Variational Iteration Method for Solving an Inverse Parabolic Problem

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Abstract. In this paper, the variational iteration method is applied to solving an inverse problem of determining more than one unknown parameters in a linear parabolic equation with Neumann boundary conditions. If one of boundary conditions is considered as unknown, it is desirable to be able to determine more than one parameter from the given data. This method is based on the use of Lagrange multipliers for identification of optimal valuse of parameters in a functional. We get a rapid convergent sequence tending to the exact solution of the inverse problem. To show the efficiency of the present method, one interesting example is presented.

Keywords: Variational iteration method · Inverse parabolic equation · Neumann boundary conditions · Lagrange multipliers

1 Introduction

In this work, we will consider the following inverse problem of simultaneously finding unknown coefficients $p(t)$, one boundary condition $q(t)$ and $u(x,t)$ from the following parabolic equation

$$u_t = u_{xx} + p(t)u + f(x,t), x \in (0,1), t \in (0,T], \tag{1}$$

with the initial-boundary conditions

$$\begin{aligned} u(x,0) &= \varphi(x), x \in (0,1), t \in (0,T], &(2)\\ u_x(0,t) &= q(t), t \in (0,T], &(3)\\ u_x(1,t) &= \mu_1(t), t \in (0,T], &(4)\\ u(1,t) &= \mu_2(t), t \in (0,T], &(5) \end{aligned}$$

and the additional specification

$$u(x^*,t) = E(t), x^* \in (0,1), t \in (0,T], \tag{6}$$

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where $f(x,t), \varphi(x), \mu_1(t), \mu_2(t)$ and $E(t) \neq 0$ are known functions, $p(t)$ and $q(t)$ are unknown function, x^* is a fixed prescribed interior point in $(0,1)$.

The determination of unknown coefficients in partial differential equations of parabolic type from additional boundary conditions (i.e., measured data taken on the boundary) is well known in literature as inverse coefficient problems (ICP). Physically, the ICP is the reconstruction of an intra property of a medium in some bounded region by using state measurements taken on the boundary. ICP for semi-linear parabolic equations have been studied by many people, for example, by Cannon and Lin [1], Emine [2], Hasanov and Liu [3], Liu [4–6], Odibat [7], Varedi, Hosseini, Rahimi, et al. [8].

The variational iteration method is introduced by He [9–11] as a modification of a general Lagrange multiplier method [12], which has been proved by many authors to be a powerful mathematical tool for various types of nonlinear problems. It was successfully applied to burger's equation and coupled equation [13], a biochemical reaction model [14], singular perturbation initial value problems [15], strongly nonlinear problems [16,17], nonlinear differential equations of fractional order [18,26], generalized nonlinear Boussinesq equation [19] and generalized KdV [20], Dehghan, Liu Jinbo, Huang Dejian and Ma Yunjie have studied the inverse problems by use of the variational iteration method [7,21,22,25,27].

In this paper, we will apply the variational iteration method to find the exact solution of a control parameter $p(t)$, a boundary condition $q(t)$ in parabolic equation.

2 The Variational Iteration Method

In this section the application of variational iteration method is discussed for solving problem (1)–(5) with over specification (6). Applying a pair of transformations [2] as follows:

$$\begin{aligned} r(t) &= \exp(-\int_0^t p(s)ds), && (7)\\ w(x,t) &= u(x,t)r(t). && (8) \end{aligned}$$

We reduce the original inverse problem (1)–(6) to the following auxiliary problem:

$$\begin{aligned} w_t &= w_{xx} + r(t)f(x,t), x \in (0,1), t \in (0,T]. && (9)\\ w(x,0) &= \varphi(x), x \in (0,1), t \in (0,T], && (10)\\ w_x(0,t) &= r(t)q(t), t \in (0,T], && (11)\\ w_x(1,t) &= r(t)\mu_1(t), t \in (0,T], && (12)\\ w(1,t) &= r(t)\mu_2(t), t \in (0,T], && (13) \end{aligned}$$

subject to

$$r(t) = \frac{w(x^*,t)}{E(t)}, t \in (0,T]. \tag{14}$$

It is easy to show that the original inverse problem (1)–(6) is equivalent to the auxiliary problem (9)–(14). Obviously, Eq. (9) has only one unknown function $w(x,t)$ [23,24] and has suitable form to apply the variational iteration method.

According to the variational iteration method, we consider the correction functional in $t-$ direction in the following form

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda(s)\{\frac{\partial w_n(x,s)}{\partial s} - \frac{\partial^2 \tilde{w}_n(x,s)}{\partial x^2} - \frac{\tilde{w}_n(x^*,s)}{E(s)} f(x,s)\}ds, \quad (15)$$

where $\lambda(t)$ is the general Lagrange multiplier, its optimal value is found by using variational theory, $w_0(x,t)$ is an initial approximation which must be chosen suitably and $\tilde{w}_n$ is the restricted variation i.e. $\delta\tilde{w}_n = 0$ [9].

To find the optimal value of $\lambda(t)$, we have

$$\delta w_{n+1}(x,t) = \delta w_n(x,t) + \delta \int_0^t \lambda(s)\{\frac{\partial w_n(x,s)}{\partial s} - \frac{\partial^2 \tilde{w}_n(x,s)}{\partial x^2} - \frac{\tilde{w}_n(x^*,s)}{E(s)} f(x,s)\}ds, \quad (16)$$

or

$$\delta w_{n+1}(x,t) = \delta w_n(x,t) + \delta \int_0^t \lambda(s)\{\frac{\partial w_n(x,s)}{\partial s}\}ds. \quad (17)$$

Using integration by parts, we have

$$\delta w_{n+1}(x,t) = \delta w_n(x,t)(1+\lambda(t)) - \int_0^t \delta w_n(x,s)\lambda'(s)ds = 0, \quad (18)$$

which yields

$$\lambda'(s) = 0|_{s=t}, \quad (19)$$

$$1+\lambda(s) = 0|_{s=t}. \quad (20)$$

Thus we have

$$\lambda(t) = -1. \quad (21)$$

and we obtain the following iteration formula

$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t \{\frac{\partial w_n(x,s)}{\partial s} - \frac{\partial^2 w_n(x,s)}{\partial x^2} - \frac{w_n(x^*,s)}{E(s)} f(x,s)\}ds. \quad (22)$$

Now using (22) we can find the solution of Eq. (9). Then we get the solutions of the original inverse problem from the following

$$u(x,t) = \frac{w(x,t)}{E(t)}, \quad (23)$$

and

$$p(t) = -\frac{r'(t)}{r(t)}, \quad (24)$$

then

$$q(t) = u_x(0,t), \tag{25}$$

where $r(t)$ is given in (7).

Also we can consider w_n as an approximation of the exact solution for sufficiently large values of n.

3 The Test Example

To show the efficiency of the present method, we consider the following example, which can be solved iteratively by using the variational iteration method.

Consider Eqs. (1)–(6) with the following conditions:

$$\begin{aligned} u(x,0) &= \cos(\pi x) + x, & (26)\\ u_x(1,t) &= \exp(t), & (27)\\ u(1,t) &= 0, & (28)\\ f(x,t) &= \pi^2 \exp(t)\cos(\pi x) - t^2 \exp(t)[\cos(\pi x) + x], & (29)\\ E(t) &= (\frac{\sqrt{2}}{2} + \frac{1}{4})\exp(t), & (30) \end{aligned}$$

with $x^* = 0.25$. The exact solution of this problem is [26]

$$u(x,t) = \exp(t)[\cos(\pi x) + x], \tag{31}$$

and

$$\begin{aligned} p(t) &= 1 + t^2, & (32)\\ q(t) &= \exp(t). & (33) \end{aligned}$$

We set from (10)

$$w_0 = \varphi(x) = \cos(\pi x) + x. \tag{34}$$

Using Eq. (22), we obtain

$$\begin{aligned} w_1(x,t) &= w_0(x,t) - \int_0^t \{\tfrac{\partial w_0(x,s)}{\partial s} - \tfrac{\partial^2 w_0(x,s)}{\partial x^2} - \tfrac{w_0(x^*,s)}{E(s)} f(x,s)\} ds\\ &= \cos(\pi x) + x - \int_0^t \{\pi^2 \cos(\pi x) - [\pi^2 \cos(\pi x) - s^2(\cos(\pi x) + x)]\} ds\\ &= [\cos(\pi x) + x](1 - \tfrac{t^3}{3})\\ &= \sum_{j=0}^{1} \tfrac{(-\frac{t^3}{3})^j}{j!}[\cos(\pi x) + x], \end{aligned} \tag{35}$$

$$\begin{aligned} w_2(x,t) &= w_1(x,t) - \int_0^t \{\tfrac{\partial w_1(x,s)}{\partial s} - \tfrac{\partial^2 w_1(x,s)}{\partial x^2} - \tfrac{w_1(x^*,s)}{E(s)} f(x,s)\} ds\\ &= [\cos(\pi x) + x](1 - \tfrac{t^3}{3}) - \int_0^t \{-[\cos(\pi x) + x]s^2 + (s^2 - \tfrac{s^5}{3})[\cos(\pi x) + x]\} ds\\ &= [\cos(\pi x) + x](1 - \tfrac{t^3}{3} + \tfrac{t^6}{18})\\ &= \sum_{j=0}^{2} \tfrac{(-\frac{t^3}{3})^j}{j!}[\cos(\pi x) + x], \end{aligned} \tag{36}$$

$$\begin{aligned}
w_3(x,t) &= w_2(x,t) - \textstyle\int_0^t \{\frac{\partial w_2(x,s)}{\partial s} - \frac{\partial^2 w_2(x,s)}{\partial x^2} - \frac{w_2(x^*,s)}{E(s)} f(x,s)\}ds \\
&= [\cos(\pi x) + x](1 - \frac{t^3}{3} + \frac{t^6}{18}) - \textstyle\int_0^t \{[\cos(\pi x) + x]\frac{s^8}{18}\}ds \\
&= [\cos(\pi x) + x](1 - \frac{t^3}{3} + \frac{t^6}{18} - \frac{t^9}{162}) \\
&= \textstyle\sum_{j=0}^{3} \frac{(-\frac{t^3}{3})^j}{j!}[\cos(\pi x) + x],
\end{aligned} \tag{37}$$

and so on.

Generally we obtain

$$w_n(x,t) = \sum_{j=0}^{n} \frac{(-\frac{t^3}{3})^j}{j!}[\cos(\pi x) + x]. \tag{38}$$

Thus the exact value of w in a closed form is

$$w(x,t) = \exp(-\frac{t^3}{3})[\cos(\pi x) + x] \quad (n \to \infty), \tag{39}$$

which results the exact solution of the problem. It can be seen that the same results are obtained using Finite difference method [25], Comparing with Finite difference method, it is easy to know that the approximation obtained by the variational iteration method converges to its exact solution faster than those of Finite difference without calculating implicit difference scheme. The results show the computation efficiency of the variational iteration method for the studied model.

4 Conclusion

In this work, the variational iteration method has been successfully applied to inverse parabolic equation with Neumann boundary conditions. Since this method solves the problem without any need to discretization of the variables, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. The example shows that this method provides the solution of the problem in a closed form without calculating implicit difference scheme, which is an advantage of the variational iteration method over Finite difference method. Thus we can say the proposed method is very simple and straightforward.

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