Ordered Fuzzy GARCH Model for Volatility Forecasting

Adam Marszałek[1(✉)] and Tadeusz Burczyński[2]

[1] Institute of Computer Science, Computational Intelligence Department, Cracow University of Technology, Warszawska 24, 31-155 Cracow, Poland
amarszalek@pk.edu.pl

[2] Institute of Fundamental Technological Research, Polish Academy of Sciences, Pawinskiego 5B, 02-106 Warsaw, Poland
tburczynski@ippt.pan.pl

Abstract. A volatility forecasting comparative study between the most popular original GARCH model and the same model defined based on concepts of Ordered Fuzzy Numbers and Ordered Fuzzy Candlsticks is presented. These approaches offer a suitable tool to handle both imprecision of measurements and uncertainty associated with financial data. Therefore, they are particularly useful for volatility forecasting, since the volatility is unobservable and a proxy for it is used (realised volatility). In presented study, based on intra-daily data of the Warsaw Stock Exchange Top 20 Index (WIG 20), one showed that based on the adjusted-R squared and several prediction measurements, the fuzzy approach does perform better than the original GARCH model and forecasts more precisely in both the in-sample and out-of-sample predictions.

Keywords: Volatility forecasting · Realized volatility · Ordered fuzzy number · Kosinski's fuzzy number · Ordered fuzzy candlestick · Ordered fuzzy GARCH model · Financial high-frequency data

1 Introduction

Volatility refers to the price fluctuation over a period of time and as an important parameter, volatility is widely applied in asset pricing, portfolio decisions, and risk management through calculation of value at risk (VaR). In financial economics, volatility is often defined as the standard deviation, although actually volatility and standard deviation are not totally the same.

The two main types of volatility are implied volatility and realized volatility (historical volatility). The implied volatility is the volatility implied by the market price of the option based on an option pricing model. In other words, it is the estimated volatility of a securities price and refers to the markets assessment of future volatility. In contrast, realized volatility measures what actually happened in the past. It directly sums the realized log-returns in a given dimension.

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Therefore, more information can be acquired if higher frequency is used. In connection with the development of information technology, data can be kept in every hour, minute, and even in every second. The extreme high frequency is called ultra high frequency data, which means that every tick is recorded for the calculation of the realized volatility.

Volatility of financial time series is often characterized by some stylized facts such as volatility clusters, persistence, leptokurtic data behavior and time-varying volatility. A popular tools for dealing with time dependent volatility in financial markets are models based on the autoregressive conditional heteroskedasticity (ARCH) model, proposed by Engle [4]. Providing a more flexible structure, Bollerslev [1] introduced the Generalized ARCH (GARCH) model, which combines the ARCH and autoregressive moving average (ARMA) models. The GARCH model estimates jointly a conditional mean and conditional variance equation, and it is regularly used in studying the daily returns of stock market data [6]. The conventional econometric approaches overlook the intrinsic imprecise nature of volatility. Therefore, in the literature many contributions have tried to apply methods based on artificial intelligence such as artificial neural network [3,5,23], fuzzy rule systems [7,20], hybrid adaptive network-based fuzzy inference system (ANFIS) [2,14,15] and fuzzy regression methods [16].

In this paper, one presents an approach where the Ordered Fuzzy GARCH model is defined in the same way as original one, but the all components of equation are replaced by Ordered Fuzzy Numbers and rates of return are modeled by Ordered Fuzzy Candlesticks. The use of them allows modeling uncertainty associated with financial data based on high-frequency data. The empirical results show the effectiveness of the proposed model are provided by modeling and forecasting the volatility of the Warsaw Stock Exchange Top 20 Index (WIG 20) from January 4, 2010 through March 31, 2017, in comparison with original GARCH model.

2 Fuzzy Background Concepts

2.1 Ordered Fuzzy Numbers (OFN)

Ordered Fuzzy Numbers (called also the Kosinski's Fuzzy Numbers) introduced by Kosiński et al. in series of papers [8–12] are defined by ordered pairs of continuous real functions defined on the interval $[0,1]$ i.e.

$$A = (f, g) \text{ with } f, g\colon [0,1] \to \mathbb{R} \text{ as continuous functions.} \tag{1}$$

Functions f and g are called the *up* and *down*-parts of the Ordered Fuzzy Number A, respectively. The continuity of both parts implies their images are bounded intervals, say UP and $DOWN$, respectively. In general, the functions f and g need not be invertible, and only continuity is required. If one assumes, however, that these functions are monotonous, i.e., invertible, and add the constant function of x on the interval $[1_A^-, 1_A^+]$ with the value equal to 1, one might define the membership function

$$\mu(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [f(0), f(1)], \\ g^{-1}(x) & \text{if } x \in [g(1), g(0)], \\ 1 & \text{if } x \in [1_A^-, 1_A^+], \end{cases} \tag{2}$$

if f is increasing and g is decreasing, and such that $f \leq g$ (pointwise). In this way, the obtained membership function $\mu(x),\ x \in \mathbb{R}$ represents a mathematical object which resembles a convex fuzzy number in the classical sense. The Ordered Fuzzy Number and Ordered Fuzzy Number as a fuzzy number in classical meaning are presented in Fig. 1.

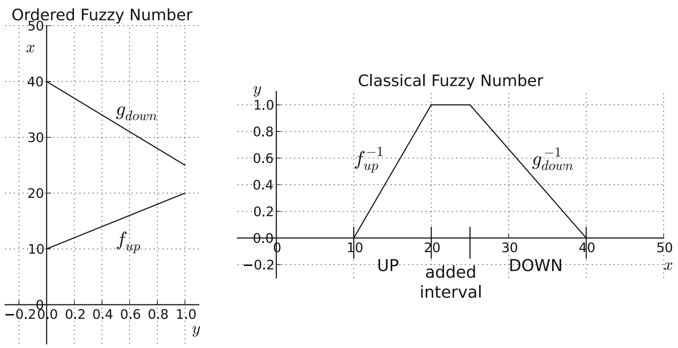

Fig. 1. Graphical interpretation of OFN and a OFN presented as fuzzy number in classical meaning

In addition, note that a pair of continuous functions (f, g) determines different Ordered Fuzzy Number than the pair (g, f). In this way, an extra feature to this object, named the orientation is appointed. Depending on the orientation, the Ordered Fuzzy Numbers can be divided into two types: a positive orientation, if the direction of Ordered Fuzzy Number is consistent with the direction of the axis Ox and a negative orientation, if the direction of the Ordered Fuzzy Number is opposite to the direction of the axis Ox.

Furthermore, the basic arithmetic operations on Ordered Fuzzy Numbers are defined as the pairwise operations of their elements. Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ are Ordered Fuzzy Numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula

$$f_C(y) = f_A(y) * f_B(y), \qquad g_C(y) = g_A(y) * g_B(y) \tag{3}$$

where $*$ works for $+$, $-$, $\cdot$ and $\div$, respectively, and where $C = A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero.

This definition leads to some useful properties. The one of them is existence of neutral elements of addition and multiplication. This fact causes that not always the result of an arithmetic operation is a fuzzy number with a larger support.

This allows to build fuzzy models based on Ordered Fuzzy Numbers in the form of the classical equations without losing the accuracy. In a similar way, basic functions such as log, exp, sqrt etc. can be defined (see [21]).

Moreover, a universe $\mathcal{O}$ of all Ordered Fuzzy Numbers can be identified with $\mathcal{C}^0([0,1]) \times \mathcal{C}^0([0,1])$, hence the space $\mathcal{O}$ is topologically a Banach space [11]. A class of defuzzification operators of Ordered Fuzzy Numbers can be defined, as linear and continuous functionals on the Banach space $\mathcal{O}$. Each of them has a representation by a sum of two Stieltjes integrals with respect to functions ν_1 and ν_2 of bounded variation [13].

An example of a nonlinear functional is *center of gravity defuzzification* functional (CoG) calculated at $A = (f_A, g_A)$

$$CoG(A) = \int_0^1 \frac{f_A(s) + g_A(s)}{2} |f_A(s) - g_A(s)| ds \left\{ \int_0^1 |f_A(s) - g_A(s)| ds \right\}^{-1} \quad (4)$$

provided $\int_0^1 |f_A(s) - g_A(s)| ds \neq 0$. Center of gravity operator defined above is equivalent to the center of gravity operator in classical fuzzy logic.

2.2 Ordered Fuzzy Candlesticks (OFC)

Concept of Ordered Fuzzy Candlesticks was proposed by the authors in [17–19]. Generally, in this approach, a fixed time interval of financial high frequency data is identified with Ordered Fuzzy Number and it is called Ordered Fuzzy Candlestick. The general idea is presented in Fig. 2. Notice, that the orientation of the Ordered Fuzzy Number shows whether the Ordered Fuzzy Candlestick is long or short. While the information about movements in the price are contained in the shape of the f and g functions.

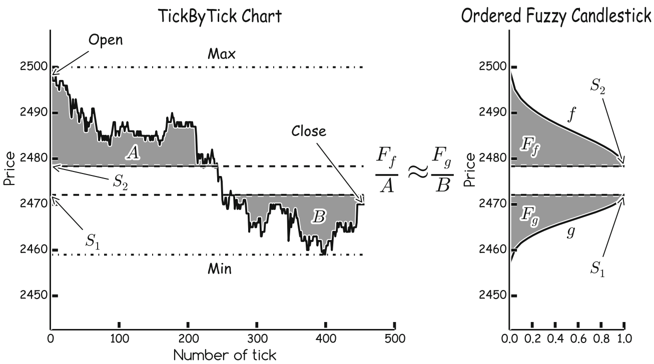

Fig. 2. Draft of general concept of Ordered Fuzzy Candlestick

In previous own works listed two cases of construction of Ordered Fuzzy Candlesticks. The first assumes that the functions f and g are functions of predetermined type, moreover, the shapes of these functions should depend on two parameters (e.g. linear, etc.). Then the Ordered Fuzzy Candlestick for given time series can be defined as follows.

Let $\{X_t \colon t \in T\}$ be a given time series and $T = \{1, 2, \ldots, n\}$. The Ordered Fuzzy Candlestick is defined as an Ordered Fuzzy Number $C = (f, g)$ which satisfies the following conditions 1–4 (for long candlestick) or 1′–4′ (for short candlestick).

1. $X_1 \leq X_n$.
2. $f \colon [0, 1] \to \mathbb{R}$ is continuous and increasing on $[0, 1]$.
3. $g \colon [0, 1] \to \mathbb{R}$ is continuous and decreasing on $[0, 1]$.
4. $S_1 < S_2,\ f(1) = S_1,\ f(0) = \min_{t \in T} X_t - C_1,\ g(1) = S_2$ and $g(0)$ is such that the ratios $\frac{F_g}{A}$ and $\frac{F_f}{B}$ are equal.

1′. $X_1 > X_n$.
2′. $f \colon [0, 1] \to \mathbb{R}$ is continuous and decreasing on $[0, 1]$.
3′. $g \colon [0, 1] \to \mathbb{R}$ is continuous and increasing on $[0, 1]$.
4′. $S_1 < S_2,\ f(1) = S_2,\ f(0) = \max_{t \in T} X_t + C_2,\ g(1) = S_1$ and $g(0)$ is such that the ratios $\frac{F_f}{A}$ and $\frac{F_g}{B}$ are equal.

In the above conditions the center of Ordered Fuzzy Candlestick (i.e. added interval) is designated by parameters $S_1,\ S_2 \in [\min_{t \in T} X_t, \max_{t \in T} X_t]$ and can be compute as different kinds of averages (e.g. arithmetic, weighted or exponential). While C_1 and C_2 are arbitrary nonnegative real numbers, which further extend the support of fuzzy numbers and can be compute e.g. as standard deviation or volatility of X_t. The parameters A and B are positive real numbers, which determine the relationship between the functions f and g. They can be calculated as the mass of the desired area with the assumed density (see Fig. 2). Numbers F_f and F_g are the fields under the graph of functions f^{-1} and g^{-1}, respectively. The examples of realizations of Trapezoid and Gaussian Ordered Fuzzy Candlesticks are defined below and presented in Fig. 3.

Example 1. *Trapezoid OFC.* Suppose that f and g are linear functions in form

$$f(x) = (b_f - a_f)\,x + a_f \quad \text{and} \quad g(x) = (b_g - a_g)\,x + a_g \tag{5}$$

then the Ordered Fuzzy Candlestick $C = (f, g)$ is called *a Trapezoid OFC*, especially if $S_1 = S_2$ then also can be called *a Triangular OFC*.

Example 2. *Gaussian OFC.* The Ordered Fuzzy Candlestick $C = (f, g)$ where the membership relation has a shape similar to the Gaussian function is called *a Gaussian OFC*. It means that f and g are given by functions

$$f(z) = \sigma_f \sqrt{-2\ln(z)} + m_f \quad \text{and} \quad g(z) = \sigma_g \sqrt{-2\ln(z)} + m_g \tag{6}$$

where e.g. $z = (1 - \alpha)x + \alpha$, α close to zero.

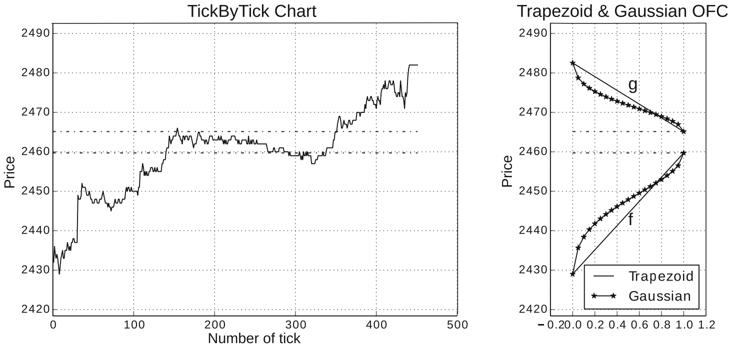

Fig. 3. Examples of Trapezoid and Gaussian OFC

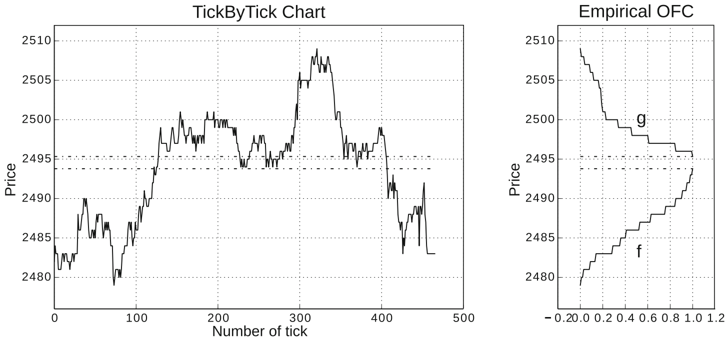

Fig. 4. Example of empirical OFC

The second case of construction of Ordered Fuzzy Candlesticks assumes that the functions f and g are defined in similar way as the empirical distribution in the statistical sciences and it is called *an Empirical OFC*.

Let $\{X_t : t \in T\}$ be a given time series and $T = \{1, 2, \ldots, n\}$. The values of parameters S_1, S_2 and C_1, C_2 are determined based on a time series X_t. The new time series Y_t is created from time series X_t by sorting in ascending. Next, the two time series $Y_t^{(1)}$ and $Y_t^{(2)}$ are created as

$$Y_t^{(1)} = \{Y_i : i \in T \wedge Y_i \leq S_1\} \qquad t \in \{0, 1, \ldots, K_1\}$$
$$Y_t^{(2)} = \{Y_i : i \in T \wedge S_2 \leq Y_i\} \qquad t \in \{0, 1, \ldots, K_2\}$$

Now, based on these time series we define the two discrete functions on interval $[0, 1]$ with step $dx = \frac{1}{M}$ (i.e. $M + 1$ points) as

$$\Psi_1(k \cdot dx) = \begin{cases} Y^{(1)}_{\lfloor \frac{k}{dx} \rfloor} - \frac{M-k}{M} C_1 & \text{if } \; k \in \{0, 1, \dots, M-1\} \\ S_1 & \text{if } k = M \end{cases}$$

$$\Psi_2(k \cdot dx) = \begin{cases} Y^{(2)}_{K_2 - \lfloor \frac{k}{dx} \rfloor} + \frac{M-k}{M} C_2 & \text{if } \; k \in \{0, 1, \dots, M-1\} \\ S_2 & \text{if } \; k = M \end{cases}$$

Then the *empirical OFC* is an Ordered Fuzzy Number $C = (f, g)$ where the functions f and g are continous approximation of functions Ψ_1 and Ψ_2, respectively for long candlestick, whilst for short candlestick Ψ_2 and Ψ_1, respectively. The example of realization of the Empirical OFC is presented in Fig. 4.

3 Fuzzy Returns and Ordered Fuzzy GARCH Model

3.1 Original GARCH Model

In traditional volatility modeling, the most often, rates of return y_t are computed as a percentage logarithmic rates of return i.e.

$$r_t = 100 \cdot (\log(P_t) - \log(P_{t-1})), \tag{7}$$

where P_t is the daily closing stock price at time t, $t = 1, \dots, T$.

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models proposed by Bollerslev [1] are used to capture the time varying behaviour of variance. These models relate the unobserved volatility/variance of data to the past variance and past observations. Hence, the conditional density of the data is a normal distribution, but the occurrence of positive or negative extreme data values depends on the past observations together with past volatility. The standard GARCH(p,q) model for $t = 1, \dots, T$ observations is defined as

$$r_t = \sqrt{h_t} \varepsilon_t, \tag{8}$$

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{9}$$

where r_t is the data with a conditional normal distribution, ε_t is a sequence of independent and identically distributed random variables with zero-mean and unit variance, h_t is the conditional variance of ε_t. Scalars p and q are the lag order for past returns and past conditional volatility in the GARCH model, respectively, and $(\alpha_0, \alpha_i, \beta_j)$ for $i = 1, \dots, p$ and $j = 1, \dots, q$ are unknown coefficients to be estimated. Sufficient conditions for positive variance h_t at every period are

$$\alpha_0 > 0, \;\; \alpha_i \geq 0, \;\; \beta_j \geq 0, \;\; \sum_{i=0}^{p} + \sum_{j=1}^{q} < 1, \;\; i = 1, \dots, p, \;\; j = 1, \dots, q, \tag{10}$$

where these restrictions also ensure a stationary variance process and the existence of a finite mean and variance of h_t.

The forecasts of the GARCH model are obtained recursively as the forecasts of an ARMA model. If we consider the GARCH(1,1) model which is the GARCH models under study at the forecast origin t, the 1-step ahead forecast of h_{t+1} is

$$h_{t+1} = \alpha_0 + \alpha_1 r_t^2 + \beta h_t. \tag{11}$$

When calculating multistep ahead forecasts the volatility Eq. (8) can be rewritten as $r_t^2 = h_t \varepsilon_t^2$.

3.2 Fuzzy Returns

In presented approach, the fuzzy percentage logarithmic rates of return are modeled based on high-frequency data (tick-by-tick data) using concept of Ordered Fuzzy Candlestick. Let X_{ti}, $i = 1, \dots, I_t$ be a time series of quotations of financial instrument for given fixed time interval t (e.g. day, week, month). The new time series R_{ti} is created from time series X_{ti} as follows

$$R_{ti} = 100 \cdot (\log(X_{ti}) - \log(X_{t1})) \qquad i = 1, \dots, I_t \tag{12}$$

Now, based on these time series the Ordered Fuzzy Candlestick $\tilde{R}_t = (f_{\tilde{R}_t}, g_{\tilde{R}_t})$ is defined and it is called fuzzy percentage logarithmic rate of return for given time interval t. The example of time series R_{ti} and associated with it the Gaussian OFC is presented in Fig. 5.

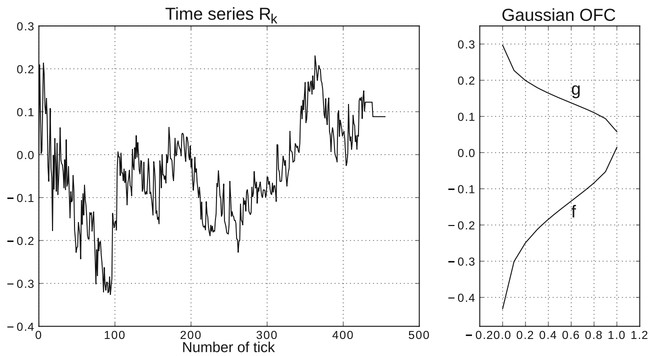

Fig. 5. Example of time series R_{ti} together with a Gaussian OFC associated with it

3.3 Ordered Fuzzy GARCH Model

Let $\tilde{R}_t$, $t = 1, \dots, T$ denote the sequences of fuzzy percentage logarithmic rates of return. Notice, that $\tilde{R}_t$ are Ordered Fuzzy Numbers. Thanks to well-defined arithmetic of OFN, the Ordered Fuzzy GARCH model can be defined as follows

$$\tilde{R}_t = \sqrt{\tilde{H}_t}\tilde{\varepsilon}_t, \tag{13}$$

$$\tilde{H}_t = \tilde{A}_0 + \sum_{i=1}^{p} \tilde{A}_i \cdot \tilde{R}_{t-i}^2 + \sum_{j=1}^{q} \tilde{B}_j \cdot \tilde{H}_{t-j}, \tag{14}$$

where $\tilde{H}_t$ is the conditional variance of $\tilde{\varepsilon}_t$ and it is OFN. Scalars p and q are the lag order for past returns and past conditional volatility in the GARCH model, respectively, and $(\tilde{A}_0, \tilde{A}_i, \tilde{B}_j)$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$ are unknown coefficients to be estimated, all are OFN. Moreover, $\tilde{\varepsilon}_t$ is a sequence of independent and identically distributed fuzzy random variables with zero-mean and unit variance, in this sense, that for each $x \in [0, 1]$ $f_{\tilde{\varepsilon}_t}(x)$ and $g_{\tilde{\varepsilon}_t}(x)$ are sequences of independent and identically distributed real random variables with zero-mean and unit variance. Notice, that for each $x \in [0, 1]$ $f_{\tilde{R}_t}(x)$ and $g_{\tilde{R}_t}(x)$ can be regarded as a original GARCH model.

The forecasts of the Ordered Fuzzy GARCH model are obtained in the same way as for original model. If we consider the Ordered Fuzzy GARCH(1,1) model, the 1-step ahead forecast of $\tilde{H}_{t+1}$ is

$$\tilde{H}_{t+1} = \tilde{A}_0 + \tilde{A}_1\tilde{R}_t^2 + \tilde{B}\tilde{H}_t. \tag{15}$$

The result of forecast is Ordered Fuzzy Number, which includes three kinds of predictions:

- **point forecast:** given by value of a defuzzification operator, e.g. CoG,
- **interval forecast:** given by subset of support of the OFN,
- **direction forecast:** given by orientation of the OFN.

4 Empirical Results

The study attempts to modeling and forecasting the future volatility of the Warsaw Stock Exchange Top 20 Index (WIG 20), solely on the basis of past index prices. For the case study we use the intra-daily quotations of WIG 20 index from January 4, 2010 through March 31, 2017.[1] For forecasting, the parameters of proposed and original GARCH model are estimated using data from January 4, 2010 through December 30, 2016, under assumption of normal distribution.[2] In order to select best lag parameters for original GARCH model, the Bayesian information criterion (BIC) were performed. According to BIC criterion the best specification for original model was $p = 1$ and $q = 1$. The same values of parameters p and q was also set to Ordered Fuzzy GARCH model. The out-of-sample forecast horizon covers period from January 2, 2017 to March 31, 2017.

In the case of Ordered Fuzzy GARCH model, three models, in which the fuzzy returns are modeled by using three kind of Ordered Fuzzy Candlesticks

[1] Data has been retrieved from the site www.bossa.pl.

[2] The estimation of parameters of proposed and original GARCH model was made by using ARCH package for Python (https://pypi.python.org/pypi/arch/4.0) [22].

(Trapezoid, Gaussian and Empirical) are considered. For each model, parameters S_1, S_2 are computed as minimum and maximum of different kinds of averages (arithmetic, weighted and exponential). The parameters C_1, C_2 are calculated as a standard deviation and the parameters A and B are calculated as the mass of the desired area with the exponential density.

To proxy for true variance, the daily realized volatility for the WIG 20 computed by using d_t-th mid-quote (tick-time sampling) is used, where d_t is chosen such that the average sampling duration is five minutes. Let r_{ti}, $i = 1, \ldots, I_t$ denote the series of percentage logarithmic rates of return on day t (see (7)). The realized volatility proxy is the estimator constructed using sums of intra-daily squared returns

$$RV_t = \sum_{i=1}^{I_t} r_{ti}^2. \tag{16}$$

The overnight return is omitted, as is often done in the literature.

Volatility forecasts comparison was conducted for 1-step, 5-step and 22-step ahead horizon in terms of adjusted-R squared, mean absolute error (MAE), mean percentage error (MPE), root mean squared error (RMSE) and the Theil inequality coefficient (TIC). These criteria are given as

$$MAE = \frac{1}{N} \sum_{i=1}^{N} |\sigma_i^2 - \hat{\sigma}_i^2|, \tag{17}$$

$$MPE = \frac{1}{N} \sum_{i=1}^{N} \frac{|\sigma_i^2 - \hat{\sigma}_i^2|}{\sigma_i^2}, \tag{18}$$

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\sigma_i^2 - \hat{\sigma}_i^2)^2}, \tag{19}$$

$$TIC = \frac{\sqrt{\frac{1}{N} \sum_{i=1}^{N} (\sigma_i^2 - \hat{\sigma}_i^2)^2}}{\sqrt{\frac{1}{N} \sum_{i=1}^{N} (\sigma_i^2)^2} + \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\hat{\sigma}_i^2)^2}}, \tag{20}$$

where N is the number of observations, σ_i^2 is the actual volatility at period i, measured as the realized volatility, and $\hat{\sigma}_i^2$ is the forecast volatility at i. In the case of fuzzy models these criteria are computed by using crisp values obtained by using centre of gravity operator.

Tables 1 and 2 contain in-sample and out-of-sample prediction results respectively. Different panels present different horizons, i.e., daily (h = 1), weekly (h = 5), monthly (h = 22). The number in bold means it is the best one of every row. It is easy to see that no matter it is in-sample or out-of-sample, the all of proposed Ordered Fuzzy GARCH models performs better than original GARCH model. The performance of fuzzy models constructed based on Gaussian and Empirical OFC are similar, model with Trapezoid OFC is worse but still better than original one. It is worth pointing out that over 85% predictions obtained by original GARCH model

Table 1. Comparison of the in-sample performance between the models

Horizon	Criterion	GARCH	OFGARCH_T	OFGARCH_G	OFGARCH_E
h = 1	adj. R^2	0.3256	0.3994	0.3997	**0.4482**
	MAE	0.6698	0.5135	**0.4285**	0.4403
	MPE	0.9349	0.6328	**0.4736**	0.5234
	RMSE	1.1469	1.0276	0.9929	**0.9557**
	TIC	0.3471	0.3239	0.3385	**0.3175**
	overestimated	87.41%	77.8%	**66.71%**	72%
h = 5	adj. R^2	0.195	0.3995	**0.3997**	0.2426
	MAE	0.7172	0.5138	**0.4288**	0.4942
	MPE	1.0196	0.6328	**0.4737**	0.6094
	RMSE	1.2534	1.0286	**0.9939**	1.1281
	TIC	0.3834	**0.3239**	0.3386	0.3816
	overestimated	86.34%	77.75%	**66.69%**	71.76%
h = 22	adj. R^2	−0.0591	0.3996	**0.3998**	0.0921
	MAE	0.7905	0.5171	**0.4312**	0.5601
	MPE	1.2139	0.6365	**0.476**	0.7688
	RMSE	1.345	1.0334	**0.9986**	1.234
	TIC	0.4252	**0.3243**	0.339	0.4383
	overestimated	85.85%	78%	**66.94%**	74.91%

Table 2. Comparison of the out-of-sample performance between the models

Horizon	Criterion	GARCH	OFGARCH_T	OFGARCH_G	OFGARCH_E
h = 1	adj. R^2	0.0798	**0.155**	0.152	0.1175
	MAE	0.4699	0.3775	0.3137	**0.3129**
	MPE	0.7091	0.5544	**0.4223**	0.4236
	RMSE	0.5339	0.4549	**0.3879**	0.3912
	TIC	0.2465	0.22	**0.2025**	0.2056
	overestimated	85.93%	76.56%	**67.19%**	68.75%
h = 5	adj. R^2	0.0831	0.0932	**0.1014**	0.0951
	MAE	0.4712	0.3875	0.3183	**0.3171**
	MPE	0.7349	0.5897	**0.4512**	0.4541
	RMSE	0.5351	0.4687	0.399	**0.3956**
	TIC	0.2464	0.2271	0.209	**0.2077**
	overestimated	87.5%	78.13%	**65.63%**	**65.63%**
h = 22	adj. R^2	−0.012	**−0.01**	−0.0134	−0.0161
	MAE	0.5468	0.4217	**0.3473**	0.3546
	MPE	0.8711	0.6533	**0.4977**	0.5221
	RMSE	0.6309	0.5037	**0.4378**	0.4425
	TIC	0.2817	0.2452	0.231	**0.229**
	overestimated	85.94%	76.56%	**64.06%**	67.19%

are overestimated, whereas for fuzzy models it is less than 68%. Figure 6 presents the out-of-sample prediction graphs for original GARCH model and Ordered Fuzzy GARCH model based on Gaussian OFC (crisp forecast) on background of forecast in form of Ordered Fuzzy Numbers. Notice, that for fuzzy model it is easy to obtain other forecasts by using different defuzzification operators.

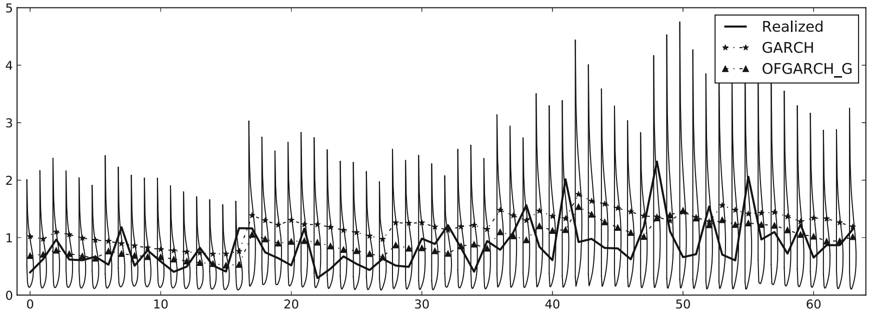

Fig. 6. Daily, WIG 20 index realized volatility, original GARCH and Ordered Fuzzy Gaussian GARCH out-of-sample forecasts

5 Conclusion

Volatility forecasting plays a central role in several financial applications like asset allocation and hedging, option pricing and risk analysis. This paper investigated a volatility forecasting problem in the complex uncertain environment. The representation of fuzzy rates of return using concept of the Ordered Fuzzy Candlesticks was described. Moreover, the Ordered Fuzzy GARCH model using well-defined arithmetic of Ordered Fuzzy Numbers in the same way as original GARCH model was defined. A numerical example based on WIG 20 index was given to demonstrate the potential of the Ordered Fuzzy GARCH approach to the problem of volatility forecasting, providing more accurate results than original GARCH model in statistical terms. For the future work, the proposed model should be compare with other fuzzy approaches or more sophisticated GARCH-type models and then should be applied to financial decision making problems related to volatility such as option pricing and risk modeling.

References

1. Bollerslev, T.: Generalized autoregressive conditional heteroskedasticity. J. Econometrics **31**, 307–327 (1986)
2. Chang, J., Wei, L., Cheng, C.: A hybrid ANFIS model based on AR and volatility for TAIEX forecasting. Appl. Soft Comput. **11**, 1388–1395 (2011)
3. Coelho, L.S., Santos, A.A.P.: A RBF neural network model with GARCH errors: application to electricity price forecasting. Eletric Power Syst. Res. **81**, 74–83 (2011)

4. Engle, R.F.: Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. Econometrica **50**, 987–1007 (1982)
5. Hamid, S.A., Iqbal, Z.: Using neural networks for forecasting volatility of S&P 500. J. Bus. Res. **57**, 1116–1125 (2004)
6. Han, H., Park, J.Y.: Time series properties of ARCH processes with persistent covariates. J. Econometrics **146**, 275–292 (2008)
7. Hung, J.: A fuzzy asymmetric GARCH model applied to stock markets. Inf. Sci. **179**, 3930–3943 (2009)
8. Kosiński, W., Prokopowicz, P., Ślęzak, D.: Drawback of fuzzy arithmetic - new intuitions and propositions. In: Burczyński, T., Cholewa, W., Moczulski, W. (eds.) Proceedings of the Methods of Artificial Intelligence, pp. 231–237. PACM, Gliwice (2002)
9. Kosiński, W., Prokopowicz, P., Ślęzak, D.: On algebraic operations on fuzzy numbers. In: Klopotek, M., Wierzchoń, S.T., Trojanowski, K. (eds.) Intelligent Information Processing and Web Mining, Proceedings of International Symposium on IIS: IIPWM 2003, Zakopane, Poland, pp. 353-362. Physica Verlag, Heidelberg (2003)
10. Kosiński, W., Prokopowicz, P., Ślęzak, D.: Ordered fuzzy numbers. Bull. Polish Acad. Sci. Ser. Sci. Math. **51**(3), 327–338 (2003)
11. Kosiński, W., Prokopowicz, P.: Algebra of fuzzy numbers. Math. Applicanda **32**(46), 37–63 (2004). Journals of the Polish Mathematical Society (in Polish)
12. Kosiński, W.: On soft computing and modelling. Image Process. Commun. **11**(1), 71–82 (2006)
13. Kosiński, W., Frischmuth, K., Wilczyńska-Sztyma, D.: A new fuzzy approach to ordinary differential equations. In: Rutkowski, L., et al. (eds.) ICAISC 2010, Part I. LNAI, vol. 6113, pp. 120–127. Springer, Heidelberg (2010)
14. Luna, I., Ballini, R.: Adaptive fuzzy system to forecast financial time series volatility. J. Intell. Fuzzy Syst. **23**, 27–38 (2012)
15. Maciel, L., Gomide, F., Ballini, R.: Evolving fuzzy-GARCH approach for financial volatility modeling and forecasting. Comput. Econ. **48**, 379–398 (2016)
16. Muzzioli, S., De Baets, B.: A comparative assessment of different fuzzy regression methods for volatility forecasting. Fuzzy Optim. Decis. Making **12**, 433–450 (2013)
17. Marszałek, A., Burczyński, T.: Financial fuzzy time series models based on ordered fuzzy numbers. In: Pedrycz, W., Chen, S.-M. (eds.) Time Series Analysis, Model and Applications. ISRL, vol. 47, pp. 77–95. Springer, Heidelberg (2013)
18. Marszałek, A., Burczyński, T.: Modelling financial high frequency data using ordered fuzzy numbers. In: Rutkowski, L., et al. (eds.) ICAISC 2013. LNAI, vol. 7894, Part I, pp. 345–352. Springer, Heidelberg (2013)
19. Marszałek, A., Burczyński, T.: Modeling and forecasting financial time series with ordered fuzzy candlesticks. Inf. Sci. **273**, 144–155 (2014)
20. Popov, A.A., Bykhanov, K.V.: Modeling volatility of time series using fuzzy GARCH models. In: Annals of the 9th Russian-Korean International Symposium on Science and Technology (2005)
21. Prokopowicz, P.: Flexible and Simple Methods of Calculations on Fuzzy Numbers with the Ordered Fuzzy Numbers Model. LNCS (LNAI), vol. 7894, pp. 365–375. Springer, Heidelberg (2013)
22. Sheppard, K.: ARCH Toolbox for Python [Data set]. Zenodo (2015). http://doi.org/10.5281/zenodo.15681
23. Tseng, C., Cheng, S., Wang, Y., Peng, J.: Artificial neural network model of the hybrid EGARCH volatility of the Taiwan stock index option prices. Phys. A: Stat. Mech. Appl. **387**, 3192–3200 (2008)