Stochastic Control for Insurance: New Problems and Methods

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Abstract Stochastic control for insurance is concerned with problems in insurance models (jump processes) and for insurance applications (constraints from supervision and market). This leads to questions of the following type:

1. How to find numerically a viscosity solution to an integro differential equation;
2. Uniqueness of viscosity solutions when boundary conditions are values of derivatives; and
3. How to solve control problems with the two objectives: dividends and ruin.

We shall present simple Euler schemes (similar to the ones in Fleming–Soner (Controlled Markov Processes and Viscosity Solutions. Stochastic Modelling and Applied Probability. Springer, New York, 2006), Chap. IX) which converge when the value function has a continuous first derivative. This method works in many univariate control problems also when value functions are without continuous second (and first) derivative. Cases with non-smooth value function arise when constraints are restrictive. Furthermore, we consider the infinite horizon problem: maximize dividend payment and minimize ruin probability. This problem is described and solved with a non-stationary approach in the classical Lundberg model.

Keywords Stochastic control • Viscosity solutions • Euler type discretisations • Multi objective problem

AMS Keys Primary 91B30, 93E20; Secondary 49l20, 49L25, 49M25

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J.A. Londoño et al. (eds.), *Actuarial Sciences and Quantitative Finance*, Springer Proceedings in Mathematics & Statistics 214, https://doi.org/10.1007/978-3-319-66536-8_5

1 Introduction

Stochastic control in finance started more than 40 years ago with Robert Merton's papers *Lifetime portfolio selection under uncertainty: the continuous-time case* (Merton 1969) and *Optimum consumption and portfolio rules in a continuous-time model* (Merton 1971), paving the ground for the famous option pricing articles by Robert Merton *Theory of rational option pricing* (Merton 1973) as well as Fischer Black and Myron Scholes *The pricing of options and corporate liabilities* (Black and Scholes 1973). By now, this is a well-established field with standard textbook such as Fleming-Rishel *Deterministic and Stochastic Optimal Control* (Fleming and Rishel 1975) and Fleming-Soner *Controlled Markov Processes and Viscosity Solutions* (Fleming and Soner 2006), as well as Merton *Continuous Finance* (Merton 1992). I would also mention Karatzas-Shreve *Methods of Mathematical Finance* (Karatzas and Shreve 1998) and the work of Bert Øksendal (2005) and Jerome Stein (2012), and this list is still far from being complete.

Surprisingly, the development of stochastic control in insurance took much longer, although the idea was present already in 1967. Karl Borch (NHH Bergen, Norway) wrote in his *The theory of risk* (Borch 1967, p. 451):

The theory of control processes seems to be tailor made for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed if actuaries and engineers had realized that they were studying the same problems and joined forces over 50 years ago. A little reflection should teach us that a highly specialized problem may, when given the proper mathematical formulation, be identical to a series of other, seemingly unrelated problems.

As the beginning of stochastic control in insurance one might choose the year 1995 in which Sid Browne's paper *Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin* (Browne 1995) appeared. Since then, this field is very active, and its group of researchers is still growing. A first monograph is Hanspeter Schmidli's book *Stochastic Control in Insurance* (Schmidli 2007) in which an extended list of references contains also earlier work. New books were written recently by Pham (2009) and Azcue and Muler (2014).

Stochastic control in insurance is concerned with control of investment, reinsurance, exposure, and product design. An objective is often the ruin probability which is a dynamic risk measure used in internal models. Minimizing ruin probability results in the reduction of solvency capital, so optimal strategies have also an economic impact. These strategies can be used in scenario generators for management decisions.

Ruin probabilities are not satisfactory when they lead to the decision to stop insurance business (which might happen in reinsurance control with interest on the reserves, or in the control of exposure). Alternatively, one can maximize the value of the company, which is the expected sum of discounted dividends. This objective is more complex, since the company value is itself the result of a control problem: one uses optimal dividend payment.

Company values have the drawback of certain ruin: if an insurer pays dividends to maximize the company value, then the with dividend ruin probability equals 1, no matter how large the initial surplus is. As an alternative we investigate a company value which has a constrained ruin probability. In this setup, only those dividend payments are allowed which lead to a given with dividend ruin probability. This quantity is even more complex since its computation involves a control problem with two objectives, its solution is work in progress.

In this paper we first consider control of constraint investment (such as no short-selling and/or no leverage) to minimize the ruin probability and present the numerical methods for the value function and optimal strategy. Next, concepts and numerical methods are presented for the computation of a company value which has a constrained ruin probability.

2 Optimal Investment for Insurers

Investment of a fixed proportion of the surplus leads to a substantial increase in ruin probability (see Kalashnikov and Norberg 2002). Optimal investment control with unconstrained investment was first given in Hipp and Plum (2000, 2003) for the classical Lundberg model. The risk process at time t is given by

$$S(t) = s + ct - X_1 - \ldots - X_{N(t)},$$

where s is the initial surplus, c the premium intensity, and $X, X_1, X_2, \ldots$ are independent identically distributed claim sizes which are independent of the claims arrival process $N(t)$ being modeled as a homogeneous Poisson process with claim frequency λ. The dynamics of the asset for investment is logarithmic Brownian motion

$$dZ(t) = \mu Z(t)dt + \sigma Z(t)dW(t), t \geq 0,$$

with a standard Wiener process independent of $S(t), t \geq 0$, and constants $\mu, \sigma > 0$. The classical Hamilton-Jacobi-Bellman equation for the minimal ruin probability $V(s)$ reads

$$0 = \inf_A\{\lambda E[V(s - X) - V(s)] + (c + A\mu)V'(s) + A^2\sigma^2 V''(s)/2\},\ s \geq 0, \quad (1)$$

where the infimum is taken over all real A representing the amount invested at surplus s. The minimizer $A(s)$ defines an optimal investment strategy given in feedback form: invest the amount $A(s)$ when surplus is s. When X has a continuous density, Eq. (1) has classical bounded solutions, and the unique solution $V(s)$ with $V(\infty) = 0, V'(0) = \lambda(V(0) - 1)/c$ is the minimal ruin probability. The optimizer $A(s)$ converges to zero for $s \to 0$ at the rate $\sqrt{s}$ which leads to $A(s)/s \to \infty$, and this shows that the corresponding investment strategy has unlimited leverage.

Since unlimited leverage strategies are not admissible for insurers, we have to restrict the set of investment strategies for each surplus $s : A \in \mathcal{A}(s)$. Possible restrictions are $\mathcal{A}(s) = (-\infty, s]$ for no leverage, $\mathcal{A}(s) = [0, \infty)$ for no short-selling, or $\mathcal{A}(s) = [0, s]$ for neither leverage nor short-selling. Such constraints change the nature of the control problem: the constraint $\mathcal{A}(s) = (-\infty, s]$ results in a control problem having no solution; other constraints yield a Hamilton-Jacobi-Bellman equation

$$0 = \sup_{A \in \mathcal{A}(s)} \{\lambda E[V(s - X) - V(s)] + (c + A\mu)V'(s) + A^2\sigma^2 V''(s)/2\},\ s \geq 0, \tag{2}$$

which does not have a solution with (continuous) second derivative.

For this situation one can use the concept of viscosity solutions described in Fleming and Soner (2006). This concept is tailor made for risk processes which are diffusions (in which ruin probabilities at zero are 1), or for dividend maximization. For ruin probabilities in Lundberg models it has to be modified: instead of two fixed values ($V(0) = 1, V(\infty) = 0$) we have boundary conditions on $V(\infty)$ and $V'(0)$. But also for this situation, the Crandall-Ishii comparison argument is valid under the additional hypothesis that the viscosity solutions to be compared have a continuous first derivative (see Hipp 2015).

The numerical solution of Eq. (2) is done with Euler type discretisations. For a step size Δ we define the function $V_\Delta(s)$ as the solution of the discretised equation, with discretised derivatives

$$V'_\Delta(s) = (V_\Delta(s) - V_\Delta(s - \Delta))/\Delta,$$

$$V''_\Delta(s) = (V'_\Delta(s + \Delta) - V'_\Delta(s))/\Delta,$$

and also the integral is discretised: for $s = k\Delta$ it is approximated by

$$G_\Delta(s) = \sum_{i=1}^{k} V_\Delta(s - i\Delta)\mathbb{P}\{(i-1)\Delta \leq X < i\Delta\}.$$

Example 1 Let $\mathcal{A}(s) = [-bs, as]$ with $a = 1, b = 60$. The other parameters are $c = 3.5, \lambda = 1, \mu = 0.3, \sigma = 1$, and the claim size has an Erlang(2) distribution with density $x \to xe^{-x}, x > 0$. Here, the supremum in (2) can be either the unrestricted maximum $A(s) = -\mu V'(s)/(\sigma^2 V''(s))$ or one of the two values $-bs, as$, whichever produces the smaller value for $V'(s)$. For a possible maximizer A in (2), $A \in \{-bs, as\}$, we have

$$V'_\Delta(s) = \frac{\lambda(V_\Delta(s) - G_\Delta(s)) - 0.5A^2\sigma^2 V'_\Delta(s - \Delta)}{(c + A\mu)\Delta + 0.5A^2\sigma^2}, \tag{3}$$

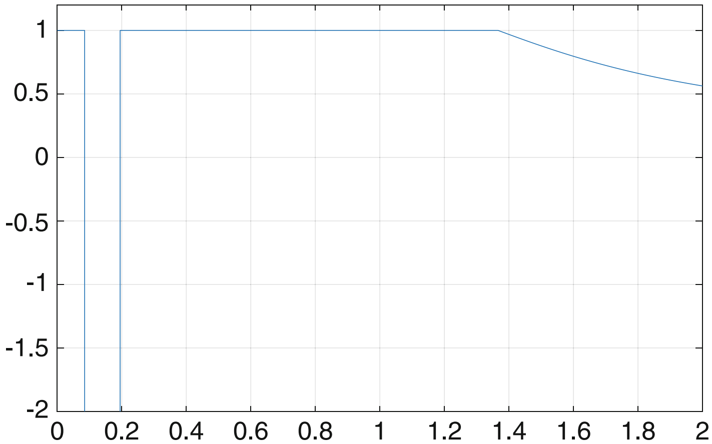

Fig. 1 Optimal proportion $A(s)/s$ invested

and for the unrestricted maximizer we obtain the following quadratic equation with $H(s) = \lambda(G_\Delta(s) - V_\Delta(s))$:

$$V'_\Delta(s)^2(c + 0.5\Delta) = V'_\Delta(s)(H(s) - cV'(s - \Delta)) - H(s)V'(s - \Delta). \tag{4}$$

We see (Fig. 1) that the optimal investment strategy jumps from the maximal admissible long position s to the maximal short position $-60s$, and back. In contrast to the Example 5.1 in Belkina et al. (2012) we have a positive safety loading ($c > \lambda E[X]$), and interest zero.

Using equicontinuity of $V'_\Delta(s)$ one can show—as in Chap. IX of Fleming and Soner (2006)—that these discretisations converge (see Edalati and Hipp 2013 and Hipp 2015; for the case of Example 1, see also Belkina et al. 2012).

Numerical experiments show that the Euler type discretisations seem to converge also in cases in which the regularity conditions for the mentioned proofs are not satisfied. In the following example the claim size distribution is purely discrete.

Example 2 We consider claims X of size 1, and $\lambda = \mu = \sigma = 1, c = 2$. The above recursions lead to the two optimal amounts invested $A(s)$ for the unconstrained case ($\mathcal{A}(s) = (-\infty, \infty)$, dashed line) and the case without leverage and short-selling ($\mathcal{A}(s) = [0, s]$, solid line). Notice that $A(1) = 0$ in both cases (Fig. 2).

3 Dividend Payment with Ruin Constraint

For the risk process, we again use a classical Lundberg model. If $D(t)$ is the accumulated dividend stream of some admissible dividend payment strategy, then the dividend value for a discount rate $\delta > 0$ is

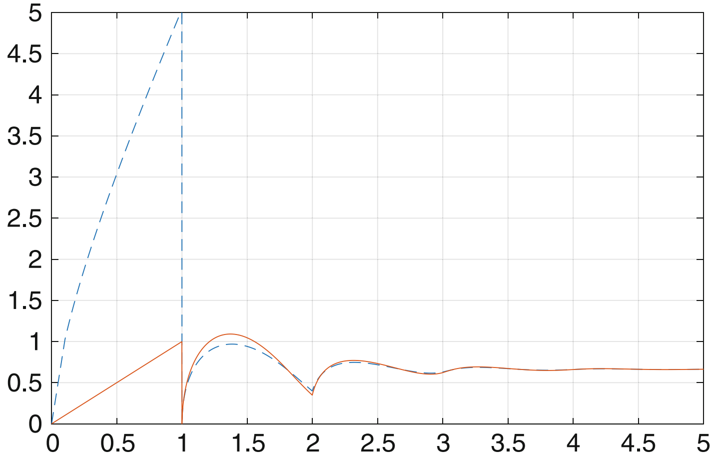

Fig. 2 Optimal amount invested $A(s)$ for $0 \leq s \leq 5$

$$V^D(s) = E\left[\int_0^\infty e^{-\delta t} dD(t)|S(0) = s\right].$$

Here we tacitly assume that no dividends are paid at or after ruin. The value of the company (without ruin constraint) is

$$V_0(s) = \sup_D V^D(s),$$

where the supremum is taken over all admissible dividend payment strategies. For many popular claim size distributions, the optimal dividend payment strategy is a barrier strategy (see Loeffen 2008) with barrier M, say. The computation of $V_0(s)$ and M is based on the dynamic equation

$$0 = \delta V_0(s) + \mathcal{G}V_0(s), \tag{5}$$

where

$$\mathcal{G}f(s) = \lambda E[f(s - X) - f(s)] + cf'(s)$$

is the infinitesimal generator of the Lundberg model. If $v(s)$ is the solution to (5) with $v(0) = v'(0) = 1$, then

$$M = \arg\min v'(s)$$

and

$$V_0(s) = v(s)/v'(M),\ s \leq M,\ V_0(s) = V_0(M) + s - M,\ s \geq M.$$

The company value with ruin constraint is

$$V(s,\alpha) = \sup_D [V^D(s) : \psi^D(s) \le \alpha],$$

where $\psi^D(s)$ is the ruin probability of the with dividend process $S^D(t)$ for initial surplus s. The corresponding problem with Lagrange multiplier L is

$$V(s,L) = \sup_D [V^D(s) - L\psi^D(s)].$$

These two concepts are not equivalent: we have

$$V(s,L) = \sup\{V(s,\alpha) - L\alpha : 0 \le \alpha \le 1\},$$

but it might happen that we have $\alpha_1 < \alpha_2$ for which

$$V(s,L) = V(s,\alpha_1) - L\alpha_1 = V(s,\alpha_2) - L\alpha_2,$$

and then for $\alpha_1 < \alpha < \alpha_2$ we cannot find any L for which

$$V(s,L) = V(s,\alpha) - L\alpha.$$

This situation is called *Lagrange gap.*

We will compute $V(s,L)$ and the corresponding with dividend ruin probability α. This way, we also obtain $V(s,\alpha)$, at least for such values of α. The computation is based on a non-stationary approach: for time t we consider dividends payment and ruin after time t, where dividends are discounted to time 0:

$$W(s,t) = \sup_D \left[E\left[\int_t^\infty e^{-\delta u} dD(u) | S(t) = s \right] - L\mathbb{P}\{\inf_{u \ge t} S^D(u) < 0 | S(t) = s\} \right].$$

The functions $W(s,t)$ satisfy the dynamic equation

$$0 = W_t(s,t) + \mathcal{G}W(s,t), s,t \ge 0. \tag{6}$$

This is the dynamic equation (5) where the term for discounting is replaced by a term for time dependence in a non-stationary model. Then $W(s,\infty) = -L\psi(s)$ leads to the following approximation: for large T we let $W(s,T) = -L\psi(s)$, and then we calculate backward the functions $W(s,t)$ to use $W(s,0)$ as an approximation for $V(s,L)$. The following numerical example has exponential claims with mean 1, premium rate $c = 2$, discount rate $\delta = 0.03$ and claim frequency $\lambda = 1$. The calculation is based on the following recursion of discretisations:

$$W_\Delta(s,t-dt) = W_\Delta(s,t) + dt\mathcal{G}W_\Delta(s,t),$$

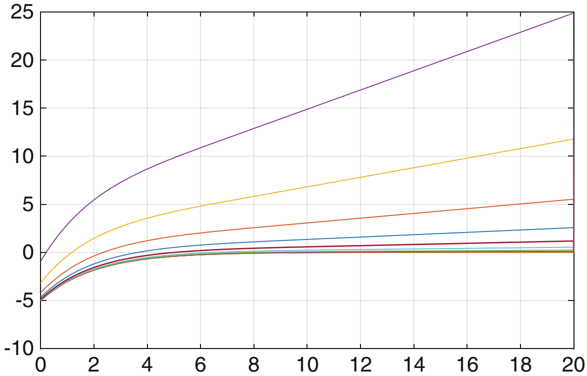

Fig. 3 The functions $W(s,t)$ for $0 \le s \le 20$

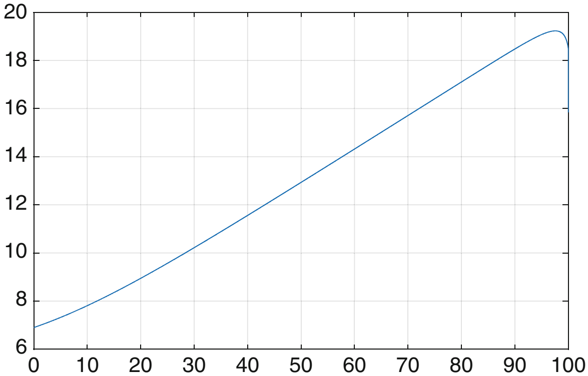

Fig. 4 The barrier $M(t)$ for $0 \le t \le 100$

where in $\mathcal{G}W_\Delta(s,t)$ we use difference ratios instead of derivatives, as in the Euler type approach above. The time dependent barrier $M(t)$ is defined as the first value s at which

$$(W_\Delta(s,t-dt) - W_\Delta(s-ds,t-dt))/\Delta < e^{-\delta t},$$

and $W_\Delta(s,t-dt)$ is linear on $[M(t),\infty)$ with slope $e^{-\delta t}$.

The following figures are calculated with $T = 100,\ ds = 0.01$ and $dt = 0.001$.

The function $M(t)$ should be increasing. It drops in Fig. 2 close to T, but this is a typical artefact caused by the definition of $W(s,T)$ (Fig. 4).

This shows that dividend values with a ruin constraint can be computed numerically. As a next step one should use this objective for the control of reinsurance and (constrained) investment.

Our last figure shows the efficiency curve for dividend values and ruin probabilities. It is computed with $s = 5,\ T = 300,\ ds = 0.01,\ dt = 0.001$ and $0 \le L \le 600$. The value without ruin constraint is $V_0(s) = 12.669$. One would conclude that there is no Lagrange gap here (Fig. 5).

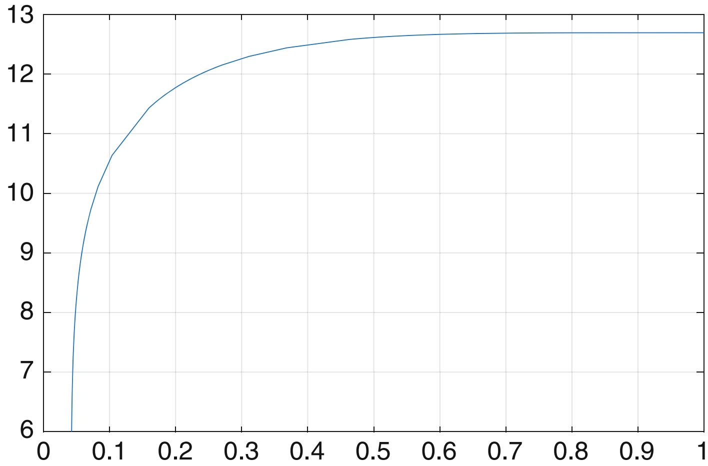

Fig. 5 $V(s, L)$ against the corresponding α-values

4 Conclusions and Future Work

For stochastic control in insurance the classical Hamilton-Jacobi-Bellman equations are still useful for infinite horizon problems; a finite horizon view is not appropriate here since insurance uses diversification in time. Viscosity solutions of these equations can be derived with simple Euler schemes, they converge under weak assumptions. This solves problems in which the ruin probability is minimized or the company value is maximized. An objective function connecting these two opposite views is a company value with a ruin constraint. For the computation of this quantity, a Lagrange approach and an appropriate discretisation are given leading to dividend strategies with a barrier which increases with time. For this given barrier, the corresponding ruin probability can be computed as well.

When the concept of a company value with ruin constraint is well understood (concerning algorithms and proof of convergence), one could and should start optimizing this objective by control of investment, reinsurance, and other control variables in insurance. If the above non-stationary approach is used, the numerics for non-linear partial differential equations will be of major importance.

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