Signed Total Domination and Mycielski Structure in Graphs

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Abstract. Let $G = (V, E)$ be a graph. The function $f : V(G) \rightarrow \{-1, 1\}$ is a signed total dominating function if for every vertex $v \in V(G)$, $\sum_{x \in N_G(v)} f(x) \geq 1$. The value of $\omega(f) = \sum_{x \in V(G)} f(x)$ is called the weight of f. The signed total domination number of G is the minimum weight of a signed total dominating function of G. In this paper, we initiate the study of the signed total domination numbers of Mycielski graphs and find some upper bounds for this parameter. We also calculate the exact value of the signed total domination number of the Mycielski graph when the underlying graph is a special graph.

Keywords: Signed total domination number · Mycielski construction

1 Introduction

All graphs considered throughout this paper are simple, finite, undirected and connected. For the terminology and notations not defined here, we refer the reader to [7]. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of a vertex v in graph G is $N_G[v] = N_G(v) \cup \{v\}$. Moreover, the *open* and *closed neighborhoods* of a subset $S \subseteq V(G)$ are $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) =\mid N_G(v) \mid$. A vertex $v \in V(G)$ is called an *odd* (*even*) vertex if $\deg_G(v)$ is odd (even). For a graph $G = (V, E)$, let V_o and V_e be the set of odd and even vertices, respectively. We denote the *maximum* degree of G with $\Delta(G)$ and its *minimum degree* with $\delta(G)$. A vertex is called *universal* if it is adjacent to all other vertices of a graph. In a complete graph, all vertices are universal.

For a function $f : V(G) \longrightarrow \{-1, 1\}$ and a subset S of $V(G)$, we define $f(S) = \sum_{x \in S} f(x)$. If $S = N_G(v)$ for $v \in V(G)$, then we denote $f(S)$ by $f[v]$. Let $C_f = \{v \in V(G) \mid f(v) \geq 1\}$. A *signed total dominating function* of G is a function $f : V(G) \longrightarrow \{-1, 1\}$ such that for all vertices v of G, $v \in C_f$. The *weight* of a signed total dominating function f is $\omega(f) = \sum_{v \in V(G)} f(v) = f(V(G))$.

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The *signed total domination number* (STDN), $\gamma_{st}(G)$, is the minimum weight of a signed total dominating function of G. A signed total dominating function of weight $\gamma_{st}(G)$ is called a $\gamma_{st}(G)$-*function*. For a signed total dominating function f of G we define $P_f = \{v \in V(G) \mid f(v) = 1\}$ and $M_f = \{v \in V(G) \mid f(v) = -1\}$.

The concept of the signed total domination number of a graph was proposed by Zelinka [8]. Henning in [4] proved that the problem of determining the signed total domination number for general graphs is NP-hard.

For a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$, let $U = \{u_1, u_2, \cdots, u_n\}$ be a disjoint copy of $V(G)$ and let w be a new vertex. The *Mycielski graph* $\mu(G)$ of G is defined as follows:

$$V(\mu(G)) = V(G) \cup U \cup \{w\},$$

$$E(\mu(G)) = E(G) \cup \{v_i u_j \mid v_i v_j \in E(G)\} \cup \{wu_i \mid 1 \le i \le n\}.$$

The vertex w is called the *root* of $\mu(G)$ and the vertex $u_i = c(v_i)$ is called the *twin* of the vertex v_i, $i = 1, 2, \cdots, n$. The Mycielski graph of a graph G was introduced by Mycielski in order to construct triangle-free graphs with an arbitrary large chromatic number [5]. In recent years, there have been results reported on Mycielski graphs related to various domination parameters. In [1], it was proved that $\gamma(\mu(G)) = \gamma(G) + 1$. This shows that the domination number of a Mycielski graph can exceed the domination number of its underlying graph G, but Ghameshlou et al. proved such a result is not true for signed domination number of Mycielski graphs [2,6].

In this paper, we initiate the study of the signed total domination numbers of Mycielski graphs. In Sect. 2, we present some preliminary results on Mycielski graphs and their signed total domination numbers. In Sect. 3, we calculate the exact value of signed total domination number of a Mycielski graph, whose underlying graph has at least one universal vertex. Then we calculate the exact values of $\gamma_{st}(\mu(G))$ when G is a star, a wheel, a fan, a Dutch windmill or a complete graph. In Sect. 4, we prove that if $\gamma_{st}(G) \ge 0$, then $\gamma_{st}(\mu(G)) \le 2\gamma_{st}(G) + 1$, otherwise $\gamma_{st}(\mu(G)) \le \gamma_{st}(G) + 3$. Finally, in Sect. 5, we calculate $\gamma_{st}(\mu(G))$ when G is a cycle, a path or a complete bipartite graph. It is worth to note that there are graphs G, such as $K_{m,n}$, when $m = 1$ or m and n are both odd, with $\gamma_{st}(\mu(G)) < \gamma_{st}(G)$.

2 Preliminary Results

Proposition 1. Let f be a signed total dominating function of $\mu(G)$. Then,

$$\omega(f) \equiv 1 \pmod 2.$$

Proposition 2. Let f be a signed total dominating function of $\mu(G)$.

1. If G has at least one vertex of degree 1, then for Mycielski graph $\mu(G)$, $w \in P_f$.
2. For $v \in V(\mu(G))$ if v is an even vertex, then $f[v] \ge 2$ while if v is an odd vertex, then $f[v] \ge 1$.

Ghameshlou et al. proved the following results for signed domination number of Mycielski graphs.

Theorem 1 [2]. *Let G be a graph of order n. If G has at least an universal vertex, then $\gamma_s(\mu(G)) \geq 3$.*

Corollary 1 [2]. For every graph $G \in \{K_n, K_{1,n}, D_3^n, W_n, F_n\}$, $\gamma_s(\mu(G)) = 3$.

Theorem 2 [2]. *For any graph G of order n,*

$$\gamma_s(\mu(G)) \leq \begin{cases} \gamma_s(G) + 2 & \text{if } \gamma_s(G) \leq -1, \\ 2\gamma_s(G) + 1 & \text{if } \gamma_s(G) \geq 0. \end{cases}$$

Furthermore, for $\gamma_s(G) \geq 0$ the bound is sharp for K_n when n is odd and for $\overline{K_n}$.

Theorem 3 [2]. *For every cycle C_n of order n,*

$$\gamma_s(\mu(C_n)) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 \pmod 8, \\ \frac{n+5}{2} & \text{if } n \equiv 1, 5 \pmod 8, \\ \frac{n}{2} + 2 & \text{if } n \equiv 2, 6 \pmod 8, \\ \frac{n+7}{2} & \text{if } n \equiv 3 \pmod 8, \\ \frac{n}{2} + 3 & \text{if } n \equiv 4 \pmod 8, \\ \frac{n+3}{2} & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Theorem 4 [2]. *For a path P_n, $n \geq 8$,*

$$\gamma_s(\mu(P_n)) = \begin{cases} \frac{n+5}{2} & \text{if } n \equiv 1 \pmod 8, \\ \frac{n+4}{2} & \text{if } n \equiv 2 \pmod 8, \\ \frac{n+3}{2} & \text{if } n \equiv 3, 7 \pmod 8, \\ \frac{n+2}{2} & \text{if } n \equiv 0, 4 \pmod 8, \\ \frac{n+1}{2} & \text{if } n \equiv 5 \pmod 8, \\ \frac{n}{2} & \text{if } n \equiv 6 \pmod 8. \end{cases}$$

Theorem 5 [2]. *For complete bipartite graph $K_{m,n}$ with $m \geq n \geq 2$,*

$$\gamma_s(\mu(K_{m,n})) = 5.$$

Theorem 6 [6]. *If G is a graph of order n, then*

$$\gamma_s(\mu(G)) \geq \begin{cases} \left\lceil \frac{(2n+1)(\delta(G)+1) - 2\Delta(G)(n-1) - n_o}{\Delta(G) + \delta(G) + 1} \right\rceil & \text{if } n \text{ is odd}, \\ \left\lceil \frac{(2n+1)(\delta(G)+1) - \Delta(G)(2n-1) - n_o}{\Delta(G) + \delta(G) + 1} \right\rceil & \text{if } n \text{ is even}. \end{cases}$$

Furthermore, this bound is sharp.

Theorem 7 [6]. *If G is a r-regular graph, then*

$$\gamma_s(\mu(G)) \geq \begin{cases} \dfrac{2n+2r+1}{2r+1} & \text{if } n, r \text{ are even,} \\ \dfrac{n+2r+1}{2r+1} & \text{if } n \text{ is even, } r \text{ is odd,} \\ \dfrac{2n+3r+1}{2r+1} & \text{if } n, r \text{ are odd.} \end{cases}$$

Moreover, this bound is sharp for complete graph K_n.

3 Graphs with Universal Vertices

In this section, we show that the signed total domination number of a Mycielski graph, whose underlying graph has at least one universal vertex, is at least 3.

Theorem 8. *Let G be a graph of order n. If G has at least an universal vertex, then*

$$\gamma_{st}(\mu(G)) \geq 3.$$

Corollary 2. For every complete graph K_n,

$$\gamma_{st}(\mu(K_n)) = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 5 & \text{if } n \text{ is even.} \end{cases}$$

Corollary 3. For every graph $G = \{K_{1,n}, K_3^m, W_n, F_n\}$, $\gamma_{st}(\mu(G)) = 5$.

4 A Relation Between $\gamma_{st}(G)$ and $\gamma_{st}(\mu(G))$

Theorem 9. *For any graph G of order n,*

$$\gamma_{st}(\mu(G)) \leq \begin{cases} 2\gamma_{st}(G) + 1 & \text{if } \gamma_{st}(G) \geq 0, \\ \gamma_{st}(G) + 3 & \text{if } \gamma_{st}(G) \leq -1. \end{cases}$$

Furthermore, for $\gamma_{st}(G) \geq 0$ the bound is sharp for F_n when n is odd, $K_{m,n}$ when m and n are odd, and K_n.

5 Cycles, Paths and Complete Bipartite Graphs

In this section we find the signed total domination number of $\mu(G)$ when G is a cycle, a path, or a complete bipartite graph.

Theorem 10. *For every cycle C_n of order n,*

$$\gamma_{st}(\mu(C_n)) = \begin{cases} n+1 & \text{if } n \equiv 0, 2 \pmod 4, \\ n & \text{if } n \equiv 1, 3 \pmod 4. \end{cases}$$

Theorem 11. *If $G = P_n$, then*

$$\gamma_{st}(\mu(P_n)) = \begin{cases} n+1 & \text{if } n \equiv 0 \pmod 4, \\ n+2 & \text{if } n \equiv 1, 3 \pmod 4, \\ n+3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Theorem 12. *For complete bipartite graph $K_{m,n}$ with $m \geq n \geq 2$,*

$$\gamma_{st}(\mu(K_{m,n})) = 5.$$

6 Conclusion

Comparing the results presented here shows that there are some underlying graphs G of order n which can be generalized to Mycielski graph $\mu(G)$ of order $2n+1$ such that $\gamma_{st}(\mu(G)) \leq \gamma_{st}(G)$; for instance, if $G \in \{K_{1,n}, K_3^m\}$, then $\gamma_{st}(\mu(G)) \leq \gamma_{st}(G)$.

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