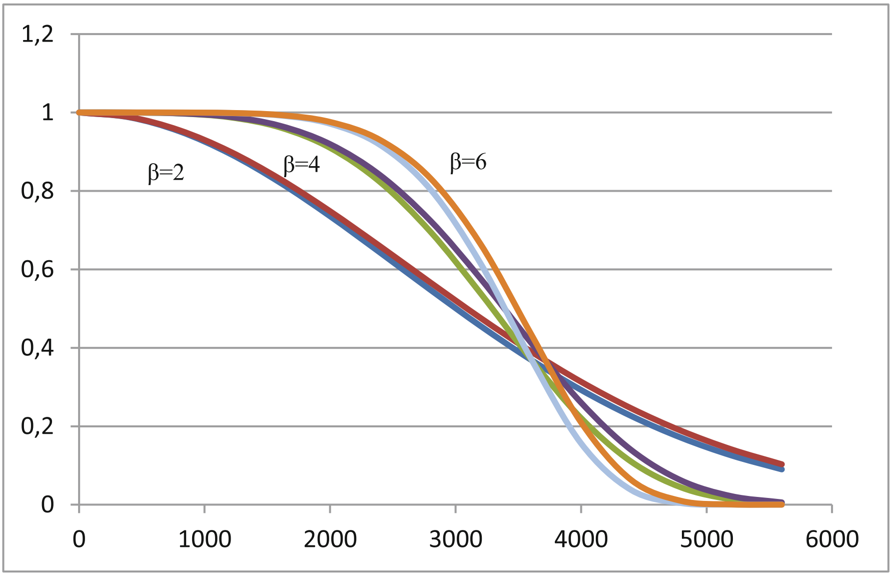

Fig. 3. The α – cut of fuzzy survival plot while β = 2, 4, 6

The membership function $\mu_{M\tilde{T}TF}$(x) of fuzzy mean time to faire is given as follows. The graphical representation is given in Fig. 4.

$$\mu_{M\tilde{T}TF}(x) = \begin{cases} 0, & x < 3580 \\ \frac{x-3580}{81}, & 3580 \leq x < 3661 \\ \frac{3742-x}{81}, & 3661 \leq x \leq 3742 \\ 0, & x > 3742 \end{cases} \tag{33}$$

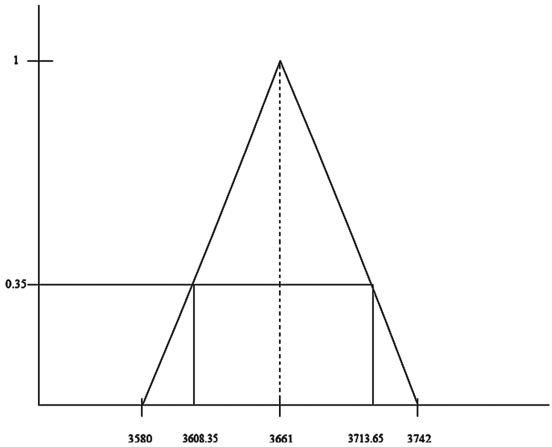

Fig. 4. The membership function of $M\tilde{T}TF$

6 Conclusion

Fuzzy reliability is based on the fuzzy sets concept and fuzzy probability theory. In this paper, the failure times of a product are determined using Weibull distribution with a fuzzy parameter. In real life, the failure can be observed at certain interval times and cannot be determined at an exact time. Therefore the time to failure includes uncertainty. For this reason, fuzzy set theory is used in this study. Also the Weibull scale parameter is considered as a fuzzy triangular number. The fuzzy life time of a component in a refrigerator with censored data is estimated via the fuzzy reliability function. According to the application results, failure times of all samples are deduced. It is examined how the result changes according to different values of the shape parameter.

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Monitoring Fraction Nonconforming in Process with Interval Type-2 Fuzzy Control Chart

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Abstract. Fuzzy set theory is particularly appropriate approach when data include imprecise. Type-2 fuzzy set theory captures ambiguity that associates the uncertainty of membership functions by incorporating footprints and models high level uncertainty. If the quality characteristic is a binary classification into conforming/non-conforming of product, this decision depends on human subjectivity that have ambiguity or vague. In this situation, monitoring the process with statistical control charts based on interval type-2 fuzzy sets, a special case of type-2 fuzzy sets, is more suitable due to the human imprecise judgments on quality characteristics. In this paper, interval type-2 fuzzy p-control chart is developed into the literature for the first time. Due to the interval type-2 fuzzy sets modelled more uncertainty for defining membership functions, in this paper interval type-2 fuzzy fraction nonconforming numbers used for handling more uncertainty in process. Real word application is implemented with developed fuzzy control chart.

Keywords: Fuzzy set theory · Interval type-2 fuzzy sets · Fraction nonconforming · Fuzzy control charts

1 Introduction

Fuzzy sets theory is developed for modelling uncertainty on data. Type-1 fuzzy sets introduce vagueness by using membership function. In this situation the membership function degree is crisp and located [0, 1]. But type-2 fuzzy set consider membership function degree as fuzzy. So, it can be defined the membership function of membership function. If the upper membership function is equal to 1, it is called interval type-2 fuzzy number. Interval type-2 fuzzy sets can be modelled more effectively uncertainty than type-1 fuzzy sets.

Fuzzy set theory was introduced firstly by Zadeh [1] in literature. Zadeh (1965) gave the information about the reasons why fuzzy set theory is needed. Control charts

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proposed by Montgomery [2] are used for monitoring and controlling the process. Fuzzy control charts are more suitable to monitor the process if data are fuzzy. Fuzzy variable and attribute control charts have been well documented in literature since 1990. But the latest articles are handled in this paper. Erginel [3] presented the fuzzy individual and moving range control charts with α-cuts. Şentürk and Erginel [4] developed α-cut fuzzy. $\tilde{\bar{X}} - R$ and $\tilde{\bar{X}} - S$ control charts together with α-level fuzzy midrange transformation techniques. Şentürk [5] presented fuzzy regression control charts based on an α-cut approximation. Kaya and Kahraman [6] are derived firstly fuzzy rule method for evaluating the fuzzy variable control charts in their paper. Gülbay et al. [7] proposed α-cut control chart for attribute with triangular fuzzy numbers. Gülbay and Kahraman [8] presented fuzzy c-control chart using a without defuzzification. Şentürk et al. [9] showed a theoretical structure of fuzzy $\tilde{u}$ control charts. Erginel [10] developed a fuzzy p control chart based on both constant sample size and variable sample size and a fuzzy np control chart using decision rules.

2 Interval Type-2 Fuzzy Sets

The arithmetic operations on interval type-2 fuzzy sets are defined by Mendel et al. [11] and given in following definitions. This calculations are made by using trapezoidal fuzzy number and their upper and lower membership functions.

Definition 1. A type-2 fuzzy set $\tilde{\tilde{A}}$ in the universe of discourse X can be represented by a type-2 membership function $\mu_{\tilde{\tilde{A}}}$ shown as follow [11, 12];

$$\tilde{\tilde{A}} = \left\{ (x, u), \mu_{\tilde{\tilde{A}}}(x, u) | \forall x \in X, \forall u \in J_x \subseteq [0, 1], 0 \leq \mu_{\tilde{\tilde{A}}}(x, u) \leq 1 \right\} \quad (1)$$

where J_x denotes an interval in [0, 1]. The type-2 fuzzy set $\tilde{\tilde{A}}$ also can be represented as follows:

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{\tilde{A}}}(x, u)/(x, u) \quad (2)$$

where $J_x \subseteq [0, 1]$ and $\iint$ denotes the union over all admissible x and u. If all $\mu_{\tilde{\tilde{A}}}(x, u) = 1$, then $\tilde{\tilde{A}}$ is called an interval type-2 fuzzy set. An interval type-2 fuzzy set can be regarded as a special case of type-2 fuzzy set, shown as follows,

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} 1/(x, u) \quad (3)$$

where $J_x \subseteq [0, 1]$.

Definition 2. The upper membership function and the lower membership function of an interval type-2 fuzzy set are type-1 membership functions, respectively [11, 12];

A trapezoidal interval type-2 fuzzy set

$$\tilde{\tilde{A}}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \begin{pmatrix} \left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right), \\ \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^L\right), H_2\left(\tilde{A}_i^L\right)\right) \end{pmatrix} \tag{4}$$

where $\tilde{A}_i^U$ and $\tilde{A}_i^L$ are type-1 fuzzy numbers; $a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U, a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L$ are the references points of the interval type-2 fuzzy $\tilde{\tilde{A}}_i$. While subscript j takes values 1 and 2 in the Eq. (4), the subscript i can take values from 1 to n. $H_j\left(\tilde{A}_i^U\right)$ present the membership value of the element $a_{i(j+1)}^U$ in the upper trapezoidal membership function $\tilde{A}_i^U$, $H_j\left(\tilde{A}_i^L\right)$ indicates the membership value of the element $a_{i(j+1)}^L$ in the lower trapezoidal membership function $\tilde{A}_i^L$ above equation and the membership functions take values in the interval [0, 1].

Definition 3. The addition operation between two trapezoidal interval type-2 fuzzy sets are defined as follows:

$$\begin{aligned} \tilde{\tilde{A}}_1 \oplus \tilde{\tilde{A}}_2 &= \left(\tilde{A}_1^U, \tilde{A}_1^L\right) \oplus \left(\tilde{A}_2^U, \tilde{A}_2^L\right) \\ &= \begin{pmatrix} a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; \\ \min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right)\min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right), \\ a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; \\ \min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right) \end{pmatrix} \end{aligned} \tag{5}$$

Definition 4. The subtraction operation between the trapezoidal interval type-2 fuzzy sets are defined as follow:

$$\begin{aligned} \tilde{\tilde{A}}_1 \ominus \tilde{\tilde{A}}_2 &= \left(\tilde{A}_1^U, \tilde{A}_1^L\right) \ominus \left(\tilde{A}_2^U, \tilde{A}_2^L\right) \\ &= \begin{pmatrix} a_{11}^U - a_{24}^U, a_{12}^U - a_{23}^U, a_{13}^U - a_{22}^U, a_{14}^U - a_{21}^U; \\ \min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right), \min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right), \\ a_{11}^L - a_{24}^L, a_{12}^L - a_{23}^L, a_{13}^L - a_{22}^L, a_{14}^L - a_{21}^L; \\ \min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right) \end{pmatrix} \end{aligned} \tag{6}$$

3 Interval Type-2 Fuzzy p-Control Chart

The theoretical structure interval type-2 fuzzy p control chart for trapezoidal fuzzy numbers are developed and given in Table 1 and following equations.

Table 1. Interval type-2 fuzzy fraction nonconforming p

p
$\tilde{\tilde{p}}_{1.} = [(a_{11}^U \quad a_{12}^U \quad a_{13}^U \quad a_{14}^U \; ; \; H_1(\tilde{A}_1^U) \quad H_2(\tilde{A}_1^U))$,
$(a_{11}^L \quad a_{12}^L \quad a_{13}^L \quad a_{14}^L \; ; \; H_1(\tilde{A}_1^L) \quad H_2(\tilde{A}_1^L))]$
$\tilde{\tilde{p}}_{2.} = [(a_{21}^U \quad a_{22}^U \quad a_{23}^U \quad a_{24}^U \; ; \; H_1(\tilde{A}_2^U) \quad H_2(\tilde{A}_2^U))$,
$(a_{21}^L \quad a_{22}^L \quad a_{23}^L \quad a_{24}^L \; ; \; H_1(\tilde{A}_2^L) \quad H_2(\tilde{A}_2^L))]$
$\vdots$
$\tilde{\tilde{p}}_{n.} = [(a_{m1}^U \quad a_{m2}^U \quad a_{m3}^U \quad a_{m4}^U \; ; \; H_1(\tilde{A}_n^U) \quad H_2(\tilde{A}_n^U))$,
$(a_{m1}^L \quad a_{m2}^L \quad a_{m3}^L \quad a_{m4}^L \; ; \; H_1(\tilde{A}_n^L) \quad H_2(\tilde{A}_n^L))]$

$$\bar{p}_{a_{.1}^U} = \frac{\sum_{i=1}^m a_{i1}^U}{m},\ \bar{p}_{a_{.2}^U} = \frac{\sum_{i=1}^m a_{i2}^U}{m},\ \bar{p}_{a_{.3}^U} = \frac{\sum_{i=1}^m a_{i3}^U}{m},\ \bar{p}_{a_{.4}^U} = \frac{\sum_{i=1}^m a_{i4}^U}{m} \tag{7}$$

$$\bar{p}_{a_{.1}^L} = \frac{\sum_{i=1}^m a_{i1}^L}{m},\ \bar{p}_{a_{.2}^L} = \frac{\sum_{i=1}^m a_{i2}^L}{m},\ \bar{p}_{a_{.3}^L} = \frac{\sum_{i=1}^m a_{i3}^L}{m}, \bar{p}_{a_{.4}^L} = \frac{\sum_{i=1}^m a_{i4}^L}{m} \tag{8}$$

The fuzzy upper control limit can be calculated by following equations;

$$UCL_p = \begin{bmatrix} \bar{p}_{a_{.1}^U} + 3\sqrt{\frac{\bar{p}_{a_{.1}^U}\left(1-\bar{p}_{a_{.1}^U}\right)}{n}}, \bar{p}_{a_{.2}^U} + 3\sqrt{\frac{\bar{p}_{a_{.2}^U}\left(1-\bar{p}_{a_{.2}^U}\right)}{n}}, \bar{p}_{a_{.3}^U} + 3\sqrt{\frac{\bar{p}_{a_{.3}^U}\left(1-\bar{p}_{a_{.3}^U}\right)}{n}}, \bar{p}_{a_{.4}^U} + 3\sqrt{\frac{\bar{p}_{a_{.4}^U}\left(1-\bar{p}_{a_{.4}^U}\right)}{n}}; min\left(H_1\left(\widetilde{A}_i^U\right), H_2\left(\widetilde{A}_i^U\right)\right) \\ \bar{p}_{a_{.1}^L} + 3\sqrt{\frac{\bar{p}_{a_{.1}^L}\left(1-\bar{p}_{a_{.1}^L}\right)}{n}}, \bar{p}_{a_{.2}^L} + 3\sqrt{\frac{\bar{p}_{a_{.2}^L}\left(1-\bar{p}_{a_{.2}^L}\right)}{n}}, \bar{p}_{a_{.3}^L} + 3\sqrt{\frac{\bar{p}_{a_{.3}^L}\left(1-\bar{p}_{a_{.3}^L}\right)}{n}}, \bar{p}_{a_{.4}^L} + 3\sqrt{\frac{\bar{p}_{a_{.4}^L}\left(1-\bar{p}_{a_{.4}^L}\right)}{n}}; min\left(H_1\left(\widetilde{A}_i^L\right), H_2\left(\widetilde{A}_i^L\right)\right) \end{bmatrix} \tag{9}$$

The fuzzy centre line and fuzzy lower control limit can be calculated by following equations;

$$CL_p = \begin{bmatrix} \bar{p}_{a_{.1}^U}, \bar{p}_{a_{.2}^U}, \bar{p}_{a_{.3}^U}, \bar{p}_{a_{.4}^U}; \min\left(H_1\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_1^U\right)\right), \\ \bar{p}_{a_{.1}^L}, \bar{p}_{a_{.2}^L}, \bar{p}_{a_{.3}^L}, \bar{p}_{a_{.4}^L}; min\left(H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right) \end{bmatrix} \tag{10}$$

$$LCL_p = \begin{bmatrix} \bar{p}_{a_{.1}^U} - 3\sqrt{\frac{\bar{p}_{a_{.4}^U}\left(1-\bar{p}_{a_{.4}^U}\right)}{n}}, \bar{p}_{a_{.2}^U} - 3\sqrt{\frac{\bar{p}_{a_{.3}^U}\left(1-\bar{p}_{a_{.3}^U}\right)}{n}}, \bar{p}_{a_{.3}^U} - 3\sqrt{\frac{\bar{p}_{a_{.2}^U}\left(1-\bar{p}_{a_{.2}^U}\right)}{n}}, \bar{p}_{a_{.4}^U} - 3\sqrt{\frac{\bar{p}_{a_{.1}^U}\left(1-\bar{p}_{a_{.1}^U}\right)}{n}}; min\left(H_1\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_1^U\right)\right) \\ \bar{p}_{a_{.1}^L} - 3\sqrt{\frac{\bar{p}_{a_{.4}^L}\left(1-\bar{p}_{a_{.4}^L}\right)}{n}}, \bar{p}_{a_{.2}^L} - 3\sqrt{\frac{\bar{p}_{a_{.3}^L}\left(1-\bar{p}_{a_{.3}^L}\right)}{n}}, \bar{p}_{a_{.3}^L} - 3\sqrt{\frac{\bar{p}_{a_{.2}^L}\left(1-\bar{p}_{a_{.2}^L}\right)}{n}}, \bar{p}_{a_{.4}^L} - 3\sqrt{\frac{\bar{p}_{a_{.1}^L}\left(1-\bar{p}_{a_{.1}^L}\right)}{n}}; min\left(H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right) \end{bmatrix} \tag{11}$$

3.1 Defuzzification Method for Interval Type-2 Fuzzy p-Control Chart

In the type-2 fuzzy sets studies, several defuzzification techniques are proposed for reduction process. Kahraman et al. [12]. are modified BNP method for trapezoidal type-2 fuzzy sets, given as follows:

$$DIT2^U_{Trap(i)} = \frac{\left(a^U_{i4} - a^U_{i1}\right) + \left(H_2\left(\tilde{A}^U_1\right)a^U_{i2} - a^U_{i1}\right) + \left(H_1\left(\tilde{A}^U_1\right)a^U_{i3} - a^U_{i1}\right)}{4} + a^U_{i1} \quad (12)$$

$$DIT2^L_{Trap(i)} = \frac{\left(a^L_{i4} - a^L_{i1}\right) + \left(H_2\left(\widetilde{A}^L_1\right)a^L_{i2} - a^L_{i1}\right) + \left(H_1\left(\widetilde{A}^L_1\right)a^L_{i3} - a^L_{i1}\right)}{4} + a^L_{i1} \quad (13)$$

$$DIT2_{Trap(i)} = \frac{DIT2^U_{Trap(i)} + DIT2^L_{Trap(i)}}{2} \quad i = 1, 2, \ldots, n \quad (14)$$

where $H_1\left(\tilde{A}^U_1\right)$ and $H_2\left(\tilde{A}^U_1\right)$ are the maximum membership degree values of upper membership function in type-2 fuzzy number; a^U_{i4} is the largest possible value of the upper membership function; a^U_{i1} is the least possible value of the upper membership function; a^U_{i2} and a^U_{i3} are the second and third parameters of the upper membership function. The same definitions valid for the lower membership function.

Modified BNP method for trapezoidal type-2 fuzzy sets are adopted to the interval type-2 fuzzy control charts for defuzzification as follows:

$$\mathrm{P\_DIT2}^{\mathrm{U}}_{\mathrm{Trap(i)}} = \frac{\left(\bar{p}_{a^U_{.4}} - \bar{p}_{a^U_{.1}}\right) + \left(H_2\left(\tilde{A}^U_1\right)\bar{p}_{a^U_{.2}} - \bar{p}_{a^U_{.1}}\right) + \left(H_1\left(\tilde{A}^U_1\right)\bar{p}_{a^U_{.3}} - \bar{p}_{a^U_{.1}}\right)}{4} + \bar{p}_{a^U_{.1}} \quad (15)$$

$$\mathrm{P\_DIT2}^{\mathrm{L}}_{\mathrm{Trap(i)}} = \frac{\left(\bar{p}_{a^L_{.4}} - \bar{p}_{a^L_{.1}}\right) + \left(\mathrm{H}_2\left(\tilde{A}^{\mathrm{L}}_1\right)\bar{p}_{a^L_{.2}} - \bar{p}_{a^L_{.1}}\right) + \left(\mathrm{H}_1\left(\tilde{A}^{\mathrm{L}}_1\right)\bar{p}_{a^L_{.3}} - \bar{p}_{a^L_{.1}}\right)}{4} + \bar{p}_{a^L_{.1}} \quad (16)$$

$$\mathrm{P\_DIT2}_{\mathrm{Trap(i)}} = \frac{\mathrm{P\_DIT2}^{\mathrm{U}}_{\mathrm{Trap(i)}} + \mathrm{P\_DIT2}^{\mathrm{L}}_{\mathrm{Trap(i)}}}{2}; i = 1, 2, 3 \quad (17)$$

where $P\_DIT2_{Trap(i)}$ represents the deffuzification value of interval type-2 fuzzy fraction nonconforming.

4 Application

The interval type-2 fuzzy p-control chart application was made in company that produces in the ceramic sector in Eskisehir Industry Region in Turkey. The case of errors in ceramic tiles produced in this company is analysed using numbers of crack in different samples with the same sample size. The process is observed at 15 different times and the numbers of cracked samples are determined. Due to the operators made visual checking, there are ambiguity on data by operators' judgments. The number of nonconforming include vagueness, so these data expressed as interval type-2 fuzzy numbers.

Table 2. The interval type-2 trapezoidal fuzzy number of fractions of cracked samples

$\tilde{\tilde{p}}_i$: *The* interval type-2 trapezoidal fuzzy number of fractions of cracked samples
$\tilde{\tilde{p}}_1 = [(0.089,0.111,0.122,0.133;1,1)(0.078,0.1,0.111,0.122;0.9,0.5)]$
$\tilde{\tilde{p}}_2 = [(0.078,0.089,0.1,0.111;1,1)\ (0.056,0.067,0.078,0.089;0.6,0.5)]$
$\tilde{\tilde{p}}_3 = [(0.089,0.1,0.1,0.122;1,1)\ (0.078,0.089,0.1,0.111;0.7,0.5)]$
$\tilde{\tilde{p}}_4 = [(0.033,0.056,0.067,0.078;1,1)\ (0.022,0.033,0.056,0.067;0.8,0.6)]$
$\tilde{\tilde{p}}_5 = [(0.111,0.133,0.144,0.156;1,1)\ (0.1,0.111,0.122,0.133;0.6,0.5)]$
$\tilde{\tilde{p}}_6 = [(0.033,0.044,0.056,0.078;1,1)\ (0.022,0.033,0.044,0.067;0.7,0.6)]$
$\tilde{\tilde{p}}_7 = [(0.1,0.111,0.111,0.122;1,1)\ (0.089,0.1,0.1,0.111;0.9,0.6)]$
$\tilde{\tilde{p}}_8 = [(0.067,0.089,0.1,0.111;1,1)\ (0.056,0.067,0.078,0.089;0.7,0.6)]$
$\tilde{\tilde{p}}_9 = [(0.156,0.178,0.189,0.2;1,1)\ (0.144,0.156,0.167,0.189;0.9,0.5)]$
$\tilde{\tilde{p}}_{10} = [(0.078,0.089,0.089,0.111;1,1)\ (0.067,0.078,0.078,0.1;0.8,0.5)]$
$\tilde{\tilde{p}}_{11} = [(0.122,0.133,0.144,0.156;1,1)\ (0.1,0.122,0.133,0.144;0.9,0.7)]$
$\tilde{\tilde{p}}_{12} = [(0.033,0.056,0.067,0.078;1,1)(0.022,0.044,0.056,0.067;0.8,0.6)]$
$\tilde{\tilde{p}}_{13} = [(0.133,0.144,0.156,0.167;1,1)(0.122,0.133,0.144,0.156;0.7,0.5)]$
$\tilde{\tilde{p}}_{14} = [(0.044,0.056,0.056,0.067;1,1)\ (0.033,0.044,0.044,0.056;0.9,0.7)]$
$\tilde{\tilde{p}}_{15} = [(0.022,0.033,0.033,0.056;1,1)\ (0.011,0.022,0.033,0.044;0.9,0.8)]$

In addition, the trapezoidal interval type-2 fuzzy fractions of cracked units for each sample are given in Table 2. The interval type-2 fuzzy arithmetic means are given as follows:

$$\bar{p}_{a^U_{.1}} = 0.079;\ \bar{p}_{a^U_{.2}} = 0.095;\ \bar{p}_{a^U_{.3}} = .0.102;\ \bar{p}_{a^U_{.4}} = 0.116$$
$$\bar{p}_{a^L_{.1}} = 0.067;\ \bar{p}_{a^L_{.2}} = .0.08;\ \bar{p}_{a^L_{.3}} = .0.089;\ \bar{p}_{a^L_{.4}} = 0.102$$

The fuzzy upper and lower limits and fuzzy centre line of interval type-2 fuzzy p-control chart are calculated according to the Eqs. (9–11) where fuzzy arithmetic operations given in Eqs. (5–7) are used.

$$U\tilde{\tilde{C}}L_p = \begin{bmatrix} (0.165, 0.187, 0.198, 0.218; 1, 1), \\ (0.146, 0.166, 0.180, 0.199; 0.6, 0.5) \end{bmatrix}$$

$$CL_p = \begin{bmatrix} (0.079, 0.095, 0.102, 0.116; 1, 1); \\ (0.067, 0.080, 0.090, 0.103; 0.6, 0.5) \end{bmatrix}$$

$$LCL_p = \begin{bmatrix} (-0.022, -0.001, 0.010, 0.031; 1, 1), \\ (-0.029, -0.010, 0.004, 0.024; 0.6, 0.5) \end{bmatrix}$$

After calculating the limits of interval type-2 fuzzy p-control chart, p-control limits are defuzzified by using proposed modified BNP method for fuzzy control charts. The operations of defuzzification for interval type-2 fuzzy control limits and centre line are performed according to the Eqs. (15–17). The results of these operations are given in Table 3.

Table 3. Defuzzified values of fuzzy control limits and centre line

$UCL\_P\_DIT2^U_{Trap(i)}$	$\mathrm{CL}\_P\_DIT2^L_{Trap(i)}$	$UCL\_P\_DIT2_{Trap(i)}$
0.192	0.167	0.179
$CL\_P\_DIT2^U_{Trap(i)}$	$\mathrm{CL}\_P\_DIT2^L_{Trap(i)}$	$CL\_P\_DIT2_{Trap(i)}$
0.098	0.081	0.090
$LCL\_P\_DIT2^U_{Trap(i)}$	$\mathrm{CL}\_P\_DIT2^L_{Trap(i)}$	$LCL\_P\_DIT2_{Trap(i)}$
0.004	–0.009	–0,002 < 0

After the control limits of p-control charts are defuzzified, fraction of cracked samples is defuzzified by using modified BNP method. The obtained results are given in Table 4.

Table 4. Defuzzified values of fraction nonconforming for each sample

$P\_DIT2^U_{Trap(i)}$	$P\_DIT2^L_{Trap(i)}$	$P\_DIT2_{Trap(i)}$
0.114	0.099	0.107
0.094	0.069	0.082
0.103	0.091	0.097
0.058	0.042	0.050
0.136	0.113	0.125
0.053	0.039	0.046
0.111	0.099	0.105
0.092	0.069	0.081
0.181	0.162	0.171
0.092	0.079	0.085
0.139	0.123	0.131
0.058	0.043	0.051
0.150	0.136	0.143
0.056	0.043	0.049
0.036	0.027	0.031

Defuzzified p-control chart limit and defuzzified observations are analysed in Fig. 1. Regard to Fig. 1, the production process of ceramic tile is under control for cracking number.

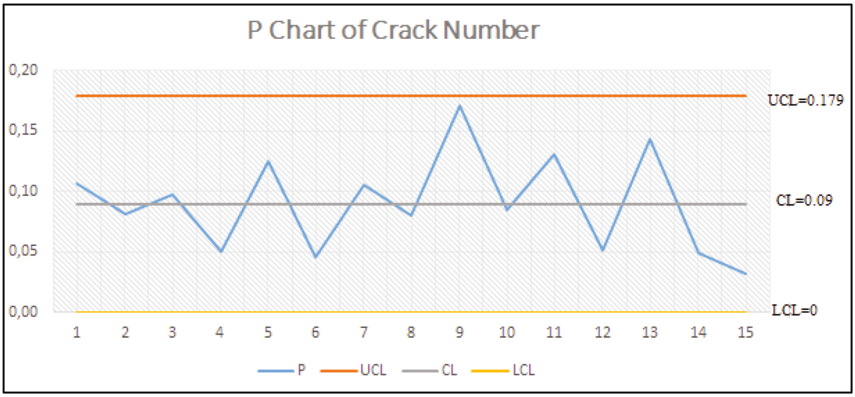

Fig. 1. Interval type-2 fuzzy p control chart for crack numbers

5 Conclusion

Since the interval type-2 fuzzy numbers consider the membership function of membership functions, they can analyse ambiguity more sensitive than type-1 fuzzy numbers for reflecting human decision. Also, decision on cracking error of ceramic tile depends on human experience and subjective. Therefore, using the interval type-2 fuzzy numbers for monitoring and evaluation production process with a fuzzy control chart is more suitable than type-1 fuzzy numbers. In this study, interval type-2 fuzzy p-control chart is developed into the literature for the first time and application is presented. The main contribution of this paper is to give a structure of interval type-2 fuzzy p-control chart for modelling more vagueness that exists inherently collecting nonconforming data from process.

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