Intuitionistic Fuzzy Rough Set Based on the Cut Sets of Intuitionistic Fuzzy Set

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Abstract. In this paper, the triple valued fuzzy set is selected as the cut set of intuitionistic fuzzy set and four kinds of cut sets of intuitionistic fuzzy sets are selected to investigate the intuitionistic fuzzy rough set. The intuitionistic fuzzy rough set is constructed by using the representation theorem of intuitionistic fuzzy set, and it is proved to be equivalent to the original intuitionistic fuzzy rough set. These works give a new perspective on intuitionistic fuzzy rough sets, which promotes the further research and development of the theory of intuitionistic fuzzy sets and rough sets.

Keywords: Fuzzy rough set · Intuitionistic fuzzy set · Cut set · Three valued fuzzy set · Rough set

1 Introduction

Atanassov expanded the fuzzy set [1] and put forward the concept of intuitionistic fuzzy sets [2] in 1986. Intuitionistic fuzzy set takes membership and non-membership into consideration, which makes it more accurate and effective in dealing with uncertainty. Rough set theory proposed by Professor Pawlak [3] in 1982 is a mathematical tool to deal with imprecise, inconsistent, incomplete information and knowledge. In 1998, Chakrabarty proposed the theory of intuitionistic fuzzy rough sets [4], which is the generalization of the concept of rough sets of Pawlak. At present, the combination of intuitionistic fuzzy sets and rough sets is a hot research topic [7–12]. In 2009, based on intuitionistic fuzzy implication operator and intuitionistic fuzzy model, L. Zhou and W.Z. Wu established the theory framework of intuitionistic fuzzy rough set by using constructive method and axiomatic method [6]. In 2011, Thomas studied Rough intuitionistic fuzzy sets in a lattice [13].

In this paper, we select triple valued fuzzy sets [5] as the cut sets of intuitionistic fuzzy sets. Taking the triple valued fuzzy sets as the theoretical basis, we use the way 'cut set—operate—bond together' to construct the intuitionistic fuzzy rough sets. Comparing with the original theory, we use the representation theorem for two cut sets, triple fuzzy set and triple fuzzy equivalence relation.

This paper is structured as follows: In the second section, the preparation knowledge required in this paper is introduced. In the third and fourth section, we provide four kinds of methods to investigate the upper and lower approximation of intuitionistic fuzzy sets by using the representation theorem.

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2 Preliminary

Definition 2.1 [1]. Let X be a set. The mapping $A : X \to [0, 1]$ is called a fuzzy subset of X.

Definition 2.2 [2]. Let X be a set. If the two functions, $\mu_A : X \to [0, 1], \nu_A : X \to [0, 1]$ satisfy $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, we call $A = (X, \mu_A, \nu_A)$ is an intuitionistic fuzzy set of X, which is denoted as $A(x) = (\mu_A(x), \nu_A(x))$. Then we have the following operations:

$$\begin{aligned} &A \subset B \Leftrightarrow (\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x)), \forall x \in X; \\ &A = B \Leftrightarrow (\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x)), \forall x \in X; \\ &A^c = <X, \nu_A, \mu_A> . \end{aligned}$$

Definition 2.3 [5]. Let $3^X = \{A | A : X \to \{0, \frac{1}{2}, 1\} \text{ is a mapping}\}$. If $A_\lambda, A_{\underline{\lambda}} \in 3^X$ and

$$A_\lambda(x) = \begin{cases} 1, & \mu_A(x) \geq \lambda; \\ \frac{1}{2}, & \mu_A(x) < \lambda \leq 1 - \nu_A(x); \\ 0, & \lambda > 1 - \nu_A(x), \end{cases} \quad A_{\underline{\lambda}}(x) = \begin{cases} 1, & \mu_A(x) > \lambda; \\ \frac{1}{2}, & \mu_A(x) \leq \lambda < 1 - \nu_A(x); \\ 0, & \lambda \geq 1 - \nu_A(x). \end{cases}$$

We call A_λ and $A_{\underline{\lambda}}$ the λ – upper cut set and λ – strong upper cut set of A.

If $A^\lambda, A^{\underline{\lambda}} \in 3^X$ and

$$A^\lambda(x) = \begin{cases} 1, & \nu_A(x) \geq \lambda; \\ \frac{1}{2}, & \nu_A(x) < \lambda \leq 1 - \mu_A(x); \\ 0, & \lambda > 1 - \mu_A(x), \end{cases} \quad A^{\underline{\lambda}}(x) = \begin{cases} 1, & \nu_A(x) > \lambda; \\ \frac{1}{2}, & \nu_A(x) \leq \lambda < 1 - \mu_A(x); \\ 0, & \lambda \geq 1 - \mu_A(x). \end{cases}$$

We call A^λ and $A^{\underline{\lambda}}$ the $\lambda -$ lower cut set and $\lambda -$ strong lower cut set of A.

If $A_{[\lambda]}, A_{[\underline{\lambda}]} \in 3^X$ and

$$A_{[\lambda]}(x) = \begin{cases} 1, & \mu_A(x) + \lambda \geq 1; \\ \frac{1}{2}, & \nu_A(x) \leq \lambda < 1 - \mu_A(x); \\ 0, & \lambda < \nu_A(x), \end{cases} \quad A_{[\underline{\lambda}]}(x) = \begin{cases} 1, & \mu_A(x) + \lambda > 1; \\ \frac{1}{2}, & \nu_A(x) < \lambda \leq 1 - \mu_A(x); \\ 0, & \lambda \leq \nu_A(x). \end{cases}$$

We call $A_{[\lambda]}$ and $A_{[\underline{\lambda}]}$ the $\lambda -$ upper quasi cut set and $\lambda -$ strong quasi upper cut set of A.

If $A^{[\lambda]}, A^{[\underline{\lambda}]} \in 3^X$ and

$$A^{[\lambda]}(x) = \begin{cases} 1, & \nu_A(x) + \lambda \geq 1; \\ \frac{1}{2}, & \mu_A(x) \leq \lambda < 1 - \nu_A(x); \\ 0, & \lambda < \mu_A(x), \end{cases} \quad A^{[\lambda]}(x) = \begin{cases} 1, & \nu_A(x) + \lambda \geq 1; \\ \frac{1}{2}, & \mu_A(x) < \lambda \leq 1 - \nu_A(x); \\ 0, & \lambda \leq \mu_A(x), \end{cases}$$

We call $A^{[\lambda]}$ and $A^{[\underline{\lambda}]}$ the $\lambda -$ lower quasi cut set and $\lambda -$ strong quasi lower cut set of A.

Definition 2.4 [5]. Let $A \in 3^X, \lambda \in [0,1]$ and $f_i : [0,1] \times 3^X \to L^X, (\lambda, A) \mapsto f_i(\lambda, A)$ be the following mappings $(i = 1, 2, \ldots, 8)$:

$$f_1(\lambda,A)(x) = \begin{cases} (0,1), & A(x) = 0; \\ (\lambda, 1-\lambda), & A(x) = 1; \\ (0, 1-\lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_2(\lambda,A)(x) = \begin{cases} (\lambda, 1-\lambda), & A(x) = 0; \\ (1,0), & A(x) = 1; \\ (\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

$$f_3(\lambda,A)(x) = \begin{cases} (1-\lambda,\lambda), & A(x) = 0; \\ (0,1), & A(x) = 1; \\ (0, \lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_4(\lambda,A)(x) = \begin{cases} (1,0), & A(x) = 0; \\ (1-\lambda,\lambda), & A(x) = 1; \\ (1-\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

$$f_5(\lambda,A)(x) = \begin{cases} (0,1), & A(x) = 0; \\ (1-\lambda,\lambda), & A(x) = 1; \\ (0, \lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_6(\lambda,A)(x) = \begin{cases} (1-\lambda,\lambda), & A(x) = 0; \\ (1,0), & A(x) = 1; \\ (1-\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

$$f_7(\lambda,A)(x) = \begin{cases} (\lambda, 1-\lambda), & A(x) = 0; \\ (0,1), & A(x) = 1; \\ (0, 1-\lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_8(\lambda,A)(x) = \begin{cases} (1,0), & A(x) = 0; \\ (\lambda, 1-\lambda), & A(x) = 1; \\ (\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

Definition 2.5 [7]. Let U be a set, R is an equivalence relation in U. That is $R \subset U \times U$, for $\forall x, y, z \in U$ satisfying:

(1) $(x, x) \in R$;
(2) $(x, y) \in R \Rightarrow (y, x) \in R$;
(3) $(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$.

Definition 2.6 [3]. Let $X \subset U$ and R be an equivalence relation in U. We call
$\overline{R}(X) = \cup\{x | x \in U, [x] \cap X \neq \emptyset\}$ the upper approximation of X
$\underline{R}(X) = \cup\{x | x \in U, [x] \subset X\}$ the lower approximation of X.

Definition 2.7 [5]. Let $H : [0,1] \to 3^X$ be a mapping. If $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supset H(\lambda_2)$, we call H a triple valued inverse order nested sets on X.

Let $H : [0,1] \to 3^X$ be a mapping. If $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subset H(\lambda_2)$, we call H a triple valued order nested sets on X.

3 The Upper Approximation of Intuitionistic Fuzzy Set

First of all, we give a kind of cut set and cut relations to construct the upper approximation of intuitionistic fuzzy sets.

Let U be a finite non empty set, $X = (\mu_X, v_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, v_R)$ is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then X_λ is the triple valued fuzzy set over U, and R_λ is the triple valued fuzzy equivalence relation over U. Let $\overline{H_1}(\lambda) = \overline{R_\lambda}(X_\lambda)$, that is for $x \in U, \overline{H_1}(\lambda)(x) = \vee_{y \in U}(X_\lambda(y) \wedge R_\lambda(x, y))$, then $\overline{H_1}(\lambda)$ is the triple valued inverse order nested sets on U. *Let* $\overline{R}(X) = \bigcup_{\lambda \in [0,1]} f_1(\lambda, \overline{H_1}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_2(\lambda, \overline{H_1}(\lambda))$. Then we get the following theorem:

Theorem 3.1. Let $X \in IF(U), x \in U$, then

$$\overline{R}(X)(x) = (\underset{y\in U}{\vee}(\mu_X(y) \wedge \mu_R(x,y)), \underset{y\in U}{\wedge}(v_X(y) \vee v_R(x,y))). \quad (1)$$

Proof

$$\overline{R}(X)(x) = \underset{\lambda\in[0,1]}{\vee} f_1(\lambda, \overline{H_1}(\lambda))(x)$$

(1) $$= \{\vee(\lambda, 1-\lambda) \big| \overline{H_1}(\lambda)(x) = 1\} \vee \{\vee(0, 1-\lambda) \big| \overline{H_1}(\lambda)(x) = \frac{1}{2}\}$$

$$= (\vee\{\lambda \big| \overline{H_1}(\lambda)(x) = 1\}, \wedge\{1-\lambda \big| \overline{H_1}(\lambda)(x) \geq \frac{1}{2}\})$$

Because

$$\begin{aligned}\overline{H_1}(\lambda)(x) = 1 &\Leftrightarrow \underset{y\in U}{\vee}(X_\lambda(y) \wedge R_\lambda(x,y)) = 1\\ &\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \wedge R_\lambda(x,y_0) = 1)\\ &\Leftrightarrow \exists y_0 \in U(\mu_X(y_0) \wedge \mu_R(x,y_0) \geq \lambda)\\ &\Leftrightarrow \underset{y\in U}{\vee}(\mu_X(y) \wedge \mu_R(x,y)) \geq \lambda\end{aligned}$$

Thus $\vee\{\lambda \big| \overline{H_1}(\lambda)(x) = 1\} = \underset{y\in U}{\vee}(\mu_X(y) \wedge \mu_R(x,y))$.

And

$$\begin{aligned}\overline{H_1}(\lambda)(x) \geq \frac{1}{2} &\Leftrightarrow \underset{y\in U}{\vee}(X_\lambda(y) \wedge R_\lambda(x,y)) \geq \frac{1}{2}\\ &\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \wedge R_\lambda(x,y_0) \geq \frac{1}{2})\\ &\Leftrightarrow \exists y_0 \in U(1 - v_X(y_0) \geq \lambda \text{ and } 1 - v_R(x,y_0) \geq \lambda)\\ &\Leftrightarrow \exists y_0 \in U(v_A(y_0) \vee v_R(x,y_0) \leq 1-\lambda)\\ &\Leftrightarrow \underset{y\in U}{\wedge}(v_X(y) \vee v_R(x,y)) \leq 1-\lambda\end{aligned}$$

Thus $\wedge\{1-\lambda \big| \underline{H_1}(\lambda)(x) \geq \frac{1}{2}\} = \underset{y\in U}{\wedge}(v_X(y) \vee v_R(x,y))$.

Therefore $\overline{R}(X)(x) = (\underset{y\in U}{\vee}(\mu_X(y) \wedge \mu_R(x,y)), \underset{y\in U}{\wedge}(v_X(y) \vee v_R(x,y)))$.

$$\overline{R}(X)(x) = \underset{\lambda\in[0,1]}{\wedge} f_2(\lambda, \overline{H_1}(\lambda))(x)$$

(2) $$= \{\wedge(\lambda, 1-\lambda) \big| \overline{H_1}(\lambda)(x) = 0\} \wedge \{\wedge(\lambda, 0) \big| \overline{H_1}(\lambda)(x) = \frac{1}{2}\}$$

$$= (\wedge\{\lambda \big| \overline{H_1}(\lambda)(x) \leq \frac{1}{2}\}, \vee\{1-\lambda \big| \overline{H_1}(\lambda)(x) = 0\})$$

Because

$$\overline{H_1}(\lambda)(x) \le \frac{1}{2} \Leftrightarrow \bigvee_{y\in U}(X_\lambda(y) \wedge R_\lambda(x,y)) \le \frac{1}{2}$$
$$\Leftrightarrow \forall y \in U(X_\lambda(y) \wedge R_\lambda(x,y) \le \frac{1}{2})$$
$$\Leftrightarrow \forall y \in U(\mu_X(y) \wedge \mu_R(x,y) < \lambda)$$
$$\Leftrightarrow \bigvee_{y\in U}(\mu_X(y) \wedge \mu_R(x,y)) < \lambda$$

Thus $\wedge\{\lambda | \overline{H_1}(\lambda)(x) \le \frac{1}{2}\} = \bigvee_{y\in U}(\mu_R(y) \wedge \mu_R(x,y))$.

And

$$\overline{H_1}(\lambda)(x) = 0 \Leftrightarrow \bigvee_{y\in U}(X_\lambda(y) \wedge R_\lambda(x,y)) = 0$$
$$\Leftrightarrow \forall y \in U, X_\lambda(y) \wedge R_\lambda(x,y) = 0$$
$$\Leftrightarrow \forall y \in U(1 - v_X(y) \le \lambda \text{ or } 1 - v_R(x,y_0) \le \lambda)$$
$$\Leftrightarrow \bigwedge_{y\in U}(v_X(y) \vee v_R(x,y)) \ge 1 - \lambda$$

Thus $\vee\{1-\lambda | \overline{H_1}(\lambda)(x) = 0\} = \bigwedge_{y\in U}(v_X(y) \vee v_R(x,y))$.

Therefore $\overline{R}(X)(x) = ((\bigvee_{y\in U}(\mu_X(y) \wedge \mu_R(x,y)), \bigwedge_{y\in U}(v_X(y) \vee v_R(x,y)))$.

In the following, we construct the upper approximation of intuitionistic fuzzy sets by three different methods.

Let U be a finite non empty set, $X = (\mu_X, v_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, v_R)$, is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then $X_{[\lambda]}, X^\lambda, X^{[\lambda]}$ are the triple valued fuzzy sets over U, and $R_{[\lambda]}, R^\lambda, R^{[\lambda]}$ are the triple valued fuzzy equivalence relations over U.

Let $\overline{H_2}(\lambda) = \overline{R^\lambda}(X^\lambda)$. That is for $x \in U$, $\overline{H_2}(\lambda)(x) = \bigwedge_{y\in U}(X^\lambda(y) \vee R^\lambda(x,y))$, thus $\overline{H_2}(\lambda)$ is the triple valued inverse order nested sets over U. Let $\overline{R}(X) = \bigcup_{\lambda\in[0,1]} f_3(\lambda, \overline{H_2}(\lambda))$ or $\bigcap_{\lambda\in[0,1]} f_4(\lambda, \overline{H_2}(\lambda))$.

In addition, let $\overline{H_3}(\lambda) = \overline{R_{[\lambda]}}(X_{[\lambda]})$. That is for $x \in U, \overline{H_3}(\lambda)(x) = \bigvee_{y\in U}(X_{[\lambda]}(y) \wedge R_{[\lambda]}(x,y))$, then $\overline{H_3}(\lambda)$ is the triple valued order nested sets over U. Let $\overline{R}(X) = \bigcup_{\lambda\in[0,1]} f_5(\lambda, \overline{H_3}(\lambda))$ or $\bigcap_{\lambda\in[0,1]} f_6(\lambda, \overline{H_3}(\lambda))$.

Similarly, let $\overline{H_4}(\lambda) = \overline{R^{[\lambda]}}(X^{[\lambda]})$. That is for $x \in U$, $\overline{H_4}(\lambda)(x) = \bigwedge_{y\in U}(X^{[\lambda]}(y) \vee R^{[\lambda]}(x,y))$, thus $\overline{H_4}(\lambda)$ is the triple valued order nested sets over U. Let $\overline{R}(X) = \bigcup_{\lambda\in[0,1]} f_7(\lambda, \overline{H_4}(\lambda))$ or $\bigcap_{\lambda\in[0,1]} f_8(\lambda, \overline{H_4}(\lambda))$.

By the three methods above, we get the same theorem:

Theorem 3.2. For $X \in IF(U), x \in U$, then

$$\overline{R}(X)(x) = (\underset{y\in U}{\vee}(\mu_X(y) \wedge \mu_R(x,y)), \underset{y\in U}{\wedge}(v_X(y) \vee v_R(x,y))).$$

The proof of Theorem 3.2 is similar Theorem 3.1.

4 The Lower Approximation of Intuitionistic Fuzzy Set

In this section, we select the corresponding cut sets and cut relations to construct the lower approximation of intuitionistic fuzzy sets.

Let U be a finite non empty set, $X = (\mu_X, v_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, v_R)$ is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then X_λ is the triple valued fuzzy set over U, and $R_{[\underline{\lambda}]}$ is the triple valued fuzzy equivalence relation over U. Let $\underline{H_1}(\lambda) = \underline{(R_{[\underline{\lambda}]})^c}(X_\lambda)$. That is for $x \in U$, $\underline{H_1}(\lambda)(x) = \underset{y\in U}{\wedge}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y))$, then $\underline{H_1}(\lambda)$ is the triple valued inverse order nested sets on U. Let $\underline{R}(X) = \bigcup\limits_{\lambda\in[0,1]} f_1(\lambda, \underline{H_1}(\lambda))$ or $\bigcap\limits_{\lambda\in[0,1]} f_2(\lambda, \underline{H_1}(\lambda))$. Then we get the following theorem:

Theorem 4.1. Let $X \in IF(U), x \in U$, then

$$\underline{R}(X)(x) = (\underset{y\in U}{\wedge}(\mu_X(y) \vee v_R(x,y)), \underset{y\in U}{\vee}(v_X(y) \wedge \mu_R(x,y))). \tag{2}$$

Proof

$$\underline{R}(X)(x) = \bigcup_{\lambda\in[0,1]} f_1(\lambda, \underline{H_1}(\lambda))(x)$$

(1) $=\{\vee(\lambda, 1-\lambda) | \underline{H_1}(\lambda)(x) = 1\} \vee \{\vee(0, 1-\lambda) | \underline{H_1}(\lambda)(x) = \frac{1}{2}\}$

$= (\vee\{\lambda | \underline{H_1}(\lambda)(x) = 1\}, \wedge\{1-\lambda | \underline{H_1}(\lambda)(x) \geq \frac{1}{2}\})$

Because

$$\begin{aligned}\underline{H_1}(\lambda)(x) = 1 &\Leftrightarrow \underset{y\in U}{\wedge}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) = 1\\ &\Leftrightarrow \forall y \in U(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y) = 1)\\ &\Leftrightarrow \forall y \in U(\mu_X(y) \geq \lambda \text{ or } v_R(x,y) \geq \lambda)\\ &\Leftrightarrow \underset{y\in U}{\wedge}(\mu_X(y) \vee \mu_R(x,y)) \geq \lambda\end{aligned}$$

Thus $\vee\{\lambda | \underline{H_1}(\lambda)(x) = 1\} = \underset{y\in U}{\wedge}(\mu_X(y) \vee v_R(x,y))$.

And

$$\begin{aligned}\underline{H_1}(\lambda)(x) \geq \frac{1}{2} &\Leftrightarrow \bigwedge_{y\in U}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) \geq \frac{1}{2}\\ &\Leftrightarrow \forall y \in U(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y) \geq \frac{1}{2})\\ &\Leftrightarrow \forall y \in U(1 - v_X(y) \geq \lambda \text{ or } 1 - \mu_R(x,y) \geq \lambda)\\ &\Leftrightarrow \forall y \in U(v_A(y) \wedge \mu_R(x,y) \leq 1 - \lambda)\\ &\Leftrightarrow \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)) \leq 1 - \lambda\end{aligned}$$

Thus $\wedge\{1 - \lambda \big| \underline{H_1}(\lambda)(x) \geq \frac{1}{2}\} = \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y))$.

Therefore $\underline{R}(X)(x) = (\bigwedge_{y\in U}(\mu_X(y) \vee v_R(x,y)), \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)))$.

$$\underline{R}(X)(x) = \bigwedge_{\lambda\in[0,1]} f_2(\lambda, \underline{H_1}(\lambda))(x)$$

(2) $=\{\wedge(\lambda, 1-\lambda)\big|\underline{H_1}(\lambda)(x) = 0\} \wedge \{\wedge(\lambda, 0)\big|\underline{H_1}(\lambda)(x) = \frac{1}{2}\}$

$= (\wedge\{\lambda\big|\underline{H_1}(\lambda)(x) \leq \frac{1}{2}\}, \vee\{1-\lambda\big|\underline{H_1}(\lambda)(x) = 0\})$

Because

$$\begin{aligned}\underline{H_1}(\lambda)(x) \leq \frac{1}{2} &\Leftrightarrow \bigwedge_{y\in U}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) \leq \frac{1}{2}\\ &\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \vee (R_{[\underline{\lambda}]})^c(x,y_0) \leq \frac{1}{2})\\ &\Leftrightarrow \exists y_0 \in U(\mu_X(y_0) \vee v_R(x,y_0) < \lambda)\\ &\Leftrightarrow \bigwedge_{y\in U}(\mu_X(y) \vee v_R(x,y)) < \lambda\end{aligned}$$

Thus $\wedge\{\lambda\big|\overline{H_1}(\lambda)(x) \leq \frac{1}{2}\} = \bigwedge_{y\in U}(\mu_R(y) \vee v_R(x,y))$.

And

$$\begin{aligned}\overline{H_1}(\lambda)(x) = 0 &\Leftrightarrow \bigwedge_{y\in U}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) = 0\\ &\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \vee (R_{[\underline{\lambda}]})^c(x,y_0) = 0)\\ &\Leftrightarrow \exists y_0 \in U(1 - v_X(y_0) < \lambda \text{ and } 1 - \mu_R(x,y_0) < \lambda)\\ &\Leftrightarrow \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)) > 1 - \lambda\end{aligned}$$

Thus $\vee\{1-\lambda\big|\overline{H_1}(\lambda)(x) = 0\} = \bigvee_{y\in U}(v_X(y) \wedge v_R(x,y))$.

Therefore $\underline{R}(X)(x) = (\bigwedge_{y\in U}(\mu_X(y) \vee v_R(x,y)), \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)))$.

In the following, we construct the lower approximation of intuitionistic fuzzy sets by three different methods.

Let U be a finite non empty set, $X = (\mu_X, v_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, v_R)$ is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then $X^{\lambda}, X_{[\lambda]}, X^{[\lambda]}$ are the triple valued fuzzy sets over U, and $R_{\underline{\lambda}}, R^{[\underline{\lambda}]}, R^{\underline{\lambda}}$ are the triple valued fuzzy equivalence relations over U.

Let $\underline{H_2}(\lambda) = \underline{(R^{\underline{\lambda}})^c}(X^{[\lambda]})$. That is for $x \in U$, $\underline{H_2}(\lambda)(x) = \bigvee_{y \in U}(X^{\lambda}(y) \wedge (R^{[\underline{\lambda}]})^c(x,y))$, thus $\underline{H_2}(\lambda)$ is the triple valued inverse order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda \in [0,1]} f_3(\lambda, \underline{H_2}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_4(\lambda, \underline{H_2}(\lambda))$.

Similarly, let $\underline{H_3}(\lambda) = \underline{(R_{\underline{\lambda}})^c}(X_{[\lambda]})$. That is for $x \in U$, $\underline{H_3}(\lambda)(x) = \bigwedge_{y \in U}(X_{[\lambda]}(y) \vee (R_{\underline{\lambda}})^c(x,y))$, thus $\underline{H_3}(\lambda)$ is the triple valued order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda \in [0,1]} f_5(\lambda, \underline{H_3}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_6(\lambda, \underline{H_3}(\lambda))$.

In addition, let $\underline{H_4}(\lambda) = \underline{(R^{\underline{\lambda}})^c}(X^{[\lambda]})$. That is for $x \in U$, $\underline{H_4}(\lambda)(x) = \bigvee_{y \in U}(X^{[\lambda]}(y) \wedge (R^{\underline{\lambda}})^c(x,y))$, thus $\underline{H_4}(\lambda)$ is the triple valued order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda \in [0,1]} f_7(\lambda, \underline{H_4}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_8(\lambda, \underline{H_4}(\lambda))$.

By the three methods above, we get the same theorem:

Theorem 4.2. Let $X \in IF(U), x \in U$. Then

$$\underline{R}(X)(x) = (\bigwedge_{y \in U}(\mu_X(y) \vee v_R(x,y)), \bigvee_{y \in U}(v_X(y) \wedge \mu_R(x,y))).$$

The proof of Theorem 4.2 is similar to Theorem 4.1.

5 Conclusion

In this paper, triple valued fuzzy sets are selected as the cut sets of intuitionistic fuzzy sets, and four methods are provided to construct the upper and lower approximations of intuitionistic fuzzy sets by using the representation theorem. Zhou et al. [6] and Zhang et al. [7] construct the upper and lower approximations of intuitionistic fuzzy sets by different intuitionistic fuzzy implicators. Jena et al. [9], Zhou et al. [10] and Samanta et al. [11] investigate intuitionistic fuzzy rough through the extension of fuzzy rough set. We get the same result as the original intuitionistic fuzzy rough set given in [6–11] and provide a theoretical support for the extension from fuzzy rough set to intuitionistic fuzzy rough sets [9–11]. From the results in this paper, we notice that (1) Cut sets and representation theorems play an important role in the research of intuitionistic fuzzy set theory and rough set, and a lot of fuzzy theories can be studied by means of nested set. (2) The triple fuzzy set is an efficient tool to study intuitionistic fuzzy rough set, which deserve our more attentions.

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