# Examining Students' Procedural and Conceptual Understanding of Eigenvectors and Eigenvalues in the Context of Inquiry-Oriented Instruction

**Khalid Bouhjar, Christine Andrews-Larson, Muhammad Haider and Michelle Zandieh**

**Abstract** This study examines students' reasoning about eigenvalues and eigenvectors as evidenced by their written responses to two open-ended response questions. This analysis draws on data taken from 126 students whose instructors received a set of supports to implement a particular inquiry-oriented instructional approach and 129 comparable students whose instructors did not use this instructional approach. In this chapter, we offer examples of student responses that provide insight into students' reasoning and summarize broad trends observed in our quantitative analysis. In general, students in both groups performed better on the procedurally oriented question than on the conceptually oriented question. The group of students whose instructors received support to implement the inquiry-oriented approach outperformed the other group of students on the conceptually oriented question and performed equally well on the procedurally oriented question.

**Keywords** Eigenvalues · Eigenvectors · Linear algebra · Inquiry-oriented instruction · Student thinking

Linear algebra is a mandatory course for many science, technology, engineering, and mathematics (STEM) students. The theoretical nature of linear algebra makes it a difficult course for many students because it may be their first time to deal with abstract and conceptual content (Carlson, 1993). Carlson (1993) also posited that this difficulty arises from the prevalence of procedural and computational emphases

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in students' coursework prior to linear algebra, and that it might therefore be difficult for students to connect new linear algebra topics and their previous knowledge. To address this issue, researchers have developed instructional materials for Inquiry-Oriented Linear Algebra (IOLA; http://iola.math.vt.edu/) and approaches to help students develop more robust, conceptual ways of reasoning about core topics in introductory linear algebra (e.g. Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, & Larson, 2013; Zandieh, Wawro, & Rasmusen, 2017).

Instructors who were not involved in the development of these kinds of research-based, inquiry-oriented instructional materials have been shown to encounter challenges when implementing such materials (Johnson, Caughman, Fredericks, & Gibson, 2013). Under an NSF-supported project Teaching Inquiry-Oriented Mathematics: Establishing Supports (TIMES), Johnson, Keene, and Andrews-Larson (2015) designed and implemented a system of instructional supports based on research in instructional change in undergraduate mathematics education, teacher learning, and professional development in settings ranging from K-20 (e.g. Henderson, Beach, & Finkelstein, 2011). These supports included sequences of student activities with implementation notes, a three-day summer workshop, and weekly online workgroups during the semester instructors implemented the materials in their teaching. This chapter examines differences in performance and reasoning of students whose instructors received these supports through the TIMES project (TIMES students) as compared to students whose instructors did not receive these supports (Non-TIMES students). In particular, we examine assessment data to identify differences in student performance and reasoning about eigenvectors and eigenvalues.

In this work we draw on data from an assessment that was developed to align with four core introductory linear algebra topic areas addressed in the IOLA instructional materials: linear independence and span; systems of linear equations; linear transformations; and eigenvalues. and eigenvectors. In the assessment, there were two questions that addressed eigenvalues and eigenvectors: question 8 and 9. Question 8 was a procedurally oriented question related to the eigenvalue of a given matrix and question 9 focused on conceptual understanding of the eigenvectors. The research questions for this analysis are:

- How does the performance of students whose instructors received TIMES instructional supports for teaching linear algebra compare to the performance of other students?
- How did students reason about eigenvectors and eigenvalues in the context of questions designed to assess aspects of students' procedural and conceptual understanding? How did reasoning differ for students of TIMES and Non-TIMES instructors?

## 1 Literature

Many have argued that the shift from predominantly computational and procedural approaches to mathematics many students experience before college to more theoretical approaches causes a lot of difficulties for students as they transition to university mathematics. Linear algebra is a course in which students struggle to develop conceptual understanding (Carlson, 1993; Dorier & Sierpinska, 2001; Dorier, Robers, Robinet, & Rogalski, 2000; Stewart & Thomas, 2009). Across the literature on the teaching and learning of eigenvalues and eigenvectors, procedural thought processes feature prominently. For example, Stewart and Thomas (2006) pointed to ways in which students often learn about eigenvalues and eigenvectors, where a formal definition is often linked to a symbolic presentation and its manipulation. For the purpose of this paper, we will draw on the following formal definition for eigenvectors and eigenvalues:

Suppose $A$ is an $n \times n$ real-valued matrix and $x$ is a non-zero vector in $\mathbb{R}^n$. We say the vector $x$ is an eigenvector of the matrix $A$ if there is some scalar $\lambda$ such that $Ax = \lambda x$. Further, in this case, we say that $\lambda$ is the eigenvalue associated with the eigenvector $x$.

Thomas and Stewart (2011) highlighted a difficulty students find when faced with formal definitions for eigenvalues and eigenvectors: these definitions contain an embedded symbolic form $(Ax = \lambda x)$, and instructors often move quickly into symbolic manipulations of algebraic and matrix representations such as transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$. Their findings that students struggle to make sense of formal definitions, struggle to make use of geometric representations of eigenvectors, and exhibit procedural orientations toward eigenvectors suggest that such treatments might not provide sufficient opportunities for students to make sense of the reasons behind these symbolic shifts (Stewart & Thomas, 2009; Thomas & Stewart, 2011).

In order to help students make sense of situations that might be modeled using eigenvectors and eigenvalues, Salgado and Trigueros (2015) developed a problem that tasked students with designing a mathematical model that describes the employment dynamics of a population and its long-term behavior. While this modeling problem does not foreground geometric interpretations, the researchers also developed other activities to subsequently establish a relationship between the algebraic and geometric interpretation of eigenvectors and eigenvalues. Drawing on analysis of data from 30 undergraduate students, Salgado and Trigueros (2015) argued that this instructional sequence supported students' learning by helping students link ideas about eigenvectors and eigenvalues to other previously learned concepts.

Schoenfeld (1995) used eigenpictures in the $2 \times 2$ case ("stroboscopic" pictures) to show $x$ and $Ax$ at the same time by using multiple line segments in the $x$–$y$-plane. He observed that graphical representations of eigenvalues and eigenvectors got little attention in the literature and that a picture may benefit more than algebraic presentations. It is also documented more generally in linear algebra that students struggle to coordinate algebraic with geometric interpretations (e.g. Larson & Zandieh, 2013; Stewart & Thomas, 2010) and the students' understanding of eigenvectors is not always well connected to concepts of other topics of linear algebra (Lapp, Nyman, & Berry, 2010).

To support students in developing a better understanding of the formal definition and associated interpretations of the eigenvalues and eigenvectors, researchers have developed a variety of instructional interventions (e.g. Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Zandieh, Wawro & Rasmussen, 2017). This paper examines student learning outcomes associated with a geometrically motivated instructional approach (see Plaxco et al. 2018; Zandieh, Wawro & Rasmussen, 2017) when paired with TIMES instructional supports; the approach will be described in the Study Design section.

## 2 Theoretical Framing

Researchers often make reference to conceptual understanding and procedural understanding when discussing students' reasoning about mathematical concepts (Hiebert, 1986). Hiebert and Lefevre (1986) defined conceptual knowledge as a "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relatonships are as prominent as the discrete pieces of information" (pp. 3–4). According to Hiebert and Lefevre (1986) students have procedural knowledge if they can combine formal language and symbolic representation systems with algorithms or rules in order to complete mathematical tasks.

In this paper we also draw on Larson and Zandieh's (2013) framework for students' mathematical thinking about matrix equations of the form $Ax = b$. This framework details three important interpretations, relationships between geometric and symbolic representations within each interpretation, and the complexity entailed in shifting among interpretations. The three interpretations this framework includes are (1) a linear combination interpretation, in which $b$ is viewed as a linear combination of the column vectors of the matrix $A$ with $x$ functioning as the set of weights on the column vectors of $A$, (2) a system of equations interpretation in which $x$ is viewed as a solution and $A$ is seen as a set of coefficients, and (3) a linear transformation interpretation in which $x$ is viewed as an input vector, $b$ as an output vector, and $A$ as the matrix that transforms $x$ into $b$.

We argue these interpretations are helpful for making sense of students' reasoning, but that the framework may need to be modified or expanded to more fully account for student reasoning in the context of eigenvalues and eigenvectors. In the context of eigenvectors and eigenvalues, students need to coordinate a transformation interpretation with the equation $Ax = \lambda x$, where the matrix $A$ transforms the vector $x$ by stretching, shrinking, and/or reversing the direction of vector $x$. Additionally, students need to shift to a systems interpretation and consider when the equivalent system $(A - \lambda I)x = 0$ has a non-trivial solution in order to make sense of standard procedures for computing eigenvalues and eigenvectors.

## 3 Study Design

In previous work, we have developed an assessment aligned with the inquiry-oriented linear algebra (IOLA) instructional materials used in the TIMES project (Haider et al., 2016). This paper-and-pencil assessment consists of 9 items, most of which include an open-ended response component. The assessment was administered at the end of the semester, and students were allocated one hour to complete the assessment.

In this analysis we examine assessment data from 126 students across six TIMES instructors and 129 students across three Non-TIMES instructors from different institutions in the US. Non-TIMES linear algebra instructors were selected from either the same institutions as TIMES instructors or a similar institution (e.g. preferably one from a similar geographic area, with similar size of student population, with similar acceptance rate) to collect assessment data for comparison of TIMES and Non-TIMES students. In this study, we focused on an in-depth analysis of students' reasoning on the assessment questions related to eigenvalues and eigenvectors. Both items are shown in Fig. 1.

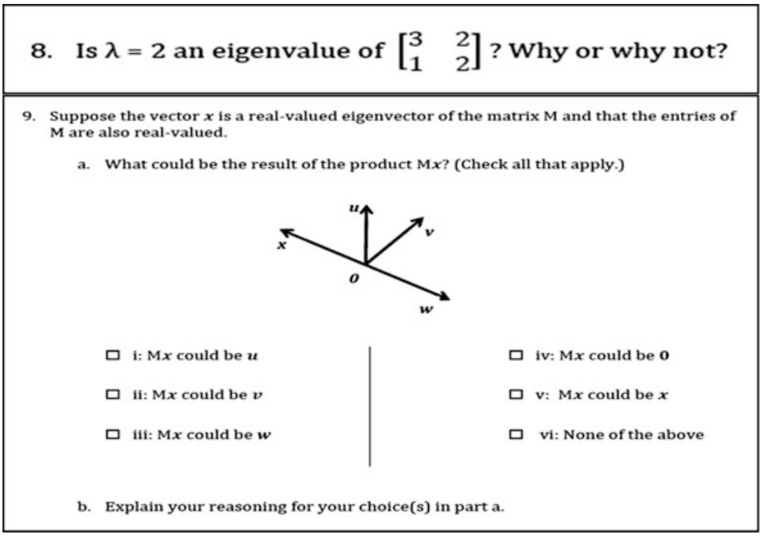


**Fig. 1** Assessment items related to eigenvectors and eigenvalues (Question 9 was retrieved from http://mathquest.carroll.edu and developed as part of an NSF-supported project entitled Project MathVote: Teaching Mathematics with Classroom Voting. For related research, see Cline, Zullo, Duncan, Stewart, & Snipes, 2013)

The inquiry-oriented approach to learn eigenvalues and eigenvectors associated with this study is characterized in detail elsewhere (Plaxco et al. 2018; Zandieh, Wawro, & Rasmusen, 2017). Briefly described, this approach supports students in first learning about eigenvalues and eigenvectors as a set of “stretch” factors and directions that can be used to more easily characterize a geometric transformation. Students work through a series of tasks, first aiming to find (using standard coordinate systems) the image of a figure in a plane under a transformation that is easily described in a non-standard coordinate system. Students then work to label points in both the pre-image and the image using the standard and the more convenient coordinate systems, find matrices that rename points from one coordinate system to the other, and find matrices corresponding to the transformation described relative to both coordinate systems. The instructor works to link this work to the matrix equation $A = PDP^{-1}$ and subsequent tasks aim to leverage this conceptual basis as students learn more traditional computational methods associated with computing eigenvalues and eigenvectors.

## 4 Methods of Analysis

To answer our research questions, our analysis has two main components. The first component of our analysis is quantitative in nature, as we aim to compare learning outcomes of students whose instructors received TIMES instructional supports to those who did not. The second component of our analysis is qualitative in nature, as we work to identify students’ ways of reasoning on both the more procedurally oriented assessment item (Q8) and the more conceptually oriented item (Q9). We follow Kwon, Rasmussen, and Allen’s (2005) approach for distinguishing assessment items that are conceptually oriented from those that are procedurally oriented. In particular we consider Q8 to be more procedurally oriented in that there is a commonly taught procedure that students can directly invoke (with some interpretation) to produce a correct answer to the question. There is no such standard procedure for Q9, so we consider it to be more conceptually oriented. In our qualitative analysis, we also look for similarities and differences that emerge from considering the two groups.

To facilitate our quantitative analysis, we needed to score students’ responses to the two assessment items. Specifically, we needed to develop a uniform system for assigning a number of points to students’ responses that provide an overall assessment of the quality of their response and the understanding reflected in that response. Question 9a required students to select which subset of 6 possible options were appropriate responses, so 1 point was awarded to each of the possible options for correctly selecting or not selecting that option. Both Question 8 and Question 9b were open-ended response questions, and both of these were scored on a scale of 0 to 3 points. Three points were awarded for a fully correct response, two points were awarded for a mostly correct response (e.g. if a minor computational error was made, 2 points would be assigned), one point was awarded if the student’s response provided evidence of some knowledge or understanding relevant to the question, and no points were awarded otherwise. A scoring scheme was developed to specify what kinds of responses received how many points. In order to ensure consistency among coders in how points were assigned, new examples were added to the scoring guide throughout the scoring process. A condensed version of the scoring scheme for assigning points to open-ended response questions can be found in Appendix. Additionally, the Appendix includes some explanation of how this scoring scheme aligns with our coding categories for *how* students reasoned, which are described in greater detail below. Student work exemplifying common ways of reasoning with explanation of points awarded are provided in the Findings section.

To ensure agreement regarding points assigned to each response, two researchers looked at every student's attempt and assigned a score independently before comparing with each other. If the two researchers assigned a different score to a particular student, they then discussed according to the codebook and agreed on a common score for that student. If both researchers disagreed about a particular score, then a third researcher was consulted to reach a consensus.

Once scores had been assigned to all student responses, descriptive statistics were generated to examine the overall performance of students on the eigenvalue and eigenvector questions and to compare TIMES students with Non-TIMES students for both questions. We were unable to control for factors such as students' mathematical background, major, and instructor's teaching experience, so this is an unavoidable limitation for our statistical analysis. However, we tried our best to choose TIMES and Non-TIMES students either from the same school or from similar schools. This helps us establish similarity of students in TIMES and non-TIMES classes. Then, we compared the mean scores of TIMES and Non-TIMES students using two-tailed t-tests to identify when differences of means were statistically significant.

In order to facilitate our qualitative analysis of students' reasoning, we examined student responses to the open-ended portions of question 8 and question 9. As noted before, we consider Q8 to be more procedurally oriented. After examining the data several times and refining the categories of the students' reasoning about item 8, we sorted students' responses into 5 broad categories: (1) reasoning about the determinant, (2) reasoning about $A - \lambda I$ without computing a determinant, (3) other, (4) students who explicitly indicated they did not know, and (5) students who left the item blank. A student's response was categorized as "reasoning about the determinant" if he or she solved the characteristic equation, plugged the possible given eigenvalue into the characteristic equation, or computed the determinant of the $A - 2I$ matrix and compared the result to 0. A students' response was categorized as "reasoning about $A - \lambda I$" if he or she solved the system of linear equations $(A - \lambda I)x = 0$, considered the linear independence of the columns of $A - \lambda I$, or considered whether $rref(A - \lambda I)$ had any free variables.

In examining students' responses to question 9, we found it helpful to distinguish responses that were conceptually aligned with the formal definition for eigenvectors and eigenvalues from those that were not. We were specifically interested in student reasoning that appropriately coordinated interpretations of

$A, x,$ and $\lambda$ in the context of the matrix equation $Ax = \lambda x$. In particular, we say a student response "uses the eigen-concept" when there is evidence a student is coordinating $M, x,$ and $\lambda$ in at least one of the following ways:

- Algebraically: The matrix $M$ is a fixed matrix that transforms the (nonzero) eigenvector $x$ in a particular way, namely such that the resulting vector $Ax$ is a scalar multiple ($\lambda$) of $x$.
- Geometrically: this can be interpreted to mean that multiplying $x$ by $A$ has the effect of
  - stretching $x$ in the same direction or opposite direction, or
  - causing the resultant vector to lie along the same line as the vector $x$.

If a student drew on a transformation interpretation to make sense of $Ax$ but did not coordinate this appropriately with $\lambda x$ in one of the ways mentioned above, we did not say that the student's response used the eigen-concept.

We grouped students' responses to question 9 into five categories: (1) responses that used the eigen-concept, (2) responses that focused on the role of the matrix $M$ in a way that did not use the eigen-concept, (3) other, (4) responses in which the student explicitly indicated he or she did not know, and (5) responses that were left blank. There were two primary kinds of responses coded as focused on the role of the matrix $M$ in a way that did not use the eigen-concept. The first one is when students focused on the role of the matrix $M$ as a transformation, but without specifying the particular way it will transform an eigenvector $x$. The second kind of response is when students suggested specific matrices $M$ that would satisfy particular equations (e.g "$Mx = x$ if $M = I$"). While this is certainly a true statement, it doesn't include evidence of understanding the special relationship between a matrix and its eigenvector(s).

After coding students' responses to Q8 and Q9, we aggregated these responses into tables, organized by the category assigned to each response and number of points awarded. We also separated TIMES from Non-TIMES students in counting the number of responses in these discrete categories. This allowed us to look for patterns in which approaches were conceptually oriented, which approaches lent themselves to arriving at correct answers, and differences in approaches taken by TIMES and Non-TIMES students.

## 5 Findings

In order to answer our research question about how TIMES students compared to Non-TIMES students, we first present our quantitative analysis of students' performance on the more procedurally-oriented question (Q8) and the more conceptually oriented question (Q9), separating students of TIMES instructors from students of Non-TIMES instructors. We then summarize findings from our coding of students' approaches to these same questions, providing examples of responses that highlight important trends in student reasoning.

## 5.1 Overview of Differences in Student Performance

We highlight three central trends from our quantitative analysis. First, TIMES students outperformed Non-TIMES on both items, with a strongly significant difference of means on the conceptual item. Second, both TIMES and Non-TIMES students did better on the procedurally oriented item than on the conceptually oriented item. Third, correlations between students' performance on both the conceptual and procedural items were weak for students in both groups, suggesting that the two items assessed relatively different aspects of student understanding. Note that the last trend is not part of answering our research questions, it is more of a side observation that emerged from our quantitative analysis.

To compare the performance of TIMES students with Non-TIMES students, we first computed the mean and standard deviation for question 8 (which was an open-ended response question with a total of 3 points possible), question 9a (which was a multiple-choice question) and 9b (which is also an open-ended response question). To make a 'cleaner' comparison, we have separately included the mean and standard deviation of part a and part b of question 9. Part a of item 9 is a multiple-choice problem with six distractors, three of which are correct choices. Per our grading scheme, students can earn a maximum of 6 points from part a, three points by selecting three correct choices and three points by not selecting incorrect choices, so chances of making a guess for correct answers are higher in 9a. We also observed that the difference in performance of TIMES and Non-TIMES students on 9a was not statistically significant with the available sample size. However, question 9b is open-ended and students can earn at most three points by providing a complete and correct explanation. Therefore, we compared question 8 with question 9b as they are naturally comparable items.

The data presented in Table 1 show that on the procedurally oriented question (Q8) the mean score of TIMES students (M = 1.98, SD = 1.24) was greater than that of Non-TIMES students (M = 1.71, SD = 1.37), but this difference of means was not statistically significant with the available sample size. Similarly, there was not a statistically significant difference in means on question 9a. However, in comparing the performance of students in both groups on question 9b (which is an open ended response style question like question 8), we noticed that TIMES students performed significantly better (M = 1.05, SD = 1.12) than the Non-TIMES students (M = 0.54, SD = 0.86). The results of the t-test indicated that this difference of means was statistically meaningful, $t(125) = 4.29$, $p < 0.001$. In this way, TIMES students outperformed Non-TIMES students on the conceptually oriented question.

Overall, students performed better on the procedurally oriented question (Q8) than the conceptually oriented question (Q9). We compared Q8 to Q9b and found

**Table 1** Summary of results of quantitative analysis

| Question | All students | TIMES students | Non-TIMES students | *p*-value (two-tailed) |
|---|---|---|---|---|
| Q8<br>3 Points | Mean: 1.85<br>SD: 1.31 | Mean: 1.98<br>SD: 1.24 | Mean: 1.71<br>SD: 1.37 | $t(125) = 1.73$<br>$p = 0.08 > 0.05$ |
| Q9 (part a only)<br>6 Points | Mean: 3.73<br>SD: 1.68 | Mean: 3.74<br>SD: 1.76 | Mean: 3.71<br>SD: 1.61 | $t(249) = 0.56$<br>$p = 0.88 > 0.05$ |
| Q9 (part b only)<br>3 Points | Mean: 0.79<br>SD: 1.03 | Mean: 1.05<br>SD: 1.12 | Mean: 0.54<br>SD: 0.86 | $t(125) = 4.29$<br>$p < 0.001$ |

that the difference of means for all students between Q8 (M = 1.85, SD = 1.31) and Q9b (M = 0.79, SD = 1.03) was also statistically meaningful with *p*-value (two-tailed) less than 0.001.

Since both problems we investigated in this study were related to eigenvectors and eigenvalues, one might think that students' performance on the two items should be correlated. However, quantitative analysis revealed a positive but weak correlation between students' performance on the two questions; the Pearson correlation coefficient was r = 0.30 for all students. Recall that a correlation coefficient measures the degree of relationship between two variables and ranges from −1 to 1, where the sign indicates the direction of the relationship and the distance from zero indicates the strength of the relationship (e.g. 1 means the two variables are highly correlated and 0 means there is very little or no correlation between the two variables). For TIMES students, the correlation between the two items was r = 0.36 as compared to the correlation for Non-TIMES which was r = 0.22. This suggests two things: first, that the two items measure *different* aspects of student understanding of eigenvalues and eigenvectors. Second, it indicates that performance on the procedurally and conceptually oriented questions was more highly correlated for TIMES students.

## 5.2 Trends in Student Reasoning on the Conceptually Oriented Question

In this section, we provide our qualitative analysis of question 8, which we consider to be the more procedurally oriented question. In particular, we highlight two common approaches to this problem: approaches that involve reasoning about the determinant, and approaches that involve reasoning about $A - \lambda I$ without computing a determinant. The majority of students who reasoned about the determinant responded correctly. Reasoning about $A - \lambda I$ was a less common approach but more frequently observed among TIMES students. Importantly, TIMES students were more often able to arrive at a correct answer by reasoning about $A - \lambda I$ than were Non-TIMES students. Further, we argue that students who reasoned about $A - \lambda I$

showed more evidence of conceptual understanding. A summary of our coding and scoring of student responses is shown in Table 2.

Reasoning about the determinant was the most common approach observed in students' responses to question 8, and students who used this kind of approach tended to do so without making conceptual errors.[1] Overall, 146 out of 255 students (57% of all students) responded to question 8 by reasoning about the determinant. We note two interesting trends within those who used this approach distinguishing TIMES from Non-TIMES students. First, more TIMES students who used determinants in their response made computational errors (usually when factoring the characteristic polynomial) than did Non-TIMES students—such errors are evidenced by 2-point responses in our coding. On the other hand, fewer TIMES students using this approach made *conceptual* errors than did Non-TIMES students —such errors are evidenced by 1-point responses in our coding. In the TIMES instructional approach (previously described under study context), the standard algorithm for finding eigenvalues and eigenvectors is intended to emerge in relation to student-invented strategies on the third of fourth day of instruction in the unit, so we conjecture Non-TIMES students may have spent more time practicing this procedure in comparison to TIMES students.

A less common approach to solve problem 8 was by reasoning about $A - \lambda I$ without computing a determinant. Overall, 48 out of 255 students (19%) used such a determinant-free approach to solve the problem. This approach was more common among TIMES students than among Non-TIMES students, and far more TIMES students successfully responded to the problem in this way without conceptual errors (evidenced by a score of 2 or 3 points in our grading scheme). Indeed, 70% (19 out of 27) of TIMES students who used this approach did so with without conceptual errors whereas only 38% (8 out of 21) Non-TIMES students who used this approach did so without conceptual errors. This indicates that more TIMES students used a determinant-free approach to solving Q8, and those who used this kind of approach did so correctly at higher rates than Non-TIMES students who used the same approach.

Students whose responses were categorized as "other" showed little or no evidence of understanding related to the definition or computation of eigenvectors and eigenvalues. We noticed that twice as many Non-TIMES students as TIMES students gave a response categorized as 'other.' However, TIMES and Non-TIMES students left the item blank at similar rates, but a larger number of Non-TIMES

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[1]We align our conceptions of conceptual and procedural errors with our definitions for conceptual and procedural understanding. We refer to an error as conceptual when there is evidence that a student does not understand an important underlying idea or relationship. We refer to an error as procedural when a student incorrectly performs a step in a mathematical process that is not central to the idea being assessed (e.g. an error in computation or algebraic manipulation). Examples of conceptual errors include incorrectly interpreting the value of the determinant to decide if something is an eigenvalue, or computing the determinant of A rather than the determinant of $A - \lambda I$. Examples of procedural errors include incorrectly factoring the characteristic polynomial or making an error when row reducing $A - \lambda I$.

**Table 2** Summary of students' approaches and scores on Q8

| Category | | TIMES, $n$ = 126 | | | | Non-TIMES, $n$ = 129 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 3 pts | 2 pts | 1 pt | 0 pts | 3 pts | 2 pts | 1 pt | 0 pts |
| Reasoning about the determinant | Solving the characteristic equation | 43 | 12 | 4 | 0 | 49 | 4 | 4 | 4 |
| | Plugging 2 into the characteristic equation | 3 | 1 | 0 | 0 | 3 | 0 | 0 | 0 |
| Total: 146 (57%)<br>TIMES: 75 (60%)<br>Non-TIMES: 71 (55%) | Finding $\det(A-2I)\neq 0$ | 10 | 2 | 0 | 0 | 5 | 2 | 0 | 0 |
| Reasoning about $A-\lambda I$ | Solving the system $(A-2I)\begin{bmatrix} x \\ y \end{bmatrix}=0$ or $Av=2v$ | 9 | 2 | 2 | 1 | 5 | 1 | 2 | 0 |
| Total: 48 (19%)<br>TIMES: 27 (21%)<br>Non-TIMES: 21 (16%) | Considering linear dependence of the set of column vectors of $(A-2I)\cong\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ | 2 | 0 | 1 | 0 | 0 | 2 | 1 | 1 |
| | Noting $rref(A-2I)$ does not have a free variable | 3 | 3 | 4 | 0 | 0 | 0 | 6 | 3 |
| Other | | 0 | 0 | 1 | 7 | 0 | 0 | 0 | 16 |
| Student indicated he/she did not know | | 0 | 0 | 0 | 5 | 0 | 0 | 0 | 9 |
| Item left blank | | 0 | 0 | 0 | 11 | 0 | 0 | 0 | 10 |

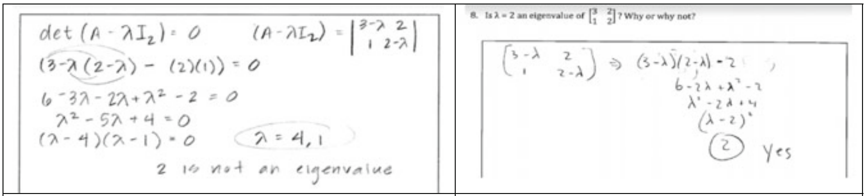
students explicitly mentioned that they "don't know" or "have no clue" how to solve this problem.

### *5.3 Examples of Student Reasoning on the Procedurally Oriented Question*

In this section we examine examples of common approaches identified in our analysis of students' responses to question 8. We provide two example responses coded as 'reasoning about the determinant' and two example responses coded as 'reasoning about $A - \lambda I$ without using the determinant.' We highlight the use of multiple representations in these responses, as well as connections between these representations and the formal definition of eigenvectors and eigenvalues. Based on these differences, we posit that responses coded as 'reasoning about $A - \lambda I$' tend to be more conceptually rich based on flexible use of multiple representations and more explicit connections between these approaches and the formal definition of eigenvectors and eigenvalues.

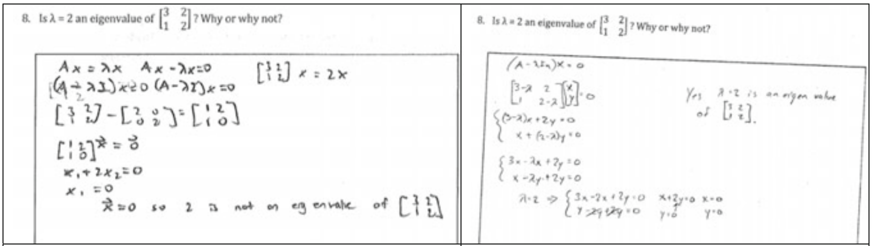
The two examples shown in Fig. 2 show typical responses to question 8 coded as "reasoning about the determinant." Response 2.a. was awarded full points because the student correctly found the roots of the characteristic polynomial, presumably noted that 2 was not one of those roots, and concluded that 2 is not an eigenvalue. The response shown in 2.b. was awarded two out of three possible points because the student made computational errors in finding the roots of the characteristic polynomial that resulted in the student concluding that two was a root of this polynomial and thus an eigenvalue. It is interesting to note that response 2.b. does not explicitly set the characteristic polynomial equal to 0 in his or her written response, but the work suggests that the student is trying to factor the polynomial in a way consistent with finding the roots.

The two examples shown in Fig. 3 show typical responses to question 8 coded as "reasoning about $A - \lambda I$ without using the determinant." We note that in response 3.a., the student began with the equation $Ax = \lambda x$, rewrote this as $Ax - \lambda x = 0$, and then factored this to write $(A - \lambda I)x = 0$. The student then computed the entries of the matrix $A - 2I$, rewrote this as a homogeneous matrix equation which he or she translated into a system of equations, correctly solved, and correctly concluded that because the solution is the zero vector that 2 **is not** an eigenvalue of the given matrix. Response 3.b. similarly considers the solution of $(A - \lambda I)x = 0$ by rewriting this matrix equation as a system of equations, substituting $\lambda = 2$ into this system, and finding the solution to this system to be when $x = 0$ and $y = 0$. However, this student incorrectly concluded from this that 2 **is** an eigenvalue. Because this is a conceptual error (thinking that finding only the trivial solution to $(A - 2I)x = 0$ means that 2 is an eigenvalue of $A$), this response was awarded only one point.




(a) Response awarded three points | (b) Response awarded two points

**Fig. 2** Responses to Q8 coded as “reasoning about the determinant”




(a) Response awarded three points | (b) Response awarded one points

**Fig. 3** Responses to Q8 coded as “reasoning about $A - \lambda I$”

In alignment with Hiebert and Lefevre’s (1986) characterization of procedural and conceptual knowledge, we claim that responses coded as reasoning about the determinant correspond to a more procedural approach to this question. We note that those who substituted 2 in the characteristic equation and those who noted that $\det(A - 2I) \neq 0$ showed some procedural flexibility indicative of conceptual aspects of their reasoning. We argue that responses coded as “reasoning about $A - \lambda I$” show more evidence of conceptual understanding of eigenvalues and eigenvectors than do responses coded as “reasoning about the determinant.” Examples of responses coded as “reasoning about $A - \lambda I$ without using the determinant” included representation of the system being solved in order to determine whether or not 2 was an eigenvalue of the given matrix, whereas the examples of responses coded as “reasoning about the determinant” typically only included representation of the computation to be executed to determine whether 2 is an eigenvalue. While this doesn’t mean these students didn’t have a conceptual understanding of eigenvalues and eigenvectors, there is not explicit evidence of this connection in their responses. In addition, both examples of responses coded as “reasoning about $A - \lambda I$” included evidence that these students could comfortably transition between matrix equations and systems of equations, a skill that has elsewhere been documented to be both difficult for students and important for their understanding (Larson & Zandieh, 2013; Selinski, Rasmussen, Wawro, & Zandieh, 2014). This can be interpreted as evidence of connectedness of ideas and representations—which others have argued to be the very definition of conceptual understanding (Vinner, 1997; Hiebert & Lafebre, 1986).

### 5.4 Trends in Student Reasoning on the Conceptually Oriented Question

We now focus on responses to question 9, the conceptually oriented question. Overall, students' responses to this item were split somewhat evenly among responses that used the eigen-concept, responses that focused on the role of the matrix $M$ without using the eigen-concept, and students who wrote that they did not know or left the answer blank. However, TIMES students' responses used the eigen-concept at much higher rates than Non-TIMES students, and with greater success. Table 3 highlights trends in the approaches of TIMES and Non-TIMES students' responses.

The most commonly observed response to Q9 involved using the eigen-concept, with 99 out of 255 (39%) total responses coded in this way. This approach was more common among TIMES students than Non-TIMES students (61/126 vs. 38/129). Further, TIMES students who used this approach gave correct responses to the question at higher rate than Non-TIMES students; the ratio of TIMES students who used the eigen-concept in fully or mostly correct ways to those who used the eigen-concept in mostly incorrect ways was 44:17 whereas that ratio for Non-TIMES students is 18:20.

The second most commonly observed trend on Q9 involved responses that focused on the role of the matrix $M$ without using the eigen-concept. We noted that students using this approach tended to be mostly or completely incorrect, and that more Non-TIMES students than TIMES students used this approach (29/126 TIMES as compared to 40/129 Non-TIMES students). We noticed that 14/29 (48%) of the TIMES students used this approach did so with some conceptual understanding but not using the eigen-concept; only 12/40 (30%) Non-TIMES students also used this approach with some conceptual understanding but not using the eigen-concept. We argue these responses indicated some conceptual understanding because they drew on appropriate transformation interpretation of a matrix times a vector. However, the understanding reflected in these responses was incomplete in that the interpretation did not explicitly use the eigen-concept by coordinating that interpretation with the result of that multiplication as corresponding to a scalar times that same vector.

There was little difference between TIMES and Non-TIMES Students who used approaches classified as 'other.' In this category, we saw no evidence of using the eigen-concept. TIMES and Non-TIMES students indicated they did not know the answer at similar rates, and more Non-TIMES students left the item blank than TIMES students.

**Table 3** Summary of students' approaches and scores on item 9

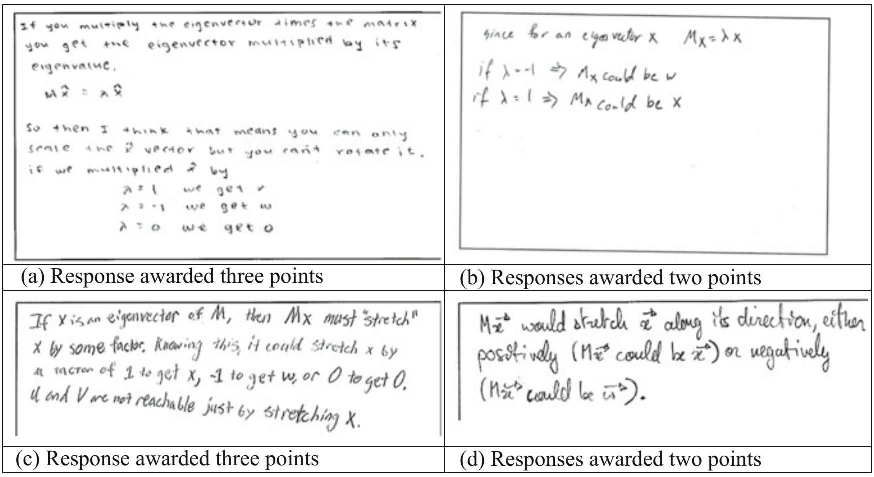
| | | TIMES, $n = 126$ | | | | Non-TIMES, $n = 129$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 3 pts | 2 pts | 1 pts | 0 pts | 3 pts | 2 pts | 1 pts | 0 pts |
| Responses using the eigen-concept<br><br>Total: 99 (39%)<br>Times: 61 (48%)<br>Non-Times: 38 (29%) | $Mx = \lambda x$ with explanations | 9 | 14 | 4 | 0 | 5 | 3 | 2 | 0 |
| | • Stretching<br>• Same direction<br>• Scalar multiple<br>• Same line as $x$ | 10 | 11 | 12 | 1 | 2 | 8 | 16 | 2 |
| Responses focusing on the role of the matrix $M$ without using the eigen-concept<br><br>Total: 69 (27%)<br>Times: 29 (22%)<br>Non-Times: 40 (31%) | Specific entries are suggested for matrix M that would yield different outputs | 0 | 0 | 12 | 8 | 0 | 0 | 11 | 15 |
| | Descriptions are given about how the matrix M transforms the input vector to yield different outputs (without suggesting specific entries of M) | 0 | 0 | 2 | 7 | 0 | 0 | 1 | 13 |
| Others | | 0 | 0 | 0 | 11 | 0 | 0 | 0 | 15 |
| I don't know | | 0 | 0 | 0 | 10 | 0 | 0 | 0 | 11 |
| Blank | | 0 | 0 | 0 | 17 | 0 | 0 | 0 | 25 |

## 5.5 *Examples of Student Reasoning on the Conceptually Oriented Question*

As with Q8, we provide examples of common approaches identified in our analysis of students' responses to question 9. Specifically, we provide four examples of responses coded as "using the eigen-concept" and two examples of responses coded as "focusing on the role of the matrix *M* without using the eigen-concept." Responses 4.a and 4.b both used the eigen-concept by writing the equation $Mx = \lambda x$ and suggesting values of $\lambda$ (e.g. 1,−1,0) that corresponded appropriately to possible outputs (Fig. 4).

Response 4.a was awarded full credit because the student linked this reasoning to all three possible outputs, whereas response 4.b was awarded just 2 out of 3 possible points due to the omission of the 0 vector as a possible output. Many students in our study who used the eigen-concept omitted the 0 vector as a possible eigenvector. We suspect this may relate to a need to distinguish the *eigenvalue* of zero from the equation $Mx = \lambda x$ having only the trivial solution when solving for the vector *x*. Responses 4.c and 4.d used the eigen-concept in a slightly different way than the previous examples. Rather than writing $Mx = \lambda x$ and suggesting appropriate values of $\lambda$, these students justified their selections of correct output vectors by describing the role of *M* as stretching the vector *x* by a factor or in its direction. Similar to the previous pair of examples, response 4.c was awarded 3 points for correctly identifying all three vectors (and even explaining that vectors *u* and *v* could not be reached by stretching *x*), whereas response 4.d was awarded just 2 points due to the omission of the 0 vector.

The next two examples presented in Fig. 5 show typical responses to question 9 that focused on the role of the matrix *M* without using the eigen-concept. Both



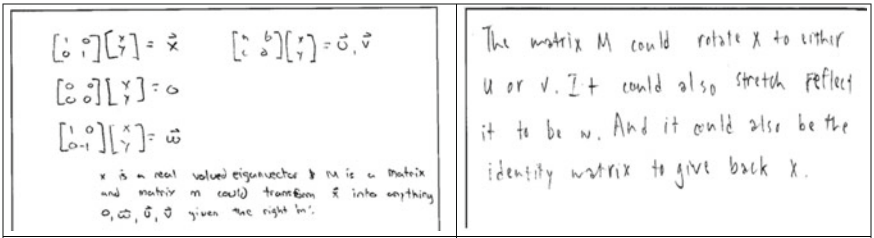

(a) Response awarded three points

(b) Responses awarded two points

(c) Response awarded three points

(d) Responses awarded two points

**Fig. 4** Responses to Q9 coded as using the eigen-concept




(a) Response awarded one point | (b) Response awarded one point

**Fig. 5** Responses to Q9 focused on the role of *M* without using the eigen-concept

responses focus on the role of the matrix $M$ as a transformation that can transform the vector $x$ in many ways (not limiting to outputs that must lie along the same line as $x$). Student 5.a's response suggests that the student sees the matrix $M$ not as a fixed matrix that transforms the eigenvector in a particular way; the student suggested different matrices that correctly produced various output vectors. The student indicated $M$ could be the identity matrix $I$ to produce $x$, $-I$ (with a sign error in one entry) to produce $w$, or the zero matrix to produce the zero vector. In addition, a matrix $M$ with generic entries was suggested as a transformation that can transform $x$ into vectors $u$ and $v$.

Response 5.b. similarly focuses on the role of the matrix $M$, arguing it could rotate $x$ to produce $u$ or $v$, "stretch reflect" to produce $w$, and that it could be the identity matrix to "give back" $x$. This combination of what the student believes the matrix could be indicates that the student did not use the eigen-concept. Responses 5.a and 5.b were both awarded 1 point because both were interpreting the matrix $M$ as a transformation and making some true statements, though in ways that did not use the eigen-concept.

We argue that interpreting matrices as transformations is an important concept that students need to make sense of eigenvectors and eigenvalues, but these responses show how that alone is not enough to ensure students are using the eigen-concept. Thinking one can choose values of the matrix $M$ is in contrast with the view that a given (fixed) matrix $M$ transforms its eigenvector $x$ in a particular way such that the resulting vector $Mx$ is a scalar multiple of $x$ and thus lies along the same line as the vector $x$. Indeed, the student whose work is shown in Fig. 5b. used the term "stretch reflect," which aligns partially with the geometric interpretation of the eigenvectors and eigenvalues concept, but the student did not limit his or her interpretation of outputs to those that appropriately correspond to eigenvectors; the student saw "stretch reflect" as just one of many possible ways the matrix $M$ could transform its eigenvector(s).

## 6 Discussion

We see this chapter contributing to the literature in three primary ways. First, we document the effectiveness of a particular instructional approach that is detailed in the literature (see Plaxco et al., 2018; Zandieh, Wawro, & Rasmusen, 2017). Second, we document aspects of students' reasoning about eigenvectors and eigenvalues (including how students draw on a transformation interpretation in ways that do and do not use the eigen-concept). Finally, we consider and discuss links between conceptual and procedural understandings of eigenvectors and eigenvalues documented in our study.

Our findings showed that both TIMES and Non-TIMES students in our study performed better on the procedurally oriented assessment question than they did on the conceptually oriented question. Further, TIMES students consistently showed evidence of more robust conceptual understanding as compared to Non-TIMES students, whereas procedural performance was similar between the two groups. This is consistent with findings of previous studies examining student learning outcomes in inquiry-oriented instructional settings at the undergraduate level (e.g., Kwon et al., 2005), though we are excited that this study was conducted on a larger scale involving instructors not involved in the development of the curricular materials. These findings are consistent with a broader body of literature documenting the benefits of student-centered approaches to learning in undergraduate mathematics (Freeman et al., 2014; Laursen Hassi, Kogan, & Weston, 2014). We conclude our paper with a discussion of the kinds of conceptual understandings observed in our analysis, and the insights these offer into what is entailed in a conceptual understanding of eigenvectors and eigenvalues.

As mentioned in our theoretical framework, conceptual understanding has been broadly defined by some in terms of the richness of connections among ideas (Hiebert & Lafevre, 1986; Vinner, 1997). More recently, Star (2005) has argued that conceptions of conceptual and procedural knowledge in mathematics education are under-articulated in a way that promotes ideological rather than empirical examination, and relationships between conceptual and procedural understandings merit greater examination. With this in mind, we now reflect on the kinds of conceptual understandings observed in our analysis, and discuss three different kinds of connections we consider to be important aspects of students' conceptual understanding of eigenvectors and eigenvalues.

First, we consider the use of appropriate interpretations of a matrix times a vector to be an important aspect of students' understanding of eigenvalues and eigenvectors. On the conceptually oriented assessment question considered in this chapter, this involved drawing on a transformation interpretation of the product of a matrix $M$ and its eigenvector $x$ consistent with the characterization given by Larson and Zandieh (2013). In our data, many students showed evidence of interpreting $Mx$, the product of a matrix $M$ and its eigenvector $x$, in ways that use the eigen-concept. A smaller number of students interpreted $Mx$ with a transformation lens, but in a way that did not use the eigen-concept in that $M$ was either thought of as a matrix that could change (to yield desired outcomes) or that the product of $M$ with the vector $x$ could be anything. This is different from a transformation interpretation that uses the eigen-concept by recognizing that the vector resulting from the multiplication by a matrix with real-valued entries $Mx$ needs to yield a vector that is a scalar multiple of $x$, or that lies on the same line as $x$, or that points in the same (or opposite) direction as $x$.

This leads to our second aspect of students' understanding of eigenvalues and eigenvectors: using the eigen-concept in the context of finding eigenvalues. While many students showed evidence of using the eigen-concept in their response to the conceptually oriented assessment item, relatively few showed evidence of using the eigen-concept on the procedurally oriented question. Indeed, one could solve our procedurally oriented assessment question by applying the standard procedure for finding eigenvalues to arrive at the correct answer without explicitly using the eigen-concept; the majority of students in both groups did just this, and most did so without error. A far smaller number of students responded to the procedurally oriented question by reasoning about $A - \lambda I$ without taking the determinant. We argue this approach provided more evidence of conceptual understanding: providing and converting between multiple representations (e.g. $Ax = \lambda x$ and $(A - \lambda I)x = 0$, written as matrix equations and systems of equations), linking those representations to the eigen-concept, and offering reasons for their conclusion in terms of a matrix equation or system of equations in their response. It is possible that a student who used the standard procedure to determine if 2 is an eigenvalue on this problem also had a deep conceptual understanding of how and why that procedure works; it is also possible that a student who used the standard procedure knew this procedure only as a sequence of steps to be executed without knowing how or why the procedure worked. Further work is needed to tease out this distinction.

This leads to the final aspect of conceptual understanding of eigenvectors and eigenvalues relevant to our analysis, which includes coordinating with the Invertible Matrix Theorem (IMT). A standard procedure for finding eigenvalues and eigenvectors draws on the argument that $Ax = \lambda x$ has a non-trivial (non-zero) solution vector $x$ for some scalar $\lambda$ if and only if the equation $(A - \lambda I)x = 0$ also has a non-trivial solution; one can argue through the IMT that this happens when $\det(A - \lambda I) = 0$. As noted above, it was often unclear from the responses of students who used the standard procedure whether they understood links among the equation $Ax = \lambda x$ used in defining eigenvectors, the solution set of $(A - \lambda I)x = 0$, and the equivalencies in the invertible matrix theorem that lead to use of the determinant as a tool for determining when the solution is non-trivial. However, among students who did not use the determinant in their response to the procedurally oriented question, there was a need to draw on equivalent ideas from the invertible matrix theorem. In these responses, we observed students noting and leveraging the following relationships:

- $(A - \lambda I)$ is invertible if and only if $(A - \lambda I)x = 0$ has a trivial solution. If $(A - \lambda I)x = 0$ has only the trivial solution, then $\lambda$ is <u>not</u> an eigenvalue of the matrix $A$.

- If the columns of $A - \lambda I$ are linearly dependent or one column is a scalar multiple of the other (in the case of a 2 × 2 matrix), then $(A - \lambda I)x = 0$ has nontrivial solution so $\lambda$ is an eigenvalue of the matrix $A$.
- If $rref(A - \lambda I)$ has no free variable then $(A - \lambda I)x = 0$ has only the trivial solution, which means $\lambda$ is not an eigenvalue of the matrix $A$.

We argue that these kinds of responses from students who did not use the previously mentioned standard procedure offer insight into conceptual connections that are both important and potentially natural for students to make as they come to make sense of standard algorithms. Students who took a procedural approach to this question typically used the determinant to decide if 2 was an eigenvalue of the matrix, without representation of the rich set of coordinations involved in these other responses, which relate interpretations of matrix equations and systems of equations, equivalencies in the Invertible Matrix Theorem, and interpretations of the eigen-concept.

Overall, students in our study correctly solved a procedural question related to eigenvalues (as in Q8) at about twice the rate they offered an appropriate conceptual understanding of $Ax = \lambda x$ (as in Q9). This suggests there is a disconnect between students' understanding of standard procedures for finding eigenvalues and the formal definition of an eigenvector and eigenvalue, and that students are more able to execute the standard procedure than draw on conceptual understandings aligned with the formal definition. If standard instructional approaches begin by introducing students to the definition of eigenvectors and eigenvalues using the equation $Ax = \lambda x$ and its algebraic and geometric interpretations but students' work is dominated by execution of procedures such as the computation of roots of the characteristic polynomial arising from $\det(A - \lambda I)$, many students may not adequately connect their results in solving these kinds of problems with the equation $Ax = \lambda x$. This points to a need to push students to think more about core understandings as they connect to procedures rather than just assess students' ability to execute standard procedures. Indeed, many connections are needed to explain why a standard procedure for finding eigenvalues and eigenvectors works and how it connects to the formal definition of eigenvalues and eigenvectors. However, we argue that there is little value in being able to compute eigenvectors and eigenvalues without being able to appropriately interpret the meaning of the result of such computations. The inquiry-oriented approach of the IOLA instructional materials taken up by instructors who received TIMES instructional supports appears to be a promising way of beginning to address this issue, but more work is needed to better understand the ways in which students come to develop and coordinate the interpretations needed for a robust understanding of eigenvectors and eigenvalues.

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## Appendix: Grading Scheme for Assigning Points to Open-Ended Response Questions 8 and 9b

| Q # | Points awarded and criteria |
|---|---|
| 8 | 3 points:<br>Method 1: Full points were awarded to students who reasoned about the determinant to arrive at the correct conclusion without making computational or conceptual errors. Examples of this kind of reasoning are shown below.<br>(i) $\det(A-\lambda I)=0$ implies $(\lambda-1)(\lambda-4)=0$ implies $\lambda=1$ or $\lambda=4$ implies $\lambda=2$ is not an eigenvalue for the matrix $A$.<br>(ii) $\det(A-2I)=-2\neq 0$ implies $\lambda=2$ is not an eigenvalue for the matrix $A$<br>(iii) $\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix}=(3-\lambda)(2-\lambda)-2=\lambda^2-5\lambda+4$. Substituting 2 in the characteristic equation gives $4-10+4=-2$ implies $\lambda=2$ is not an eigenvalue for the matrix $A$.<br>Method 2: Full points were awarded to students who reasoned about $A-\lambda I$ without using the determinant to arrive at the correct conclusion without making any computational or conceptual errors. Examples are shown below.<br>(i) $(A-2I)\begin{bmatrix} x \\ y \end{bmatrix}=0$ implies $x=0$ and $y=0$ which is the trivial solution, so $\lambda=2$ is not an eigenvector for the matrix $A$.<br>(ii) $(A-2I)\cong\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, and the column vectors of this matrix are not linearly dependent, so $\lambda=2$ is not an eigenvalue.<br>(iii) $rref(A-2I)$ does not have a free variable, so $\lambda=2$ is not an eigenvalue.<br>(iv) The first column of $(A-2I)$ is not a scalar multiple of the second column so $\lambda=2$ is not an eigenvalue so $\lambda=2$ is not an eigenvalue. |
| | 2 points: Two points were awarded to students to students who take a conceptually correct approach (either by reasoning about the determinant or by reasoning about $A-\lambda I$ without using the determinant) but either<br>• made a computational error (e.g. factoring the characteristic polynomial incorrectly) or<br>• did not offer a clear conclusion about whether 2 is an eigenvalue or not, or<br>• arrived at the correct conclusion without a full explanation of why |
| | 1 point: One point was awarded to students whose response included some evidence of conceptual understanding, but who made a conceptual error (which might be accompanied by a computational error). |
| | 0 points: No points were awarded to students who left the page blank, or whose response: (i) gave no evidence of conceptual understanding, or (ii) said something like "I don't know." Example of responses we considered to include no evidence of conceptual understanding are "Yes, because $A = PDP^{-1}$" and "I say it is… because… there are 2's in the problem." |

(continued)

(continued)

| Q # | Points awarded and criteria |
|---|---|
| 9b | 3 points: Three points were awarded to students whose response appropriately coordinated with the eigen-concept, referenced (either by directly naming or by explicitly referring to their work shown in 9a) all three correct vectors, and provided a correct rationale for this selection. |
| | 2 points: Two points were awarded to students whose response provided at least two correct explanations (e.g. $Mx = \lambda x$ is written and student writes that "an eigenvector tells you the direction of stretching") but did not identify and explicitly describe what happens to all three correct vectors. |
| | 1 point: One point was awarded to students who either<br>(i) Provided one correct explanation (e.g. by either writing "$Mx = \lambda x$" or "an eigenvector tells you the direction of stretching") and explicitly connected this explanation to at most one correctly selected vector<br>(ii) Suggested components of $M$ that would transform x into one of the given choices, such as $M = I,\ -I,\ or\ 0$. |
| | 0 point: No points were awarded to responses that do not coordinate with the eigen-concept, do not suggest components of $M$ that would transform x into one of the given choices, says I don't know, or leaves the page blank. An example of student response to question 9 which was awarded 0 point was "all are the same size." |

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