On Invariant Measures on Intuitionistic Fuzzy Sets

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Abstract. Very well known and important is the theory of the existence and uniqueness of measures invariant under a shift on a group (so-called Haar measure) in some groups. It was studied in many spaces and transformations. Such measure m is defined on a family $\mathcal{F}$ of sets and such that $m(T^{-1}(A)) = m(A)$ for any $A \in \mathcal{F}$. In the paper instead of sets intuitionistic fuzzy sets (IF-sets) are studied. As a special case the theory of invariant measures on fuzzy sets can be obtained.

Keywords: Compact group · Intuitionistic fuzzy sets · Haar measure

1 Introduction

In the classical measure theory it is known theory about Haar measure [4] stating that in every compact Abelian group there exists a probability measure invariant under shifts.

Let $(G, +)$ be a compact Abelian topological group, $\mathcal{C}$ be the family of all compact subsets of $G, \sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$. Then there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0, 1]$ such that

$$P(A + a) = P(A)$$

for any $A \in \sigma(\mathcal{C})$ and any $a \in G$ (see e.g. [4]). The measure P is usually called the Haar measure or invariant probability measure. Recall that in [8] a version of the existence of invariant measure has been proved for semigroups, and in [9] for IP-loops.

It is natural to consider fuzzy sets instead of sets [10]. In the paper we shall study the theory of invariant measures on families of intuitionistic fuzzy sets [1]. Recall that in the paper [5] there was considered a special case, the group $(R, +)$, and the shitf $T_a : [0, 1) \to [0, 1)$ given by the prescription $T_a(x) = x + a(mod 1)$.

J. Kacprzyk et al. (eds.), *Advances in Fuzzy Logic and Technology 2017*,
Advances in Intelligent Systems and Computing 642,
DOI 10.1007/978-3-319-66824-6_46

Then the theorem about the existence of an invariant IF-states on real numbers was proved. In this paper we will make an extension of this theory to compact Abelian topological group and we will use more general invariant transformation.

In the paper we shall prove the existence of an invariant measures on the family of intuitionistic fuzzy sets [1].

An intutionistic fuzzy set (IFS) is a pair $A = (\mu_A, \nu_A)$ of functions $\mu_A, \nu_A : G \to [0, 1]$ such that

$$\mu_A + \nu_A \leq 1.$$

If $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$, then we write

$$A \leq B$$

if and only if

$$\mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Here $(0_G, 1_G) \leq (\mu_A, \nu_A) \leq (1_G, 0_G)$ for all $A = (\mu_A, \nu_A)$. We shall write

$$A_n = (\mu_{A_n}, \nu_{A_n}) \nearrow (\mu_A, \nu_A) = A,$$

if and only if

$$\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A.$$

Denote by $\triangle$ the set

$$\triangle = \{(a, b) \in R^2; 0 \leq a, b \leq 1, a + b \leq 1\}.$$

Then an IF-set is a mapping $A : G \to \triangle$. If we put $\nu_A = 1 - \mu_A$, then we obtain a fuzzy set $A : G \to [0, 1]$. If $A : G \to \{0, 1\}$, then we obtain a crisp subset $A_0 \subset G$, where $\omega \in A_0$ if and only if $A(\omega) = 1$, hence A can be identified with the indicator χ_{A_0}.

In the paper we shall work with the family $\mathcal{F}$ of all $A = (\mu_A, \nu_A) : G \to \triangle$ with μ_A, ν_A continuous. The Lukasiewicz binary operations are defined on $\mathcal{F}$ by the following way

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1),$$

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0).$$

By a state on $\mathcal{F}$ we consider a mapping $m : \mathcal{F} \to [0, 1]$ satisfying the following conditions:

1. $m((0_G, 1_G)) = 0, m((1_G, 0_G)) = 1;$
2. $A \odot B = (0_G, 1_G) \Longrightarrow m(A \oplus B) = m(A) + m(B);$
3. $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$

Theorem 1. *To any state $m : \mathcal{F} \to [0,1]$ there exists a probability measure P defined on the σ-algebra $\sigma(\mathcal{C})$ and there exists $\alpha \in [0,1]$ such that*

$$m(A) = \int_G \mu_A dP + \alpha \left(1 - \int_G (\mu_A + \nu_A)\, dP\right)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

Proof. See [2,3,6,7].

2 Invariant States

Consider $a \in G$ and define the transformation

$$\tau_a : \mathcal{F} \to \mathcal{F}$$

by the formula

$$\tau_a(A)(\omega) = (\mu_A(\omega + a), \nu_A(\omega + a)).$$

For this transformation we will use the notation

$$\tau_a(A)(\omega) = (\mu_A \circ T_a, \nu_A \circ T_a).$$

Example 1. Let us define the transformation $T_a : [0,1) \to [0,1)$ by the formula

$$T_a(\omega) = \omega + a(mod 1),$$

i.e.

$$T_a(\omega) = \begin{cases} \omega + a, & \text{if } \omega + a < 1, \\ \omega + a - 1, & \text{if } \omega + a \geq 1. \end{cases}$$

Then the function

$$\tau_a(A)(\omega) = (\mu_A \circ T_a, \nu_A \circ T_a)$$

is an example of the mentioned transformation on $\mathcal{F}$. The function T_a represents the moving around the circle with the circuit equal to one.

Our main result is contained in the following theorem.

Theorem 2. *To any $\beta \in [0,1]$ there exists exactly one state $m : \mathcal{F} \to [0,1]$ such that*

$$m(\tau_a(A)) = m(A)$$

for any $A \in \mathcal{F}$ and any $a \in G$ and such that

$$m((0_G, 0_G)) = \beta.$$

Proof. Let $A = (\mu_A, \nu_A) \in \mathcal{F}$. Let $P : \sigma(\mathcal{C}) \to [0,1]$ be the invariant probability measure, i.e. $P(B + a) = P(B)$ for any $B \in \sigma(\mathcal{C})$ and any $a \in G$. Put

$$m(A) = (1-\beta) \int \mu_A dP + \beta \left(1 - \int \nu_A dP\right).$$

Then

$$\begin{aligned} m(\tau_a(A)) &= (1-\beta) \int \tau_a(\mu_A) dP + \beta \left(1 - \int \tau_a(\nu_A) dP\right) \\ &= (1-\beta) \int \mu_A \circ T_a dP + \beta \left(1 - \int \nu_A \circ T_a dP\right) \\ &= (1-\beta) \int \mu_A dP + \beta \left(1 - \int \nu_A dP\right) = m(A). \end{aligned}$$

for any $A \in \mathcal{F}$. We have proved the existence of an invariant state $m : \mathcal{F} \to [0,1]$. Evidently $m((0_G, 0_G)) = \beta$.
We shall prove the uniqueness. Let $\lambda : \mathcal{F} \to [0,1]$ be any invariant state such that $\lambda(0_G, 0_G) = \beta$. Then by Theorem 1 there exist $\alpha \in [0,1]$ and a probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$\lambda(A) = \int_G \mu_A dP + \alpha \left(1 - \int_G (\mu_A + \nu_A) dP\right)$$

for any $A \in \mathcal{F}$.

Put $\mu_A = 0_G, \nu_A = 0_G$. Then

$$\beta = \lambda(0_G, 0_G)) = 0 + \alpha(1 - 0) = \alpha$$

hence $\alpha = \beta$.

First let $\alpha = 0$. Then

$$\lambda(A) = \int_G \mu_A dP.$$

Of course, also

$$\lambda(\tau_a(A)) = \int_G \mu_A \circ T_a dP,$$

since λ is the invariant probability measure then

$$\int_G \mu_A dP = \int_G \mu_A \circ T_a dP$$

for any $A \in \mathcal{F}, a \in G$. For any $B \in \sigma(\mathcal{C})$ put $\mu_A = \chi_B$. It follows

$$P(B) = \int_G \mu_A dP = \int_G \mu_A \circ T_a dP = \int_G \chi_{T_a^{-1}(B)} dP = P(\tau_a^{-1}(B)),$$

hence $P : \sigma(\mathcal{C})) \to [0,1]$ is invariant. Moreover,

$$P(G) = \int_G 1_G dP = \lambda((1_G, 0_G)) = 1,$$

hence P is an invariant probability measure, and it is determined uniquely. Let now $\alpha \in (0, 1]$. Then

$$\lambda(A) = \int_G \mu_A dP + \alpha \left(1 - \int_G (\mu_A + \nu_A) dP\right).$$

Evidently

$$\lambda((0_G, 0_G)) = \alpha(1 - 0),$$

hence

$$\alpha = \lambda((0_G, 0_G)).$$

Moreover,

$$\lambda(\tau_a(A)) = \int_G \mu_A \circ T_a dP + \alpha \left(1 - \int_G (\mu_A \circ T_a + \nu_A \circ T_a) dP\right).$$

Put $A = (0_G, \nu_A)$. Then

$$\begin{aligned} 0 + \alpha \left(1 - \textstyle\int_G (0 + \nu_A \circ T_a) dP\right) &= \lambda(\tau_a(A)) \\ = \lambda(A) &= 0 + \alpha \left(1 - \textstyle\int_G (0 + \nu_A) dP\right), \end{aligned}$$

hence

$$\int_G \nu_A \circ T_a dP = \int_G \nu_A dP$$

for any $A \in \mathcal{F}$ and any $a \in G$. It is clear that $P : \sigma(\mathcal{C})) \to [0, 1]$ is an invariant measure. Moreover,

$$0 = \lambda((0_G, 1_G)) = \alpha \left(1 - \int_G 1_G dP\right).$$

Since $\alpha > 0$, we have

$$P(G) = \int_G 1_G dP = 1,$$

hence $P : \sigma(\mathcal{C}) \to [0, 1]$ is the unique invariant probability measure. □

3 Conclusion

We have proved for any real number $\alpha \in [0, 1]$ the existence of a unique state $m : \mathcal{F} \to [0, 1]$ invariant with respect to the group transformations

$$\tau_a((\mu_A, \nu_A))(\omega) = (\mu_A(\omega + a), \nu_A(\omega + a)),$$

and such that

$$m((0_G, 0_G)) = \alpha.$$

Of course, for different numbers α we can obtained different states m.
On the other hand for fuzzy sets [10] we have $\nu_A = 1 - \mu_A$, hence

$$m(A) = \int_G \mu_A dP + \alpha \left(1 - \int_G (\mu_A + \nu_A) dP\right) = \int_G \mu_A dP,$$

and

$$m(\tau_a(A)) = \int_G \mu_A \circ T_a dP = \int_G \mu_A dP = m(A).$$

We have obtained the existence of an invariant fuzzy state m, and even unique, it does not depend on α.

So from IF-invariant theory one can obtain the fuzzy invariant theory [11], but the opposite direction is not possible, the family of IF states is more rich. Hence the result for IF sets is not a corollary of the existence of fuzzy invariant state.

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