Bipolar Fuzzy BRK-ideals in BRK-algebras

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Abstract. In this paper, we investigated bipolar fuzzy BRK-ideals in BRK-algebras and discussed related properties. We presented some results on images and pre-images of bipolar fuzzy BRK-ideals in BRK-algebras. Finally, we introduced translation, extension and multiplications of bipolar fuzzy BRK-ideals in BRK-algebras and discussed related results.

Keywords: Bipolar fuzzy BRK-ideal · BRK-algebra · Images · Pre-images · Translations · Extensions · Multiplications

1 Introduction

The fundamental concept of fuzzy set, popularized by Zadeh [1], was used to generalize several basic concepts of algebra. Fuzzy sets are extremely useful to deal with the many problems in applied mathematics, control engineering, information sciences, expert systems and theory of automata etc. Although, there are many generalizations of fuzzy sets but none of these deal with the problems related to the contrary characteristics of the members having membership degree 0. Lee [2] handled this problem by introducing the concept of bipolar fuzzy (BF) sets. The BF set theory has been widely applied to solve real life problems. The sweet taste of foodstuffs is a BF set. Assuming that sweet taste of foodstuff as a positive membership value then bitter taste of foodstuffs as a negative membership value. The remaining foodstuffs of taste like acidic, saline, chilly etc. are extraneous to the sweet and bitter foodstuffs. Thus, these foodstuffs are accepted as zero membership values. Notice that every matter has two sides and bipolarity as well as fuzziness, is an inherent and internal part of human thinking [3,4]. A BF set is a pair of fuzzy sets, namely a membership and a non-membership function, which represent positive and negative aspects of the given information.

Imai and Iseki investigated two classes of abstract algebras: BCI-algebras and BCK-algebras [5]. In 2002, Neggers et al. [6], presented B-algebra

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and discussed related properties. The generalization of B-algebra called BF-algebra and BG-algebra proposed by Walendziak [7], and Kim [8], respectively. Recently, Bandaru [9], investigated BRK-algebra which is a generalization of $BCK/BCI/Q$-algebras. In [10,11], El-Gendy introduced fuzzy BRK-ideal of BRK-algebra and cubic BRK-ideal of BRK-algebra. Some properties of n-dimensional fuzzy subalgebra in BRK-algebras investigated by Zulfiqar [12]. Fuzzy translations and fuzzy multiplications of BCK/BCI-algebras presented in [13]. As a generalization of fuzzy set theory, BF set theory makes descriptions of the objective world more realistic, practical and very accurate in many cases, making it very promising. In the past few decades, BF set theory has been successfully applied to various algebraic structures. Lee [14], introduced BF ideals of BCK/BCI-algebras and Akram [15] introduced BF graphs. Recently, Hayat et al. [16,17], characterized himi-rings by their BF h-ideals and BAF h-ideals.

The contents of the present paper are organized as follows: In Sect. 2, we presented some basic definitions and preliminaries. In Sect. 3, we investigated bipolar fuzzy BRK-ideals in BRK-algebras and discussed related properties. In Sect. 4, we presented some results on images and pre-images of bipolar fuzzy BRK-ideals in BRK-algebras. In Sect. 5, we introduced translation, extension and multiplications of bipolar fuzzy BRK-ideals in BRK-algebras and discussed related results. Finally, we presented some conclusions and future work.

2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

Definition 2.1. *A BRK-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following conditions:*

(BRK_1) $x * 0 = x$,
(BRK_2) $(x * y) * x = 0 * y$ *for all* $x, y \in X$.

A partial ordered relation $\preceq$ can be defined by $x \preceq y$ if and only if $x * y = 0$. Throughout this paper, X denotes BRK-algebra.

Definition 2.2 [9]. *If $(X, *, 0)$ is a BRK-algebra, the following conditions hold:*

(BRK_3) $x * x = 0$,
(BRK_4) $(x * y) = 0$ *implies* $0 * x = 0 * y$ *for all* $x, y \in X$,
(BRK_5) $0 * (a * b) = (0 * a) * (0 * b)$ *for all* $a, b \in X$.

Definition 2.3 [11,12]. *A subset S of a BRK-algebra X is said to be BRK-subalgebra of X, if $x, y \in S$, implies $x * y \in S$.*

Definition 2.4 [11]. *A subset S of a BRK-algebra X is said to be a BRK-ideal of X (briefly $S_{BRK}X$) if it satisfies:*

(i) $0 \in S$,
(ii) $0 * (x * y) \in S$ *and* $0 * y \in S \implies 0 * x \in S$ *for all* $x, y \in X$.

Example 1. Consider a set $X = \{0, a_1, a_2, a_3\}$. We define "$*$" on X as the following table:

$*$	0	a_1	a_2	a_3
0	0	a_2	a_2	0
a_1	a_1	0	0	a_2
a_2	a_2	0	0	a_2
a_3	a_3	a_1	a_1	0

Clearly, X is a BRK-algebra. Then $S = \{0, a_1, a_2\}_{BRK} X$.

Definition 2.5 [11]. *Let $(X_1, *, 0)$ and $(X_2, *', 0')$ be two BRK-algebras. A mapping $\psi : X_1 \longrightarrow X_2$ is said to be a homomorphism if $\psi(x * y) = \psi(x) *' \psi(y)$, for all $x, y \in X_1$.*

Definition 2.6 [3]. *A bipolar fuzzy set is a pair (λ^+, λ^-), where $\lambda^+ : X \longrightarrow [0, 1]$, and $\lambda^- : X \longrightarrow [-1, 0]$ are any mappings.*

Definition 2.7 [3]. Let $B_1 = (\lambda^+, \lambda^-)$ and $B_2 = (\mu^+, \mu^-)$ be two BF sets in X. $Then following conditions hold:$

(i) $B_1 \leq B_2$ *if and only if* $\lambda^+ \leq \mu^+$ *and* $\lambda^- \geq \mu^-$.
(ii) $\max\{B_1, B_2\} = (\max\{\lambda^+, \mu^+\}, \min\{\lambda^-, \mu^-\})$,
$\min\{B_1, B_2\} = (\min\{\lambda^+, \mu^+\}, \max\{\lambda^-, \mu^-\})$.

3 Bipolar Fuzzy BRK-ideals in BRK-algebras

In this section, we introduced BF BRK-ideals in BRK-algebras and discussed related properties.

Definition 3.1. *A BF set $B = (\lambda^+, \lambda^-)$ of X is called BF BRK-ideal of X if it satisfies following conditions hold:*

(BF$_1$) $\lambda^+(0) \geq \lambda^+(x), \lambda^-(0) \leq \lambda^-(x)$,
(BF$_2$) $\lambda^+(0 * x) \geq \min\{\lambda^+(0 * (x * y)), \lambda^+(0 * y)\}$,
$\lambda^-(0 * x) \leq \max\{\lambda^-(0 * (x * y)), \lambda^-(0 * y)\}$.

Example 2. Consider a set $X = \{0, a_1, a_2, a_3\}$. We define "$*$" on X as the following table:

$*$	0	a_1	a_2	a_3
0	0	a_1	0	a_1
a_1	a_1	0	a_1	0
a_2	a_2	a_1	0	a_1
a_3	a_3	a_2	a_3	0

Clearly, X is a BRK-algebra. Define a bipolar fuzzy set,

	0	a_1	a_2	a_3
λ^+	t_1	t_1	t_2	t_2
λ^-	s_1	s_1	s_2	s_3

where $t_1, t_2 \in [0,1]$ and $s_1, s_2, s_3 \in [-1,0]$ with $t_1 > t_2$ and $s_1 < s_2 < s_3$, routine calculation gives that $B = (\lambda^+, \lambda^-)$ is a BF BRK-ideal of X.

Lemma 3.1. *Let B be a BF BRK-ideal of BRK-algebra X. If $y * x \leq x$ holds in X, then $\lambda^+ (0 * y) \geq \lambda^+ (0 * x)$ and $\lambda^- (0 * y) \leq \lambda^- (0 * x)$.*

Proof. Assume that $y * x \leq x$ holds in X. Then $(y * x) * x = 0$. By (BRK$_2$),

$$\begin{aligned} \lambda^+ (0 * y) &\geq \min \{\lambda^+ (0 * (y * x)), \lambda^+ (0 * x)\}, \\ \lambda^- (0 * y) &\leq \max \{\lambda^- (0 * (y * x)), \lambda^- (0 * x)\}. \end{aligned}$$

Also,

$$\begin{aligned} \lambda^+ (0 * (y * x)) &\geq \min \left\{\lambda^+ ((0 * (y * x)) * x), \lambda^+ (0 * x)\right\} \\ &= \min \left\{\lambda^+ (0), \lambda^+ (0 * x)\right\} \\ &= \lambda^+ (0 * x), \end{aligned}$$

and

$$\begin{aligned} \lambda^- (0 * (y * x)) &\leq \max \left\{\lambda^- ((0 * (y * x)) * x), \lambda^- (0 * x)\right\} \\ &= \max \left\{\lambda^- (0), \lambda^- (0 * x)\right\} \\ &= \lambda^- (0 * x). \end{aligned}$$

Hence $\lambda^+ (0 * y) \geq \lambda^+ (0 * x)$ and $\lambda^- (0 * y) \leq \lambda^- (0 * x)$.

Lemma 3.2. *Let B be a BF BRK-ideal of BRK-algebra X. If $x \leq y$ holds in X, then $\lambda^+ (0 * x) \geq \lambda^+ (0 * y)$ and $\lambda^- (0 * x) \leq \lambda^- (0 * y)$.*

Theorem 3.1. *Let $B_i = \{(\lambda_i^+, \lambda_i^-) : i \in \Omega\}$ be a family of BF BRK-ideals in X. Then $B = \underset{i\in\Omega}{\wedge} B_i$ is also a BF BRK-ideal in X, where $B = (\lambda^+, \lambda^-)$ that is $\lambda^+ = \underset{i\in\Omega}{\wedge} \lambda_i^+$ and $\lambda^- = \underset{i\in\Omega}{\vee} \lambda_i^-$ ($\lambda^+ \leq \lambda_i^+, \lambda^- \geq \lambda_i^-$ $\forall\ i \in \Omega$).*

Definition 3.2. *Let $B = (\mu^+, \mu^-)$ be BF set in BRK-algebra X and $(\alpha, \beta) \in [-1,0] \times [0,1]$, then*

(1) *The set $\widetilde{B}_\beta^+ = \{x \in R : \mu^+ (x) \geq \beta\}$ is called positive β-cut of B.*
(2) *The set $B_\alpha^- = \{x \in R : \mu^- (x) \leq \alpha\}$ is called negative α-cut of B.*
(3) *The set $B_{(\alpha,\beta)} = \{x \in R : \mu^- (x) \leq \alpha$ and $\mu^+ (x) \geq \beta\}$ is called (α, β)-cut of B.*

For every $\gamma \in (0,1]$ and $B_\gamma^+ \cap B_{-\gamma}^-$ is called γ-cut of B.

Theorem 3.2. *A BF set $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X iff the followings hold:*

(i) *For all $\beta \in [0,1]$, B_β^+ is non-empty this implies B_β^+ is a BRK-ideal of X.*
(ii) *For all $\alpha \in [-1,0]$, B_α^- is non-empty this implies B_α^- is a BRK-ideal of X.*

Proof. Let $B = (\mu^+, \mu^-)$ be a BF BRK-ideal in X. For $x \in B_\beta^+$ so $\mu^+(x) \geq \beta$ where $\beta \in [0,1]$. Now $\mu^+(0) \geq \mu^+(x) \geq \beta$. This implies that $0 \in B_\beta^+$.

Next, let $0*(x*y) \in B_\beta^+$ and $0*y \in B_\beta^+$, this means that $\mu^+(0*(x*y)) \geq \beta$ and $\mu^+(0*y) \geq \beta$. Then $\lambda^+(0*x) \geq \min\{\lambda^+(0*(x*y)), \lambda^+(0*y)\} \geq \beta$. Hence B_β^+ is a BRK-ideal of X.

Analogously, we can prove that B_α^- is a BRK-ideal of X.

Corollary 3.1. *If $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X, then the sets $\widetilde{B}^+_{\mu^+(0)}$ and $\widetilde{B}^-_{\mu^-(0)}$ are BRK-ideals of X.*

Corollary 3.2. *Let $B = (\mu^+, \mu^-)$ be BF set in X. If $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X, then for all $\gamma \in [0,1]$ the γ-cut of B is a BRK-ideal of X.*

Proof. It is analogous to the proof of Theorem 3.2.

Corollary 3.3. *If $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X, then $\widetilde{B}_{(\alpha,\beta)}$ is a BRK-ideal in X, $\forall (\alpha,\beta) \in [-1,0] \times [0,1]$. In particular, γ-cut of B is a BRK-ideal in X, for all $\gamma \in [0,1]$.*

4 Images and Pre-images of a BF BRK-ideal

In this section, we introduced images and pre-images of BF BRK-ideals and discussed some theorems.

Definition 4.1. *Let $\psi : X_1 \to X_2$ be a mapping of BRK-algebras. If $B = (\mu^+, \mu^-)$ and $V = (\upsilon^+, \upsilon^-)$ are BF set of X_1 and X_2 respectively. Then*

$$\mu^+\left(\psi^{-1}(y)\right) = \upsilon^+(y) = \begin{cases} \bigvee\limits_{x \in \psi^{-1}(y)} \mu^+(x), & \text{if } \psi^{-1}(y) \neq \emptyset, \\ 1, & \text{Otherwise}, \end{cases},$$

and

$$\mu^-\left(\psi^{-1}(y)\right) = \upsilon^-(y) = \begin{cases} \bigwedge\limits_{x \in \psi^{-1}(y)} \mu^-(x), & \text{if } \psi^{-1}(y) \neq \emptyset, \\ -1, & \text{Otherwise}, \end{cases}$$

for all $x \in R_2$ is called image of $B = (\mu^+, \mu^-)$ under ψ, where $\psi^{-1}(y) = \{x \in X_1 \psi(x) = y\}$. Also the pre-image $B = V \circ \psi$ in X_1 defined as, $\upsilon^+(\psi(x)) = \mu^+(x)$, and $\upsilon^-(\psi(x)) = \mu^-(x)$.

Theorem 4.1. *An into homomorphic pre-image of a BF BRK-ideal is also a BF BRK-ideal.*

Proof. Let $\psi : X_1 \to X_2$ be an into homomorphism of BRK-algebras. Suppose that $V = (\upsilon^+, \upsilon^-)$ be a BF BRK-ideal in $(X_2, *', 0')$ and $B = (\mu^+, \mu^-)$ be a BF BRK-ideal in $(X_1, *, 0)$. Then for all $x \in X_1$, $\mu^+(0) = \upsilon^+(\psi(0)) \geq \upsilon^+(\psi(x)) = \mu^+(x)$ and $\mu^-(0) = \upsilon^-(\psi(0)) \leq \upsilon^-(\psi(x)) = \mu^-(x)$.

Now, let $x, y \in X_1$. Then

$$\begin{aligned}\mu^+(0 * x) &= \upsilon^+(\psi(0 * x)) \\ &= \upsilon^+(\psi(0) *' \psi(x)) \\ &\geq \min\{\upsilon^+(\psi(0) *' (\psi(x) *' \psi(y))), \upsilon^+(\psi(0) *' \psi(y))\} \\ &= \min\{\upsilon^+(\psi(0 * (x * y))), \upsilon^+(\psi(0 * y))\} \\ &= \min\{\mu^+(0 * (x * y)), \mu^+(0 * y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^-(0 * x) &= \upsilon^-(\psi(0 * x)) \\ &= \upsilon^-(\psi(0) *' \psi(x)) \\ &\leq \max\{\upsilon^-(\psi(0) *' (\psi(x) *' \psi(y))), \upsilon^-(\psi(0) *' \psi(y))\} \\ &= \max\{\upsilon^-(\psi(0 * (x * y))), \upsilon^-(\psi(0 * y))\} \\ &= \max\{\mu^-(0 * (x * y)), \mu^-(0 * y)\}.\end{aligned}$$

Hence pre-image of a BF BRK-ideal is also a BF BRK-ideal.

Definition 4.2. *Let $B = (\mu^+, \mu^-)$ be a BF set in X. Then for $K \subseteq X_1$ there exist $m, n \in K$ such that $\mu^+(m) = \underset{m \in K}{\vee} \mu^+(m)$ and $\mu^-(n) = \underset{n \in K}{\wedge} \mu^-(n)$.*

Theorem 4.2. *An onto homomorphic image of a BF BRK-ideal is also a BF BRK-ideal.*

Proof. Let $\psi : X_1 \to X_2$ be an onto homomorphism of BRK-algebras and $V = (\upsilon^+, \upsilon^-)$ be a BF BRK-ideal in $(X_2, *', 0')$. Let $B = (\mu^+, \mu^-)$ be a BF BRK-ideal in $(X_1, *, 0)$ with *sup* and *inf* properties. By Definition 4.2, we get $\upsilon^+(y') = \mu^+\left(\psi^{-1}(y')\right) = \underset{x \in \psi^{-1}(y)}{\vee} \mu^+(x)$ and $\upsilon^-(y') = \mu^-\left(\psi^{-1}(y')\right) = \underset{x \in \psi^{-1}(y)}{\wedge} \mu^-(x)$ for all $y' \in X_2$. Since $B = (\mu^+, \mu^-)$ be BF BRK-ideal in X_1, we have $\mu^+(0) \geq \mu^+(x)$ and $\mu^-(0) \leq \mu^-(x)$. Note that $0 \in \psi^{-1}(0')$. Thus $\upsilon^+(0') = \mu^+\left(\psi^{-1}(0')\right) = \underset{a \in \psi^{-1}(0')}{\vee} \mu^+(a) = \mu^+(0) \geq \mu^+(x)$ and $\upsilon^-(0') = \mu^-\left(\psi^{-1}(0')\right) = \underset{a \in \psi^{-1}(0')}{\wedge} \mu^-(a) = \mu^-(0) \leq \mu^-(x)$. This implies that $\upsilon^+(0') \geq \underset{a \in \psi^{-1}(x')}{\vee} \mu^+(a) = \upsilon^+(x')$ and $\upsilon^-(0') \leq \underset{a \in \psi^{-1}(x')}{\wedge} \mu^-(a) = \upsilon^-(x')$for all $x' \in X_2$.

Now, let $x', y', z' \in X_2$ and $0_0 \in \psi^{-1}(0')$, $x_0 \in \psi^{-1}(x')$, $y_0 \in \psi^{-1}(y')$ be such that $\mu^+(0_0 * x_0) = \underset{a \in \psi^{-1}(0' * x')}{\vee} \mu^+(a)$ and $\mu^+(0_0 * y_0) = \underset{a \in \psi^{-1}(0' * y')}{\vee} \mu^+(a)$ and $\mu^+(0_0 * (x_0 * y_0)) = \upsilon^+(\psi(0_0 * (x_0 * y_0))) = \upsilon^+(0' * (x' * y')) = \underset{0_0 * (x_0 * y_0) \in \psi^{-1}(0' * (x' * y'))}{\vee} \mu^+(0_0 * (x_0 * y_0)) = \underset{a \in \psi^{-1}(0' * (x' * y'))}{\vee} \mu^+(a)$.

Thus

$$\begin{aligned}
\upsilon^{+}(0'*x') &= \bigvee_{a\in\psi^{-1}(0'*x')}\mu^{+}(a)\\
&= \mu^{+}(0_0 * x_0)\\
&\geq \min\{\mu^{+}(0_0*(x_0*y_0)),\mu^{+}(0_0*y_0)\}\\
&= \min\{\bigvee_{a\in\psi^{-1}(0'*(x'*y'))}\mu^{+}(a), \bigvee_{a\in\psi^{-1}(0'*x')}\mu^{+}(a)\}\\
&= \min\{\upsilon^{+}(0'*(x'*y')),\upsilon^{+}(0'*y')\}.
\end{aligned}$$

On the other hand, we have $\mu^{-}(0_0*x_0) = \bigwedge_{a\in\psi^{-1}(0'*x')}\mu^{-}(a)$ and $\mu^{-}(0_0*y_0) = \bigwedge_{a\in\psi^{-1}(0'*y')}\mu^{-}(a)$ and $\mu^{-}(0_0*(x_0*y_0)) = \upsilon^{-}(\psi(0_0*(x_0*y_0))) = \upsilon^{-}(0'*(x'*y')) = \bigwedge_{0_0*(x_0*y_0)\in\psi^{-1}(0'*(x'*y'))}\mu^{-}(0_0*(x_0*y_0)) = \bigwedge_{a\in\psi^{-1}(0'*(x'*y'))}\mu^{-}(a)$.

Thus,

$$\begin{aligned}
\upsilon^{-}(0'*x') &= \bigwedge_{a\in\psi^{-1}(0'*x')}\mu^{-}(a)\\
&= \mu^{-}(0_0 * x_0)\\
&\leq \max\{\mu^{-}(0_0*(x_0*y_0)),\mu^{-}(0_0*y_0)\}\\
&= \max\{\bigwedge_{a\in\psi^{-1}(0'*(x'*y'))}\mu^{-}(a), \bigwedge_{a\in\psi^{-1}(0'*x')}\mu^{-}(a)\}\\
&= \max\{\upsilon^{-}(0'*(x'*y')),\upsilon^{-}(0'*y')\}.
\end{aligned}$$

Hence onto homomorphic image of a BF BRK-ideal is also a BF BRK-ideal.

5 BF Translation and BF Extension on BF BRK-Ideals

For any BF set $B = (\lambda^{+}, \lambda^{-})$ in X, we denote $\top = 1 - \sup\{\lambda^{+}(x) \mid x \in X\}$ and $\perp = -1 - \inf\{\lambda^{-}(x) \mid x \in X\}$.

Definition 5.1. *Let $B = (\lambda^{+}, \lambda^{-})$ be a BF set in X and $(\gamma, \delta) \in [0, \top] \times [\perp, 0]$. By a BF (γ, δ)-translation of B we mean a BF set $B^{T}_{(\gamma,\delta)} = (\lambda^{+}_{(\gamma,T)}, \lambda^{-}_{(\delta,T)})$ where*

$$\begin{aligned}
\lambda^{+}_{(\gamma,T)} &: X \longrightarrow [0,1],\ x \longrightarrow \lambda^{+}(x) + \gamma,\\
\lambda^{-}_{(\delta,T)} &: X \longrightarrow [-1,0],\ x \longrightarrow \lambda^{-}(x) + \delta.
\end{aligned}$$

Theorem 5.1. *If $B = (\lambda^{+}, \lambda^{-})$ is a BF BRK-ideal in X, then the BF (γ, δ)-translation $B^{T}_{(\gamma,\delta)} = (\lambda^{+}_{(\gamma,T)}, \lambda^{-}_{(\delta,T)})$ of $B$is a BF BRK-ideal in X for all $(\gamma, \delta) \in [0, \top] \times [\perp, 0]$.*

Proof. Let $a \in X$. Then $\lambda^{+}_{(\gamma,T)}(0) = \lambda^{+}(0) + \gamma \geq \lambda^{+}(a) + \gamma = \lambda^{+}_{(\gamma,T)}(a)$, and $\lambda^{-}_{(\delta,T)}(0) = \lambda^{-}(0) + \delta \leq \lambda^{-}(a) + \delta = \lambda^{-}_{(\delta,T)}(a)$. Now, let

$$\begin{aligned}
\lambda^{+}_{(\gamma,T)}(0*x) &= \lambda^{+}(0*x) + \gamma\\
&\geq \min\{\lambda^{+}(0*(x*y)),\lambda^{+}(0*y)\} + \gamma\\
&= \min\{\lambda^{+}(0*(x*y)) + \gamma,\lambda^{+}(0*y) + \gamma\}\\
&= \min\{\lambda^{+}_{(\gamma,T)}(0*(x*y)),\lambda^{+}_{(\gamma,T)}(0*y)\},
\end{aligned}$$

and

$$\begin{aligned}\lambda^-_{(\delta,T)}(0*x) &= \lambda^-(0*x)+\delta\\ &\leq \max\{\lambda^-(0*(x*y)),\lambda^-(0*y)\}+\delta\\ &= \max\{\lambda^-(0*(x*y))+\delta,\lambda^-(0*y)+\delta\}\\ &= \max\{\lambda^-_{(\delta,T)}(0*(x*y)),\lambda^-_{(\delta,T)}(0*y)\}.\end{aligned}$$

Hence $B=(\lambda^+,\lambda^-)$ is a BF BRK-ideal in X.

Theorem 5.2. *Let $B=(\lambda^+,\lambda^-)$ be a BF set in X such that the BF $(\gamma,\ \delta)$-translation $B^T_{(\gamma,\delta)}=(\lambda^+_{(\gamma,T)},\lambda^-_{(\delta,T)})$ of B is a BF BRK-ideal in X for some $(\gamma,$ $\delta)\in[0,\top]\times[\bot,0]$. Then $B=(\lambda^+,\lambda^-)$ is a BF BRK-ideal in X.*

Proof. Let $a\in X$. Then $\lambda^+(0)+\gamma=\lambda^+_{(\gamma,T)}(0)\geq\lambda^+_{(\gamma,T)}(a)=\lambda^+(a)+\gamma$, and $\lambda^-(0)+\delta=\lambda^-_{(\delta,T)}(0)\leq\lambda^-_{(\delta,T)}(a)=\lambda^-(a)+\delta$. Thus $\lambda^+(0)\geq\lambda^+(a)$ and $\lambda^-(0)\leq\lambda^-(a)$. Now, let

$$\begin{aligned}\lambda^+(0*x)+\gamma &= \lambda^+_{(\gamma,T)}(0*x)\\ &\geq \min\{\lambda^+_{(\gamma,T)}(0*(x*y)),\lambda^+_{(\gamma,T)}(0*y)\}\\ &= \min\{\lambda^+(0*(x*y))+\gamma,\lambda^+(0*y)+\gamma\}\\ &= \min\{\lambda^+(0*(x*y)),\lambda^+(0*y)\}+\gamma,\end{aligned}$$

and

$$\begin{aligned}\lambda^-(0*x)+\delta &= \lambda^-_{(\delta,T)}(0*x)\\ &\leq \max\{\lambda^-_{(\delta,T)}(0*(x*y)),\lambda^-_{(\delta,T)}(0*y)\}\\ &= \max\{\lambda^-(0*(x*y))+\delta,\lambda^-(0*y)+\delta\}\\ &= \max\{\lambda^-(0*(x*y)),\lambda^-(0*y)\}+\delta.\end{aligned}$$

Thus $\lambda^+(0*x)\geq\min\{\lambda^+(0*(x*y)),\lambda^+(0*y)\}$ and $\lambda^-(0*x)\leq\max\{\lambda^-(0*(x*y)),\lambda^-(0*y)\}$. Hence $B=(\lambda^+,\lambda^-)$ is a BF BRK-ideal in X.

Definition 5.2. *Let $B_1=(\lambda^+,\lambda^-)$ and $B_2=(\mu^+,\mu^-)$ be two BF sets in X. If $\lambda^+(x)\leq\mu^+(x)$ and $\lambda^-(x)\geq\mu^-(x)$ for all $x\in X$, then we say that B_2 is a BF extension of B_1.*

Definition 5.3. *Let $B_1=(\lambda^+,\lambda^-)$ and $B_2=(\mu^+,\mu^-)$ be two BF sets in X. Then B_2 is called a BF BRK-ideal extension of B_1 if following conditions are holds:*

(i) *B_2 is called a BF extension of B_1,*
(ii) *If B_2 is a BF BRK-ideal extension of X then B_1 is also a BF BRK-ideal extension of X.*

By definition $\lambda^+_{(\gamma,T)}\geq\lambda^+(x)$ and $\lambda^-_{(\delta,T)}\leq\lambda^-(x)$ for all $x\in X$. Therefore, we have following theorem.

Theorem 5.3. *If $B = (\lambda^+, \lambda^-)$ is a BF BRK-ideal of X, then the BF (γ, δ)-translation $B^T_{(\gamma,\delta)} = (\lambda^+_{(\gamma,T)}, \lambda^-_{(\delta,T)})$ of Bis a BF extension of X for all $(\gamma, \delta) \in [0, \top] \times [\bot, 0]$.*

Converse of above theorem is not true in general as seen in following Example.

Example 3. Consider BRK-algebra $X = \{0, a_1, a_2, a_3\}$ defined in Example 2. Let $B_1 = (\lambda^+, \lambda^-)$ be a BF set in X defined by

	0	a_1	a_2	a_3
λ^+	0.4	0.4	0.2	0.2
λ^-	−0.2	−0.2	−0.1	−0.1

Clearly, B is a BF BRK-ideal of X. Let $B_2 = (\mu^+, \mu^-)$ be a BF set of X defined by

	0	a_1	a_2	a_3
μ^+	0.42	0.42	0.4	0.65
μ^-	−0.23	−0.23	−0.23	−0.66

Then B_2 is a BF BRK-ideal extension of B_1 but it is not BF (γ, δ)-translation of B_1, for all $(\gamma, \delta) \in [0, \top] \times [\bot, 0]$.

Definition 5.4. *Let $B = (\mu^+, \mu^-)$ be a BF set in X, $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, $\gamma \in [0, \top]$ and $\delta \in [\bot, 0]$. We define $B^{+T}_{(\beta,\gamma)} = \{x \in X : \mu^+(x) \geq \beta - \gamma\}$, $B^{-T}_{(\alpha,\delta)} = \{x \in X : \mu^-(x) \leq \alpha - \delta\}$ and $B^T_{((\alpha,\beta),(\gamma,\delta))} = \{x \in X : \mu^-(x) \leq \alpha - \gamma$ and $\mu^+(x) \geq \beta - \delta\}$.*

Theorem 5.4. *If $B = (\mu^+, \mu^-)$ is BF BRK-ideal of X, then $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are BRK-ideal of X for all $\alpha \in Im(\mu^-)$ and $\beta \in Im(\mu^+)$ with $\beta \geq \gamma$ and $\alpha \leq \delta$.*

If we do not take the condition that $B = (\mu^+, \mu^-)$ is BF BRK-ideal of X, then either both $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are not BRK-ideals of X, or one of them is not a BRK-ideal of X.

Example 4. Consider BRK-algebra $X = \{0, 1, 2\}$ with the following operation

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Let $B = (\mu^+, \mu^-)$ be a BF set of X defined by

	0	1	2
μ^+	0.5	0.2	0.4
μ^-	−0.6	−0.4	−0.5

Clearly, B is a BF BRK-ideal of X. Take $\beta = 0.32$, $\alpha = -0.35$, $\gamma = 0.05$ and $\delta = -0.04$. Then $B^{+T}_{(\beta,\gamma)} = \{0, 2\}$ and $B^{-T}_{(\alpha,\delta)} = \{0, 1, 2\}$ are BRK-ideals of X. On the other hand, BF set $B = (\mu'^+, \mu'^-)$ defined by

	0	2	1
μ'^{+}	0.6	0.5	0.7
μ'^{-}	-0.9	-0.7	-0.5

is not a BF BRK-ideal of X. Take $\beta = 0.55$, $\alpha = -0.8$, $\gamma = 0.03$ and $\delta = -0.07$. Then $B^{+T}_{(\beta,\gamma)} = \{0, 1\}$ is not a BRK-ideal of X, but $B^{-T}_{(\alpha,\delta)} = \{0\}$ is a BRK-ideal of X.

Corollary 5.1. *If $B = (\mu^+, \mu^-)$ is BF BRK-ideal of X, then $B^T_{((\alpha,\beta),(\gamma,\delta))}$ is a BRK-ideal of X, for all $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ and $(\gamma,\ \delta) \in [0, \top] \times [\perp, 0]$.*

Theorem 5.5. *Let $B = (\mu^+, \mu^-)$ be a BF set of X. Then the BF $(\gamma,\ \delta)$-translation of B is a BF BRK-ideal of X if and only if $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are BRK-ideal of X for all $\beta \in Im(\mu^+)$, $\alpha \in Im(\mu^-)$ and $(\gamma,\ \delta) \in [0, \top] \times [\perp, 0]$ with $\alpha < \delta$ and $\beta > \gamma$.*

Proof. Assume that $B^T_{(\gamma,\delta)} = (\mu^+_{(\gamma,T)}, \mu^-_{(\delta,T)})$ is a BF BRK-ideal of X. Let $x \in B^{+T}_{(\beta,\gamma)}$. Then $\mu^+(x) \geq \beta - \gamma$ such that $\mu^+_{(\gamma,T)}(0) \geq \mu^+_{(\gamma,T)}(x) \geq \beta$. This shows that $\mu^+(0) + \gamma \geq \beta$. Thus $\mu^+(0) \geq \beta - \gamma$, which implies that $0 \in B^{+T}_{(\beta,\gamma)}$. Now, let $0 * (a * b) \in B^{+T}_{(\beta,\gamma)}$ and $0 * b \in B^{+T}_{(\beta,\gamma)}$ for all $a, b \in X$. Since $\mu^+(0 * a) \geq \min\{\mu^+_{(\gamma,T)}(0 * (a * b)), \mu^+_{(\gamma,T)}(0 * b)\} \geq \beta$. This shows that $\mu^+(0 * a) + \gamma \geq \beta$. Thus $\mu^+(0 * a) \geq \beta - \gamma$, which implies that $0 * a \in B^{+T}_{(\beta,\gamma)}$. Hence $B^{+T}_{(\beta,\gamma)}$ is a BRK-ideal of X. Analogously, we can prove that $B^{-T}_{(\alpha,\delta)}$ is a BRK-ideal of X.

Conversely, suppose $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are BRK-ideals of X for all $\beta \in Im(\mu^+)$, $\alpha \in Im(\mu^-)$ and $(\gamma,\ \delta) \in [0, \top] \times [\perp, 0]$ with $\alpha < \delta$ and $\beta > \gamma$. Assume that there exist $a \in X$, such that $\mu^+_{(\gamma,T)}(0) < \beta' < \mu^+_{(\gamma,T)}(a)$ and $\mu^-_{\delta,T)}(0) > \alpha' > \mu^-_{\delta,T)}(a)$. Then $\mu^+(0) > \beta' - \gamma$, $\mu^+(0) > \beta' - \gamma$, $\mu^-(0) < \alpha' - \delta$ and $\mu^-(0) < \alpha' - \delta$. This shows that $a \in B^{+T}_{(\beta,\gamma)}$ and $a \in B^{-T}_{(\alpha,\delta)}$, but $0 \notin B^{+T}_{(\beta,\gamma)}$ and $0 \notin B^{-T}_{(\alpha,\delta)}$ which is contradiction. Hence $\mu^+_{(\gamma,T)}(0) \geq \mu^+_{(\gamma,T)}(a)$ and $\mu^-_{(\delta,T)}(0) \leq \mu^-_{(\delta,T)}(a)$. Similarly, we can prove that $\mu^+_{(\gamma,T)}(0 * a) \geq \min\{\mu^+_{(\gamma,T)}(0 * (a * b)), \mu^+_{(\gamma,T)}(0 * b)\}$, $\mu^-_{(\delta,T)}(0 * a) \leq \max\{\mu^-_{(\delta,T)}(0 * (a * b)), \mu^-_{(\delta,T)}(0 * b)\}$. Hence BF $(\gamma,\ \delta)$-translation of B is a BF BRK-ideal of X.

Theorem 5.6. *Let $B = (\lambda^+, \lambda^-)$ be a BF BRK-ideal in X, $(\gamma,\ \delta) \in [0, \top] \times [\perp, 0]$ and $(\gamma',\ \delta') \in [0, \top] \times [\perp, 0]$. If $(\gamma,\ \delta) \geq (\gamma',\ \delta')$, then BF $(\gamma,\ \delta)$-translation $B^T_{(\gamma,\delta)}$ of B is a BF BRK-ideal extension of $(\gamma',\ \delta')$-translation $B^T_{(\gamma',\delta')}$ of B.*

Theorem 5.7. *Let $B = (\lambda^+, \lambda^-)$ be a BF BRK-ideal in X and $(\gamma,\ \delta) \in [0, \top] \times [\perp, 0]$. For every BF BRK-ideals extension $B' = (\nu^+, \nu^-)$ of the BF $(\gamma,\ \delta)$-translation $B^T_{(\gamma,\delta)}$ of B, there exist $(\gamma',\ \delta') \in [0, \top] \times [\perp, 0]$ such that $(\gamma,\ \delta) \geq (\gamma',\ \delta')$ and B' is a BF BRK-ideal extension of the BF $(\gamma',\ \delta')$-translation $B^T_{(\gamma',\delta')}$ of B.*

Proof. Assume that for every BF BRK-ideal extension $B' = (\nu^+, \nu^-)$ of $B^T_{(\gamma,\delta)}$, there does not exist $(\gamma', \delta') \in [0, \top] \times [\perp, 0]$ $(\gamma, \delta) \geq (\gamma', \delta')$ such that B' is not BF BRK-ideal extension of $B^T_{(\gamma',\delta')}$. Then $\nu^+(x) > \mu^+_{(\gamma',T)}(x) = \mu^+(x) + \gamma'$ and $\nu^-(x) < \mu^-_{(\delta',T)}(x) = \mu^-(x) + \delta'$ for $x \in R$. Since, $(\gamma, \delta) < (\gamma', \delta')$ so that $\nu^+(x) > \mu^+(x) + \gamma' > \mu^+(x) + \gamma$ and $\nu^-(x) < \mu^-(x) + \delta' < \mu^-(x) + \delta$. This shows that $\nu^+(x) > \mu^+_{(\gamma,T)}(x)$ and $\nu^-(x) < \mu^-_{(\delta,T)}(x)$ which is contradiction. Thus, there exist $(\gamma', \delta') \in [0, \top] \times [\perp, 0]$ such that $(\gamma, \delta) \geq (\gamma', \delta')$ and B' is a BF BRK-ideal extension of $B^T_{(\gamma',\delta')}$.

Definition 5.5. *Let $B = (\lambda^+, \lambda^-)$ be a BF set of X and $\zeta, \eta \in [0, 1]$. By a BF (ζ, η)-multiplication of B we mean a BF set $B^m_{(\zeta,\eta)} = (\lambda^{+m}_\zeta, \lambda^{-m}_\eta)$ express as*

$$\lambda^{+m}_\zeta : X \longrightarrow [0, 1],\ x \longrightarrow \lambda^+(x)\,\zeta,$$
$$\lambda^{-m}_\eta : X \longrightarrow [-1, 0],\ x \longrightarrow \lambda^-(x)\,\eta.$$

For a BF BRK-ideal, BF (0, 0)-multiplication $B^m_{(0,0)}$ is a BF BRK-ideal of X.

Theorem 5.8. *If $B = (\lambda^+, \lambda^-)$ is a BF BRK-ideal of X, then BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X.*

Theorem 5.9. *Let $B = (\lambda^+, \lambda^-)$ be a BF set of X. Then BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X if and only if B is a BF BRK-ideal of X for some $\zeta, \eta \in [0, 1]$.*

Proof. Necessity obtains from Theorem 5.8. Let $\zeta, \eta \in [0, 1]$ such that BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X. Let $x \in X$. Then $\lambda^+(0)\,\zeta = \lambda^{+m}_\zeta(0) \geq \lambda^{+m}_\zeta(x) \geq \lambda^+(x)\,\zeta$, $\lambda^-(0)\,\eta = \lambda^{+m}_\eta(0) \leq \lambda^{+m}_\eta(x) \leq \lambda^+(x)\,\eta$ which implies that $\lambda^+(0) \geq \lambda^+(x)$. Analogously, $\lambda^-(0) \leq \lambda^-(x)$. Next, let for $x, y \in X$

$$\begin{aligned}
\lambda^+(0 * x)\,\zeta &= \lambda^{+m}_\zeta(0 * x) \\
&\geq \min\{\lambda^{+m}_\zeta(0 * (x * y)), \lambda^{+m}_\zeta(0 * y)\} \\
&= \min\{\lambda^+(0 * (x * y))\,\zeta, \lambda^+(0 * y)\,\zeta\} \\
&= \min\{\lambda^+(0 * (x * y)), \lambda^+(0 * y)\}\zeta,
\end{aligned}$$

which implies that $\lambda^+(x) \geq \min\{\lambda^+(0 * (x * y)), \lambda^+(0 * y)\}$. And,

$$\begin{aligned}
\lambda^-(0 * x)\,\eta &= \lambda^{-m}_\eta(0 * x) \\
&\leq \max\{\lambda^{-m}_\eta(0 * (x * y)), \lambda^{-m}_\eta(0 * y)\} \\
&= \max\{\lambda^-(0 * (x * y))\,\eta, \lambda^-(0 * y)\,\eta\} \\
&= \max\{\lambda^-(0 * (x * y)), \lambda^-(0 * y)\}\eta,
\end{aligned}$$

which implies that $\lambda^-(x) \leq \max\{\lambda^-(0 * (x * y)), \lambda^-(0 * y)\}$ for $x, y \in X$. Hence B is a BF BRK-ideal of X.

Theorem 5.10. *Let $B = (\lambda^+, \lambda^-)$ be a BF set of X, $(\gamma,\ \delta) \in [0, \top] \times [\bot, 0]$ and $\zeta,\ \eta \in (0, 1]$. Then every BF $(\gamma,\ \delta)$-translation $B^T_{(\gamma,\delta)}$ of B is a BF BRK-ideal extension of the $(\zeta,\ \eta)$-multiplication $B^m_{(\zeta,\eta)}$ of B.*

Proof. For all $a \in R$, we have $\mu^+_{(\gamma,T)}(a) = \mu^+(a) + \gamma \geq \mu^+(a) \geq \mu^+(a)\zeta = \mu^{+m}_{\zeta}(a)$ and $\mu^-_{(\delta,T)}(a) = \mu^-(a) + \delta \leq \mu^-(a) \leq \mu^-(a)\eta = \mu^{-m}_{\eta}(a)$. Thus $B^T_{(\gamma,\delta)}$ is a BF extension of the $(\zeta,\ \eta)$-multiplication $B^m_{(\zeta,\eta)}$ of B. Assume that BF $(\zeta,\ \eta)$-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X. Then by Theorem 5.9, B is a BF BRK-ideal of X. By Theorem 5.1, $B^T_{(\gamma,\delta)}$ is a BF BRK-ideal of X. Thus every BF $(\gamma,\ \delta)$-translation is a BF BRK-ideal extension of the $(\zeta,\ \eta)$-multiplication of B.

6 Conclusion

In this study, we investigated BF BRK-ideals in BRK-algebras and discussed related properties. We introduced translation, extension and multiplications of BF BRK-ideals in BRK-algebras and discussed related results. As an extension of above results, one could study bipolar anti fuzzy BRK-algebras and BF ideals in other algebraic structures. Applications of the method to consider the related problems in machines learning, decision makings, information sciences, cognitive science, intelligent decision-making system, and so on.

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