The Lattice of L-fuzzy Filters in a Given R_0-algebra

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Abstract. In the present paper, the L-fuzzy filter theory on R_0-algebras is further studied. Some new properties of L-fuzzy filters are given. Representation theorem of L-fuzzy filter which is generated by a fuzzy set is established. It is proved that the set consisting of all L-fuzzy filters on a given R_0-algebra, under the L-fuzzy set-inclusion order $\Subset$, forms a complete distributive lattice.

Keywords: Fuzzy logic · R_0-algebra · L-fuzzy filter · Complete distributive lattice

1 Introduction

To make the computers simulate beings in dealing with certainty and uncertainty in information is one important task of artificial intelligence. Logic appears in a "sacred" (resp., a "profane") form which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold– as a tool for applications in both areas, and a technique for laying the foundations. Nonclassical logic [1] including many-valued logic and fuzzy logic takes the advantage of classical logic to handle information with various facets of uncertainty [2], such as fuzziness and randomness. At present, nonclassical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. R_0-algebra is an important class of non-classical fuzzy logical algebras which was introduced by Wang in [3] by providing an algebra proof of the completeness theorem of the formal deductive system $\mathcal{L}^*$. From then, R_0-algebras has been extensively investigated by many researchers. Among them, Jun and Liu studied the theory of filters in R_0-algebras in [4]. The concept of fuzzy sets is introduced firstly by Zadeh in [5]. Liu and Li in [6] proposed the concept of fuzzy filters of R_0-algebras and discussed some their properties by using fuzzy sets theory. As an extension of the concept of fuzzy filter, in [7] the author and Xu propose the notion of L-fuzzy filters of R_0-algebras in terms of the notion of L-fuzzy set in [8], where the prefix L a lattice. In this paper, we will further research the properties of L-fuzzy filters in R_0-algebras. The lattice structural feature of the set containing all of L-fuzzy filters in a given R_0-algebra is investigated. It should be noticed that when $L = [0, 1]$, then [0, 1]-fuzzy sets

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are originally meant fuzzy sets. Since [0, 1] is a special completely distributive lattice, to investigate properties of L-fuzzy filters, sometimes we assume that the prefix L is a completely distributive lattice.

2 Preliminaries

Definition 1 (cf. [3]). Let M be an algebra of type $(\neg, \vee, \rightarrow)$, where $\neg$ is a unary operation, $\vee$ and $\rightarrow$ are binary operations. $(M, \neg, \vee, \rightarrow, 1)$ is called an R_0-algebra if there is a partial order $\leqslant$ such that $(M, \leqslant, 1)$ is a bounded distributive lattice with the greatest element 1, $\vee$ is the supremum operation with respect to $\leqslant$, $\neg$ is an order-reversing involution, and the following conditions hold for every $a, b, c \in M$:

(M1) $\neg a \rightarrow \neg b = b \rightarrow a$;
(M2) $1 \rightarrow a = a, a \rightarrow a = 1$;
(M3) $b \rightarrow c \leqslant (a \rightarrow b) \rightarrow (a \rightarrow c)$;
(M4) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$;
(M5) $a \rightarrow (b \vee c) = (a \rightarrow b) \vee (a \rightarrow c), a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$;
(M6) $(a \rightarrow b) \vee ((a \rightarrow b) \rightarrow (\neg a \vee b)) = 1$.

Lemma 1 (cf. [3]). Let M be an R_0-algebra, $a, b, c \in M$. Then the following properties hold.

(P1) $a \leqslant b$ if and only if $a \rightarrow b = 1$;
(P2) $a \leqslant b \rightarrow c$ if and only if $b \leqslant a \rightarrow c$;
(P3) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c), (a \wedge b) \rightarrow c = (a \rightarrow c) \vee (b \rightarrow c)$;
(P4) If $b \leqslant c$, then $a \rightarrow b \leqslant a \rightarrow c$, and if $a \leqslant b$, then $b \rightarrow c \leqslant a \rightarrow c$;
(P5) $a \rightarrow b \geqslant \neg a \vee b$ and $a \wedge \neg a \leqslant b \vee \neg b$;
(P6) $(a \rightarrow b) \vee (b \rightarrow a) = 1$ and $a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$;
(P7) $a \rightarrow (b \rightarrow a) = 1$ and $a \rightarrow (\neg a \rightarrow b) = 1$;
(P8) $a \rightarrow b \leqslant a \vee c \rightarrow b \vee c$ and $a \rightarrow b \leqslant a \wedge c \rightarrow b \wedge c$;
(P9) $a \rightarrow b \leqslant (a \rightarrow c) \vee (c \rightarrow b)$.

Lemma 2 (cf. [3]). Let M be an R_0-algebra. Define a new operator $\otimes$ on M such that $a \otimes b = \neg(a \rightarrow \neg b)$, for every $a, b, c \in M$. Then the following properties hold.

(P10) $(M, \otimes, 1)$ is a commutative monoid with the multiplicative unit element 1;
(P11) If $a \leqslant b$, then $a \otimes c \leqslant b \otimes c$;
(P12) $0 \otimes a = 0$ and $a \otimes \neg a = 0$;
(P13) $a \otimes b \leqslant a \wedge b$ and $a \otimes (a \rightarrow b) \leqslant b$ and $a \leqslant b \rightarrow (a \otimes b)$;
(P14) $a \otimes b \rightarrow c = a \rightarrow (b \rightarrow c)$ and $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$.

Let X be a non-empty set and L a lattice. A map $\mathscr{A} : X \rightarrow L$ is called an L-fuzzy subset on X. The set of all L-fuzzy subsets on X is denoted by $\mathcal{F}_L(X)$. (cf. [8]). Let $\mathscr{A}$ and $\mathscr{B}$ be two L-fuzzy subsets on X. We define $\mathscr{A} \Cap \mathscr{B}$, $\mathscr{A} \Cup \mathscr{B}$, $\mathscr{A} \Subset \mathscr{B}$ and $\mathscr{A} = \mathscr{B}$ as follows:

(i) $(\mathscr{A} \Cap \mathscr{B})(x) = \mathscr{A}(x) \wedge \mathscr{B}(x)$, for all $x \in X$;
(ii) $(\mathscr{A} \Cup \mathscr{B})(x) = \mathscr{A}(x) \vee \mathscr{B}(x)$, for all $x \in X$;
(iii) $\mathscr{A} \Subset \mathscr{B} \Longleftrightarrow \mathscr{A}(x) \leqslant \mathscr{B}(x)$, for all $x \in X$;
(iv) $\mathscr{A} = \mathscr{B} \Longleftrightarrow (\mathscr{A} \Subset \mathscr{B}$ and $\mathscr{B} \Subset \mathscr{A})$.

3 On L-fuzzy Filters in R_0-algebras

In this section, we recall the definition of L-fuzzy filters and give their some new properties.

Definition 2 (cf. [7])**.** Let M be an R_0-algebra and L a lattice. An L-fuzzy subset $\mathscr{A}$ on M is said to be an L-fuzzy filter of M, if it satisfies the following conditions:

(LF1) $\mathscr{A}(1) \geqslant \mathscr{A}(a)$ for all $a \in M$;
(LF2) $\mathscr{A}(b) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(a \to b)$ for all $a, b \in M$.

The set of all L-fuzzy filters of M is denoted by $\mathbf{LFil}(M)$.

Theorem 1. Let M be an R_0-algebra, L a lattice and $\mathscr{A}$ an L-fuzzy subset on M. Then $\mathscr{A} \in \mathbf{LFil}(M)$ if and only if it satisfies the following conditions:

(LF3) $a \leqslant b$ implies $\mathscr{A}(b) \geqslant \mathscr{A}(a)$ for all $a, b \in M$;
(LF4) $\mathscr{A}(a \otimes b) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(b)$ for all $a, b \in M$.

Proof. Assume that $\mathscr{A} \in \mathbf{LFil}(M)$. From Theorem 6 in [7], we know that $\mathscr{A}$ satisfies the condition (LF3). Let $a, b \in M$, since $a \leqslant b \to (a \otimes b)$, by (LF2) and (LF3)), we have that $\mathscr{A}(a \otimes b) \geqslant \mathscr{A}(b) \wedge \mathscr{A}(b \to (a \otimes b)) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(b)$. Thus $\mathscr{A}$ also satisfies the condition (LF4). Conversely, Assume that $\mathscr{A}$ satisfies the condition (LF3) and (LF4). since $a \leqslant 1$, by (LF3) we have $\mathscr{A}(1) \geqslant \mathscr{A}(a)$. Thus $\mathscr{A}$ satisfies the condition (LF1). From $a \otimes (a \to b) \leqslant b$, (LF3) and (LF4), it follows that $\mathscr{A}(b) \geqslant \mathscr{A}(a \otimes (a \to b)) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(a \to b)$. Thus $\mathscr{A}$ satisfies the condition (LF2). Therefore $\mathscr{A} \in \mathbf{LFil}(M)$ by Definition 2.

Definition 3. Let M be an R_0-algebra, L a lattice and $\mathscr{A}$ an L-fuzzy subset on M. An L-fuzzy subset $\mathscr{A}^\lambda$ on M is defined as follows:

$$\mathscr{A}^\lambda(a) = \begin{cases} \mathscr{A}(a), & a \neq 1, \\ \mathscr{A}(1) \vee \lambda, & a = 1, \end{cases} \tag{1}$$

for all $a \in M$, where $\lambda \in L$.

Theorem 2. Let M be an R_0-algebra, L a lattice and $\mathscr{A} \in \mathbf{LFil}(M)$. Then $\mathscr{A}^\lambda \in \mathbf{LFil}(M)$ for all $\lambda \in L$.

Proof. Firstly, for all $a, b \in M$, let $a \leqslant b$, we consider the following two cases:

(i) Assume that $b = 1$. If $a = 1$, we have that $\mathscr{A}^\lambda(b) = \mathscr{A}(1) \vee \lambda = \mathscr{A}^\lambda(a)$. If $a \neq 1$, by using $\mathscr{A} \in \mathbf{LFil}(M)$ and (LF1), we have that $\mathscr{A}^\lambda(b)=\mathscr{A}(1) \vee \lambda \geqslant \mathscr{A}(1) \geqslant \mathscr{A}(a) = \mathscr{A}^\lambda(a)$.

(ii) Assume that $b \neq 1$, then $a \neq 1$. It follows that $\mathscr{A}^{\lambda}(b) = \mathscr{A}(b) \geqslant \mathscr{A}(a) = \mathscr{A}^{\lambda}(a)$ from $\mathscr{A} \in \mathbf{LFil}(M)$ and (LF3).

Summarize above two cases, we conclude that $a \leqslant b$ implies $\mathscr{A}^{\lambda}(b) \geqslant \mathscr{A}^{\lambda}(a)$, for all $a, b \in M$. That is, $\mathscr{A}^{\lambda}$ satisfies (LF3).

Secondly, for all $a, b \in M$, we consider the following two cases:

(i) Assume that $a \otimes b = 1$. If $a = b = 1$, it is obvious that

$$\mathscr{A}^{\lambda}(a \otimes b) = \mathscr{A}(1) \vee \lambda = \mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b).$$

If $a = 1, b \neq 1$ or $a \neq 1, b = 1$, then $a \otimes b \neq 1$, it is a contradiction.

If $a \neq 1$ and $b \neq 1$, it follows that $\mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b) = \mathscr{A}(a) \wedge \mathscr{A}(b) \leqslant \mathscr{A}(a \otimes b) = \mathscr{A}(1) \leqslant \mathscr{A}(1) \vee \lambda = \mathscr{A}^{\lambda}(a \otimes b)$ from $\mathscr{A} \in \mathbf{LFil}(M)$, (LF4) and (1).

(ii) Assume that $a \otimes b \neq 1$. If $a = b = 1$, it is obvious a contradiction.

If $a = 1, b \neq 1$ or $a \neq 1, b = 1$, let's assume $a = 1, b \neq 1$, then $a \otimes b = \neg(1 \to \neg b) = b$, and so $\mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b) \leqslant \mathscr{A}^{\lambda}(b) = \mathscr{A}(b) = \mathscr{A}(a \otimes b) = \mathscr{A}^{\lambda}(a \otimes b)$.

If $a \neq 1$ and $b \neq 1$, it follows that $\mathscr{A}^{\lambda}(a \otimes b) = \mathscr{A}(a \otimes b) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(b) = \mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b)$ from $\mathscr{A} \in \mathbf{LFil}(M)$ and (LF4).

Summarize above two cases, we conclude that $\mathscr{A}^{\lambda}(a \otimes b) \geqslant \mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b)$, for all $a, b \in M$. That is, $\mathscr{A}^{\lambda}$ satisfies (LF4).

Thus it follows that $\mathscr{A}^{\lambda} \in \mathbf{LFil}(M)$ from Theorem 1.

Definition 4. Let M be an R_0-algebra, L a lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. Defined L-fuzzy subsets $\mathscr{A}^{\mathscr{B}}$ and $\mathscr{B}^{\mathscr{A}}$ on M as follows: for all $a \in M$,

$$\mathscr{A}^{\mathscr{B}}(a) = \begin{cases} \mathscr{A}(a), & a \neq 1, \\ \mathscr{A}(1) \vee \mathscr{B}(1), & a = 1, \end{cases} \quad \text{and} \quad \mathscr{B}^{\mathscr{A}}(a) = \begin{cases} \mathscr{B}(a), & a \neq 1, \\ \mathscr{B}(1) \vee \mathscr{A}(1), & a = 1. \end{cases} \tag{2}$$

Corollary 1. Let M be an R_0-algebra, L a lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. If $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$. Then $\mathscr{A}^{\mathscr{B}}, \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$.

Definition 5. Let M be an R_0-algebra, L a completely lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. An L-fuzzy set $\mathscr{A} \uplus \mathscr{B}$ on M is defined as follows: for all $a, x, y \in M$,

$$(\mathscr{A} \uplus \mathscr{B})(a) = \bigvee_{x \otimes y \leqslant a} [\mathscr{A}(x) \wedge \mathscr{B}(y)]. \tag{3}$$

Theorem 3. Let M be an R_0-algebra, L a completely distributive lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. If $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$. Then $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$.

Proof. Firstly, for all $a, b \in M$, let $a \leqslant b$, then $\{x \otimes y | x \otimes y \leqslant a\} \subseteq \{x \otimes y | x \otimes y \leqslant b\}$, and so

$$\begin{aligned} \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(b) &= \bigvee_{x \otimes y \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\ &\geqslant \bigvee_{x \otimes y \leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a). \end{aligned}$$

Hence $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ satisfies (LF3). Secondly, for all $a, b \in M$, we have that

$$\begin{aligned}
&\left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a \otimes b) \\
&= \bigvee_{x \otimes y \leqslant a \otimes b} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\
&\geqslant \bigvee_{x_1 \otimes x_2 \leqslant a \text{ and } y_1 \otimes y_2 \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(x_1 \otimes y_1) \wedge \mathscr{B}^{\mathscr{A}}(x_2 \otimes y_2)\right] \\
&\geqslant \bigvee_{x_1 \otimes x_2 \leqslant a \text{ and } y_1 \otimes y_2 \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(x_1) \wedge \mathscr{A}^{\mathscr{B}}(y_1) \wedge \mathscr{B}^{\mathscr{A}}(x_2) \wedge \mathscr{B}^{\mathscr{A}}(y_2)\right] \\
&= \bigvee_{x_1 \otimes x_2 \leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x_1) \wedge \mathscr{B}^{\mathscr{A}}(x_2)\right] \wedge \bigvee_{y_1 \otimes y_2 \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(y_1) \wedge \mathscr{B}^{\mathscr{A}}(y_2)\right] \\
&= \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a) \wedge \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(b),
\end{aligned}$$

and so $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ also satisfies (LF4). Hence $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$ by Theorem 1.

4 Generated *L*-fuzzy Filter by an *L*-fuzzy Subset

In this section, we give the notion of generated L-fuzzy filter by an L-fuzzy subset and establish its representation theorem.

Definition 6. Let M be an R_0-algebra, L a lattice and $\mathscr{A}$ an L-fuzzy subset on M. An L-fuzzy filter $\mathscr{B}$ of M is called the generated L-fuzzy filter by $\mathscr{A}$, denoted $\langle \mathscr{A} \rangle$, if $\mathscr{A} \Subset \mathscr{B}$ and for any $\mathscr{C} \in \mathbf{LFil}(M)$, $\mathscr{A} \Subset \mathscr{C}$ implies $\mathscr{B} \Subset \mathscr{C}$.

Theorem 4. Let M be an R_0-algebra, L a completely distributive lattice and $\mathscr{A}$ an L-fuzzy subset on M. An L-fuzzy subset $\mathscr{B}$ on M is defined as follows:

$$\mathscr{B}(a) = \bigvee \left\{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes \cdots \otimes x_n \leqslant a\right\}, \tag{4}$$

for all $a \in M$. Then $\mathscr{B} = \langle \mathscr{A} \rangle$.

Proof. Firstly, we prove that $\mathscr{B} \in \mathbf{LFil}(M)$. For all $a, b \in M$, let $a \leqslant b$. Then

$$\begin{aligned}
\mathscr{A}(a) &= \bigvee \left\{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\right\} \\
&\leqslant \bigvee \left\{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes \cdots \otimes x_n \leqslant b\right\} = \mathscr{B}(b).
\end{aligned}$$

Thus $\mathscr{B}$ satisfies (LF3). Assume that there are $x_1, x_2, \cdots, x_n \in M$ and $y_1, \cdots, y_m \in M$ such that $x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a$ and $y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant b$, we have that $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant a \otimes b$ by (P11). Thus, we can

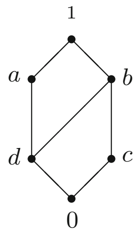

Fig. 1. The Hasse diagram of M

Table 1. Def. of "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

obtain that

$$\begin{aligned}
&\mathscr{B}(a)\wedge\mathscr{B}(b)\\
=&\bigvee\{\mathscr{A}(x_1)\wedge\cdots\mathscr{A}(x_n)|x_1,x_2,\cdots,x_n\in M \text{ and } x_1\otimes x_2\otimes\cdots\otimes x_n\leqslant a\}\\
&\qquad\wedge\bigvee\{\mathscr{A}(y_1)\wedge\cdots\mathscr{A}(y_m)|y_1,y_2,\cdots,y_m\in M \text{ and } y_1\otimes y_2\otimes\cdots\otimes y_m\leqslant b\}\\
=&\bigvee\{\mathscr{A}(x_1)\wedge\cdots\wedge\mathscr{A}(x_n)\wedge\mathscr{A}(y_1)\wedge\cdots\wedge\mathscr{A}(y_m)|x_1,\cdots,x_n,y_1,\cdots,y_m\in M\\
&\qquad\qquad\text{such that } x_1\otimes x_2\otimes\cdots\otimes x_n\leqslant a \text{ and } y_1\otimes y_2\otimes\cdots\otimes y_m\leqslant b\}\\
\leqslant&\bigvee\{\mathscr{A}(x_1)\wedge\cdots\wedge\mathscr{A}(x_n)\wedge\mathscr{A}(y_1)\wedge\cdots\wedge\mathscr{A}(y_m)|x_1,\cdots,x_n,y_1,\cdots,y_m\in M\\
&\qquad\qquad\text{such that } x_1\otimes x_2\otimes\cdots\otimes x_n\otimes y_1\otimes y_2\otimes\cdots\otimes y_m\leqslant a\otimes b\}\\
\leqslant&\bigvee\{\mathscr{A}(z_1)\wedge\cdots\mathscr{A}(z_k)|z_1,z_2,\cdots,z_k\in M \text{ and } z_1\otimes\cdots\otimes z_k\leqslant a\otimes b\}\\
=&\mathscr{B}(a\otimes b).
\end{aligned}$$

Hence $\mathscr{B}$ also satisfies (LF4). It follows from Theorem 1 that $\mathscr{B}\in\mathbf{LFil}(M)$.

Secondly, For any $a\in M$, it follows from $a\leqslant a$ and the definition of $\mathscr{B}$ that $\mathscr{A}(a)\leqslant\mathscr{B}(a)$. This means that $\mathscr{A}\Subset\mathscr{B}$.

Finally, assume that $\mathscr{C}\in\mathbf{LFil}(M)$ with $\mathscr{A}\Subset\mathscr{C}$. Then for any $a\in M$, we have

$$\begin{aligned}
\mathscr{B}(a)=&\bigvee\{\mathscr{A}(x_1)\wedge\cdots\wedge\mathscr{A}(x_n)|x_1,x_2,\cdots,x_n\in M \text{ and } x_1\otimes x_2\otimes\cdots\otimes x_n\leqslant a\}\\
\leqslant&\bigvee\{\mathscr{C}(x_1)\wedge\cdots\wedge\mathscr{C}(x_n)|x_1,x_2,\cdots,x_n\in M \text{ and } x_1\otimes x_2\otimes\cdots\otimes x_n\leqslant a\}\\
\leqslant&\bigvee\{\mathscr{C}(x_1\otimes\cdots\otimes x_n)|x_1,x_2,\cdots,x_n\in M \text{ and } x_1\otimes x_2\otimes\cdots\otimes x_n\leqslant a\}\\
\leqslant&\bigvee\{\mathscr{C}(a)\}=\mathscr{C}(a).
\end{aligned}$$

Hence $\mathscr{B}\Subset\mathscr{C}$ holds. To sum up, we have that $\mathscr{B}=\langle\mathscr{A}\rangle$.

Example 1. Let $M=\{0,a,b,c,d,1\}$, $\neg 0=1,\neg a=c,\neg b=d,\neg c=a,\neg d=b,\neg 1=0$, the Hasse diagram of lattice $(M,\vee,\wedge,\leqslant)$ be defined as Fig. 1, and the binary operator $\to$ of M be defined as Table 1.

Then $(M,\neg,\vee,\to,1)$ is an R_0-algebra. Take $L=([0,1],\max,\min)$ and define an [0,1]-fuzzy subset $\mathscr{A}$ on M by $\mathscr{A}(1)=\mathscr{A}(c)=\alpha,\mathscr{A}(a)=\mathscr{A}(b)=\mathscr{A}(d)=$

$\mathscr{A}(0) = \beta$, $0 \leqslant \beta < \alpha \leqslant 1$. Since $c \leqslant b$ but $\mathscr{A}(b) = \beta \ngeqslant \alpha = \mathscr{A}(c)$, we know that $\mathscr{A} \notin \mathbf{LFil}(M)$. It is easy to verify that $\langle \mathscr{A} \rangle \in \mathbf{LFil}(M)$ from Theorem 4, where $\langle \mathscr{A} \rangle(1) = \langle \mathscr{A} \rangle(b) = \langle \mathscr{A} \rangle(c) = \alpha, \langle \mathscr{A} \rangle(a) = \langle \mathscr{A} \rangle(d) = \langle \mathscr{A} \rangle(0) = \beta$.

5 The Lattice of L-fuzzy Filters in a Given R_0-algebra

In this section, we investigate the lattice structural feature of the set $\mathbf{LFil}(M)$ under the L-fuzzy set-inclusion order $\Subset$.

Theorem 5. Let M be an R_0-algebra and L a complete lattice. Then $(\mathbf{LFil}(M), \Subset)$ is a complete lattice.

Proof. For any $\{\mathscr{A}_\alpha\}_{\alpha\in\Lambda} \subseteq \mathbf{LFil}(M)$, where Λ is an indexed set. It is easy to verify that $\Cap_{\alpha\in\Lambda}\mathscr{A}_\alpha \in \mathbf{LFil}(M)$ is infimum of $\{\mathscr{A}_\alpha\}_{\alpha\in\Lambda}$, where $(\Cap_{\alpha\in\Lambda}\mathscr{A}_\alpha)(a) = \bigwedge\limits_{\alpha\in\Lambda} \mathscr{A}_\alpha(a)$ for all $a \in M$. i.e., $\bigwedge\limits_{\alpha\in\Lambda} \mathscr{A}_\alpha = \Cap_{\alpha\in\Lambda}\mathscr{A}_\alpha$. Define $\Cup_{\alpha\in\Lambda}\mathscr{A}_\alpha$ such that $(\Cup_{\alpha\in\Lambda}\mathscr{A}_\alpha)(a) = \bigvee\limits_{\alpha\in\Lambda} \mathscr{A}_\alpha(a)$ for all $a \in M$. Then $\langle \Cup_{\alpha\in\Lambda}\mathscr{A}_\alpha \rangle$ is supermun of $\{\mathscr{A}_\alpha\}_{\alpha\in\Lambda}$, where $\langle \Cup_{\alpha\in\Lambda}\mathscr{A}_\alpha \rangle$ is the L-fuzzy filter generated by $\Cup_{\alpha\in\Lambda}\mathscr{A}_\alpha$ of M. i.e., $\bigvee\limits_{\alpha\in\Lambda} \mathscr{A}_\alpha = \langle \Cup_{\alpha\in\Lambda}\mathscr{A}_\alpha \rangle$. Therefor $(\mathbf{LFil}(M), \Subset)$ is a complete lattice. The proof is completed.

Remark 1. Let M be an R_0-algebra and L a complete lattice. For all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$, by Theorem 5 we know that $\mathscr{A} \wedge \mathscr{B} = \mathscr{A} \Cap \mathscr{B}$ and $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B} \rangle$.

Theorem 6. Let M be an R_0-algebra and L a completely distributive lattice. Then for all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$, $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B} \rangle = \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ in the complete lattice $(\mathbf{LFil}(M), \Subset)$.

Proof. For all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$, it is obvious that $\mathscr{A} \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ and $\mathscr{B} \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$, that is, $\mathscr{A}(a) \leqslant \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a)$ and $\mathscr{B}(a) \leqslant \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a)$ for all $a \in M$. Thus$(\mathscr{A} \Cup \mathscr{B})(a) = \mathscr{A}(a) \vee \mathscr{B}(a) \leqslant \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a)$, that is, $\mathscr{A} \Cup \mathscr{B} \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$, and thus $\langle \mathscr{A} \Cup \mathscr{B} \rangle \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$ by Theorem 3. Let $\mathscr{C} \in \mathbf{LFil}(M)$ such that $\mathscr{A} \Cup \mathscr{B} \Subset \mathscr{C}$. For all $a \in M$, we consider the following two cases:

(i) If $a = 1$, then $\left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(1) = \bigvee\limits_{x\otimes y\leqslant 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \mathscr{A}^{\mathscr{B}}(1) \wedge \mathscr{B}^{\mathscr{A}}(1) = \mathscr{A}(1) \vee \mathscr{B}(1) = (\mathscr{A} \Cup \mathscr{B})(1) \leqslant \mathscr{C}(1)$.

(ii) If $a < 1$, then we have

$$\begin{aligned}
&\left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a) = \bigvee_{x\otimes y\leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\
&= \bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \vee \bigvee_{x\leqslant a} \{\mathscr{A}(x) \wedge [\mathscr{A}(1) \vee \mathscr{B}(1)]\} \\
&\qquad \vee \bigvee_{y\leqslant a} \{[\mathscr{A}(1) \vee \mathscr{B}(1)] \wedge \mathscr{B}(y)\} \\
&= \bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \vee \left[\bigvee_{x\leqslant a} \mathscr{A}(x)\right] \vee \left[\bigvee_{y\leqslant a} \mathscr{B}(y)\right] \\
&\leqslant \bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} [\mathscr{C}(x) \wedge \mathscr{C}(y)] \vee \left[\bigvee_{x\leqslant a} \mathscr{C}(x)\right] \vee \left[\bigvee_{y\leqslant a} \mathscr{C}(y)\right] \\
&= \bigvee_{x\otimes y\leqslant a} [\mathscr{C}(x) \wedge \mathscr{C}(y)] \leqslant \bigvee_{x\otimes y\leqslant a} \mathscr{C}(x\otimes y) \leqslant \mathscr{C}(a),
\end{aligned}$$

thus $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \Subset \mathscr{C}$ for above two cases.

By Definition 6 and Theorem 4 we have that $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B}\rangle = \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$.

Theorem 7. Let M be an R_0-algebra and L a completely distributive lattice. Then $(\mathbf{LFil}(M), \Subset)$ is a distributive lattice, where, $\mathscr{A} \wedge \mathscr{B} = \mathscr{A} \Cap \mathscr{B}$ and $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B}\rangle$, for all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$.

Proof. To finish the proof, it suffices to show that $\mathscr{C} \wedge (\mathscr{A} \vee \mathscr{B}) = (\mathscr{C} \wedge \mathscr{A}) \vee (\mathscr{C} \wedge \mathscr{B})$, for all $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbf{LFil}(M)$. Since the inequality $(\mathscr{C} \wedge \mathscr{A}) \vee (\mathscr{C} \wedge \mathscr{B}) \Subset \mathscr{C} \wedge (\mathscr{A} \vee \mathscr{B})$ holds automatically in a lattice, we need only to show the inequality $\mathscr{C} \wedge (\mathscr{A} \vee \mathscr{B}) \Subset (\mathscr{C} \wedge \mathscr{A}) \vee (\mathscr{C} \wedge \mathscr{B})$. i.e., we need only to show that $\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) \leqslant \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}\right)(a)$, for all $a \in M$. For these, we consider the following two cases:

(i) If $a = 1$, we have

$$\begin{aligned}
&\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(1) = \mathscr{C}(1) \wedge \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(1) \\
&= \mathscr{C}(1) \wedge \bigvee_{x\otimes y\leqslant 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \mathscr{C}(1) \wedge \left[\mathscr{A}^{\mathscr{B}}(1) \wedge \mathscr{B}^{\mathscr{A}}(1)\right] \\
&= [\mathscr{C}(1) \wedge \mathscr{A}(1)] \vee [\mathscr{C}(1) \wedge \mathscr{B}(1)] = (\mathscr{C} \Cap \mathscr{A})(1) \vee (\mathscr{C} \Cap \mathscr{B})(1) \\
&= (\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(1) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(1) \\
&= \bigvee_{x\otimes y\leqslant 1} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(y)\right] \\
&= \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}\right)(1).
\end{aligned}$$

(ii) If $a < 1$, we have

$$
\begin{aligned}
&\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) = \mathscr{C}(a) \wedge \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a) \\
=&\mathscr{C}(a) \wedge \bigvee_{x\otimes y\leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \bigvee_{x\otimes y\leqslant a} \left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\
=&\bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \vee \\
&\bigvee_{y\leqslant a} \left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(1) \wedge \mathscr{B}(y)\right] \vee \bigvee_{x\leqslant a} \left[\mathscr{C}(a) \wedge \mathscr{A}(x) \wedge \mathscr{B}^{\mathscr{A}}(1)\right] \\
=&\bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \{[\mathscr{C}(a) \wedge \mathscr{A}(x)] \wedge [\mathscr{C}(a) \wedge \mathscr{B}(y)]\} \vee \\
&\bigvee_{y\leqslant a} \left\{\left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(1)\right] \wedge [\mathscr{C}(a) \wedge \mathscr{B}(y)]\right\} \vee \\
&\bigvee_{x\leqslant a} \left\{[\mathscr{C}(a) \wedge \mathscr{A}(x)] \wedge \left[\mathscr{C}(a) \wedge \mathscr{B}^{\mathscr{A}}(1)\right]\right\} \\
\leqslant&\bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \{[\mathscr{C}(a\vee x) \wedge \mathscr{A}(a\vee x)] \wedge [\mathscr{C}(a\vee y) \wedge \mathscr{B}(a\vee y)]\} \vee \\
&\bigvee_{y\leqslant a} \{[\mathscr{C}(1) \wedge (\mathscr{A}(1) \vee \mathscr{B}(1))] \wedge [\mathscr{C}(a\vee y) \wedge \mathscr{B}(a\vee y)]\} \vee \\
&\bigvee_{x\leqslant a} \{[\mathscr{C}(a\vee x) \wedge \mathscr{A}(a\vee x)] \wedge [\mathscr{C}(1) \wedge (\mathscr{B}(1) \vee \mathscr{A}(1))]\} \\
=&\bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} [(\mathscr{C} \Cap \mathscr{A})(a\vee x) \wedge (\mathscr{C} \Cap \mathscr{B})(a\vee y)] \vee \\
&\bigvee_{y\leqslant a} \{[(\mathscr{C} \Cap \mathscr{A})(1) \vee (\mathscr{C} \Cap \mathscr{B})(1)] \wedge (\mathscr{C} \Cap \mathscr{B})(a\vee y)\} \vee \\
&\bigvee_{x\leqslant a} \{(\mathscr{C} \Cap \mathscr{A})(a\vee x) \wedge [(\mathscr{C} \Cap \mathscr{B})(1) \vee (\mathscr{C} \Cap \mathscr{A})(1)]\} \\
=&\bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(a\vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(a\vee y)\right] \vee \\
&\bigvee_{y\leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(1) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(a\vee y)\right] \vee \\
&\bigvee_{x\leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(a\vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(1)\right] \\
=&\bigvee_{x\otimes y\leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(a\vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(a\vee y)\right].
\end{aligned}
$$

Let $a \vee x = u$ and $a \vee y = v$, since $x \otimes y \leqslant a$, using Lemma 2 we get that

$$\begin{aligned} u \otimes v = (a \vee x) \otimes (a \vee y) =& ((a \vee x) \otimes a) \vee ((a \vee x) \otimes y) \\ =& (a \otimes a) \vee (a \otimes x) \vee (a \otimes y) \vee (x \otimes y) \\ \leqslant& a \vee a \vee a \vee (x \otimes y) \\ =& a \vee (x \otimes y) \leqslant a \vee a = a. \end{aligned}$$

Hence we can conclude that

$$\begin{aligned} \left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) &\leqslant \bigvee_{x \otimes y \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(a \vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(a \vee y)\right] \\ &\leqslant \bigvee_{u \otimes v \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(u) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(v)\right] \\ &= \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}\right)(a). \end{aligned}$$

To sum up, we have that

$$\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) \leqslant \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}\right)(a),$$

for all $a \in M$. The proof is completed.

6 Conclusion

As well known, filters is an important concept for studying the structural features of R_0-algebras. In this paper, the L-fuzzy filter theory in R_0-algebras is further studied. Some new properties of L-fuzzy filters are given. Representation theorem of L-fuzzy filter which is generated by an L-fuzzy subset is established. It is proved that the set consisting of all L-fuzzy filters in a given R_0-algebra, under the L-fuzzy set-inclusion order $\Subset$, forms a complete distributive lattice. Results obtained in this paper not only enrich the content of L-fuzzy filters theory in R_0-algebras, but also show interactions of algebraic technique and L-fuzzy sets method in the studying of logic problems. We hope that more links of fuzzy sets and logics emerge by the stipulating of this work.

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