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Bing-Yuan Cao *Editor*

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Preface

We had three conferences held successfully in 2016.

The International Conference on Mathematics and Decision Science, ICMDS 2016, September 12–15, 2016, was arranged at Guangzhou University, Guangzhou, China (www.icodm2020.com). Chairman Prof. Bing-Yuan Cao, Chairman of the Program Committee Prof. Seyed Hadi Nasseri, and Chairman of the Iranian Operations Research Society Prof. Nezamedin Mahdavi-Amiri ensured that the conference contained top-level presentations of great general interest. The researchers giving presentations at the conference represented researchers from many countries and parts of the world. The beautiful Guangzhou University and parks gave a relaxing discussion climate and many new meetings, more than 50 experts and professors and students attended the meeting. Delegates from China, Iran, Sweden, Poland, Canada, Pakistan, and other countries, nearly 20 people, read out academic papers.

Academic Conference on 30th Anniversary of Fuzzy Geometric Programming and 40th Education Year by and of Professor Cao Bingyuan (ACFGPAEC) has been held at Guangzhou University from July 30 to August 1, 2016. Professor Yu-bin Zhong, the Vice President and Secretary-General of Fuzzy Information and Engineering Society and Dean Assistant for School of Mathematics and Information Science of Guangzhou University gave a message to Professor Cao. The celebration was presided over by Dr. Ji-hui Yang, an Associate Professor from Shenyang Agricultural University, with more than 30 scholars and postdoctoral students from home and abroad attending it. More than 10 of Professor Cao's postdoctoral, doctoral (including Irans and Pakistans) and master reported their work in recent years, among whom many people had published, papers in the Fuzzy Sets and Systems, IEEE T on Fuzzy Systems, Information Sciences etc. magazine.

The third annual meeting of Guangdong Operational Research Society (TAMGORS) was held on October 22–23, 2016, in Foshan University, Guangdong. Honorary Chair of the Society, Academician Jing-zhong Zhang of the Chinese Academy of Sciences attended and made an academic report on the new thinking of calculus. Professor Hao Zhifeng, Vice Chairman of the Society and President of Foshan University, Professor Cao Bingyuan, respectively, made their work reports and important speeches. Members of the institute, more than 40 entrepreneurs, and related students from Foshan University attended the meeting. More than 10 people from the Institute and entrepreneurs, persons from Iran, Pakistan, represented their papers.

We carefully organized them, adopting experts' recommendation and critical way of reviews, collected, and published the papers in Intelligent Systems and Computing. The Advance focuses on five main topics as follows:

I. Mathematics and Fuzziness;
II. Decision and Fuzziness;
III. Fuzzy geometric programming and Optimization;
IV. Fuzzy Systems & Operations Research and Management;
V. Others.

Here, Topic I II, from ICMDS 2016, one of the world's continuous International Conference Papers 18 (the Conference was held in Iran after the first International Conference on Mathematics). We've chosen nine papers on Mathematics and Fuzziness. Topic III collected papers on AMPFGPAT2016 Fuzzy geometric programming and Optimization 9. Topic IV V is selected from TAMGORS's 17 paper. We carefully selected 44 papers to form the book.

Heartfelt thanks to Dr. Seyed Hadi Nasseri University of Mazandaran, Iran, who has done a lot of work for launching the ICMDS 2016, conference organization and reviewers. Thanks to Dr. Xue-gang Zhou and Dr. Ji-hui Yang for organizing such a good meeting with AMPFGPAT. Appreciation to Guangzhou University for its great support for the conference and for the preparation and support of the Foshan University for the TAMGORS. Thanks to Lu Shu-quan for his full help.

Finally, I would like to thank the publisher, Springer editors, for publishing the proceeding as Advance in Intelligent Systems and Computing.

December 2016 Bing-yuan Cao

Organization

The International Conference on Mathematics and Decision Science (ICMDS 2016)

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Cao Bing-yuan

Contents

Part II: Decision and Fuzziness

Part III: Fuzzy Geometric Programming and Optimization

Part I: Mathematics and Fuzziness

Bipolar Fuzzy BRK-ideals in BRK-algebras

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Abstract. In this paper, we investigated bipolar fuzzy BRK-ideals in BRK-algebras and discussed related properties. We presented some results on images and pre-images of bipolar fuzzy BRK-ideals in BRK-algebras. Finally, we introduced translation, extension and multiplications of bipolar fuzzy BRK-ideals in BRK-algebras and discussed related results.

Keywords: Bipolar fuzzy BRK-ideal · BRK-algebra · Images · Pre-images · Translations · Extensions · Multiplications

1 Introduction

The fundamental concept of fuzzy set, popularized by Zadeh [1], was used to generalize several basic concepts of algebra. Fuzzy sets are extremely useful to deal with the many problems in applied mathematics, control engineering, information sciences, expert systems and theory of automata etc. Although, there are many generalizations of fuzzy sets but none of these deal with the problems related to the contrary characteristics of the members having membership degree 0. Lee [2] handled this problem by introducing the concept of bipolar fuzzy (BF) sets. The BF set theory has been widely applied to solve real life problems. The sweet taste of foodstuffs is a BF set. Assuming that sweet taste of foodstuff as a positive membership value then bitter taste of foodstuffs as a negative membership value. The remaining foodstuffs of taste like acidic, saline, chilly etc. are extraneous to the sweet and bitter foodstuffs. Thus, these foodstuffs are accepted as zero membership values. Notice that every matter has two sides and bipolarity as well as fuzziness, is an inherent and internal part of human thinking [3,4]. A BF set is a pair of fuzzy sets, namely a membership and a non-membership function, which represent positive and negative aspects of the given information.

Imai and Iseki investigated two classes of abstract algebras: BCI-algebras and BCK-algebras [5]. In 2002, Neggers et al. [6], presented B-algebra

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and discussed related properties. The generalization of B-algebra called BF-algebra and BG-algebra proposed by Walendziak [7], and Kim [8], respectively. Recently, Bandaru [9], investigated BRK-algebra which is a generalization of $BCK/BCI/Q$-algebras. In [10,11], El-Gendy introduced fuzzy BRK-ideal of BRK-algebra and cubic BRK-ideal of BRK-algebra. Some properties of n-dimensional fuzzy subalgebra in BRK-algebras investigated by Zulfiqar [12]. Fuzzy translations and fuzzy multiplications of BCK/BCI-algebras presented in [13]. As a generalization of fuzzy set theory, BF set theory makes descriptions of the objective world more realistic, practical and very accurate in many cases, making it very promising. In the past few decades, BF set theory has been successfully applied to various algebraic structures. Lee [14], introduced BF ideals of BCK/BCI-algebras and Akram [15] introduced BF graphs. Recently, Hayat et al. [16,17], characterized himi-rings by their BF h-ideals and BAF h-ideals.

The contents of the present paper are organized as follows: In Sect. 2, we presented some basic definitions and preliminaries. In Sect. 3, we investigated bipolar fuzzy BRK-ideals in BRK-algebras and discussed related properties. In Sect. 4, we presented some results on images and pre-images of bipolar fuzzy BRK-ideals in BRK-algebras. In Sect. 5, we introduced translation, extension and multiplications of bipolar fuzzy BRK-ideals in BRK-algebras and discussed related results. Finally, we presented some conclusions and future work.

2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

Definition 2.1. *A BRK-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following conditions:*

(BRK_1) $x * 0 = x$,
(BRK_2) $(x * y) * x = 0 * y$ *for all* $x, y \in X$.

A partial ordered relation $\preceq$ can be defined by $x \preceq y$ if and only if $x * y = 0$. Throughout this paper, X denotes BRK-algebra.

Definition 2.2 [9]. *If $(X, *, 0)$ is a BRK-algebra, the following conditions hold:*

(BRK_3) $x * x = 0$,
(BRK_4) $(x * y) = 0$ *implies* $0 * x = 0 * y$ *for all* $x, y \in X$,
(BRK_5) $0 * (a * b) = (0 * a) * (0 * b)$ *for all* $a, b \in X$.

Definition 2.3 [11,12]. *A subset S of a BRK-algebra X is said to be BRK-subalgebra of X, if $x, y \in S$, implies $x * y \in S$.*

Definition 2.4 [11]. *A subset S of a BRK-algebra X is said to be a BRK-ideal of X (briefly $S_{BRK}X$) if it satisfies:*

(i) $0 \in S$,
(ii) $0 * (x * y) \in S$ *and* $0 * y \in S \implies 0 * x \in S$ *for all* $x, y \in X$.

Example 1. Consider a set $X = \{0, a_1, a_2, a_3\}$. We define "$*$" on X as the following table:

$*$	0	a_1	a_2	a_3
0	0	a_2	a_2	0
a_1	a_1	0	0	a_2
a_2	a_2	0	0	a_2
a_3	a_3	a_1	a_1	0

Clearly, X is a BRK-algebra. Then $S = \{0, a_1, a_2\}_{BRK} X$.

Definition 2.5 [11]. *Let $(X_1, *, 0)$ and $(X_2, *', 0')$ be two BRK-algebras. A mapping $\psi : X_1 \longrightarrow X_2$ is said to be a homomorphism if $\psi(x * y) = \psi(x) *' \psi(y)$, for all $x, y \in X_1$.*

Definition 2.6 [3]. *A bipolar fuzzy set is a pair (λ^+, λ^-), where $\lambda^+ : X \longrightarrow [0, 1]$, and $\lambda^- : X \longrightarrow [-1, 0]$ are any mappings.*

Definition 2.7 [3]. Let $B_1 = (\lambda^+, \lambda^-)$ and $B_2 = (\mu^+, \mu^-)$ be two BF sets in X. $Then following conditions hold:$

(i) $B_1 \leq B_2$ *if and only if* $\lambda^+ \leq \mu^+$ *and* $\lambda^- \geq \mu^-$.
(ii) $\max\{B_1, B_2\} = (\max\{\lambda^+, \mu^+\}, \min\{\lambda^-, \mu^-\}),$
$\min\{B_1, B_2\} = (\min\{\lambda^+, \mu^+\}, \max\{\lambda^-, \mu^-\}).$

3 Bipolar Fuzzy *BRK*-ideals in *BRK*-algebras

In this section, we introduced BF BRK-ideals in BRK-algebras and discussed related properties.

Definition 3.1. *A BF set $B = (\lambda^+, \lambda^-)$ of X is called BF BRK-ideal of X if it satisfies following conditions hold:*

(BF$_1$) $\lambda^+(0) \geq \lambda^+(x), \lambda^-(0) \leq \lambda^-(x),$
(BF$_2$) $\lambda^+(0 * x) \geq \min\{\lambda^+(0 * (x * y)), \lambda^+(0 * y)\},$
$\lambda^-(0 * x) \leq \max\{\lambda^-(0 * (x * y)), \lambda^-(0 * y)\}.$

Example 2. Consider a set $X = \{0, a_1, a_2, a_3\}$. We define "$*$" on X as the following table:

$*$	0	a_1	a_2	a_3
0	0	a_1	0	a_1
a_1	a_1	0	a_1	0
a_2	a_2	a_1	0	a_1
a_3	a_3	a_2	a_3	0

Clearly, X is a BRK-algebra. Define a bipolar fuzzy set,

	0	a_1	a_2	a_3
λ^+	t_1	t_1	t_2	t_2
λ^-	s_1	s_1	s_2	s_3

where $t_1, t_2 \in [0,1]$ and $s_1, s_2, s_3 \in [-1,0]$ with $t_1 > t_2$ and $s_1 < s_2 < s_3$, routine calculation gives that $B = (\lambda^+, \lambda^-)$ is a BF BRK-ideal of X.

Lemma 3.1. *Let B be a BF BRK-ideal of BRK-algebra X. If $y * x \leq x$ holds in X, then $\lambda^+(0 * y) \geq \lambda^+(0 * x)$ and $\lambda^-(0 * y) \leq \lambda^-(0 * x)$.*

Proof. Assume that $y * x \leq x$ holds in X. Then $(y * x) * x = 0$. By (BRK_2),

$$\begin{aligned}\lambda^+(0 * y) &\geq \min\{\lambda^+(0 * (y * x)), \lambda^+(0 * x)\},\\ \lambda^-(0 * y) &\leq \max\{\lambda^-(0 * (y * x)), \lambda^-(0 * x)\}.\end{aligned}$$

Also,

$$\begin{aligned}\lambda^+(0 * (y * x)) &\geq \min\left\{\lambda^+((0 * (y * x)) * x), \lambda^+(0 * x)\right\}\\ &= \min\left\{\lambda^+(0), \lambda^+(0 * x)\right\}\\ &= \lambda^+(0 * x),\end{aligned}$$

and

$$\begin{aligned}\lambda^-(0 * (y * x)) &\leq \max\left\{\lambda^-((0 * (y * x)) * x), \lambda^-(0 * x)\right\}\\ &= \max\left\{\lambda^-(0), \lambda^-(0 * x)\right\}\\ &= \lambda^-(0 * x).\end{aligned}$$

Hence $\lambda^+(0 * y) \geq \lambda^+(0 * x)$ and $\lambda^-(0 * y) \leq \lambda^-(0 * x)$.

Lemma 3.2. *Let B be a BF BRK-ideal of BRK-algebra X. If $x \leq y$ holds in X, then $\lambda^+(0 * x) \geq \lambda^+(0 * y)$ and $\lambda^-(0 * x) \leq \lambda^-(0 * y)$.*

Theorem 3.1. *Let $B_i = \{(\lambda_i^+, \lambda_i^-) : i \in \Omega\}$ be a family of BF BRK-ideals in X. Then $B = \underset{i\in\Omega}{\wedge} B_i$ is also a BF BRK-ideal in X, where $B = (\lambda^+, \lambda^-)$ that is $\lambda^+ = \underset{i\in\Omega}{\wedge} \lambda_i^+$ and $\lambda^- = \underset{i\in\Omega}{\vee} \lambda_i^-$ ($\lambda^+ \leq \lambda_i^+, \lambda^- \geq \lambda_i^-$ $\forall\ i \in \Omega$).*

Definition 3.2. *Let $B = (\mu^+, \mu^-)$ be BF set in BRK-algebra X and $(\alpha, \beta) \in [-1,0] \times [0,1]$, then*

1. *The set $\widetilde{B}_\beta^+ = \{x \in R : \mu^+(x) \geq \beta\}$ is called positive β-cut of B.*
2. *The set $B_\alpha^- = \{x \in R : \mu^-(x) \leq \alpha\}$ is called negative α-cut of B.*
3. *The set $B_{(\alpha,\beta)} = \{x \in R : \mu^-(x) \leq \alpha$ and $\mu^+(x) \geq \beta\}$ is called (α, β)-cut of B.*

For every $\gamma \in (0,1]$ and $B_\gamma^+ \cap B_{-\gamma}^-$ is called γ-cut of B.

Theorem 3.2. *A BF set $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X iff the followings hold:*

(i) *For all $\beta \in [0,1]$, B^+_β is non-empty this implies B^+_β is a BRK-ideal of X.*
(ii) *For all $\alpha \in [-1,0]$, B^-_α is non-empty this implies B^-_α is a BRK-ideal of X.*

Proof. Let $B = (\mu^+, \mu^-)$ be a BF BRK-ideal in X. For $x \in B^+_\beta$ so $\mu^+(x) \geq \beta$ where $\beta \in [0,1]$. Now $\mu^+(0) \geq \mu^+(x) \geq \beta$. This implies that $0 \in B^+_\beta$.

Next, let $0*(x*y) \in B^+_\beta$ and $0*y \in B^+_\beta$, this means that $\mu^+(0*(x*y)) \geq \beta$ and $\mu^+(0*y) \geq \beta$. Then $\lambda^+(0*x) \geq \min\{\lambda^+(0*(x*y)), \lambda^+(0*y)\} \geq \beta$. Hence B^+_β is a BRK-ideal of X.

Analogously, we can prove that B^-_α is a BRK-ideal of X.

Corollary 3.1. *If $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X, then the sets $\widetilde{B}^+_{\mu^+(0)}$ and $\widetilde{B}^-_{\mu^-(0)}$ are BRK-ideals of X.*

Corollary 3.2. *Let $B = (\mu^+, \mu^-)$ be BF set in X. If $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X, then for all $\gamma \in [0,1]$ the γ-cut of B is a BRK-ideal of X.*

Proof. It is analogous to the proof of Theorem 3.2.

Corollary 3.3. *If $B = (\mu^+, \mu^-)$ is a BF BRK-ideal in X, then $\widetilde{B}_{(\alpha,\beta)}$ is a BRK-ideal in X, $\forall (\alpha, \beta) \in [-1,0] \times [0,1]$. In particular, γ-cut of B is a BRK-ideal in X, for all $\gamma \in [0,1]$.*

4 Images and Pre-images of a BF BRK-ideal

In this section, we introduced images and pre-images of BF BRK-ideals and discussed some theorems.

Definition 4.1. *Let $\psi : X_1 \to X_2$ be a mapping of BRK-algebras. If $B = (\mu^+, \mu^-)$ and $V = (\upsilon^+, \upsilon^-)$ are BF set of X_1 and X_2 respectively. Then*

$$\mu^+\left(\psi^{-1}(y)\right) = \upsilon^+(y) = \begin{cases} \bigvee_{x \in \psi^{-1}(y)} \mu^+(x), & \text{if } \psi^{-1}(y) \neq \emptyset, \\ 1, & \text{Otherwise}, \end{cases},$$

and

$$\mu^-\left(\psi^{-1}(y)\right) = \upsilon^-(y) = \begin{cases} \bigwedge_{x \in \psi^{-1}(y)} \mu^-(x), & \text{if } \psi^{-1}(y) \neq \emptyset, \\ -1, & \text{Otherwise}, \end{cases}$$

for all $x \in R_2$ is called image of $B = (\mu^+, \mu^-)$ under ψ, where $\psi^{-1}(y) = \{x \in X_1 \psi(x) = y\}$. Also the pre-image $B = V \circ \psi$ in X_1 defined as, $\upsilon^+(\psi(x)) = \mu^+(x)$, and $\upsilon^-(\psi(x)) = \mu^-(x)$.

Theorem 4.1. *An into homomorphic pre-image of a BF BRK-ideal is also a BF BRK-ideal.*

Proof. Let $\psi : X_1 \to X_2$ be an into homomorphism of BRK-algebras. Suppose that $V = (\upsilon^+, \upsilon^-)$ be a BF BRK-ideal in $(X_2, *', 0')$ and $B = (\mu^+, \mu^-)$ be a BF BRK-ideal in $(X_1, *, 0)$. Then for all $x \in X_1$, $\mu^+(0) = \upsilon^+(\psi(0)) \geq \upsilon^+(\psi(x)) = \mu^+(x)$ and $\mu^-(0) = \upsilon^-(\psi(0)) \leq \upsilon^-(\psi(x)) = \mu^-(x)$.

Now, let $x, y \in X_1$. Then

$$\begin{aligned}\mu^+(0 * x) &= \upsilon^+(\psi(0 * x)) \\ &= \upsilon^+(\psi(0) *' \psi(x)) \\ &\geq \min\{\upsilon^+(\psi(0) *' (\psi(x) *' \psi(y))), \upsilon^+(\psi(0) *' \psi(y))\} \\ &= \min\{\upsilon^+(\psi(0 * (x * y))), \upsilon^+(\psi(0 * y))\} \\ &= \min\{\mu^+(0 * (x * y)), \mu^+(0 * y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^-(0 * x) &= \upsilon^-(\psi(0 * x)) \\ &= \upsilon^-(\psi(0) *' \psi(x)) \\ &\leq \max\{\upsilon^-(\psi(0) *' (\psi(x) *' \psi(y))), \upsilon^-(\psi(0) *' \psi(y))\} \\ &= \max\{\upsilon^-(\psi(0 * (x * y))), \upsilon^-(\psi(0 * y))\} \\ &= \max\{\mu^-(0 * (x * y)), \mu^-(0 * y)\}.\end{aligned}$$

Hence pre-image of a BF BRK-ideal is also a BF BRK-ideal.

Definition 4.2. *Let $B = (\mu^+, \mu^-)$ be a BF set in X. Then for $K \subseteq X_1$ there exist $m, n \in K$ such that $\mu^+(m) = \bigvee_{m \in K} \mu^+(m)$ and $\mu^-(n) = \bigwedge_{n \in K} \mu^-(n)$.*

Theorem 4.2. *An onto homomorphic image of a BF BRK-ideal is also a BF BRK-ideal.*

Proof. Let $\psi : X_1 \to X_2$ be an onto homomorphism of BRK-algebras and $V = (\upsilon^+, \upsilon^-)$ be a BF BRK-ideal in $(X_2, *', 0')$. Let $B = (\mu^+, \mu^-)$ be a BF BRK-ideal in $(X_1, *, 0)$ with *sup* and *inf* properties. By Definition 4.2, we get $\upsilon^+(y') = \mu^+(\psi^{-1}(y')) = \bigvee_{x \in \psi^{-1}(y)} \mu^+(x)$ and $\upsilon^-(y') = \mu^-(\psi^{-1}(y')) = \bigwedge_{x \in \psi^{-1}(y)} \mu^-(x)$ for all $y' \in X_2$. Since $B = (\mu^+, \mu^-)$ be BF BRK-ideal in X_1, we have $\mu^+(0) \geq \mu^+(x)$ and $\mu^-(0) \leq \mu^-(x)$. Note that $0 \in \psi^{-1}(0')$. Thus $\upsilon^+(0') = \mu^+(\psi^{-1}(0')) = \bigvee_{a \in \psi^{-1}(0')} \mu^+(a) = \mu^+(0) \geq \mu^+(x)$ and $\upsilon^-(0') = \mu^-(\psi^{-1}(0')) = \bigwedge_{a \in \psi^{-1}(0')} \mu^-(a) = \mu^-(0) \leq \mu^-(x)$. This implies that $\upsilon^+(0') \geq \bigvee_{a \in \psi^{-1}(x')} \mu^+(a) = \upsilon^+(x')$ and $\upsilon^-(0') \leq \bigwedge_{a \in \psi^{-1}(x')} \mu^-(a) = \upsilon^-(x')$for all $x' \in X_2$.

Now, let $x', y', z' \in X_2$ and $0_0 \in \psi^{-1}(0')$, $x_0 \in \psi^{-1}(x')$, $y_0 \in \psi^{-1}(y')$ be such that $\mu^+(0_0 * x_0) = \bigvee_{a \in \psi^{-1}(0' * x')} \mu^+(a)$ and $\mu^+(0_0 * y_0) = \bigvee_{a \in \psi^{-1}(0' * y')} \mu^+(a)$ and $\mu^+(0_0 * (x_0 * y_0)) = \upsilon^+(\psi(0_0 * (x_0 * y_0))) = \upsilon^+(0' * (x' * y')) = \bigvee_{0_0 * (x_0 * y_0) \in \psi^{-1}(0' * (x' * y'))} \mu^+(0_0 * (x_0 * y_0)) = \bigvee_{a \in \psi^{-1}(0' * (x' * y'))} \mu^+(a)$.

Thus

$$\begin{aligned} \upsilon^{+}(0' * x') &= \bigvee_{a\in\psi^{-1}(0'*x')} \mu^{+}(a) \\ &= \mu^{+}(0_0 * x_0) \\ &\geq \min\{\mu^{+}(0_0 * (x_0 * y_0)), \mu^{+}(0_0 * y_0)\} \\ &= \min\{\bigvee_{a\in\psi^{-1}(0'*(x'*y'))} \mu^{+}(a), \bigvee_{a\in\psi^{-1}(0'*x')} \mu^{+}(a)\} \\ &= \min\{\upsilon^{+}(0' * (x' * y')), \upsilon^{+}(0' * y')\}. \end{aligned}$$

On the other hand, we have $\mu^{-}(0_0 * x_0) = \bigwedge_{a\in\psi^{-1}(0'*x')} \mu^{-}(a)$ and $\mu^{-}(0_0 * y_0) = \bigwedge_{a\in\psi^{-1}(0'*y')} \mu^{-}(a)$ and $\mu^{-}(0_0 * (x_0 * y_0)) = \upsilon^{-}(\psi(0_0 * (x_0 * y_0))) = \upsilon^{-}(0' * (x' * y')) = \bigwedge_{0_0*(x_0*y_0)\in\psi^{-1}(0'*(x'*y'))} \mu^{-}(0_0 * (x_0 * y_0)) = \bigwedge_{a\in\psi^{-1}(0'*(x'*y'))} \mu^{-}(a)$.

Thus,

$$\begin{aligned} \upsilon^{-}(0' * x') &= \bigwedge_{a\in\psi^{-1}(0'*x')} \mu^{-}(a) \\ &= \mu^{-}(0_0 * x_0) \\ &\leq \max\{\mu^{-}(0_0 * (x_0 * y_0)), \mu^{-}(0_0 * y_0)\} \\ &= \max\{\bigwedge_{a\in\psi^{-1}(0'*(x'*y'))} \mu^{-}(a), \bigwedge_{a\in\psi^{-1}(0'*x')} \mu^{-}(a)\} \\ &= \max\{\upsilon^{-}(0' * (x' * y')), \upsilon^{-}(0' * y')\}. \end{aligned}$$

Hence onto homomorphic image of a BF BRK-ideal is also a BF BRK-ideal.

5 BF Translation and BF Extension on BF BRK-Ideals

For any BF set $B = (\lambda^{+}, \lambda^{-})$ in X, we denote $\top = 1 - \sup\{\lambda^{+}(x) \mid x \in X\}$ and $\perp = -1 - \inf\{\lambda^{-}(x) \mid x \in X\}$.

Definition 5.1. *Let $B = (\lambda^{+}, \lambda^{-})$ be a BF set in X and $(\gamma, \delta) \in [0, \top] \times [\perp, 0]$. By a BF (γ, δ)-translation of B we mean a BF set $B^{T}_{(\gamma,\delta)} = (\lambda^{+}_{(\gamma,T)}, \lambda^{-}_{(\delta,T)})$ where*

$$\lambda^{+}_{(\gamma,T)} : X \longrightarrow [0, 1],\ x \longrightarrow \lambda^{+}(x) + \gamma,$$
$$\lambda^{-}_{(\delta,T)} : X \longrightarrow [-1, 0],\ x \longrightarrow \lambda^{-}(x) + \delta.$$

Theorem 5.1. *If $B = (\lambda^{+}, \lambda^{-})$ is a BF BRK-ideal in X, then the BF (γ, δ)-translation $B^{T}_{(\gamma,\delta)} = (\lambda^{+}_{(\gamma,T)}, \lambda^{-}_{(\delta,T)})$ of Bis a BF BRK-ideal in X for all $(\gamma, \delta) \in [0, \top] \times [\perp, 0]$.*

Proof. Let $a \in X$. Then $\lambda^{+}_{(\gamma,T)}(0) = \lambda^{+}(0) + \gamma \geq \lambda^{+}(a) + \gamma = \lambda^{+}_{(\gamma,T)}(a)$, and $\lambda^{-}_{(\delta,T)}(0) = \lambda^{-}(0) + \delta \leq \lambda^{-}(a) + \delta = \lambda^{-}_{(\delta,T)}(a)$. Now, let

$$\begin{aligned} \lambda^{+}_{(\gamma,T)}(0 * x) &= \lambda^{+}(0 * x) + \gamma \\ &\geq \min\{\lambda^{+}(0 * (x * y)), \lambda^{+}(0 * y)\} + \gamma \\ &= \min\{\lambda^{+}(0 * (x * y)) + \gamma, \lambda^{+}(0 * y) + \gamma\} \\ &= \min\{\lambda^{+}_{(\gamma,T)}(0 * (x * y)), \lambda^{+}_{(\gamma,T)}(0 * y)\}, \end{aligned}$$

and

$$\begin{aligned}\lambda^-_{(\delta,T)}(0*x) &= \lambda^-(0*x)+\delta \\ &\leq \max\{\lambda^-(0*(x*y)), \lambda^-(0*y)\}+\delta \\ &= \max\{\lambda^-(0*(x*y))+\delta, \lambda^-(0*y)+\delta\} \\ &= \max\{\lambda^-_{(\delta,T)}(0*(x*y)), \lambda^-_{(\delta,T)}(0*y)\}.\end{aligned}$$

Hence $B=(\lambda^+,\lambda^-)$ is a BF BRK-ideal in X.

Theorem 5.2. *Let $B=(\lambda^+,\lambda^-)$ be a BF set in X such that the BF (γ,δ)-translation $B^T_{(\gamma,\delta)}=(\lambda^+_{(\gamma,T)},\lambda^-_{(\delta,T)})$ of B is a BF BRK-ideal in X for some $(\gamma,\delta)\in[0,\top]\times[\bot,0]$. Then $B=(\lambda^+,\lambda^-)$ is a BF BRK-ideal in X.*

Proof. Let $a\in X$. Then $\lambda^+(0)+\gamma=\lambda^+_{(\gamma,T)}(0)\geq\lambda^+_{(\gamma,T)}(a)=\lambda^+(a)+\gamma$, and $\lambda^-(0)+\delta=\lambda^-_{(\delta,T)}(0)\leq\lambda^-_{(\delta,T)}(a)=\lambda^-(a)+\delta$. Thus $\lambda^+(0)\geq\lambda^+(a)$ and $\lambda^-(0)\leq\lambda^-(a)$. Now, let

$$\begin{aligned}\lambda^+(0*x)+\gamma &= \lambda^+_{(\gamma,T)}(0*x) \\ &\geq \min\{\lambda^+_{(\gamma,T)}(0*(x*y)), \lambda^+_{(\gamma,T)}(0*y)\} \\ &= \min\{\lambda^+(0*(x*y))+\gamma, \lambda^+(0*y)+\gamma\} \\ &= \min\{\lambda^+(0*(x*y)), \lambda^+(0*y)\}+\gamma,\end{aligned}$$

and

$$\begin{aligned}\lambda^-(0*x)+\delta &= \lambda^-_{(\delta,T)}(0*x) \\ &\leq \max\{\lambda^-_{(\delta,T)}(0*(x*y)), \lambda^-_{(\delta,T)}(0*y)\} \\ &= \max\{\lambda^-(0*(x*y))+\delta, \lambda^-(0*y)+\delta\} \\ &= \max\{\lambda^-(0*(x*y)), \lambda^-(0*y)\}+\delta.\end{aligned}$$

Thus $\lambda^+(0*x)\geq\min\{\lambda^+(0*(x*y)),\lambda^+(0*y)\}$ and $\lambda^-(0*x)\leq\max\{\lambda^-(0*(x*y)),\lambda^-(0*y)\}$. Hence $B=(\lambda^+,\lambda^-)$ is a BF BRK-ideal in X.

Definition 5.2. *Let $B_1=(\lambda^+,\lambda^-)$ and $B_2=(\mu^+,\mu^-)$ be two BF sets in X. If $\lambda^+(x)\leq\mu^+(x)$ and $\lambda^-(x)\geq\mu^-(x)$ for all $x\in X$, then we say that B_2 is a BF extension of B_1.*

Definition 5.3. *Let $B_1=(\lambda^+,\lambda^-)$ and $B_2=(\mu^+,\mu^-)$ be two BF sets in X. Then B_2 is called a BF BRK-ideal extension of B_1 if following conditions are holds:*

(i) *B_2 is called a BF extension of B_1,*
(ii) *If B_2 is a BF BRK-ideal extension of X then B_1 is also a BF BRK-ideal extension of X.*

By definition $\lambda^+_{(\gamma,T)}\geq\lambda^+(x)$ and $\lambda^-_{(\delta,T)}\leq\lambda^-(x)$ for all $x\in X$. Therefore, we have following theorem.

Theorem 5.3. *If $B = (\lambda^+, \lambda^-)$ is a BF BRK-ideal of X, then the BF (γ, δ)-translation $B^T_{(\gamma,\delta)} = (\lambda^+_{(\gamma,T)}, \lambda^-_{(\delta,T)})$ of Bis a BF extension of X for all $(\gamma, \delta) \in [0, \top] \times [\bot, 0]$.*

Converse of above theorem is not true in general as seen in following Example.

Example 3. Consider BRK-algebra $X = \{0, a_1, a_2, a_3\}$ defined in Example 2. Let $B_1 = (\lambda^+, \lambda^-)$ be a BF set in X defined by

	0	a_1	a_2	a_3
λ^+	0.4	0.4	0.2	0.2
λ^-	−0.2	−0.2	−0.1	−0.1

Clearly, B is a BF BRK-ideal of X. Let $B_2 = (\mu^+, \mu^-)$ be a BF set of X defined by

	0	a_1	a_2	a_3
μ^+	0.42	0.42	0.4	0.65
μ^-	−0.23	−0.23	−0.23	−0.66

Then B_2 is a BF BRK-ideal extension of B_1 but it is not BF (γ, δ)-translation of B_1, for all $(\gamma, \delta) \in [0, \top] \times [\bot, 0]$.

Definition 5.4. *Let $B = (\mu^+, \mu^-)$ be a BF set in X, $(\alpha, \beta) \in [-1, 0] \times [0, 1]$, $\gamma \in [0, \top]$ and $\delta \in [\bot, 0]$. We define $B^{+T}_{(\beta,\gamma)} = \{x \in X : \mu^+(x) \geq \beta - \gamma\}$, $B^{-T}_{(\alpha,\delta)} = \{x \in X : \mu^-(x) \leq \alpha - \delta\}$ and $B^T_{((\alpha,\beta),(\gamma,\delta))} = \{x \in X : \mu^-(x) \leq \alpha - \gamma$ and $\mu^+(x) \geq \beta - \delta\}$.*

Theorem 5.4. *If $B = (\mu^+, \mu^-)$ is BF BRK-ideal of X, then $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are BRK-ideal of X for all $\alpha \in Im(\mu^-)$ and $\beta \in Im(\mu^+)$ with $\beta \geq \gamma$ and $\alpha \leq \delta$.*

If we do not take the condition that $B = (\mu^+, \mu^-)$ is BF BRK-ideal of X, then either both $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are not BRK-ideals of X, or one of them is not a BRK-ideal of X.

Example 4. Consider BRK-algebra $X = \{0, 1, 2\}$ with the following operation

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Let $B = (\mu^+, \mu^-)$ be a BF set of X defined by

	0	1	2
μ^+	0.5	0.2	0.4
μ^-	−0.6	−0.4	−0.5

Clearly, B is a BF BRK-ideal of X. Take $\beta = 0.32$, $\alpha = -0.35$, $\gamma = 0.05$ and $\delta = -0.04$. Then $B^{+T}_{(\beta,\gamma)} = \{0, 2\}$ and $B^{-T}_{(\alpha,\delta)} = \{0, 1, 2\}$ are BRK-ideals of X. On the other hand, BF set $B = (\mu'^+, \mu'^-)$ defined by

	0	2	1
μ'^{+}	0.6	0.5	0.7
μ'^{-}	−0.9	−0.7	−0.5

is not a BF BRK-ideal of X. Take $\beta = 0.55$, $\alpha = -0.8$, $\gamma = 0.03$ and $\delta = -0.07$. Then $B^{+T}_{(\beta,\gamma)} = \{0, 1\}$ is not a BRK-ideal of X, but $B^{-T}_{(\alpha,\delta)} = \{0\}$ is a BRK-ideal of X.

Corollary 5.1. *If $B = (\mu^+, \mu^-)$ is BF BRK-ideal of X, then $B^T_{((\alpha,\beta),(\gamma,\delta))}$ is a BRK-ideal of X, for all $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ and $(\gamma,\ \delta) \in [0, \top] \times [\bot, 0]$.*

Theorem 5.5. *Let $B = (\mu^+, \mu^-)$ be a BF set of X. Then the BF $(\gamma,\ \delta)$-translation of B is a BF BRK-ideal of X if and only if $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are BRK-ideal of X for all $\beta \in Im(\mu^+)$, $\alpha \in Im(\mu^-)$ and $(\gamma,\ \delta) \in [0, \top] \times [\bot, 0]$ with $\alpha < \delta$and $\beta > \gamma$.*

Proof. Assume that $B^T_{(\gamma,\delta)} = (\mu^+_{(\gamma,T)}, \mu^-_{(\delta,T)})$ is a BF BRK-ideal of X. Let $x \in B^{+T}_{(\beta,\gamma)}$. Then $\mu^+(x) \geq \beta - \gamma$ such that $\mu^+_{(\gamma,T)}(0) \geq \mu^+_{(\gamma,T)}(x) \geq \beta$. This shows that $\mu^+(0) + \gamma \geq \beta$. Thus $\mu^+(0) \geq \beta - \gamma$, which implies that $0 \in B^{+T}_{(\beta,\gamma)}$. Now, let $0 * (a * b) \in B^{+T}_{(\beta,\gamma)}$ and $0 * b \in B^{+T}_{(\beta,\gamma)}$ for all $a, b \in X$. Since $\mu^+(0 * a) \geq \min\{\mu^+_{(\gamma,T)}(0 * (a * b)), \mu^+_{(\gamma,T)}(0 * b)\} \geq \beta$. This shows that $\mu^+(0 * a) + \gamma \geq \beta$. Thus $\mu^+(0 * a) \geq \beta - \gamma$, which implies that $0 * a \in B^{+T}_{(\beta,\gamma)}$. Hence $B^{+T}_{(\beta,\gamma)}$ is a BRK-ideal of X. Analogously, we can prove that $B^{-T}_{(\alpha,\delta)}$ is a BRK-ideal of X.

Conversely, suppose $B^{+T}_{(\beta,\gamma)}$ and $B^{-T}_{(\alpha,\delta)}$ are BRK-ideals of X for all $\beta \in Im(\mu^+)$, $\alpha \in Im(\mu^-)$ and $(\gamma,\ \delta) \in [0, \top] \times [\bot, 0]$ with $\alpha < \delta$ and $\beta > \gamma$. Assume that there exist $a \in X$, such that $\mu^+_{(\gamma,T)}(0) < \beta' < \mu^+_{(\gamma,T)}(a)$ and $\mu^-_{\delta,T)}(0) > \alpha' > \mu^-_{\delta,T)}(a)$. Then $\mu^+(0) > \beta' - \gamma$, $\mu^+(0) > \beta' - \gamma$, $\mu^-(0) < \alpha' - \delta$ and $\mu^-(0) < \alpha' - \delta$. This shows that $a \in B^{+T}_{(\beta,\gamma)}$ and $a \in B^{-T}_{(\alpha,\delta)}$, but $0 \notin B^{+T}_{(\beta,\gamma)}$ and $0 \notin B^{-T}_{(\alpha,\delta)}$ which is contradiction. Hence $\mu^+_{(\gamma,T)}(0) \geq \mu^+_{(\gamma,T)}(a)$ and $\mu^-_{(\delta,T)}(0) \leq \mu^-_{(\delta,T)}(a)$. Similarly, we can prove that $\mu^+_{(\gamma,T)}(0 * a) \geq \min\{\mu^+_{(\gamma,T)}(0 * (a * b)), \mu^+_{(\gamma,T)}(0 * b)\}$, $\mu^-_{(\delta,T)}(0 * a) \leq \max\{\mu^-_{(\delta,T)}(0 * (a * b)), \mu^-_{(\delta,T)}(0 * b)\}$. Hence BF $(\gamma,\ \delta)$-translation of B is a BF BRK-ideal of X.

Theorem 5.6. *Let $B = (\lambda^+, \lambda^-)$ be a BF BRK-ideal in X, $(\gamma,\ \delta) \in [0, \top] \times [\bot, 0]$ and $(\gamma',\ \delta') \in [0, \top] \times [\bot, 0]$. If $(\gamma,\ \delta) \geq (\gamma',\ \delta')$, then BF $(\gamma,\ \delta)$-translation $B^T_{(\gamma,\delta)}$ of B is a BF BRK-ideal extension of $(\gamma',\ \delta')$-translation $B^T_{(\gamma',\delta')}$ of B.*

Theorem 5.7. *Let $B = (\lambda^+, \lambda^-)$ be a BF BRK-ideal in X and $(\gamma,\ \delta) \in [0, \top] \times [\bot, 0]$. For every BF BRK-ideals extension $B' = (\nu^+, \nu^-)$ of the BF $(\gamma,\ \delta)$-translation $B^T_{(\gamma,\delta)}$ of B, there exist $(\gamma',\ \delta') \in [0, \top] \times [\bot, 0]$ such that $(\gamma,\ \delta) \geq (\gamma',\ \delta')$ and B' is a BF BRK-ideal extension of the BF $(\gamma',\ \delta')$-translation $B^T_{(\gamma',\delta')}$ of B.*

Proof. Assume that for every BF BRK-ideal extension $B' = (\nu^+, \nu^-)$ of $B^T_{(\gamma,\delta)}$, there does not exist $(\gamma', \delta') \in [0, \top] \times [\bot, 0]$ $(\gamma, \delta) \geq (\gamma', \delta')$ such that B' is not BF BRK-ideal extension of $B^T_{(\gamma',\delta')}$. Then $\nu^+(x) > \mu^+_{(\gamma',T)}(x) = \mu^+(x) + \gamma'$ and $\nu^-(x) < \mu^-_{(\delta',T)}(x) = \mu^-(x) + \delta'$ for $x \in R$. Since, $(\gamma, \delta) < (\gamma', \delta')$ so that $\nu^+(x) > \mu^+(x) + \gamma' > \mu^+(x) + \gamma$ and $\nu^-(x) < \mu^-(x) + \delta' < \mu^-(x) + \delta$. This shows that $\nu^+(x) > \mu^+_{(\gamma,T)}(x)$ and $\nu^-(x) < \mu^-_{(\delta,T)}(x)$ which is contradiction. Thus, there exist $(\gamma', \delta') \in [0, \top] \times [\bot, 0]$ such that $(\gamma, \delta) \geq (\gamma', \delta')$ and B' is a BF BRK-ideal extension of $B^T_{(\gamma',\delta')}$.

Definition 5.5. *Let $B = (\lambda^+, \lambda^-)$ be a BF set of X and $\zeta, \eta \in [0, 1]$. By a BF (ζ, η)-multiplication of B we mean a BF set $B^m_{(\zeta,\eta)} = (\lambda^{+m}_\zeta, \lambda^{-m}_\eta)$ express as*

$$\lambda^{+m}_\zeta : X \longrightarrow [0, 1],\ x \longrightarrow \lambda^+(x)\,\zeta,$$
$$\lambda^{-m}_\eta : X \longrightarrow [-1, 0],\ x \longrightarrow \lambda^-(x)\,\eta.$$

For a BF BRK-ideal, BF (0, 0)-multiplication $B^m_{(0,0)}$ is a BF BRK-ideal of X.

Theorem 5.8. *If $B = (\lambda^+, \lambda^-)$ is a BF BRK-ideal of X, then BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X.*

Theorem 5.9. *Let $B = (\lambda^+, \lambda^-)$ be a BF set of X. Then BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X if and only if B is a BF BRK-ideal of X for some $\zeta, \eta \in [0, 1]$.*

Proof. Necessity obtains from Theorem 5.8. Let $\zeta, \eta \in [0, 1]$ such that BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X. Let $x \in X$. Then $\lambda^+(0)\,\zeta = \lambda^{+m}_\zeta(0) \geq \lambda^{+m}_\zeta(x) \geq \lambda^+(x)\,\zeta$, $\lambda^-(0)\,\eta = \lambda^{+m}_\eta(0) \leq \lambda^{+m}_\eta(x) \leq \lambda^+(x)\,\eta$ which implies that $\lambda^+(0) \geq \lambda^+(x)$. Analogously, $\lambda^-(0) \leq \lambda^-(x)$. Next, let for $x, y \in X$

$$\begin{aligned}
\lambda^+(0 * x)\,\zeta &= \lambda^{+m}_\zeta(0 * x) \\
&\geq \min\{\lambda^{+m}_\zeta(0 * (x * y)), \lambda^{+m}_\zeta(0 * y)\} \\
&= \min\{\lambda^+(0 * (x * y))\,\zeta, \lambda^+(0 * y)\,\zeta\} \\
&= \min\{\lambda^+(0 * (x * y)), \lambda^+(0 * y)\}\zeta,
\end{aligned}$$

which implies that $\lambda^+(x) \geq \min\{\lambda^+(0 * (x * y)), \lambda^+(0 * y)\}$. And,

$$\begin{aligned}
\lambda^-(0 * x)\,\eta &= \lambda^{-m}_\eta(0 * x) \\
&\leq \max\{\lambda^{-m}_\eta(0 * (x * y)), \lambda^{-m}_\eta(0 * y)\} \\
&= \max\{\lambda^-(0 * (x * y))\,\eta, \lambda^-(0 * y)\,\eta\} \\
&= \max\{\lambda^-(0 * (x * y)), \lambda^-(0 * y)\}\eta,
\end{aligned}$$

which implies that $\lambda^-(x) \leq \max\{\lambda^-(0 * (x * y)), \lambda^-(0 * y)\}$ for $x, y \in X$. Hence B is a BF BRK-ideal of X.

Theorem 5.10. *Let $B = (\lambda^+, \lambda^-)$ be a BF set of X, $(\gamma, \delta) \in [0, \top] \times [\bot, 0]$ and $\zeta, \eta \in (0, 1]$. Then every BF (γ, δ)-translation $B^T_{(\gamma,\delta)}$ of B is a BF BRK-ideal extension of the (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B.*

Proof. For all $a \in R$, we have $\mu^+_{(\gamma,T)}(a) = \mu^+(a) + \gamma \geq \mu^+(a) \geq \mu^+(a)\zeta = \mu^{+m}_{\zeta}(a)$ and $\mu^-_{(\delta,T)}(a) = \mu^-(a) + \delta \leq \mu^-(a) \leq \mu^-(a)\eta = \mu^{-m}_{\eta}(a)$. Thus $B^T_{(\gamma,\delta)}$ is a BF extension of the (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B. Assume that BF (ζ, η)-multiplication $B^m_{(\zeta,\eta)}$ of B is a BF BRK-ideal of X. Then by Theorem 5.9, B is a BF BRK-ideal of X. By Theorem 5.1, $B^T_{(\gamma,\delta)}$ is a BF BRK-ideal of X. Thus every BF (γ, δ)-translation is a BF BRK-ideal extension of the (ζ, η)-multiplication of B.

6 Conclusion

In this study, we investigated BF BRK-ideals in BRK-algebras and discussed related properties. We introduced translation, extension and multiplications of BF BRK-ideals in BRK-algebras and discussed related results. As an extension of above results, one could study bipolar anti fuzzy BRK-algebras and BF ideals in other algebraic structures. Applications of the method to consider the related problems in machines learning, decision makings, information sciences, cognitive science, intelligent decision-making system, and so on.

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Recommender: Academic Conference on 30th anniversary of fuzzy geometric programming and 40th education year by and of Professor Cao Bingyuan.

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A New Approach for Solving Fuzzy Supplier Selection Problems Under Volume Discount

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Abstract. In order to achieve a compromised solution for a multi-objective linear programming with fuzzy right hand sides, Tchebycheff norm and a new approach based on α-cut is suggested to minimize the distance from the current estimate of the objective values from the ideal point. Since the obtained solutions by the Tchebycheff approach are weakly efficient for multi-objective problems. Hence, an augmented weighted Tchebycheff norm has been proposed. Here, the satisficing tradeoff algorithm is used to solve the augmented weighted Tchebycheff problems. Since the supplier selection problem is usually a multi-objective problem, the augmented weighted Tchebycheff method is applied for obtaining its solutions.

Keywords: Fuzzy multi-objective linear programming · Augmented weighted Tchebycheff norm · α-cut approach · Satisficing tradeoff algorithm

1 Introduction

Companies have to work with several suppliers in order to supply their raw material. More than 70% of product's final price is related to raw material's cost. Because of this reason buying management is one of the most important parts in supply chain. In such circumstances the purchasing department can play a key role in cost reduction, and supplier selection is one of the most important functions of purchasing management [1]. Several factors may affect a supplier's performance. Dickson [2] identified 23 different criteria for vendor selection including quality, delivery, performance history, warranties, price, technical capability and financial position. Selecting the best suppliers and quota allocations to them reduces purchasing costs, improves competitiveness, and improving quality and flexibility to meet the requirements of the end consumer [3].

Basically there are two kinds of supplier selection problem based on the number of suppliers:

(1) Single sourcing,
(2) Multiple sourcing,

In the first kind of supplier selection, one supplier can satisfy all the buyer's needs. The management needs to make only one decision: which supplier is the best? In the second type, no supplier can satisfy all the buyer's requirements. That means, the buyers makes balance between suppliers and its overall demand is bought from several

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suppliers. The decision maker should make two decision: which suppliers are the best? And how much should be purchased from each selected supplier?

Based on the number of objective functions, the supplier selection programming is going to be divided into two clusters:

(1) Single objective;
(2) Multi objectives.

First cluster is consists of problems that have one objective function. This objective function can be considered as cost, quality or delivery on time, etc. In second cluster, several objective functions are supposed as objectives of decision making.

In reality most input data is not accurate. In the way that, most of these data can be mention as verbal variables such as high, low, tall and so on. Crisp models can't consider this inaccurate data. Fuzzy logic is one of the strong ways to manage this inaccurate data [4].

Here, we introduce some existed methods and criteria for supplier selection problem. We can point to these criteria as the most important ones: coast, quality of products, service aspects, delivery time, risk factors and trade restrictions. Some of the most important criteria are used for supplier selection problem from 1966 until now are summarized in Table 1.

Table 1. Literature review for supplier selection criteria

Author	Criteria				
	Cost	Quality	Delivery	Capacity	Warranty period
Lin [6]	✓	✓	✓		
Chen [7]		✓			
Chan [8]	✓	✓	✓	✓	
Ghodsypour [9]	✓	✓	✓	✓	
Stavropolous [10]	✓				
Min [11]	✓	✓	✓		
Weber [12]	✓	✓	✓	✓	
Abratt [13]	✓				
Lehmann [14]	✓		✓		✓
Dickson [2]	✓	✓	✓	✓	✓

In the previous works, several methods applied to solve supplier selection and order allocation program. Here, we introduce some of them.

Gaballa [15] is the first author who applied mathematical programming to supplier selection in a real case. He used mixed integer programming to minimize the total discounted price of allocated items to the suppliers. He also formulated a single-objective, mixed-integer programming to minimize the sum of purchasing, transportation and inventory costs by considering multiple items, multiple time periods, vendors' quality, delivery and capacity. Weber and Current [16] used a multi objective approach to systematically analyze the trade-offs between conflicting criteria in supplier selection problems. Ghodsypour and O'Brien [17] developed a Decision Support

System (DSS) for reducing the number of suppliers according to supply based optimization strategy. They used an integrated Analytical Hierarchy Process (AHP) with mixed-integer programming and considered suppliers' capacity constraint and the buyers' limitations on budget and quality etc. Ghodsypour and O'Brien [1] developed an integrated AHP and linear programming model to consider both qualitative and quantitative factors in purchasing activity. Wang et al. [18] provided an AHP method to choose from agile/lean supply chain strategies and then used Pre-emptive Goal Programming (PGP) to obtain the optimal order quantity from their suppliers. Xia and Wu [5] introduced rough sets theory to improve AHP and integrated multi-objective mixed integer programming to determine which suppliers should be selected and the quantity that should be allocated to them while considering volume discount policy.

Zadeh [19] initiated the fuzzy set theory. Bellman and Zadeh [20] presented some applications of fuzzy theories to the various decision-making processes in a fuzzy environment. Zimmerman [21, 22] presented a fuzzy optimization technique to linear programming problem with single and multiple objectives. Since then the fuzzy set theory has been applied to formulate and solve the problems in various areas such as artificial intelligence, image processing, robotics, pattern recognition, etc. Narsimhan [23] proposed a Fuzzy Goal Programming (FGP) technique to specify imprecise aspiration levels of the fuzzy goals. Yang, Ignizio and Kim [24] formulated the FGP with nonlinear membership functions.

This article is divided into the following sections: In Sect. 2, we introduce a multi objective linear programming model, and then we consider right hand side values as fuzzy term. In Sect. 3, we use α-cut approach to change fuzzy model into crisp type. After that, we use a method to solve crisp model. In Sect. 4, we use from the model that represented by Xia [5] and based on the data set adopted from a case company, and then formulate the supplier selection and order allocation model. Numerical findings are applied to show the usage of the suggested method. In Sect. 5, the conclusions are presented.

2 Fuzzy Multi Objective Linear Programming

Many of the decision problems in real life are multi objective. That mean, there are several objectives that each of them should be optimal at the same time.

Generally, a multi objective programming with p objective and is as follow:

$$\begin{array}{ll} \min & F(x) = [f_1(x), f_2(x), \ldots, f_p(x)] \\ \text{S.t.} & g(x) \leq b, \\ & x \geq 0, \end{array} \tag{1}$$

where $g(x)$ is linear function. In a real-life situation for a supplier selection problem, many input information related to the various supplier are not known with certainty such as capacity, quality, delivery time, etc. Such vagueness in the critical information cannot be captured in a deterministic problem and therefore the optimal results of these deterministic formulations may not serve the real purpose of modeling the problem.

Due to this, we have considered the model as a fuzzy model. Fuzzy mathematical programming has the capability to handle both multi objective problems and vagueness.

A multi objective programming with fuzzy resource can be formulated as:

$$\begin{array}{ll} \min & F(x) = [f_1(x), f_2(x), \ldots, f_p(x)] \\ \text{S.t.} & g(x) \leq \widetilde{b}, \\ & x \geq 0, \end{array} \tag{2}$$

Where the fuzzy number $\widetilde{b}$ is in the fuzzy region of $[b, b + u]$ with given fuzzy tolerance u, Assume that the fuzzy tolerance u for the fuzzy constraint is known. Then, $\widetilde{b}$ is equivalent to $(b + \theta u)$, where θ is in $[0, 1]$. In this case, a fuzzy constraint problem is transformed to be a crisp parametric programming problem. The following section, we apply Verdegay α-cut approach to transform fuzzy constraint to crisp constraint.

3 Propose Method

In this section, we first introduce Verdegay α-cut approach for transforming fuzzy constraint to crisp constraint. Then, we present a method in order to solve crisp equivalent multi objective programming.

3.1 Verdegay α-cut Approach [25]

For dealing problem (2), Verdegay considered if the membership function of the fuzzy constraint (shown in Fig. 1) has the following form:

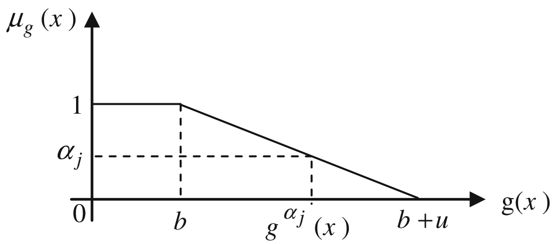

Fig. 1. Membership function of $\mu_g(x)$, with level α_j-cut

then,

$$\mu_g(x) = \begin{cases} 1, & g(x) \leq b, \\ 1 - \frac{g(x)-b}{u}, & b \leq g(x) \leq b + u, \\ 0, & g(x) > b + u, \end{cases} \tag{3}$$

Simultaneously, the membership functions of $\mu_g(x)$, is continuous and monotonic function and trade-off between this fuzzy constraint is allowed; then, problem (2) is equivalent to the following:

$$\begin{array}{ll} \min & F(x) = [f_1(x), f_2(x), \ldots, f_p(x)] \\ \text{S.t.} & x \in X_\alpha, \end{array} \tag{4}$$

where $X_\alpha = \{x \mid \mu_g(x) \leq \alpha, x \geq 0\}$, for each $\alpha \in [0, 1]$.

This is the fundamental concepts of α-cuts method of fuzzy mathematical programming. One can then substitute (3) into (4) and obtain the following formulation:

$$\begin{array}{ll} \min & F(x) = [f_1(x), f_2(x), \ldots, f_p(x)] \\ \text{S.t.} & \quad g(x) \leq b + (1 - \alpha)u, \\ & x \geq 0, \end{array} \tag{5}$$

where $\alpha \in [0, 1]$. Thus the problem given in (5) is equivalent to a crisp parametric programming formulation. For each α, one will have an optimal solution. In the following we represent a method to solve problem (5).

Now, we explain augmented weighted Tchebycheff approach to solve multi-objective linear programming that obtained from above.

3.2 Augmented Weighted Tchebycheff Approach [26]

A common method for solving multi objective problems is augmented weighted Tchebycheff approach. Next, we introduce this approach and an algorithm to solve multi objective programming.

A multi objective programming consider as follow [26]:

$$\begin{array}{ll} \min & F(x) = [f_1(x), f_2(x), \ldots, f_p(x)] \\ \text{S.t.} & x \in X, \end{array} \tag{6}$$

where X is feasible region. Suppose that $v_k \geq 0, k = 1, 2, \ldots, p$, are nonnegative weights such that $\sum_{k=1}^{p} v_k = 1$. So, augmented weighted Tchebycheff norm related to $F(x) \in \mathbb{R}^p$ define as follow:

$$\|F(x)\|_\rho^v = \|F(x)\|_\infty^v + \rho\|F(x)\|_1, \tag{7}$$

where,

$$\|F(x)\|_\infty^v = \max_{k=1,2,\ldots,p} \{v_k |f_k(x)|\} \tag{8}$$

and,

$$\|F(x)\|_1 = |f_1(x)| + |f_2(x)| + \ldots + |f_p(x)| \tag{9}$$

and, ρ is a nonnegative scalar that usually is a small number between 0.01 and 0.0001.

We apply augmented weighted Tchebycheff norm to find minimum distance between objective functions and vector ideal solutions. So, we have to solve an optimization problem as fellow:

$$\begin{array}{ll} \min & \|F(x) - F^I\|_\infty^v \\ S.t. & x \in X. \end{array} \tag{10}$$

The equivalent program is obtaining from (7), (8) and (9). Therefore, we have:

$$\begin{array}{l} \min\{\max\limits_{k=1,2,\ldots,p} v_k(f_k(x) - f_k^I)\} + \rho \sum\limits_{k=1}^{p} (f_k - f_k^I) \\ S.t. \quad x \in X, \end{array} \tag{11}$$

In order to solve (11) by using to linear programming techniques, we reformulate it into a linear programming as follow:

$$\begin{array}{l} \min\{\beta + \rho \sum\limits_{k=1}^{p} (f_k - f_k^I)\} \\ S.t. \quad \beta \geq v_k(f_k(x) - f_k^I), \quad k = 1, 2, \ldots, p, \\ \qquad x \in X. \end{array} \tag{12}$$

One of the iterative algorithms to solve (12) is Satisficing Trade of Method (STOM). Next, we introduce steps of STOM algorithm.

STOM algorithm [27]:

Step 1. Obtain ideal solution f_k^I for each f_k objective function as follow:

$$\begin{array}{ll} \min & f_k \\ S.t & x \in X, \end{array} \tag{13}$$

f_k^I is equals to optimal value form (13).

Step 2. The decision maker has to specify aspiration levels for each function (the aspiration levels are determined by decision maker such that $f_k^+ > f_k^I$, where f_k^+ is aspiration level for f_k objective function).

Step 3. The relative weights determine as follow:

$$v_k = \frac{1}{f_k^+ - f_k^I}. \tag{14}$$

In fact, v_k is reverse of distance between ideal value and aspiration levels for f_k objective function.
Then, solve (12).
Step 4. The solutions whose obtain in step 3 offer to decision maker. The decision maker is asked to classify the objective functions into three classes:

i. The unacceptable objective functions whose values should be improved.
ii. The objective functions whose values may weakly.
iii. The acceptable objective functions whose values are acceptable as they are.

If no objective function is in the group (i) then, STOP. This solution is optimal. Otherwise, the decision maker has to specify new aspiration levels for functions in group (i) and (ii) then, go to step 3.

If the new program is infeasible then, the decision maker has determines more weakly aspiration levels. This process continues until the new program being feasible.

The optimal points whose obtain from STOM algorithm is a Pareto solution of (11).

In order to illustrate the performance of the propose approach we apply it on a case study from a drilling company.

4 Case Study

The proposed approach to solve supplier selection model was implemented in a drilling company. There are three sources and three raw materials for purchasing. Decision maker has to select the best sources and decide how many material buy from them. Four objectives are considered by decision maker in this company for select suppliers and order allocation. These objectives are cost, quality, on time delivery, suppliers score. There are three kinds of commodity, Pipe (P), Gravel (G), Bentonite powder (B) for purchasing.

We applied proposed method to solve case study model. In order to determine distance between ideal solution and current solution, we use metric function that represent by Steuer in [27] as follow:

$$D_K(\lambda, p) = [\sum_{i=1}^{p} \lambda_i^K (1 - d_i)^K]^{\frac{1}{K}}, \tag{15}$$

Where d_i indicates the degree of closeness between obtained solution $Z_i(x^*)$ and their ideal solution $Z_i^I(x)$ and obtain as follow:

When the i-th objective is maximized as:

$$\frac{Z_i(x^*)}{Z_i^I(x)}. \tag{16}$$

Otherwise,

$$\frac{Z_i^I(x)}{Z_i(x^*)}. \tag{17}$$

Also, λ is unit vector of aspiration levels for objective function. K is distance parameter such that, $1 \leq K \leq \infty$.

Here, we use distance function (19) for obtain distance between obtained solution and ideal solution for $K = 2$.

We compare the obtained compromise solution by proposed method with the weighted additive approach. These results are shown in Fig. 2.

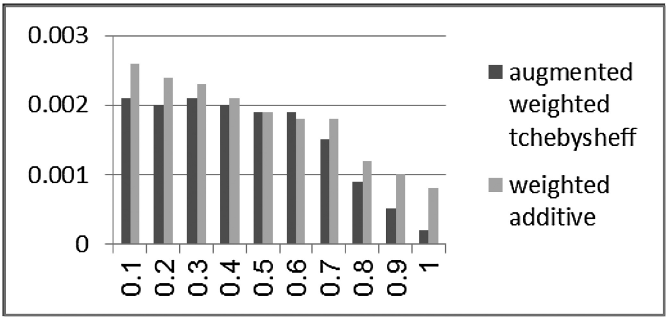

Fig. 2. Comparison of distance between obtained solution and ideal solution

As seen in Fig. 2, the compromise solution that obtain by proposed method has less distance from ideal solution related to weighted additive method for each α-cut, $\alpha = 0.1, 0.2, \ldots, 1$.

So, the result obtained from proposed method is better than weighted additive approach for each α-cut, $\alpha = 0.1, 0.2, \ldots, 1$.

5 Conclusion

Supplier selection is a complex multi objective decision-making problem. Since each supplier has its own advantages and disadvantages in terms of cost, quality, delivery and the technology, a flexibility model is required. In this paper, we use Xia_s model for formulating our case study. Since, many information of firm is not precise in real life so, we consider fuzzy number for show this information. We proposed an interactive approach by using - cut method, augmented weighted Tchebycheff norm and STOM algorithm. We compare proposed method with weighted additive approach by using a distance function. According to Eq. (10) proposed method from ideal solution has less distance related to weighted additive approach. Future studies may like to use stochastic variable instead of fuzzy variable. Moreover, using different norm to minimize the distance between the obtain solution of the objectives and the ideal solution.

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The Lattice of L-fuzzy Filters in a Given R_0-algebra

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Abstract. In the present paper, the L-fuzzy filter theory on R_0-algebras is further studied. Some new properties of L-fuzzy filters are given. Representation theorem of L-fuzzy filter which is generated by a fuzzy set is established. It is proved that the set consisting of all L-fuzzy filters on a given R_0-algebra, under the L-fuzzy set-inclusion order $\Subset$, forms a complete distributive lattice.

Keywords: Fuzzy logic · R_0-algebra · L-fuzzy filter · Complete distributive lattice

1 Introduction

To make the computers simulate beings in dealing with certainty and uncertainty in information is one important task of artificial intelligence. Logic appears in a "sacred" (resp., a "profane") form which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold– as a tool for applications in both areas, and a technique for laying the foundations. Nonclassical logic [1] including many-valued logic and fuzzy logic takes the advantage of classical logic to handle information with various facets of uncertainty [2], such as fuzziness and randomness. At present, nonclassical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. R_0-algebra is an important class of non-classical fuzzy logical algebras which was introduced by Wang in [3] by providing an algebra proof of the completeness theorem of the formal deductive system $\mathcal{L}^*$. From then, R_0-algebras has been extensively investigated by many researchers. Among them, Jun and Liu studied the theory of filters in R_0-algebras in [4]. The concept of fuzzy sets is introduced firstly by Zadeh in [5]. Liu and Li in [6] proposed the concept of fuzzy filters of R_0-algebras and discussed some their properties by using fuzzy sets theory. As an extension of the concept of fuzzy filter, in [7] the author and Xu propose the notion of L-fuzzy filters of R_0-algebras in terms of the notion of L-fuzzy set in [8], where the prefix L a lattice. In this paper, we will further research the properties of L-fuzzy filters in R_0-algebras. The lattice structural feature of the set containing all of L-fuzzy filters in a given R_0-algebra is investigated. It should be noticed that when $L = [0, 1]$, then [0, 1]-fuzzy sets

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are originally meant fuzzy sets. Since [0, 1] is a special completely distributive lattice, to investigate properties of L-fuzzy filters, sometimes we assume that the prefix L is a completely distributive lattice.

2 Preliminaries

Definition 1 (cf. [3]). Let M be an algebra of type $(\neg, \vee, \rightarrow)$, where $\neg$ is a unary operation, $\vee$ and $\rightarrow$ are binary operations. $(M, \neg, \vee, \rightarrow, 1)$ is called an R_0-algebra if there is a partial order $\leqslant$ such that $(M, \leqslant, 1)$ is a bounded distributive lattice with the greatest element 1, $\vee$ is the supremum operation with respect to $\leqslant$, $\neg$ is an order-reversing involution, and the following conditions hold for every $a, b, c \in M$:

(M1) $\neg a \rightarrow \neg b = b \rightarrow a$;
(M2) $1 \rightarrow a = a, a \rightarrow a = 1$;
(M3) $b \rightarrow c \leqslant (a \rightarrow b) \rightarrow (a \rightarrow c)$;
(M4) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$;
(M5) $a \rightarrow (b \vee c) = (a \rightarrow b) \vee (a \rightarrow c), a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$;
(M6) $(a \rightarrow b) \vee ((a \rightarrow b) \rightarrow (\neg a \vee b)) = 1$.

Lemma 1 (cf. [3]). Let M be an R_0-algebra, $a, b, c \in M$. Then the following properties hold.

(P1) $a \leqslant b$ if and only if $a \rightarrow b = 1$;
(P2) $a \leqslant b \rightarrow c$ if and only if $b \leqslant a \rightarrow c$;
(P3) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c), (a \wedge b) \rightarrow c = (a \rightarrow c) \vee (b \rightarrow c)$;
(P4) If $b \leqslant c$, then $a \rightarrow b \leqslant a \rightarrow c$, and if $a \leqslant b$, then $b \rightarrow c \leqslant a \rightarrow c$;
(P5) $a \rightarrow b \geqslant \neg a \vee b$ and $a \wedge \neg a \leqslant b \vee \neg b$;
(P6) $(a \rightarrow b) \vee (b \rightarrow a) = 1$ and $a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$;
(P7) $a \rightarrow (b \rightarrow a) = 1$ and $a \rightarrow (\neg a \rightarrow b) = 1$;
(P8) $a \rightarrow b \leqslant a \vee c \rightarrow b \vee c$ and $a \rightarrow b \leqslant a \wedge c \rightarrow b \wedge c$;
(P9) $a \rightarrow b \leqslant (a \rightarrow c) \vee (c \rightarrow b)$.

Lemma 2 (cf. [3]). Let M be an R_0-algebra. Define a new operator $\otimes$ on M such that $a \otimes b = \neg(a \rightarrow \neg b)$, for every $a, b, c \in M$. Then the following properties hold.

(P10) $(M, \otimes, 1)$ is a commutative monoid with the multiplicative unit element 1;
(P11) If $a \leqslant b$, then $a \otimes c \leqslant b \otimes c$;
(P12) $0 \otimes a = 0$ and $a \otimes \neg a = 0$;
(P13) $a \otimes b \leqslant a \wedge b$ and $a \otimes (a \rightarrow b) \leqslant b$ and $a \leqslant b \rightarrow (a \otimes b)$;
(P14) $a \otimes b \rightarrow c = a \rightarrow (b \rightarrow c)$ and $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$.

Let X be a non-empty set and L a lattice. A map $\mathscr{A} : X \rightarrow L$ is called an L-fuzzy subset on X. The set of all L-fuzzy subsets on X is denoted by $\mathcal{F}_L(X)$. (cf. [8]). Let $\mathscr{A}$ and $\mathscr{B}$ be two L-fuzzy subsets on X. We define $\mathscr{A} \Cap \mathscr{B}$, $\mathscr{A} \Cup \mathscr{B}$, $\mathscr{A} \Subset \mathscr{B}$ and $\mathscr{A} = \mathscr{B}$ as follows:

(i) $(\mathscr{A} \Cap \mathscr{B})(x) = \mathscr{A}(x) \wedge \mathscr{B}(x)$, for all $x \in X$;
(ii) $(\mathscr{A} \Cup \mathscr{B})(x) = \mathscr{A}(x) \vee \mathscr{B}(x)$, for all $x \in X$;
(iii) $\mathscr{A} \Subset \mathscr{B} \Longleftrightarrow \mathscr{A}(x) \leqslant \mathscr{B}(x)$, for all $x \in X$;
(iv) $\mathscr{A} = \mathscr{B} \Longleftrightarrow (\mathscr{A} \Subset \mathscr{B}$ and $\mathscr{B} \Subset \mathscr{A})$.

3 On L-fuzzy Filters in R_0-algebras

In this section, we recall the definition of L-fuzzy filters and give their some new properties.

Definition 2 (cf. [7])**.** Let M be an R_0-algebra and L a lattice. An L-fuzzy subset $\mathscr{A}$ on M is said to be an L-fuzzy filter of M, if it satisfies the following conditions:

(LF1) $\mathscr{A}(1) \geqslant \mathscr{A}(a)$ for all $a \in M$;
(LF2) $\mathscr{A}(b) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(a \to b)$ for all $a, b \in M$.

The set of all L-fuzzy filters of M is denoted by $\mathbf{LFil}(M)$.

Theorem 1. Let M be an R_0-algebra, L a lattice and $\mathscr{A}$ an L-fuzzy subset on M. Then $\mathscr{A} \in \mathbf{LFil}(M)$ if and only if it satisfies the following conditions:

(LF3) $a \leqslant b$ implies $\mathscr{A}(b) \geqslant \mathscr{A}(a)$ for all $a, b \in M$;
(LF4) $\mathscr{A}(a \otimes b) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(b)$ for all $a, b \in M$.

Proof. Assume that $\mathscr{A} \in \mathbf{LFil}(M)$. From Theorem 6 in [7], we know that $\mathscr{A}$ satisfies the condition (LF3). Let $a, b \in M$, since $a \leqslant b \to (a \otimes b)$, by (LF2) and (LF3)), we have that $\mathscr{A}(a \otimes b) \geqslant \mathscr{A}(b) \wedge \mathscr{A}(b \to (a \otimes b)) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(b)$. Thus $\mathscr{A}$ also satisfies the condition (LF4). Conversely, Assume that $\mathscr{A}$ satisfies the condition (LF3) and (LF4). since $a \leqslant 1$, by (LF3) we have $\mathscr{A}(1) \geqslant \mathscr{A}(a)$. Thus $\mathscr{A}$ satisfies the condition (LF1). From $a \otimes (a \to b) \leqslant b$, (LF3) and (LF4), it follows that $\mathscr{A}(b) \geqslant \mathscr{A}(a \otimes (a \to b)) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(a \to b)$. Thus $\mathscr{A}$ satisfies the condition (LF2). Therefore $\mathscr{A} \in \mathbf{LFil}(M)$ by Definition 2.

Definition 3. Let M be an R_0-algebra, L a lattice and $\mathscr{A}$ an L-fuzzy subset on M. An L-fuzzy subset $\mathscr{A}^\lambda$ on M is defined as follows:

$$\mathscr{A}^\lambda(a) = \begin{cases} \mathscr{A}(a), & a \neq 1, \\ \mathscr{A}(1) \vee \lambda, & a = 1, \end{cases} \tag{1}$$

for all $a \in M$, where $\lambda \in L$.

Theorem 2. Let M be an R_0-algebra, L a lattice and $\mathscr{A} \in \mathbf{LFil}(M)$. Then $\mathscr{A}^\lambda \in \mathbf{LFil}(M)$ for all $\lambda \in L$.

Proof. Firstly, for all $a, b \in M$, let $a \leqslant b$, we consider the following two cases:

(i) Assume that $b = 1$. If $a = 1$, we have that $\mathscr{A}^\lambda(b) = \mathscr{A}(1) \vee \lambda = \mathscr{A}^\lambda(a)$. If $a \neq 1$, by using $\mathscr{A} \in \mathbf{LFil}(M)$ and (LF1), we have that $\mathscr{A}^\lambda(b) = \mathscr{A}(1) \vee \lambda \geqslant \mathscr{A}(1) \geqslant \mathscr{A}(a) = \mathscr{A}^\lambda(a)$.

(ii) Assume that $b \neq 1$, then $a \neq 1$. It follows that $\mathscr{A}^{\lambda}(b) = \mathscr{A}(b) \geqslant \mathscr{A}(a) = \mathscr{A}^{\lambda}(a)$ from $\mathscr{A} \in \mathbf{LFil}(M)$ and (LF3).

Summarize above two cases, we conclude that $a \leqslant b$ implies $\mathscr{A}^{\lambda}(b) \geqslant \mathscr{A}^{\lambda}(a)$, for all $a, b \in M$. That is, $\mathscr{A}^{\lambda}$ satisfies (LF3).

Secondly, for all $a, b \in M$, we consider the following two cases:

(i) Assume that $a \otimes b = 1$. If $a = b = 1$, it is obvious that

$$\mathscr{A}^{\lambda}(a \otimes b) = \mathscr{A}(1) \vee \lambda = \mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b).$$

If $a = 1, b \neq 1$ or $a \neq 1, b = 1$, then $a \otimes b \neq 1$, it is a contradiction.

If $a \neq 1$ and $b \neq 1$, it follows that $\mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b) = \mathscr{A}(a) \wedge \mathscr{A}(b) \leqslant \mathscr{A}(a \otimes b) = \mathscr{A}(1) \leqslant \mathscr{A}(1) \vee \lambda = \mathscr{A}^{\lambda}(a \otimes b)$ from $\mathscr{A} \in \mathbf{LFil}(M)$, (LF4) and (1).

(ii) Assume that $a \otimes b \neq 1$. If $a = b = 1$, it is obvious a contradiction.

If $a = 1, b \neq 1$ or $a \neq 1, b = 1$, let's assume $a = 1, b \neq 1$, then $a \otimes b = \neg(1 \to \neg b) = b$, and so $\mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b) \leqslant \mathscr{A}^{\lambda}(b) = \mathscr{A}(b) = \mathscr{A}(a \otimes b) = \mathscr{A}^{\lambda}(a \otimes b)$.

If $a \neq 1$ and $b \neq 1$, it follows that $\mathscr{A}^{\lambda}(a \otimes b) = \mathscr{A}(a \otimes b) \geqslant \mathscr{A}(a) \wedge \mathscr{A}(b) = \mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b)$ from $\mathscr{A} \in \mathbf{LFil}(M)$ and (LF4).

Summarize above two cases, we conclude that $\mathscr{A}^{\lambda}(a \otimes b) \geqslant \mathscr{A}^{\lambda}(a) \wedge \mathscr{A}^{\lambda}(b)$, for all $a, b \in M$. That is, $\mathscr{A}^{\lambda}$ satisfies (LF4).

Thus it follows that $\mathscr{A}^{\lambda} \in \mathbf{LFil}(M)$ from Theorem 1.

Definition 4. Let M be an R_0-algebra, L a lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. Defined L-fuzzy subsets $\mathscr{A}^{\mathscr{B}}$ and $\mathscr{B}^{\mathscr{A}}$ on M as follows: for all $a \in M$,

$$\mathscr{A}^{\mathscr{B}}(a) = \begin{cases} \mathscr{A}(a), & a \neq 1, \\ \mathscr{A}(1) \vee \mathscr{B}(1), & a = 1, \end{cases} \quad \text{and} \quad \mathscr{B}^{\mathscr{A}}(a) = \begin{cases} \mathscr{B}(a), & a \neq 1, \\ \mathscr{B}(1) \vee \mathscr{A}(1), & a = 1. \end{cases} \tag{2}$$

Corollary 1. Let M be an R_0-algebra, L a lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. If $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$. Then $\mathscr{A}^{\mathscr{B}}, \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$.

Definition 5. Let M be an R_0-algebra, L a completely lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. An L-fuzzy set $\mathscr{A} \uplus \mathscr{B}$ on M is defined as follows: for all $a, x, y \in M$,

$$(\mathscr{A} \uplus \mathscr{B})(a) = \bigvee_{x \otimes y \leqslant a} [\mathscr{A}(x) \wedge \mathscr{B}(y)]. \tag{3}$$

Theorem 3. Let M be an R_0-algebra, L a completely distributive lattice and $\mathscr{A}, \mathscr{B}$ two L-fuzzy subsets on M. If $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$. Then $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$.

Proof. Firstly, for all $a, b \in M$, let $a \leqslant b$, then $\{x \otimes y | x \otimes y \leqslant a\} \subseteq \{x \otimes y | x \otimes y \leqslant b\}$, and so

$$\begin{aligned} \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(b) &= \bigvee_{x \otimes y \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\ &\geqslant \bigvee_{x \otimes y \leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a). \end{aligned}$$

Hence $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ satisfies (LF3). Secondly, for all $a, b \in M$, we have that

$$\begin{aligned}
&\left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a \otimes b) \\
&= \bigvee_{x \otimes y \leqslant a \otimes b} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\
&\geqslant \bigvee_{x_1 \otimes x_2 \leqslant a \text{ and } y_1 \otimes y_2 \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(x_1 \otimes y_1) \wedge \mathscr{B}^{\mathscr{A}}(x_2 \otimes y_2)\right] \\
&\geqslant \bigvee_{x_1 \otimes x_2 \leqslant a \text{ and } y_1 \otimes y_2 \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(x_1) \wedge \mathscr{A}^{\mathscr{B}}(y_1) \wedge \mathscr{B}^{\mathscr{A}}(x_2) \wedge \mathscr{B}^{\mathscr{A}}(y_2)\right] \\
&= \bigvee_{x_1 \otimes x_2 \leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x_1) \wedge \mathscr{B}^{\mathscr{A}}(x_2)\right] \wedge \bigvee_{y_1 \otimes y_2 \leqslant b} \left[\mathscr{A}^{\mathscr{B}}(y_1) \wedge \mathscr{B}^{\mathscr{A}}(y_2)\right] \\
&= \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a) \wedge \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(b),
\end{aligned}$$

and so $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ also satisfies (LF4). Hence $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$ by Theorem 1.

4 Generated *L*-fuzzy Filter by an *L*-fuzzy Subset

In this section, we give the notion of generated L-fuzzy filter by an L-fuzzy subset and establish its representation theorem.

Definition 6. Let M be an R_0-algebra, L a lattice and $\mathscr{A}$ an L-fuzzy subset on M. An L-fuzzy filter $\mathscr{B}$ of M is called the generated L-fuzzy filter by $\mathscr{A}$, denoted $\langle \mathscr{A} \rangle$, if $\mathscr{A} \Subset \mathscr{B}$ and for any $\mathscr{C} \in \mathbf{LFil}(M)$, $\mathscr{A} \Subset \mathscr{C}$ implies $\mathscr{B} \Subset \mathscr{C}$.

Theorem 4. Let M be an R_0-algebra, L a completely distributive lattice and $\mathscr{A}$ an L-fuzzy subset on M. An L-fuzzy subset $\mathscr{B}$ on M is defined as follows:

$$\mathscr{B}(a) = \bigvee \left\{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes \cdots \otimes x_n \leqslant a\right\}, \tag{4}$$

for all $a \in M$. Then $\mathscr{B} = \langle \mathscr{A} \rangle$.

Proof. Firstly, we prove that $\mathscr{B} \in \mathbf{LFil}(M)$. For all $a, b \in M$, let $a \leqslant b$. Then

$$\begin{aligned}
\mathscr{A}(a) &= \bigvee \{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\} \\
&\leqslant \bigvee \{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes \cdots \otimes x_n \leqslant b\} = \mathscr{B}(b).
\end{aligned}$$

Thus $\mathscr{B}$ satisfies (LF3). Assume that there are $x_1, x_2, \cdots, x_n \in M$ and $y_1, \cdots, y_m \in M$ such that $x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a$ and $y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant b$, we have that $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant a \otimes b$ by (P11). Thus, we can

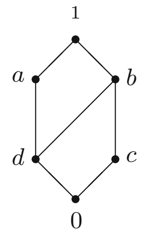

Fig. 1. The Hasse diagram of M

Table 1. Def. of "→"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

obtain that

$$\begin{aligned}
&\mathscr{B}(a) \wedge \mathscr{B}(b)\\
=&\bigvee \{\mathscr{A}(x_1) \wedge \cdots \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\}\\
&\quad \wedge \bigvee \{\mathscr{A}(y_1) \wedge \cdots \mathscr{A}(y_m) | y_1, y_2, \cdots, y_m \in M \text{ and } y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant b\}\\
=&\bigvee \{\mathscr{A}(x_1) \wedge \cdots \wedge \mathscr{A}(x_n) \wedge \mathscr{A}(y_1) \wedge \cdots \wedge \mathscr{A}(y_m) | x_1, \cdots, x_n, y_1, \cdots, y_m \in M\\
&\quad \text{such that } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a \text{ and } y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant b\}\\
\leqslant&\bigvee \{\mathscr{A}(x_1) \wedge \cdots \wedge \mathscr{A}(x_n) \wedge \mathscr{A}(y_1) \wedge \cdots \wedge \mathscr{A}(y_m) | x_1, \cdots, x_n, y_1, \cdots, y_m \in M\\
&\quad \text{such that } x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes y_1 \otimes y_2 \otimes \cdots \otimes y_m \leqslant a \otimes b\}\\
\leqslant&\bigvee \{\mathscr{A}(z_1) \wedge \cdots \mathscr{A}(z_k) | z_1, z_2, \cdots, z_k \in M \text{ and } z_1 \otimes \cdots \otimes z_k \leqslant a \otimes b\}\\
=&\mathscr{B}(a \otimes b).
\end{aligned}$$

Hence $\mathscr{B}$ also satisfies (LF4). It follows from Theorem 1 that $\mathscr{B} \in \mathbf{LFil}(M)$.

Secondly, For any $a \in M$, it follows from $a \leqslant a$ and the definition of $\mathscr{B}$ that $\mathscr{A}(a) \leqslant \mathscr{B}(a)$. This means that $\mathscr{A} \Subset \mathscr{B}$.

Finally, assume that $\mathscr{C} \in \mathbf{LFil}(M)$ with $\mathscr{A} \Subset \mathscr{C}$. Then for any $a \in M$, we have

$$\begin{aligned}
\mathscr{B}(a) =&\bigvee \{\mathscr{A}(x_1) \wedge \cdots \wedge \mathscr{A}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\}\\
\leqslant&\bigvee \{\mathscr{C}(x_1) \wedge \cdots \wedge \mathscr{C}(x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\}\\
\leqslant&\bigvee \{\mathscr{C}(x_1 \otimes \cdots \otimes x_n) | x_1, x_2, \cdots, x_n \in M \text{ and } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leqslant a\}\\
\leqslant&\bigvee \{\mathscr{C}(a)\} = \mathscr{C}(a).
\end{aligned}$$

Hence $\mathscr{B} \Subset \mathscr{C}$ holds. To sum up, we have that $\mathscr{B} = \langle \mathscr{A} \rangle$.

Example 1. Let $M = \{0, a, b, c, d, 1\}$, $\neg 0 = 1, \neg a = c, \neg b = d, \neg c = a, \neg d = b, \neg 1 = 0$, the Hasse diagram of lattice $(M, \vee, \wedge, \leqslant)$ be defined as Fig. 1, and the binary operator $\to$ of M be defined as Table 1.

Then $(M, \neg, \vee, \to, 1)$ is an R_0-algebra. Take $L = ([0,1], \max, \min)$ and define an [0,1]-fuzzy subset $\mathscr{A}$ on M by $\mathscr{A}(1) = \mathscr{A}(c) = \alpha, \mathscr{A}(a) = \mathscr{A}(b) = \mathscr{A}(d) =$

$\mathscr{A}(0) = \beta$, $0 \leqslant \beta < \alpha \leqslant 1$. Since $c \leqslant b$ but $\mathscr{A}(b) = \beta \ngeqslant \alpha = \mathscr{A}(c)$, we know that $\mathscr{A} \notin \mathbf{LFil}(M)$. It is easy to verify that $\langle \mathscr{A} \rangle \in \mathbf{LFil}(M)$ from Theorem 4, where $\langle \mathscr{A} \rangle(1) = \langle \mathscr{A} \rangle(b) = \langle \mathscr{A} \rangle(c) = \alpha, \langle \mathscr{A} \rangle(a) = \langle \mathscr{A} \rangle(d) = \langle \mathscr{A} \rangle(0) = \beta$.

5 The Lattice of L-fuzzy Filters in a Given R_0-algebra

In this section, we investigate the lattice structural feature of the set $\mathbf{LFil}(M)$ under the L-fuzzy set-inclusion order $\Subset$.

Theorem 5. Let M be an R_0-algebra and L a complete lattice. Then $(\mathbf{LFil}(M), \Subset)$ is a complete lattice.

Proof. For any $\{\mathscr{A}_\alpha\}_{\alpha \in \Lambda} \subseteq \mathbf{LFil}(M)$, where Λ is an indexed set. It is easy to verify that $\Cap_{\alpha \in \Lambda} \mathscr{A}_\alpha \in \mathbf{LFil}(M)$ is infimum of $\{\mathscr{A}_\alpha\}_{\alpha \in \Lambda}$, where $(\Cap_{\alpha \in \Lambda} \mathscr{A}_\alpha)(a) = \bigwedge\limits_{\alpha \in \Lambda} \mathscr{A}_\alpha(a)$ for all $a \in M$. i.e., $\bigwedge\limits_{\alpha \in \Lambda} \mathscr{A}_\alpha = \Cap_{\alpha \in \Lambda} \mathscr{A}_\alpha$. Define $\Cup_{\alpha \in \Lambda} \mathscr{A}_\alpha$ such that $(\Cup_{\alpha \in \Lambda} \mathscr{A}_\alpha)(a) = \bigvee\limits_{\alpha \in \Lambda} \mathscr{A}_\alpha(a)$ for all $a \in M$. Then $\langle \Cup_{\alpha \in \Lambda} \mathscr{A}_\alpha \rangle$ is supermun of $\{\mathscr{A}_\alpha\}_{\alpha \in \Lambda}$, where $\langle \Cup_{\alpha \in \Lambda} \mathscr{A}_\alpha \rangle$ is the L-fuzzy filter generated by $\Cup_{\alpha \in \Lambda} \mathscr{A}_\alpha$ of M. i.e., $\bigvee\limits_{\alpha \in \Lambda} \mathscr{A}_\alpha = \langle \Cup_{\alpha \in \Lambda} \mathscr{A}_\alpha \rangle$. Therefor $(\mathbf{LFil}(M), \Subset)$ is a complete lattice. The proof is completed.

Remark 1. Let M be an R_0-algebra and L a complete lattice. For all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$, by Theorem 5 we know that $\mathscr{A} \wedge \mathscr{B} = \mathscr{A} \Cap \mathscr{B}$ and $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B} \rangle$.

Theorem 6. Let M be an R_0-algebra and L a completely distributive lattice. Then for all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$, $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B} \rangle = \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ in the complete lattice $(\mathbf{LFil}(M), \Subset)$.

Proof. For all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$, it is obvious that $\mathscr{A} \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$ and $\mathscr{B} \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$, that is, $\mathscr{A}(a) \leqslant \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a)$ and $\mathscr{B}(a) \leqslant \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a)$ for all $a \in M$. Thus$(\mathscr{A} \Cup \mathscr{B})(a) = \mathscr{A}(a) \vee \mathscr{B}(a) \leqslant \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a)$, that is, $\mathscr{A} \Cup \mathscr{B} \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$, and thus $\langle \mathscr{A} \Cup \mathscr{B} \rangle \Subset \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \in \mathbf{LFil}(M)$ by Theorem 3. Let $\mathscr{C} \in \mathbf{LFil}(M)$ such that $\mathscr{A} \Cup \mathscr{B} \Subset \mathscr{C}$. For all $a \in M$, we consider the following two cases:

(i) If $a = 1$, then $\left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(1) = \bigvee\limits_{x \otimes y \leqslant 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \mathscr{A}^{\mathscr{B}}(1) \wedge \mathscr{B}^{\mathscr{A}}(1) = \mathscr{A}(1) \vee \mathscr{B}(1) = (\mathscr{A} \Cup \mathscr{B})(1) \leqslant \mathscr{C}(1)$.

(ii) If $a < 1$, then we have

$$\begin{aligned}
&\left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a) = \bigvee_{x\otimes y\leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\
&= \bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \vee \bigvee_{x\leqslant a} \{\mathscr{A}(x) \wedge [\mathscr{A}(1) \vee \mathscr{B}(1)]\} \\
&\qquad \vee \bigvee_{y\leqslant a} \{[\mathscr{A}(1) \vee \mathscr{B}(1)] \wedge \mathscr{B}(y)\} \\
&= \bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \vee \left[\bigvee_{x\leqslant a} \mathscr{A}(x)\right] \vee \left[\bigvee_{y\leqslant a} \mathscr{B}(y)\right] \\
&\leqslant \bigvee_{x\otimes y\leqslant a, x\neq 1, y\neq 1} [\mathscr{C}(x) \wedge \mathscr{C}(y)] \vee \left[\bigvee_{x\leqslant a} \mathscr{C}(x)\right] \vee \left[\bigvee_{y\leqslant a} \mathscr{C}(y)\right] \\
&= \bigvee_{x\otimes y\leqslant a} [\mathscr{C}(x) \wedge \mathscr{C}(y)] \leqslant \bigvee_{x\otimes y\leqslant a} \mathscr{C}(x\otimes y) \leqslant \mathscr{C}(a),
\end{aligned}$$

thus $\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}} \Subset \mathscr{C}$ for above two cases.

By Definition 6 and Theorem 4 we have that $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B} \rangle = \mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}$.

Theorem 7. Let M be an R_0-algebra and L a completely distributive lattice. Then $(\mathbf{LFil}(M), \Subset)$ is a distributive lattice, where, $\mathscr{A} \wedge \mathscr{B} = \mathscr{A} \Cap \mathscr{B}$ and $\mathscr{A} \vee \mathscr{B} = \langle \mathscr{A} \Cup \mathscr{B} \rangle$, for all $\mathscr{A}, \mathscr{B} \in \mathbf{LFil}(M)$.

Proof. To finish the proof, it suffices to show that $\mathscr{C} \wedge (\mathscr{A} \vee \mathscr{B}) = (\mathscr{C} \wedge \mathscr{A}) \vee (\mathscr{C} \wedge \mathscr{B})$, for all $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbf{LFil}(M)$. Since the inequality $(\mathscr{C} \wedge \mathscr{A}) \vee (\mathscr{C} \wedge \mathscr{B}) \Subset \mathscr{C} \wedge (\mathscr{A} \vee \mathscr{B})$ holds automatically in a lattice, we need only to show the inequality $\mathscr{C} \wedge (\mathscr{A} \vee \mathscr{B}) \Subset (\mathscr{C} \wedge \mathscr{A}) \vee (\mathscr{C} \wedge \mathscr{B})$. i.e., we need only to show that $\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) \leqslant \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}\right)(a)$, for all $a \in M$. For these, we consider the following two cases:

(i) If $a = 1$, we have

$$\begin{aligned}
&\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(1) = \mathscr{C}(1) \wedge \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(1) \\
&= \mathscr{C}(1) \wedge \bigvee_{x\otimes y\leqslant 1} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \mathscr{C}(1) \wedge \left[\mathscr{A}^{\mathscr{B}}(1) \wedge \mathscr{B}^{\mathscr{A}}(1)\right] \\
&= [\mathscr{C}(1) \wedge \mathscr{A}(1)] \vee [\mathscr{C}(1) \wedge \mathscr{B}(1)] = (\mathscr{C} \Cap \mathscr{A})(1) \vee (\mathscr{C} \Cap \mathscr{B})(1) \\
&= (\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(1) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(1) \\
&= \bigvee_{x\otimes y\leqslant 1} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}}(x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}(y)\right] \\
&= \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C}\Cap\mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C}\Cap\mathscr{A}}\right)(1).
\end{aligned}$$

(ii) If $a < 1$, we have

$$
\begin{aligned}
&\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) = \mathscr{C}(a) \wedge \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)(a) \\
=&\mathscr{C}(a) \wedge \bigvee_{x \otimes y \leqslant a} \left[\mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] = \bigvee_{x \otimes y \leqslant a} \left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \\
=& \bigvee_{x \otimes y \leqslant a, x \neq 1, y \neq 1} \left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(x) \wedge \mathscr{B}^{\mathscr{A}}(y)\right] \vee \\
& \bigvee_{y \leqslant a} \left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(1) \wedge \mathscr{B}(y)\right] \vee \bigvee_{x \leqslant a} \left[\mathscr{C}(a) \wedge \mathscr{A}(x) \wedge \mathscr{B}^{\mathscr{A}}(1)\right] \\
=& \bigvee_{x \otimes y \leqslant a, x \neq 1, y \neq 1} \{[\mathscr{C}(a) \wedge \mathscr{A}(x)] \wedge [\mathscr{C}(a) \wedge \mathscr{B}(y)]\} \vee \\
& \bigvee_{y \leqslant a} \left\{\left[\mathscr{C}(a) \wedge \mathscr{A}^{\mathscr{B}}(1)\right] \wedge [\mathscr{C}(a) \wedge \mathscr{B}(y)]\right\} \vee \\
& \bigvee_{x \leqslant a} \left\{[\mathscr{C}(a) \wedge \mathscr{A}(x)] \wedge \left[\mathscr{C}(a) \wedge \mathscr{B}^{\mathscr{A}}(1)\right]\right\} \\
\leqslant& \bigvee_{x \otimes y \leqslant a, x \neq 1, y \neq 1} \{[\mathscr{C}(a \vee x) \wedge \mathscr{A}(a \vee x)] \wedge [\mathscr{C}(a \vee y) \wedge \mathscr{B}(a \vee y)]\} \vee \\
& \bigvee_{y \leqslant a} \{[\mathscr{C}(1) \wedge (\mathscr{A}(1) \vee \mathscr{B}(1))] \wedge [\mathscr{C}(a \vee y) \wedge \mathscr{B}(a \vee y)]\} \vee \\
& \bigvee_{x \leqslant a} \{[\mathscr{C}(a \vee x) \wedge \mathscr{A}(a \vee x)] \wedge [\mathscr{C}(1) \wedge (\mathscr{B}(1) \vee \mathscr{A}(1))]\} \\
=& \bigvee_{x \otimes y \leqslant a, x \neq 1, y \neq 1} [(\mathscr{C} \Cap \mathscr{A})(a \vee x) \wedge (\mathscr{C} \Cap \mathscr{B})(a \vee y)] \vee \\
& \bigvee_{y \leqslant a} \{[(\mathscr{C} \Cap \mathscr{A})(1) \vee (\mathscr{C} \Cap \mathscr{B})(1)] \wedge (\mathscr{C} \Cap \mathscr{B})(a \vee y)\} \vee \\
& \bigvee_{x \leqslant a} \{(\mathscr{C} \Cap \mathscr{A})(a \vee x) \wedge [(\mathscr{C} \Cap \mathscr{B})(1) \vee (\mathscr{C} \Cap \mathscr{A})(1)]\} \\
=& \bigvee_{x \otimes y \leqslant a, x \neq 1, y \neq 1} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(a \vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(a \vee y)\right] \vee \\
& \bigvee_{y \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(1) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(a \vee y)\right] \vee \\
& \bigvee_{x \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(a \vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(1)\right] \\
=& \bigvee_{x \otimes y \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(a \vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(a \vee y)\right].
\end{aligned}
$$

Let $a \vee x = u$ and $a \vee y = v$, since $x \otimes y \leqslant a$, using Lemma 2 we get that

$$\begin{aligned} u \otimes v = (a \vee x) \otimes (a \vee y) =& ((a \vee x) \otimes a) \vee ((a \vee x) \otimes y) \\ =& (a \otimes a) \vee (a \otimes x) \vee (a \otimes y) \vee (x \otimes y) \\ \leqslant& a \vee a \vee a \vee (x \otimes y) \\ =& a \vee (x \otimes y) \leqslant a \vee a = a. \end{aligned}$$

Hence we can conclude that

$$\begin{aligned} \left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) &\leqslant \bigvee_{x \otimes y \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(a \vee x) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(a \vee y)\right] \\ &\leqslant \bigvee_{u \otimes v \leqslant a} \left[(\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}}(u) \wedge (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}(v)\right] \\ &= \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}\right)(a). \end{aligned}$$

To sum up, we have that

$$\left(\mathscr{C} \Cap \left(\mathscr{A}^{\mathscr{B}} \uplus \mathscr{B}^{\mathscr{A}}\right)\right)(a) \leqslant \left((\mathscr{C} \Cap \mathscr{A})^{\mathscr{C} \Cap \mathscr{B}} \uplus (\mathscr{C} \Cap \mathscr{B})^{\mathscr{C} \Cap \mathscr{A}}\right)(a),$$

for all $a \in M$. The proof is completed.

6 Conclusion

As well known, filters is an important concept for studying the structural features of R_0-algebras. In this paper, the L-fuzzy filter theory in R_0-algebras is further studied. Some new properties of L-fuzzy filters are given. Representation theorem of L-fuzzy filter which is generated by an L-fuzzy subset is established. It is proved that the set consisting of all L-fuzzy filters in a given R_0-algebra, under the L-fuzzy set-inclusion order $\Subset$, forms a complete distributive lattice. Results obtained in this paper not only enrich the content of L-fuzzy filters theory in R_0-algebras, but also show interactions of algebraic technique and L-fuzzy sets method in the studying of logic problems. We hope that more links of fuzzy sets and logics emerge by the stipulating of this work.

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Intuitionistic Fuzzy Rough Set Based on the Cut Sets of Intuitionistic Fuzzy Set

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Abstract. In this paper, the triple valued fuzzy set is selected as the cut set of intuitionistic fuzzy set and four kinds of cut sets of intuitionistic fuzzy sets are selected to investigate the intuitionistic fuzzy rough set. The intuitionistic fuzzy rough set is constructed by using the representation theorem of intuitionistic fuzzy set, and it is proved to be equivalent to the original intuitionistic fuzzy rough set. These works give a new perspective on intuitionistic fuzzy rough sets, which promotes the further research and development of the theory of intuitionistic fuzzy sets and rough sets.

Keywords: Fuzzy rough set · Intuitionistic fuzzy set · Cut set · Three valued fuzzy set · Rough set

1 Introduction

Atanassov expanded the fuzzy set [1] and put forward the concept of intuitionistic fuzzy sets [2] in 1986. Intuitionistic fuzzy set takes membership and non-membership into consideration, which makes it more accurate and effective in dealing with uncertainty. Rough set theory proposed by Professor Pawlak [3] in 1982 is a mathematical tool to deal with imprecise, inconsistent, incomplete information and knowledge. In 1998, Chakrabarty proposed the theory of intuitionistic fuzzy rough sets [4], which is the generalization of the concept of rough sets of Pawlak. At present, the combination of intuitionistic fuzzy sets and rough sets is a hot research topic [7–12]. In 2009, based on intuitionistic fuzzy implication operator and intuitionistic fuzzy model, L. Zhou and W.Z. Wu established the theory framework of intuitionistic fuzzy rough set by using constructive method and axiomatic method [6]. In 2011, Thomas studied Rough intuitionistic fuzzy sets in a lattice [13].

In this paper, we select triple valued fuzzy sets [5] as the cut sets of intuitionistic fuzzy sets. Taking the triple valued fuzzy sets as the theoretical basis, we use the way 'cut set—operate—bond together' to construct the intuitionistic fuzzy rough sets. Comparing with the original theory, we use the representation theorem for two cut sets, triple fuzzy set and triple fuzzy equivalence relation.

This paper is structured as follows: In the second section, the preparation knowledge required in this paper is introduced. In the third and fourth section, we provide four kinds of methods to investigate the upper and lower approximation of intuitionistic fuzzy sets by using the representation theorem.

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2 Preliminary

Definition 2.1 [1]. Let X be a set. The mapping $A: X \to [0,1]$ is called a fuzzy subset of X.

Definition 2.2 [2]. Let X be a set. If the two functions, $\mu_A : X \to [0,1]$, $v_A : X \to [0,1]$ satisfy $\mu_A(x) + v_A(x) \le 1$, for all $x \in X$, we call $A = (X, \mu_A, v_A)$ is an intuitionistic fuzzy set of X, which is denoted as $A(x) = (\mu_A(x), v_A(x))$. Then we have the following operations:

$$A \subset B \Leftrightarrow (\mu_A(x) \le \mu_B(x), v_A(x) \ge v_B(x)), \forall x \in X;$$
$$A = B \Leftrightarrow (\mu_A(x) = \mu_B(x), v_A(x) = v_B(x)), \forall x \in X;$$
$$A^c = \,<X, v_A, \mu_A>.$$

Definition 2.3 [5]. Let $3^X = \{A | A : X \to \{0, \frac{1}{2}, 1\}$ is a mapping$\}$. If $A_\lambda, A_{\underline{\lambda}} \in 3^X$ and

$$A_\lambda(x) = \begin{cases} 1, & \mu_A(x) \ge \lambda; \\ \frac{1}{2}, & \mu_A(x) < \lambda \le 1 - v_A(x); \\ 0, & \lambda > 1 - v_A(x), \end{cases} \quad A_{\underline{\lambda}}(x) = \begin{cases} 1, & \mu_A(x) > \lambda; \\ \frac{1}{2}, & \mu_A(x) \le \lambda < 1 - v_A(x); \\ 0, & \lambda \ge 1 - v_A(x). \end{cases}$$

We call A_λ and $A_{\underline{\lambda}}$ the λ – upper cut set and λ – strong upper cut set of A.

If $A^\lambda, A^{\underline{\lambda}} \in 3^X$ and

$$A^\lambda(x) = \begin{cases} 1, & v_A(x) \ge \lambda; \\ \frac{1}{2}, & v_A(x) < \lambda \le 1 - \mu_A(x); \\ 0, & \lambda > 1 - \mu_A(x), \end{cases} \quad A^{\underline{\lambda}}(x) = \begin{cases} 1, & v_A(x) > \lambda; \\ \frac{1}{2}, & v_A(x) \le \lambda < 1 - \mu_A(x); \\ 0, & \lambda \ge 1 - \mu_A(x). \end{cases}$$

We call A^λ and $A^{\underline{\lambda}}$ the $\lambda -$ lower cut set and $\lambda -$ strong lower cut set of A.

If $A_{[\lambda]}, A_{[\underline{\lambda}]} \in 3^X$ and

$$A_{[\lambda]}(x) = \begin{cases} 1, & \mu_A(x) + \lambda \ge 1; \\ \frac{1}{2}, & v_A(x) \le \lambda < 1 - \mu_A(x); \\ 0, & \lambda < v_A(x), \end{cases} \quad A_{[\underline{\lambda}]}(x) = \begin{cases} 1, & \mu_A(x) + \lambda > 1; \\ \frac{1}{2}, & v_A(x) < \lambda \le 1 - \mu_A(x); \\ 0, & \lambda \le v_A(x). \end{cases}$$

We call $A_{[\lambda]}$ and $A_{[\underline{\lambda}]}$ the $\lambda -$ upper quasi cut set and $\lambda -$ strong quasi upper cut set of A.

If $A^{[\lambda]}, A^{[\underline{\lambda}]} \in 3^X$ and

$$A^{[\lambda]}(x) = \begin{cases} 1, & v_A(x) + \lambda \ge 1; \\ \frac{1}{2}, & \mu_A(x) \le \lambda < 1 - v_A(x); \\ 0, & \lambda < \mu_A(x), \end{cases} \quad A^{[\lambda]}(x) = \begin{cases} 1, & v_A(x) + \lambda \ge 1; \\ \frac{1}{2}, & \mu_A(x) < \lambda \le 1 - v_A(x); \\ 0, & \lambda \le \mu_A(x), \end{cases}$$

We call $A^{[\lambda]}$ and $A^{[\underline{\lambda}]}$ the $\lambda -$ lower quasi cut set and $\lambda -$ strong quasi lower cut set of A.

Definition 2.4 [5]. Let $A \in 3^X, \lambda \in [0,1]$ and $f_i : [0,1] \times 3^X \to L^X, (\lambda, A) \mapsto f_i(\lambda, A)$ be the following mappings $(i = 1, 2, \ldots, 8)$:

$$f_1(\lambda,A)(x) = \begin{cases} (0,1), & A(x) = 0; \\ (\lambda, 1-\lambda), & A(x) = 1; \\ (0, 1-\lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_2(\lambda,A)(x) = \begin{cases} (\lambda, 1-\lambda), & A(x) = 0; \\ (1,0), & A(x) = 1; \\ (\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

$$f_3(\lambda,A)(x) = \begin{cases} (1-\lambda,\lambda), & A(x) = 0; \\ (0,1), & A(x) = 1; \\ (0, \lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_4(\lambda,A)(x) = \begin{cases} (1,0), & A(x) = 0; \\ (1-\lambda,\lambda), & A(x) = 1; \\ (1-\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

$$f_5(\lambda,A)(x) = \begin{cases} (0,1), & A(x) = 0; \\ (1-\lambda,\lambda), & A(x) = 1; \\ (0, \lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_6(\lambda,A)(x) = \begin{cases} (1-\lambda,\lambda), & A(x) = 0; \\ (1,0), & A(x) = 1; \\ (1-\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

$$f_7(\lambda,A)(x) = \begin{cases} (\lambda,1-\lambda), & A(x) = 0; \\ (0,1), & A(x) = 1; \\ (0, 1-\lambda), & A(x) = \frac{1}{2}. \end{cases} \quad f_8(\lambda,A)(x) = \begin{cases} (1,0), & A(x) = 0; \\ (\lambda,1-\lambda), & A(x) = 1; \\ (\lambda, 0), & A(x) = \frac{1}{2}. \end{cases}$$

Definition 2.5 [7]. Let U be a set, R is an equivalence relation in U. That is $R \subset U \times U$, for $\forall x, y, z \in U$ satisfying:

(1) $(x, x) \in R$;
(2) $(x, y) \in R \Rightarrow (y, x) \in R$;
(3) $(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$.

Definition 2.6 [3]. Let $X \subset U$ and R be an equivalence relation in U. We call
$\overline{R}(X) = \cup \{x | x \in U, [x] \cap X \neq \emptyset\}$ the upper approximation of X
$\underline{R}(X) = \cup \{x | x \in U, [x] \subset X\}$ the lower approximation of X.

Definition 2.7 [5]. Let $H : [0,1] \to 3^X$ be a mapping. If $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supset H(\lambda_2)$, we call H a triple valued inverse order nested sets on X.

Let $H : [0,1] \to 3^X$ be a mapping. If $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subset H(\lambda_2)$, we call H a triple valued order nested sets on X.

3 The Upper Approximation of Intuitionistic Fuzzy Set

First of all, we give a kind of cut set and cut relations to construct the upper approximation of intuitionistic fuzzy sets.

Let U be a finite non empty set, $X = (\mu_X, \nu_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then X_λ is the triple valued fuzzy set over U, and R_λ is the triple valued fuzzy equivalence relation over U. Let $\overline{H_1}(\lambda) = \overline{R_\lambda}(X_\lambda)$, that is for $x \in U, \overline{H_1}(\lambda)(x) = \vee_{y \in U}(X_\lambda(y) \wedge R_\lambda(x, y))$, then $\overline{H_1}(\lambda)$ is the triple valued inverse order nested sets on U. *Let* $\overline{R}(X) = \bigcup_{\lambda \in [0,1]} f_1(\lambda, \overline{H_1}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_2(\lambda, \overline{H_1}(\lambda))$. Then we get the following theorem:

Theorem 3.1. Let $X \in IF(U), x \in U$, then

$$\overline{R}(X)(x) = (\bigvee_{y\in U}(\mu_X(y) \wedge \mu_R(x,y)), \bigwedge_{y\in U}(\nu_X(y) \vee \nu_R(x,y))). \qquad (1)$$

Proof

$$\overline{R}(X)(x) = \bigvee_{\lambda\in[0,1]} f_1(\lambda, \overline{H_1}(\lambda))(x)$$

(1) $$=\{\vee(\lambda, 1-\lambda)\big|\overline{H_1}(\lambda)(x) = 1\} \vee \{\vee(0, 1-\lambda)\big|\overline{H_1}(\lambda)(x) = \frac{1}{2}\}$$

$$= (\vee\{\lambda\big|\overline{H_1}(\lambda)(x) = 1\}, \wedge\{1-\lambda\big|\overline{H_1}(\lambda)(x) \geq \frac{1}{2}\})$$

Because

$$\begin{aligned}\overline{H_1}(\lambda)(x) = 1 &\Leftrightarrow \bigvee_{y\in U}(X_\lambda(y) \wedge R_\lambda(x,y)) = 1\\ &\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \wedge R_\lambda(x,y_0) = 1)\\ &\Leftrightarrow \exists y_0 \in U(\mu_X(y_0) \wedge \mu_R(x,y_0) \geq \lambda)\\ &\Leftrightarrow \bigvee_{y\in U}(\mu_X(y) \wedge \mu_R(x,y)) \geq \lambda\end{aligned}$$

Thus $\vee\{\lambda\big|\overline{H_1}(\lambda)(x) = 1\} = \bigvee_{y\in U}(\mu_X(y) \wedge \mu_R(x,y))$.

And

$$\begin{aligned}\overline{H_1}(\lambda)(x) \geq \frac{1}{2} &\Leftrightarrow \bigvee_{y\in U}(X_\lambda(y) \wedge R_\lambda(x,y)) \geq \frac{1}{2}\\ &\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \wedge R_\lambda(x,y_0) \geq \frac{1}{2})\\ &\Leftrightarrow \exists y_0 \in U(1-\nu_X(y_0) \geq \lambda \text{ and } 1-\nu_R(x,y_0) \geq \lambda)\\ &\Leftrightarrow \exists y_0 \in U(\nu_A(y_0) \vee \nu_R(x,y_0) \leq 1-\lambda)\\ &\Leftrightarrow \bigwedge_{y\in U}(\nu_X(y) \vee \nu_R(x,y)) \leq 1-\lambda\end{aligned}$$

Thus $\wedge\{1-\lambda\big|\underline{H_1}(\lambda)(x) \geq \frac{1}{2}\} = \bigwedge_{y\in U}(\nu_X(y) \vee \nu_R(x,y))$.

Therefore $\overline{R}(X)(x) = (\bigvee_{y\in U}(\mu_X(y) \wedge \mu_R(x,y)), \bigwedge_{y\in U}(\nu_X(y) \vee \nu_R(x,y)))$.

$$\overline{R}(X)(x) = \bigwedge_{\lambda\in[0,1]} f_2(\lambda, \overline{H_1}(\lambda))(x)$$

(2) $$=\{\wedge(\lambda, 1-\lambda)\big|\overline{H_1}(\lambda)(x) = 0\} \wedge \{\wedge(\lambda, 0)\big|\overline{H_1}(\lambda)(x) = \frac{1}{2}\}$$

$$= (\wedge\{\lambda\big|\overline{H_1}(\lambda)(x) \leq \frac{1}{2}\}, \vee\{1-\lambda\big|\overline{H_1}(\lambda)(x) = 0\})$$

Because

$$\overline{H_1}(\lambda)(x) \leq \frac{1}{2} \Leftrightarrow \bigvee_{y \in U} (X_\lambda(y) \wedge R_\lambda(x,y)) \leq \frac{1}{2}$$
$$\Leftrightarrow \forall y \in U(X_\lambda(y) \wedge R_\lambda(x,y) \leq \frac{1}{2})$$
$$\Leftrightarrow \forall y \in U(\mu_X(y) \wedge \mu_R(x,y) < \lambda)$$
$$\Leftrightarrow \bigvee_{y \in U} (\mu_X(y) \wedge \mu_R(x,y)) < \lambda$$

Thus $\wedge\{\lambda \big| \overline{H_1}(\lambda)(x) \leq \frac{1}{2}\} = \bigvee_{y \in U} (\mu_R(y) \wedge \mu_R(x,y))$.

And

$$\overline{H_1}(\lambda)(x) = 0 \Leftrightarrow \bigvee_{y \in U} (X_\lambda(y) \wedge R_\lambda(x,y)) = 0$$
$$\Leftrightarrow \forall y \in U, X_\lambda(y) \wedge R_\lambda(x,y) = 0$$
$$\Leftrightarrow \forall y \in U(1 - v_X(y) \leq \lambda \text{ or } 1 - v_R(x,y_0) \leq \lambda)$$
$$\Leftrightarrow \bigwedge_{y \in U} (v_X(y) \vee v_R(x,y)) \geq 1 - \lambda$$

Thus $\vee\{1 - \lambda \big| \overline{H_1}(\lambda)(x) = 0\} = \bigwedge_{y \in U} (v_X(y) \vee v_R(x,y))$.

Therefore $\overline{R}(X)(x) = ((\bigvee_{y \in U} (\mu_X(y) \wedge \mu_R(x,y)), \bigwedge_{y \in U} (v_X(y) \vee v_R(x,y)))$.

In the following, we construct the upper approximation of intuitionistic fuzzy sets by three different methods.

Let U be a finite non empty set, $X = (\mu_X, v_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, v_R)$, is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then $X_{[\lambda]}, X^\lambda, X^{[\lambda]}$ are the triple valued fuzzy sets over U, and $R_{[\lambda]}, R^\lambda, R^{[\lambda]}$ are the triple valued fuzzy equivalence relations over U.

Let $\overline{H_2}(\lambda) = \overline{R^\lambda}(X^\lambda)$. That is for $x \in U$, $\overline{H_2}(\lambda)(x) = \bigwedge_{y \in U} (X^\lambda(y) \vee R^\lambda(x,y))$, thus $\overline{H_2}(\lambda)$ is the triple valued inverse order nested sets over U. Let $\overline{R}(X) = \bigcup_{\lambda \in [0,1]} f_3(\lambda, \overline{H_2}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_4(\lambda, \overline{H_2}(\lambda))$.

In addition, let $\overline{H_3}(\lambda) = \overline{R_{[\lambda]}}(X_{[\lambda]})$. That is for $x \in U, \overline{H_3}(\lambda)(x) = \bigvee_{y \in U} (X_{[\lambda]}(y) \wedge R_{[\lambda]}(x,y))$, then $\overline{H_3}(\lambda)$ is the triple valued order nested sets over U. Let $\overline{R}(X) = \bigcup_{\lambda \in [0,1]} f_5(\lambda, \overline{H_3}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_6(\lambda, \overline{H_3}(\lambda))$.

Similarly, let $\overline{H_4}(\lambda) = \overline{R^{[\lambda]}}(X^{[\lambda]})$. That is for $x \in U$, $\overline{H_4}(\lambda)(x) = \bigwedge_{y \in U} (X^{[\lambda]}(y) \vee R^{[\lambda]}(x,y))$, thus $\overline{H_4}(\lambda)$ is the triple valued order nested sets over U. Let $\overline{R}(X) = \bigcup_{\lambda \in [0,1]} f_7(\lambda, \overline{H_4}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_8(\lambda, \overline{H_4}(\lambda))$.

By the three methods above, we get the same theorem:

Theorem 3.2. For $X \in IF(U), x \in U$, then

$$\overline{R}(X)(x) = (\underset{y\in U}{\vee}(\mu_X(y) \wedge \mu_R(x,y)), \underset{y\in U}{\wedge}(\nu_X(y) \vee \nu_R(x,y))).$$

The proof of Theorem 3.2 is similar Theorem 3.1.

4 The Lower Approximation of Intuitionistic Fuzzy Set

In this section, we select the corresponding cut sets and cut relations to construct the lower approximation of intuitionistic fuzzy sets.

Let U be a finite non empty set, $X = (\mu_X, \nu_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0,1]$, then X_λ is the triple valued fuzzy set over U, and $R_{[\underline{\lambda}]}$ is the triple valued fuzzy equivalence relation over U. Let $\underline{H_1}(\lambda) = \underline{(R_{[\underline{\lambda}]})^c}(X_\lambda)$. That is for $x \in U$, $\underline{H_1}(\lambda)(x) = \underset{y\in U}{\wedge}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y))$, then $\underline{H_1}(\lambda)$ is the triple valued inverse order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda\in[0,1]} f_1(\lambda, \underline{H_1}(\lambda))$ or $\bigcap_{\lambda\in[0,1]} f_2(\lambda, \underline{H_1}(\lambda))$. Then we get the following theorem:

Theorem 4.1. Let $X \in IF(U), x \in U$, then

$$\underline{R}(X)(x) = (\underset{y\in U}{\wedge}(\mu_X(y) \vee \nu_R(x,y)), \underset{y\in U}{\vee}(\nu_X(y) \wedge \mu_R(x,y))). \quad (2)$$

Proof

$$\underline{R}(X)(x) = \bigcup_{\lambda\in[0,1]} f_1(\lambda, \underline{H_1}(\lambda))(x)$$

(1) $$=\{\vee(\lambda, 1-\lambda)\big|\underline{H_1}(\lambda)(x) = 1\} \vee \{\vee(0, 1-\lambda)\big|\underline{H_1}(\lambda)(x) = \frac{1}{2}\}$$

$$= (\vee\{\lambda\big|\underline{H_1}(\lambda)(x) = 1\}, \wedge\{1-\lambda\big|\underline{H_1}(\lambda)(x) \geq \frac{1}{2}\})$$

Because

$$\begin{aligned}\underline{H_1}(\lambda)(x) = 1 &\Leftrightarrow \underset{y\in U}{\wedge}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) = 1\\ &\Leftrightarrow \forall y \in U(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y) = 1)\\ &\Leftrightarrow \forall y \in U(\mu_X(y) \geq \lambda \text{ or } \nu_R(x,y) \geq \lambda)\\ &\Leftrightarrow \underset{y\in U}{\wedge}(\mu_X(y) \vee \mu_R(x,y)) \geq \lambda\end{aligned}$$

Thus $\vee\{\lambda\big|\underline{H_1}(\lambda)(x) = 1\} = \underset{y\in U}{\wedge}(\mu_X(y) \vee \nu_R(x,y))$.

And

$$\underline{H_1}(\lambda)(x) \geq \frac{1}{2} \Leftrightarrow \bigwedge_{y\in U}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) \geq \frac{1}{2}$$
$$\Leftrightarrow \forall y \in U(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y) \geq \frac{1}{2})$$
$$\Leftrightarrow \forall y \in U(1 - v_X(y) \geq \lambda \text{ or } 1 - \mu_R(x,y) \geq \lambda)$$
$$\Leftrightarrow \forall y \in U(v_A(y) \wedge \mu_R(x,y) \leq 1 - \lambda)$$
$$\Leftrightarrow \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)) \leq 1 - \lambda$$

Thus $\wedge\{1-\lambda \big| \underline{H_1}(\lambda)(x) \geq \frac{1}{2}\} = \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y))$.

Therefore $\underline{R}(X)(x) = (\bigwedge_{y\in U}(\mu_X(y) \vee v_R(x,y)), \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)))$.

$$\underline{R}(X)(x) = \bigwedge_{\lambda\in[0,1]} f_2(\lambda, \underline{H_1}(\lambda))(x)$$

(2) $=\{\wedge(\lambda, 1-\lambda)\big|\underline{H_1}(\lambda)(x) = 0\} \wedge \{\wedge(\lambda, 0)\big|\underline{H_1}(\lambda)(x) = \frac{1}{2}\}$

$$= (\wedge\{\lambda\big|\underline{H_1}(\lambda)(x) \leq \frac{1}{2}\}, \vee\{1-\lambda\big|\underline{H_1}(\lambda)(x) = 0\})$$

Because

$$\underline{H_1}(\lambda)(x) \leq \frac{1}{2} \Leftrightarrow \bigwedge_{y\in U}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) \leq \frac{1}{2}$$
$$\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \vee (R_{[\underline{\lambda}]})^c(x,y_0) \leq \frac{1}{2})$$
$$\Leftrightarrow \exists y_0 \in U(\mu_X(y_0) \vee v_R(x,y_0) < \lambda)$$
$$\Leftrightarrow \bigwedge_{y\in U}(\mu_X(y) \vee v_R(x,y)) < \lambda$$

Thus $\wedge\{\lambda\big|\overline{H_1}(\lambda)(x) \leq \frac{1}{2}\} = \bigwedge_{y\in U}(\mu_R(y) \vee v_R(x,y))$.

And

$$\overline{H_1}(\lambda)(x) = 0 \Leftrightarrow \bigwedge_{y\in U}(X_\lambda(y) \vee (R_{[\underline{\lambda}]})^c(x,y)) = 0$$
$$\Leftrightarrow \exists y_0 \in U(X_\lambda(y_0) \vee (R_{[\underline{\lambda}]})^c(x,y_0) = 0)$$
$$\Leftrightarrow \exists y_0 \in U(1 - v_X(y_0) < \lambda \text{ and } 1 - \mu_R(x,y_0) < \lambda)$$
$$\Leftrightarrow \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)) > 1 - \lambda$$

Thus $\vee\{1-\lambda\big|\overline{H_1}(\lambda)(x) = 0\} = \bigvee_{y\in U}(v_X(y) \wedge v_R(x,y))$.

Therefore $\underline{R}(X)(x) = (\bigwedge_{y\in U}(\mu_X(y) \vee v_R(x,y)), \bigvee_{y\in U}(v_X(y) \wedge \mu_R(x,y)))$.

In the following, we construct the lower approximation of intuitionistic fuzzy sets by three different methods.

Let U be a finite non empty set, $X = (\mu_X, \nu_X)$ is an intuitionistic fuzzy subset over $U, R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy equivalence relation over U, for $\lambda \in [0, 1]$, then $X^{\lambda}, X_{[\lambda]}, X^{[\lambda]}$ are the triple valued fuzzy sets over U, and $R_{\underline{\lambda}}, R^{[\underline{\lambda}]}, R^{\underline{\lambda}}$ are the triple valued fuzzy equivalence relations over U.

Let $\underline{H_2}(\lambda) = \underline{(R^{\underline{\lambda}})^c}(X^{[\lambda]})$. That is for $x \in U$, $\underline{H_2}(\lambda)(x) = \bigvee_{y \in U}(X^{\lambda}(y) \wedge (R^{[\underline{\lambda}]})^c(x, y))$, thus $\underline{H_2}(\lambda)$ is the triple valued inverse order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda \in [0,1]} f_3(\lambda, \underline{H_2}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_4(\lambda, \underline{H_2}(\lambda))$.

Similarly, let $\underline{H_3}(\lambda) = \underline{(R_{\underline{\lambda}})^c}(X_{[\lambda]})$. That is for $x \in U$, $\underline{H_3}(\lambda)(x) = \bigwedge_{y \in U}(X_{[\lambda]}(y) \vee (R_{\underline{\lambda}})^c(x, y))$, thus $\underline{H_3}(\lambda)$ is the triple valued order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda \in [0,1]} f_5(\lambda, \underline{H_3}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_6(\lambda, \underline{H_3}(\lambda))$.

In addition, let $\underline{H_4}(\lambda) = \underline{(R^{\underline{\lambda}})^c}(X^{[\lambda]})$. That is for $x \in U$, $\underline{H_4}(\lambda)(x) = \bigvee_{y \in U}(X^{[\lambda]}(y) \wedge (R^{\underline{\lambda}})^c(x, y))$, thus $\underline{H_4}(\lambda)$ is the triple valued order nested sets on U. Let $\underline{R}(X) = \bigcup_{\lambda \in [0,1]} f_7(\lambda, \underline{H_4}(\lambda))$ or $\bigcap_{\lambda \in [0,1]} f_8(\lambda, \underline{H_4}(\lambda))$.

By the three methods above, we get the same theorem:

Theorem 4.2. Let $X \in IF(U), x \in U$. Then

$$\underline{R}(X)(x) = \left(\bigwedge_{y \in U}(\mu_X(y) \vee \nu_R(x, y)), \bigvee_{y \in U}(\nu_X(y) \wedge \mu_R(x, y))\right).$$

The proof of Theorem 4.2 is similar to Theorem 4.1.

5 Conclusion

In this paper, triple valued fuzzy sets are selected as the cut sets of intuitionistic fuzzy sets, and four methods are provided to construct the upper and lower approximations of intuitionistic fuzzy sets by using the representation theorem. Zhou et al. [6] and Zhang et al. [7] construct the upper and lower approximations of intuitionistic fuzzy sets by different intuitionistic fuzzy implicators. Jena et al. [9], Zhou et al. [10] and Samanta et al. [11] investigate intuitionistic fuzzy rough through the extension of fuzzy rough set. We get the same result as the original intuitionistic fuzzy rough set given in [6–11] and provide a theoretical support for the extension from fuzzy rough set to intuitionistic fuzzy rough sets [9–11]. From the results in this paper, we notice that (1) Cut sets and representation theorems play an important role in the research of intuitionistic fuzzy set theory and rough set, and a lot of fuzzy theories can be studied by means of nested set. (2) The triple fuzzy set is an efficient tool to study intuitionistic fuzzy rough set, which deserve our more attentions.

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Applications and Mathematical Modeling in Operations Research

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Abstract. Theoretical understanding of the relevant problem structure and consistent mathematical modeling are necessary keys to formulating operations research models to be used for optimization of decisions in real applications. The numbers of alternative models, methods and applications of operations research are very large. This paper presents fundamental and general decision and information structures, theories and examples that can be expanded and modified in several directions. The discussed methods and examples are motivated from the points of view of empirical relevance and computability.

Keywords: Operations research · Mathematical modeling · Optimization

1 Introduction

Operations research is a very large area. In this paper, we will focus on operations research in connection to optimization of decisions, with one or more decision maker(s). The classical analytical methods of optimization and comparative statics analysis, basic economic theory and fundamental linear programming are well presented in Chiang [3].

Mathematical modeling is central to operations research. Usually, in applied problems, there are many different ways to define the mathematical models representing the components of the system under analysis. The reference book of the software package LINGO [1] contains large numbers of alternative operations research models and applications with numerical solutions.

A particular applied problem should, if possible, be analyzed with a problem relevant operations research method, using a problem relevant set of mathematical models. This may seem obvious to the reader, but it is far from trivial to determine the problem relevant method and models.

The two books by Winston, references [16, 17], give a good and rather complete presentation of most operations research methods, algorithms and typical applications. The operations research literature contains large numbers of alternative methods and models, applied to very similar types of applied problems. In many cases, the optimal decisions that are the results of the analyses, differ considerably.

For instance, if we want to determine the optimal decision in a particular problem, we may define it as a one dimensional optimization problem, or as a multidimensional problem where we simultaneously optimize several decisions that may be linked in

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different ways. We may also consider constraints of different sorts. In most problems, present decisions have consequences for the future development of the system under analysis. Hence, multi period analysis is often relevant. Weintraub et al. [15] contains many dynamic operations research problems and solutions from different natural resource sectors. Then, we realize that the future state of the world can change for several reasons. In resource management problems, for instance, we often want to determine optimal present extraction of some resource, such as coal or oil. If we take more today, we have to take less in the future. The present and future prices are very important parameters in such decision problems and we usually have to agree that the future prices are not perfectly known today. Price changes may occur because of technical innovations, political changes and many other reasons. We simply have to accept that future prices can never be perfectly predicted. Hence, the stochastic properties of prices have to be analyzed and used in the operations research studies in order to determine optimal present decisions. Many types of resources are continuously used, thanks to biological growth. Braun [2] gives a very good presentation of ordinary differential equations, which is key to the understanding and modeling of dynamical systems, including biological resources of all kinds. In agriculture, fishing, forestry, wildlife management and hunting, resources are used for many different purposes, including food, building materials, paper, energy and much more. In order to determine optimal present decisions in such industries, it is necessary to develop and use dynamic models that describe how the biological resources grow and how the growth is affected by present harvesting and other management decisions. Clark [4] contains several examples and solutions of deterministic optimal control theory problems in natural resource sectors.

The degree of unexplained variation in the future state of the resource is often considerable. Many crops are sensitive to extreme rains, heat, floods, parasites and pests. Forests are sensitive to storms and hurricanes, fires etc. Obviously, risk is of central importance to modeling and applied problem solving in these sectors. Grimmet and Stirzaker [6] contains most of the important theory of probability and random processes. Fleming and Rishel [5] contains the general theory of deterministic and stochastic optimal control. Sethi and Thompson [12] cover a field very similar to [5], but is more focused on applied derivations. Lohmander [8, 9] shows how dynamic and stochastic management decisions can be optimized with different methods, including different versions of stochastic dynamic programming. Lohmander [10] develops methodology for optimization of large scale energy production under risk, using stochastic dynamic programming with a quadratic programming subroutine. Deterministic systems are not necessarily predictable. Tung [13] is a fantastic book that contains many kinds of mathematical modeling topics and applications, including modern chaos theory and examples. Such theories and methods are also relevant to rational decision making in resource management problems. Until now, we have only considered problems with one decision maker. In reality, we often find many decision makers that all influence the development of the same system. In such cases, we can model this situation using game theory. Luce and Raiffa [11] gives a very good coverage of the classical field. In games without cooperation, the Nash equilibrium theory is very useful. Each player maximizes his/her own objective given that the other player maximizes his/her objective. Washburn [14] focuses on such games and the important and often quite relevant subset “two person zero sum games”. In such games, linear programming finds many relevant applications. Isaacs [7] describes and analyses several games of this nature, but in continuous time, with the method differential games. This manuscript could have been expanded in the direction of dynamic and stochastic games. The present format limitation however makes this impossible. Let us conclude this section with the finding that mathematical modeling in operations research is a rich field with an almost unlimited number of applications.

2 Analysis

Let us investigate alternative specifications of operations research models and discuss the properties. We may consider (1) as a general representation of linear constraints, as we find them in most logistics problems, manufacturing problems and many other applied problems. We assume that a feasible set exists and know that the feasible set obtained with linear constraints is convex. In a production problem, x_k is the production volume of product k and the constraints are capacity constraints, where C_l is the total capacity of resource l.

$$\begin{cases} \alpha_{11}x_1 + \ldots + \alpha_{1K}x_K \leq C_1 \\ \qquad \ldots \\ \alpha_{L1}x_1 + \ldots + \alpha_{LK}x_K \leq C_L \end{cases} \tag{1}$$

In case we have a linear objective function, such as the total profit, π, we may express that as (2).

$$\pi(x_1, \ldots, x_K) = p_0 + p_1x_1 + \ldots + p_Kx_K \tag{2}$$

Linear programming is a relevant optimization method if we want to maximize (2) subject to (1). The simplex algorithm will give the optimal solution in a finite number of iterations. In many applied problems, such as production optimization problems, it is also important to be able to handle the fact that market prices often are decreasing functions of the produced and sold quantities of different products. Furthermore, the production volume of one product may affect the prices of other products, the marginal production costs of different products may be linked and so on. Then, the objective function of the company may be approximated as a quadratic function (3). (Note that (3) may be further simplified.)

$$\begin{aligned} \pi(x_1, \ldots, x_K) = & p_0 + p_1x_1 + \ldots + p_Kx_K + \\ & + r_{11}x_1^2 + r_{12}x_1x_2 + \ldots + r_{1(K-1)}x_1x_{K-1} + r_{1K}x_1x_K + \\ & + \ldots \\ & + r_{K1}x_Kx_1 + r_{K2}x_Kx_2 + \ldots + r_{K(K-1)}x_Kx_{K-1} + r_{KK}x_K^2 \end{aligned} \tag{3}$$

With a quadratic objective function and linear constraints, we have a quadratic programming problem (4). Efficient quadratic programming computer codes are available, that have several similarities to the simplex algorithm for linear programming. The Kuhn-Tucker conditions can be considered as linear constraints and in [1, 16], many such examples are solved.

$$\begin{aligned}
&\max \pi(x_1, \ldots, x_K) \\
&\quad s.t. \\
&\quad \alpha_{11}x_1 + \ldots + \alpha_{1K}x_K \le C_1 \\
&\quad \ldots \\
&\quad \alpha_{L1}x_1 + \ldots + \alpha_{LK}x_K \le C_L
\end{aligned} \tag{4}$$

In real applications, we are often interested to handle the sequential nature of information. Market prices usually have to be regarded as partially stochastic. We may influence the price level via our production and sales volumes. Still, there is usually a considerable price variation outside the control of the producer. Then, we can optimize our decisions via stochastic dynamic programming, as shown in the example in (5) and (6). Let us consider the optimal extraction over time from a limited oil reserve. In every period t until we reach the planning horizon T, we maximize the expected present value, $f(.)$, for every possible level of the remaining reserve, s, and for every market state, m. $f(.) = 0$ for $t = T + 1$, which is shown in (6). In all earlier periods, the values of $f(.)$ are maximized for all possible reserve and market levels, via the control u, the extraction level. In a period t, before we reach $t = T + 1$, the control u is selected so that the sum of the present value of instant extraction $\pi(.)$ and the expected present value of future extraction $\sum_n \tau(n|m) f(t+1, s-u, n)$ is maximized. $\tau(n|m)$ denotes the transition probability from market state m to market state n from one period to the next. The control u has to belong to the set of feasible controls $U(.)$ which is a function of t, s and m. Equations (5) and (6) summarize the principles and the recursive structure.

$$\begin{aligned}
f(t,s,m) = \max_{u \in U(t,s,m)} &\left(\pi(u;t,s,m) + \sum_n \tau(n|m) f(t+1, s-u, n) \right) \\
\forall (t,s,m) | (0 \le t \le T)&
\end{aligned} \tag{5}$$

$$f(T+1, s, m) = 0 \quad \forall (s, m) \tag{6}$$

With the stochastic dynamic programming method as a general tool, we may again consider the detailed production and/or logistics problem (4). Now, we can solve many such problems, (4), as sub problems, within the general stochastic dynamic programming formulation (5), (6). Hence, for each state and stage, we solve the relevant sub problems.

Now, the capacity levels (7) may be defined as functions of the control decisions, time, the remaining reserve and the market state. Furthermore, all other “parameters”, may be considered as functions, as described in (8), (9) and (10). As a result, we may describe the sub problems as (11) or even as (12).

$$C_l = C_l(u,t,s,m) \quad \forall l \tag{7}$$

$$\alpha_{lk} = \alpha_{lk}(u,t,s,m) \quad \forall (l,k) \tag{8}$$

$$p_k = p_k(u,t,s,m) \quad \forall k \tag{9}$$

$$r_{k_1k_2} = r_{k_1k_2}(u,t,s,m) \quad \forall (k_1,k_2) \tag{10}$$

$$\begin{aligned} &\max \pi(x_1,\ldots,x_K;u,t,s,m) \\ &\quad s.t. \\ &\quad \alpha_{11}x_1 + \ldots + \alpha_{1K}x_K \le C_1 \\ &\quad \ldots \\ &\quad \alpha_{L1}x_1 + \ldots + \alpha_{LK}x_K \le C_L \end{aligned} \tag{11}$$

$$\begin{aligned} &\max \pi(x_1,\ldots,x_K;u,t,s,m) \\ &\quad s.t \\ &\quad \alpha_{11}(u,t,s,m)x_1 + \ldots + \alpha_{1K}(u,t,s,m)x_K \le C_1(u,t,s,m) \\ &\quad \ldots \\ &\quad \alpha_{L1}(u,t,s,m)x_1 + \ldots + \alpha_{LK}(u,t,s,m)x_K \le C_L(u,t,s,m) \end{aligned} \tag{12}$$

Now, we include the sub problems in the stochastic dynamic programming recursion Eq. (13). A problem of this kind is defined and numerically solved using LINGO software [1] by Lohmander [10].

$$f(t,s,m) = \max_{u \in U(t,s,m)} \left(\left(\begin{array}{l} \max \pi(x_1,\ldots,x_K;u,t,s,m) \\ s.t. \\ \alpha_{11}x_1 + \ldots + \alpha_{1K}x_K \le C_1 \\ \ldots \\ \alpha_{L1}x_1 + \ldots + \alpha_{LK}x_K \le C_L \end{array} \right) + \sum_n \tau(n|m) f(t+1, s-u, n) \right) \quad \forall (t,s,m)|(0 \le t \le T) \tag{13}$$

Observe that (13) represents a very general and flexible way to formulate and solve applied stochastic multi period production and logistics problems of many kinds. The true sequential nature of decisions and information is explicitly handled, stochastic market prices and very large numbers of decision variables and constraints may be consistently considered. Furthermore, many other stochastic phenomena may be consistently handled with this approach. Several examples of how different kinds of stochastic disturbances may be included in optimal dynamic decisions are found in Lohmander [8, 9].

In the game theory literature, [7, 11, 14], we find many examples of two player constant sum games. In (14), we find such an example, with one objective function. The value of the game, Z, is what we obtain when one player maximizes and one player minimizes the same objective function $Q(\phi, \varphi)$. The maximizing player, A, determines control φ and the minimizing player, B, determines control ϕ. $Q(\phi, \varphi)$ can, for instance, represent the difference in profit or resources between two companies or countries, during a conflict over a particular economic market, a geographical territory or something else. During a period of conflict, it may be relevant to define this as a constant sum game. (In other cases, con-constant sum games are sometimes more relevant, but then it is not always the case that strictly mathematical definitions of the game can be defined and explicitly solved.) Of course, φ and ϕ may represent vectors.

$$\mathrm{Z} = \min_{\phi} \max_{\varphi} Q(\phi, \varphi) = Q(\overline{\phi}, \overline{\varphi}) \tag{14}$$

We may develop and analyze constant sum games in a similar way as the earlier discussed problems, via the stochastic dynamic programming framework. In (15) and (16), one player maximizes and one player minimizes the value of the game. The maximizing player A controls u and x and the minimizing player B controls v and y. The resources of A and B at time t are s_{At} and s_{Bt}. Stochastic exogenous disturbances influence the development of the system via the transition probabilities $\tau(n|m)$. The state in the next period is considered as a general function of decisions of both players and of other variables and parameters. In simple situations, continuous time versions of dynamic game problems can be defined as differential games, as reported by Isaacs [7]. With a higher level of detail, we usually have to use discrete time and state space. Several interesting discrete examples are found in Washburn [14].

$$Z(t, s_{At}, s_{Bt}, m) = \min_{v \in V(t, s_{Bt}, m)} \max_{u \in U(t, s_{At}, m)} \left(\begin{array}{c} \left(\begin{array}{l} \min\limits_{y \in Y(t, s_{Bt}, u, v, m)} \max\limits_{x \in X(t, s_{At}, u, v, m)} Q(x, y; u, v, t, s_{At}, s_{Bt}, m) \\ s.t. \\ F_{1, f_1}(x, y) \le 0 \ \forall f_1 \\ F_{2, f_2}(x, y) \ge 0 \forall f_2 \\ F_{3, f_3}(x, y) = 0 \forall f_3 \end{array} \right) \\ + \sum_n \tau(n|m) Z(t+1, s_{A(t+1)}(s_{At}, t, m, v, u), s_{B(t+1)}(s_{Bt}, t, m, v, u), n) \end{array} \right)$$
$$\forall (t, s_{At}, s_{Bt}, m) | (0 \le t \le T) \tag{15}$$

$$Z(T+1, s_{At}, s_{Bt}, m) = 0 \quad \forall (s_{At}, s_{Bt}, m) \tag{16}$$

Note that the specification of the structure described by (15) and (16) can be adjusted to specific applications. This structure can be regarded as a generalization of many problems in [7, 14].

The control decisions u and v, may represent key decisions, such as total use of constrained resources. As seen in (15), these decisions also influence the options and game values in future periods. The other control decisions, x and y, where x and y may be vectors, can represent the decisions of A and B in very high resolution. Linear or quadratic programming as a tool in the sub problems makes this possible. Furthermore, the stochastic dynamic main program can provide solutions with almost unlimited resolution in the time dimension. The recursive structure of problem solving does not make it necessary to store all results in the internal memory. Of course, computation time increases with resolution.

3 Main Results

Operations research contains a large number of alternative approaches. With logically consistent mathematical modeling, relevant method selection and good empirical data, the best possible decisions can be obtained. This paper has presented arguments for using some particular combinations of methods that often are empirically motivated and computationally feasible (Fig. 1).

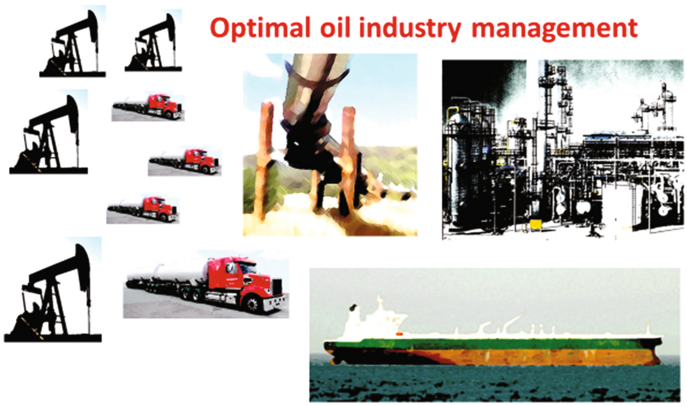

Fig. 1. The optimal oil industry management problem includes finding the optimal combination of oil extraction in different fields, domestic crude oil transport, refining and international logistics. All of this should be done with consideration of stochastic world market prices and possibly other stochastic events. Source: Lohmander [10]. Equations (13) and (6) are useful to solve this problem.

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Covering Topology Countability Based on a Subbasis

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Abstract. Covering topology is induced by covering rough sets, and its topological property is worth researching. In this paper, covering topology countability is studied by a subbasis. At first, basic definitions and properties are achieved for the covering topology countability based on a subbasis, including the first and second countability. Then, the relevant connections between countability and separability are revealed. Finally, three examples are given for illustration. This study establishes subbasis -based countability to deepen covering topology.

Keywords: Rough set · Covering topology · Countability · Subbasis

1 Introduction

Rough sets theory, introduced by Pawlak, is a useful mathematical approach for dealing with vague and uncertain information [1]. Rough set approximation operators are a bidirectional approximate description [2]. Rough sets have a basic relationship with the topological structure, so it is significance for the research of the combination of rough sets and topology. In this view, some authors extend the topology model of Pawlak rough set theory, and prove that the Pawlak rough set model is the nature of the special topology promotion instance [3]. Several authors have studied the relationship between the covering rough sets and topology [4–11]. Reference [12] studies the basic topology. A later paper [13] studies the similarity of binary relations based on rough set theory and topology to give an application for topological structures of matroids. Reference [14] discusses the interior and closure under the subbasis, where the subbasis is a classification under the rough sets and is also a subbasis of topology. On the basis of Refs. [14,15] further researches the topology separability based on a subbasis.

Covering rough sets can naturally induce covering topology, and the latter's property is significant. This article aims to investigate the covering topology based on a subbasis, and we mainly utilize and promote relevant results of Refs. [14,15]. Section 2 reviews some preliminaries about rough sets theory and

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subbases of topology space. Section 3 introduces the first and second countable spaces. Section 4 offers the connection of separability and countability. Section 5 finally provides three examples for relevant illustration.

2 Preliminaries

In this section, we first review some basic concepts in covering rough set theory, which can also be referred to [1,14]. Furthermore, some preliminaries about subbases of covering topological space will be reviewed by Refs. [12,14,15].

2.1 Covering Rough Set Theory

Definition 2.1 [1]**.** Let $K = (U, R)$ be an approximation space. With each subset $X \subseteq U$ and an equivalence relation $R \in IND(K)$, we associate two subsets:

$$\underline{R}(X) = \bigcup\{Y \in U/R | Y \subseteq X\}, \tag{1}$$

$$\overline{R}(X) = \bigcup\{Y \in U/R | Y \bigcap X \neq \emptyset\}, \tag{2}$$

and they are called the $R - lower$ and $R - upper$ approximations of X respectively, where R is an equivalence relation on U and U/R represents a set of all equivalence classes of R.

From [7], let U be a finite set $U \neq \emptyset$, and R is an equivalence relation on U. Set $\beta = U/R = \{[x]_R | x \in U\}$, then β is a subbasis of topology on U (it is actually a basis).

Definition 2.2 [2]**.** Let U be a finite set and $U \neq \emptyset$, then let E be a covering of U. We say that order pairs (U, E) is an approximation space. Let $X \subseteq U$, Set

$$\underline{E}(X) = \{x \in X | \forall B \in E,\ x \in B,\ \text{then } B \subseteq X\}, \tag{3}$$

$$\overline{E}(X) = \bigcup\{B \in E | B \bigcap X \neq \emptyset\}, \tag{4}$$

and they are called the weak lower approximation set and weak upper approximation set of X respectively.

From [7], let U be a finite set $U \neq \emptyset$ and $E \subseteq U$, where E is a covering of U. If E is a subbasis of topology $\mathscr{T}$ on U, then the weak lower approximation set and weak upper approximation set of X in approximation space (U, E) correspond to the interior $i_\beta(A)$ and closure cl_β, respectively, which are about E in topological space $(U, \mathscr{T})$.

2.2 A Subbasis of the Covering Topological Space

Definition 2.3 [14]**.** Let $(U, \mathscr{T})$ be a topological space and β be a subbasis of $\mathscr{T}$. Two set

$$i_\beta(A) = \{x \in A | \forall B \in \beta, x \in B,\ \text{then } B \subseteq A\}, \tag{5}$$

$$cl_\beta(A) = \bigcup\{B \in \beta | B \bigcap A \neq \emptyset\}, \tag{6}$$

are called the interior and closure of A about β respectively.

Let U be a nonempty set. We say that $(U, \mathscr{T}, \beta)$ is a topological space about β, where β is a subbasis of $\mathscr{T}$ and $\mathscr{T}$ is a topology of U.

Definition 2.4 [12]**.** We say that the space is the first countable space if there are countable neighborhoods bases for any point.

Definition 2.5 [12]**.** We say that the space is the second countable space if there are countable topological bases.

Remark. In this paper, we specify that "*about subbasis*" recorded as "$\beta-$".

Definition 2.6 [15]**.** Let $(U_1, \mathscr{T}_1, \beta)$, $(U_2, \mathscr{T}_2, \alpha)$ be the topological space and $f : U_1 \rightarrow U_2$. We say that f is (β, α) continuous mapping if the original image of each $\alpha-$open set of U_2 is a $\beta-$open set of U_1.

3 Covering Topology Countability Based on a Subbasis

In this section, the first and second countable spaces about subbases are introduced. Furthermore, related properties will be discussed.

Definition 3.1. Let $(U, \mathscr{T}, \beta)$ be a topological space. We say that the $(U, \mathscr{T}, \beta)$ is a $\beta-$first countable space if there exists $\{V_i\}_{i \in Z^+}$ of x for every $x \in U$ such that

$$\{V_i\}_{i \in Z^+} \in U, V_{n+1} \subseteq V_n,$$

where $\{V_i\}_{i \in Z^+}$ is the family of $\beta-$open set and U is the $\beta-$open neighborhood of x. The $\beta-$first countable space is recorded as $\beta - C_1$ space.

It is clear that the $\beta - C_1$ space is a C_1 space.

Definition 3.2. Let $x \in U$, we say that the collection of all the $\beta-$neighborhoods of x are the family of $\beta-$neighborhoods of x, which is recorded as $N_\beta(x)$. Then we say that W is a basis of $\beta-$neighborhoods of x, where W is a $\beta-$subset of $N_\beta(x)$.

Theorem 3.1. If U has the countable basis of $\beta-$neighborhoods at x, then x has the countable basis of $\beta-$neighborhoods $\{V_n\}$ such that

$$V_m \subseteq V_n, m > n.$$

Proof. Let $\{W_n\}$ be the countable basis of $\beta-$neighborhoods of x. Suppose that

$$V_n = \bigcap W_i (i = 1, 2, ..., n), \forall n \in N^+,$$

then

$$V_n \subseteq cl_\beta(V_n) \subseteq W_n.$$

Thus V_n becomes the countable basis of $\beta-$neighborhoods of x. It is clear that $V_m \subseteq V_n$ when $m > n$.

Theorem 3.2. Let $(U, \mathscr{T}, \beta)$ be a topological space. If U is a $\beta - C_1$ space and $A \in U$, $x \in cl_\beta(A)$, then there exists the $\beta-$convergent sequence in A that convergence to x.

Proof. Let $\{V_i\}_{i \in Z^+}$ be the countable basis of $\beta-$neighborhoods of x. Use Theorem 3.1 with

$$V_m \subseteq V_n, m > n.$$

Since

$$x \in cl_\beta(A),$$

then

$$\{V_n\} \bigcap A \neq \emptyset.$$

Pick

$$\{x_i\}_{i \in Z^+} \in (\{V_n\} \bigcap A), \ \forall n \in N,$$

then we obtain the sequence $\{x_i\}_{i \in Z^+}$ in A. Pick any W which is the $\beta-$open neighborhood of x, then there exists $n \in N$ such that

$$\{x_i\}_{i \in Z^+} \subseteq V_n \subseteq W.$$

Thus

$$V_m \subseteq cl_\beta(V_m) \subseteq W, \forall m \geq n.$$

Then

$$x_m \in U, \forall m \geq n.$$

According to the convergence definition, we have

$$\{x_i\}_{i \in Z^+} \to x.$$

Thus

$$x_i \to_\beta x (i \to \infty).$$

Theorem 3.3. Let $(U_1, \mathscr{T}_1, \beta)$,$(U_2, \mathscr{T}_2, \alpha)$ be the topological space. If $f : U_1 \to U_2$ is the map in (β, α) such that the sequence $f\{(x_i)\}_{i \in Z^+}$ in U_2 $\alpha-$converges to $f(x_0)$ when $\{x_i\}_{i \in Z^+}\beta-$ converges to x_0. Then we say that f is continuous on x_0.

Proof. Suppose that f is not continuous on x_0. For a $\alpha-$neighborhood V of sequence $f\{(x_i)\}_{i \in Z^+}$.

Then there exists $f^{-1}(V)$ and $f^{-1}(V)$ is not a $\beta-$neighborhood of x_0, i.e. $x_0 \in cl_\beta((f^{-1}(V))^c)$.

It follows from Theorem 3.2 that there exists a $\beta-$sequence $\{x_i\}_{i \in Z^+}$ in $(f^{-1}(V))^c$ and $x_i \to_\beta x_0$.

Since $f\{(x_i)\}_{i \in Z^+}$ in U_2 $\alpha-$converges to $f(x_0)$, then

$$f\{(x_i)\}_{i \in Z^+} \in V, \{x_i\}_{i \in Z^+} \in f^{-1}(V), \ \forall n \in N^+.$$

It is a contradiction.

Definition 3.3. Let $\mathscr{B}^*$ be a family of $\beta-$subset of U. Set a new family of $\beta-$subset

$$\overline{\mathscr{B}}^* = \{W \subseteq U | \forall x \in W, \exists B \in \mathscr{B}^* \text{ such that } x \in B \subseteq W\},$$

then we say that $\overline{\mathscr{B}}^*$ is a family of $\beta-$subset which is generated by $\mathscr{B}^*$. Obviously $\mathscr{B}^* \subseteq \overline{\mathscr{B}}^*$, $\emptyset \in \overline{\mathscr{B}}^*$.

Definition 3.4. Let $(U, \mathscr{T}, \beta)$ be a topological space. We say that a family of $\beta-$subset $\overline{\mathscr{B}}^*$ in U is a basis of $\beta-$topology in U. If $\mathscr{B}^*$ is a $\beta-$topology in U. We say that the family of $\beta-$subset $\mathscr{B}^*$ of (U, $\mathscr{T}, \beta$) is a basis of $\beta-$topology in $(U, \mathscr{T}, \beta)$ if $\overline{\mathscr{B}}^* = \mathscr{T}$.

Definition 3.5. Let $(U, \mathscr{T}, \beta)$ be a topological space. We say $\mathscr{B}^*$ is a $\beta-$second countable space if there exists a basis of $\beta-$topology $\mathscr{B}^* = \{\beta_i\}_{0 \leq i \leq \infty}$ in U, which is recorded as the $\beta - C_2$ space.

It is clear that the $\beta - C_2$ space is a C_2 space and is also a $\beta - C_1$ space.

Theorem 3.4. A $\beta-$detachable metric space is a $\beta - C_2$ space.

Proof. Let (U, d, β) be a $\beta-$detachable metric space. Suppose A is a countable and dense $\beta-$subset of (U, d, β). Set up

$$\mathscr{B}^* = \{B_\beta(a, \frac{1}{n}) | a \in A,\ n \in N^+\},$$

then $\mathscr{B}^*$ is a countable family of $\beta-$open set.

Let's verify that $\mathscr{B}^*$ is a basis of $\beta-$topology of (U, d, β). Thus, it suffices to check that there exist $x \in A$ and $n \in N^+$, such that

$$x \in \{B_\beta(a, \frac{1}{n})\} \subseteq W, \forall W \in \mathscr{B}^*, \forall x \in W.$$

Fix $\varepsilon > 0$ such that

$$B_\beta(x, \varepsilon) \subseteq W.$$

Pick $n > \frac{2}{\varepsilon}$ and $a \in A$ such that

$$d(x, a) < \frac{1}{n}, \text{ then } x \in B_\beta(a, \frac{1}{n}).$$

If $y \in \{B_\beta(a, \frac{1}{n})\}$, then

$$d(a, y) < \frac{1}{n}.$$

From triangle inequality, we can know

$$d(x, y) < \frac{2}{n} < \varepsilon.$$

Thus

$$y \in B_\beta(x, \varepsilon).$$

Finally

$$B_\beta(a, \frac{1}{n}) \subseteq B_\beta(x, \varepsilon) \subseteq W.$$

4 Connections Between Separability and Countability

In this section, we study the relationship between the $\beta - C_1$ space and $\beta - C_2$ space. At the same time, we also study the connection between separability and countability.

Theorem 4.1. Let $(U, \mathscr{T}, \beta)$ be a $\beta - C_1$ space. The $\beta-$sequence of $(U, \mathscr{T}, \beta)$ has no more than one $\beta-$limit point if and only if $(U, \mathscr{T}, \beta)$ is a $\beta - T_2$ space.

Proof. Suppose x_n is a $\beta-$sequence of $(U, \mathscr{T}, \beta)$. We may suppose that

$$\{x_n\} \to_\beta x_0, \{x_n\} \to_\beta y_0.$$

First consider the necessity. Since $(U, \mathscr{T}, \beta)$ is a $\beta - C_1$ space and is also a $\beta - T_2$ space. There exist $\beta-$open neighborhoods W and V if

$$x_0 \neq y_0, \text{ where } W \in N_\beta(x), V \in N_\beta(y).$$

Thus we have

$$x_0 \in W, y_0 \in V, W \bigcap V = \emptyset.$$

Since

$$\{x_n\} \to_\beta x_0,$$

there exists $N_1 > 0$ such that

$$\{x_n\} \in W, \forall n > N_1.$$

In a similar way, since

$$\{x_n\} \to_\beta y_0,$$

there exists $N_2 > 0$ such that

$$\{x_n\} \in V, \forall n > N_2.$$

Finally, pick $N = N_1 + N_2$, and we can see

$$\{x_n\} \in W \bigcap V, \forall n > N.$$

It is a contradiction.

Next focus on the sufficiency. Suppose $(U, \mathscr{T}, \beta)$ is not a $\beta - T_2$ space. Then there exist

$$x, y \in U, \ x \neq y, W \in N_\beta(x), V \in N_\beta(y), W \bigcap V \neq \emptyset.$$

Since $(U, \mathscr{T}, \beta)$ is a $\beta-C_1$ space, then $\{W_n\}$ is a countable wide-down basis of $\beta-$neighborhood of x. $\{V_n\}$ is a countable wide-down basis of $\beta-$neighborhood of y. So

$$\{x_n\} \in \{W_n\} \bigcap \{V_n\} \neq \emptyset,$$

then

$$\{x_n\} \to_\beta x, \{x_n\} \to_\beta y.$$

We obtain

$$x = y.$$

It contradicts with the suppose $x \neq y$. Finally, the $(U, \mathscr{T}, \beta)$ is a $\beta-T_2$ space.

Theorem 4.2. If $(U, \mathscr{T}, \beta)$ is a $\beta - C_2$ space and is also a $\beta - T_3$ space. Then $(U, \mathscr{T}, \beta)$ is a $\beta - T_4$ space.

Proof. Pick a countable basis of $\beta-$ topology $\mathscr{B}^*$ of U. Suppose that A and A' are the $\beta-$closed set, and they are non-intersect. For any $x \in A$, then $x \notin A'$. We may therefore apply $\beta - T_3$ space, and so we have $\beta-$neighborhood G and G', where $G \in N_\beta(x)$,$G' \in N_\beta(A')$.

Then

$$cl_\beta(W) \bigcap A' = \emptyset.$$

Pick $B \in \mathscr{B}^*$ such that

$$x \in B \subseteq G.$$

Then

$$cl_\beta(B) \bigcap A' = \emptyset.$$

Suppose $\{B_1, B_2, ...\}$ is the member of all the closure in $\mathscr{B}^*$ but not in A'. It has been proved that

$$A \subseteq \bigcup B_n,\ n = 1, 2, 3, ...$$

We say that $\{B'_1, B'_2, ...\}$ is the member of all the closure in $\mathscr{B}^*$ but not in A. Then

$$A' \subseteq \bigcup B'_n,\ n = 1, 2, 3, ...$$

$$\text{Suppose } W_n = B_n \backslash \bigcup cl'_\beta(B_n), V_n = B'_n \backslash \bigcup cl_\beta(B_n),\ n = 1, 2, 3, ...$$

Then W_n and V_n are all the $\beta-$open sets and

$$W_n \bigcap V_m = \emptyset, \forall m, n.$$

Set up

$$W = \bigcup W_n, V = \bigcup V_n,\ n = 1, 2, 3, ...,$$

then

$$W \bigcap V = \bigcup (W_n \bigcap V_m) = \emptyset,\ n, m = 1, 2, 3, ...$$

Suppose $x \in A$, then there exists n such that

$$x \in B_n.$$

Thus

$$x \in W_n \subseteq W.$$

So W is a $\beta-$open neighborhood of A. In the same way, the V is a $\beta-$open neighborhood of A'. So W and V are the $\beta-$neighborhood and respectively belong to A and A'. So $(U, \mathscr{T}, \beta)$ is a $\beta - T_4$ space.

5 Examples

In this section, we use three examples to illustrate the previous conclusion.

Example 5.1. $(\mathbf{R}, \mathscr{T}, \beta)$ is not a $\beta - C_1$ space. Suppose $x \in \mathbf{R}$, where $\mathbf{R}$ is the real number set. Any countable system of $\beta-$neighborhood $N_\beta(x)$ of x is not a basis of $\beta-$neighborhood of x.

Note. Any $W \subseteq N_\beta(x)$ is a complementary set of $\beta-$finit set. Then $\bigcup W^c$ where $W \in N_\beta(x)$ is a $\beta-$countable set.

Fix

$$y \notin \bigcup W^c, W \in N_\beta(x), y \neq x.$$

Then

$$\forall W \subseteq N_\beta(x), y \in W.$$

Thus $R\backslash\{y\}$ is a $\beta-$open neighborhood of x and it does not contain any $W \subseteq N_\beta(x)$.

Example 5.2. Let $\mathscr{B}^* = \{[a,b)|a < b\}$ and $(\mathbf{R}, \overline{\mathscr{B}}^*, \beta)$ be a topological space. Then $(\mathbf{R}, \overline{\mathscr{B}}^*, \beta)$ is not a $\beta - C_2$ space.

Note. Suppose μ is a basis of $\beta-$topology of $(\mathbf{R}, \overline{\mathscr{B}}^*, \beta)$, and $a, b \in R$. Then $[a, a+1)$ is a $\beta-$open set. Thus there exist the $\beta-$open neighborhoods W_a, W_b in μ.

Then

$$a \in cl_\beta(W_a) \subseteq [a, a+1),$$

and a is the smallest member in W_a.

Obviously,

$$W_a \neq W_b, a \neq b.$$

Thus the member of μ is uncountable. Then there exists a countless basis of $\beta-$neighborhood in $(\mathbf{R}, \overline{\mathscr{B}}^*, \beta)$.

So $(\mathbf{R}, \overline{\mathscr{B}}^*, \beta)$ is not a $\beta - C_2$ space.

Example 5.3. Let S be a collection of all the irrational numbers. Suppose the family of $\beta-$subset $\mathscr{T} = \{W \setminus A | W\ is\ a\ \beta - open\ set\ of\ \mathbf{E}^1,\ A \subseteq S\}$.

Then:

(1) $(\mathbf{R}, \mathscr{T}, \beta)$ is a separable space such that $(\mathbf{R}, \mathscr{T}, \beta)$ is a $\beta - C_1$ space.
(2) $(\mathbf{R}, \mathscr{T}, \beta)$ is not a $\beta - C_2$ space.

Note. (1) Since $(\mathbf{R}, \mathscr{T}, \beta)$ is a $\beta - C_1$ space.

Then pick

$$W_n = \{x\} \bigcup ((x - \frac{1}{n}, x + \frac{1}{n}) \bigcap \mathbf{Q}), \forall x \in \mathbf{R}.$$

Thus $\{W_n\}$ is a countable basis of $\beta-$neighborhood of x.

So $(\mathbf{R}, \mathscr{T}, \beta)$ is separable.

(2) Suppose $(\mathbf{R}, \mathscr{T}, \beta)$ is a $\beta - C_2$ space. Let $\mathscr{T}$ be a $\beta-$subspace on x and $\mathscr{T}_S$ be a discrete topology.

Suppose $A \subseteq S$. Since $\mathbf{R} \setminus (S \setminus A)$ is a $\beta-$open set of $(\mathbf{R}, \mathscr{T}, \beta)$, then $(\mathbf{R} \setminus (S \setminus A)) \bigcap S = A$ is a $\beta-$open set of $(S, \mathscr{T}_S, \beta)$.

It is suggested that every $\beta-$subset of S is the $\beta-$open set of $(S, \mathscr{T}_S, \beta)$.

Thus $(S, \mathscr{T}_S, \beta)$ is not separable.

If $(\mathbf{R}, \mathscr{T}, \beta)$ is a $\beta - C_2$ space, then $(S, \mathscr{T}_S, \beta)$ is also a $\beta - C_2$ space.

Thus $(S, \mathscr{T}_S, \beta)$ should be separable.

It is a contradiction. So $(\mathbf{R}, \mathscr{T}, \beta)$ is not a $\beta - C_2$ space.

Obviously, $\beta-$metric space is a $\beta - C_1$ space; $\{B(x, q)|q \in Q^+\}$ is a countable basis of $\beta-$neighborhood of x; If

$$\mathscr{T} = \{(-\infty, a)| - \infty \leq a \leq +\infty\},$$

then $(\mathbf{R}, \mathscr{T}, \beta)$ is a $\beta - C_2$ space.

6 Conclusion

This paper focuses on covering topology and its subbasis to define the $\beta-$first countable space and the $\beta-$second countable space, i.e. $\beta - C_1$ space and $\beta - C_2$ space. At the same time, it also defines the basis of $\beta-$neighborhood, the system of $\beta-$neighborhood, the family of $\beta-$subset and the basis of $\beta-$topology. Then, we study the relationship between $\beta - C_1$ space and $\beta - C_2$ space. Finally, we study the connections of the countable space and separable space. In summary, this study offers the covering topology countability based on a subbasis, as well as its relationships with separability. Other topological properties are worth deeply researching for covering topology.

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A New Type of Soft Subincline of Incline

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Abstract. Firstly, this paper presents the new concept of soft subincline. Then some equivalent conditions of it and two operations "RESTRICTED INTERSECT" and "AND" on it are discussed. After that the relationship between soft subincline and the dual of soft set based on the method of the dual of soft set are studied. In addition, the concepts and properties of maps between soft subincline are given. Finally, the chain condition of H which consists of all of the soft subinclines is introduced and obtain a necessary and sufficient condition for H is Artinian or Noetherian.

Keywords: Incline-algebra · Soft incline · Soft sets · Dual soft sets · Chain condition

1 Introduction

As the fuzzy set theory was proposed, mathematical tools dealing with incomplete and uncertain problems were also presented. In particular, Pawlak and Atanassov proposed the rough set theory [1] and intuitionistic fuzzy set theory [2]. However, these mathematical theories are lack of parameter tools. Therefore, in order to solve this problem, Molodtsov gave the concept of soft sets innovatively in 1999 [3]. After that, Aktas and Cagman propose the definition of soft group in 2007 [4] which created a new field of soft algebra. A few years afterwards, many scholars had done a series of researches in soft algebra [5–13].

The notion of incline algebra was proposed by Cao in 1981 [14]. He also published a monograph about incline algebra with other two scholars [15]. In 2001, Jun applied fuzzy sets to incline algebra and proposed the concept of fuzzy subincline [16].

Liao applied soft sets to incline algebra and proposed the concept of soft incline in 2012 [17]. The concept of fuzzy soft incline and $(\in, \in \vee q)-$ fuzzy soft incline were proposed by Alshehri in 2012 [18]. The study of inclines and incline matrices is significant both in theory and in practice, they have good foreground of applications in many areas including automation theory, decision theory, cybernetics, graph theory and nervous system [15]. At present, the theories of incline algebras and incline matrices are highly utilized by computer science applications [19–21].

In 2008, Yuan and Wen introduced algebraic structures in parameter set and obtained a new algebraic structure of soft sets [22]. They introduced a soft algebra

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structure which can be reduced to L-fuzzy algebra by using the concept of dual soft sets, where L = P(X) (P (X) is the power set of the common universe X) is a Boolean algebra. In general the element in lattice L has no structure. However, the elements over L = P(X) is a set which can also have many elements and algebraic structure. Therefore, more meaningful results can be obtained than general L-fuzzy algebra.

In this paper, by using the idea above, we give the new concept of soft subincline. The difference between our new concept of soft subincline and the concept of soft incline proposed in literature [17] is that: the parameter set of a soft subincline is a fixed incline in this paper, while it is a subincline of a certain incline in literature [17]. Furthermore, we investigate some algebraic properties of our new type of soft subincline and introduced some properties of the new type of soft subincline of incline under the chain condition. These results enrich the theory of soft algebra.

2 Preliminary Notes

Definition 2.1 [14]. An inline (algebra) is a set K with two binary operations denoted by "·" and "·" Satisfying the following axioms: for all $x, y, z \in K$,

(1) $x + y = y + x$;
(2) $(x + y) + z = x + (y + z)$;
(3) $(xy)z = x(yz)$;
(4) $x(y + z) = xy + xz$;
(5) $(y + z)x = yx + zx$;
(6) $x + x = x$;
(7) $x + xy = x$;
(8) $y + xy = y$.

For convenience, we pronounce " + " (resp."·") as addition (resp. multiplication).

Every distributive lattice is an incline. An incline is a distributive lattice if and only if $xx = x$ for all $x \in K$.

Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$.

A subincline of an incline K is a non-empty subset M of K which is closed under addition and multiplication. *A* subincline M is said to be an ideal of an incline K if $x \in M$ and $y \leq x$ then $y \in M$. By a homomorphism of inclines we shall mean a mapping f from an incline K into an incline L such that $f(x + y) = f(x)f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in K$.

Definition 2.2 (Cartesian product). Let A and B be two non-empty set, then $A \times B = \{(x, y) | x \in A, y \in B\}$ is called a Cartesian product over A and B.

Theorem 2.1 [14]. Let K_1 and K_2 be incline algebras, then their Cartesian product is an incline algebra if for all $(x_1, x_2), (y_1, y_2) \in K_1 \times K_2$:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$
$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Definition 2.3 [14]. A pair (F, A) is called a soft set (over X) if and only if F is a mapping of E into the set of all subsets of the set X.

Definition 2.4 [23] (Restricted intersection operation of two soft sets). Let (F, A) and (G, B) be two soft sets over a common universe X. If the soft set (H, C) satisfy $C = A \cap B$ and for any $e \in C, H(e) = F(e) \cap G(e)$. We call (H, C) is the restricted intersection of (F, A) and (G, B), and denote (H, C)=$(F, A) \cap (G, B)$.

Definition 2.5 [24] (AND operation on two soft sets). Let (F, A) and (G, B) be two soft sets, then "(F, A) and (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined to be $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta), \forall (\alpha, \beta) \in A \times B$.

Definition 2.6 [22] (The duality of soft sets)

$A_H : X \to P(E), x \mapsto A_H = \{g | x \in H(g)\}$ is called the duality soft set of H if $H : E \to P(X), g \mapsto H(g)$ is a soft set over K.

$H_A : E \to P(X), g \mapsto H_{(g)} = \{x | g \in A(x)\}$ is called the duality soft set of A if $A : X \to P(E)$ is a soft set over X.

Definition 2.7 [22] (The Extension Principle) let X be the common universe. Let f be defined by $f : K_1 \to K_2$ and let $H_1 : K_1 \to P(X)$ and $H_2 : K_2 \to P(X)$ are soft sets over K_1 and K_2 respectively. Define soft sets $f(H_1)$ over K_1 and $f^{-1}(H_2)$ over K_2 by $\forall g_2 \in K_2$, $f(H_1)(g_2) = \begin{cases} \cup_{f(g_1)=g_2} H_1(g_1) & f^{-1}(g_2) \neq \varnothing \\ \varnothing & f^{-1}(g_2) = \varnothing \end{cases}$ and $\forall g_1 \in K_1$,$f^{-1}(H_2)(g_1) = H_2$. Then $f(H_1)$ is said to be the image of H_1 and $f^{-1}(H_2)$ is said to be the preimage of K_2.

Definition 2.8 [17]. Let K be an incline algebra. A pair (F, A) is called a soft incline over K if $F(x)$ is a subincline of K for all $x \in A$.

3 A New Type of Soft Subincline of Incline

Definition 3.1. Let K be an incline algebra and X be the common universe, $H : K \to P(X)$ is a soft set. H is called a new type of soft subincline of incline if it satisfies the following conditions: for all $g_1, g_2 \in K$,

(1) $H(g_1 + g_2) \supseteq H(g_1) \cap H(g_2)$;

(2) $H(g_1 g_2) \supseteq H(g_1) \cap H(g_2)$.

Example 3.1. Let $K = \{0, a, b, 1\}$ be an incline with the operation tables given in Table. Let $X = \{0, a, b, 1\}$ and $H : K \to P(X)$ be a soft set defined $H(0) = \{0, a\}$ $H(a) = \{0, 1\}, H(b) = \{a, b\}, H(1) = \{b\}$. Clearly, H is a new type of soft subincline of incline over K and it could be verified by Definition 2.1. Because $H(b) = \{a, b\}$ is not a subincline of K, so H is not a soft incline over K, then the new type of soft subincline is a new algebraic structure.

The operation tables of the incline

+	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

·	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Theorem 3.1. Let K_1 and K_2 be two subinclines of K and let H_1 and H_2 be new type of soft subinclines of incline over K_1 and K_2 respectively, if $K_1 \cap K_2 \neq \varnothing$ and $(H_1, K_1 \cap K_2) = (H, K_1) \cap (H, K_2)$, then H is a new type of soft incline of incline over $K_1 \cap K_2$.

Proof. Because of K_1 and K_2 be two subinclines of K and $K_1 \cap K_2 \neq \varnothing$, it is easy to say that $K_1 \cap K_2$ is also a subincline of K. For any $g_1, g_2 \in K_1 \cap K_2$, we have

$$\begin{aligned} H(g_1g_2) &= H_1(g_1g_2) \cap H_2(g_1g_2) \supseteq [H_1(g_1) \cap H_1(g_2)] \cap [H_2(g_1) \cap H_2(g_2)] \\ &= [H_1(g_1) \cap H_2(g_1)] \cap [H_1(g_2) \cap H_2(g_2)] = H(g_1) \cap H(g_2). \end{aligned} Z$$

Similarly, $H(g_1 + g_2) \supseteq H(g_1) \cap H(g_2), \forall g_1, g_2 \in K_1 \cap K_2$.

Therefore, H is a new type of soft subincline of incline of $K_1 \cap K_2$.

Theorem 3.2. Let K_1 and K_2 be two inclines and let H_1 and H_2 be new type of soft subinclines of incline over K_1 and K_2 respectively, if $K = K_1 \times K_2$ and $(H, K) = (H_1, K_1) \wedge (H_2, K_2)$, then H is a new type of soft subincline of incline of K.

Proof. Since K_1 and K_2 are two inclines, by Theorem 2.1, it is sufficient to show that $K_1 \times K_2$ is also an incline. Then clearly $H[(x_1, y_1)(x_2, y_2)] = H(x_1x_2, y_1y_2) = H_1(x_1x_2) \cap H_2(y_1y_2) \supseteq [H_1(x_1) \cap H_1(x_2)] \cap [H_2(y_1) \cap H_2(y_2)] = [H_1(x_1) \cap H_2(y_1)] \cap [H_1(x_2) \cap H_2(y_2)] = H(x_1, y_1) \cap H(x_2, y_2)$ for all $(x_1, y_1), (x_2, y_2) \in K$.

Similarly, $H[(x_1, y_1) + (x_2, y_2)] \supseteq H(x_1, y_1) \cap H(x_2, y_2)$ for all (x_1, y_1), $(x_2, y_2) \in K$.

Therefore, H is a new type of soft subincline of incline of K.

Theorem 3.3. Let K be an incline, then the following are equivalent:

(i) H is a new type of soft subincline of incline of K.
(ii) For all $x \in X, A_H(x) \neq \emptyset$ is a subincline over K.

Proof. $(i) \Rightarrow (ii)$ For all $g_1, g_2(x) \in A_H(x)$, we have $x \in H(g_1)$ and $x \in H(g_2)$, therefore $x \in H(g_1) \cap H(g_2)$. Since H is a new type of soft subincline of incline of K, it follows that $H(g_1) \cap H(g_2) \subseteq H(g_1g_2)$, then $x \in H(g_1g_2)$. Hence $g_1g_2 \in A_H(x)$.

Similarly, $g_1 + g_2 \in A_H(x)$ for all $g_1, g_2 \in A_H(x)$.

Therefore, $A_H(x)$ is a subincline over K for all $x \in X$.

$(ii) \Rightarrow (i)$: For all $g_1 + g_2 \in K$, if $H(g_1) \cap H(g_2) = \varnothing$, then $H(g_1) \cap H(g_2) = \varnothing \subseteq H(g_2)$; if $H(g_1) \cap H(g_2) \neq \varnothing$, assume that $x \in H(g_1) \cap H(g_2)$, then $x \in H(g_1)$ and $x \in H(g_2)$, hence $g_1, g_2 \in A_H(x)$ Note that $A_H(x)$ is a subincline over K, then $g_1g_2 \in A_H(x)$ and so $x \in H(g_1g_2)$. Thus $H(g_1) \cap H(g_2) \subseteq H(g_1g_2)$.

Similarly, $H(g_1)\cap H(g_2)\subseteq H(g_1+g_2)$ for all $x\in H(g_1)\cap H(g_2)$.

Therefore, H is a new type of soft subincline of incline of K.

Theorem 3.4. Let K be an incline, then the following are equivalent:

(i) Let A be defined by $A: X\to P(K)$, for all $x\in X, A_H(x)\neq\varnothing$ is a subincline over K.

(ii) H_A is a new type of soft subincline of incline over K.

Proof. $(i)\Rightarrow(ii)$: Assume that $x\in H_A(g_1)\cap H_A(g_2)$, then $g_1\in A(x)$ and $g_2\in A(x)$. Note that $A(x)$ is a subincline of K, then $g_1g_2\in A(x)$ and $g_1+g_2\in A(x)$. Clearly $x\in H_A(g_1g_2)$ and $x\in H_A(g_1+g_2)$. Thus $H_A(g_1)\cap H_A(g_2)\subseteq H_A(g_1g_2)$ and $H_A(g_1)\cap H_A(g_2)\subseteq H_A(g_1g_2)$.

Therefore, H_A is a new type of soft subincline of incline over K.

$(ii)\Rightarrow(i)$: For any $g_1,g_2\in A(x)$, we have $x\in H_A(g_1)$ and $x\in H_A(g_2)$, and so $x\in H_A(g_1)\cap H_A(g_2)$. Since H_A is a new type of soft subincline of incline over K, it follows that $H(g_1)\cap H(g_2)\subseteq H(g_1g_2)$ and $H(g_1)\cap H(g_2)\subseteq H(g_1+g_2)$, then $g_1g_2\in A(x)$ and $g_1+g_2\in A(x)$. Therefore,$A(x)$ is a subincline over K.

Theorem 3.5. Let K_1 and K_2 be two inclines and let X be the common universe. Let $f: K_1\to K_2$ be a hemimorphic mapping. Let $H_1: K_1\to P(X)$ and $H_2: K_2\to P(X)$ be soft sets over K_1 and K_2 respectively. Then $f(H_1)$ is a new type of soft incline of incline over K_2 if H_1 is a new type of soft subincline of incline over K_1.

Proof. For all $g_2,g_2'\in K_2$.

Case1: Assume $f^{-1}(g_2)\neq\varnothing$ and $f^{-1}(g_2')\neq\varnothing$. if $f(H_1)(g_2)\cap f(H_1)(g_2')=\varnothing$, then $f(H_1)(g_2)\cap f(H_1)(g_2')\subseteq f(H_1)(g_2+g_2')$ if $f(H_1)(g_2)\cap f(H_1)(g_2')\neq\varnothing$ then $\forall x\in f(H_1)(g_2)\cap f(H_1)(g_2')$ we have $x\in\ \cup_{f(g_1)=g_2}H_1(g_1)$ and $x\in\ \cup_{f(g_1')=g_2'}H_1(g_1')$. Thus there exists $g_1\in K_1$ such that $x\in H_1(g_1)$ and $f(g_1)=g_2$. There also exists $g_1'\in K_1$ such that $x\in H_1(g_1')$ and $f(g_1')=g_2'$. Hence $x\in H_1(g_1)\cap H_1(g_1')$. Since H_1 is a new type of soft subincline of incline over K_1, we can get $H_1(g_1)\cap H_1(g_1')\subseteq H_1(g_1+g_1')$, then $x\in H_1(g_1+g_1')$. Note that f is a homomorphic mapping, so $f(g_1+g_1')=f(g_1)+f(g_1')=g_2+g_2'$ where $g_1+g_1'\in K_1$. Therefore, $x\in H_1(g_1+g_1')\subseteq\ \cup_{f(g)=g_2+g_2'}H_1(g)=f(H_1)(g_2+g_2')$.

Case2: If $f^{-1}(g_2)=\varnothing$ or $f^{-1}(g_2')=\varnothing$, then $f(H_1)(g_2)=\varnothing$ or $f(H_1)(g_2')=\varnothing$, and so $f(H_1)(g_2)\cap f(H_1)(g_2')=\emptyset\subseteq f(H_1)(g_2+g_2')$.

Similarly, $f(H_1)(g_2)\cap f(H_1)(g_2')\subseteq f(H_1)(g_2g_2'),\forall g_2,g_2'\in K_2$.

Therefore, $f(H_1)$ is a new type of soft incline of incline over K_2 .

Theorem 3.6. Let K_1 and K_2 be two inclines and let X be the common universe. Let be $f: K_1\to K_2$ be a homomorphic mapping. $H_1: K_1\to P(X)$ and $H_2: K_2\to P(X)$ are soft sets over K_1 and K_2 respectively. Then $f^{-1}(H_2)$ is a new type of soft subincline of incline of K_1, if H_2 is a new type of soft subincline of incline over K_2.

Proof. For any $g_1, g'_1 \in K_1$, we have

$$f^{-1}(H_2)(g_1) \cap f^{-1}(H_2)(g'_1) = H_2(f(g_1)) \cap H_2(f(g'_1)) \subseteq H_2(f(g_1) + f(g'_1)) = H_2(f(g_1 + g'_1)) = f^{-1}(H_2)(g_1 + g'_1).$$

Similarly, $f^{-1}(H_2)(g_1) \cap f^{-1}(H_2)(g'_1) \subseteq f^{-1}(H_2)(g_1 g'_1), \forall g_1, g'_1 \in K_1$.
Therefore, $f^{-1}(H_2)$ is a new type of soft subincline of incline of K_1.

Definition 3.2. Let K_1 and K_2 be two inclines and let H_1 be a new type of soft subincline of incline over K_1. Let $f : K_1 \to K_2$ be a map. For all $x, y \in K_1$, if $f(x) = f(y)$, we have $H_1(x) = H_1(y)$, then H_1 is said to be f-invariant.

Theorem 3.7. Let K_1 and K_2 be two inclines and let X be the common universe. If f is a homomorphic mapping from K_1 to K_2, $H_1 : K_1 \to P(X)$ is a soft set over K_1 and H_1 is f-invariant. Then the following are equivalent:

(i) H_1 is a new type of soft subincline of incline over K_1.
(ii) $f(H_1)$ is a new type of soft subincline of incline over K_2.

Proof. $(i) \Rightarrow (ii)$: Following Theorem 3.5, it is sufficient to show that the conclusion is correct.

$(ii) \Rightarrow (i)$: For any $g_1, g'_1 \in K_1$ and $x \in H_1(g_1) \cap H_1(g'_1)$, we have $x \in H_1(g_1)$ and $x \in H_1(g'_1)$. Assume that $f(g_1) = g_2$ and $f(g'_1) = g'_2 \in K_2$, then $x \in \cup_{f(g)=g_2} H_1(g) = f(H_1)(g_2)$ and $x \in \cup_{f(g')=g'_2} H_1(g) = f(H_1)(g'_2)$, and so $x \in f(H_1)(g_2) \cap f(H_1)(g'_2)$. since $f(H_1)$ is a new type of soft subincline of incline over K_2, hence $f(H_1)(g_2) \cap f(H_1)(g'_2) \subseteq f(H_1)(g_2 g'_2)$, then $x \in f(H_1)(g_2 g'_2) = \cup_{f(g)=g_2 g'_2} H_1(g)$

So clearly there exists $g \in K_1$ such that $f(g) = g_2 g'_2$ and $x \in H_1(g)$. Since f is a homomorphic mapping, then $f(g) = f(g_1) f(g'_1) = f(g_1 g'_1)$. Also note that H_1 is a f-invariant, then $H_1(g) = H_1(g_1 g'_1)$, so $x \in H_1(g_1 g'_1)$.

Hence $H_1(g_1) \cap H_1(g'_1) \subseteq H_1(g_1 g'_1)$.

Similarly, $H_1(g_1) \cap H_1(g'_1) \subseteq H_1(g_1 + g'_1), \forall g_1, g'_1 \in K$.

Therefore, H_1 is a new type of soft subincline of incline of K_1.

Theorem 3.8. Let K_1 and K_2 be two inclines and let X be the common universe. If f is a homomorphic mapping from K_1 to K_2 and $H_2 : K_2 \to P(X)$ is a soft set over K_2. Then the following are equivalent:

(i) H_2 is a new type of soft subincline of incline over K_2.
(ii) $f^{-1}(H_2)$ is a new type of soft subincline of incline over K_1.

Proof. $(i) \Rightarrow (ii)$: Following Theorem 3.6, it is sufficient to show that the conclusion is correct.

$(ii) \Rightarrow (i)$: For any $g_2, g'_2 \in K_2$, note that f is a homomorphic mapping, so there exists $g_1, g'_1 \in K_1$ such that $g_2 = f(g_1), g'_2 = f(g'_1)$ and $g_2 g'_2 = f(g_1) f(g'_1) = f(g_1 g'_1)$, then $H_2(g_2) \cap H_2(g'_2) = H_2(f(g_1)) \cap H_2(f(g'_1)) = f^{-1}(H_2)(g_1) \cap f^{-1}(H_2)(g'_1)$ since $f^{-1}(H_2)$ is a new type of soft subincline of incline over K_1, we get $f^{-1}(H_2)(g_1)$

$\cap f^{-1}(H_2)(g_1')\subseteq f^{-1}(H_2)(g_1g_1') = H_2(f(g_1g_1')) = H_2(g_1g_1')$. Hence $H_2(g_2)\cap H_2(g_2')$ $\subseteq H_2(g_2g_2')$.

Similarly, $H_1(g_2)\cap H_1(g_2')\subseteq H_1(g_2+g_2'), \forall g_2, g_2' \in K_2$.

Therefore, H_2 is a new type of soft subincline of incline over K_2.

4 The Chain Condition of Incline

Definition 4.1. Let K be an incline algebra and H is the set of all new type of soft subincline of incline over K. Suppose H_1 and H_2 are elements of H. Define a binary relation "$\leq$" over H as follows: $H_1 \leq H_2 \Leftrightarrow H_1(g)\subseteq H_2(g), \forall g \in K$.

Definition 4.2. Let K be an incline algebra, H is the set of all new type of soft subincline of incline over K, H_1 and H_2 are elements of H. Define a binary relation "=" over H as follows: $H_1 = H_2 \Leftrightarrow H_1(g) = H_2(g), \forall g \in K$.

Theorem 4.1. $(H, \leq)$ is a partially ordered set.

Proof. For any $H_1 \in H$, we have $H_1(g) \in H_1(g)$ for all $g \in K$, hence $H_1 \leq H_1$.
For any $H_1, H_2, H_3 \in H$, assume that $H_1 \leq H_2$ and $H_2 \leq H_3$. Clearly for any $g \in K$ we have $H_1(g)\subseteq H_2(g)$ and $H_2(g)\subseteq H_3(g)$, then $H_1(g)\subseteq H_3(g)$. Hence $H_1 \leq H_3$.

For any $H_1, H_2 \in H$, assume that $H_1 \leq H_2$ and $H_2 \leq H_1$. Clearly for any $g \in K$, we have $H_1(g)\subseteq H_2(g)$ and $H_2(g)\subseteq H_1(g)$, then $H_2(g) = H_1(g)$.

Hence $H_2 = H_1$.

Therefore, $(H, \leq)$ is a partially ordered set.

Theorem 4.2. K is an incline, H_1 and H_2 are new type of soft subinclines of incline over K. Then $H_1 \leq H_2$ if and only if $A_{H_1}(x)\subseteq A_{H_2}(x)$ for all $x \in X$.

Proof. Necessity: For any $g \in A_{H_1(x)}$, we have $H_1(g)\subseteq H_1(g)$. Since $H_1 \leq H_2$, then $H_1(g)\subseteq H_2(g)$. This implies that $x \in H_2(g)$, so that $g \in A_{H_2}(x)$. Therefore $A_{H_1}(x)\subseteq A_{H_2}(x)$.

Sufficiency: For any $g \in K$ and $x \in H_1(g)$, we have $g \in A_{H_1}(x)$. Since $A_{H_1}(x)\subseteq A_{H_2}(x)$ then $g \in A_{H_2}(x)$. This implies that $x \in H_2(g)$, so that $H_2(g)\subseteq H_2(g)$. Therefore $H_1 \leq H_2$.

Corollary 4.1. K is an incline, H_1 and H_2 are new type of soft subinclines of incline over K. Then $H_1 = H_2$ if and only if $A_{H_1}(x) = A_{H_2}(x)$ for all $x \in X$.

Definition 4.3. K is an incline, $\Omega(K)$ is a subincline family of K. For any ascending chain of $\Omega(K), K_1\subseteq K_2\subseteq \cdots \subseteq K_n\subseteq \cdots$, if there exists a positive integer n such that $K_n = K_m$ for all m > n, $\Omega(K)$ is called Noetherian. K is called a Noetherian incline if $\Omega(K)$ is the set of all subinclines over K. The number min $\{i|K_i = K_{i+1}, i = 1, 2, \cdots\}$ is called the stabilize index of Noetherian and denoted by $m_{\{K_i\}}$.

Definition 4.4. K is an incline, $\Omega(K)$ is a subincline family of K. For any descending chain of $\Omega(K), K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$, if there exists a positive integer n such that $K_n = K_m$ *for all* $m > n$, $\Omega(K)$ is called Artinian. K is called a Artinian incline if $\Omega(K)$

is the set of all subinclines over K. The number min $\{i|K_i = K_{i+1}, i = 1, 2, \cdots\}$ is called the stabilize index of Artinian and denoted by $n_{\{K_i\}}$.

Definition 4.5. K is an incline, $\sum$ is the set of all subinclines over K. K is said to satisfy the maximal condition if every nonempty set over $\sum$ has a maximal element.

Definition 4.6. K is an incline, $\sum$ is the set of all subinclines over K. K is said to satisfy the minimal condition if every nonempty set over $\sum$ has a minimal element.

Theorem 4.3. K is an incline, K is a Noetherian incline if and only if K satisfies the maximal condition.

Proof. Necessity: Let $\sum$ be the set of all subinclines over K. Assume that nonempty subset $\sum'$ over $\sum$ without maximal element, then for any $K_i \in \sum'$, there exists K_{i+1} such that $K_i \subseteq K_{i+1}$. Hence there exists an infinite ascending chain $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq K_{i+1} \subseteq \cdots$, a contradiction. Therefore, K satisfies the maximal condition.

Theorem 4.4. K is an incline, K is an Artinian incline if and only if K satisfy the minimal condition.

Proof. The proof of this theorem is similar to the proof of Theorem 4.3 and so is omitted.

Definition 4.7. K is an incline, let A be a subincline over K. A is said to be reducible if there exist B and C, which are subincline over K, properly including A, such that $A = B \cap C$. If $A = B \cap C$, there must have $A = B$ or $A = C$, then A is said to be irreducible.

Theorem 4.5. K is a Noetherian incline, then every subincline over K can be expressed as intersectiob of finite number of subinclines which are irreducible.

Proof. Let $\sum_1$ be the set of subinclines over K which are cannot be expressed as intersection of a finite number of irreducible subinclines. Assume that $\sum_1 \neq \varnothing$. Since K is a Noetherian incline, clearly we know K satisfies the maximal condition. Hence there exists a maximal element in $\sum_1$ and denoted by a. Because the elements in $\sum_1$ are reducible, there exist b and c, which are subinclines over K, such that $a = b \cap c$ where $a \subset b$ and $a \subset c$. For a is the maximal element, then $b, c \notin \sum_1$, and so b, c can be expressed as intersection of a finite number of irreducible subinclines, denoted by $b = b_1 \cap \cdots \cap b_m, c = c_1 \cap \cdots \cap c_n$ (the b_i, c_j are irreduclble). Clearly, $a = b \cap c = b_1 \cap \cdots \cap b_m \cap c_1 \cap \cdots \cap c_n$. a can be expressed as intersection of a finite number of irreducible subinclines, a contradiction. So we have $\sum_1 = \emptyset$, hence every subincline over K can be expressed as intersection of finite number of subinclines which are irreducible.

Definition 4.8. K is a Noetherian incline, H is the set of all new type of soft subinclines of incline over K. For any ascending chain of new type of soft subinclines of incline $H_1 \leq H_2 \leq H_3 \leq \cdots$, if there exists a positive integer n such that $H_m = H_n$ for all $m > n, H$ is said to have the ascending chain condition, or we say H is Noetherian.

Definition 4.9. K is an incline, H is the set of all new type of soft subinclines of incline over K. For any descending chain of new type of soft subinclines of incline $H_1 \geq H_2 \geq H_3 \geq \cdots$, if there exists a positive integer n such that $H_m = H_n$ for all $m > n, H$ is said to have the descending chain condition, or we say H is Artinian.

Theorem 4.6. K is an incline, H is the set of all new type of soft subinclines of incline over K. H is Noetherian if and only if $\Omega(K)(x) \triangleq \{A_{H_i}(x) | H_i \in H\}$ is Noetherian for all $x \in X$ and $\sup\{m_{\{A_{H_i(x)}\}} | x \in X\}$ is finited.

Proof. Necessity: Following Theorem 3.3, it is sufficient to show that $A_{H_i}(x)$ is a subincline of K. Then $\Omega(K)(x)$ is a subincline family. Let $A_{H_1}(x) \subseteq A_{H_2}(x) \subseteq \cdots A_{H_n}(x) \subseteq \cdots$ be an ascending chain over $\Omega(K)(x)$, according to Theorem 4.2, it follows that $H_1 \leq H_2 \leq \cdots \leq H_n \leq \cdots$ For H is Noetherian, so there exists a positive integer n such that $H_m = H_n$ for all $m > n$. According to Corollrry 4.1, it follows that $A_{H_m}(x) = A_{H_n}(x)$ for all $x \in X$.

Consequently, we infer that $\Omega(K)(x)$ is Noetherian for all $x \in X$ and $\sup\{m_{\{A_{H_i(x)}\}} | x \in X\}$ is finited.

Sufficiency: Let $H_1 \leq H_2 \leq \cdots \leq H_n \leq \cdots$ be an ascending chain over H, according to Theorem 4.2, we have that $A_{H_1}(x) \subseteq A_{H_2}(x) \subseteq \cdots A_{H_n}(x) \subseteq \cdots$ for all $x \in X$. Let $\sup\{m_{\{A_{H_i(x)}\}} | x \in X\} = n$, since $\Omega(K)(x)$ is Noetherian, so $A_{H_m}(x) = A_{H_n}(x)$ for all $x \in X$ if $m > n$. According to Corollary 4.1, it follows that $H_m = H_n$ if $m > n$. Therefore H is Noetherian.

Theorem 4.7. K is an incline, H is the set of all new type of soft subinclines of incline over K. H is Artinian if and only if $\Omega(K)(x) \triangleq \{A_{H_i}(x) | H_i \in H\}$ is Artinian for all $x \in X$ and $\sup\{n_{\{A_{H_i(x)}\}} | x \in X\}$ is finited.

Proof. The proof of this theorem is similar to the proof of Theorem 4.6 and so is omitted.

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Bidirectional Quantum Teleportation with 5-Qubit States

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Abstract. In this paper, a bidirectional teleportation scheme is proposed, in which Alice wants to transmit an single qubit state to Bob and Bob wants to teleport a single qubit state to Alice too. They are shared a set of entangled 5-qubit sates as the quantum channel. All the operations in this scheme are given in the paper.

Keywords: Bidirectional teleportation · Partially entangled · GHZ

1 Introduction

Teleportation is one of the important applications of quantum information theory. In 1993, the first quantum teleportation scheme was proposed by Bennett [1]. In the scheme, Alice want to transmit an unknown quantum state to Bob with maximally entangled Einstein-Podolsky-Rosen states. Later, Karlsson and Bourennane [2] proposed the first controlled quantum teleportation by using maximally entangled GHZ state as quantum channel. Actually, this schemes of controlled teleportation are the same as quantum state sharing [3–7]. From then on, many theoretical schemes of quantum teleportation [8–14] have been given by using differently entangled states. At the same time, experimental development of quantum teleportation has also been reported [15,16]. Recently, Zha [17] demonstrated that some cluster state can be used as quantum channel for bidirectional quantum teleportation. In this type of teleportation schemes, Alice and Bob can simultaneously transmit an single quantum state each other after performing some appropriate locally operators. Up to now, various Bidirectional quantum teleportation schemes have been given with entangled states [18–23].

As to teleportation, the entangled qubit states, such as GHZ states [24–26], W states [27,28] and other entangled state [29,30], play a pivotal role in quantum schemes. In general, those entangled states, which arc used as quantum channel,

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are maximally entangled. However, those maximally entangled states are difficult to be generated for the coupling of the quantum states. If the quantum states are partially entangled in scheme, this schemes of quantum teleportation are called probabilistic teleportation [31–33], which are not almost realized perfect but implemented with a probability less than unit. However, some of partially entangled states [34,35] are found that they can be utilized as quantum channel for an optimal teleportation just as the biggest entangled states work in the scheme. Now, it is very natural to ask the following question: Whether bidirectional teleportation can also be implemented with probability unit if the quantum states, worked as quantum channel, are partially entangled for some reason? Based on those works, we propose a bidirectional controlled quantum teleportation with non-maximally entangled states in the paper.

The organization of this paper is outlined as follows. In Sect. 2, we firstly illustrate how to generate a 5-qubit entangled state from a normal GHZ state, utilized as quantum channel in the following scheme. In Sect. 3, we propose a scheme of bidirectional controlled quantum teleportation. Finally, discussions and conclusions about our scheme are given.

2 Bidirectional Controlled Teleportation

Before describing our scheme, we discuss how to generate the non-maximally entangled GHZ-type state $|\phi_1\rangle$ (Eq. 1) from a GHZ state, which will be employed in our teleportation scheme. The state $|\phi_1\rangle$ can be expressed as

$$\begin{aligned}|\phi_1\rangle = \frac{1}{2}(&sin\theta|00000\rangle + sin\theta|00110\rangle + sin\theta|01001\rangle + sin\theta|01111\rangle \\ &+ cos\theta|11100\rangle - cos\theta|11010\rangle - cos\theta|10101\rangle + cos\theta|10011\rangle).\end{aligned} \tag{1}$$

As showed in Fig. 1, the state input into the circuit is GHZ state

$$|\phi_0\rangle = sin\theta|00000\rangle + cos\theta|11111\rangle. \tag{2}$$

When two Hadamard operations are implemented on the fourth and the fifth qubit of EQ(2), the generalized GHZ state $|\phi_0\rangle$ is transformed into the following state (Fig. 1)

$$\begin{aligned}|\phi_0'\rangle &= sin\theta|000++\rangle + cos\theta|111--\rangle \\ &= \frac{1}{2}(sin\theta|00000\rangle + sin\theta|00010\rangle + sin\theta|00001\rangle + sin\theta|00011\rangle \\ &+ cos\theta|11100\rangle - cos\theta|11110\rangle - cos\theta|11101\rangle + cos\theta|11111\rangle).\end{aligned}$$

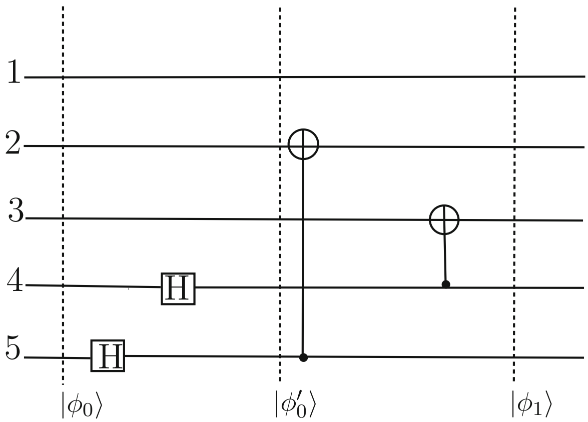

Fig. 1. A quantum circuit to generate 5-qubit partially entangled GHZ-type states.

Then, implementing a CNOT gate on the second qubit with the fifth qubit as the control qubit and another CNOT gate on the third qubit with the fourth qubit as the control qubit, we have generated the partially entangled state $|\phi_1\rangle$ (Eq. 1). We use the GME-concurrence [36] to analyze the entanglement properties of the state $|\phi_1\rangle$, which is given as

$$C_{GME}(|\Omega\rangle) := \min_{r_i \in r} \sqrt{2[1 - Tr(\rho^2_{A_{r_i}})]},$$

then we have $C_{GME}(|\phi_1\rangle) = |sin2\theta|$, which varies from 0 to 1. When $C_{GME} = 0$, the states

$$|\phi\rangle = \frac{1}{2}|1\rangle(|1100\rangle - |1010\rangle - |0101\rangle + |0011\rangle)$$

are biseparable. When $C_{GME} = 1$, the states

$$\begin{aligned}|\phi_1\rangle =& \frac{1}{2\sqrt{2}}(|00000\rangle + |00110\rangle + |01001\rangle + |01111\rangle \\ &+ |11100\rangle - |11010\rangle - |10101\rangle + |10011\rangle)\end{aligned}$$

are maximally entangled.

We supposed the three parties in the scheme are Alice, Bob and Charlie and the 5-qubit partially entangled state $|\phi_1\rangle$, used as quantum channel, are shared between Alice and Bob. Now, Alice wants to transmit an unknown qubit state $|\mu_A\rangle$ to Bob, and Bob also wants to transmit an unknown qubit state $|\mu_B\rangle$ to Alice. Charlie, as a controller, decides whether Alice and Bob, in the scheme, can attain the qubit state successfully from each other. The two qubit states transmitted from Alice and Bob, which are known nothing by everyone, are given by

$$|\mu_A\rangle = a_0|0\rangle + a_1|1\rangle, \quad |\mu_B\rangle = b_0|0\rangle + b_1|1\rangle, \tag{3}$$

where $|a_0|^2 + |a_1|^2 = 1, \ \ |b_0|^2 + |b_1|^2 = 1$. The total state of physical system can be shown as

$$|\Omega\rangle = |\mu_A\rangle_A \otimes |\phi_1\rangle_{12345} \otimes |\mu_B\rangle_B. \tag{4}$$

where the qubits $A, 3, 5$ held by Alice, qubits $B, 2, 4$ by Bob and the qubit 1 belongs to Charlie. At first, a Bell-states measurement is performed by Alice and Bob on their own qubit respectively. then they public their outcomes each other by sending two bits of classical information. Thus, the whole states (Eq. 4) can be rewritten as

$$\begin{aligned}
|\Omega\rangle_{B2A3145} = &|\Psi^x\rangle_{B2}|\Psi^y\rangle_{A3}[(a_0b_0sin\theta|000\rangle + (-1)^x a_0b_1sin\theta|001\rangle + \\
&(-1)^y a_1b_0sin\theta|010\rangle + (-1)^{x+y} a_1b_1sin\theta|011\rangle + a_0b_0cos\theta|111\rangle - \\
&(-1)^x a_0b_1cos\theta|110\rangle - (-1)^y a_1b_0cos\theta|101\rangle + (-1)^{x+y} a_1b_1cos\theta|100\rangle] \\
&+ |\Psi^x\rangle_{B2}|\Phi^y\rangle_{A3}[(a_0b_0sin\theta|010\rangle + (-1)^x a_0b_1sin\theta|011\rangle + \\
&(-1)^y a_1b_0sin\theta|000\rangle + (-1)^{x+y} a_1b_1sin\theta|001\rangle - a_0b_0cos\theta|101\rangle + \\
&(-1)^x a_0b_1cos\theta|100\rangle + (-1)^y a_1b_0cos\theta|111\rangle - (-1)^{x+y} a_1b_1cos\theta|110\rangle] \\
&+ |\Phi^x\rangle_{B2}|\Psi^y\rangle_{A3}[(a_0b_0sin\theta|001\rangle + (-1)^x a_0b_1sin\theta|000\rangle + \\
&(-1)^y a_1b_0sin\theta|011\rangle + (-1)^{x+y} a_1b_1sin\theta|010\rangle - a_0b_0cos\theta|110\rangle + \\
&(-1)^x a_0b_1cos\theta|111\rangle + (-1)^y a_1b_0cos\theta|100\rangle - (-1)^{x+y} a_1b_1cos\theta|101\rangle] \\
&+ |\Phi^x\rangle_{B2}|\Phi^y\rangle_{A3}[(a_0b_0sin\theta|011\rangle + (-1)^x a_0b_1sin\theta|010\rangle + \\
&(-1)^y a_1b_0sin\theta|001\rangle + (-1)^{x+y} a_1b_1sin\theta|000\rangle + a_0b_0cos\theta|100\rangle - \\
&(-1)^x a_0b_1cos\theta|101\rangle - (-1)^y a_1b_0cos\theta|110\rangle + (-1)^{x+y} a_1b_1cos\theta|111\rangle],
\end{aligned} \tag{5}$$

where $|\Psi^0\rangle = |\Psi^+\rangle, \ |\Psi^1\rangle = |\Psi^-\rangle, \ |\Phi^0\rangle = |\Phi^+\rangle, |\Phi^1\rangle = |\Phi^-\rangle$, and the four states of $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are so called Bell-states. It is clear that, when Alice and Bob have finished their measurement, the whole physical states will collapse to one of 16 results. At this time, if Charlie does not want to implement the communication about quantum information between Alice and Bob, she can do nothing on her own particle to terminate this scheme. While she needs to take a measurement on the qubit 1 under bases $\{|0\rangle, \ |1\rangle\}$ and tell the others of her measurement outcome by sending one bit of classical information. Both of Alice and Bob can recover the right states transformed from the other when they receive all of the measurement outcomes informed by the others. For example, supposed the measurement results of Alice and Bob are $|\Psi^1\rangle_{B2}|\Phi^0\rangle_{A3}$, the system state collapses into the following state

$$\begin{aligned}
|\Omega\rangle_{145} = &a_0b_0sin\theta|010\rangle - a_0b_1sin\theta|011\rangle + a_1b_0sin\theta|000\rangle - a_1b_1sin\theta|001\rangle \\
&- a_0b_0cos\theta|101\rangle - a_0b_1cos\theta|100\rangle + a_1b_0cos\theta|111\rangle + a_1b_1cos\theta|110\rangle.
\end{aligned} \tag{6}$$

Now, we also assume Charlie allows Alice and Bob to exchange their quantum information. Thus, Charlie will take a classical measurement on her qubit 1 and tell Alice and Bob of her outcome. In term of Charlie's measurement result, the state (Eq. 6) of physic system can be rewritten as follows

$$\begin{aligned} |\Omega\rangle_{145} = & sin\theta|0\rangle(a_0|1\rangle + a_1|0\rangle) \otimes (b_0|0\rangle - b_1|1\rangle) \\ & + cos\theta|1\rangle(a_1|1\rangle - a_0|0\rangle) \otimes (b_0|1\rangle + b_1|0\rangle). \end{aligned} \tag{7}$$

If Charlie's measurement result is $|0\rangle$, the composed states hold by Alice and Bob will be

$$|\Omega\rangle_{45} = (a_1|0\rangle + a_0|1\rangle) \otimes (b_0|0\rangle - b_1|1\rangle), \tag{8}$$

the above state (Eq. 8) hold by Alice and Bob is not absolutely entangled but biseparable states. Thus, it is possible for them to recover the information transmitting from the others by taking some locally unitary operations. Lets come back to the above example, when Alice and Bob have performed the two locally unitary operations U_A and U_B on the quibt 5 and qubit 4 respectively, they can recover the states $|\mu_B\rangle$ and $|\mu_A\rangle$, that is $U_A|\Omega\rangle_5 = |\mu_B\rangle_5$ and $U_B|\Omega\rangle_4 = |\mu_A\rangle_4$. The two unit operations are given by $U_A = \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, $U_B = \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, where σ_x, σ_z are the Pauli operations.

Moreover, if Alice and Bob got any other results of their measurements and Charlie agree they to recover their qubits, Alice and Bob can recover the qubit state too. On the basis of the measuring results, all of the appropriate operations, performed by Alice and Bob, are listed in Table1.

3 Conclusion

In this paper, a scheme of bidirectional controlled quantum teleportation via a non-maximally entangled GHZ-type state, which can be transformed by the generalized 5-qubit GHZ states, are proposed. As to the mean of technology, our scheme can be implemented deterministically with only two Bell-state measurements and a classical measurement.

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Recommender: Pang Yicheng, Associate professor, School of Mathematics and Statistics, Guizhou University of Finance and Economics.

Appendix

Table 1. Bob, Alice and Charlie's possible measuring result, final states by Bob and Alice, and the corresponding locally operations performed by Bob and Alice respectively.

Bob's result	Alice's result	Charlie's result	Final states hold in Bob and Alice	Locally operation $U_B \otimes U_A$
$\|\Psi^+\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$I \otimes I$
$\|\Psi^+\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$\sigma_z \otimes I$
$\|\Psi^+\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$\sigma_x \otimes I$
$\|\Psi^+\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$\sigma_z\sigma_x \otimes I$
$\|\Psi^-\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$I \otimes \sigma_z$
$\|\Psi^-\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$\sigma_z \otimes \sigma_z$
$\|\Psi^-\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$\sigma_x \otimes \sigma_z$
$\|\Psi^-\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$\sigma_z\sigma_x \otimes \sigma_z$
$\|\Phi^+\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$I \otimes \sigma_x$
$\|\Phi^+\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$\sigma_z \otimes \sigma_x$
$\|\Phi^+\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$\sigma_x \otimes \sigma_x$
$\|\Phi^+\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$\sigma_z\sigma_x \otimes \sigma_x$
$\|\Phi^-\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$I \otimes \sigma_z\sigma_x$
$\|\Phi^-\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$\sigma_z \otimes \sigma_z\sigma_x$
$\|\Phi^-\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$\sigma_x \otimes \sigma_z\sigma_x$
$\|\Phi^-\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|0\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$\sigma_z\sigma_x \otimes \sigma_z\sigma_x$
$\|\Psi^+\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$\sigma_z\sigma_x \otimes \sigma_z\sigma_x$
$\|\Psi^+\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$\sigma_x \otimes \sigma_z\sigma_x$
$\|\Psi^+\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$\sigma_z \otimes \sigma_z\sigma_x$
$\|\Psi^+\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|1\rangle - b_1\|0\rangle)_5$	$I \otimes \sigma_z\sigma_x$
$\|\Psi^-\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$\sigma_z\sigma_x \otimes \sigma_x$
$\|\Psi^-\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$\sigma_x \otimes \sigma_x$
$\|\Psi^-\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$\sigma_z \otimes \sigma_x$
$\|\Psi^-\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|1\rangle + b_1\|0\rangle)_5$	$I \otimes \sigma_x$
$\|\Phi^+\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$\sigma_z\sigma_x \otimes \sigma_z$
$\|\Phi^+\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$\sigma_x \otimes \sigma_z$
$\|\Phi^+\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$\sigma_z \otimes \sigma_z$
$\|\Phi^+\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|0\rangle - b_1\|1\rangle)_5$	$I \otimes \sigma_z$
$\|\Phi^-\rangle_{B2}$	$\|\Psi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle - a_1\|0\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$\sigma_z\sigma_x \otimes I$
$\|\Phi^-\rangle_{B2}$	$\|\Psi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|1\rangle + a_1\|0\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$\sigma_x \otimes I$
$\|\Phi^-\rangle_{B2}$	$\|\Phi^+\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle - a_1\|1\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$\sigma_z \otimes I$
$\|\Phi^-\rangle_{B2}$	$\|\Phi^-\rangle_{A3}$	$\|1\rangle_1$	$(a_0\|0\rangle + a_1\|1\rangle)_4 \otimes (b_0\|0\rangle + b_1\|1\rangle)_5$	$I \otimes I$

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Infinitely Small Quantity and Infinitely Large Quantity of Fuzzy Valued Functions for Linear Generation of Structural Elements

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Abstract. A kind of fuzzy distance gives a new definition of the limit of fuzzy valued function for linear generation of structural elements. Then the limit definition is used to define infinitely small quantity and infinitely large quantity of fuzzy valued functions for linear generation of structural elements. Simultaneously, the connected properties of low order the infinitely small quantity, the higher order infinitely small quantity and the equivalent infinitely small quantity are studied.

Keywords: Structural element · Fuzzy valued function · Infinitely small quantity · Infinitely large quantity

1 Introduction

The notion of fuzzy valued function for linear generation of structural elements is proposed by Guo in the literature [10]. For the limit of the fuzzy valued function for linear generation of structural elements, it has the different form of expression because of the different form of fuzzy distance. In this paper, the limit of fuzzy valued function is defined by this fuzzy distance which is given in the literature [11]. Then we let the limit of fuzzy valued function for linear generation of structural elements define the infinitely small quantity and infinitely large quantity of fuzzy valued functions for linear generation of structural elements. Simultaneously, the related properties about the infinitely small quantity and infinitely large quantity of fuzzy valued functions for linear generation of structural elements are discussed.

2 Brief Introduction of the Fuzzy Valued Function of Linear Construction Theory

Definition 2.1 [9]**.** E is the fuzzy structure element on the R of real number field. If its membership function E(x)($x \in R$)has the following properties:

(1) E(0)=1, E(1+0)=E(-1-0)=0;

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(2) $x \in [-1, 0)$, E(x) monotone increasing right continuous function; $x \in (0, 1]$, E(x) monotone decreasing left continuous function, respectively;
(3) $x \in (-\infty, -1) \cup (1, +\infty)$, E(x)=0. Obviously, the fuzzy structured element is a regular convex fuzzy set on the R, which is a bounded closed fuzzy number.

Definition 2.2 [10]**.** $\widetilde{A}$ is finite fuzzy number. If there is a fuzzy structured element E and finite real number $a \in R, r \in R^+$, such that $\widetilde{A} = a + rE(r \to 0^+)$, $\widetilde{A}$ is a fuzzy number linear generated by fuzzy structured element E. The entire number of fuzzy numbers linear generated by the E is denoted as $\varepsilon(E) = \{\widetilde{A} \mid \widetilde{A} = a + rE, \forall\ a \in R,\ r \in R^+\}$.

All in this paper, $\widetilde{A} \in \varepsilon(E)$, on account of the decomposition theorem of fuzzy sets, $\widetilde{A} = \bigcup\limits_{\lambda \in [0,1]} \lambda A_\lambda = \bigcup\limits_{\lambda \in [0,1]} \lambda[a + rE_\lambda^-, a + rE_\lambda^+]$.

Definition 2.3 [11]**.** Let X, Y is the two set of real numbers. $\widetilde{N}(f)$ is the whole set of fuzzy sets on Y. $\widetilde{f}$ is a mapping from X to $\widetilde{N}(f)$. In other words, for arbitrary x $\in$ X, there exists only $\widetilde{f} \in \widetilde{N}(f)$ with it correspondence. It is remembered as $\widetilde{y} = \widetilde{f}(x)$. Then $\widetilde{f}(x)$ is called a fuzzy value function on X. If E is a regular fuzzy structure element on $\widetilde{N}(f)$. It is said that $\widetilde{f}(x)$=$h(x) + \omega(x)E$ is a fuzzy valued function linear generated by E for X ($X \subseteq R$). Where h(x), $\omega(x)$ in the X on the bounded, also $\omega(x) > 0$. All of the bounded fuzzy functions linear generated by E are denoted as $N(E_f) = \{\widetilde{f}(x) \mid \widetilde{f}(x) = h(x) + \omega(x)E,\ \forall\ x \in X,\ \omega(x) > 0\}$.

All in this paper $\widetilde{f}(x) \in N(E_f)$, on account of the decomposition theorem of fuzzy sets, $\widetilde{f}(x) = \bigcup\limits_{\lambda \in [0,1]} \lambda \widetilde{f}_\lambda(x) = \bigcup\limits_{\lambda \in [0,1]} (h(x) + \omega(x)E_\lambda) = \bigcup\limits_{\lambda \in [0,1]} [h(x) + \omega(x)e_\lambda^-, h(x) + \omega(x)e_\lambda^+]$.

Definition 2.4 [12]**.** $\forall\ \widetilde{a},\ \widetilde{b} \in E$, The distance of $\widetilde{a}$ and $\widetilde{b}$ was identified as $\widetilde{d}(\widetilde{a}, \widetilde{b}) = \bigcup\limits_{\lambda \in [0,1]} \lambda[\sup\limits_{\lambda \le \mu \le 1} |\widetilde{a}_\mu^- - \widetilde{b}_\mu^-|, \sup\limits_{0 \le \lambda \le \mu} (|\widetilde{a}_\mu^- - \widetilde{b}_\mu^-| \vee |\widetilde{a}_\mu^+ - \widetilde{b}_\mu^+|)]$.

3 A New Definition of the Limit of $\widetilde{f}(x)$

Definition 3.1 (The limit of $\widetilde{f}(x)$ when x tends to $+\infty$)**.** Let the definition domain of $\widetilde{f}(x)$ to $[a, +\infty)$. If $\forall\ \varepsilon > 0$, there exists positive $M(\ge a)$ such that $x > M$, then $d(\widetilde{f}(x), \widetilde{A}) < \varepsilon$. So call $\widetilde{f}(x)$ when x tends to $+\infty$ to $\widetilde{A}$ as the limit. It is recorded as $\lim\limits_{x \to +\infty} \widetilde{f}(x) = \widetilde{A}$.

We can define similarly $\widetilde{f}(x)$ limit of $x \to -\infty$, $x \to \infty$. They are denoted respectively as $\lim\limits_{x \to -\infty} \widetilde{f}(x)$=$\widetilde{A}$; $\lim\limits_{x \to \infty} \widetilde{f}(x)$=$\widetilde{A}$.

Theorem 3.1. If $\widetilde{f}(x)$ domain is defined as R, then there are $\lim_{x\to\infty}\widetilde{f}(x)=\widetilde{A}\Leftrightarrow \lim_{x\to-\infty}\widetilde{f}(x)=\lim_{x\to+\infty}\widetilde{f}(x)=\widetilde{A}$.

Easy to prove by Definition 2.1.

Definition 3.2 ($\varepsilon-\delta$ definition of the limit of $\widetilde{f}(x)$)**.** Let the definition domain of $\widetilde{f}(x)$ to $U^0(x_0;\delta')$ $\widetilde{A}\in\varepsilon(E)$. If $\forall\ \varepsilon>0$, there exists positive $\delta(<\delta')$ such that $0<|\ x-x_0\ |<\delta$, then $d(\widetilde{f}(x),\widetilde{A})<\varepsilon$. So call $\widetilde{f}(x)$ when x tends to x_0 to $\widetilde{A}$ as the limit. It is recorded as $\lim_{x\to x_0}\widetilde{f}(x)=\widetilde{A}$.

We can define similarly $\widetilde{f}(x)$ limit of $x\to x_0^-$, $x\to x_0^+$, They are denoted respectively as $\lim_{x\to x_0^-}\widetilde{f}(x)=\widetilde{A}$; $\lim_{x\to x_0^+}\widetilde{f}(x)=\widetilde{A}$.

Easy to prove by Definition 2.2 and Theorem 2.1.

Theorem 3.2. If $\widetilde{f}(x)$ is defined on $U^o(x_0)$, then there is $\lim_{x\to x_0}\widetilde{f}(x)=\widetilde{A}\Leftrightarrow \lim_{x\to x_0^-}\widetilde{f}(x)=\lim_{x\to x_0^+}\widetilde{f}(x)=\widetilde{A}$.

The six definition of $\lim_{x\to\infty}\widetilde{f}(x)$, $\lim_{x\to+\infty}\widetilde{f}(x)$, $\lim_{x\to-\infty}\widetilde{f}(x)$, $\lim_{x\to x_0}\widetilde{f}(x)$, $\lim_{x\to x_0^+}\widetilde{f}(x)$ and $\lim_{x\to x_0^-}\widetilde{f}(x)$ are similar in nature.

Next we take $\lim_{x\to x_0}\widetilde{f}(x)$ as an example to discuss, the rest can only be modified accordingly.

4 Infinitely Small Quantity of $\widetilde{f}(x)$

Definition 4.1. Let the definition domain of $\widetilde{f}(x)$ to $U^0(x_0)$, If $\lim_{x\to x_0}\widetilde{f}(x)=0$, So $\widetilde{f}(x)$ is called the infinitely small quantity of $x\to x_0$.

Theorem 4.1. Two the infinitely small quantity sum is still infinitesimal.

Proof. Let $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ be two the infinitely small quantity of $x\to x_0$, and $\widetilde{f}(x)=\widetilde{f}_1(x)+\widetilde{f}_2(x)$. $\forall\ \varepsilon>0$, $\widetilde{f}_1(x)$ is the infinitely small quantity of $x\to x_0$. So for any $\frac{\varepsilon}{2}>0$, there exists $\delta_1>0$, whenever $0<|\ x-x_0\ |<\delta_1$, hence $|\ \widetilde{f}_1(x)\ |<\frac{\varepsilon}{2}$. $\widetilde{f}_2(x)$ is the infinitely small quantity of $x\to x_0$, so for any $\frac{\varepsilon}{2}>0$, there exists $\delta_2>0$, whenever $0<|\ x-x_0\ |<\delta_2$, hence $|\ \widetilde{f}_2(x)\ |<\frac{\varepsilon}{2}$. Take $\delta=\min\{\delta_1,\delta_2\}$, when $0<|\ x-x_0\ |<\delta$, we have $|\ \widetilde{f}(x)\ |=|\ \widetilde{f}_1(x)+\widetilde{f}_2(x)\ |\leq|\ \widetilde{f}_1(x)\ |+|\ \widetilde{f}_2(x)\ |=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon$. Therefore, $\lim_{x\to x_0}\widetilde{f}(x)=0$. That has been proved $\widetilde{f}(x)$ is the infinitely small quantity of $x\to x_0$.

Corollary 4.1. Two infinitely small quantity subtraction is still infinitely small quantity.

Theorem 4.2. In the same change process of the independent variable $x \to x_0$, $\lim\limits_{x \to x_0} \widetilde{f}(x) = \widetilde{A} \Leftrightarrow \widetilde{f}(x) = \widetilde{A} + \widetilde{a}$, where $\widetilde{a}$ is infinitely small quantity.

Proof. Necessity. Let $\widetilde{f}(x) = \widetilde{A} + \widetilde{a}$. So there is $| \widetilde{f}(x) - \widetilde{A} | = | \widetilde{a} |$. Because $\widetilde{a}$ is the infinitely small quantity of $x \to x_0$, for each $\varepsilon > 0$, there exist $\delta > 0$, whenever $0 <| x - x_0 |< \delta$, hence $| \widetilde{a} |< \varepsilon$. Namely $| \widetilde{f}(x) - \widetilde{A} |< \varepsilon$. Tt has been proved that $\lim\limits_{x \to x_0} \widetilde{f}(x) = \widetilde{A}$.

Sufficiency. Let $\lim\limits_{x \to x_0} \widetilde{f}(x) = \widetilde{A}$. As a result, for each $\varepsilon > 0$, there exist $\delta > 0$, whenever $0 <| x - x_0 |< \delta$, hence $| \widetilde{f}(x) - \widetilde{A} |< \varepsilon$. Order $\widetilde{a} = \widetilde{f}(x) - \widetilde{A}$,so that $\widetilde{a}$ is the infinitely small quantity of $x \to x_0$. And $\widetilde{A} \Leftrightarrow \widetilde{f}(x) = \widetilde{A} + \widetilde{a}$. This proves that $\widetilde{f}(x)$ is equal to its limit with an infinitely small quantity.

Theorem 4.3. The product of infinitely small quantity and bounded quantity is infinitely small quantity.

Proof. Let $\widetilde{f_1}(x)$ be in the definition domain $U^0(x_0; \delta_1)$ within the bounded, so presence of $\widetilde{A'} > 0$ makes $| \widetilde{f_1}(x) |< \widetilde{A'}$, for all the $x \in U^0(x_0)$ are set up. Then set $\widetilde{f_2}(x)$ is the infinitely small quantity of $x \to x_0$. so $\forall\, \varepsilon > 0$, there exist $\delta_2 > 0$, whenever $0 <| x - x_0 |< \delta_2$, hence $| \widetilde{f_2}(x) |< \frac{\varepsilon}{\widetilde{A'}}$. Take $\delta = \min\{\delta_1, \delta_2\}$, when $0 <| x - x_0 |< \delta$, we have $| \widetilde{f_1}(x) \cdot \widetilde{f_2}(x) | = | \widetilde{f_1}(x) | \cdot | \widetilde{f_2}(x) | < \frac{\varepsilon}{\widetilde{A'}} \cdot \widetilde{A'} = \varepsilon$. Tt is to prove that $\lim\limits_{x \to x_0} (\widetilde{f_1}(x) \cdot \widetilde{f_2}(x)) = 0$. Therefore, $\widetilde{f_1}(x) \cdot \widetilde{f_2}(x)$ is the infinitely small quantity of $x \to x_0$.

Definition 4.2. $\widetilde{f_1}(x)$ and $\widetilde{f_2}(x)$ be two the infinitely small quantity of $x \to x_0$, also supp$\widetilde{f_2}(x) \neq 0$.

(1) If $\lim\limits_{x \to x_0} \frac{\widehat{f_1}(x)}{\widehat{f_2}(x)} = 0$, it is said that the $\widehat{f_1}(x)$ is higher order than the $\widehat{f_2}(x)$ of the infinitely small quantity of $x \to x_0$, or the $\widehat{f_2}(x)$ is smaller order than the $\widehat{f_1}(x)$ of the infinitely small quantity of $x \to x_0$, which is denoted as $\widetilde{f_1}(x) = 0(\widetilde{f_2}(x))(x \to x_0)$.
(2) If there are positive numbers a, b $\in R$, when $x \in U^0(x_0)$, there is $a \leq \lim\limits_{x \to x_0} \frac{\widehat{f_1}(x)}{\widehat{f_2}(x)} \leq b$. So we call $\widehat{f_1}(x)$ and $\widehat{f_2}(x)$ for the same order infinitely small quantity of $x \to x_0$.

Especially,

(1) If $\lim\limits_{x \to x_0} \frac{\widehat{f_1}(x)}{\widehat{f_2}(x)} = a \neq 0$, $\widehat{f_1}(x)$ and $\widehat{f_2}(x)$ are absolutely the same order infinitely small quantity of $x \to x_0$;
(2) when $x \in U^0(x_0)$, there is $\lim\limits_{x \to x_0} \frac{\widehat{f_1}(x)}{\widehat{f_2}(x)} \leq b$. Hence it is denoted as $\widetilde{f_1}(x) = 0(\widetilde{f_2}(x))(x \to x_0)$;
(3) If $\widetilde{f}(x)$ is bounded in $U^0(x_0)$, then it is denoted as $\widetilde{f_1}(x) = 0(1)(x \to x_0)$.

(3) If $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{(\widehat{f}_2(x))^k}=a\neq 0$, where $k>0$, a $\in R$. so we say that $\widehat{f}_1(x)$ is about K's $\widehat{f}_2(x)$ order infinitely small quantity of $x\to x_0$.

(4) If $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}=1$, it is said that $\widehat{f}_1(x)$ and $\widehat{f}_2(x)$ are equivalent infinitely small quantity of $x\to x_0$, which is denoted as $\widetilde{f}_1(x)\sim\widetilde{f}_2(x)(x\to x_0)$.

Theorem 4.4. $\widehat{f}_1(x)$ and $\widehat{f}_2(x)$ are equivalent infinitely small quantity $\Leftrightarrow$ $\widehat{f}_1(x)=\widehat{f}_2(x)+0(\widehat{f}_2(x))$.

Proof. Necessity. Let $\widetilde{f}_1(x)\sim\widetilde{f}_2(x)(x\to x_0)$, that is $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}=1$. then $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)-\widehat{f}_2(x)}{\widehat{f}_2(x)}=\lim\limits_{x\to x_0}(\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}-1)=\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}-1=0$. Consequently, $\widehat{f}_1(x)-\widehat{f}_2(x)=0(\widehat{f}_2(x))$, the same result, $\widehat{f}_1(x)=\widehat{f}_2(x)+0(\widehat{f}_2(x))$.

Sufficiency. Let $\widehat{f}_1(x)=\widehat{f}_2(x)+0(\widehat{f}_2(x))$. Then $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}=\lim\limits_{x\to x_0}\frac{\widehat{f}_2(x)+0(\widehat{f}_2(x))}{\widehat{f}_2(x)}=\lim\limits_{x\to x_0}(1-\frac{\widehat{f}_2(x)}{\widehat{f}_2(x)})=1+\lim\limits_{x\to x_0}\frac{0(\widehat{f}_2(x))}{\widehat{f}_2(x)}=1+0=1$. Therefore, $\widetilde{f}_1(x)\sim\widetilde{f}_2(x)(x\to x_0)$.

This theorem is fully proved.

Theorem 4.5. Let the definition domain of $\widetilde{f}_1(x)$, $\widetilde{f}_2(x)$, $\widetilde{f}_3(x)$ to $U^0(x_0)$, also $\widetilde{f}_2(x)\sim\widetilde{f}_3(x)(x\to x_0)$.

(1) If $\lim\limits_{x\to x_0}(\widetilde{f}_1(x)\cdot\widetilde{f}_2(x))=a$, where a $\in R$. so $\lim\limits_{x\to x_0}(\widetilde{f}_1(x)\cdot\widetilde{f}_3(x))=a$;

(2) If $\lim\limits_{x\to x_0}\frac{\widetilde{f}_1(x)}{\widetilde{f}_2(x)}=b$, where b $\in R$. so $\lim\limits_{x\to x_0}\frac{\widetilde{f}_1(x)}{\widetilde{f}_3(x)}=b$.

Proof. (1) $\lim\limits_{x\to x_0}(\widetilde{f}_1(x)\cdot\widetilde{f}_3(x))=\lim\limits_{x\to x_0}\frac{\widetilde{f}_3(x)}{\widetilde{f}_2(x)}\cdot\lim\limits_{x\to x_0}(\widetilde{f}_1(x)\cdot\widetilde{f}_2(x))=1\cdot a=a$.

(2) $\lim\limits_{x\to x_0}\frac{\widetilde{f}_1(x)}{\widetilde{f}_3(x)}=\lim\limits_{x\to x_0}\frac{\widetilde{f}_1(x)}{\widetilde{f}_2(x)}\cdot\lim\limits_{x\to x_0}\frac{\widetilde{f}_2(x)}{\widetilde{f}_3(x)}=b\cdot 1=b$.

Theorem 4.6. Let $\widetilde{f}_1(x)\sim\widetilde{f}_1'(x),\widetilde{f}_2(x)\sim\widetilde{f}_2'(x)(x\to x_0)$. If $\lim\limits_{x\to x_0}\frac{\widehat{f}_1'(x)}{\widehat{f}_2'(x)}$ exists, so $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}=\lim\limits_{x\to x_0}\frac{\widehat{f}_1'(x)}{\widehat{f}_2'(x)}$.

Proof. Because $\widetilde{f}_1(x)\sim\widetilde{f}_1'(x),\widetilde{f}_2(x)\sim\widetilde{f}_2'(x)(x\to x_0)$, $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_1'(x)}=1$, $\lim\limits_{x\to x_0}\frac{\widehat{f}_2(x)}{\widehat{f}_2'(x)}=1$, hence $\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)}=\lim\limits_{x\to x_0}(\frac{\widehat{f}_1'(x)}{\widehat{f}_2'(x)}\cdot\frac{\widehat{f}_2'(x)}{\widehat{f}_1'(x)}\cdot\frac{\widehat{f}_1(x)}{\widehat{f}_2(x)})=\lim\limits_{x\to x_0}\frac{\widehat{f}_1'(x)}{\widehat{f}_2'(x)}\cdot\lim\limits_{x\to x_0}\frac{\widehat{f}_2'(x)}{\widehat{f}_2(x)}\cdot\lim\limits_{x\to x_0}\frac{\widehat{f}_1(x)}{\widehat{f}_1'(x)}=\lim\limits_{x\to x_0}\frac{\widehat{f}_1'(x)}{\widehat{f}_2'(x)}$.

5 Infinitely Large Quantity of $\widetilde{f}(x)$

Definition 5.1. Let the definition domain of $\widetilde{f}(x)$ to $U^0(x_0)$, If for any given $\widetilde{A}>0$, there exists positive $\delta>0$ such that $x\in U^0(x_0;\delta)(\subset U^0(x_0))$, hence|

$\widetilde{f}(x) \mid > \widetilde{A}$, so it is said that $\widetilde{f}(x)$ is the non normal limit ∞ of $x \to x_0$. which is denoted as $\widetilde{f}(x) = \infty$. If the $\mid \widetilde{f}(x) \mid > \widetilde{A}$ change is written in $\widetilde{f}(x) > \widetilde{A}$ or $\widetilde{f}(x) < -\widetilde{A}$, then it is said that $\widetilde{f}(x)$ is the non normal limit $+\infty$ or $-\infty$ of $x \to x_0$. which is respectively denoted as $\widetilde{f}(x) = +\infty$ or $\widetilde{f}(x) = -\infty$. For the independent variable $x \in U^0(x_0)$, all with as the limit of ∞, $+\infty$ or $-\infty$, all the $\widetilde{f}(x)$ are defined as infinitely large quantity of $x \to x_0$.

Definition 5.2. Let $\widetilde{f_1}(x)$ and $\widetilde{f_2}(x)$ be two the infinitely small quantity of $x \to x_0$, also supp$\widetilde{f_2}(x) \neq 0$. If $\lim\limits_{x \to x_0} \frac{\widehat{f_1}(x)}{\widehat{f_2}(x)} = \infty$, so it is said that the $\widehat{f_1}(x)$ is lower order than the $\widehat{f_2}(x)$ of the infinitely small quantity of $x \to x_0$.

Theorem 5.1. Let $\widetilde{f}(x)$ is defined in $U^0(x_0)$, also supp$\widetilde{f}(x) \neq 0$.

(1) If $\widetilde{f}(x)$ is the infinitely small quantity of $x \to x_0$, so $\frac{1}{\widetilde{f}(x)}$ is the infinitely large quantity of $x \to x_0$.
(2) If $\widetilde{f}(x)$ is the infinitely large quantity of $x \to x_0$, so $\frac{1}{\widetilde{f}(x)}$ is the infinitely small quantity of $x \to x_0$.

Proof. (1) Let $\lim\limits_{x \to x_0} \widetilde{f}(x) = 0$, also $\widetilde{f}(x) \neq 0$. $\forall\ a > 0$, a $\in R$. According to the definition of the infinitely small quantity, about $\varepsilon = \frac{1}{a}$, there exist $\delta > 0$, whenever $0 <\mid x - x_0 \mid < \delta$, hence $\mid \widetilde{f}(x) \mid < \varepsilon = \frac{1}{a}$. Because when $0 <\mid x - x_0 \mid < \delta$ has $\widetilde{f}(x) \neq 0$, $\mid \frac{1}{\widetilde{f}(x)} \mid > \frac{1}{\varepsilon} = a$. Therefore, $\frac{1}{\widetilde{f}(x)}$ is the infinitely large quantity of $x \to x_0$.

(2) $\forall\ b > 0$, b $\in R$. According to the definition of the infinitely large quantity, about $b = \frac{1}{\varepsilon}$, there exist $\delta > 0$, whenever $0 <\mid x - x_0 \mid < \delta$, hence $\mid \widetilde{f}(x) \mid > b = \frac{1}{\varepsilon}$. Consequently, $\mid \frac{1}{\widetilde{f}(x)} \mid < \frac{1}{b} = \varepsilon$. In other words, $\frac{1}{\widetilde{f}(x)}$ is the infinitely small quantity of $x \to x_0$.

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Part II:
Decision and Fuzziness

Hesitant Fuzzy Group Decision Making Under Incomplete Information

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Abstract. To make full use of data information, In this paper, we proposed that using hesitant fuzzy set as feedback extension of concept. Then, the relationship and operation of the hesitant fuzzy sets, that considered the hesitancy degree, was given; Based on this operation, the extension envelope of fuzzy concept is defined, and the preference information is gathered by using it. In order to give a hesitant fuzzy group decision making method under incomplete information, this paper also presents a method for filling the missing information. At last, the group decision making procedure is given and the above theory is applied in example.

Keywords: Factor space · Representation extension · Feedback extension · Hesitant fuzzy set · Incomplete information · Group decision making

1 Introduction

In the 1980s, the theory of factors space [1, 2], Professor Wang Peizhuang proposed, provided a universal coordinate frame for the knowledge representation in artificial intelligence. This theory as well as formal concept analysis [3], rough set [4] played an important role in the wave of intelligent mathematics. At present, the theory has been applied successfully in the fuzzy information processing [5, 6], data mining [7], bioinformatics [8], and other fields.

As the carrier of knowledge, the concept is the basis for reasoning and decision-making. As to the problem of concept representation, the theory of factor space defined the representation extension of the concept using the Zadeh's extension principle [9], and gave the method of concept representation based on representation extension. In the literature [10], the feedback extension of the concept was constructed with the help of representation extension, and in order to improve the accuracy, that the envelope of feedback extension which was an "external" approach to the concept extension has been defined. Then, in the literature [11], the min-type representation extension of the concept has been defined and discussed based on the minimal extension principle, in addition, the outer envelope of min-type feedback extension has been constructed by using the feedback extension of representation extension. In another literature [12], in the point of the opposite concept, the outer envelope of feedback extension has been defined using Zadeh's maximal extension principle. As an extension form of fuzzy sets, hesitant fuzzy sets was been proposed by Torra [13, 14],

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etc., and they pointed out that an element belongs to a sets' membership can be formed by multiple possible values, and discussed the difference and relation among the hesitant fuzzy sets, intuitionistic fuzzy sets, type-2 fuzzy sets, and fuzzy multiple sets.

Group decision making is a common decision method, and has been brought into wide focus by many researchers in recent years. However, in many cases, the limitation of decision makers' own knowledge and the fuzziness and uncertainty of practical problems background, often leads to incomplete decision information. Usually there were two ways of solving decision making problems of incomplete information. One way was that the incomplete information system would be complete by using of deleting data and completing scheme or property of containing vacancy value, but it would cause that the subjective factors of decision makers destroyed the original information system in different degrees. The other was that decision would be made by using of the theory of evidence and the theory of utility directly without completing system, but it did not involve the impact of incomplete information system on the result of decision making.

To make full use of data information, In this paper, we proposed that using hesitant fuzzy set as feedback extension of concept. Then, the relationship and operation of the hesitant fuzzy sets, that considered the hesitancy degree, was given; Based on this operation, the extension envelope of fuzzy concept is defined, and the preference information is gathered by using it. In order to give a hesitant fuzzy group decision making method under incomplete information, this paper also presents a method for filling the missing information. At last, the group decision making procedure is given and the above theory is applied in example.

2 Preliminaries

In this paper, the symbols were derived from the literature [2].

In this paper, U represents an object set (also called a domain); $F(U)$ represents all of the fuzzy set on U; $\forall A \in F(U)$, $A(u)$ is the membership function of A, V represents the factor set. If there is the optional $u \in U$, all the factors associated with u will be in V, then $(U, V]$ is called a left pair.

Definition 2.1 [2]. The left pair $(U, V]$ is given, $F \subset V$, then set family $\{X(f)\}_{f \in F}$ is called a factor space on U. If the following axiom is satisfied:

(1) $F = F(\vee, \wedge, c, 1, 0)$ is fully Boolean Algebra;
(2) $X(0) = \{\theta\}$, in which θ represents a null state;
(3) $\forall T \subset F$, if $(\forall s, t \in T)(s \neq t \Rightarrow s \wedge t = 0)$, then $X(\vee_{f \in T} f) = \prod_{f \in T} X(f)$.

F is called a factor set, $f \in F$ is called factor, $X(f)$ is the state space of the factor f, 1 is called full factor, $X(1)$ is called full space.

Definition 2.2 [2]. The concept group $C = \{\alpha, \beta, \gamma, \cdots\}$ is given. Its domain is recorded as U, and the factor family V is taken, then a left pair $(U, V]$ is made up of U and V. Then the factor set $F \subset V$ is taken, and F is sufficient for U. That is $\forall u_1, u_2 \in U$

is satisfied, $\exists f \in F$, and it makes that $f(u_1) \neq f(u_2)$, therefore the triplet $(U, C, F]$ or $(U, C, \{X(f)\}_{f \in F}]$ is a description frame of C.

If the description frame $(U, C, \{X(f)\}_{f \in F}]$ is given, and $\alpha \in C$ is taken, the extension of α is $A \in F(U), \forall f \in F$, noted: $f(A) : X(f) \to [0, 1]$, $x \to f(A)(x) = \max_{f(u)=x}\{A(u)\}$

$$f < A > \; : X(f) \to [0, 1], x \to f < A > (x) = \min_{f(u)=x}\{A(u)\}$$

Then the fuzzy sets $f(A)$ and $f < A >$ is called the representation extension of the concept α which is based on the strong (weak) extension principle in the representation theory domain $X(f)$.

Noted: $f^{-1}(f(A)) : U \to [0, 1], u \to f^{-1}(f(A))(u) = f(A)(x) = \max_{f(u)=x}\{A(u)\}$

$$f^{-1}(f < A >) : U \to [0, 1], u \to f^{-1}(f < A >)(u) = f < A > (x) = \min_{f(u)=x}\{A(u)\}$$

Then the fuzzy sets $f^{-1}(f(A))$ and $f^{-1}(f < A >)$ is called the feedback extension of the concept α regarding the factor f based on the strong (weak) extension principle.

Definition 2.3 [2]. The factor space is given, and let $f, g \in F$ and $f \geq g$ be true.

If B is any fuzzy set of $X(f)$, it will be noted that $\downarrow_g^f X(f) \to X(g)$, $B \to \downarrow_g^f B$, In which $(\downarrow_g^f B)(x) = \bigvee_{y \in X(f-g)} B(x, y)$, $x \in X(g)$, and $\downarrow_g^f$ is called the projection to g for f. If B is any fuzzy set of $X(g)$, it will be noted that $\uparrow_g^f X(g) \to X(f)$, $B \to \uparrow_g^f B$, In which $(\uparrow_g^f B)(x, y) = B(x)$, $(x, y) \in X(g) \times X(f - g)$, and $\uparrow_g^f$ is called the column extension to f for g.

Definition 2.4 [2]. The description frame $(U, C, \{X(f)\}_{f \in F}]$ is given. If $G \subset F$, and noted:

$$A[G] = \bigcap_{f \in G} f^{-1}(f(A)), A\langle G \rangle = \bigcup_{f \in G} f^{-1}(f < A >)$$

$A[G]$, $A\langle G \rangle$ will be called the inner (outer) envelope of feedback extension correlated to G for A.

Obviously, $A[G]$ and $A\langle G \rangle$ are actually the two approaches to A from the outside and the inside of the A. So there will be $A\langle G \rangle \subseteq A \subseteq A[G]$.

As a new extension of fuzzy sets, the concept of hesitant fuzzyset was proposed by Torra [13, 14].

Definition 2.5 [13, 14]. Let X be a given set, and the hesitant fuzzyset H on X is defined

$H = \{ < h_H(x), x > | x \in X\}$. In which $h_H(x)$ is the set of several different numerical values on the interval $[0, 1]$, which represented several possible degree that the element x in X belongs to the set H.

$h_H(x)$ is called the hesitant fuzzy cellular by Xu and Xia [15]. In order to facilitate the representation, in this paper, it is noted that $l(h_H(x))$ is the number of the values in the hesitant fuzzy cellular $h_H(x)$, and these values are arranged in ascending order. $h_H^{\sigma(j)}(x)$ represented the j-th value in $h_H(x)$.

Definition 2.6 [16, 17]. Let h_1 and h_2 be two hesitant fuzzy cellular. $l(h_1)$ 和 $l(h_2)$ represented the number of the values in h_1 和 h_2. $l_{\max} = \{l(h_1), l(h_2)\ \}$, The probability that h_1 is greater than h_2 is defined

$$p(h_1 \geq h_2) = \frac{1}{l_{\max}} \sum_{j=1}^{l_{\max}} f(h_1^{\sigma(j)}, h_2^{\sigma(j)}), \text{ In which} f(x,y) = \begin{cases} 1, & x > y, \\ 0.5, & x = y, \ ((x, y \geq 0)). \\ 0, & x < y. \end{cases}$$

3 New Representation of Concept Extension Based on Factor Space

Inspired by the concept of hesitant fuzzyset, the representation extension of the existing concept α in the representation theory domain $X(f)$, and the feedback extension of the concept α regarding the factor f, the definition of them has been expanded in this paper.

Definition 3.1. The description frame $(U, C, \{X(f)\}_{f \in F}]$ is given. The concept $\alpha \in C$ is taken, and the extension A of it is the fuzzy set on the domain U, noted:

$$f_H(A) : X(f) \rightarrow [0,1],\ x \rightarrow f_H(A)(x) = \bigcup_{f(u)=x} \{A(u)\} \tag{5}$$

$$f_H^{-1}(f_H(A)) : U \rightarrow [0,1], u \rightarrow f_H^{-1}(f_H(A))(u) = f_H(A)(x) \tag{6}$$

Then $f_H(A)$ is called the representation extension of the concept α in the representation theory domain $X(f)$, and $f_H^{-1}(f_H(A))$ is called the feedback extension of the concept α regarding the factor f.

By the Definition 3.1, we can easily know that $f_H(A)$ and $f_H^{-1}(f_H(A))$ are the hesitant fuzzysets. For example, let the concept α be 'Colds', and the extension of it is A, let the factor f be 'Temperature'. The domain U is for a part of people, u_1, u_2, u_3 is the three people in the domain U, and the temperature of them is the same, that is $f(u_1) = f(u_2) = f(u_3) = x = 38\,°\text{C}$. But u_1, u_2, u_3 all had a cold in different degree, that is $A(u_1) = 0.5$, $A(u_2) = 0.8$, $A(u_3) = 0.9$. According to the maximal extension principle, we can get that the degree of colds of 'the people of the temperature $x = 38\,°\text{C}$' belongs to $f^{-1}(f(A))(u) = 0.9$. However, according to the minimal extension principle, we can get that the degree of coldsof 'the people of the temperature $x = 38°$ C' belongs to $f^{-1}(f < A >\ )(u) = 0.5$. According to the representation extension and the feedback extension formed by the maximal(minimal) extension principle, the above is actually 'choosing one in the numerous' on the choice of the degree of membership, that will result in a loss of information. So the representation extension and the feedback extension is regarded as hesitant fuzzyset, that is more close to the real extension, while the utilization of information is higher. So using the definition 3.1 we can get that the degree of coldsof 'the people of the temperature $x = 38\,^{\circ}\mathrm{C}$' belongs to $f_H^{-1}(f_H(A))(u) = \{0.5, 0.8, 0.9\}$, apparently, the feedback extension we can get does not have information loss in the given domain.

Definition 3.2. For the two hesitant fuzzy cellular $h_{H_1}(x), h_{H_2}(x)$, if $l(h_{H_1}(x)) \leq l(h_{H_2}(x))$, then $m(h_{H_1}(x)) \leq m(h_{H_2}(x))$, in which $l(h_H(x))$ is the number of the values in the hesitant fuzzy cellular $h_H(x)$ and $m(h_{H_1}(x)), m(h_{H_2}(x))$ respectively represented the hesitant degree of the hesitant fuzzy cellular $h_{H_1}(x), h_{H_2}(x)$.

For example: $h_{H_1}(x) = \{0.3, 0.5\}, h_{H_2}(x) = \{0.3, 0.6, 0.8\}$, The membership degree that x belongs to H_1 is between 0.3 and 0.5. The membership degree that x belongs to H_2 is among 0.3, 0.6 and 0.5. Thus, the hesitant degree of $h_{H_2}(x)$ is greater than $h_{H_1}(x)$.

For the feedback extension of a concept, the lower the hesitant degree, the more close to the real extension. Based on that, the relationship between two hesitant fuzzy sets is defined as follows.

Definition 3.3. Let H_1, H_2 be the hesitant fuzzysets on domains U. If there is $h_{H_1}(u) \subseteq h_{H_2}(u)$ for $\forall u \in U$, then $H_1 \subseteq H_2$.

Principle 3.1. The description frame is given. The optional concept $\alpha \in C$ is taken, and its extension is the fuzzy sets A of U. If it is satisfied that $f \geq g$ for $\forall f, g \in F$, then $A \subset f_H^{-1}(f_H(A)) \subset g_H^{-1}(g_H(A))$.

Proof. For $\forall u \in U$, there is

$$\begin{aligned} f_H^{-1}(f_H(A))(u) &= f_H(A)(f(u)) = \bigcup_{f(u')=f(u)} \{A(u')\} \\ g_H^{-1}(g_H(A))(u) &= g_H(A)(g(u)) = \bigcup_{g(u')=g(u)} \{A(u')\} \\ f \geq g &\Rightarrow \{u' \in U | f(u') = f(u)\} \subseteq \{u' \in U | g(u') = g(u)\} \\ &\Rightarrow \bigcup_{f(u')=f(u)} \{A(u')\} \subseteq \bigcup_{g(u')=g(u)} \{A(u')\} \end{aligned}$$

According to the Definition 3.3, we can get that $f_H^{-1}(f_H(A)) \subset g_H^{-1}(g_H(A))$; The all-factor 1 is injection, then $1^{-1}(1(A)) = A$. There is the all-factor $1 \geq f$, then $A \subset f_H^{-1}(f_H(A)) \subset g_H^{-1}(g_H(A))$.

The proof is finished.

The Principle 3.1 showed that the more "big" (the more complex) the factor, the more close to the real extension the feedback extension.

Definition 3.4. Let h_1 and h_2 be the hesitant fuzzy cellular, $h = \{\gamma_1 \cup \gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\}$, and the elements in h are arranged in ascending order, $h^{\tau(i)} = \gamma_i$ represented the i-th element in h. Now some of their basic operations are given as follows:

(1) $h_1 \cap h_2 = \{h^{\tau(i)} | h^{\tau(i)} \in h\}$, in which $i = 1, 2, \ldots, \min\{l(h_1), l(h_2)\}$;
(2) $h_1 \cup h_2 = \{h^{\tau(i)} | h^{\tau(i)} \in h\}$,

In which, $i = l(h) - \max\{l(h_1), l(h_2)\} + 1, l(h) - \max\{l(h_1), l(h_2)\} + 2, \ldots, l(h)$.

Explanation: in fact, these operations are that all the elements in h_1, h_2 are put in the set h, then select the small ones and $\min\{l(h_1), l(h_2)$ of them constitutes the set $h_1 \cap h_2$, select the big ones and $\max\{l(h_1), l(h_2)\}$ of them constitutes $h_1 \cup h_2$.

The proposed operation by the Definition 3.4 is different from the literature [13, 14], and the operation is not only considering the change of membership degree, but also the degree of hesitation. For example: let $h_1 = \{0.2, 0.3, 0.5\}, h_2 = \{0.4, 0.5, 0.7, 0.8\}$, According to the above definition, we can get that $h_1 \cup h_2 = \{0.4, 0.5, 0.7, 0.8\}$, $h_1 \cap h_2 = \{0.2, 0.3, 0.4\}$.

The feedback extension provides the direct theoretical basis and operation method for the concept extension expression. Because of the complexity of all-factor, it is often not directly to get the representation extension $B(1)$ of the concept α in the complete representation theory domain $X(1)$. Similar to the literature [2], the complex factors are decomposed into the simple factors, then seeking out the representation extension of the concept α for these simple factors, the representation extension would 'synthesize'. The 'synthesis' method is in the definition as following:

Definition 3.5. The description frame $(U, C, \{X(f)\}_{f \in F}]$ is given. The optional concept $\alpha \in C$ is taken, and the extension is A. $G \subset F$ is taken, and the factors in G are independent of each other, then it is noted that $A[G] \triangleq \bigcap_{f \in G} f_H^{-1}(f_H(A))$. $A[G]$ is called the feedback extension G-envelope of A.

Notes:

(1) $A[G]$ is the hesitant fuzzyset. The feedback extension got from each factor by A, is obtained by the operation of definition 4;
(2) When $A[G]$ is degraded into the fuzzy set, $A[G] = A$.

4 The Hesitant Fuzzy Method of Decision-Making Based on Incomplete Information of Factor Space

The left pair $(U, V]$ was given, and it was specified that there was a relationship R between U and $V : R(u,f) = [0, 1]$. If the value of $R(u,f)$ was more than one, R would be called the hesitant fuzzy relation. If the factor $f \in V$ was seen as a mapping, it acted on an object $u \in U$ and a certain state $f(u) = x$ would be obtained, then $B(f)(x)$ could be a hesitant fuzzy element.

Let $\{X(f)\}_{f \in F}$ be a factor space on U. The triplet $S = (U, V, \{X(f)\}_{f \in F})$ is an incomplete information system, in which $U = \{u_1, u_2, \cdots, u_m\}$ is a non-empty finite object set, $F \subset V$ and $F = \{f_1, f_2, \cdots f_n\}$ is a non-empty finite factor set. Let $B(f_j)(f(u_i)) = h_{ij}$, h_{ij} be a hesitant fuzzy element, if $B(f_j)(f(u_i))$ is empty, we can use "h_{ij}^*" to represent the missing data.

4.1 Filled Methods of Missing Value

The performance value of the domain $U = \{u_1, u_2, \cdots, u_m\}$ under the factor f_j was divided into two categories: complete information set and incomplete information set, noted:

$$B^{+}(f_j)(f(u_i)) = h_{ij} = \{h_{ij} | i = r_1, r_2, \cdots, r_t\}$$

$$B^{*}(f_j)(f(u_i)) = h^{*}_{ij} = \{h^{*}_{ij} | i = r_{t+1}, r_{t+2}, \cdots, r_m\};$$

In which $r_1, r_2, \cdots, r_m$ was the arrangement of $1, 2, \cdots, m$.

For the convenience of research, it is required that a complete information value under this factor was selected to fill, for the incomplete information under the factor f_j. Then the filled methods of missing value were defined as follows:

$$h^{*}_{ij} = HFWA(h_{1j}, h_{2j}, \cdots, h_{r_t j}) = \overset{r_t}{\underset{k=1}{\oplus}} w_k h_{kj} = \bigcup\nolimits_{\gamma_1 \in h_{1j}, \gamma_2 \in h_{2j}, \cdots \gamma_{r_t} \in h_{r_t j}} \{\sum_{k=1}^{r_t} w_k \gamma_k\}. \quad (7)$$

In which, $w = \{w_1, w_2, \cdots, w_{r_t}\}$ was the weight vector of $h_{1j}, h_{2j}, \cdots, h_{r_t j}$. For convenience in computation, it could be taken that $w_1 = w_2 = \cdots = w_{r_t} = 1/r_t$.

For example: there were the information values $\{h_{1j}, h_{2j}, h^{*}_{3j}\}$ under the factor f_j, in which $h_{1j} = \{0.2, 0.3\}$, $h_{2j} = \{0.3, 0.4\}$. The missing value h^{*}_{3j} was filled using the complete information set $\{h_{1j}, h_{2j}\}$. From the formula(7), we can get that $h^{*}_{3j} = \{0.25, 0.3, 0.35\}$, in which 0.3 in h^{*}_{3j} repeated for two times.

Notes:

(1) The complete information set was regarded as a vector set, and the missing value was a combination consisted of one element from each vector. Then the elements of the new combination were weighted average, and there were many combinations, there were many elements in missing values. The missing value is a new hesitant fuzzy number.

(2) It could be considered that the opportunity to take all the values on the representation theory domain was equal, in a certain factor. So we could make the weight of each vector of the complete information set (vector set)equal. That was $w_1 = w_2 = \cdots = w_{r_t} = 1/r_t$.

4.2 The Hesitant Fuzzy Group Method of Decision-Making Based on the Incomplete Information

In the group decision-making problem, let $U = \{u_1, u_2, \cdots, u_m\}$ be the alternative set, in which m was the number of the alternatives. $F = \{f_1, f_2, \cdots, f_n\}$, in which F was the factors group related to decision making, and n was the number of the factors. $D = \{d_1, d_2, \cdots, d_s\}$ was the expert set, and s was the number of the experts. The decision maker $d_k \in D (k = 1, 2, \ldots, s)$ evaluated the alternative through a number of factors $F_k = \{f^{(k)}_1, f^{(k)}_2, \ldots, f^{(k)}_{l(F_k)}\}$, in which $F_k \subset F$, $l(F_k)$ represented the number of the factors in F_k. Let $w^{(k)} = (w_1^{(k)}, w_2^{(k)}, \ldots, w_{l(F_k)}^{(k)})$ $(k = 1, 2, \ldots, s)$ be the weight vector of the factors group $F_k = \{f_1^{(k)}, f_2^{(k)}, \ldots, f_{l(F_k)}^{(k)}\}$ considered by the decision maker d_k, and then the evaluation value of the alternative u_i by the decision maker $d_k (k = 1, 2, \ldots, s)$ was $f_j^{(k)}(u_i)$ under the factor $f_j \in F_k (j = 1, 2, \ldots, l(F_k))$, we could get the decision-making matrix $R^{(k)} = [f_j^{(k)}(u_i)]_{m \times l(F_K)} (k = 1, 2, \ldots, s)$.

Based on the above decision-making matrix, the specific decision-making steps have been given:

Step 1: According to the evaluation of the decision maker $d_k \in D (k = 1, 2, \ldots, s)$, we could get the hesitant fuzzy decision-making matrix $R^{(k)} = [B(f_j)(f_j^{(k)}(u_i))]_{m \times l(F_k)}$ $(k = 1, 2, \ldots, s)$, For convenience of representing, let $B(f_j)(f_j^{(k)}(u_i)) = h_{ij}^{(k)}$, and then $R^{(k)} = [h_{ij}^{(k)}]_{m \times l(F_k)} (k = 1, 2, \ldots, s)$, the fuzziness and uncertainty of the actual problem background often lead to the incomplete decision-making information, that is, there is a missing value in the decision-making matrix.

Step 2: The above hesitant fuzzy decision-making matrix was completely filled and weighted using the formula (7), and we could get the new decision-making matrix $R'^{(k)}$, that is: $R'^{(k)} = [h_{ij}'^{(k)}]_{m \times l(F_k)}$, in which $h_{ij}'^{(k)} = w_j^{(k)} \cdot h_{ij}^{(k)}$;

Step 3: According to $A[F_k](u_i) \triangleq \bigcap_{f_j \in F_k} f_j^{-1}(f_j(A))(u_i) = \bigcap_{f_j \in F_k} w_j^{(k)} \cdot B(f_j)(f_j^{(k)}(u_i)) = \bigcap_{f_j \in F_k} h_{ij}'^{(k)}$, aggregating the row vector information of the matrix $R'^{(k)}$, we could get the comprehensive preferences information $A[F_k](u_i)$ of the alternative u_i by the decision maker d_k.

Step 4: Using the union operation of the hesitant fuzzy set defined by definition4, synthesizing the preference information of the solution u_i from s decision makers, we can get that

$$A[G](u_i) = \bigcup_{k=1}^{s} A[F_k](u_i), \text{ in which } G = \bigcup_{k=1}^{s} F_k;$$

Step 5: Construct a comparison table of the hesitant fuzzy set. This comparison table was a square table with equal number of rows and columns, its rower and column were the object names in the domain $U = \{u_1, u_2, \cdots, u_m\}$, the element of the table was C_{ij}. Based on Definitions 2.5 and 2.6 we compute and get: $C_{ij} = p(A[G](u_i) \geq A[G](u_j))$;

Step 6: Compute and compare the row sum of the table $P_i = \sum_{j=1}^{m} C_{ij}$, as well as the column sum of the table $Q_i = \sum_{j=1}^{m} C_{ji}$, then we can obtain the score $S_i = P_i - Q_i$ of the alternative u_i. At last, we can determine the order of the alternative based on the score.

5 Examples Analysis

Let 3 experts (decision makers) be d_1, d_2, d_3, and they evaluated three sets of houses $U = \{u_1, u_2, u_3\}$ from price f_1, appearance f_2, traffic f_3, environment f_4 and area f_5. If the main factors considered by the experts d_1, d_2, d_3 were $F_1 = \{f_1, f_2, f_3\}, F_2 = \{f_2, f_3, f_4\}$ and $F_3 = \{f_3, f_4, f_5\}$, the evaluated result of each expert was given in the form of the preference value that was the hesitant fuzzy number, "h_{ij}^*" represented the missing values, the concrete results are shown in the table below (Tables 1, 2 and 3).

Table 1. Evaluation value of the expert d_1

U	f_1	f_2	f_3
u_1	{0.2,0.5,0.8}	{0.1,0.2,0.3}	{0.4,0.5,0.6}
u_2	h_{21}^*	{0.2,0.4,0.6}	{0.5,0.6,0.7}
u_3	{0.3,0.6,0.7}	{0.1,0.5}	{0.3,0.5}

Table 2. Evaluation value of the expert d_2

U	f_2	f_3	f_4
u_1	{0.5,0.7}	{0.3,0.7}	{0.2,0.5,0.9}
u_2	{0.6,0.8}	{0.1,0.3}	{0.1,0.6}
u_3	{0.3,0.6}	h_{33}^*	{0.3,0.7}

Table 3. Evaluation value of the expert d_3

U	f_3	f_4	f_5
u_1	{0.5,0.6}	{0.5,0.6}	{0.3,0.6}
u_2	{0.5,0.8}	{0.4,0.7}	{0.2,0.7}
u_3	{0.4,0.5}	{0.4,0.5,0.7}	h_{35}^*

Three decision makers $d_k(k = 1, 2, 3)$ gave different weights $w^{(1)} = (0.4, 0.3, 0.3), w^{(2)} = (0.2, 0.4, 0.4), w^{(3)} = (0.4, 0.4, 0.2)$, to the factors $F_k(k = 1, 2, 3)$ considered. According to the formula (7), completely fill the missing value in the table, and weight the above hesitant fuzzy value using weight information, the information after weight was shown in the following table (Tables 4, 5 and 6).

According to the step 3, the preference value of each alternative given by each decision maker:

Table 4. Evaluation value of the expert d_1

U	f_1	f_2	f_3
u_1	{0.08,0.2,0.32}	{0.03,0.06,0.09}	{0.12,0.15,0.18}
u_2	{0.1,0.16,0.18,0.22,0.24, 0.28,0.3}	{0.03,0.12,0.18}	{0.15,0.18,0.21}
u_3	{0.12,0.24,0.28}	{0.03,0.15}	{0.09,0.15}

Table 5. Evaluation value of the expert d_2

U	f_2	f_3	f_4
u_1	{0.2,0.14}	{0.12,0.28}	{0.08,0.2,0.36}
u_2	{0.12,0.16}	{0.04,0.12}	{0.04,0.24}
u_3	{0.06,0.12}	{0.08,0.12,0.16,0.2}	{0.12,0.28}

Table 6. Evaluation value of the expert d_3

U	f_3	f_4	f_5
u_1	{0.2,0.24}	{0.2,0.24}	{0.06,0.12}
u_2	{0.2,0.32}	{0.16,0.28}	{0.04,0.14}
u_3	{0.16,0.2}	{0.16,0.2,0.28}	{0.05,0.08,0.1,0.13}

$$A[F_1](u_1) = \{0.03, 0.06, 0.08\},\ A[F_1](u_2) = \{0.03, 0.1, 0.12\},\ A[F_1](u_3) = \{0.03, 0.09\};$$
$$A[F_2](u_1) = \{0.08, 0.12\},\ A[F_2](u_2) = \{0.04, 0.12\},\ A[F_2](u_3) = \{0.06, 0.08\};$$
$$A[F_3](u_1) = \{0.06, 0.12\},\ A[F_3](u_2) = \{0.04, 0.14\},\ A[F_3](u_3) = \{0.05, 0.08\}.$$

Using step 4, synthesize the preference information of the three decision makers to obtain the final preference value:

$$A[G](u_1) = \{0.06, 0.08, 0.12\},\ A[G](u_2) = \{0.1, 0.12, 0.14\},\ A[G](u_3) = \{0.06, 0.08, 0.09\}$$

in which $G = \bigcup_{k=1}^{3} F_k$;

According to step 5, the calculated scores of each solution are: $S_1 = -0.66$, $S_2 = 2S_3 = -1.34$;

According to the final scores, we can determine u_2 was the best.

6 Conclusion

In this paper, based on the theory of factors space the feedback extension of the concept was presented, regarding as hesitant fuzzy set, and the new definition of relation and operation was given among the hesitant fuzzy sets considering the hesitancy degree.

Based on this definition, the operation of extension envelopes and group decision making method under incomplete information was given, and through the example the above theories method was applied. The results show that this method can utilize more efficiently the data and information, the process of gathering information is simple, and the results are valid.

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Signed Total Domination and Mycielski Structure in Graphs

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Abstract. Let $G = (V, E)$ be a graph. The function $f : V(G) \to \{-1, 1\}$ is a signed total dominating function if for every vertex $v \in V(G)$, $\sum_{x \in N_G(v)} f(x) \geq 1$. The value of $\omega(f) = \sum_{x \in V(G)} f(x)$ is called the weight of f. The signed total domination number of G is the minimum weight of a signed total dominating function of G. In this paper, we initiate the study of the signed total domination numbers of Mycielski graphs and find some upper bounds for this parameter. We also calculate the exact value of the signed total domination number of the Mycielski graph when the underlying graph is a special graph.

Keywords: Signed total domination number · Mycielski construction

1 Introduction

All graphs considered throughout this paper are simple, finite, undirected and connected. For the terminology and notations not defined here, we refer the reader to [7]. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of a vertex v in graph G is $N_G[v] = N_G(v) \cup \{v\}$. Moreover, the *open* and *closed neighborhoods* of a subset $S \subseteq V(G)$ are $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) =| N_G(v) |$. A vertex $v \in V(G)$ is called an *odd* (*even*) vertex if $\deg_G(v)$ is odd (even). For a graph $G = (V, E)$, let V_o and V_e be the set of odd and even vertices, respectively. We denote the *maximum* degree of G with $\Delta(G)$ and its *minimum degree* with $\delta(G)$. A vertex is called *universal* if it is adjacent to all other vertices of a graph. In a complete graph, all vertices are universal.

For a function $f : V(G) \longrightarrow \{-1, 1\}$ and a subset S of $V(G)$, we define $f(S) = \sum_{x \in S} f(x)$. If $S = N_G(v)$ for $v \in V(G)$, then we denote $f(S)$ by $f[v]$. Let $C_f = \{v \in V(G) \mid f(v) \geq 1\}$. A *signed total dominating function* of G is a function $f : V(G) \longrightarrow \{-1, 1\}$ such that for all vertices v of G, $v \in C_f$. The *weight* of a signed total dominating function f is $\omega(f) = \sum_{v \in V(G)} f(v) = f(V(G))$.

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The *signed total domination number* (STDN), $\gamma_{st}(G)$, is the minimum weight of a signed total dominating function of G. A signed total dominating function of weight $\gamma_{st}(G)$ is called a $\gamma_{st}(G)$-*function*. For a signed total dominating function f of G we define $P_f = \{v \in V(G) \mid f(v) = 1\}$ and $M_f = \{v \in V(G) \mid f(v) = -1\}$.

The concept of the signed total domination number of a graph was proposed by Zelinka [8]. Henning in [4] proved that the problem of determining the signed total domination number for general graphs is NP-hard.

For a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$, let $U = \{u_1, u_2, \cdots, u_n\}$ be a disjoint copy of $V(G)$ and let w be a new vertex. The *Mycielski graph* $\mu(G)$ of G is defined as follows:

$$V(\mu(G)) = V(G) \cup U \cup \{w\},$$

$$E(\mu(G)) = E(G) \cup \{v_i u_j \mid v_i v_j \in E(G)\} \cup \{wu_i \mid 1 \leq i \leq n\}.$$

The vertex w is called the *root* of $\mu(G)$ and the vertex $u_i = c(v_i)$ is called the *twin* of the vertex v_i, $i = 1, 2, \cdots, n$. The Mycielski graph of a graph G was introduced by Mycielski in order to construct triangle-free graphs with an arbitrary large chromatic number [5]. In recent years, there have been results reported on Mycielski graphs related to various domination parameters. In [1], it was proved that $\gamma(\mu(G)) = \gamma(G) + 1$. This shows that the domination number of a Mycielski graph can exceed the domination number of its underlying graph G, but Ghameshlou et al. proved such a result is not true for signed domination number of Mycielski graphs [2,6].

In this paper, we initiate the study of the signed total domination numbers of Mycielski graphs. In Sect. 2, we present some preliminary results on Mycielski graphs and their signed total domination numbers. In Sect. 3, we calculate the exact value of signed total domination number of a Mycielski graph, whose underlying graph has at least one universal vertex. Then we calculate the exact values of $\gamma_{st}(\mu(G))$ when G is a star, a wheel, a fan, a Dutch windmill or a complete graph. In Sect. 4, we prove that if $\gamma_{st}(G) \geq 0$, then $\gamma_{st}(\mu(G)) \leq 2\gamma_{st}(G) + 1$, otherwise $\gamma_{st}(\mu(G)) \leq \gamma_{st}(G) + 3$. Finally, in Sect. 5, we calculate $\gamma_{st}(\mu(G))$ when G is a cycle, a path or a complete bipartite graph. It is worth to note that there are graphs G, such as $K_{m,n}$, when $m = 1$ or m and n are both odd, with $\gamma_{st}(\mu(G)) < \gamma_{st}(G)$.

2 Preliminary Results

Proposition 1. Let f be a signed total dominating function of $\mu(G)$. Then,

$$\omega(f) \equiv 1 \pmod 2.$$

Proposition 2. Let f be a signed total dominating function of $\mu(G)$.

1. If G has at least one vertex of degree 1, then for Mycielski graph $\mu(G)$, $w \in P_f$.
2. For $v \in V(\mu(G))$ if v is an even vertex, then $f[v] \geq 2$ while if v is an odd vertex, then $f[v] \geq 1$.

Ghameshlou et al. proved the following results for signed domination number of Mycielski graphs.

Theorem 1 [2]. *Let G be a graph of order n. If G has at least an universal vertex, then* $\gamma_s(\mu(G)) \geq 3$.

Corollary 1 [2]. For every graph $G \in \{K_n, K_{1,n}, D_3^n, W_n, F_n\}$, $\gamma_s(\mu(G)) = 3$.

Theorem 2 [2]. *For any graph G of order n,*

$$\gamma_s(\mu(G)) \leq \begin{cases} \gamma_s(G) + 2 & \text{if } \gamma_s(G) \leq -1, \\ 2\gamma_s(G) + 1 & \text{if } \gamma_s(G) \geq 0. \end{cases}$$

Furthermore, for $\gamma_s(G) \geq 0$ the bound is sharp for K_n when n is odd and for $\overline{K_n}$.

Theorem 3 [2]. *For every cycle C_n of order n,*

$$\gamma_s(\mu(C_n)) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 \pmod 8, \\ \frac{n+5}{2} & \text{if } n \equiv 1, 5 \pmod 8, \\ \frac{n}{2} + 2 & \text{if } n \equiv 2, 6 \pmod 8, \\ \frac{n+7}{2} & \text{if } n \equiv 3 \pmod 8, \\ \frac{n}{2} + 3 & \text{if } n \equiv 4 \pmod 8, \\ \frac{n+3}{2} & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Theorem 4 [2]. *For a path P_n, $n \geq 8$,*

$$\gamma_s(\mu(P_n)) = \begin{cases} \frac{n+5}{2} & \text{if } n \equiv 1 \pmod 8, \\ \frac{n+4}{2} & \text{if } n \equiv 2 \pmod 8, \\ \frac{n+3}{2} & \text{if } n \equiv 3, 7 \pmod 8, \\ \frac{n+2}{2} & \text{if } n \equiv 0, 4 \pmod 8, \\ \frac{n+1}{2} & \text{if } n \equiv 5 \pmod 8, \\ \frac{n}{2} & \text{if } n \equiv 6 \pmod 8. \end{cases}$$

Theorem 5 [2]. *For complete bipartite graph $K_{m,n}$ with $m \geq n \geq 2$,*

$$\gamma_s(\mu(K_{m,n})) = 5.$$

Theorem 6 [6]. *If G is a graph of order n, then*

$$\gamma_s(\mu(G)) \geq \begin{cases} \left\lceil \frac{(2n+1)(\delta(G)+1) - 2\Delta(G)(n-1) - n_o}{\Delta(G) + \delta(G) + 1} \right\rceil & \text{if } n \text{ is odd}, \\ \left\lceil \frac{(2n+1)(\delta(G)+1) - \Delta(G)(2n-1) - n_o}{\Delta(G) + \delta(G) + 1} \right\rceil & \text{if } n \text{ is even}. \end{cases}$$

Furthermore, this bound is sharp.

Theorem 7 [6]. *If G is a r-regular graph, then*

$$\gamma_s(\mu(G)) \geq \begin{cases} \dfrac{2n+2r+1}{2r+1} & \text{if } n, r \text{ are even,} \\ \dfrac{n+2r+1}{2r+1} & \text{if } n \text{ is even, } r \text{ is odd,} \\ \dfrac{2n+3r+1}{2r+1} & \text{if } n, r \text{ are odd.} \end{cases}$$

Moreover, this bound is sharp for complete graph K_n.

3 Graphs with Universal Vertices

In this section, we show that the signed total domination number of a Mycielski graph, whose underlying graph has at least one universal vertex, is at least 3.

Theorem 8. *Let G be a graph of order n. If G has at least an universal vertex, then*

$$\gamma_{st}(\mu(G)) \geq 3.$$

Corollary 2. For every complete graph K_n,

$$\gamma_{st}(\mu(K_n)) = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 5 & \text{if } n \text{ is even.} \end{cases}$$

Corollary 3. For every graph $G = \{K_{1,n}, K_3^m, W_n, F_n\}$, $\gamma_{st}(\mu(G)) = 5$.

4 A Relation Between $\gamma_{st}(G)$ and $\gamma_{st}(\mu(G))$

Theorem 9. *For any graph G of order n,*

$$\gamma_{st}(\mu(G)) \leq \begin{cases} 2\gamma_{st}(G) + 1 & \text{if } \gamma_{st}(G) \geq 0, \\ \gamma_{st}(G) + 3 & \text{if } \gamma_{st}(G) \leq -1. \end{cases}$$

Furthermore, for $\gamma_{st}(G) \geq 0$ the bound is sharp for F_n when n is odd, $K_{m,n}$ when m and n are odd, and K_n.

5 Cycles, Paths and Complete Bipartite Graphs

In this section we find the signed total domination number of $\mu(G)$ when G is a cycle, a path, or a complete bipartite graph.

Theorem 10. *For every cycle C_n of order n,*

$$\gamma_{st}(\mu(C_n)) = \begin{cases} n+1 & \text{if } n \equiv 0, 2 \pmod 4, \\ n & \text{if } n \equiv 1, 3 \pmod 4. \end{cases}$$

Theorem 11. *If $G = P_n$, then*

$$\gamma_{st}(\mu(P_n)) = \begin{cases} n+1 & \text{if } n \equiv 0 \pmod 4, \\ n+2 & \text{if } n \equiv 1, 3 \pmod 4, \\ n+3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Theorem 12. *For complete bipartite graph $K_{m,n}$ with $m \geq n \geq 2$,*

$$\gamma_{st}(\mu(K_{m,n})) = 5.$$

6 Conclusion

Comparing the results presented here shows that there are some underlying graphs G of order n which can be generalized to Mycielski graph $\mu(G)$ of order $2n+1$ such that $\gamma_{st}(\mu(G)) \leq \gamma_{st}(G)$; for instance, if $G \in \{K_{1,n}, K_3^m\}$, then $\gamma_{st}(\mu(G)) \leq \gamma_{st}(G)$.

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An Arbitrated Quantum Signature Scheme Based on W States

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Abstract. Arbitrated quantum signature(AQS) is a cryptographic scenario. There are three participants in this scheme. Sender(signer) Alice generates the signature of a message. Receiver(verifier) Bob verifies the signature. A trusted arbitrator helps Bob verify the signature. In this paper, we propose an arbitrated quantum signature scheme with W states. The W states are used for quantum signature and verification. The W states have stronger robustness than the GHZ states in the loss of the quantum bits. Finally, we also discuss its security against forgery and disavowal.

Keywords: Quantum cryptography · Quantum signature · Arbitrated quantum signature · W states

1 Introduction

Quantum cryptography is new cross subject with the combination of classic cryptography and quantum information. It is a new type of cryptographic system that uses quantum effects to realize the information exchange of unconditional security. The ideology of quantum cryptography can be traced back to the earliest Wiesner Stephen article in 1983 [1]. Bennet et al. designed the first quantum cryptography scheme named BB84 [2]. Since then, quantum cryptography has developed rapidly. Quite a few branches of quantum cryptography have been pointed out, including quantum key distribution(QKD) [3–7], quantum secure direct communication(QSDC) [8–11], quantum secret sharing(QSS) [12–15] and so on.

The principle of quantum signature is a combination of quantum theory and the principle of digital signature. Gotteman et al. [16] and Buhrman et al. [17] proposed quantum digital signatures in 2001. Zeng and Keitel proposed and designed the first arbitration quantum signature scheme by using the classical signature and the entanglement of the Greenberger-Horne-Zeilinger(GHZ) triplet states [18]. Li et al. modified the signature of Zeng and Keitel by using Bell states instead of GHZ states, which is more efficient and more convenient [19]. Zou and Qiu proposed an AQS scheme with a public board which can avoid being disavowed for the integrality of the signature by Bob [20]. With the continuous

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development and application of the arbitration quantum signature, many practical quantum signature protocols have been put forward, such as quantum proxy signature [21,22] ,quantum group signature [23,24], quantum blind signature [25,26], quantum multi signature [27,28], etc.

In 2000, D$\ddot{u}$r et al. proposed a new entangled state, and found that the W states have stronger robustness than the GHZ states in the loss of the quantum bits [29]. In the case of the loss of particles, the W states can maintain the quantum entanglement properties well. In this paper, we propose an arbitrated quantum signature scheme based on W states with public board. And we also discuss its security against forgery and disavowal.

This paper is arranged as follows. In Sect. 2, we introduce the general principle we demand for this AQS scheme. In Sect. 3, we describe the basic scheme including an initial phase, a signing phase and a verifying phase. In Sect. 4, we make security analyses on the proposed scheme to show neither to be disavowed by the signatory nor to be deniable for the receiver. In Sect. 5, we give a brief conclusion.

2 Preliminaries

There are four Bell basis shown as below

$$\begin{aligned} |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned} \tag{1}$$

There are three participants in the protocol, the signer Alice, the receiver Bob and the arbitrator Trent. Alice need to sign the message $|P\rangle$ with a appropriate signature $|S\rangle$. We assume n qubits in the string, such that $|P\rangle = (|p_1\rangle, |p_2\rangle, \cdots, |p_n\rangle)$. Any qubit $|p_i\rangle$ can be expressed as below

$$|p_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle \tag{2}$$

where α_i, β_i are complex numbers with $|\alpha_i|^2 + |\beta_i|^2 = 1$. And $|P\rangle$ can be known or unknown. In advance, three participants share a three-particle W state

$$|\varphi\rangle_{ATB} = \frac{1}{2}(|000\rangle + |110\rangle + |101\rangle + |011\rangle)_{ATB} \tag{3}$$

where the subscripts A correspond to Alice, T correspond to Trent and B correspond to Bob. Alice implements a Bell measurement on $|p_i\rangle$ and the particle she owns in W state, the system is expressed as follows

$$\begin{aligned}|\Psi\rangle_{iATB} = |p_i\rangle \otimes |\varphi\rangle_{ATB} \\ =\frac{1}{2\sqrt{2}}\{|\phi^+\rangle_A[\alpha_i(|00\rangle + |11\rangle)_{TB} + \beta_i(|10\rangle + |01\rangle)_{TB}] \\ + |\phi^-\rangle_A[\alpha_i(|00\rangle + |11\rangle)_{TB} - \beta_i(|10\rangle + |01\rangle)_{TB}] \\ + |\psi^+\rangle_A[\alpha_i(|10\rangle + |01\rangle)_{TB} + \beta_i(|00\rangle + |11\rangle)_{TB}] \\ + |\psi^-\rangle_A[\alpha_i(|10\rangle + |01\rangle)_{TB} - \beta_i(|00\rangle + |11\rangle)_{TB}]\}\end{aligned} \tag{4}$$

where $|\phi^+\rangle_A, |\phi^-\rangle_A, |\psi^+\rangle_A, |\psi^-\rangle_A$ represent the Bell states in Eq. (1). At present, Trent uses $\{|0\rangle, |1\rangle\}$ in the basis to implement a single-measurement, and sends the outcomes to Bob. Then, Bob can apply a proper unitary operation to recover the message.

Suppose Alice's measurement result is $|\phi^+\rangle_A$. After the Trent's measurement, the particles of Trent and Bob collapse into the state as follows

$$|0\rangle_T(\alpha_i|0\rangle + \beta_i|1\rangle)_B + |1\rangle_T(\alpha_i|1\rangle + \beta_i|0\rangle)_B \tag{5}$$

If Trent's measurement result is $|0\rangle$, Bob's particle will be $\alpha_i|0\rangle + \beta_i|1\rangle$. Bob can use local unitary operation I to recover the message $|p_i\rangle$. If Trent's measurement result is $|1\rangle$, Bob's particle will be $\alpha_i|1\rangle + \beta_i|0\rangle$. Bob can use unitary operation σ_x to recover the message $|p_i\rangle$, where

$$\begin{aligned} I &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ \sigma_x &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ i\sigma_y &= |0\rangle\langle 1| - |1\rangle\langle 0| \\ \sigma_z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned} \tag{6}$$

All possibilities of the scheme are shown in Table 1. $|M_A\rangle$ means Alice's measurement results in Table 1. $|M_T\rangle$ means Trent's measurement result. $|\phi_B\rangle$ means Bob's collapse state and U_B means the unitary operation which Bob needs to recover the Alice's message.

Table 1. Relation between the local unitary operations and measurement results

$\lvert M_A\rangle$	$\lvert M_T\rangle$	$\lvert\phi_B\rangle$	U_B
$\lvert\phi^+\rangle_A$	$\lvert 0\rangle_T/\lvert 1\rangle_T$	$\alpha\lvert 0\rangle + \beta\lvert 1\rangle/\alpha\lvert 1\rangle + \beta\lvert 0\rangle$	I/σ_x
$\lvert\phi^-\rangle_A$	$\lvert 0\rangle_T/\lvert 1\rangle_T$	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle/\alpha\lvert 1\rangle - \beta\lvert 0\rangle$	$\sigma_z/i\sigma_y$
$\lvert\psi^+\rangle_A$	$\lvert 0\rangle_T/\lvert 1\rangle_T$	$\alpha\lvert 1\rangle + \beta\lvert 0\rangle/\alpha\lvert 0\rangle + \beta\lvert 1\rangle$	σ_x/I
$\lvert\psi^-\rangle_A$	$\lvert 0\rangle_T/\lvert 1\rangle_T$	$\alpha\lvert 1\rangle - \beta\lvert 0\rangle/\alpha\lvert 0\rangle - \beta\lvert 1\rangle$	$i\sigma_y/\sigma_z$

3 Arbitrated Quantum Signature Based on W States

There are three participants in the protocol, the signer Alice, the receiver Bob and the arbitrator Trent. Trent is absolutely trusted by Alice and Bob. The two sides share classical keys with arbitrator respectively. The key is stored by the communication terminal, which can be used for a long time. We also use public board to avoid being disavowed by Bob. The presented scheme includes three phases, initializing phase, signing phase, and verifying phase.

3.1 Initializing Phase

Step $I1$. Alice shares the secret keys K_A with arbitrator Trent through the quantum key distribution [3–7], which were proved to be unconditionally secure [7,30]. Similarly, Bob shares the secret keys K_B with Trent.

Step $I2$. Trent generates n W triplet states $|\varphi\rangle_{ATB} = (|\varphi_1\rangle, |\varphi_2\rangle, \cdots, |\varphi_n\rangle).|\varphi_i\rangle$ is the same as Eq.(3).

$$|\varphi_i\rangle_{ATB} = \frac{1}{2}(|000\rangle + |110\rangle + |101\rangle + |011\rangle)_{ATB} \tag{7}$$

where the subscripts A, T and B correspond to Alice, Trent and Bob. Trent distributes corresponding particles to Alice and Bob.

Step $S1$. Alice need to sign a qubit string $|P\rangle = (|p_1\rangle, |p_2\rangle, \cdots, |p_n\rangle)$ related to the message with $|p_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$. Alice prepares three copies of $|P\rangle$ necessarily. Then, Alice uses four unitary operators on the $|P\rangle$ for local operation.

$$|P'\rangle = \sigma|P\rangle = (\sigma_1|p_1\rangle, \sigma_2|p_2\rangle, \cdots, \sigma_n|p_n\rangle) \tag{8}$$

where $\sigma_i \in \{I, \sigma_x, i\sigma_y, \sigma_z\}, i = 1, 2, \cdots, n$. Here notice that $|P'\rangle$ return to the original states perfectly because of Hermitian conjugate operators of unitary operators, while measurement operations are not usually reversible.

Step $S2$. Alice transforms the qubit string $|P'\rangle$ into a secret qubit string $|R_A\rangle$ in terms of the key K_A.

$$|R_A\rangle = E_{K_A}|P'\rangle \tag{9}$$

For example, assume that the key K_A is related to a collection of unitary operators $R_{K_A} = (R^1_{K^1_A}, R^2_{K^2_A}, \cdots, R^n_{K^n_A})$. If $R^i_{K^i_A} = 0$, Alice applies the unitary operation σ_x, namely, $R^i_{K^i_A} = \sigma_x$. If $R^i_{K^i_A} = 1$, Alice applies the unitary operation σ_z, namely, $R^i_{K^i_A} = \sigma_z$. So $|R_A\rangle = R_{K_A}(P) = (|r_1\rangle, |r_2\rangle, \cdots, |r_n\rangle)$ with $|r_i\rangle = M^i_{K^i_A}(p_i)$.

Step $S3$. Alice combines each secret message state $|P'\rangle$ and the W states. Then, she implements a Bell measurement on her particles. It shows in Eq.(4). And she can obtain $|M_A\rangle = (|M^1_A\rangle, |M^2_A\rangle, \cdots, |M^n_A\rangle)$, where $|M^i_A\rangle$ represents one of the four Bell states in Eq.(1).

Step $S4$. Alice generates the signature $|S'\rangle = E_{K_A}(|M_A\rangle, |R_A\rangle)$ of the message $|P'\rangle$ with the secret key K_A by using the quantum one-time pad algorithm.

Step $S5$. Alice transmits the signature $|S'\rangle$ and $|P'\rangle$ to Bob.

3.2 Verifying Phase

Step $V1$. Bob encrypts $|S'\rangle$ and $|P'\rangle$ with the secret key K_B and sends the resultant outcomes $|Y_B\rangle = E_{K_B}(|S'\rangle, |P'\rangle)$ to the arbitrator Trent.

Step $V2$. Trent decrypts with K_B and gets $|S\prime\rangle$ and $|P'\rangle$. Then he decrypts $|S'\rangle$ with K_A and gets $|M_A\rangle$ and $|R_A\rangle$. Trent encrypts $|P'\rangle$ by using K_A and gets $|R'_A\rangle$. The operation is same as Alice in Step $S2$. Then Trent compares $|R_A\rangle$ with $|R'_A\rangle$ through swap [17]. If $R_A\rangle = |R'_A\rangle$, Trent sets the verification parameter $r = 1$; otherwise, he sets $r = 0$.

Step $V3$. Trent implements a measurement in the basis $\{|0\rangle, |1\rangle\}$ and obtains $|M_T\rangle = (|M_T^1\rangle, |M_T^2\rangle, \cdots, |M_T^n\rangle)$. All possibilities of the measurement results are shown in Table 1.

Step $V4$. Trent sends the encrypted results $|Y_T\rangle = E_{K_B}(|S'\rangle, |P'\rangle, |R'_A\rangle, |M_T\rangle, r)$ to Bob.

Step $V5$. Bob decrypts $|Y_T\rangle$ and gets $|S'\rangle, |P'\rangle, |R'_A\rangle, |M_T\rangle$ and r. If $r = 0$, obviously the signature has been forged and Bob rejects it directly. If $r = 1$, Bob goes on the next step.

Step $V6$. Bob combines the $|R'_A\rangle$ and $|M_T\rangle$ and implements the corresponding unitary operation according to Table 1. Bob obtains $|P'_B$. He makes comparisons between $|P'_B\rangle$ and $|P'\rangle$. This method is still swap [17]. If $|P'_B\rangle \neq |P'\rangle$, Bob rejects the signature; otherwise he informs Alice by the public board to publish σ, which Alice used in Eq.(8).

Step $V7$. Alice publishes σ by the public board.

Step $V8$. Bob gets back $|P\rangle$ from $|P'\rangle$ and holds $|S\rangle = (|S'\rangle, \sigma)$ as Alice's signature for quantum message $|P\rangle$.

The communications in this AQS scheme are described in Fig. 1.

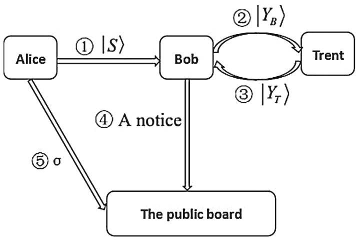

Fig. 1. The communications of the AQS scheme

4 Security Analysis and Discussion

A secure quantum signature scheme should satisfy two requirements: the signature should not be forged by the attacker(including the malicious receiver) and the signature should not be disavowed by the signatory and the receiver. We discuss security of the proposed AQS scheme to against the two attacks.

4.1 Impossibility of Forgery

If the attacker Eve tries to forge Alice's signature $|S'\rangle = E_{K_A}(|M_A\rangle, |R_A\rangle)$ for his own benefit, she has to know Alice's secret keys K_A. However, this is impossible due to the unconditionally security of quantum key distribution [7,30]. Besides, the use of quantum one-time pad algorithm enhances the security. Subsequently the parameter r used in verifying phase will not pass the test.

In the worse situation, for instance, the secret key is exposed to attacker, attacker still cannot forge the signature, since she cannot create appropriate $|M_A\rangle$ and $|M_T\rangle$. Bob would find such forgery, because the further verification about $|P'_B\rangle = |P'\rangle$ could not hold without the correct $|M_A\rangle$ and $|M_T\rangle$.

If the malicious receiver Bob wants to forge Alice's signature $|S'\rangle = E_{K_A}(|M_A\rangle, |R_A\rangle)$ for his own sake, he also should know Alice's secret K_A. It's also impossible because of the unconditionally security of quantum key distribution.

4.2 Impossibility of Disavowal by Signatory and Receiver

Suppose that Alice disavows her signature for her own benefits. In this case, the arbitrator Trent can confirm that Alice has signed the message since Alice's initial secret key k_A in the signature $|S'\rangle = E_{K_A}(|M_A\rangle, |R_A\rangle)$. Thus Alice cannot deny signing the message$|P\rangle$.

Similarly, suppose Bob repudiates the receipt of the signature. Then Trent also can confirm that Bob has received the signature since he needs the assistance of Trent to verify the signature. And if Bob wants to deny the signature by saying $|P'_B\rangle \neq |P'\rangle$, he cannot get σ to recover the message $|P\rangle$. This means that Bob cannot disavow the signature.

5 Conclusion

We have investigated an AQS based on W states in three phases, including initialing phased, signing phase and verifying phase. In the case of the loss of particles, the W states can maintain the quantum entanglement properties well. To avoid being disavowed by Bob, Bob has to ask Alice to publish the encryption key σ which means Bob has no chance to repudiate the signature.

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Optimal Stochastic Dynamic Control of Spatially Distributed Interdependent Production Units

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Abstract. Stochastic dynamic programming, SDP, is often the optimal method. SDP can be extended to handle very large dimensionality in the decision space, as long as the dimensionality of the state space is not too large, since SDP can be combined with linear or quadratic programming subroutines for every state and stage. When the number of decision variables is large and the optimal decisions are dependent on detailed information in a state space of large dimensionality, SDP cannot be applied. Then, optimal control functions for local decisions may be defined and the parameters can be determined via stochastic full system simulation and multidimensional regression analysis. This paper includes an approach to determination of all local decisions based on locally relevant state space information within stochastic dynamic and spatially explicit production. The expected present value of all harvests, over time and space, in a forest area, is maximized. Each tree is affected by competition from neighbor trees. The harvest decisions, for each tree, are functions of the price in the stochastic market, the dimensions and qualities of the individual trees and the local competition. The expected present value of the forest is an increasing function of the level of price risk.

Keywords: Stochastic dynamic control · Spatial optimization

1 Introduction

The ambition of this study is to develop a general method for optimization of stochastic and dynamic decision problems with spatial dependencies that cannot be neglected and where the need to use a multidimensional state space in high resolution makes it computationally and economically impossible to apply the otherwise relevant method stochastic dynamic programming. Applications can be found in most sectors of the economies. One of the most obvious cases, where useful and statistically estimated functions already exist, is the forest sector. We start with a forest area with 1000 trees of different sizes, as shown in Fig. 1. The initial locations and sizes of the trees are simulated.

The problem is to determine an adaptive control function to be used in this forest, giving the maximum of the total expected present value of all activities over time. The annual increment of each tree is a function of tree size and competition from neighbor

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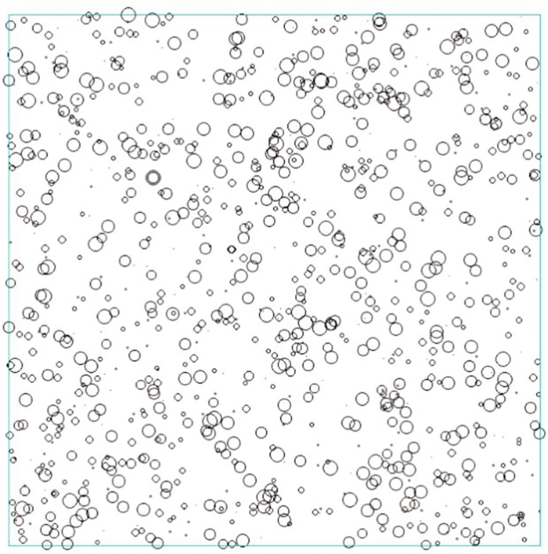

Fig. 1. Spatial map of initial conditions at t = −1 (years from the present time). The locations of the circle centers are the locations of the trees. The circle diameters are proportional to the tree diameters. The square represents one hectare (100 m * 100 m).

trees. The different trees have different wood qualities, initially randomly assigned to the individuals. The market value of a tree is a function of size, quality and stochastic price variations. The variable harvesting cost of a tree is size dependent. Every five years, the trees in the forest are inspected. Then, depending on market prices, tree sizes, competition, quality etc., it is possible that some or many trees are harvested. The optimized control function is used to make all of these decisions. Figure 2 shows the structure of the forest directly after optimal harvesting at t = 0. Obviously, a considerable number of large trees have been removed. Many new seedlings are however found on the land, in random positions. The trees continue to grow and Fig. 3 illustrates the situation 35 years later. 69 years after the first harvest, trees of considerable sizes exist (Fig. 4). The total number of large trees in year 69 is however much lower than before the harvest during year 0. Several large trees are harvested in year 70 (Fig. 5). This type of stochastic dynamic and spatial forest development is sustainable. Furthermore, there are always trees in the forest. We have a system of “optimal continuous cover forestry”.

Lohmander [1] describes several alternative methods to optimize forest management decisions at higher levels. Lohmander and Mohammadi [2] determine optimal harvest levels in beech forests in Iran, using stochastic dynamic programming. Then, however, the tree selection decisions were never analyzed.

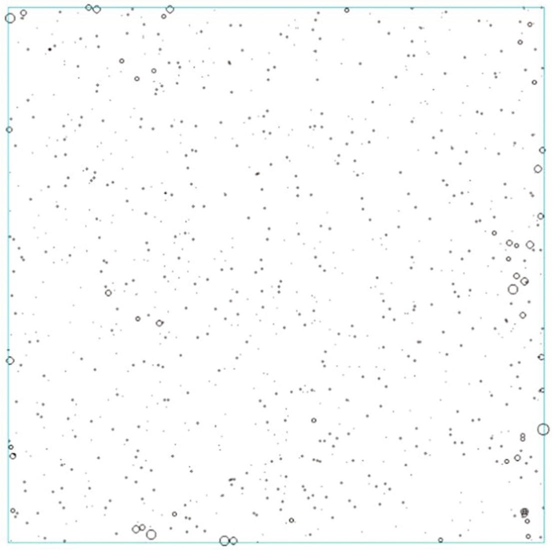

Fig. 2. The state after the first application of the optimized control function at t = 0. Most of the largest trees have been removed.

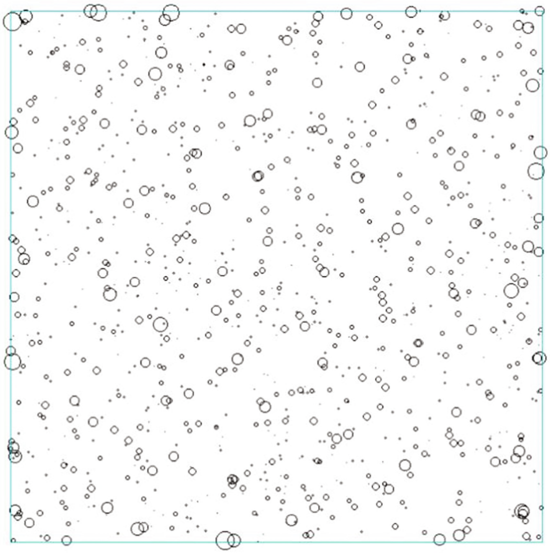

Fig. 3. The forest at t = 35.

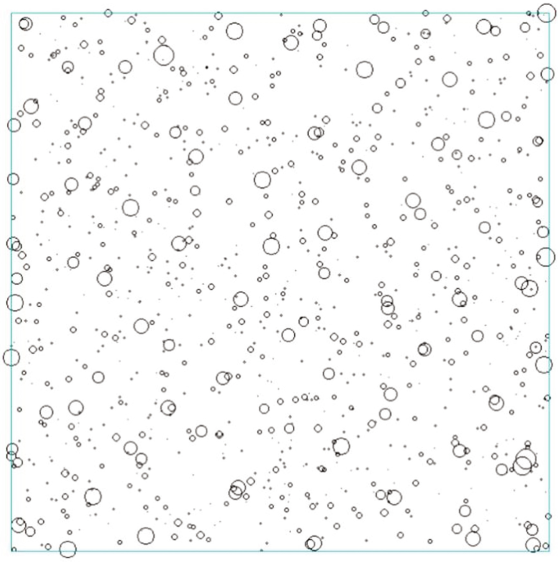

Fig. 4. The forest at t = 69.

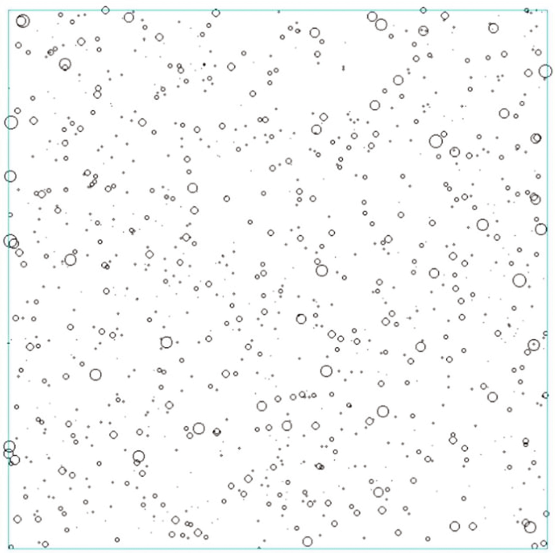

Fig. 5. The forest at t = 70.

2 Analysis

The optimal decisions for each tree, i, at time t, is determined by the diameter limit function $d_L(i,t)$. If the diameter is larger than the diameter limit, then the tree should be harvested. Otherwise, it should be left for continued production.

$$d_L(i,t) = d_0 + d_c C(i) + d_q Q(i) + d_p \Delta P(t) \tag{1}$$

The parameters $\left(d_0, d_c, d_q, d_p\right)$ are optimized in this study. In the graphs and software, they are denoted (dlim_0, dlim_c, dlim_q and dlim_p).

$(C(i), Q(i), \Delta P(t))$ denote competition index for tree i, quality of tree i and the stochastic deviation of the market price from the expected price, at time t. The stochastic price deviations are i.i.d. and have uniform pdf on the interval -10 to +10 EURO/cubic meter.

The objective function is the total expected present value of all revenues minus all costs from year 0 until year 200. The real rate of interest is set to 3%. The computer model includes functions for tree height as a function of diameter, functions used in tree volume calculations, set up costs, tree size dependent revenues and variable harvesting costs etc.

The trees grow according to a modified version of the function reported by Schütz [3]. The modification is that in [3], competition is assumed to come from all parts of the forest area, also far away from the individual tree. In the function applied in this new analysis, only competition from trees at distances ten meters or closer, is considered. Furthermore, in the Schütz function, each tree is only affected by competition from trees with larger diameters. In the present study, also competition from trees with smaller diameters is considered. However, it is probably the case that trees with smaller diameter give a lower degree of competition. The motivation for the new function, used here, is that competition for light, water and nutrients, obviously is stronger from neighbor trees than from trees far away. Furthermore, also smaller trees use some of the available light, water and nutrients. $C(i)$ is now expressed as the basal area per hectare of larger competing trees plus the basal area of smaller competing trees divided by 2 (all within the 10 m radius circle). In future studies, the competition function should be estimated with locally relevant data.

$$I(i) = b_0 + b_1 LN(d(i)) + b_2 (C(i))^3 \tag{2}$$

$I(i)$ is the diameter increment of tree i and $d(i)$ is the diameter. (b_0, b_1, b_2) is a set of empirically estimated parameters, published by Schütz [3], for beech in Germany.

The optimization of the total expected present value, via the parameters of the adaptive control function, contained the following steps: A software code was constructed and tested in QB64. The objective function was estimated for a set of combinations of the control function parameters $\left(d_0, d_c, d_q, d_p\right)$. For this purpose, a four dimensional loop with alternative parameter values was run. Preliminary iterative studies were first made to determine interesting parameter intervals. Then, a 3 * 3 * 3 * 4 loop was used, which gave 108 parameter combinations. For each parameter combination, the total expected present value during 200 years was estimated for 10 different forest areas of one hectare, each with 1000 initial random trees. That analysis took approximately 8 h on an Acer Aspire V personal computer with an Intel Core i5 processor. Next, the parameter values of (d_c, d_q) determined in the "108-loop", were considered optimal and fixed. A more detailed analysis, with higher resolution, of the parameters (d_0, d_p) was made.

3 Main Results

The adaptive control function parameters (d_0, d_c, d_q, d_p) were determined in a general loop. 108 combinations were evaluated. This is the adaptive control function:

$$d_{L,a}(i,t) = 0.60 - 0.0030\,C(i) + 0.020\,Q(i) - 0.020\,\Delta P(t) \tag{3}$$

The optimal objective function value was estimated to 2571 EURO/hectare. Next, the parameter values of $(d_c, d_q) = (-0.003, 0.020)$ determined in the "108-loop", were considered optimal and fixed. A more detailed analysis, with higher resolution, of the parameters $(d_0, d_p) = (\mathrm{dlim\_0}, \mathrm{dlim\_p})$ was made. Figure 6 shows the objective function and in Fig. 7, the objective function level curves are given.

Multiple regression analysis and the data presented in Fig. 6 were used to estimate a quadratic approximation of the objective function, Z. Let $(\mathrm{x}, \mathrm{y}) = (d_0, d_p)$.

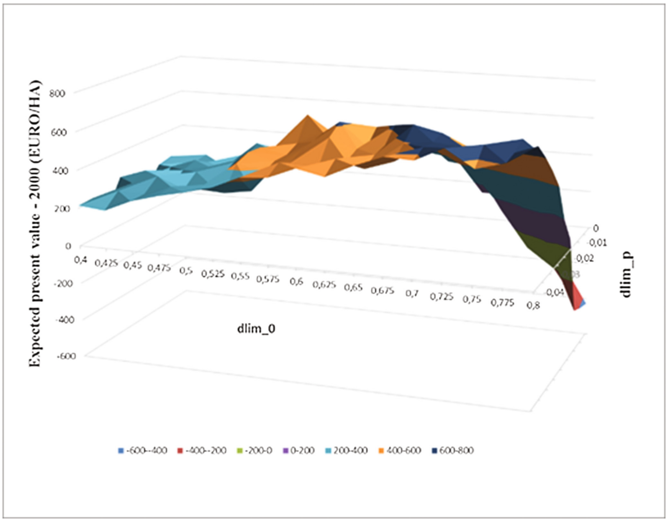

Fig. 6. The objective function reduced by a constant as a function of the parameters dlim_0 and dlim_p, for optimal values of the other parameters, namely dlim_c = −0.003 and dlim_q = 0.02.

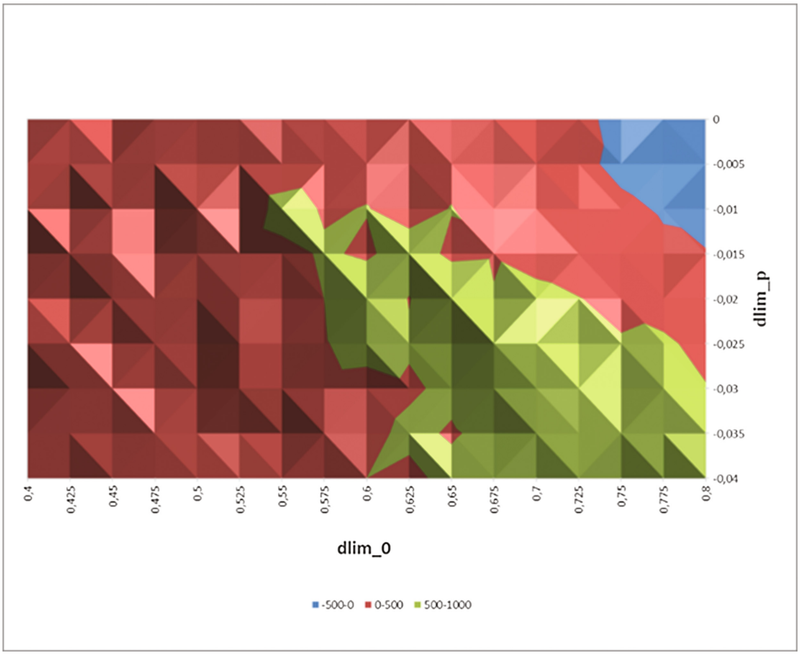

Fig. 7. The level curves of the objective function as a function of the parameters dlim_0 and dlim_p, when the other parameters were held constant at their optimal values.

$$Z = 8694\text{x} + 22248\text{y} - 8170\text{x}^2 - 235019y^2 - 65389xy \tag{4}$$

The R2 value of the regression was 0.999 and all coefficients were statistically significant, with p-values below 0.00003. The first order optimum conditions are:

$$\frac{dZ}{dx} = -16340x - 65389y + 8694 = 0 \tag{5}$$

$$\frac{dZ}{dy} = -65389x - 470038y + 22248 = 0 \tag{6}$$

The equation system $\begin{bmatrix} -16340 & -65389 \\ -65389 & -470038 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8694 \\ -22248 \end{bmatrix}$ gives this unique solution:

$(x, y) \approx (0.773, -0.0602)$. Now, the objective function value is 2690 EURO/hectare.

The derived optimum is a unique maximum, which is confirmed by:

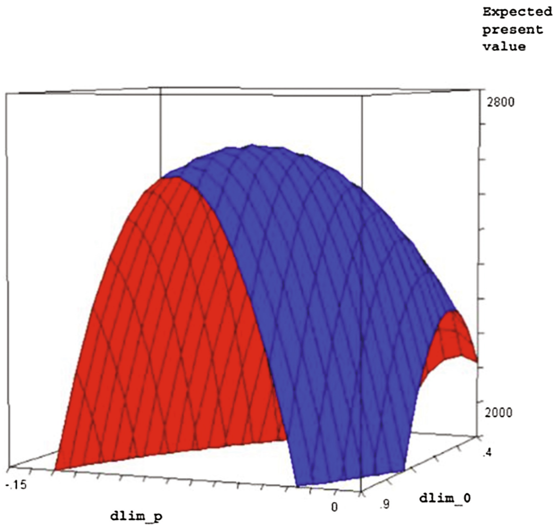

Fig. 8. The objective function as a function of the parameters dlim_0 and dlim_p, according to the quadratic approximation, when the other parameters were held constant at their optimal values.

$$|-16340| = -16340 < 0 \quad \wedge \quad \begin{vmatrix} -16340 & -65389 \\ -65389 & -470038 \end{vmatrix} \approx 3.405 \cdot 10^9 > 0 \tag{7}$$

The quadratic approximation gave this control function (Fig. 8):

$$d_{L,b}(i,t) = 0.773 - 0.0030\,C(i) + 0.020\,Q(i) - 0.0602\,\Delta P(t) \tag{8}$$

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Analysis for the Presence of Quantum Noise on the Teleportation

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Abstract. Quantum teleportation has provided us with an interesting way to transmit an arbitrary quantum state using one maximal entangled state and two classical bits of information. A variety of theoretical suggestions and experimental efforts have been made in this realm. In practical implementations of the teleportation protocol, quantum noise is an unavoidable factor. In this paper, we investigate the probabilistic quantum teleportation of two-particle. The fidelity of quantum state is calculated in detail, after suffering from the quantum noise. The relationship between quantum noise, quantum channel and quantum state fidelity is obtained. The effect of noise on the teleportation is analyzed.

Keywords: Quantum teleportation · Quantum noise · Fidelity · Quantum channel

1 Introduction

Quantum teleportation is a technique for the direct transmission of quantum states between the correspondents. In 1993, Bennett et al. [1] proposed the concept of quantum teleportation. At present, the technology has made numerous achievements in both theory [2–15] and experiment [16–18]. It has been studied from the original single particle quantum teleportation to many particle quantum teleportation and continuous variable quantum teleportation; from accurate quantum teleportation (the value of fidelity is 1) to the precise quantum teleportation (the value of fidelity is less than 1). All quantum teleportation protocols need establish a quantum channel by the entangled state. In this process, the particles of the entangled channel are susceptible to the interference by the quantum noise.

In this paper, we take the probability teleportation of two particle entangled state [19] as an example. and analyze the effects of several common quantum noises [20, 21] on the quantum channel. We have obtained the relationship between the quantum fidelity, the parameters of noise and the state parameters of channel. It has an important reference value for the practical application of the quantum teleportation.

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2 Preparation

2.1 Fidelity and Quantum Noise

To quantify the efficiency of the probabilistic teleportation protocol we use the fidelity. The fidelity is

$$F = Tr[\rho_{Target}\rho_{out}] = \langle\psi|\rho_{out}|\psi\rangle.$$

The action of the noise on the qubit, described by the density matrix ρ, is (Table 1)

$$\rho \rightarrow \rho_{out} = \sum_{j=1}^{n} E_j \rho E_j^{\dagger}. \ \sum_{j=1}^{n} E_j E_j^{\dagger} = I.$$

Table 1. Four types of noise

Types of noise	Kraus operators	
Bit flip	$E_1 = \sqrt{1-p}I$	$E_2 = \sqrt{p}\sigma_x$
Phase flip	$E_1 = \sqrt{1-p}I$	$E_2 = \sqrt{p}\sigma_z$
Depolarizing	$E_1 = \sqrt{1-3p/4}I$	$E_2 = \sqrt{p/4}\sigma_x$
	$E_3 = \sqrt{p/4}\sigma_y$	$E_4 = \sqrt{p/4}\sigma_z$
Amplitude damping	$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$	$E_1 = \begin{pmatrix} 1 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$

2.2 Examples of Quantum Teleportation

The quantum teleportation of the participant are Alice and Bob. The quantum state that Alice wants to transmit is $|\varphi\rangle_{12}$. And the quantum entanglement channel is $|\phi\rangle_{34}$.

$$|\varphi\rangle_{12} = x|00\rangle + y|11\rangle, \tag{1}$$

$$|\phi\rangle_{34} = a|00\rangle + b|11\rangle, \tag{2}$$

where x and y are non negative real numbers, and $x^2 + y^2 = 1$; also a and b are non negative real numbers, and $a^2 + b^2 = 1$. We can suppose $0 < b \leq a < 1$.

Suppose Alice prepared entangled state $|\phi\rangle_{34}$, then sends Bob the qubit 4 by optical fiber to set up communication channel. In this process, the qubit 4 is easier affected by noise. So when noise act on the qubit 4, we study the influence of noise for the quantum teleportation.

Without noise, composite system composed of qubits 1, 2, 3, 4 is:

$$|\Phi\rangle_{1234} = |\varphi\rangle_{12} \otimes |\phi\rangle_{34}; \tag{3}$$

the quantum state of Eq. (3) can be written as

$$\begin{aligned}|\Phi\rangle_{1234} = \frac{1}{\sqrt{2}}[&|\phi^+\rangle_{23}(ax|00\rangle_{14} + by|11\rangle_{14}) + |\phi^-\rangle_{23}(ax|00\rangle_{14} - by|11\rangle_{14}) \\ &+|\psi^+\rangle_{23}(bx|01\rangle_{14} + ay|10\rangle_{14}) + |\psi^-\rangle_{23}(bx|01\rangle_{14} - ay|10\rangle_{14})];\end{aligned} \tag{4}$$

where $|\phi^{\pm}\rangle$ and $|\psi^{\pm}\rangle$ are Bell states.

Without noise, the quantum channel is $|\phi\rangle_{34}$. And the density matrix is

$$\rho_{in} = \begin{pmatrix} a^2 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & b^2 \end{pmatrix}.$$

3 The Presence of Quantum Noise on the Teleportation

3.1 Bit Flip

After suffering from the noise named bit flip, the density matrix of $|\phi\rangle_{34}$ is

$$\begin{aligned}\rho_{out} &= (I \otimes E_1^\dagger)\rho_{in}(I \otimes E_1) + (I \otimes E_2^\dagger)\rho_{in}(I \otimes E_2) \\ &= (1-p)\begin{pmatrix} a^2 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & b^2 \end{pmatrix} + p\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^2 & ab & 0 \\ 0 & ab & b^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};\end{aligned}$$

so the quantum channel has changed as

$$|\phi\rangle'_{34} = \sqrt{1-p}(a|00\rangle_{34} + b|11\rangle_{34}) + \sqrt{p}(a|01\rangle_{34} + b|10\rangle_{34}); \tag{5}$$

and the composite system has changed as

$$\begin{aligned}|\Phi\rangle'_{1234} =& |\varphi\rangle_{12} \otimes |\phi\rangle'_{34} \\ =& \frac{1}{\sqrt{2}}\sqrt{1-p}[|\phi^+\rangle_{23}(ax|00\rangle_{14} + by|11\rangle_{14}) + |\phi^-\rangle_{23}(ax|00\rangle_{14} - by|11\rangle_{14}) \\ &+|\psi^+\rangle_{23}(bx|01\rangle_{14} + ay|10\rangle_{14}) + |\psi^-\rangle_{23}(bx|01\rangle_{14} - ay|10\rangle_{14})] \\ &+\frac{1}{\sqrt{2}}\sqrt{p}[|\phi^+\rangle_{23}(ax|01\rangle_{14} + by|10\rangle_{14}) + |\phi^-\rangle_{23}(ax|01\rangle_{14} - by|10\rangle_{14}) \\ &+|\psi^+\rangle_{23}(bx|00\rangle_{14} + ay|11\rangle_{14}) + |\psi^-\rangle_{23}(bx|00\rangle_{14} - ay|11\rangle_{14})];\end{aligned} \tag{6}$$

obviously, the fidelity of the composite system is $F_1 = (1-p) \times 1 + p \times 0 = 1-p$.

3.2 Phase Flip

After suffering from the noise named phase flip, the density matrix of $|\phi\rangle_{34}$ is

$$\rho_{out} = (I \otimes E_1^\dagger)\rho_{in}(I \otimes E_1) + (I \otimes E_2^\dagger)\rho_{in}(I \otimes E_2) \\ = (1-p)\begin{pmatrix} a^2 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & b^2 \end{pmatrix} + p\begin{pmatrix} a^2 & 0 & 0 & -ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab & 0 & 0 & b^2 \end{pmatrix};$$

so the quantum channel has changed as

$$|\phi\rangle'_{34} = \sqrt{1-p}(a|00\rangle_{34} + b|11\rangle_{34}) + \sqrt{p}(a|00\rangle_{34} - b|11\rangle_{34}); \tag{7}$$

and the composite system has changed as

$$\begin{aligned} |\Phi\rangle'_{1234} =& |\varphi\rangle_{12} \otimes |\phi\rangle'_{34} \\ =& \frac{1}{\sqrt{2}}\sqrt{1-p}[|\phi^+\rangle_{23}(ax|00\rangle_{14} + by|11\rangle_{14}) + |\phi^-\rangle_{23}(ax|00\rangle_{14} - by|11\rangle_{14}) \\ &+ |\psi^+\rangle_{23}(bx|01\rangle_{14} + ay|10\rangle_{14}) + |\psi^-\rangle_{23}(bx|01\rangle_{14} - ay|10\rangle_{14})] \\ &+ \frac{1}{\sqrt{2}}\sqrt{p}[|\phi^+\rangle_{23}(ax|00\rangle_{14} - by|11\rangle_{14}) + |\phi^-\rangle_{23}(ax|00\rangle_{14} + by|11\rangle_{14}) \\ &+ |\psi^+\rangle_{23}(-bx|01\rangle_{14} + ay|10\rangle_{14}) + |\psi^-\rangle_{23}(-bx|01\rangle_{14} - ay|10\rangle_{14})]; \end{aligned} \tag{8}$$

obviously, the fidelity of the composite system is

$$\begin{aligned} F_2 =& (1-p) \times 1 + p \times \frac{1}{4}[|\ (a^2x^2 - b^2y^2) + (a^2x^2 - b^2x^2) + (-b^2x^2 + a^2y^2) + (-b^2x^2 + a^2y^2)\ |] \\ =& (1-p) \times 1 + \frac{1}{2}p \times |\ a^2 - b^2\ | \\ =& 1 + (a^2 - \frac{3}{2})p. \end{aligned} \tag{9}$$

3.3 Depolarizing

After suffering from the noise named depolarizing, the density matrix of $|\phi\rangle_{34}$ is

$$\rho_{out} = (I \otimes E_1^\dagger)\rho_{in}(I \otimes E_1) + (I \otimes E_2^\dagger)\rho_{in}(I \otimes E_2) + (I \otimes E_3^\dagger)\rho_{in}(I \otimes E_3) + (I \otimes E_4^\dagger)\rho_{in}(I \otimes E_4) \\ = (1 - \frac{3p}{4})\begin{pmatrix} a^2 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & b^2 \end{pmatrix} + \frac{p}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^2 & ab & 0 \\ 0 & ab & b^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{p}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a^2 & -ab & 0 \\ 0 & -ab & -b^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{p}{4}\begin{pmatrix} a^2 & 0 & 0 & -ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab & 0 & 0 & b^2 \end{pmatrix};$$

so the quantum channel has changed as

$$\begin{aligned} |\phi\rangle'_{34} =& \sqrt{1 - \frac{3p}{4}}(a|00\rangle_{34} + b|11\rangle_{34}) + \sqrt{\frac{p}{4}}(a|01\rangle_{34} + b|10\rangle_{34}) \\ &+ \sqrt{\frac{p}{4}}(-ai|01\rangle_{34} - bi|10\rangle_{34}) + \sqrt{\frac{p}{4}}(a|00\rangle_{34} - b|11\rangle_{34}); \end{aligned} \tag{10}$$

and the composite system has changed as

$$\begin{aligned}
|\Phi\rangle'_{1234} =&|\varphi\rangle_{12}\otimes|\phi\rangle'_{34}\\
=&\frac{1}{\sqrt{2}}\sqrt{\frac{3p}{4}}[|\phi^+\rangle_{23}(ax|00\rangle_{14}+by|11\rangle_{14})+|\phi^-\rangle_{23}(ax|00\rangle_{14}-by|11\rangle_{14})\\
&+|\psi^+\rangle_{23}(bx|01\rangle_{14}+ay|10\rangle_{14})+|\psi^-\rangle_{23}(bx|01\rangle_{14}-ay|10\rangle_{14})]\\
&+\frac{1}{\sqrt{2}}\sqrt{\frac{p}{4}}[|\phi^+\rangle_{23}(ax|01\rangle_{14}+by|10\rangle_{14})+|\phi^-\rangle_{23}(ax|01\rangle_{14}-by|10\rangle_{14})\\
&+|\psi^+\rangle_{23}(bx|00\rangle_{14}+ay|11\rangle_{14})+|\psi^-\rangle_{23}(bx|00\rangle_{14}-ay|11\rangle_{14})]\\
&-\frac{1}{\sqrt{2}}i\sqrt{\frac{p}{4}}[|\phi^+\rangle_{23}(ax|01\rangle_{14}+by|10\rangle_{14})+|\phi^-\rangle_{23}(ax|01\rangle_{14}-by|10\rangle_{14})\\
&+|\psi^+\rangle_{23}(bx|00\rangle_{14}+ay|11\rangle_{14})+|\psi^-\rangle_{23}(bx|00\rangle_{14}-ay|11\rangle_{14})]\\
&+\frac{1}{\sqrt{2}}\sqrt{\frac{p}{4}}[|\phi^+\rangle_{23}(ax|00\rangle_{14}-by|11\rangle_{14})+|\phi^-\rangle_{23}(ax|00\rangle_{14}+by|11\rangle_{14})\\
&+|\psi^+\rangle_{23}(-bx|01\rangle_{14}+ay|10\rangle_{14})+|\psi^-\rangle_{23}(-bx|01\rangle_{14}-ay|10\rangle_{14})];
\end{aligned} \tag{11}$$

obviously, the fidelity of the composite system is

$$\begin{aligned}
F_3 =&(1-\frac{3p}{4})\times 1+\frac{p}{4}\times 0+\frac{p}{4}\times 0+\frac{p}{8}\times(a^2-b^2)\\
=&1-\frac{1}{8}(7-a^2)p.
\end{aligned} \tag{12}$$

3.4 Amplitude Damping

After suffering from the noise named amplitude damping, the density matrix of $|\phi\rangle_{34}$ is

$$\begin{aligned}
\rho_{out} &= (I\otimes E_1^\dagger)\rho_{in}(I\otimes E_1)+(I\otimes E_2^\dagger)\rho_{in}(I\otimes E_2)\\
&=\begin{pmatrix} a^2 & 0 & 0 & ab\sqrt{1-p}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ ab\sqrt{1-p} & 0 & 0 & b^2(1-p)\end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & b^2p & 0\\ 0 & 0 & 0 & 0\end{pmatrix};
\end{aligned}$$

so the quantum channel has changed as

$$|\phi\rangle'_{34} = (a|00\rangle_{34}+b\sqrt{1-p}|11\rangle_{34})+b\sqrt{p}|10\rangle_{34}; \tag{13}$$

and the composite system has changed as

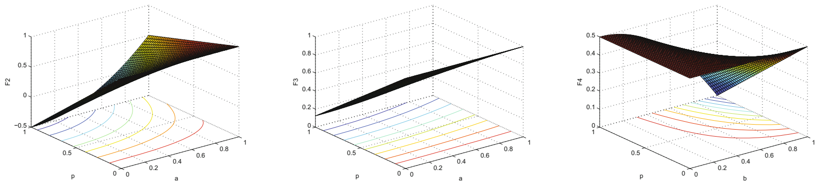
$$\begin{aligned}
|\Phi\rangle'_{1234} =&|\varphi\rangle_{12}\otimes|\phi\rangle'_{34}\\
=&\frac{1}{\sqrt{2}}[|\phi^+\rangle_{23}(ax|00\rangle_{14}+by\sqrt{1-p}|11\rangle_{14})+|\phi^-\rangle_{23}(ax|00\rangle_{14}-by\sqrt{1-p}|11\rangle_{14})\\
&+|\psi^+\rangle_{23}(bx\sqrt{1-p}|01\rangle_{14}+ay|10\rangle_{14})+|\psi^-\rangle_{23}(bx\sqrt{1-p}|01\rangle_{14}-ay|10\rangle_{14})]\\
&+\frac{1}{\sqrt{2}}[|\phi^+\rangle_{23}(by\sqrt{p}|10\rangle_{14})+|\phi^-\rangle_{23}(-by\sqrt{p}|10\rangle_{14})\\
&+|\psi^+\rangle_{23}(bx\sqrt{p}|00\rangle_{14})+|\psi^-\rangle_{23}(bx\sqrt{p}|00\rangle_{14})];
\end{aligned} \tag{14}$$

obviously, the fidelity of the composite system is

$$\begin{aligned} F_4 =& \frac{1}{4}[a^2x^2 + b^2y^2(1-p)] + [a^2x^2 + b^2y^2(1-p)] + [b^2x^2(1-p) + a^2y^2] + [b^2x^2(1-p) + a^2y^2] \\ =& \frac{1}{2}(1-b^2p). \end{aligned} \tag{15}$$

4 Analysis

In the process of quantum teleportation, the fidelity of the composite system is $F_1 = 1 - p$, after the channel suffering from the noise named bit flip. So, we clearly known the fidelity F_1 decreases with the increase of the noise parameters p. If $p = 0$, means there is no noise, the value of fidelity is 1.



(a) Phase flip (b) Depolarizing (c) Amplitude damping

In the process of quantum teleportation, the fidelity of the composite system is $F_2 = 1 + (a^2 - \frac{3}{2})p$, after the channel suffering from the noise named phase flip. In this equation, we known $0 < b \leq a < 1$, and $a^2 + b^2 = 1$. So we get $\frac{\sqrt{2}}{2} \leq a < 1$. According to the figure (a), suppose a is constant, the fidelity F_2 decreases with the increase of the noise parameters p; suppose p is constant, the fidelity F_2 increases with the increase of a. However, if a close to 1, the entangled state will be unstable. similarly, if $p = 0$, the value of fidelity is 1.

In the process of quantum teleportation, the fidelity of the composite system is $F_3 = 1 - \frac{1}{8}(7 - a^2)p$, after the channel suffering from the noise named depolarizing. In this equation, we known $\frac{\sqrt{2}}{2} \leq a < 1$ and $0 \leq p \leq 1$. According to the figure (b), suppose a is constant, the fidelity F_3 decreases with the increase of the noise parameters p; suppose p is constant, the fidelity F_3 increases with the increase of a. Also a can not be close to 1, and if $p = 0$, the value of fidelity is 1.

In the process of quantum teleportation, the fidelity of the composite system is $F_4 = \frac{1}{2}(1 - b^2p)$, after the channel suffering from the noise named amplitude damping. In this equation, we known $0 < b \leq \frac{\sqrt{2}}{2}$ and $0 < p \leq 1$. In another case, if $p = 0$, the value of fidelity F_4 is 1. According to the figure (c), suppose b is constant, the fidelity F_4 decreases with the increase of the noise parameters p; suppose p is constant, the fidelity F_4 decreases with the increase of b. And b can not be close to 0.

5 Conclusion

We have calculated the fidelity of quantum state, and analysis of the effect of quantum noise on the quantum teleportation. It is useful for the practical application of the quantum teleportation. Next, we will study the situations that the quantum states suffer from the different noise at the same time.

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Design of An Active Control Method for Complete Stabilization of Unknown Fractional-Order Non-autonomous Systems

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Abstract. This paper introduces an active control technique for stabilization of fractional-order nonlinear non-autonomous systems. The main characteristic of the proposed control technique is the fast convergence to the origin. The proposed control scheme is theoretically designed based on the fractional Lyapunov stability theory. The robustness against system uncertainties and the ability of control of non-autonomous fractional-order complex systems are the other features of proposed method. Furthermore, numerical examples are included to highlight the applicability and usefulness of the proposed method in suppress chaotic/hyperchaotic behaviors of fractional-order complex systems. It is worth to notice that the proposed active control approach can be employed for control and stabilization of a vast class of uncertain nonlinear fractional-order complex systems.

Keywords: Fractional-order system · Complex systems · Active control · Stabilization · Lyapunov stability theorem

1 Introduction

Nowadays one of the main challenges of science is the faithful modeling of natural phenomena by the possible simplest equations. In this regard, Fractional Calculus provides a new frame to this field. The Fractional Calculus establishes the differentiation and integration of arbitrary orders [1,2]. Although this theory has a long history, it has been utilized in engineering and physics during the past three decades [3]. Moreover, it has been known that many systems in various

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fields such as viscoelastic materials [4], rotor-bearing system [5], energy systems [6], thermoelectric systems [7], electromechanical systems [8], finance systems [9], biological system [10], medicine [11], epidemiology mechanisms [12] and plasma [13] can be efficiently described by fractional-order differential equations.

Due to the high sensitivity to the initial conditions and chaotic behaviors, the fractional-order systems are very interesting to investigate [14]. Therefore, some researchers have proposed various methods to stabilize these systems [15–18]. Up to now, several methods for controlling and stabilization of fractional-order systems have been designed which includes sliding mode control [19–22], optimal control [23], robust control [24–26], adaptive control [27–29], PID control [30,31]. However, there are few related results reported on the controllers for fractional-order complex systems.

On the other hand, stability proving of most of mentioned methods are performed by inappropriate lyapunov methods [32–34]. Despite the fact that, these techniquees are useful in linear cases, in nonlinear cases it leads to a problem with more dimensions and increase the capacity of calculations. Although Lyapunov approach is interesting in the formulation of LMI conditions, has not yet received acceptable solutions, and more particularly in the non-linear case. In addition, stability proving of Mittag-Leffler method, which is used by many of researchers, depends on a unsatisfactory definition of fractional order systems [35].

In this paper, the problem of designing an active control scheme for stabilizing uncertain fractional-order complex systems is investigated. The effects of model uncertainties and external disturbances are fully taken into account. In this regard, we use the fractional version of the Lyapunov stability theory to introduce an active controller for guaranteeing the convergence of the fractional-order complex system's trajectories to the equilibrium point. Moreover, the stability of the closed-loop system is mathematically proved. In addition, fractional-order permanent magnet synchronous motor system and fractional-order four wing system are stabilized to show the efficacy of the designed control scheme in industry and engineering.

The paper is presented as follows: in Sect. 2, basic preliminaries of fractional calculus and notations are given. In Sect. 3, the general form of fractional-order complex systems and the suggested active control scheme are presented. Two numerical examples are simulated, in Sect. 4, to illustrate the usefullness of the proposed control approache. Finally, concluding remarks are given in Sect. 5.

2 Basic Concepts

Some basic definition of the fractional calculus and a necessary stability theorem are given below.

Definition 1. *The Riemann-Liouville fractional integration of order α is presented by [36]*

$$ {}_{t_0}I_t = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{1} $$

where t_0 is initial time. Also $\Gamma(.)$ is the Gamma function.

Definition 2. *Let $m-1 < \alpha \leq m$ and $m \in N$. The Riemann-Liouville fractional derivative of order α of function $f(t)$ is defined as follows* [36]

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{d^{\alpha}}{dt^{\alpha}} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{\alpha}}{dt^{\alpha}} \int_{t_0}^{t} f(\tau)(t-\tau)^{m-\alpha-1} d\tau. \tag{2} $$

Definition 3. *The αth order Caputo fractional derivative of a continuous function $f(t) : R^{+} \longrightarrow R$ is defined as* [36]

$$ {}_{t_0}D_t^{\alpha} f(t) = {}_{t_0}D_t^{-(m-\alpha)} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} f^{(m)}(\tau)(t-\tau)^{m-\alpha-1} d\tau. \tag{3} $$

It should be noted that, the initial conditions for the fractional differential equations (FDEs) with the Caputo derivative are in the same form as for integer-order derivatives which have clear physical meaning. So, the Caputo fractional derivative is more popular than the Riemann-Liouville fractional derivative, when modeling real-world phenomena with FDEs. Therefore, the Caputo derivative is used and in the rest of this paper, the notation D^{α} indicates the Caputo fractional derivative.

Definition 4. *Suppose that $h(t)$ be the impulse response of a linear system. The diffusive representation of $h(t)$ is called $\mu(\omega)$ with relation as follows* [35]

$$ h(t) = \int_{0}^{\infty} \mu(\omega)\, e^{-\omega t}\, d\omega. \tag{4} $$

Remark 1. The fractional order integral (1) can rewrite as [35]

$$ {}_{t_0}I_t^{\alpha} f(t) = h(t) * f(t), \tag{5} $$

where $*$ is the convolution operator and $h(t)$ define as $h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.
The diffusive representation of $h(t)$ is defined as

$$ \mu(\omega) = \frac{\sin(\alpha\pi)}{\pi} \omega^{-\alpha}. \tag{6} $$

Definition 5. *Consider the following nonlinear FDE* [35]

$$ {}_{t_0}D_t^{\alpha} X = f(X(t)). \tag{7} $$

According to the continuous frequency distributed model of the fractional integrator, the nonlinear system (7) can be written as:

$$ \begin{cases} \frac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(X(t)), \\ x(t) = \int_0^{\infty} \mu(\omega)\, z(\omega,t)\, d\omega, \end{cases} \tag{8} $$

while $\mu(\omega)$ is the same as (6).

Theorem 1. *Consider $w_1 = \int_0^{\infty} \mu(\omega)\omega z^2(\omega,t)\mathrm{d}\omega$ and $w_2 = ax^2$. Then the quadratic form $w = w_1 + w_2$ is positive definite if and only if $a > 0$* [35].

3 The Method and Main Results

Consider the following N-dimensional non-autonomous uncertain complex fractional-order system:

$$D^{\alpha}X = F(X,t) + \Delta(X,t) + d(t) - U(t), \tag{9}$$

where $\alpha \in (0,1)$ is the order of the system and $X(t) = [x_1, x_2, ..., x_n]^T \in R^n$ is the vector of states, $F(X,t) = f_1(X,t), f_2(X,t), ..., f_n(X,t)]^T$ is nonlinear function of X and t, $\Delta F(X,t) = [\Delta f_1(X,t), \Delta f_2(X,t), ..., \Delta f_n(X,t)]^T$ and $d(t) = [d_1(t), d_2(t), ..., d_n(t)]^T$ represent unknown model of uncertainty and external disturbances of the system, respectively. And finally, $U(t) = [u_1(t), u_2(t), ..., u_n(t)]^T$ is the control input, which is designed later.

Assumption 1: Since the trajectories of complex fractional-order systems are always bounded, then the unknown uncertainties $\Delta f_i(X,t), i = 1, 2, ..., n$ and external disturbances $d_i(t), i = 1, 2, ..., n$ are assumed to be bounded. Therefore, there exist appropriate positive constants δ_i and β_i such that

$$|\Delta f_i(X,t)| < \delta_i, \qquad i = 1, 2, ..., n \tag{10}$$

and

$$|d_i(t)| < \beta_i, \qquad i = 1, 2, ..., n. \tag{11}$$

So, as a result from (10) and (11), one has

$$|\Delta f_i(X,t)| + |d_i(t)| < \delta_i + \beta_i < \rho_i, \qquad i = 1, 2, ..., n. \tag{12}$$

Here, we propose the active control law as follows:

$$U(t) = \lambda \xi sign(X) + F(X,t) + \rho, \tag{13}$$

where $\xi = q\|X\|$ also λ and q are possitive constant vectors in R^n and $\|.\|$ is a standard norm and $\rho = [\rho_1, \rho_2, ..., \rho_n]^T$.

Theorem 2. *Consider the fractional-order complex system* (9). *If this system is controlled by the control signal* (13)*, then the system trajectories.*

Proof. According to Definition 5 and (8), we can rewrite the Eq.(9) as

$$\begin{cases} \frac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(X(t)), \\ x(t) = \int_0^{\infty} \mu(\omega)\, z(\omega,t)\, d\omega. \end{cases} \tag{14}$$

Now, we define two Lyapunov function where the first is $v(\omega,t)=\frac{z^2(\omega,t)}{2}$. For $v(\omega,t)$ one can has $\frac{\partial v(\omega,t)}{\partial z(\omega,t)}=z(\omega,t)$ and by using (456) we can obtain

$$\begin{aligned}\frac{\partial v(\omega,t)}{\partial t} &= \frac{\partial v(\omega,t)}{\partial z(\omega,t)}\cdot\frac{\partial z(\omega,t)}{\partial t}\\ &= z(\omega,t)[-\omega z(\omega,t)+F(X,t)+\Delta f(X,t)+d(t)-U(t)]\\ &\le z(\omega,t)[-\omega z(\omega,t)+F(X,t)+\underbrace{|\Delta f(X,t)|+|d(t)|}_{<\rho}-U(t)]\\ &< z(\omega,t)[-\omega z(\omega,t)+F(X,t)+\rho-\underbrace{(\lambda\xi sign(X)+\rho+F(X,t))}_{U(t)}]\\ &< z(\omega,t)[-\omega z(\omega,t)-\lambda\xi sign(X)]\\ &= -\omega z^2(\omega,t)-\lambda\xi sign(X)z(\omega,t).\end{aligned}$$

Therefore, we have

$$\frac{\partial v(\omega,t)}{\partial t} < -\omega z^2(\omega,t)-\lambda\xi sign(X)z(\omega,t). \tag{15}$$

Now, we introduce the main Lyapunov function as follows:

$$V(t)=\int_0^\infty \mu(\omega)\,v(\omega,t)\,d\omega=\frac{1}{2}\int_0^\infty \mu(\omega)\,z^2(\omega,t)\,d\omega. \tag{16}$$

Obviously $V(t)>0$ and according to the Lyapunove stability theorem, we must demonstrate that $\frac{dV}{dt}<0$. Therefore, by attention to (16), we can obtain

$$\begin{aligned}\frac{dV}{dt} &= \int_0^\infty \mu(\omega)\,\frac{\partial v(\omega,t)}{\partial t}\,d\omega\\ &\le \int_0^\infty \mu(\omega)\,[-\omega z^2(\omega,t)-\lambda\xi sign(X)z(\omega,t)]\,d\omega\\ &< -\int_0^\infty \mu(\omega)\omega z^2(\omega,t)\,d\omega-\lambda\xi sign(X)\underbrace{\int_0^\infty \mu(\omega)z(\omega,t)\,d\omega}_{X}\\ &< -\int_0^\infty \mu(\omega)\omega z^2(\omega,t)\,d\omega-\lambda\underbrace{q\|X\|}_{\xi}\|X\|\\ &< -(\int_0^\infty \mu(\omega)\omega z^2(\omega,t)\,d\omega+\lambda q\|X\|^2).\end{aligned}$$

Since the constants λ and q are positive, according to Theorem 1, we have $\frac{dV}{dt}<0$. And this completes the proof.

Remark 2. According to Eq. (13), the control method depends on the constant and positive values of λ_i, q_i and $\rho_i, i=1,2,...,n$, respectively. This means that the control law is proportional to the values of λ_i, q_i and ρ_i. Hence, for q_i and larger λ_i and ρ_i, $i=1,2,...,n$ lead to appropriate control effort and vice versa.

4 Numerical Simulations

In this section, to validate of proposed control scheme, two well-known fractional-order complex systems are concerned. The fractional-order chaotic permanent magnet synchronous motor system and fractional-order hyper chaotic four-wing system are adopted here. A modification of Adams-Bashforth-Moulton algorithm, which is proposed by Shahbazi Asl and Javidi in [37], is utilized to solve FDEs. Also, the MATLAB software is applied to numerical simulations with a step time of 0.001.

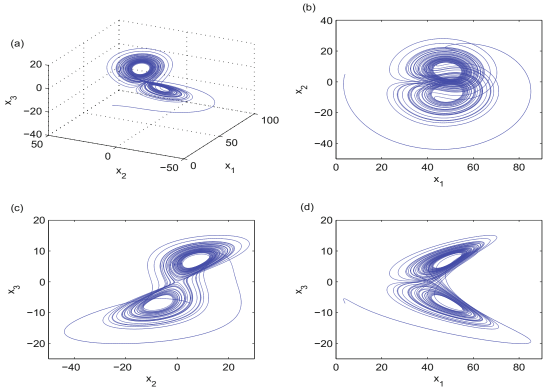

Fig. 1. Strange attractor in FOPMSM system (17) for $\alpha = 0.97$. (a) 3D view in the $x_1 - x_2 - x_3$ space. (b) Projection on the $x_1 - x_2$ plane. (c) Projection on the $x_2 - x_3$ plane. (d) Projection on the $x_1 - x_3$ plane.

4.1 Stabilization of Fractional-Order Permanent Magnet Synchronous Motor

Fractional-order permanent magnet synchronous motor (FOPMSM) is extremely utilized in high performance applications because of simple structure, high power density, small size, and high efficiency [38]. Here, we verify the effectiveness of the proposed active controller (13) in the control and stabilization of chaotic FOPMSM.

Consider the following uncertain FOPMSM [39]

$$FOPMSM = \begin{cases} D^\alpha x_1 = -x_1 + x_2x_3 + \Delta f_1(X,t) + d_1(t) - u_1(t), \\ D^\alpha x_2 = -x_2 - x_1x_3 + 50x_3 + \Delta f_2(X,t) + d_2(t) - u_2(t), \\ D^\alpha x_3 = 4(x_2 - x_3) + \Delta f_3(X,t) + d_3(t) - u_3(t), \end{cases} \quad (17)$$

where the uncertainty terms of the above system are selected as

$$\begin{aligned} \Delta f_1(X,t) + d_1(t) &= 0.2\cos(2t)x_1 + 0.15\sin(5t), \\ \Delta f_2(X,t) + d_2(t) &= 0.15\sin(3t)x_2 - 0.1\cos(2t), \\ \Delta f_3(X,t) + d_3(t) &= 0.24\cos(4t)x_3 + 0.12\sin(4t). \end{aligned} \quad (18)$$

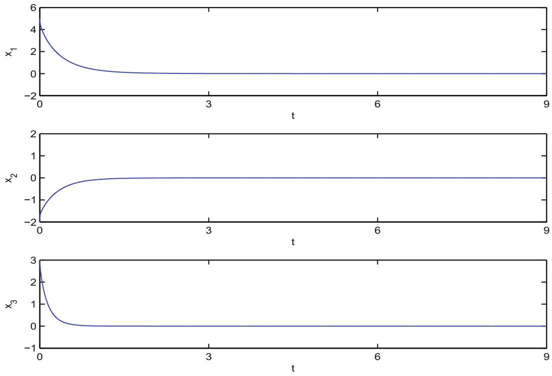

Fig. 2. State trajectories of the FOPMSM system (17) controlled with (13).

One can see the chaotic attractor of the uncertain FOPMSM system (17) for $\alpha = 0.97$ in Fig. 1. In order to suppress the chaotic behavior of the system (17), we apply the suggested active control method (13). The control parameters are selected as $\lambda_1 = 5$, $\lambda_2 = 5$, $\lambda_3 = 2$ and $q_1 = 2$, $q_2 = 3$, $q_3 = 3$, and $\rho = [0.4, 0.4, 0.45]^T$. Also, $x_1(0) = 1$, $x_2(0) =?$ and $x_3(0) = 3$ are chosen as initial conditions.

The stabilization of the FOPMSM system is illustrated in Fig. 2. It can be observed that the state trajectories of the uncertain FOPMSM system converge to equilibrium point, which indicates that the fractional-order permanent magnet synchronous motor system is indeed stabilized. The time response of the active control input (13) is shown in Fig. 3. Obviously, the control signal is implementable in practice.

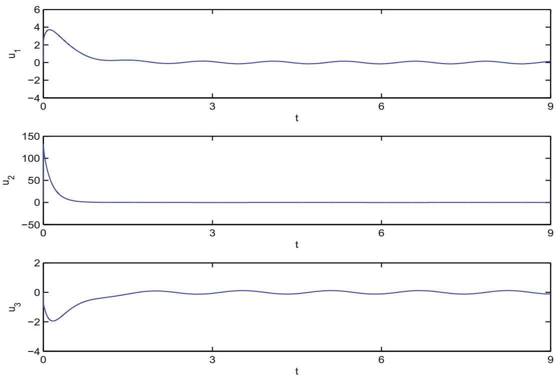

Fig. 3. Time history of the control input (13) applied to the FOPMSM system (17).

4.2 Stabilization of Fractional-Order Four-Wing Hyperchaotic System

This example validates the appropriateness of the proposed active controller (13) in stabilizing a fractional-order hyperchaotic system. Recently, Dadras et al. [40], have been introduced a 4D fractional-order system with two interesting property. First, this system can lead to a four-wing hyperchaotic behavior. The second one, this system has only one equilibrium point at the origin, while we know that four-wing hyperchaotic systems have usually four or more unstable equilibrium point.

Consider the following uncertain Fractional-order Four-Wing hyperchaotic system [40]

$$FO\,Four-Wing=\begin{cases} D^{\alpha}x_1 = 8x_1 + x_2x_3 + x_4 + \Delta f_1(X,t) + d_1(t) - u_1(t), \\ D^{\alpha}x_2 = -40x_2 + x_1x_3 + \Delta f_2(X,t) + d_2(t) - u_2(t), \\ D^{\alpha}x_3 = x_1x_2 - 49x_3 + x_1x_4 + \Delta f_3(X,t) + d_3(t) - u_3(t), \\ D^{\alpha}x_4 = -x_2 + \Delta f_4(X,t) + d_4(t) - u_4(t), \end{cases} \tag{19}$$

where the following uncertainties and external noises are adopted.

$$\begin{aligned} \Delta f_1(X,t) + d_1(t) &= 0.2\sin(5t)x_4 + 0.25\cos(3t), \\ \Delta f_2(X,t) + d_2(t) &= 0.3\cos(3t)x_1x_3 - 0.15\cos(t), \\ \Delta f_3(X,t) + d_3(t) &= 0.25\sin(4t)x_3 + 0.2\sin(3t), \\ \Delta f_4(X,t) + d_4(t) &= 0.25\sin(t)x_4 + 0.15\cos(4t). \end{aligned} \tag{20}$$

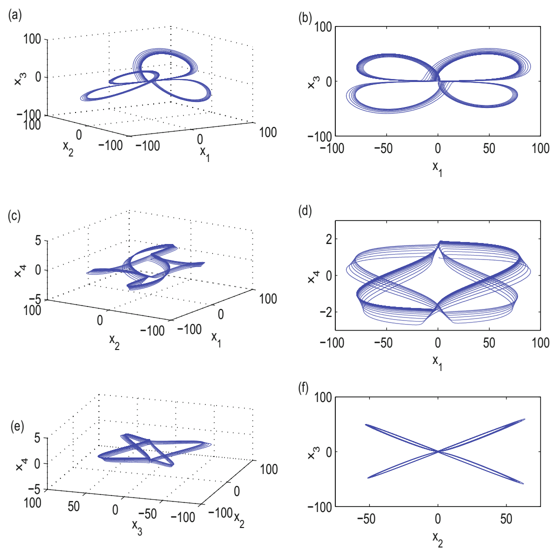

Fig. 4. Hyperchaotic behaviour of FOfour-wing system (19) for $\alpha = 0.98$. (a) Strange attractor in the $x_1 - x_2 - x_3$ space. (b) Projection on the $x_1 - x_3$ plane. (c) Strange attractor in the $x_1 - x_2 - x_4$ space. (d) Projection on the $x_1 - x_4$ plane. (e) Strange attractor in the $x_2 - x_3 - x_4$ space. (f) Projection on the $x_2 - x_3$ plane.

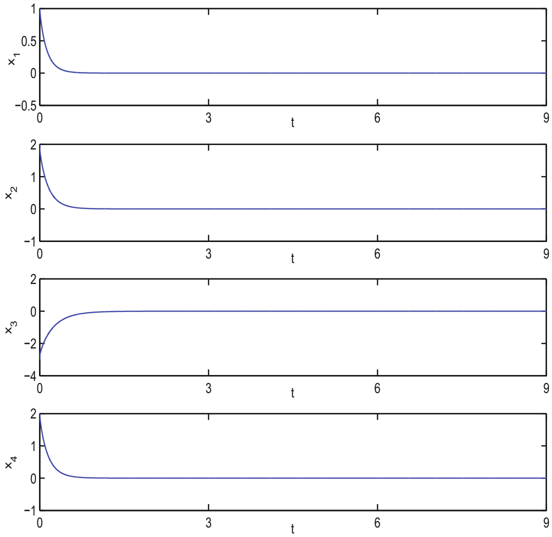

Fig. 5. State trajectories of the FO Four-Wing hyperchaotic system (19), controlled with (13).

The hyperchaotic behavior of the fractional-order Four-Wing hyperchaotic system (19) is shown in Fig. 4, where $x_1(0) = 1$, $x_2(0) =?$, $x_3(0) = 3$, $x_4(0) = 2$

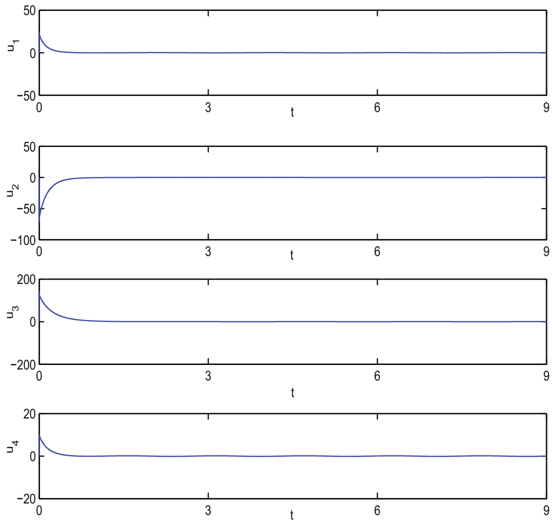

Fig. 6. Time history of the control input (13) applied to the FO Four-Wing hyperchaotic system (19).

and $\alpha = 0.98$ are selected as initial values and fractional order of the system, respectively. Based on the introduced control method (13), we set $\lambda_1 = 1.3$, $\lambda_2 = 2$, $\lambda_3 = 1.3$, $\lambda_4 = 1.3$ and $q_1 = 3$, $q_2 = 3$, $q_3 = 3$, $q_4 = 3$ and $\rho = [0.45, 0.45, 0.45, 0.45]^T$ to suppress the chaos of the fractional-order Four-Wing hyperchaotic system (19). Figures 5 and 6 show the state trajectories of the system (19) and the time history of the active control input (13), respectively. It can be seen that the state trajectories converge to equilibrium point and the control signal is feasible in practice. This means that the proposed fractional active controller can effectively stabilize the FO Four-Wing hyperchaotic system.

5 Conclusions

In this paper, the problem of control and stabilization of uncertain fractional-order complex systems is studied. An efficient active controller is presented, on the basis of fractional version of Lyapunov stability theory. Also, the stability of the suggested approach is mathematically proved. Numerical examples demonstrate that the proposed control technique can stabilize uncertain fractional-order chaotic/hyperchaotic systems, especially when the whole dynamics of the system is disturbed by unknown uncertainties and external disturbances. It is worth to notice that the introduced active control method can be applied to control a large class of uncertain nonlinear fractional-order dynamical systems.

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Three Uncertainty Measures in Neighborhood Systems

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Abstract. Uncertainty measures are important. In neighborhood systems, neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy can be used for evaluating uncertainty. However, they can not provide enough information in some situations. The paper depends on knowledge granulation to propose three uncertainty measures. Firstly, in neighborhood systems, neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation are defined; Secondly, some important properties of the three new measures are studied, such as granulation monotonicity; Finally, two examples are designed to make illustration and comparison, and the three new measures achieve better performance. Neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation are reasonable and effective.

Keywords: Neighborhood systems · Uncertainty measures · Knowledge granulation · Neighborhood accuracy · Neighborhood roughness · Neighborhood approximation accuracy

1 Introduction

Rough set theory introduced by Pawlak [1] can deal well with imprecise, vague and uncertain information, and it has been successfully applied in many research fields, such as machine learning, pattern recognition, knowledge discovery and data analysis [2,3].

In rough set theory, uncertainty measures are important. Uncertainty measures can provide us with principled methodologies to analyze uncertain data, so we can unveil the substantive characteristics of the data sets. Accuracy, roughness and approximation accuracy were proposed by Pawlak [4], and they are main tools to deal with uncertainty measures issues in rough set theory. Many authors have studied uncertainty measures of date sets from several viewpoints [6–8]. Closely associated with uncertainty measures, several measures on

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knowledge granulation in an information system were proposed and the relationships between these measures were discussed in [5].

Traditional rough set theory is suitable for discrete data rather than continuous data since only equivalence class and equivalence relation are considered. The requirement of equivalence relation is too restrictive for many practical data sets and applications. To overcome the defects, neighborhood spaces are more general than equivalence spaces, so the neighborhood relation is introduced into rough set theory [10]. In neighborhood rough set theory, uncertainty measures are significant. In recent years, many scholars have done some studies on the uncertainty measures in neighborhood systems [11–14]. In neighborhood systems, neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy are proposed in [9], and they can be used for evaluating uncertainty to a certain degree. However, they do not provide enough information in some situations.

In this paper, the three existing measures (i.e., neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy) are modified in neighborhood systems. Concretely, knowledge granulation is introduced to propose three new measures: neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation. Moreover, some significant propositions of the three new measures are studied, such as granulation monotonicity. In the end, two examples are designed to illustrate and compare them, and the results show that three new measures can achieve better performance. Neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation are actually reasonable and effective.

2 Neighborhood Rough Sets

In this section, we review some basic concepts. In [9], Chen proposed three uncertainty measures for evaluating uncertainty of an information system or a decision system in neighborhood rough set theory, and they are neighborhood accuracy and neighborhood roughness for an information system as well as neighborhood approximation accuracy for a decision system.

Definition 1 [15]**.** Let $IS = (U, C, V, f, \delta)$ be a neighborhood information system. Herein, U is a nonempty finite set of objects called the universe; C is a nonempty finite set of attributes; V is the union of attribute domains such that $V = \cup_{c \in C} V_c$; for any $c \in C$, there exists a mapping $U \rightarrow V_c$, where V_c is the set of values and it is normalized between 0 and 1, δ is a neighborhood parameter $[0, 1]$. More specially, $DS = (U, C \cup D, V, f, \delta)$ is called a neighborhood decision system, where C is a set of condition attributes, D is a decision attribute.

Definition 2 [15]**.** Let x and y be two objects in N-dimensional feature space $C = \{a_1, a_2, \cdots, a_N\}$, $v(x, a_i)$ denotes the value of sample x in the ith dimension a_i, then a general metric, named Minkowsky distance, is defined by:

$$d_C(x, y) = (\sum_{i=1}^{N} |v(x, a_i) - v(y, a_i)|^p)^{\frac{1}{p}}. \quad (1)$$

Herein, the distance can be called Manhattan distance if $p = 1$; it is called Euclidean distance if $p = 2$; while it is called Chebychev distance if $p = \infty$.

This paper mainly uses the Euclidean distance.

Definition 3 [15]**.** Give $IS = (U, C, V, f, \delta)$ and a distance function $d : U \times U \longrightarrow [0, 1]$. For any attribute subset $B \subseteq C$ and a neighborhood parameter $\delta \in [0, 1]$, B determines a similarity relation denoted by $NR_\delta(B)$ in the following form

$$NR_\delta(B) = \{(x, y) \in U \times U | d_B(x, y) \leq \delta\}.$$

The neighborhood class $n_B^\delta(x)$ of $x \in U$ in the subset B is defined by:

$$n_B^\delta(x) = \{y | x, y \in U, d_B(x, y) \leq \delta\}.$$

The $d_B(x, y)$ is a distance function, and satisfies the following four conditions:

(1) $d_B(x, y) \geq 0$ (Distances cannot be negative);
(2) $d_B(x, y) = 0$, if and only if $x = y$;
(3) $d_B(x, y) = d_B(y, x)$ (Distance is symmetric);
(4) $d_B(x, y) + d_B(y, z) \geq d_B(x, z)$, which means the triangular inequality.

Obviously, the neighborhood relation is a similarity relation, which satisfies reflexivity and symmetry. Specially, $n_B^\delta(x)$ is an equivalence class and $NR_\delta(B)$ is an equivalence relation if $\delta = 0$, and this case is applicable to categorical data.

Proposition 1 [9]**.** Give $IS = (U, C, V, f, \delta)$. For $P, Q \subseteq C$, and $x \in U$, we have:

(1) if $Q \subseteq P$, then $n_P^\delta(x) \subseteq n_Q^\delta(x)$;
(2) if $0 \leq \gamma \leq \delta \leq 1$, then $n_P^\gamma(x) \subseteq n_P^\delta(x)$;
(3) $n_P^\delta(x) = \bigcap_{p \in P} n_p^\delta(x)$;
(4) $n_P^\delta(x) \neq 0$, and $\bigcup_{x \in U} n_p^\delta(x) = U$.

Definition 4 [9]**.** Give $IS = (U, C, V, f, \delta)$. For a subset $X \subseteq U$, and an attribute subset $B \subseteq C$, the B-lower and B-upper approximations of X are defined, respectively, as follows:

$$\underline{B}^\delta(X) = \{x \in U | n_B^\delta(x) \subseteq X\},$$

$$\overline{B}^\delta(X) = \{x \in U | n_B^\delta(x) \cap X \neq \emptyset\}.$$

Definition 5 [9]**.** Give $IS = (U, C, V, f, \delta)$. For a domain subset $X \subseteq U$, an attribute subset $B \subseteq C$, accuracy and roughness of X with respect to B under neighborhood system are defined, respectively, as follows:

$$\alpha_B^\delta(X) = \frac{|\underline{B}^\delta(X)|}{|\overline{B}^\delta(X)|},$$

$$\rho_B^\delta(X) = 1 - \alpha_B^\delta(X).$$

Definition 6 [9]**.** Give $DS = (U, C \cup D, V, f, \delta)$, let $D = \{D_1, D_2, \cdots, D_m\}$ be equivalence classes constituted by decision attribute D on the universe U, and the condition attribute subset $B \subseteq C$. The approximation accuracy of D with respect to B under neighborhood system is defined as

$$\gamma_B^\delta(D) = \frac{\sum_{D_j \in D} |\underline{B}^\delta(D_j)|}{\sum_{D_j \in D} |\overline{B}^\delta(D_j)|}. \tag{2}$$

Definitions 5 and 6 proposed three uncertainty existing measures in neighborhood systems, i.e., neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy.

Proposition 2. Give $IS = (U, C, V, f, \delta)$, for $A, B \subseteq C$, $X \subseteq U$, if $\forall x_i \in U, n_A^\delta(x_i) = n_B^\delta(x_i)$, the following property holds:

$$\alpha_A^\delta(X) = \alpha_B^\delta(X).$$

Proof. If $\forall x_i \in U, n_A^\delta(x_i) = n_B^\delta(x_i)$, so $\underline{A}^\delta(X) = \underline{B}^\delta(X)$, and $\overline{A}^\delta(X) = \overline{B}^\delta(X)$, and then $\alpha_A^\delta(X) = \alpha_B^\delta(X)$.

However, the reverse of Proposition 2 is not true, which can be verified by results of Example 4.2.

Definition 7 [13]**.** Give $DS = (U, C \cup D, V, f, \delta)$. An attribute subset $B \subseteq C$, $x_i \in U$, $n_B^\delta(x_i)$ is a neighborhood class. The knowledge granulation of DS with respect to B is defined as

$$GK_\delta(B) = \frac{1}{|U|^2} \sum_{i=1}^{|U|} |n_B^\delta(x_i)|. \tag{3}$$

Definition 7 proposed the definition of knowledge granulation, which is the basis of three new uncertainty existing measures in neighborhood systems.

3 Three Uncertainty Measures in Neighborhood Systems

In this section, we introduce knowledge granulation and propose three new measures. Our basic method is to modify the three existing measures of neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy.

Definition 8. Give $IS = (U, C, V, f, \delta)$. For $X \subseteq U$, $B \subseteq C$, the neighborhood accuracy and neighborhood roughness based on knowledge granulation of X with respect to B are defined, respectively, as follows:

$$\alpha ac_B^\delta(X) = 1 - (1 - \alpha_B^\delta(X))GK_\delta(B), \tag{4}$$

$$\rho ro_B^\delta(X) = 1 - \alpha ac_B^\delta(X) = \rho_B^\delta(X)GK_\delta(B). \tag{5}$$

Definition 9. Give $DS = (U, C \cup D, V, f, \delta)$, let $D = \{D_1, D_2, \cdots, D_m\}$, and give $B \subseteq C$. The neighborhood approximation accuracy based on knowledge granulation of X with respect to B is defined by:

$$\gamma ac_B^\delta(D) = 1 - (1 - \gamma_B^\delta(D))GK_\delta(B). \tag{6}$$

In the Definitions 8 and 9, we give the neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation. Next, we discuss some relevant propositions.

Proposition 3. Give $IS = (U, C, V, f, \delta)$, $A \subseteq B \subseteq C$ and $X \subseteq U$, then

$$\alpha ac_A^\delta(X) \leq \alpha ac_B^\delta(X),$$

$$\rho ro_B^\delta(X) \leq \rho ro_A^\delta(X).$$

Proof. Let $A \subseteq B$. By Proposition 1, for $\forall x \in U$, we can know that $n_B^\delta(x) \subseteq n_A^\delta(x)$, then $|n_B^\delta(x)| \leq |n_A^\delta(x)|$. So $GK_\delta(B) \leq GK_\delta(A)$, and $\rho_B^\delta(X) \leq \rho_A^\delta(X)$, and then $\rho ro_B^\delta(X) = \rho_B^\delta(X)GK_\delta(B) \leq \rho_A^\delta(X)GK_\delta(A) = \rho ro_A^\delta(X)$. It is easy to obtain $\alpha ac_A^\delta(X) \leq \alpha ac_B^\delta(X)$.

According to Proposition 3, the neighborhood accuracy based on knowledge granulation of X with respect to B increases, and neighborhood roughness based on knowledge granulation of X with respect to B decreases, where B is finer.

Proposition 4. Give $IS = (U, C, V, f, \delta)$. $B \subseteq C$, $0 \leq \gamma \leq \delta \leq 1$, $X \subseteq U$, then

$$\alpha ac_B^\gamma(X) \geq \alpha ac_B^\delta(X),$$

$$\rho ro_B^\gamma(X) \leq \rho ro_B^\delta(X).$$

Proof. By $0 \leq \gamma \leq \delta \leq 1$, for any $x_i \in U$, we have $n_B^\gamma(x_i) \subseteq n_B^\delta(x_i)$, then $|n_B^\gamma(x_i)| \leq |n_B^\delta(x_i)|$, $\underline{B}^\gamma(X) \supseteq \underline{B}^\delta(X)$, $\overline{B}^\gamma(X) \subseteq \overline{B}^\delta(X)$, so $|\underline{B}^\gamma(X)| \geq |\underline{B}^\delta(X)|$, $|\overline{B}^\gamma(X)| \leq |\overline{B}^\delta(X)|$, and $\rho ro_B^\gamma(X) = \rho_B^\gamma(X)GK_\gamma(B)$ $= (1 - \frac{|\underline{B}^\gamma(X)|}{|\overline{B}^\gamma(X)|})\frac{1}{|U|^2}\sum_{i=1}^{|U|}|n_B^\gamma(x_i)| \leq (1 - \frac{|\underline{B}^\delta(X)|}{|\overline{B}^\delta(X)|})\frac{1}{|U|^2}\sum_{i=1}^{|U|}|n_B^\delta(x_i)| = \rho ro_B^\delta(X)$. So $\alpha ac_B^\gamma(X) \geq \alpha ac_B^\delta(X)$.

According to Proposition 4, the neighborhood accuracy based on knowledge granulation of X with respect to B decreases, and neighborhood roughness based on knowledge granulation of X with respect to B increases, where δ is bigger than γ. The results will be verified by Examples 4.1 and 4.2 in next section. Thus Proposition 4 reflects the parameter monotonicity.

Proposition 5. Let $DS = (U, C \cup D, V, f, \delta)$ be a neighborhood decision system, and $D = \{D_1, D_2, \cdots, D_m\}$, $B \subseteq C$, $0 \leq \gamma \leq \delta \leq 1$, then

$$\gamma ac_B^{\gamma}(D) \geq \gamma ac_B^{\delta}(D).$$

Proof. By $0 \leq \gamma \leq \delta \leq 1$, for any $x_i \in U$, $D_j \in D$, we have $n_B^{\gamma}(x_i) \subseteq n_B^{\delta}(x_i)$, then $|n_B^{\gamma}(x_i)| \leq |n_B^{\delta}(x_i)|$, $\underline{B}^{\gamma}(D_j) \supseteq \underline{B}^{\delta}(D_j)$, $\overline{B}^{\gamma}(D_j) \subseteq \overline{B}^{\delta}(D_j)$, so $\sum_{D_j \in D} |\underline{B}^{\gamma}(D_j)| \geq \sum_{D_j \in D} |\underline{B}^{\delta}(D_j)|$, $\sum_{D_j \in D} |\overline{B}^{\gamma}(D_j)| \leq \sum_{D_j \in D} |\overline{B}^{\delta}(D_j)|$, and so $\gamma_B^{\delta}(D) \leq \gamma_B^{\gamma}(D)$, $GK_{\delta}(B) \geq GK_{\gamma}(B)$. Then $\gamma ac_B^{\gamma}(D) \geq \gamma ac_B^{\delta}(D)$.

According to Proposition 5, the neighborhood approximation accuracy based on knowledge granulation of D with respect to B increases, when δ becomes smaller. The results are verified by Examples 4.1 and 4.2.

Proposition 6. Give $IS = (U, C, V, f, \delta)$. $A, B \subseteq C$, if $\forall x_i \in U, n_A^{\delta}(x_i) = n_B^{\delta}(x_i)$, then

$$\alpha ac_A^{\delta}(X) = \alpha ac_B^{\delta}(X).$$

Proposition 7. Give $DS = (U, C \cup D, V, f, \delta)$. Let $D = \{D_1, D_2, \cdots, D_m\}$, $A, B \subseteq C$, if $\forall x_i \in U, n_A^{\delta}(x_i) = n_B^{\delta}(x_i)$, then

$$\gamma ac_A^{\delta}(D) = \gamma ac_B^{\delta}(D).$$

Propositions 6 and 7 are similar to proof of Proposition 2.

Proposition 8 (Minimum, Maximum)**.** Give $IS = (U, C, V, f, \delta)$. $B \subseteq C$, $X \subseteq U$.

(1) The minimum neighborhood accuracy based on knowledge granulation of X with respect to B is zero ($\alpha ac_B^{\delta}(X) = 0$), if and only if $\forall x \in U$, $n_B^{\delta}(x) = U$;
(2) The maximum neighborhood accuracy based on knowledge granulation of X with respect to B is one ($\alpha ac_B^{\delta}(X) = 1$), if and only if $\forall x \in U$, $n_B^{\delta}(x) = \{x\}$.

Obviously, let $IS = (U, C, V, f, \delta)$. $0 \leq \alpha ac_B^{\delta}(X) \leq 1$ for any subset B of C.

Proposition 9 (Minimum, Maximum)**.** Give $DS = (U, C \cup D, V, f, \delta)$. Let $D = \{D_1, D_2, \cdots, D_m\}$, $B \subseteq C$.

(1) The minimum neighborhood approximation accuracy based on knowledge granulation of D with respect to B is zero ($\gamma ac_B^{\delta}(D) = 0$), if and only if $\forall x \in U$, $n_B^{\delta}(x) = U$;

(2) The maximum neighborhood approximation accuracy based on knowledge granulation of D with respect to B is one ($\gamma ac_B^\delta(D) = 1$), if and only if $\sum_{D_j \in D} |\underline{B}^\delta(D_j)| = \sum_{D_j \in D} |\overline{B}^\delta D_j|$.

Obviously, let $DS = (U, C \cup D, V, f, \delta)$. $0 \le \gamma ac_B^\delta(D) \le 1$ for any subset B of C.

4 Two Illustrative Examples

Now, we give two examples to calculate neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation, and analyze monotonicity of the three uncertainty measures.

Table 1. A neighborhood decision system

U	c_1	c_2	c_3	D
x_1	0.8	1	0.8	Y
x_2	0.6	1	0.7	Y
x_3	1	0.8	1	Y
x_4	0.3	0.7	0	N
x_5	1	0	0.3	Y
x_6	0.2	0.3	0.6	N
x_7	0	0.4	0.2	N

Example 4.1. Suppose a neighborhood decision system $DS = (U, C \cup D, V, f, \delta)$ shown in Table 1, where $U = \{x_1, x_2, \cdots, x_7\}$, $C = \{c_1, c_2, c_3\}$, $D = \{D_1, D_2\} = \{\{x_1, x_2, x_3, x_5\}, \{x_4, x_6, x_7\}\}$. It has three real-valued condition attributes and a single categorical decision attribute. In the table, all the value of condition attributes are normalized between 0 and 1, and let $\delta = 0.4$.

Suppose $A = \{c_1\}$, $B = \{c_1, c_2\}$, $X = \{x_4, x_6, x_7\}$. The neighborhood lower approximation and upper approximation of X with respect to the condition attribute sets A, B are generated as follows:

$$\underline{A}^{0.4}(X) = \{x_7\}, \qquad \overline{A}^{0.4}(X) = \{x_2, x_4, x_6, x_7\},$$
$$\underline{B}^{0.4}(X) = \{x_6, x_7\}, \qquad \overline{B}^{0.4}(X) = \{x_2, x_4, x_6, x_7\}.$$

The neighborhood accuracy and neighborhood roughness of X with respect to A, B are computed respectively:

$$\alpha_A^{0.4}(X) = \frac{|\underline{B}^{0.4}(X)|}{|\overline{B}^{0.4}(X)|} = \frac{1}{4}, \qquad \rho_A^{0.4}(X) = 1 - \alpha_A^{0.4}(X) = \frac{3}{4},$$
$$\alpha_B^{0.4}(X) = \frac{|\underline{B}^{0.4}(X)|}{|\overline{B}^{0.4}(X)|} = \frac{1}{2}, \qquad \rho_B^{0.4}(X) = 1 - \alpha_B^{0.4}(X) = \frac{1}{2}.$$

The neighborhood lower approximation and upper approximation of D with respect to A, B are generated as follows:

$\underline{A}^{0.4}(D) = \underline{A}^{0.4}(D_1) \cup \underline{A}^{0.4}(D_2) = \{x_1, x_3, x_5\} \cup \{x_7\}$,
$\overline{A}^{0.4}(D) = \overline{A}^{0.4}(D_1) \cup \overline{A}^{0.4}(D_2) = \{x_1, x_2, x_3, x_4, x_5, x_6\} \cup \{x_2, x_4, x_6, x_7\}$,
$\underline{B}^{0.4}(D) = \underline{A}^{0.4}(D_1) \cup \underline{A}^{0.4}(D_2) = \{x_1, x_3, x_5\} \cup \{x_6, x_7\}$,
$\overline{B}^{0.4}(D) = \overline{B}^{0.4}(D_1) \cup \overline{B}^{0.4}(D_2) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{x_2, x_4, x_6, x_7\}$.
Thus, we can obtain
$\gamma_A^\delta(D) = \frac{|\underline{A}^{0.4}(D_1)|+|\underline{A}^{0.4}(D_2)|}{|\overline{A}^{0.4}(D_1)|+|\overline{A}^{0.4}(D_2)|} = \frac{2}{5}$,
$\gamma_B^\delta(D) = \frac{|\underline{B}^{0.4}(D_1)|+|\underline{B}^{0.4}(D_2)|}{|\overline{B}^{0.4}(D_1)|+|\overline{B}^{0.4}(D_2)|} = \frac{5}{9}$.
Obviously, if $A \subseteq B$, we have the following results:
$\alpha_A^\delta(X) \leq \alpha_B^\delta(X)$, $\rho_B^\delta(X) \leq \rho_A^\delta(X)$, $\gamma_A^\delta(D) \leq \gamma_B^\delta(D)$.

When B becoming finer, we can know that neighborhood accuracy and neighborhood approximation accuracy X with respect to B increase, while the neighborhood roughness of X with respect to B decreases.

$GK^{0.4}(A) = \frac{29}{49}$, $GK^{0.4}(B) = \frac{3}{7}$,
$\alpha ac_A^{0.4}(X) = 0.5561$, $\alpha ac_B^{0.4}(X) = 0.7857$,
$\rho ro_A^{0.4}(X) = 0.4439$, $\rho ro_B^{0.4}(X) = 0.2143$,
$\gamma ac_A^{0.4}(D) = 0.6449$, $\gamma ac_B^{0.4}(D) = 0.8095$.
Obviously, if $A \subseteq B$, we have the following results from the example:
$\alpha ac_A^\delta(X) \leq \alpha ac_B^\delta(X)$, $\rho ro_B^\delta(X) \leq \rho ro_A^\delta(X)$, $\gamma ac_A^\delta(D) \leq \gamma ac_B^\delta(D)$.

From Example 4.1, when B becomes finer, we can know that neighborhood accuracy and neighborhood approximation accuracy based on knowledge granulation X with respect to B increase, while the neighborhood roughness based on knowledge granulation of X with respect to B decreases.

In generally, three existing uncertainty measures and three new uncertainty measures have same monotonicity. However, in some situations, neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy, all can not provide enough information. The limitations are revealed by the following example.

Example 4.2 (continue Example 4.1)**.** Let $\delta = 0.3$.
Suppose $A = \{c_1\}$, $B = \{c_1, c_2\}$, $X = \{x_4, x_6, x_7\}$.

The neighborhood classes of objects with respect to A, B are generated as follows:

$$
\begin{array}{ll}
n_A^{0.3}(x_1) = \{x_1, x_2, x_3, x_5\}, & n_A^{0.3}(x_2) = \{x_1, x_2, x_4\}, \\
n_A^{0.3}(x_3) = n_A^{0.3}(x_5) = \{x_1, x_3, x_5\} & n_A^{0.3}(x_4) = \{x_2, x_4, x_6, x_7\}, \\
n_A^{0.3}(x_6) = n_A^{0.3}(x_7) = \{x_4, x_6, x_7\}, & \\
n_B^{0.3}(x_1) = \{x_1, x_2, x_3\} & n_B^{0.3}(x_2) = \{x_1, x_2, x_4\}, \\
n_B^{0.3}(x_3) = \{x_1, x_3\}, & n_B^{0.3}(x_4) = \{x_2, x_4, x_7\}, \\
n_B^{0.3}(x_5) = \{x_5\}, & n_B^{0.3}(x_6) = \{x_6, x_7\}, \\
n_B^{0.3}(x_7) = \{x_4, x_6, x_7\}. &
\end{array}
$$

The neighborhood lower approximation and upper approximation of X with respect to A, B are generated as follows:

$$
\begin{array}{ll}
\underline{A}^{0.3}(X) = \{x_6, x_7\}, & \overline{A}^{0.3}(X) = \{x_2, x_4, x_6, x_7\}, \\
\underline{B}^{0.3}(X) = \{x_6, x_7\}, & \overline{B}^{0.3}(X) = \{x_2, x_4, x_6, x_7\}.
\end{array}
$$

The neighborhood accuracy and neighborhood roughness of X with respect to A, B are computed respectively:

$$
\alpha_A^{0.3}(X) = \alpha_B^{0.3}(X) = \tfrac{1}{2}, \qquad \rho_A^{0.3}(X) = \rho_B^{0.3}(X) = \tfrac{1}{2}.
$$

The neighborhood lower approximation and upper approximation of D with respect to A, B are generated as follows:

$$
\begin{array}{l}
\underline{A}^{0.3}(D) = \underline{A}^{0.3}(D_1) \cup \underline{A}^{0.3}(D_2) = \{x_1, x_3, x_5\} \cup \{x_6, x_7\}, \\
\overline{A}^{0.3}(D) = \overline{A}^{0.3}(D_1) \cup \overline{A}^{0.3}(D_2) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{x_2, x_4, x_6, x_7\}, \\
\underline{B}^{0.3}(D) = \underline{B}^{0.3}(D_1) \cup \underline{B}^{0.3}(D_2) = \{x_1, x_3, x_5\} \cup \{x_6, x_7\}, \\
\overline{B}^{0.3}(D) = \overline{B}^{0.3}(D_1) \cup \overline{B}^{0.3}(D_2) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{x_2, x_4, x_6, x_7\}.
\end{array}
$$

The neighborhood approximation accuracy of D with respect to A, B are computed respectively:

$$
\gamma_A^{0.3}(D) = \frac{|\underline{A}^{0.3}(D_1)|+|\underline{A}^{0.3}(D_2)|}{|\overline{A}^{0.3}(D_1)|+|\overline{A}^{0.3}(D_2)|} = \frac{5}{9}, \qquad \gamma_B^{0.3}(D) = \frac{|\underline{B}^{0.3}(D_1)|+|\underline{B}^{0.3}(D_2)|}{|\overline{B}^{0.3}(D_1)|+|\overline{B}^{0.3}(D_2)|} = \frac{5}{9}.
$$

The neighborhood accuracy and roughness based on knowledge granulation of X with respect to B, neighborhood approximation accuracy based on knowledge granulation of D with respect to B, respectively, are computed as follows:

$$
\begin{array}{ll}
GK^{0.3}(A) = \frac{23}{49}, & GK^{0.3}(B) = \frac{17}{49}, \\
\alpha ac_A^{0.3}(X) = 0.7653, & \alpha ac_B^{0.3}(X) = 0.8265, \\
\rho ro_A^{0.3}(X) = 0.2347, & \rho ro_B^{0.3}(X) = 0.1735, \\
\gamma ac_A^{0.3}(D) = 0.7914, & \gamma ac_B^{0.3}(D) = 0.8458.
\end{array}
$$

In Example 4.2, by comparing, neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation can be more well used for evaluating uncertainty in neighborhood systems, so they can more precisely describe the monotonicity. The results show that neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation are more reasonable and effective.

5 Conclusion

In this paper, on the basis of neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy, we introduce knowledge granulation and propose three new measures: neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation. They can be used more well for evaluating uncertainty in neighborhood systems. Moreover, we studied some important propositions of the three new measures, such as granulation monotonicity. Finally, two illustrative examples show that the proposed new uncertainty measures can be used more precision for evaluating uncertainty in neighborhood systems. By illustrating comparing the classification accuracy of three new measures, the results show that neighborhood accuracy, neighborhood roughness and neighborhood approximation accuracy based on knowledge granulation can achieve better performance. In summary, the three new uncertainty measures become more reasonable and effective.

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An Approach in Solving Data Envelopment Analysis with Stochastic Data

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Abstract. The importance and necessity of the data envelopment analysis as a relevant and effective instrument in investigation of the performance of units studied such as banks and so forth is an evident issue. One of the main issues we encounter is envelope analysis of data with random variable values. In this research, by explaining the general model of the data envelopment analysis models and inspired by how to work on interval input/output, there would a solution for random state provided. In fact, the interval the random variable varies in is considered and using envelope analysis on interval data, we will determine the effective unit.

Keywords: Data envelopment analysis · Stochastic programing · Stochastic input and output · Interval data

1 Introduction

Data envelopment analysis is a decision making instrument and approach about the organizations performance. In this structure, the efficiency and effectiveness of entities would be studied. Throughout the world either in developed or developing countries, in a long-term planning for the future, there have always been 3 substantial principles of the efficiency increasing and enhancement so that they can at least reach their minimum economic growth. As theories developed, getting together by economies and elimination of their distance caused a competition in production and world trading. In this structure, undoubtedly, making use of different data in line with best selecting the entities for investment is a necessary issue. In this regard, lack of how to select the best companies would be eliminated by data envelopment analysis. Hence, the data envelopment analysis is called a mathematical model or planning provides the decision maker with the best selection based on available data. The importance of study in this area can be seen in different situations. For example, in a portfolio containing the risk assets such as stocks, the chance in structure of each model planning is subjected to issues such as calculations. In this article, we use available approaches in planning the data envelopment analysis issues for envelope analysis modeling using random data. In fact in this paper, we study the random data in data envelopment analysis modeling

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structure. Studies conducted on data envelopment analysis has begun since 1950 and early 1960s. In fact, the first model of planning was announced by Farrell in 1975 whose structure was non-parametric and had one input and output. After Farrell, one can mention the Cupper, Charles and Rhodes (1978). This research fixated the studies in the area of data envelopment analysis. Their model was the expansion of Farrell's model with input/output variables. Their model is called CCR [1]. Using the definition of efficiency (P) as:

$$P = \frac{U * Y}{V * X}. \tag{1}$$

Where U, Y, V and X are output weight, output value, input weight and input value, respectively, they planned the chance constraint programming in case of a certain event occurrence [2]. This programming holds each chance in an event like λ. Land et al. extended this model to obtain the systems' performance with the only random output [3]. Olsen and Petersen used the Land et al.'s idea in CCR model [4]. Kwakernaak was the first to use the random concept with mixed fuzzy in data envelopment analysis [5]. Other articles in this regard are Jing Liang et al.'s research worked on a random environment model DEA which can measure the environmental performance under random conditions [6]. In next chapter, the CCR model, the interval solutions, random problem and necessary tools are investigated. Then the proposed state is provided and the interval solution and averaging procedures are proposed. Finally, the numerical results are analyzed.

2 The Basic Methods

In this chapter, 2 mathematical expectation and interval envelope analysis are investigated which are instruments to solve the random problems used.

2.1 Data Envelopment Analysis

The first data overage analysis model as a comprehensive one was proposed by Edward Roders in his PhD dissertation. In this thesis, the achievement of the students of Carnegie in USA was investigated and the CCR model was provided for the teacher's performance [7]:

$$\begin{aligned}
& Min\,\theta \\
& s.t. \\
& \sum_{t=1}^{n} \lambda_t X_t \le \theta X_k \; t = 1, \ldots, n \;\; t \ne k, \\
& \sum_{t=1}^{n} \lambda_t Y_t \ge Y_k \; t = 1, \ldots, n \;\; t \ne k, \\
& \lambda_t \ge 0 \;\; t = 1, \ldots, n.
\end{aligned} \tag{2}$$

In this structure, the efficiency value of unit k (DMU_k) is obtained. After CCR programming, the BCC model was build adding $\sum_{t=1}^{n} \lambda_t = 1$ to above model which is used by Charles used in order to make the efficiency fixed relative to the scale [8]. Above problem is as follow:

$$\begin{aligned} & MAX\ U^T Y_k \\ & s.t. \\ & V^T X_k = 1, \\ & U^T Y_t - V^T X_t \leq 0 \quad t = 1, \ldots, n, t \neq k, \\ & U, V \geq 0. \end{aligned} \tag{3}$$

2.1.1 Interval Data Envelopment Analysis

In interval data envelopment analysis method, the value of each data is in an interval and can be variable in this interval. If each of n units available uses m input to produce s output, then K^{th} unit performance is DMU_K which is from $\{X_{jk} | j = 1, \ldots, n\}$ in order to generate $\{Y_{ik} | i = 1, \ldots, s\}$. Now, if input and output are interval ones and have lower and upper bounds, then for unit k, the input j is as $\left[X_{jk}^L, X_{jk}^U\right]$ and output i is as $\left[Y_{ik}^L, Y_{ik}^U\right]$ denoted where L is lower bound and U is upper bound of interval and $X_{jk}^L, Y_{ik}^L \geq 0$ cannot be neglected. The model considered for interval state is written in 2 forms [9]. For upper bound calculation it is given as:

$$\begin{aligned} & MAX \sum_{i=1}^{s} U_i^T Y_{ik}^U \\ & s.t. \\ & \sum_{j=1}^{m} V_j^T X_{jk}^L = 1, \\ & \sum_{i=1}^{s} U_i^T Y_{it}^U - \sum_{j=1}^{m} V_j^T X_{jt}^L = 1 \, t = 1, \ldots, n, t \neq k, \\ & U_i, V_j \geq 0 \, i = 1, \ldots, m j = 1, \ldots, s. \end{aligned} \tag{4}$$

And, for lower bound calculation it is given as:

$$\begin{aligned} & MAX \sum_{i=1}^{s} U_i^T Y_{ik}^L \\ & s.t. \\ & \sum_{j=1}^{m} V_j^T X_{jk}^L = 1, \\ & \sum_{i=1}^{s} U_i^T Y_{it}^U - \sum_{j=1}^{m} V_j^T X_{jt}^L = 1 \, t = 1, \ldots, n \, t \neq k, \\ & U_i, V_j \geq 0 \, i = 1, \ldots, m j = 1, \ldots, s. \end{aligned} \tag{5}$$

2.2 Mathematical Expectation

The set S including all possible events of an experiment is called the sample space. This set can be finite or infinite, countable or uncountable. Suppose F is a member of subsets of S with following features:

1. If $A \in F$, then F includes the complement $A\,(A^c)$ too.
2. $S \in F$,
3. $\cup_{i \in I} A_i \in F$ in which I is the index set and for $i \in I$, we have $A_i \in F$.

The mapping $P : F \to [01]$ is called a probable value if:

1. $P(S) = 1$,
2. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ in which for each i, $A_i \in F$ for $i \neq j$ and $A_i \neq A_j$.

The pair $(S,\ F)$ is called measure space and (S,F,P) is probable space.

In probable space, the mapping $X : (S, F) \to (\mathbb{R}, \beta)$ in which β is the set of all open sets in real digits set and is called random variable when $X^{-1}(B) \in F$ in which $B \in \beta$. Considering the probable space (S, F, P) and random variable X, the mathematical expectation, average and expected values or considered value expected from a random variable equals sum of multiplication results in their probability which is denoted as $E[X]$. In finite discrete state, the expectancy is denoted as $E[X] = \sum_{i=1}^{n} \chi_i p(\chi_i)$. In continuous state we will have $E[X] = \int \chi dP(\chi)$ if $f(x)$ is the distribution function of this random variable, $f(x) * dx = dP(x)$. Therefore, we will have $E[X] = \int \chi f(\chi) d\chi$ [10].

In this chapter, at first the data envelopment analysis and programming like envelope and multiple ones are investigated and then the interval model is introduced and using the classical approach, solving methodology would be provided and finally, the mathematical expectation is defined. All mentioned issues are solutions for programing problem solving with random data in data envelopment analysis which are explained in next chapter.

3 Solving Data Envelopment Analysis with Stochastic Data

In previous sections, the approaches used for random programming in data envelopment analysis were investigated. At first, we state the problem. Suppose in a structure with n unit, each unit needs for producing s output ad m input. In fact the data of this programming is as follow (Table 1):

Table 1. Input and output of system

DMU_1	DMU_2	·	$DMU_{(n-1)}$	DMU_n
X_{11}	X_{12}	·	$X_{1(n-1)}$	X_{1n}
X_{21}	X_{22}	·	$X_{2(n-1)}$	X_{2n}
X_{31}	X_{32}	·	$X_{3(n-1)}$	X_{3n}
·	·		·	·
·	·		·	·
·	·		·	·
$X_{(m-2)1}$	$X_{(m-2)2}$	·	$X_{(m-2)(n-1)}$	$X_{(m-2)n}$
$X_{(m-1)1}$	$X_{(m-1)2}$	·	$X_{(m-1)(n-1)}$	$X_{(m-1)n}$
X_{m1}	X_{m2}	·	$X_{m(n-1)}$	X_{mn}
Input	Input		Input	Input
Y_{11}	Y_{12}	·	$Y_{1(n-1)}$	Y_{1n}
Y_{21}	Y_{22}	·	$Y_{2(n-1)}$	Y_{2n}
Y_{31}	Y_{32}	·	$Y_{3(n-1)}$	Y_{3n}
·	·		·	·
·	·		·	·
·	·		·	·
$Y_{(s-2)1}$	$Y_{(s-2)2}$	·	$Y_{(s-2)(n-1)}$	$Y_{(s-2)n}$
$Y_{(s-1)1}$	$Y_{(s-1)2}$	·	$Y_{(s-1)(n-1)}$	$Y_{(s-1)n}$
Y_{s1}	Y_{s2}	·	$Y_{s(n-1)}$	Y_{sn}
Output	Output		Output	Output

In above table, information of n unit of firms is available. Now, the random state is explained. At first, the input variable X_{ij} is considered. This variable is random and has distribution function of $f_{ij}(x)$ or in discrete state, suppose there is a natural digit w for each X_{ij} such that (Table 2):

Table 2. Probability of system's input

	→	$P^{ij}_{1}\left(X^{1}_{ij}\right)=\alpha^{ij}1$
		$P^{ij}_{2}\left(X^{2}_{ij}\right)=\alpha^{ij}2$
		. . .
		$P^{ij}_{(w-1)}\left(X^{(w-1)}_{ij}\right)=\alpha^{ij}(w-1)$
		$P^{ij}_{w}\left(X^{w}_{ij}\right)=\alpha^{ij}w$

Also, there is a natural digit z for each Y_{ij} such that (Table 3):

Table 3. Probability of system's Output

Y_{ij}	→	$P^{ij}_{1}\left(Y^{1}_{ij}\right)=\gamma^{ij}1$
		$P^{ij}_{2}\left(Y^{2}_{ij}\right)=\gamma^{ij}2$
		. . .
		$P^{ij}_{(z-1)}\left(Y^{(z-1)}_{ij}\right)=\gamma^{ij}(z-1)$
		$P^{ij}_{z}\left(Y^{z}_{ij}\right)=\gamma^{ij}z$

After statement of the problem, these are solved. Now, we will explain 2 solutions for these problems based on input/output variables. The first one is to use mathematical expectation. In fact, we calculate the average value of these input/output random variables and them we will solve the problem and obtain the best DMU. The discrete state for each of X_{ij} is given as:

$$X_{ij} = \sum_{t=1}^{w} P(X_{ij}^{t})X_{ij}^{t} = \sum_{t=1}^{w} \alpha_{ij}^{t} X_{ij}^{t}. \tag{6}$$

And for each Y_{ij} is given as:

$$Y_{ij} = \sum_{t=1}^{z} P\left(Y_{ij}^{t}\right) Y_{ij}^{t} = \sum_{t=1}^{z} \gamma^{ij} t * Y_{ij}^{t}. \tag{7}$$

In continuous state with distribution function $f_{ij}(x)$ for each X_{ij}, the random state would be certain as follow:

$$X_{ij} = \int x f_{ij}(x) dx. \tag{8}$$

Also for outputs, each would be as follow with its distribution function:

$$Y_{ij} = \int y f_{ij}(y) dy. \tag{9}$$

By this, the problem would be solves as reference model (abovementioned). In mathematical expectation model, we consider other solutions due to large volume of the calculations particularly in continuous form. The other approach considered for this problem is interval approach. In this structure, we consider the minimum and maximum possible for each input/output random variable. Then, using these value whose lower and upper bounds are minimum and maximum random variables, respectively, we will solve the problem by converting using the interval approach. In discrete finite state, for each X_{ij} the intervals are given as:

$$a_{ij} = Min\left\{X_{ij}^{1}, X_{ij}^{2}, X_{ij}^{3}, \ldots, X_{ij}^{(w-2)}, X_{ij}^{(w-1)}, X_{ij}^{w}\right\}, \tag{10}$$

$$b_{ij} = Max\left\{X_{ij}^{1}, X_{ij}^{2}, X_{ij}^{3}, \ldots, X_{ij}^{(w-2)}, X_{ij}^{(w-1)}, X_{ij}^{w}\right\}. \tag{11}$$

Now, we substitute X_{ij} with $\left[a_{ij}, b_{ij}\right]$. For Y_{ij}, we will have:

$$a'_{ij} = Min\left\{Y_{ij}^{1}, Y_{ij}^{2}, Y_{ij}^{3}, \ldots, Y_{ij}^{(z-2)}, Y_{ij}^{(z-1)}, Y_{ij}^{z}\right\}, \tag{12}$$

$$b'_{ij} = Max\left\{Y_{ij}^{1}, Y_{ij}^{2}, Y_{ij}^{3}, \ldots, Y_{ij}^{(z-2)}, Y_{ij}^{(z-1)}, Y_{ij}^{z}\right\}. \tag{13}$$

Here as above, Y_{ij} is substituted with $\left[a'_{ij}, b'_{ij}\right]$.

These intervals include all possible states and based on classical interval approach of data envelopment analysis, the efficiency obtained will be for all points. Theo other significant case is the random processes; i.e. each input/output follows a random variable. In this case, using the approaches related to each, the maximum and minimum would be obtained and the interval considered is determined. An approach for estimation of maximum and minimum proposed I that using historical data related to those processes (such as past prices of stocks which follows a particular process) and discretization of time, at first the interval considered is determined until the end and finding the maximum and minimum in each section of time and averaging them and the problem is solved. It is necessary to note that this is a proposed approach and one can use other approached to make intervals based on the nature and structure of process. Now, the tables (reference) include the intervals instead of random variables and using the model (reference) DMUs considered are obtained and the units are analyzed. In following, the numerical results are presented.

3.1 Numerical Result

Here, we defied several random paths (Brownian motion multiple-process absolute value) as input and output of each unit and then, based on above approach, we investigate the efficiency of each unit (Table 4).

Table 4. Estimate of input and output with end method of last part

$DMU1$		$DMU2$		$DMU3$	
[0.0001	1.0219]	[0.0014	0.6167]	[0.0046	1.4008]
[0.0009	1.0534]	[0.0012	0.7511]	[0.0111	1.7530]
Input	Input	Input	Input	Input	Input
[0.0014	1.4095]	[0.0011	0.6177]	[0.0001	0.9569]
Output	Output	Output	Output	Output	Output

After solving the obtained models, the efficiency interval $[E_k^L, E_k^U]$ is obtained. Based on this, there is no unit being efficient for all values available in interval. As well, units 1 and 3 would not be efficient for none of the values in these intervals.

4 Conclusion

Based on studies on random data envelopment analysis and considering the classical approaches, we solved the data envelopment analysis problem. In this model, we had random input and output. Then by defining the probable space and mathematical expectation, we proposed an approach for solving the random problem by changing it to certain one. Also, for solving these problems the interval approach was used in which the beginning and end of the maximum and minimum intervals was considered for variables and solved with classical interval approach. In further studies, one can analyze the sensitivity by investigating the intervals and their effect on solution.

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Why Do Young People Hate on the Internet?

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Abstract. The paper describes hating practices among young people. First a review of current literature was made and then a study in Poland was conducted. This study was the first phase of research including young people from Poland, India, Romania and Bulgaria. Results prove that hating activities are present among youngsters and, although some information about dealing with them is commonly known, there is still a large need for child and teenager' education in this field.

Keywords: Haters · Social media · Cyber aggression

1 Introduction

The internet and social media play a crucial role in present day young people's lives. It is a powerful force that affects social space and social context and influences whole communities as well as individuals (Nowak and Krejtz 2006). Though it seems obvious that the internet modifies industry, economy, society and almost each aspect of everyday life (Castells 2009), this particular relationship between youngsters and cyberspace should be talked about and studied as it may influence their personal development, providing new opportunities as well as threats and dangers.

Exploring the internet can be perceived as a process performed by a core part of society: people work, study, shop, entertain and socialize online. Still, there are some differences, some of them usually related to age. Each generation (X, Y, Z, alpha) explore and experience the meanderings of cyberspace more deeply and strongly than older ones. In Prensky (2001) coined the terms 'digital immigrants' and 'digital natives'. According to the Prensky's vision, the first term described people aware of the Internet and modern technology, but born in the days before their widespread deployment. Digital immigrants encountered the Internet at some stage of life. The implication of this situation is the fact that - regardless of proficiency in the use of information technology - they treat real life and cyberspace as a separate sphere. Adapting to a new cyber 'environment', they remain the stranger, not treating it as their own. Prensky describes this phenomenon with the term 'accent', which is to be a

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parallel to the process of learning foreign languages and to emphasize the fact that full assimilation in the new digital environment for them is impossible. On the contrary, digital natives are the people born in the time of the dynamic dissemination of information technologies. For them, the Internet existed 'forever'. For these young people immersion in virtual reality takes the strongest form. As a result, digital natives often treat real and virtual space as one and the same. The symbolic birth date to distinguish immigrants and natives is 1980. People born in this period and later are digital natives, and before that time - immigrants. This division is to emphasize how people at different ages may perceive the Internet and its role in their everyday life, but does not refer to proficiency in the use of this medium.

Among the main patterns of a young person's internet usage the following can be mentioned: immersion in cyberspace, being almost always online, creating new aspects of their own personality in a cyber reality, transferring social life to the Internet. They can be named digital natives, generation Y or Z or – based on their lifestyle not date of birth – generation C, V or L (Hatalska 2008). Their experience with the internet can be generalized and classified but for every one of them it constitutes a unique form of participation in the online reality and creation of their own e-personality.

With the beginning of 21st century, Social Networking Services (SNSs) services increased users' participation in cyberspace. By allowing two-way real time communication and publishing content with ease, they changed the internet form Web 1.0 read-only model to read-write called Web 2.0. (Jabłońska 2015). As a result, every user can publish and comment easily reaching a broad audience. The spread of SNSs has resulted in a steep rise in socializing online. This process can be very beneficial for an individual as it was proven to be positively correlated with physical and mental health, happiness, self-esteem, earnings and business performance (Sabatini and Sarracino 2014). It was a huge change for online socialization; research from the beginning of century reported major concerns about the correlation between internet usage and distraction from social interactions in real life and the impoverishment of human relations (Nie et al. 2002; Wellman and Hampton 2001), but after the spread of SNSs these conclusions have started to change. Lee (2008), Steinfield et al. (2008), Bauernschuster et al. (2010), Näsi et al. (2012) and others have proven that social media positively influence community life and social capital.

Electronic aggression is generally defined as a form of aggressive behavior with the use of communication technologies (Pyżalski 2009). It is the broadest meaning of offensive behavior, going beyond cyberspace. A slightly narrower range belongs to the term of harassment online, which already refers strictly to a particular medium and actions taken through it in order to harm a specific person. The narrowest significance belongs to cyber bullying that is a deliberate and repeated aggressive behavior occurring in a particular social group, which includes both the victim and the assailant (Pyżalski 2009). Hate speech and verbal aggression, which is the main subject of this study, may be perceived as a form of cyber bullying.

Hate speech is a form of verbal aggression. It goes far beyond constructive criticism of expression, content, person, group or social phenomenon. Unsupported by logical argumentation, it is focused on the desire to hurt, damage and besmirch the dignity of a victim. It often takes the form of vulgar, harassing, offensive, violent, disrespectful, disdainful, filled with hatred, jealous or envious comments. This kind of online aggression takes the form of comments and statements that are offensive, insulting, intimidating, threatening, harassing and incite to violence, aggression, hatred, or discrimination (Erjavec and Kovačič 2012a). Some works suggest that the internet (Cohen-Almagor 2011) and SNSs (Oksanen et al. 2014) plays a significant role in distributing hate and in transforming speech into action as well as providing a new environment for making hateful material more visible for a broad audience. Imaginable consequences of this phenomenon seem to be so crucial and threatening that hate speech is considered a major concern in today's society (Harris et al. 2009). Erjavec and Kovačič (2012b) express the necessity of further studies on this issue.

This paper aims to investigate hate speech among young adolescents. This cohort is known as digital natives and they use technology at higher rates than previous generations. Fluent in internet usage, they massively explore the world of social media. Due to the fact that a great part of their lives is concerned with cyberspace, it is more probable that they can become victims of hate speech as well as aggressors (haters). Since information and communication technologies are developing in particular countries at a different pace, behaviours correlated with social interactions through the internet may vary. That is why five countries have been chosen to conduct a study: Poland, Romania, India, Hungary and Bulgaria. In its first phase, described in this paper, data from Poland have been collected and analyzed.

The remainder of the paper is organized as follows. Section 2 briefly reviews the related literature. Section 3 describes hater attitude and motives. The next Section is dedicated to an explanation of methodology and construction of the study. Findings and conclusions are given in Sects. 5 and 6, respectively.

2 Hate Speech in Current Studies

The phenomenon of hate speech has evolved into numerous definitions. It is intended to injure, dehumanize, harass, debase, degrade, and/or victimise and refers to abusive, insulting, offensive, hate-laden, intimidating, bias-motivated, hostile, malicious, harassing comments. It incites to violence, threatening a person or a group of people, is used for sharing ideology, propaganda and involves the advocacy of hatred and discrimination on the basis of race, colour, ethnicity, gender, national origin, religious beliefs, sexual orientation, physical condition, disability, political conviction or other status (Coliver 1992; Walker 1994; Wentraub-Reiter 1998; Boyle 2001; Cohen-Almagor 2011; Erjavec and Kovačič 2012a; Cohen-Almagor 2014). The core factors of online hate speech are presented in Fig. 1.

With the advent of social media, hate speech has spread significantly, still there is a significant gap in understanding the nature of hate speech on SNSs (Silva et al. 2016). Belief that the social nature of the internet has contributed to an accumulative rise in the number of online hate activities was expressed by i.e. Perry and Olsson (2009) and Banks (2011).

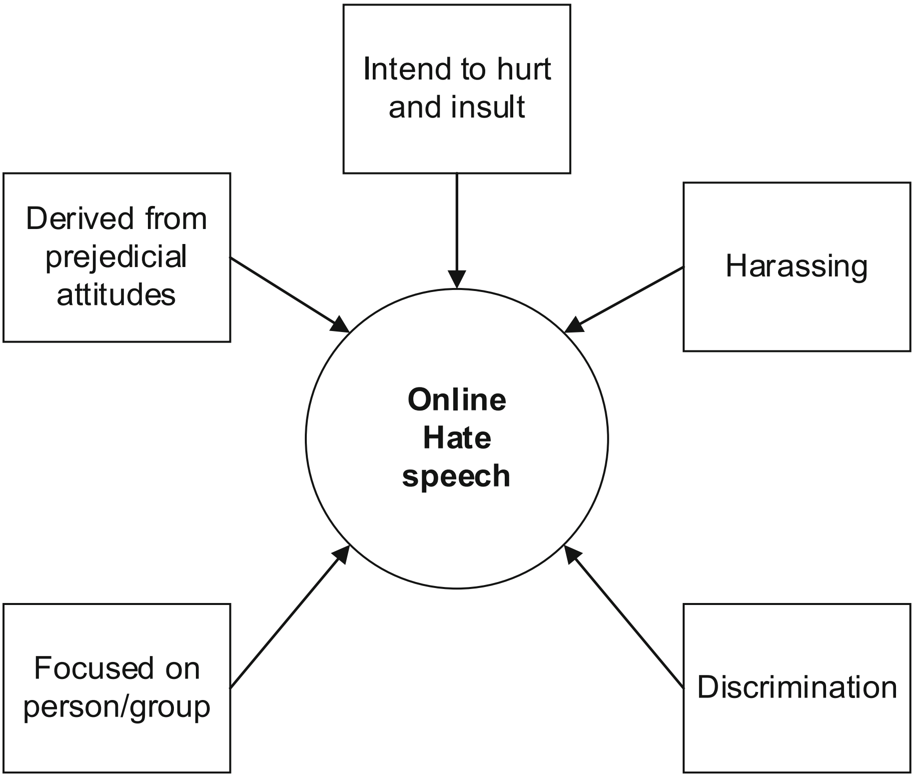

Fig. 1. Online hate speech key factors, own elaboration.

Several works were dedicated to legal aspects of hate speech online (Schieb and Preuss 2016; Cohen-Almagor 2011; Harris et al. 2009; Erjavec and Kovačič 2012a; Jabłońska 2016), proving that a growing number of countries perceive online hate speech as a peril and discern a necessity to search for new solutions in legal responsibility.

The motivations of aggressors using the internet were studied and described by McDevitt et al. (2002) and Cammaerts (2009). Suler (2004) defines a disinhibition effect that may influence people's online behaviours, while Erjavec and Kovačič (2012a) present a classification of haters.

Lincoln and Wilson (2005) describe three types of websites dedicated to hatred messages: in-your-face, misleading and ambiguous. The first type uses a specific frame of reference in posting its message, the second depicts itself as an appropriate and legitimate source of factual information, while the third one practices a more tempered style of messages and is more refined, sophisticated and covert in its delivery. Cohen-Almagor (2011) suggests methods of dealing with this form of hate speech by publishing sites' names, highlighting their content, locations and ISPs or attempting to curtail their activities.

Admittedly, websites dedicated to hatred messages are a powerful tool of hate, discrimination, prejudice or propaganda. Still, in Web 2.0 times, SNSs have becoming a main arena for distributing hatred comments. Erjavec and Kovačič (2012b) distinguish three features of social media responsible for such a situation: anonymity, quick publication and greater autonomy of writing. For the aims of this paper, this way of spreading hate speech is the most important as the authors concentrate on youngsters and this cohort of people is extremely engaged in this form of socialization. SNSs are extremely popular among young people so they have provided new methods for making hateful material increasingly visible to millions of young adolescents (Okansen et al. 2014). Figure 2 presents possible ways of combating hate speech spread through comments on SNSs.

Fighting hateful activities on social media
- Filtering, monitoring and auditing tools to provide a safer internet environment, - Installing computer blocking programs at work and school, - Developing standards for responsible and acceptable practices by web-hosting companies

Fig. 2. Possible ways of combating hate speech spread through comments on SNSs, own elaboration based on (Cohen-Almagor 2011).

Activities aiming at combating hate speech on SNSs should be taken especially by adults, parents or teachers but also local online communities gathering around particular SNSs.

Schieb and Preuss (2016) mention hate speech effects on victims and society, including: deepening prejudice and stereotypes, influencing mental health as well as the emotional well-being of victimized groups and/or individuals and inciting violent acts in real life. A massive part of the attacks may intend to hurt young people as they are heavy internet users. Teimouri et al. (2014) names risks in cyberspace to which this cohort is especially exposed: cyber-bullying, cybergrooming, hacking and downloading illegal content, identity theft and providing personal information, meeting online acquaintances in the real world as well as pornography and risky sexual behaviours. Baboo et al. (2013) emphasizes the fact that the negative effects of the online activity might influence young people's behavioural and social development.

3 Haters' Motives and Emotions

Hate can be perceived as a distinct emotion or be closely linked to a variation of pride, rage or anger (Sternberg 2005). It is a complex mental state which is still not fully understood (Aumer-Ryan and Hatfield 2007). Incited with pride, disgrace, embarrassment, fear, irritation or fury, the hater wishes to humiliate, discredit, hurt, defeat and annihilate a victim (Aumer-Ryan and Hatfield 2007).

According to Nietzsche, hate is born from fear. Still, there can be numerous factors causing haters to perform their actions on the internet. The psychological background for such actions performed by the aggressors may be joy and satisfaction from inflicting harm, a desire to create their own image or positioning in the group, a desire to discharge frustration, anger or resentment, which aims to improve well-being. Sometimes attackers take their actions with revenge, feeling attacked and harmed, regarding themselves as victims.

Hate speech may be derived from comparing and aggressors tend to abuse comparisons. They can be indeed useful or even motivating when the process is impartial and aims at improving someone's performance by endeavouring to resemble a person perceived as a perfect example. Comparing in this situation may be positive and even support personal development (Wawrzyniak 2015). Though, if used as a tool of disgrace and embarrassment, comparisons may become a prelude of hate speech.

The internet itself releases some aspects of e-personality that can cause hate speech. The lack of the physical presence of a victim and witnesses, hiding behind false accounts and numerous nicknames help haters feel anonymous. In 1969, George Philip Zimbardo conducted a study on the role of a sense of anonymity in the process of exchanging opinions. Participants of the experiment were divided into two groups. Representatives of one of them were dressed in robes with large hoods, preventing identification, which encouraged them to express less popular opinions. This study involved a process called deindividuation, the phenomenon of getting lost in the crowd, where self-consciousness and the fear of others' reactions are weakened (Zimbardo 1969).

The important role of anonymity in the process of interpersonal communication through the internet was also highlighted by Hayne and Rice (1997). They distinguish two forms of anonymity: social and technical. The first one is the inability to identify the unit as a result of the lack of visibility of personality traits, i.e. appearance, voice, characteristic gestures or personality, while the second one refers to the lack of information uniquely identifying a unit (document numbers, personal data, address and check-in, date of birth, telephone number or IP address). The impact of anonymity on hate speech was widely described by Zimmerman and Ybarra (2012). Disinhibition revealed in cyberspace under the often illusory sense of anonymity is an important factor of hate speech, as it often is an incentive for the aggressor.

Among other features of cyberspace that may support aggressive behaviours the following should be distinguished: dissociative anonymity, invisibility, asynchronicity, solipsistic introjections, dissociative imagination and minimising authority. All these may lead to the so-called disinhibition effect. It means that some people self-disclose or act out more frequently or intensely online than they would in person (Suler 2004).

In Yudofsky et al. (1986) and his team found that verbal aggression is one of the key forms of confrontational behavior. Ten years later verbal aggression was marked as a form of electronic aggression (Thompen and Foulger 1996) and another year later, Katz (1997) stressed that the Internet is a place of constant confrontation, misinformation and insult. In Kellner (1998) published a work in which he stated that the Internet broadens the scope of the discussions carried out, regardless of their nature (creative or critical). Despite its features, the internet still remains a tool. It may intensify some aspects of e-personality changing people into haters but the majority of society may remain resistant and communicate without a tone of hatred.

4 Research Methodology

The first phase of the study tended toward examining Polish internet users and their attitudes as well as experiences with online hating by collecting data in a questionnaire. First, the authors made a literature review on haters and their victims and analyzed common beliefs and views on online hating.

Then another questionnaire was prepared. It comprised of questions about performing and witnessing online hate. The research tool was a questionnaire with a high level of standardization. Standardization signifies that all interviewees were asked the same questions in recurring order. The questions were predominantly closed ones, in which the respondent had to select answers from a list of options prepared by the investigators. In the case of open questions, the interviewee could answer without any guidance from the authors.

Making an allowance for the method of filling in questionnaires, generally two fundamental techniques in quantitative research may be distinguished: a questionnaire which is filled autonomously by the respondent and interview questionnaire with questions asked and written down by the interviewer. Due to the online nature of the conducted study, the authors decided to implement the first one, CAWI – Computer Assisted Web Interviewing, and published the poll online.

5 Results

In the study 100 Polish respondents participated, 54 female and 46 male. They were interviewed about their attitudes and practices on online hating. The average respondent's vita may be characterised by age, education level and place of residence. The study's results present the following average interviewee profile, based on the most common answers: a young person in his or her early twenties with secondary education or bachelor's degree, living in a city with more than 500,000 inhabitants. Detailed demographic statistics are presented in Table 1.

When asked about witnessing online hating behaviours, 84% of interviewees declared having observed such practices in cyberspace. The three most popular online environments where such behaviours occur are: Facebook, YouTube and online games (Table 2).

Among the most encountered activities of online aggressors, harassing a particular person or a group and criticizing events may be included as presented on Fig. 3. 80% of respondents claim that the most experienced form of hating is anonymous, without the aggressor's identity being revealed.

While analyzing attitudes and reactions on the witnessed hating, the most frequent feedback was to ignore a hater (41,77% of answers), respond politely (20,25%) and report (16,45%). Interviewees also admitted performing some actions that may escalate aggressive reactions, namely to jeer a hater (14,55%) and respond aggressively (5,69%), though these behaviours were less disseminated.

Table 1. Demographic statistics, own elaboration.

Place of residence		Education level	
Village	22	Primary education	6
City to 100,000 inhabitants	21	Vocational education	1
City from 100,001 to 500,000 inhabitants	9	Secondary education	52
City above 500,000 inhabitants	48	Bachelor's degree	24
		Master's degree	10
		Completed postgraduate studies	1
Age		The academic title	7
Under 18	6		
18–26	80		
27–35	8	Gender	
36–44	5	Female	54
Above 45	1	Male	46

Table 2. The most common places of haters' attacks (multiple choice question), own elaboration.

Facebook	81	Instagram	13
YouTube	66	Twitter	11
Online game	37	LinkedIn	1
Forum	33	GoldenLine	1
In comments above published content	31	Tumblr	1
	18	Other	1
	14		

31% of respondents fall prey to haters' attacks, while 53% believe that these practices are common and 37% claim that hating on the internet is ubiquitous. Only 2% perceive this form of online aggression as something marginal (Table 3). The frequent belief among interviewees is to perceive hating as a dangerous and threatening phenomenon (82%).

Table 3. Frequency of hating behaviours in respondents' opinions, own elaboration.

In your opinion internet hating is:	
Ubiquitous	37
Common	53
Sporadic	9
Marginal	2

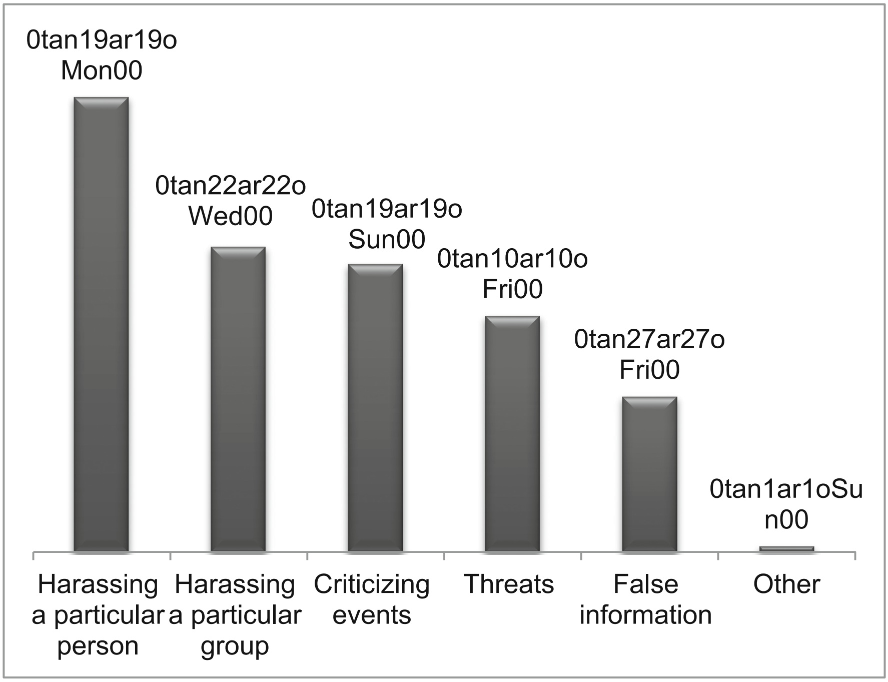

Fig. 3. Forms of hating actions witnessed, own elaboration.

Defining the most vulnerable groups as probable victims of hating, subjects named first celebrities (77 answers), then politicians (58), random internet users (56), the followers of a particular religion or ideology (55), persons engaged in internet activities, i.e. bloggers (52), children and teenagers (19) and particular social groups, i.e. homosexuals, LGBT (10).

When interviewed about their knowledge on the legal responsibility of hating, almost two thirds (61%) of subjects admitted that they were not aware of this concern. Only 20 respondents tried to define actions that would decrease the number of such aggressive attacks. Among their suggestions, the most common one was educating young people about the consequences of hating for the attacker and victim. Among the others the following can be mentioned: improving skills to ignore such practices, comments moderation, accounts validations and blocking aggressive users, financial penalties or even imprisonment in extreme cases.

After determining respondents' attitudes towards hating, they were asked about performing such activities and 18% claimed to be a hater. Among the motivations underlying their online aggressive behaviours, the following were listed: indignation about a particular comment or situation, aversion to a person or group, jealousy and bad mood. After performing hating attacks, investigated haters felt satisfaction (4 persons), relief (3) or even joy (1). Seven subjects declared frustration and no one fear. The most common frequency of hating practices was occasionally, only two interviews claimed they present such behaviours every day. Four haters feel better after publishing aggressive comments, but the rest of the subjects do not feel cheered up.

6 Discussion and Summary

As mentioned in Sect. 2, hating is a threatening phenomenon and with the advent and massive dissemination of social media, it is spreading widely. The described results seem to prove this fact and also agree with the most harassed possible groups of victims, i.e. celebrities, politicians or discrimination based on race, gender, national origin, religious beliefs, sexual orientation, physical condition and disability. Also youngsters are more vulnerable to haters' attacks as they spend massive amounts of time online.

The authors of the paper also agree with the most common motivations for haters. Anger, aversion, discrimination or jealousy born from comparing are all present in both the literature review in Sect. 3 and the study's results. Almost one-fifth of subjects declared themselves as haters, feeling mostly satisfaction and relief, but also frustration after conducting an attack.

The limitation of the study is its small total number of respondents. Still, the research is in the first stage with further data on larger sample.

Acknowledgments. This article was written as part of the project "Studies of Security Solutions for Business on WEB" completed in Poland, India, Romania and Bulgaria. As hating is perceived as one of the core threats in cyberspace, a study for understanding and evaluating this phenomenon in the above countries was needed.

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Part III: Fuzzy Geometric Programming and Optimization

Properties of Fuzzy Relation Inequalities with Addition-Min Composition

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Abstract. Fuzzy relation inequalities with addition-min composition could be used to describe the Peer-to-Peer (P2P) file sharing system. In this paper we study some properties of a system of addition-min fuzzy relation inequalities. The complete solution set of such system is verified to be convex. Besides, it is found that the complete solution set is fully determined by a unique maximum solution and a number of minimal ones.

Keywords: Fuzzy relation inequality · Addition-min composition · P2P network system · Solution set · Convexity

1 Introduction

Fuzzy relation equation with max-min composition was first introduced by E. Sanchez [1–3]. He pointed out some basic properties and practical applications of fuzzy relation equation. Method for solving all the solutions is one of the most important aspects in the theoretical research on fuzzy relation equation. Many researchers has focused on such topic [4–12]. Besides, application of fuzzy relation equations or inequalities with various kinds of composition were also investigated [13–21]. Fuzzy relation equations or inequalities were usually applied to describe a real-word system [22–24]. Furthermore, based on some practical considerations, corresponding fuzzy relation programming problems were established and investigated [23–26]. In fact, optimization problem subject to fuzzy relation system was an interesting research topic [27–35], since P.-Z. Wang [36] proposed and studied the relevant fuzzy relation latticized linear programming.

In 2012, fuzzy relation inequality with addition-min composition was introduced by J.-X. Li and S.-J. Yang [22] to describe the data transmission mechanism in BitTorrent-like Peer-to-Peer (P2P) file sharing system. Motivated by this application, S.-J. Yang [25] investigated the corresponding fuzzy relation

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linear programming problem. In [25], the authors defined the concept of pseudo-minimal indexes (PMI) and presented a PMI algorithm to deal with the proposed problem. Based on the pseudo-minimal indexes, the main problem was decomposed into some simple linear programming problems. Recently, X.-P. Yang et al. [26] improved the objective function presented in [25], based on some considerations of the application background. They proposed the min-max programming with addition-min fuzzy relation inequalities constraint and solved the problem by converting it into several single subproblems [26].

Due to the important application background of the addition-min fuzzy relation inequalities, we will further study some properties of such system, including the convexity and structure of its complete solution set. The rest of this paper is organized as follows. Section 2 is some basic concepts and results on the addition-min fuzzy relation system. In Sect. 3, we study the convexity of the solution set, while in Sect. 4, structure of the solution set is presented. Simple conclusion is given in Sect. 5.

2 Preliminaries

Let $I = \{1, 2, \cdots, m\}$ and $J = \{1, 2, \cdots, n\}$ be two index sets and denote $X = [0, 1]^n$.

Definition 1 (See [22,25]). Let $x^1 = (x_1^1, x_2^1, \cdots, x_n^1)$, $x^2 = (x_1^2, x_2^2, \cdots, x_n^2) \in X$, we define:

(i) $x^1 \le x^2$ if $x_j^1 \le x_j^2$, $\forall j \in J$;
(ii) $x^1 < x^2$ if $x^1 \le x^2$ and there are some $j \in J$ such that $x_j^1 < x_j^2$.

In what follows we denote the dual of order relation '$<$' and '$\le$' by the symbol '$>$' and '$\ge$', respectively. Obviously, the operator '$\le$' forms a partial order relation on X and $(X, \le)$ becomes a lattice.

A system of addition-min FRI is formulated as

$$\begin{cases} a_{11} \wedge x_1 + a_{12} \wedge x_2 + \cdots + a_{1n} \wedge x_n \ge b_1, \\ a_{21} \wedge x_1 + a_{22} \wedge x_2 + \cdots + a_{2n} \wedge x_n \ge b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1} \wedge x_1 + a_{m2} \wedge x_2 + \cdots + a_{mn} \wedge x_n \ge b_m. \end{cases} \tag{1}$$

where $a_{ij}, x_j \in [0, 1]$, $b_i > 0$, $(i \in I, j \in J)$, and the operation '+' represents the ordinary addition, $a_{ij} \wedge x_j = \min\{a_{ij}, x_j\}$. System (1) can be written as

$$\sum_{j \in J} a_{ij} \wedge x_j \ge b_i, \ \forall i \in I, \tag{2}$$

or

$$A \odot x^T \ge b^T \tag{3}$$

where $A = (a_{ij})_{m \times n}$, $x = (x_1, x_2, \cdots, x_n)$ and $b = (b_1, b_2, \cdots, b_m)$.

The solution set of system (1) is denoted by

$$X(A,b) = \{x \in X | A \odot x^T \geq b^T\}.$$

Definition 2. System (1) is said to be consistent (compatible or solvable) if $X(A,b) \neq \emptyset$. Otherwise, it is said to be inconsistent.

Theorem 1 (See [22,25])**.** *For system* (1)*, we have:*

(i) (1) *is consistent if and only if* $\sum_{j\in J} a_{ij} \geq b_i$ *for arbitrary* $i \in I$*;*
(ii) Let $x^* \in X(A,b)$*,* $x \in X$*.* $x^* \leq x$ *implies* $x \in X(A,b)$*;*
(iii) Let $x', x \in X$ *and* $x \leq x'$*.* $x' \notin X(A,b)$ *implies* $x \notin X(A,b)$*;*
(iv) Let $x \in X(A,b)$*. If* $\sum_{j\in J} a_{ij} = b_i$ *for some* $i \in I$*, then* $(a_{i1}, a_{i2}, \cdots, a_{in}) \leq x$*.*

Definition 3. In system (1), $\dot{\hat{x}} \in X(A,b)$ is called the maximum (or greatest) solution if and only if $x \leq \dot{\hat{x}}$ for all $x \in X(A,b)$, and $\check{x} \in X(A,b)$ is called a minimal solution if and only if $x \leq \check{x}$ implies $x = \check{x}$ for any $x \in X(A,b)$.

Denote $\dot{\hat{x}} = (1, 1, \cdots, 1)$. It is easy to check the following Remark 1.

Remark 1. *System* (1) *is consistent if and only if* $\dot{\hat{x}}$ *is its maximum solution.*

Furthermore, when system (1) is consistent, its solution set is exactly determined by the unique maximum solution and a number of minimal solution(s), i.e.,

$$X(A,b) = \bigcup_{\check{x}\in\check{X}(A,b)} \{x \in X | \check{x} \leq x \leq \dot{\hat{x}}\},$$

where $\check{X}(A,b)$ represents the minimal solution set of system (1).

Theorem 2 (See [22,25])**.** *Let* $x \in X(A,b)$ *be a solution of system* (1)*. Then we have:*

(i) $x > 0$*;*
(ii) For arbitrary $i \in I$*,* $j \in J$*,*

$$x_j \geq b_i - \sum_{k\in J-\{j\}} a_{ik} \wedge x_k \geq b_i - \sum_{k\in J-\{j\}} a_{ik};$$

(iii) For arbitrary $i \in I$*,* $j \in J$*,*

$$a_{ij} \geq b_i - \sum_{k\in J-\{j\}} a_{ik} \wedge x_k \geq b_i - \sum_{k\in J-\{j\}} a_{ik}.$$

Denote $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_n)$, where

$$\begin{aligned} \hat{\alpha}_{ij} &= \max\{0, b_i - \sum_{k\in J-\{j\}} a_{ik}\}, \\ \hat{\alpha}_j &= \max\{\hat{\alpha}_{ij} | i \in I\}, \end{aligned} \tag{4}$$

$i \in I, j \in J$.

Theorem 3 (See [25])**.** *System* (1) *has the unique minimal solution if and only if* $\hat{\alpha}$ *is a solution of* (1)*, i.e.* $\hat{\alpha} \in X(A,b)$*. In particular, when* (1) *has the unique minimal solution,* $\hat{\alpha}$ *is the unique minimal solution of system* (1)*.*

3 Convexity of the Complete Solution Set

Let $y_{ij} = a_{ij} \wedge x_j$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Based on system (1), we construct the following system of linear inequalities:

$$\begin{cases} y_{i1} + y_{i2} + \cdots + y_{in} \geq b_i, \quad i = 1, 2, \cdots, m, \\ y_{ij} \leq a_{ij}, \quad i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, \\ 0 \leq y_{ij} \leq x_j \leq 1, \quad i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, \end{cases} \tag{5}$$

with variables $x_1, x_2, \cdots, x_n, y_{11}, y_{12}, \cdots, y_{1n}, y_{21}, y_{22}, \cdots, y_{2n}, \cdots, y_{m1}, y_{m2}, \cdots, y_{mn}$. System (5) can be written as its matrix form, i.e.,

$$\overline{A} \cdot z \geq \bar{b},$$

where $\overline{A}$ is the coefficient matrix, $\bar{b}$ is the right-side vector, and $z = (x, y_1, y_2, \cdots, y_m)$, $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})$, $i = 1, 2, \cdots, m$. We denote the solution set of system (5) by

$$X(\overline{A}, \bar{b}).$$

It is clear that $X(\overline{A}, \bar{b})$ is a convex set when it is not empty.

Proposition 1. *If $z' = (x', y'_1, y'_2, \cdots, y'_m)$ is a solution of system (5), then x' is a solution of system (1).*

Proof. Suppose that $z' = (x', y'_1, y'_2, \cdots, y'_m)$ is a solution of system (5). According to the inequalities in system (5), it is obvious that $x'_j \in [0, 1]$, $\forall j \in J$. Besides we have $y'_{ij} \leq a_{ij}$ and $y'_{ij} \leq x'_j$, which indicate

$$y'_{ij} \leq a_{ij} \wedge x'_j,$$

$\forall i \in I, j \in J$. And then it follows that

$$a'_{i1} \wedge x'_1 + a'_{i2} \wedge x'_2 + \cdots + a'_{in} \wedge x'_n \geq y'_{i1} + y'_{i2} + \cdots + y'_{in} \geq b_i,$$

$\forall i \in I$. Consequently, x' is a solution of system (1).

Proposition 2. *If $x' = (x'_1, x'_2, \cdots, x'_n)$ is a solution of system (1), then $z' = (x', y'_1, y'_2, \cdots, y'_m)$ is a solution of system (5), where $y'_i = (a_{i1} \wedge x'_1, a_{i2} \wedge x'_2, \cdots, a_{in} \wedge x'_n), i = 1, 2, \cdots, m$.*

Proof. Let

$$y'_{ij} = a_{ij} \wedge x'_j, \tag{6}$$

$\forall i \in I, j \in J$. Then $y'_i = (y'_{i1}, y'_i, \cdots, y'_{in})$, $\forall i \in I$. Inequality (6) indicates

$$y'_{ij} \leq a_{ij}, \tag{7}$$

and

$$y'_{ij} \leq x'_j, \tag{8}$$

$\forall i \in I, j \in J$. Since $x' = (x'_1, x'_2, \cdots, x'_n)$ is a solution of system (1), we get

$$0 \leq x'_j \leq 1, \quad \forall j \in J, \tag{9}$$

and

$$y'_{i1} + y'_{i2} + \cdots + y'_{in} = a_{i1} \wedge x'_1 + a_{i2} \wedge x'_2 + \cdots + a_{in} \wedge x'_n \geq b_i, \quad \forall i \in I. \tag{10}$$

Moreover, considering $0 \leq a_{ij} \leq 1$ and Inequalities (8) and (9), it is easy to check that

$$0 \leq y'_{ij} = a_{ij} \wedge x'_j \leq x'_j \leq 1, \quad \forall i \in I, j \in J. \tag{11}$$

From Inequalities (7), (10) and (11), it is cleat that $z' = (x', y'_1, y'_2, \cdots, y'_m)$ is a solution of system (5).

Definition 4. Let $\emptyset \neq C \subseteq R^n, n \in Z^+$. Then C is said to be a convex set if and only if

$$\lambda x^1 + (1 - \lambda)x^2 \in C$$

holds for any $x^1, x^2 \in C$ and $\lambda \in [0, 1]$.

Theorem 4. *If the solution set of system* (1) *is nonempty, then it is a convex set.*

Proof. Obviously $\emptyset \neq X(A, b) \subseteq R^n$. Let $x^1 = (x^1_1, x^1_2, \cdots, x^1_n), x^2 = (x^2_1, x^2_2, \cdots, x^2_n) \in X(A, b)$ be two arbitrary solution of system (1). We construct vector z^1, z^2 as follows:

$$z^k = (x^k, y^k_1, y^k_2, \cdots, y^k_m), \quad k = 1, 2,$$

where

$$y^k_i = (a_{i1} \wedge x^k_1, a_{i2} \wedge x^k_2, \cdots, a_{in} \wedge x^k_n), \quad k = 1, 2, \ i = 1, 2, \cdots, m.$$

According to Proposition 2, z^1, z^2 are solutions of system (5), i.e. $z^1, z^2 \in X(\overline{A}, \overline{b})$. Observe that system (5) is exactly a system of linear inequalities. It is well known that the nonempty solution set of a system of linear inequalities should be convex set. Hence $X(\overline{A}, \overline{b})$ is a convex set. According to Definition 4, we have

$$\lambda z^1 + (1 - \lambda)z^2 \in X(\overline{A}, \overline{b}), \tag{12}$$

for any $\lambda \in [0, 1]$. On the other hand,

$$\begin{aligned} \lambda z^1 + (1 - \lambda)z^2 &= \lambda(x^1, y^1_1, y^1_2, \cdots, y^1_m) + (1 - \lambda)(x^2, y^2_1, y^2_2, \cdots, y^2_m) \\ &= (\lambda x^1 + (1 - \lambda)x^2, \lambda y^1_1 + (1 - \lambda)y^2_1, \cdots, \lambda y^1_m + (1 - \lambda)y^2_m). \end{aligned} \tag{13}$$

According to Proposition 1, (12) and (13) imply that $\lambda x^1 + (1 - \lambda)x^2 \in X(A, b)$. By Definition 4 it follows that $X(A, b)$ is a convex set.

4 Structure of Complete Solution Set

It is well-known that the complete solution set of a system of max-min (or max-product) fuzzy relation inequalities (or equations) was fully determined by a unique maximum solution and a finite number of minimal solutions. Naturally, one would be interested in the structure of the solution set of the addition-min system, with comparison to that of the classical max-min or max-product system. In this section we will shown that these tow kinds of system have the similar structure in their solution sets.

Theorem 5. *If $X(A, b) \neq \emptyset$, then*

$$X(A,b) = \bigcup_{\check{x} \in \check{X}(A,b)} \{x \in X | \check{x} \leq x \leq \hat{x}\} = \bigcup_{\check{x} \in \check{X}(A,b)} [\check{x}, \hat{x}],$$

where $\hat{x} = (1, 1, \cdots, 1)$ is the maximum solution of (1).

Proof. According to Theorem 1, it is clear that $\bigcup_{\check{x} \in \check{X}(A,b)} \{x \in X | \check{x} \leq x \leq \hat{x}\} \subseteq X(A, b)$. Thus we just need to prove that $X(A, b) \subseteq \bigcup_{\check{x} \in \check{X}(A,b)} \{x \in X | \check{x} \leq x \leq \hat{x}\}$.

For any $y \in X(A, b)$, we define $X_y = \{x | x \leq y, x \in X(A, b)\}$. Then $y \in X_y \neq \emptyset$ and $(X_y, \leq)$ forms a partial order set.

Choose an arbitrary chain $\{y^1, y^2, \cdots\} \subseteq X_y$ such that $y^1 \geq y^2 \geq \cdots$. It is obvious that $y \geq y^1 \geq y^2 \geq \cdots$ and $y^1, y^2, \cdots \in X(A, b) \subseteq X$. Since X is a closed bounded set, there exists $y^0 \in X$ such that $\lim_{k \to \infty} y^k = y^0$. y^0 is a lower bound of the chain $\{y^1, y^2, \cdots\}$. Furthermore,

$$y^0 = \lim_{k \to \infty} y^k \leq y,$$

and

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \odot {y^k}^T \geq b_i, \quad i \in I, k = 1, 2, \cdots,$$

i.e.

$$a_{i1} \wedge y_1^k + a_{i2} \wedge y_2^k + \cdots + a_{in} \wedge y_n^k \geq b_i, \quad i \in I, k = 1, 2, \cdots,$$

$$\lim_{k \to \infty} (a_{i1} \wedge y_1^k + a_{i2} \wedge y_2^k + \cdots + a_{in} \wedge y_n^k) \geq b_i, \quad i \in I,$$

$$a_{i1} \wedge \lim_{k \to \infty} y_1^k + a_{i2} \wedge \lim_{k \to \infty} y_2^k + \cdots + a_{in} \wedge \lim_{k \to \infty} y_n^k \geq b_i, \quad i \in I,$$

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \odot (\lim_{k \to \infty} y_1^k, \lim_{k \to \infty} y_2^k, \cdots, \lim_{k \to \infty} y_n^k)^T \geq b_i, \quad i \in I,$$

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \odot {y^0}^T \geq b_i, \quad i \in I.$$

Hence, $y^0 \in X_y$. That is to say, any chain of $(X_y, \leq)$ has a lower bound in X_y.

By Zorn's Lemma, there exists a minimum element $\check{y} \in X_y$. We may get $\check{y} \leq y \leq \hat{x}$ for any $y \in X(A, b)$ and $\check{y} \in X_y$. In order to show the conclusion, we have to verify $\check{y} \in \check{X}(A, b)$. Firstly, $\check{y} \in X_y \subseteq X(A, b)$. Secondly, for any $x \in X(A, b)$, if $x \leq \check{y}$, then $x \leq y$. So we get $x \in X_y$. Since $\check{y}$ is the minimum element in X_y and $x \leq \check{y}$, we have $x = \check{y}$. Consequently, $\check{y}$ is a minimum solution of (1), i.e., $\check{y} \in \check{X}(A, b)$.

Theorem 5 indicates the solution set of system (1) is also determined by its unique maximum solution and a number of minimal solutions. As shown in [26], the minimal solution set of system (1) might be infinite set (see Example 1 in [26]). This is much different from that of the classical max-T fuzzy relation system, where T is a continuous triangular norm.

5 Conclusion

Considering the application in BitTorrent-like Peer-to-Peer file sharing system, we study some properties of a system of addition-min fuzzy relation inequalities. As know to everyone, the complete solution set of a group of consistent fuzzy relation equations (or inequalities) with classical max-T (including max-min and max-product, and T represents a continuous triangular norm) composition, is usually non-convex set. However, it is found in this paper that the solution set is convex in a consistent system of addition-min relation inequalities. Besides, we have shown that the structure of the solution set of addition min fuzzy relation inequalities is similar to that of the classical max-T fuzzy relation inequalities or equations. However, the numbers of their minimal solutions might be different. The classical max-T fuzzy relation system should have finite number of minimal solutions, while the addition-min one might have infinite number of minimal solutions.

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Recommender: Academic Conference on 30th anniversary of fuzzy geometric programming and 40th education year by and of Professor Cao Bingyuan.

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Geometric Programming with Intuitionistic Fuzzy Coefficient

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Abstract. An geometric programming model is presented with the intuitionistic fuzzy coefficient, and then the model is turned into a crisp geometric programming based on centain accuracy degree of intuitionistic fuzzy sets, the duality theory is used to solve the crisp geometric programming. And finally, two numerical examples are given to illustrate the feasibility and effectiveness.

Keywords: Intuitionistic fuzzy sets · Geometric programming · Accuracy degree · The duality theory · Optimal solution

1 Introduction

Geometric programming (GP) is an important optimization type, it was founded in 1961 [1,2]. GP has been applied in more than a dozen fields. It include communication system, civil engineering, mechanical engineering, structural design and optimization, chemical engineering, optimal control, decision making, network flows, theory of inventory, balance of machinery, analog circuitry, design theory, transportation, fiscal and monetary, management science, electrical engineering, electronic engineering, environmental engineering, nuclear engineering, technical economical analysis, and so on [3]. In 1965, L.A. Zadeh developed the concept of fuzzy sets [4]. In 1987, Prof. Cao developed fuzzy geometric programming (FGP), FGP is enlarge to GP, FGP has also been applied in power system, environmental engineering, postal services, economical analysis, transportation, inventory theory, engineering design, civil Engineering, etc. [5]. In 1983, Atanassov developed intuitionistic fuzzy sets [6]. The theory of intuitionistic fuzzy sets is the generalization of the theory of fuzzy sets. It is well suited to dealing with vagueness. Intuitionistic fuzzy sets have been used to build soft decision making models, such as medical diagnosis [7], electronic engineering [8],

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image processing and pattern recognition [9], etc. This paper presents a geometric programming with intuitionistic fuzzy coefficient, it will expand the range of the geometric programming under uncertain environment [10–12].

The rest of this paper is organized as follows: In Sect. 2 brief states the intuitionistic fuzzy sets and geometric programming. In Sect. 3, geometric programming with intuitionistic fuzzy Coefficient are described and a numerical algorithm is developed. In Sect. 4, we have used two numerical examples to explain the effectiveness of the algorithm. In Sect. 5, some concluding remarks are given.

2 Intuitionistic Fuzzy Sets and Geometric Programming

Definition 2.1 [13]. Let X be a universal set. An intuitionistic fuzzy set A in X is an triple having the following form

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2.1}$$

where: $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that

$$0 \le \mu_A(x) + \nu_A(x) \le 1$$

The functions $\mu_A(x), \nu_A(x) \in [0, 1]$ is called the degree of membership and non-membership of the element $x \in X$ to the set $A \subseteq X$, respectively.

For convenience of notation, we abbreviate intuitionistic fuzzy set to IFS and represent $IFS(X)$ as all the IFS in X.

Definition 2.2 [13]. For each intuitionistic fuzzy set A in X, we call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2.2}$$

an intuitionistic fuzzy index of $x \in A$ and it is a hesitation degree of whether x belongs to A or not.

It is very easy to see that $0 \le \pi_A(x) \le 1$, for each $x \in X$.

For every $A, B \in IFS(X)$, the operations of IFS can be defined as follows:

(1) $A \le B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all x in X.
(2) $A = B$ if and only if $A \le B$ and $A \ge B$.
(3) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) | x \in X\}$.
(4) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) | x \in X\}$.

Definition 2.3 [13]. $A^C = \{(x, \nu_A(x), \mu_A(x)) | x \in X\}$ is called the complementary of the intuitionistic fuzzy set A.

Definition 2.4 [13]. The $\alpha = (u_\alpha, v_\alpha)$ is called an intuitionistic fuzzy number, where $u_\alpha \in [0, 1], v_\alpha \in [0, 1], u_\alpha + v_\alpha \le 1$.

Definition 2.5. Let $\alpha = (u_\alpha, v_\alpha)$ be an intuitionistic fuzzy number, a score function S of α can be represented as follows [14]:

$$S(\alpha) = u_\alpha - v_\alpha, S(\alpha) \in [-1, 1].$$

Definition 2.6. Let $\alpha = (u_\alpha, v_\alpha)$ be an intuitionistic fuzzy number, an accuracy function H of α can be represented as follows [15]:

$$H(\alpha) = u_\alpha + v_\alpha, H(\alpha) \in [0, 1].$$

Definition 2.7. Let $\alpha = (u_\alpha, v_\alpha)$ and $\beta = (u_\beta, v_\beta)$ be two intuitionistic fuzzy numbers, $S(\alpha) = u_\alpha - v_\alpha$ and $S(\beta) = u_\beta - v_\beta$ be the scores of α and β, respectively, if let $H(\alpha) = u_\alpha + v_\alpha$ and $H(\beta) = u_\beta + v_\beta$ be the accuracy degrees of α and β, respectively, if $S(\alpha) < S(\beta)$, then α is smaller than β, denoted by $\alpha < \beta$. If $S(\alpha) = S(\beta)$, when $H(\alpha) = H(\beta)$, α and β have the same information, denoted by $\alpha = \beta$; when $H(\alpha) < H(\beta)$, α is smaller than β, denoted by $\alpha < \beta$.

Definition 2.8 [13]. We call $\alpha = ([a_\alpha, b_\alpha], [c_\alpha, d_\alpha])$ is an interval intuitionistic fuzzy number, where $[a_\alpha, b_\alpha] \subset [0, 1], [c_\alpha, d_\alpha] \subset [0, 1], b_\alpha + d_\alpha \leq 1$.

Definition 2.9. Let $\alpha = ([a_\alpha, b_\alpha], [c_\alpha, d_\alpha])$ be an interval intuitionistic fuzzy number, a score function S of α can be represented as follows [16]:

$$S(\alpha) = \frac{a_\alpha - c_\alpha + b_\alpha - d_\alpha}{2}, S(\alpha) \in [-1, 1].$$

Definition 2.10. Let $\alpha = ([a_\alpha, b_\alpha], [c_\alpha, d_\alpha])$ be an interval intuitionistic fuzzy number, an accuracy function H of α can be represented as follows [16]:

$$H(\alpha) = \frac{a_\alpha + +b_\alpha + c_\alpha + d_\alpha}{2}, H(\alpha) \in [0, 1].$$

Definition 2.11. Let $\alpha = ([a_\alpha, b_\alpha], [c_\alpha, d_\alpha])$ and $\beta = ([a_\beta, b_\beta], [c_\beta, d_\beta])$ be two interval intuitionistic fuzzy numbers, $S(\alpha) = \frac{a_\alpha - c_\alpha + b_\alpha - d_\alpha}{2}$ and $S(\beta) = \frac{a_\beta - c_\beta + b_\beta - d_\beta}{2}$ be the scores of α and β, respectively, if let $H(\alpha) = \frac{a_\alpha + +b_\alpha + c_\alpha + d_\alpha}{2}$ and $H(\beta) = \frac{a_\beta + +b_\beta + c_\beta + d_\beta}{2}$ be the accuracy degrees of α and β, respectively, if $S(\alpha) < S(\beta)$, then α is smaller than β, denoted by $\alpha < \beta$. If $S(\alpha) = S(\beta)$, when $H(\alpha) = H(\beta)$, α and β have the same information, denoted by $\alpha = \beta$; when $H(\alpha) < H(\beta)$, α is smaller than β, denoted by $\alpha < \beta$ [16].

Definition 2.12 [17]. The following standard form

$$\begin{array}{lll} (GP) & \min & f_0(x) \\ & s.\ t. & f_i(x) \leq 1, (1 \leq i \leq p) \\ & & l_j(x) \leq 1, (1 \leq j \leq q) \\ & & x > 0, \end{array} \tag{2.3}$$

is called geometric programming (GP), where

$$f_i(x) = \sum_{k=1}^{J_i} f_{ik}(x) = \sum_{k=1}^{J_i} c_{ik} \prod_{l=1}^{n} x_l^{\gamma_{ikl}} (0 \leq i \leq p) \tag{2.4}$$

is posynomial function of variable x.

$$l_j(x) = c_j \prod_{l=1}^{n} x_l^{\gamma_{jl}} (1 \le j \le q) \tag{2.5}$$

is monomial function of variable x, and coefficient $c_{ik} > 0, c_j > 0$, variable $x = (x_1, x_2, \cdots, x_n)^T > 0$, exponent γ_{ikl} $(1 \le k \le J_i, 0 \le i \le p, 1 \le l \le n)$, γ_{jl} $(1 \le j \le q, 1 \le l \le n)$ is arbitrary real number.

3 Geometric Programming with Intuitionistic Fuzzy Coefficient

In this real world most of geometric programming problems take place in a fuzzy environment. The coefficient of objective and constraints function are difficult to be determined accurately. The coefficient can be described through intuitionistic fuzzy set, which can bring convenience for dealing with practical problems. The geometric programming with intuitionistic fuzzy coefficient can be expressed as follows:

Definition 3.1. The following standard form

$$\begin{array}{lll} (GP) & \min & \tilde{g}_0(x) \\ & s.\ t. & \tilde{g}_i(x) \le 1, (1 \le i \le p) \\ & & \tilde{h}_j(x) \le 1, (1 \le j \le q) \\ & & x > 0, \end{array} \tag{3.1}$$

is called geometric programming with intuitionistic fuzzy coefficient or intuitionistic fuzzy Geometric programming (IFGP), where

$$\tilde{g}_i(x) = \sum_{k=1}^{J_i} \tilde{g}_{ik}(x) = \sum_{k=1}^{J_i} \tilde{c_{ik}} \prod_{l=1}^{n} x_l^{\gamma_{ikl}} (0 \le i \le p) \tag{3.2}$$

is intuitionistic fuzzy posynomial function of variable x.

$$\tilde{h}_j(x) = \tilde{c_j} \prod_{l=1}^{n} x_l^{\gamma_{jl}} (1 \le j \le q) \tag{3.3}$$

is intuitionistic fuzzy monomial function of variable x, and coefficient $\tilde{c_{ik}}$ and $\tilde{c_j}$ are the intuitionistic fuzzy numbers or the interval ones, variable $x = (x_1, x_2, \cdots, x_n)^T > 0$, exponent γ_{ikl} $(1 \le k \le J_i, 0 \le i \le p, 1 \le l \le n)$, γ_{jl} $(1 \le j \le q, 1 \le l \le n)$ is arbitrary real number.

4 The Solution of Geometric Programming with Intuitionistic Fuzzy Coefficient

Definition 4.1. The following geometric programming

$$\begin{array}{lll} (GP) & \min & g_0(x) \\ & s.\ t. & g_i(x) \le 1, (1 \le i \le p) \\ & & h_j(x) \le 1, (1 \le j \le q) \\ & & x > 0, \end{array} \tag{4.1}$$

is called crisp geometric programming of IFGP (3.1), where

$$g_i(x) = \sum_{k=1}^{J_i} g_{ik}(x) = \sum_{k=1}^{J_i} a_{ik} \prod_{l=1}^{n} x_l^{\gamma_{ikl}} (0 \le i \le p) \tag{4.2}$$

$$h_j(x) = a_j \prod_{l=1}^{n} x_l^{\gamma_{jl}} (1 \le j \le q) \tag{4.3}$$

and coefficient $a_{ik} = c_{ik} H(c_{ik})$, $a_j = c_j H(c_j)$, variable $x = (x_1, x_2, \cdots, x_n)^T > 0$, exponent γ_{ikl} $(1 \le k \le J_i, 0 \le i \le p, 1 \le l \le n)$, γ_{jl} $(1 \le j \le q, 1 \le l \le n)$ is arbitrary real number.

The programming (4.1) can usually be solved based on the theory of convex programming and geometric programming [18,19].

Definition 4.2. The solution x^* of crisp geometric programming (4.1) is called the solution of intuitionistic fuzzy geometric programming (3.1).

In order to illustrate the relation between intuitionistic fuzzy coefficients and the solution x^* of intuitionistic fuzzy geometric programming (3.1), some accuracy degrees of the solution x^* of intuitionistic fuzzy geometric programming (3.1) is defined as follows:

Definition 4.3. Let

$$H(x^*) = (\bigwedge_{i=0}^{p} \bigwedge_{k=1}^{J_i} H(\tilde{c_{ik}})) \wedge (\bigwedge_{j=1}^{q} H(\tilde{c_j})), \tag{4.4}$$

the $H(x^*)$ is called the minimal accuracy degree of the solution x^* of intuitionistic fuzzy geometric programming (3.1).

Definition 4.4. Let

$$H(x^*) = \sqrt[w]{(\prod_{i=0}^{p} \prod_{k=1}^{J_i} H(\tilde{c_{ik}})) \prod_{j=1}^{q} H(\tilde{c_j})}, \tag{4.5}$$

where $w = \sum_{i=0}^{p} J_i + q$, the $H(x^*)$ is called the geometric mean accuracy degree of the solution x^* of intuitionistic fuzzy geometric programming (3.1).

Definition 4.5. Let

$$H(x^*) = \frac{(\sum_{i=0}^{p} \sum_{k=1}^{J_i} H(\tilde{c_{ik}})) + (\sum_{j=1}^{q} H(\tilde{c_j}))}{w}, \tag{4.6}$$

where $w = \sum_{i=0}^{p} J_i + q$, the $H(x^*)$ is called the arithmetic mean accuracy degree of the solution x^* of intuitionistic fuzzy geometric programming (3.1).

Definition 4.6. Let

$$H(x^*) = (\bigvee_{i=0}^{p} \bigvee_{k=1}^{J_i} H(\tilde{c_{ik}})) \vee (\bigvee_{j=1}^{q} H(\tilde{c_j})), \tag{4.7}$$

the $H(x^*)$ is called the maximal accuracy degree of the solution x^* of intuitionistic fuzzy geometric programming (3.1).

The above some accuracy degrees can depicts the solution x^* of intuitionistic fuzzy geometric programming (3.1) from different side.

Based on the above discussion, we are ready to present an algorithm to find an optimal solution for intuitionistic fuzzy geometric programming (3.1).

Algorithm 4.1

Step 1 Through the fuzzy information processing technology, the coefficient are denoted by the intuitionistic fuzzy numbers or the interval ones.
Step 2 Establish intuitionistic fuzzy geometric programming (3.1).
Step 3 Establish crisp geometric programming (4.1) based on intuitionistic fuzzy geometric programming (3.1).
Step 4 Solving crisp geometric programming by duality theory.
Step 5 Solving the optimal solution x^* and the optimal value $g(x^*)$ of crisp geometric programming (4.1).
Step 6 Utilize definition (4.4)–(4.7) to calculate the accuracy degrees of the optimal solution x^*.

5 Numerical Example

In this section, two optimization examples of geometric programming with intuitionistic fuzzy coefficient are provided [19,20]. The first example is geometric programming with intuitionistic fuzzy number coefficient. The second example is geometric programming with interval intuitionistic fuzzy number. The optimal solution can be obtained and the accuracy degrees of the optimal solution can be analyzed by the Algorithm 4.1.

Example 1

$$\begin{aligned}
\min\ & \tilde{g}_0(x) = \widetilde{200}x_1x_2 + \tilde{\tfrac{5}{9}}x_1x_2^{\frac{2}{3}}x_3^3 + \widetilde{\tfrac{41}{45}}x_1x_2^{\frac{2}{3}}x_3^3x_4,\\
\text{s.t}\ & \tilde{g}_1(x) = \widetilde{2.74\cdot 10^6}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \le 1,\\
& \tilde{g}_2(x) = \widetilde{1\tfrac{1}{9}}x_4 + \widetilde{\tfrac{67}{80}\cdot 10^{-6}}x_2x_3x_4 \le 1,\\
& x_1, x_2, x_3, x_4 > 0.
\end{aligned}$$

where $\widetilde{200} = (0.6, 0.4)$, $\tilde{\frac{5}{9}} = (0.8, 0.1)$, $\widetilde{\frac{41}{45}} = (0.7, 0.2)$ $\widetilde{2.74\cdot 10^6} = (0.7, 0.3)$, $\widetilde{1\frac{1}{9}} = (0.5, 0.4)$, $\widetilde{\frac{67}{80}\cdot 10^{-6}} = (0.6, 0.2)$.

The intuitionistic fuzzy geometric programming can be changed into crisp geometric programming as follows:

$$\begin{aligned}
\min\ & g_0(x) = 200x_1x_2 + 0.5x_1x_2^{\frac{2}{3}}x_3^3 + 0.82x_1x_2^{\frac{2}{3}}x_3^3x_4,\\
\text{s.t}\ & g_1(x) = 2.74\cdot 10^6x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \le 1,\\
& g_2(x) = x_4 + 0.67\cdot 10^{-6}x_2x_3x_4 \le 1,\\
& x_1, x_2, x_3, x_4 > 0.
\end{aligned}$$

The duality programming is as follows:

$$\begin{aligned}
\max\ & D(\delta) = (\tfrac{200}{\delta_{01}})^{\delta_{01}}(\tfrac{0.50}{\delta_{02}})^{\delta_{02}}(\tfrac{0.82}{\delta_{03}})^{\delta_{03}}(2.74\cdot 10^6)^{\delta_{11}}(\tfrac{\delta_{20}}{\delta_{21}})^{\delta_{21}}(\tfrac{0.67\cdot 10^{-6}\delta_{20}}{\delta_{22}})^{\delta_{22}},\\
\text{s.t}\ & \delta_{01} + \delta_{02} + \delta_{03} = 1,\\
& \delta_{01} + \delta_{02} + \delta_{03} - \delta_{11} = 0,\\
& \delta_{01} + \tfrac{2}{3}\delta_{02} + \tfrac{2}{3}\delta_{03} - \delta_{11} + \delta_{22} = 0,\\
& 3\delta_{02} + 3\delta_{03} - \delta_{11} + \delta_{22} = 0,\\
& \delta_{03} - \delta_{11} + \delta_{21} + \delta_{22} = 0,\\
& \delta_{01}, \delta_{02}, \delta_{03}, \delta_{11}, \delta_{20}, \delta_{21}, \delta_{22} \ge 0.
\end{aligned}$$

Solving the duality programming, the optimal solutions of duality variables can be obtained:

$$\delta_{01}^* = 0.700, \delta_{02}^* = 0.125, \delta_{03}^* = 0.175, \delta_{11}^* = 1.000, \delta_{20}^* = 0.825, \delta_{21}^* = 0.725, \delta_{22}^* = 0.100.$$

Further, the optimal solutions of primal variables can be obtained:

$$x_1^* = 15, x_2^* = 16960, x_3^* = 12.200, x_4^* = 0.80.$$

The optimal value $g_0(x^*) = 73.26$.
The minimal accuracy degree of the solution x^* is $H(x^*) = 0.8$.
The geometric mean accuracy degree of the solution x^* is $H(x^*) = 0.91$.
The arithmetic mean accuracy degree of the solution x^* is $H(x^*) = 0.92$.
The maximal accuracy degree of the solution x^* is $H(x^*) = 1$.

Example 2

$$\begin{aligned}
\min\ & \tilde{g_0}(x) = \widetilde{\tfrac{10}{7}} x_1^{-4} x_2^{-1} x_4^2 + \tilde{6} x_1^2 x_2^{-2},\\
\text{s.t}\ & \tilde{g_1}(x) = \widetilde{\tfrac{5}{2}} x_2 x_3 + \widetilde{\tfrac{30}{7}} x_1^{\frac{-1}{2}} x_2^{\frac{-3}{4}} x_3^{-1} + \widetilde{\tfrac{15}{2}} x_2^{\frac{1}{2}} x_3^{-1} x_4^{\frac{-1}{2}} \le 1,\\
& x_1, x_2, x_3, x_4 > 0.
\end{aligned}$$

Where

$$\begin{aligned}
\widetilde{\frac{10}{7}} &= ([0.6, 0.1], [0.5, 0.2]), \tilde{6} = ([0.4, 0.2], [0.3, 0.1]),\\
\widetilde{\frac{5}{2}} &= ([0.6, 0.1], [0.7, 0.2]), \widetilde{\frac{30}{7}} = ([0.4, 0.1], [0.6, 0.3]),\\
\widetilde{\frac{15}{2}} &= ([0.5, 0.2], [0.4, 0.1]).
\end{aligned}$$

The intuitionistic fuzzy geometric programming can be changed into crisp geometric programming as follows:

$$\begin{aligned}
\min\ & g_0(x) = x_1^{-4} x_2^{-1} x_4^2 + 3 x_1^2 x_2^{-2},\\
\text{s.t}\ & g_1(x) = \tfrac{1}{3} x_2 x_3 + 3 x_1^{\frac{-1}{2}} x_2^{\frac{-3}{4}} x_3^{-1} + \tfrac{9}{2} x_2^{\frac{1}{2}} x_3^{-1} x_4^{\frac{-1}{2}} \le 1,\\
& x_1, x_2, x_3, x_4 > 0.
\end{aligned}$$

The duality programming is as follows:

$$\begin{aligned}
\max\ & D(\delta) = (\tfrac{1}{\delta_{01}})^{\delta_{01}} (\tfrac{3}{\delta_{02}})^{\delta_{02}} (\tfrac{\frac{1}{3}\delta_{10}}{\delta_{11}})^{\delta_{11}} (\tfrac{3\delta_{10}}{\delta_{12}})^{\delta_{12}} (\tfrac{9\delta_{10}}{\delta_{13}})^{\delta_{13}},\\
\text{s.t}\ & \delta_{01} + \delta_{02} = 1,\\
& -4\delta_{01} + 2\delta_{02} - \tfrac{1}{2}\delta_{12} = 0,\\
& -\delta_{01} - 2\delta_{02} + \delta_{11} - \tfrac{3}{4}\delta_{12} + \tfrac{1}{2}\delta_{13} = 0,\\
& \delta_{11} - \delta_{12} - \delta_{13} = 0,\\
& 2\delta_{01} - \tfrac{1}{2}\delta_{13} = 0,\\
& \delta_{01}, \delta_{02}, \delta_{11}, \delta_{12}, \delta_{13} \ge 0.
\end{aligned}$$

Where $\delta_{10} = \delta_{11} + \delta_{12} + \delta_{13}$.

Solving the duality programming, the optimal solutions of duality variables can be obtained:

$$\delta_{01}^* = \frac{1}{4}, \delta_{02}^* = \frac{3}{4}, \delta_{11}^* = 2, \delta_{12}^* = 1, \delta_{13}^* = 1, \delta_{10}^* = 4.$$

Further, the optimal solutions of primal variables can be obtained:

$$x_1^* = 418, x_2^* = 42.7, x_3^* = 0.035, x_4^* = 11184810.$$

The optimal value $g_0(x^*) = 384$.
The minimal accuracy degree of the solution x^* is $H(x^*) = 0.5$.
The geometric mean accuracy degree of the solution x^* is $H(x^*) = 0.65$.
The arithmetic mean accuracy degree of the solution x^* is $H(x^*) = 0.66$.
The maximal accuracy degree of the solution x^* is $H(x^*) = 0.8$.

6 Conclusion

In this paper, we studied the geometric programming with intuitionistic fuzzy coefficient. We have changed intuitionistic fuzzy geometric programming into crisp geometric programming based on centain accuracy degree, we can obtain optimal solution by solving crisp geometric programming, the optimal solution has been analyzed based on some accuracy degree. At last, two numerical examples is given to illustrate the proposed algorithm.

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A New Method for Solving Fully Fuzzy Monomial Geometric Programming with Trapezoidal Fuzzy Parameters

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Abstract. Geometric Programming (GP) problem is very considered problem in many fields these days. In the literature, many studies have been focusing on the different types of GP. Many methods were proposed and developed during these years to find the optimal solution easier. In this paper, we propose a method to solve one type of geometric programming so called Monomial Geometric Programming problem respect to Trapezoidal Fuzzy Numbers (MGPTFN). We try to keep the fuzzy numbers in the fuzzy form during the whole solution and use negative fuzzy numbers in the problem. We want to show that in this type of Fuzzy Geometric Programming problem (FGP), for the problem with negative coefficients or exponents, optimal solution and optimal value can be computed and for multi-objective geometric programming problem or GP with polynomial objective function, this method holds. For illustrate the method we use numerical examples.

Keywords: Fully fuzzy monomial geometric programming · Trapezoidal fuzzy variables · Negative fuzzy numbers

1 Introduction

Fuzzy set theory is very popular in engineering system and science management. For the first time, Tanaka et al. proposed the fuzzy mathematical programming problem [19]. Afterwards, Zimmermann [18] proposed the first formulation of Fuzzy Linear Programming (FLP). Geometric programming is a type of nonlinear programming problem. In fact geometric programming mainly, is extension comprehension of linear programming applications and constitutionally classified in many types of nonlinear sets. In the beginning, GP was applied in engineering and sciences. At first the most applications of geometric programming were

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in chemical and mechanical engineering, statistics and probability, economics, wireless networking and etc. [4,10,24].

Nowadays fuzzy geometric programming is very important in each field, such as engineering, statistics, economic, management. Many authors worked on it and tried to find the best method to solve FGP's problem and found out the optimal value.

In this paper, a new method is proposed to find the optimal fuzzy solution and optimal fuzzy value of fuzzy monomial geometric programming problem with equality constraints. This method has two general parts. The first part is to convert fuzzy monomial geometric programming to linear one, and the second step is to solve the fuzzy linear programming without changing fuzzy number to real one in whole solution. In some papers the method is to change the fuzzy number to real number by using Rank on objective function or constraints [20,22], but we try to keep it fuzzy number and we show that there is no limitation for numbers to be negative or positive.

The rest of this paper is organized as follows: Sect. 2 review some basic definitions of geometric programming and trapezoidal fuzzy number's arithmetic. In Sect. 3, we first introduce the monomial fuzzy geometric programming problem and some depending theorems and then devotes to presenting a method to solve it. In Sect. 4, we illustrate the algorithm by three different types of numerical examples to show that there is no boundary for objective function and negative fuzzy numbers exist in this method. The conclusions are discussed in Sect. 5.

2 Preliminaries

In this section, we review some basic and necessary definitions and notices.

Definition 1. *The subset $\tilde{A}$ in set X defined as $\tilde{A} = \{(\mu_{\tilde{A}}(x), x) | x \in X\}$, where $\mu_{\tilde{A}}(x)$ is a real number belong to interval [0,1]. $\mu_{\tilde{A}}(x)$ is degree of membership x in $\tilde{A}$ and call*

$$\mu_{\tilde{A}} : X \to [0,1],$$
$$x \to \mu_{\tilde{A}}(x)$$

a membership function in fuzzy set $\tilde{A}$.

Definition 2. *We denote the trapezoidal fuzzy number as $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})$ and show the set of all trapezoidal fuzzy numbers with $F(\mathbb{R})$.*

Definition 3. *Fuzzy number $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})$ is said to be a trapezoidal fuzzy number, if its membership function defined as follows:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0\,; & x < a^- - \underline{a}, x > a^+ + \overline{a}, \\ 1 - \frac{a^- - x}{\underline{a}}\,; & a^- - \underline{a} \leqslant x < a^-, \\ 1\,; & a^- \leqslant x \leqslant a^+, \\ 1 - \frac{x - a^+}{\overline{a}}\,; & a^+ < x \leqslant a^+ + \overline{a}. \end{cases}$$

Definition 4 [28]. *L-R fuzzy number $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})_{LR}$ can be presented as follows:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0\,; & x < a^- - \underline{a}, x > a^+ + \overline{a}, \\ L(\frac{a^- - x}{\underline{a}})\,; & a^- - \underline{a} \leqslant x < a^-, \\ 1\,; & a^- \leqslant x \leqslant a^+, \\ R(\frac{x - a^+}{\overline{a}})\,; & a^+ < x \leqslant a^+ + \overline{a}, \end{cases}$$

where $a^- < a^+$ and $\underline{a} > 0$ and $\overline{a} > 0$ are left and right spreads, respectively. Functions $L(\frac{a^- - x}{\underline{a}})$ and $R(\frac{x - a^+}{\overline{a}})$ are continuous and strictly decreasing functions in closed interval $[0, 1]$ and satisfy $L(x) = R(x) = 0$ for $x \geqslant 1$, and $L(x) = R(x) = 1$ for $x \leqslant 0$.

Remark 1 [5]. In Definition 4 particularly when $L(x) = R(x) = 1 - x$, the L-R fuzzy number $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})_{LR}$ becomes a trapezoidal fuzzy number $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})$.

Definition 5 [5,11,14]. *Let $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})$ and $\tilde{B} = (b^-, b^+, \underline{b}, \overline{b})$ be two trapezoidal fuzzy numbers. The arithmetic operations properties on trapezoidal fuzzy numbers denote as follows:*

$$(\textbf{\textit{1}})\tilde{A} + \tilde{B} = (a^- + b^-, a^+ + b^+, \underline{a} + \underline{b}, \overline{a} + \overline{b}).$$

From Definition 3, let $L(\frac{a^- - x}{\underline{a}}) = L(\frac{b^- - y}{\underline{b}}) = v, (v \in [0, 1])$, and $R(\frac{x - a^+}{\overline{a}}) = R(\frac{y - b^+}{\overline{b}}) = v, (v \in [0, 1])$. So $x = a^- - \underline{a}L^{-1}(v) = a^+ + \overline{a}R^{-1}(v)$ and $y = b^- - \underline{b}L^{-1}(v) = b^+ + \overline{b}R^{-1}(v)$. It shows that $t = x + y = (a^- + b^-) - (\underline{a} + \underline{b})L^{-1}(v)$. So it holds and the same for $R^{-1}(v)$, i.e., $t = x + y = (a^+ + b^+) + (\overline{a} + \overline{b})R^{-1}(v)$. So the formula of addition can be proved. For opposite of $\tilde{A}$, we have

$$(\textbf{\textit{2}}) - \tilde{A} = (-a^+, -a^-, \overline{a}, \underline{a}),$$

$$(\textbf{\textit{3}})c \geqslant 0, c \in \mathbb{R}; \quad c\tilde{A} = (ca^-, ca^+, c\underline{a}, c\overline{a}),$$

$$(\textbf{\textit{4}})c < 0, c \in \mathbb{R}; \quad c\tilde{A} = (ca^+, ca^-, -c\overline{a}, -c\underline{a}),$$

$$(\textbf{\textit{5}})\tilde{A} - \tilde{B} = (a^- - b^+, a^+ - b^-, \underline{a} + \overline{b}, \overline{a} + \underline{b}).$$

For multiplication of two trapezoidal fuzzy numbers, suppose that $\tilde{A} = (a^-, a^+, \underline{a}, \overline{a})$ and $\tilde{B} = (b^-, b^+, \underline{b}, \overline{b})$ are positive numbers. Considering the basic operation of this type to LR fuzzy numbers, the computation of multiplication is similar to computation of two fuzzy numbers addition. Following formula for $\tilde{A}, \tilde{B}$ holds:

$t = xy = (a^- - \underline{a}L^{-1}(v))(b^- - \underline{b}L^{-1}(v)) = a^-b^- - (a^-\underline{b} + \underline{a}b^-)L^{-1}(v) + \underline{a}\underline{b}(L^{-1}(v))^2$.

$t = xy = (a^+ + \overline{a}R^{-1}(v))(b^+ + \overline{b}R^{-1}(v)) = a^+b^+ + (a^+\overline{b} + \overline{a}b^+)R^{-1}(v) + \overline{a}\overline{b}(R^{-1}(v))^2$.

Consider to $L^{-1}(v)$ and $R^{-1}(v)$, without losing generality we can eliminate terms $(a^-\underline{b} + \underline{a}b^-)L^{-1}(v)$ and $(a^+\overline{b} + \overline{a}b^+)R^{-1}(v)$ and construct trapezoidal fuzzy number.

The same as $L^{-1}(v)$, *we can also compute approximate formula for the multiplication by* $R^{-1}(v)$.

$t = -xy = (-a^- + \underline{a}L^{-1}(v))(b^- - \underline{b}L^{-1}(v)) = -a^-b^- + (a^-\underline{b} + \underline{a}b^-)L^{-1}(v) - \underline{ab}(L^{-1}(v))^2$, *and its similar for* $R^{-1}(v)$.

For trapezoidal fuzzy numbers $\tilde{A} < 0$ *and* $\tilde{B} > 0$, *can get their formula for multiplication similarly. For extended multiplication of these LR fuzzy numbers, the approximate formulas obtain as follows:*

$$(\boldsymbol{6})\tilde{A} > 0, \tilde{B} > 0; \qquad \tilde{A} \times \tilde{B} = (a^-b^-, a^+b^+, \underline{ab}, \overline{a}\overline{b}),$$

$$(\boldsymbol{7})\tilde{A} < 0, \tilde{B} > 0; \qquad \tilde{A} \times \tilde{B} = (-a^-b^-, -a^+b^+, -\underline{ab}, -\overline{a}\overline{b}).$$

More information can be found in [11,14,16].

3 Monomial Geometric Programming

In this section, we express Geometric Programming (GP) and the type of GP, called Monomial Fuzzy Geometric Programming (MFGP) and then will discuss some definitions and theorems to demonstrate the sufficient and necessary condition for feasible and optimal solution in posynomial fuzzy geometric programming problem, after that propose the method to solve MFGP with trapezoidal fuzzy numbers in all constants and variables.

Definition 6 [5]. *Call*

$$\min \quad \sum_{k=1}^{J_0} k_{0k} \prod_{j=1}^{m} x_j^{\alpha_{0kj}} \tag{1}$$

$$s.t. \quad \sum_{k=1}^{J_i} k_{ik} \prod_{j=1}^{m} x_j^{\alpha_{ikj}} \leqslant t_i, \quad (1 \leqslant i \leqslant n),$$

$$x > 0,$$

the posynomial geometric programming of x, *where* $x = (x_1, ..., x_m)^T$ *is an* m-*dimensional variable vector,* $k_{ik} > 0$ *is a coefficient real number and* $\alpha_{ikj} > 0$ *is an arbitrary real number.*

Definition 7. *We define fully fuzzy monomial geometric program as follows:*

$$\min \quad \tilde{k}_0 \prod_{j=1}^{m} \tilde{x}_j^{\tilde{\alpha}_{0j}} \tag{2}$$

$$s.t. \quad \tilde{k}_i \prod_{j=1}^{m} \tilde{x}_j^{\tilde{\alpha}_{ij}} \leqslant \tilde{t}_i, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{x} > \tilde{0},$$

where $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_m)^T$ *is* m-*dimensional fuzzy variable vector, Here* $x_i = (x^-, x^+, \underline{x}, \overline{x})$ *is a trapezoidal fuzzy number,* $\tilde{k}_i > 0$ *is a coefficient fuzzy number,* $\tilde{t}_i > 0$ *is a fuzzy number and* $\tilde{\alpha}_{ij}$ *is an arbitrary fuzzy number.*

Degree of Difficulty. In geometric programing problem, the term $N - n - 1$ indicate the degree of difficulty. In posynomial geometric programming problem, N represents the number of all the terms of posynomials in geometric programming i.e., N represent numbers of terms in objective function plus numbers of the terms in constraints and n is the number of design variables.

Note. If the geometric programming problem has a zero-degree-of-difficulty, the solution is unique. So this is the sufficient condition of feasible solution for geometric programming problem. For more information see [25].

Theorem 1. *Each fuzzy posynomial geometric programming* (1) *can turn into fuzzy convex programming* [7].

Proof. Let $x_j = e^{z_j}$ for $1 \leqslant j \leqslant m$. Then

$$\sum_{k=1}^{J_i} k_{ik} \prod_{j=1}^{m} x_j^{\alpha_{ikj}} = \sum_{k=1}^{J_i} k_{ik} e^{\sum_{j=1}^{m} z_j \alpha_{ikj}} = H_i(z), \quad 0 \leqslant i \leqslant n. \tag{3}$$

From [2] the conclusion of the theorem holds.

Remark 2 [5]. Each fuzzy posynomial geometric programming problem (1) can turn into a monomial fuzzy geometric programming problem.

Theorem 2. *Each monomial fuzzy posynomial geometric programming (2) can turn into a fuzzy linear programming with the optimal solution:*

$$\min \quad \ln\tilde{k}_0 + \sum_{j=1}^{m} \tilde{\alpha}_{0j}\tilde{z}_j \tag{4}$$

$$s.t. \quad \ln\tilde{k}_i + \sum_{j=1}^{m} \tilde{\alpha}_{ij}\tilde{z}_j \leqslant \ln\tilde{ti}, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > \tilde{0}.$$

Proof. Apply "ln" on Eq. (2), it turns to

$$\min \quad \ln\tilde{k}_0 + \sum_{j=1}^{m} \tilde{\alpha}_{0j}\ln\tilde{x}_j \tag{5}$$

$$s.t. \quad \ln\tilde{k}_i + \sum_{j=1}^{m} \tilde{\alpha}_{ij}\ln\tilde{x}_j \leqslant \ln\tilde{ti}, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > \tilde{0}.$$

Let $\tilde{z}_j = \ln\tilde{x}_j$. So we obtain

$$\min \quad \ln\tilde{k}_0 + \sum_{j=1}^{m} \tilde{\alpha}_{0j}\tilde{z}_j \tag{6}$$

$$s.t. \quad \ln\tilde{k}_i + \sum_{j=1}^{m} \tilde{\alpha}_{ij}\tilde{z}_j \leqslant \ln\tilde{ti}, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > \tilde{0}.$$

From Theorem 1, Eq. (2) is a convex programming, so by [2] it has fuzzy optimal solution.

Theorem 3. *Each fuzzy posynomial geometric programming (FGP) (2) can be turned into fuzzy linear programming (FLP) (5).*

Proof. The theorem holds by Remark 2 and Theorem 2.

Definition 8 [1]. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in T$ be feasible solution. If $x^* \in T$ and $f(x) \geqslant f(x^*), \forall x \in T$, then we called x^*, an optimal solution.*

Theorem 4 [1]. *Suppose that $\phi \neq T \in \mathbb{R}^n$ be a open convex set and $f : T \to \mathbb{R}$ be differentiable on T, then f is convex iff $\forall x_1, x_2 \in T$, we have*

$$(\nabla f(x_2) - \nabla f(x_1))^t (x_2 - x_1) \geqslant 0.$$

Definition 9. *Suppose that $\phi \neq T \in \mathbb{R}^n$ be a convex set and $f : T \to \mathbb{R}$. Function F is convex if for $x, y \in T$*

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y), \quad 0 \leqslant \lambda \leqslant 1.$$

Theorem 5. *Suppose that $\phi \neq T \in \mathbb{R}^n$ be a convex set and $f : T \to \mathbb{R}$ be convex on T. For minimize $f(x)$ objective problem, suppose that $x^* \in T$ be a local optimal solution, then*

(1) x^* is a global optimal solution.
(2) If x^* is a strict local minimum or f is strictly convex, then x^* is the uniqe global optimal solution.

Proof. Suppose that x^* be a local optimal solution, so there exists a ϵ-neighborhood $N_\epsilon(x^*)$ for x^*, where $f(x^*) \leqslant f(x), \forall x \in T \cap N_\epsilon(x^*)$.
Use contradiction for solving. Suppose that x^* is not a global optimal solution, so there exists some $\overline{x} \in T$, such that $f(\overline{x}) \leqslant f(x^*)$. Again, f is a convex function on T, so for $0 \leqslant \lambda \leqslant 1$, we have

$$f(\lambda\overline{x} + (1-\lambda)x^*) \leqslant \lambda f(\overline{x}) + (1-\lambda) f(x^*) \leqslant \lambda f(x^*) + (1-\lambda) f(x^*) = f(x^*) \in T \cap N_\epsilon(x^*).$$

So $\lambda\overline{x} + (1-\lambda)x^* \in T \cap N_\epsilon(x^*)$. It's clear that $f(\lambda\overline{x} + (1-\lambda)x^*) \leqslant f(x^*)$ has contradicts with $f(x^*) \leqslant f(x), \forall x \in T \cap N_\epsilon(x^*)$.

For proving part 2 see [1].

Corollary 1. *From the theorems and definitions mentioned above, we can conclude that the necessary condition to have optimal solution in minimization objective problem is the objective function to be a convex function.*

Remark 3. In solving Eq. (2) there is no limitation for $\tilde{z}_j$, i.e., $\tilde{z}_j$ can be positive or negative number, because from Theorem 1, $\tilde{x}_j = e^{\tilde{z}_j}$, so $\tilde{x}_j > 0$ always holds.

As considering, the monomial geometric programming change into linear programming problem. Now we can solve it with the suitable method in type of linear programming.

3.1 Algorithm

Here by using the definitions and theorem mentioned in Sect. 2, we would like to explain the algorithm step by step.

(1) For the first step is to change Eq. (2) to (5) based on Theorem 2.
(2) Write the Eq. (5) as follows:

$$\min \quad \ln\tilde{k}_0 + \sum_{j=1}^{m} \tilde{\alpha}_{0j} \times \tilde{z}_j \tag{7}$$

$$s.t. \quad \ln\tilde{k}_i + \sum_{j=1}^{m} \tilde{\alpha}_{ij} \times \tilde{z}_j \leqslant \ln\tilde{ti}, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > 0.$$

(3) Substitute all of the trapezoidal fuzzy parameters $\tilde{k}_i = (k_i^-, k_i^+, \underline{k}_i, \overline{k}_i)$, $\tilde{\alpha}_{ij} = (\alpha_{ij}^-, \alpha_{ij}^+, \underline{\alpha}_{ij}, \overline{\alpha}_{ij})$, $\tilde{z}_j = (z_j^-, z_j^+, \underline{z}_j, \overline{z}_j)$ and $\tilde{t}_i = (t_i^-, t_i^+, \underline{t}_i, \overline{t}_i)$ in Eq. (7). We obtain:

$$\min \quad \ln(k_i^-, k_i^+, \underline{k}_i, \overline{k}_i) + \sum_{j=1}^{m} (\alpha_{0j}^-, \alpha_{0j}^+, \underline{\alpha}_{0j}, \overline{\alpha}_{0j}) \times (z_j^-, z_j^+, \underline{z}_j, \overline{z}_j)$$

$$s.t. \quad \ln(k_i^-, k_i^+, \underline{k}_i, \overline{k}_i) + \sum_{j=1}^{m} (\alpha_{ij}^-, \alpha_{ij}^+, \underline{\alpha}_{ij}, \overline{\alpha}_{ij}) \times (z_j^-, z_j^+, \underline{z}_j, \overline{z}_j) \leqslant \ln(t_i^-, t_i^+, \underline{t}_i, \overline{t}_i), (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > 0.$$

(4) Let $(\alpha_{ij}^-, \alpha_{ij}^+, \underline{\alpha}_{ij}, \overline{\alpha}_{ij}) \times (z_j^-, z_j^+, \underline{z}_j, \overline{z}_j) = (m_{ij}^-, m_{ij}^+, \underline{m}_{ij}, \overline{m}_{ij}) = \tilde{m}_{ij}, (1 \leqslant j \leqslant m), (0 \leqslant i \leqslant n)$ by using the arithmetic operation properties, defined in preliminaries section, the fuzzy geometric programming problem in Step 3 change into the following problem:

$$\min \quad \ln\tilde{k}_i + \sum_{j=1}^{m} \tilde{m}_{ij}$$

$$s.t. \quad \ln k_i^- + \sum_{j=1}^{m} m_{ij}^- = \ln t_i^-, \quad (1 \leqslant i \leqslant n),$$

$$\ln k_i^+ + \sum_{j=1}^{m} m_{ij}^+ = \ln t_i^+, \quad (1 \leqslant i \leqslant n),$$

$$\ln \underline{k}_i + \sum_{j=1}^{m} \underline{m}_{ij} = \ln \underline{t}_i, \quad (1 \leqslant i \leqslant n),$$

$$\ln \overline{k}_i + \sum_{j=1}^{m} \overline{m}_{ij} = \ln \overline{t}_i, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > \tilde{0}.$$

(5) Keep the parameters in one side and transfer the given fuzzy numbers to another side. Let $\ln t_i - \ln k_i = l_i$ for each for constraints

$$\min \quad \ln \tilde{k}_i + \sum_{j=1}^{m} \tilde{m}_{ij}$$

$$s.t. \quad \sum_{j=1}^{m} m_{ij}^- = l_i^-, \quad (1 \leqslant i \leqslant n),$$

$$\sum_{j=1}^{m} m_{ij}^+ = l_i^+, \quad (1 \leqslant i \leqslant n),$$

$$\sum_{j=1}^{m} \underline{m}_{ij} = \underline{l}_i, \quad (1 \leqslant i \leqslant n),$$

$$\sum_{j=1}^{m} \overline{m}_{ij} = \overline{l}_i, \quad (1 \leqslant i \leqslant n),$$

$$\tilde{z}_j > \tilde{0}.$$

(6) Now we find the solution $z_j^-, z_j^+, \underline{z}_j$ and $\overline{z}_j$ easily from the equations in step5.
(7) Substitute the values of step 5 in $z_j^- = \ln x_j^-, z_j^+ = \ln x_j^+, \underline{z}_j = \ln \underline{x}_j$ and $\overline{z}_j = \ln \overline{x}_j$ and obtain optimal solution $x_j^-, x_j^+, \underline{x}_j$ and $\overline{x}_j$.
(8) Obtain fuzzy optimal solution by putting $x_j^-, x_j^+, \underline{x}_j$ and $\overline{x}_j$ in $\tilde{x}_j^* = (x_j^-, x_j^+, \underline{x}_j, \overline{x}_j), (1 \leqslant j \leqslant m)$.
(9) Find fuzzy optimal value by substitute fuzzy optimal solution $\tilde{x}_j^*$ in Eq. (2).

The proposed algorithm will be explained by solving illustration examples in the next section.

4 Numerical Example

In this section, we illustrate the method by helping numerical example. We solve three different type of examples. First is only for positive fuzzy numbers. Example 1 exhibit the process of the algorithm for solving fully fuzzy monomial geometric programming problem step-by-step. The method on how to reach the optimal solution will be fully described with this simple example. Following the first example, we provide Example 2 to explain when negative fuzzy numbers appear in objective function and constraints in monomial geometric programming problem, this method can be utilized for obtaining the optimal solution. Finally in Example 3, we try to extend monomial geometric programming to posynomial geometric programming problem. This example shows that if the objective function become a posynomial GP function and constraints be monomial GP function, for positive and negative fuzzy numbers we can use this method to compute optimal solution and optimal value.

So these three example illustrate the accuracy and impact of the algorithm and clearly explain the examples diversity.

Example 1. Solve the fully fuzzy monomial geometric programming problem with the help of algorithm method

$$\begin{aligned} \min \quad & \tilde{x}_1^{\tilde{1}} \tilde{x}_2^{\tilde{2}} \\ s.t. \quad & \tilde{x}_1^{\tilde{1}} \tilde{x}_2^{\tilde{1}} = e^{\tilde{5}}, \\ & \tilde{x}_1^{\tilde{3}} \tilde{x}_2^{\tilde{2}} = e^{\tilde{12}}, \\ & \tilde{x}_1, \tilde{x}_2 > \tilde{0}, \end{aligned}$$

where $\tilde{1} = (1,2,1,1), \tilde{2} = (2,3,1,2), \tilde{3} = (3,4,2,3), \tilde{5} = (5,14,3,5), \tilde{12} = (12,24,4,12)$ and $\tilde{0} = (0,0,0,0)$.

Solution. At change the problem by applying Theorem 2, we obtain

$$\begin{aligned} \min \quad & \tilde{1} \times \tilde{z}_1 + \tilde{2} \times \tilde{z}_2 \\ s.t. \quad & \tilde{1} \times \tilde{z}_1 + \tilde{1} \times \tilde{z}_2 = \tilde{5} \times 1, \\ & \tilde{3} \times \tilde{z}_1 + \tilde{2} \times \tilde{z}_2 = \tilde{12} \times 1, \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

Now substitute the values $\tilde{1}, \tilde{2}, \tilde{3}, \tilde{5}, \tilde{12}$ and variables $\tilde{z}_1 = (z_1^-, z_1^+, \underline{z}_1, \overline{z}_1), \tilde{z}_2 = (z_2^-, z_2^+, \underline{z}_2, \overline{z}_2)$ in the above equation.

$$\begin{aligned} \min \quad & (1,2,1,1) \times (z_1^-, z_1^+, \underline{z}_1, \overline{z}_1) + (2,3,1,2) \times (z_2^-, z_2^+, \underline{z}_2, \overline{z}_2) \\ s.t. \quad & (1,2,1,1) \times (z_1^-, z_1^+, \underline{z}_1, \overline{z}_1) + (1,2,1,1) \times (z_2^-, z_2^+, \underline{z}_2, \overline{z}_2) = (5,14,3,5), \\ & (3,4,2,3) \times (z_1^-, z_1^+, \underline{z}_1, \overline{z}_1) + (2,3,1,2) \times (z_2^-, z_2^+, \underline{z}_2, \overline{z}_2) = (12,24,4,12), \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

By using the arithmetic properties of trapezoidal fuzzy numbers, obtain

$$\begin{aligned} \min \quad & (z_1^-, 2z_1^+, \underline{z}_1, \overline{z}_1) + (2z_2^-, 3z_2^+, \underline{z}_2, 2\overline{z}_2) \\ s.t. \quad & (z_1^-, 2z_1^+, \underline{z}_1, \overline{z}_1) + (z_2^-, 2z_2^+, \underline{z}_2, \overline{z}_2) = (5, 14, 3, 5), \\ & (3z_1^-, 4z_1^+, 2\underline{z}_1, 3\overline{z}_1) + (2z_2^-, 3z_2^+, \underline{z}_2, 2\overline{z}_2) = (12, 24, 4, 12), \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

The above fuzzy linear programming problem change into the equations in Step 4 as follows:

$$\begin{aligned} \min \quad & (z_1^- + 2z_2^-, 2z_1^+ + 3z_2^+, \underline{z}_1 + \underline{z}_2, \overline{z}_1 + 2\overline{z}_2) \\ s.t. \quad & z_1^- + z_2^- = 5, \\ & 2z_1^+ + 2z_2^+ = 14, \\ & \underline{z}_1 + \underline{z}_2 = 3, \\ & \overline{z}_1 + \overline{z}_2 = 5, \\ & 3z_1^- + 2z_2^- = 12, \\ & 4z_1^+ + 3z_2^+ = 24, \\ & 2\underline{z}_1 + \underline{z}_2 = 4, \\ & 3\overline{z}_1 + 2\overline{z}_2 = 12, \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

By solving equations, we obtain $z_1^- = 2, z_1^+ = 3, \underline{z}_1 = 1, \overline{z}_1 = 2$ and $z_2^- = 3, z_2^+ = 4, \underline{z}_2 = 2, \overline{z}_2 = 3$, so $\tilde{z}_1 = (2, 3, 1, 2)$, $\tilde{z}_2 = (3, 4, 2, 3)$ and $(z_1^- + 2z_2^-, 2z_1^+ + 3z_2^+, \underline{z}_1 + \underline{z}_2, \overline{z}_1 + 2\overline{z}_2) = (8, 18, 3, 8)$. Now $\tilde{x}_1^* = (e^2, e^3, e^1, e^2)$ and $\tilde{x}_2^* = (e^3, e^4, e^2, e^3)$ are the optimal solution of the problem and the optimal value is (e^8, e^{18}, e^3, e^8).

Example 2. Solve the following fuzzy geometric programming problem with negative exponents by helping algorithm method.

$$\begin{aligned} \min \quad & \tilde{x}_1^{\tilde{1}} \tilde{x}_2^{\tilde{2}} \\ s.t. \quad & \tilde{x}_1^{\tilde{-1}} \tilde{x}_2^{\tilde{1}} = e^{\tilde{1}}, \\ & \tilde{x}_1^{\tilde{4}} \tilde{x}_2^{\tilde{-1}} = e^{\tilde{5}}, \\ & \tilde{x}_1, \tilde{x}_2 > \tilde{0}, \end{aligned}$$

where $\tilde{1} = (1, 2, 1, 1), \tilde{2} = (2, 3, 1, 2), \tilde{4} = (4, 5, 3, 4), \tilde{5} = (5, 7, 1, 5)$ and $\tilde{0} = (0, 0, 0, 0)$.

Solution. At change the problem by applying Theorem 2:

$$\begin{aligned} \min \quad & \tilde{1} \times \tilde{z}_1 + \tilde{2} \times \tilde{z}_2 \\ s.t. \quad & \tilde{-1} \times \tilde{z}_1 + \tilde{1} \times \tilde{z}_2 = \tilde{1} \times 1, \\ & \tilde{4} \times \tilde{z}_1 + \tilde{-1} \times \tilde{z}_2 = \tilde{5} \times 1, \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

Now substitute the values $\tilde{1}, \tilde{2}, \tilde{4}, \tilde{5}$ and variables $\tilde{z}_1 = (z_1^-, z_1^+, \underline{z}_1, \overline{z}_1), \tilde{z}_2 = (z_2^-, z_2^+, \underline{z}_2, \overline{z}_2)$ in above equation and by using the arithmetic properties of trapezoidal fuzzy numbers, we obtain

$$\begin{aligned} \min \quad & (z_1^-, 2z_1^+, \underline{z}_1, \overline{z}_1) + (2z_2^-, 3z_2^+, \underline{z}_2, 2\overline{z}_2) \\ s.t. \quad & (-z_1^-, -2z_1^+, -\underline{z}_1, -\overline{z}_1) + (z_2^-, 2z_2^+, \underline{z}_2, \overline{z}_2) = (1, 2, 1, 1), \\ & (4z_1^-, 5z_1^+, 3\underline{z}_1, 4\overline{z}_1) + (-1z_2^-, -2z_2^+, -\underline{z}_2, -\overline{z}_2) = (5, 7, 1, 5), \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

This fuzzy linear programming problem can be changed into the equations in Step 4 as follows:

$$\begin{aligned} \min \quad & (z_1^- + 2z_2^-, 2z_1^+ + 3z_2^+, \underline{z}_1 + \underline{z}_2, \overline{z}_1 + 2\overline{z}_2) \\ s.t. \quad & -z_1^- + z_2^- = 1, \\ & -2z_1^+ + 2z_2^+ = 2, \\ & -\underline{z}_1 + \underline{z}_2 = 1, \\ & -\overline{z}_1 + \overline{z}_2 = 1, \\ & 4z_1^- - z_2^- = 5, \\ & 5z_1^+ - 2z_2^+ = 7, \\ & 3\underline{z}_1 - \underline{z}_2 = 1, \\ & 4\overline{z}_1 - \overline{z}_2 = 5, \\ & \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

By solving equations, $z_1^- = 2, z_1^+ = 3, \underline{z}_1 = 1, \overline{z}_1 = 2$ and $z_2^- = 3, z_2^+ = 4, \underline{z}_2 = 2, \overline{z}_2 = 3$, so the fuzzy solution for linear programming is $\tilde{z}_1 = (2, 3, 1, 2)$ and $\tilde{z}_2 = (3, 4, 2, 3)$ and optimal value is $(z_1^- + 2z_2^-, 2z_1^+ + 3z_2^+, \underline{z}_1 + \underline{z}_2, \overline{z}_1 + 2\overline{z}_2) = (8, 18, 3, 8)$. Now $\tilde{x}_1^* = (e^2, e^3, e^1, e^2)$ and $\tilde{x}_2^* = (e^3, e^4, e^2, e^3)$ are the fuzzy optimal solution of the monomial geometric programming problem and, (e^8, e^{18}, e^3, e^8) is the optimal value of it. It's wonderful that to see

$\tilde{x}_1^{\tilde{1}} \tilde{x}_2^{\tilde{2}} = (e^2, e^6, e^1, e^2) \times (e^6, e^{12}, e^2, e^6) = (e^8, e^{18}, e^3, e^8)$.

Therefore we can have following remark.

Remark 4. In geometric programming problem if the objective function is polynomial function, the algorithm holds and optimal solution is not concerned to objective function. The constraints are very important, so objective function can each be polynomial geometric programming problem. Consider the following example.

Example 3. Solve the fully fuzzy polynomial geometric programming problem with negative fuzzy exponents in objective function and constraints

$$\begin{aligned} \min \quad & \tilde{2}\tilde{x}_1^{\tilde{1}}\tilde{x}_2^{\tilde{2}} + \tilde{x}_1^{\tilde{-2}}\tilde{x}_2^{\tilde{1}} \\ s.t. \quad & \tilde{x}_1^{\tilde{-1}}\tilde{x}_2^{\tilde{1}} = e^{\tilde{1}}, \\ & \tilde{x}_1^{\tilde{4}}\tilde{x}_2^{\tilde{-1}} = e^{\tilde{5}}, \\ & \tilde{x}_1, \tilde{x}_2 > \tilde{0}, \end{aligned}$$

where $\tilde{1} = (1,2,1,1), \tilde{2} = (2,3,1,2), \tilde{4} = (4,5,3,4), \tilde{5} = (5,7,1,5)$ and $\tilde{0} = (0,0,0,0)$.

Solution. By applying Theorem 2, the problem change into:

$$\begin{aligned} \min \quad & \ln(\tilde{2}\tilde{x}_1^{\tilde{1}}\tilde{x}_2^{\tilde{2}} + \tilde{x}_1^{\tilde{-2}}\tilde{x}_2^{\tilde{1}}) \\ s.t. \quad & \tilde{-1} \times \tilde{z}_1 + \tilde{1} \times \tilde{z}_2 = \tilde{1} \times 1, \\ & \tilde{4} \times \tilde{z}_1 + \tilde{-1} \times \tilde{z}_2 = \tilde{5} \times 1, \\ & \tilde{x}_1, \tilde{x}_2, \tilde{z}_1, \tilde{z}_2 > \tilde{0}. \end{aligned}$$

Now substitute the value of $\tilde{1}, \tilde{2}, \tilde{4}, \tilde{5}$ and variables $\tilde{z}_1 = (z_1^-, z_1^+, \underline{z}_1, \overline{z}_1), \tilde{z}_2 = (z_2^-, z_2^+, \underline{z}_2, \overline{z}_2)$ in above equation.

$$s.t. \quad (-z_1^-, -2z_1^+, -\underline{z}_1, -\overline{z}_1) + (z_2^-, 2z_2^+, \underline{z}_2, \overline{z}_2) = (1,2,1,1),$$

$$(4z_1^-, 5z_1^+, 3\underline{z}_1, 4\overline{z}_1) + (-1z_2^-, -2z_2^+, -\underline{z}_2, -\overline{z}_2) = (5,7,1,5),$$

$$\tilde{z}_1, \tilde{z}_2 > \tilde{0}.$$

The constraints of the above equation is the same as Example 2. As we see the fuzzy optimal solution of this problem is $\tilde{x}_1^* = (e^2, e^3, e^1, e^2)$ and $\tilde{x}_2^* = (e^3, e^4, e^2, e^3)$. Just substitute the fuzzy optimal solution in objective function. Obtain

$\tilde{2}\tilde{x}_1^{\tilde{1}}\tilde{x}_2^{\tilde{2}} + \tilde{x}_1^{\tilde{-2}}\tilde{x}_2^{\tilde{1}} = (2,3,1,2) \times (e^2, e^6, e^1, e^2) \times (e^6, e^{12}, e^2, e^6) + (e^{-4}, e^{-9}, e^{-1}, e^{-4}) \times (e^3, e^8, e^2, e^3) = (2e^8 + e^{-1}, 3e^{18} + e^{-1}, e^3 + e, 2e^8 + e^1)$.

Therefore the optimal value of the problem is $(2e^8 + e^{-1}, 3e^{18} + e^{-1}, e^3 + e, 2e^8 + e^1)$.

Note. We can solve fully fuzzy posynomial geometric programming problem in monomial constraints by this method. As we see either coefficients or exponents can be negative or positive, the solution is the same and the method holds.

5 Conclusion

In this paper we proposed a new method to solve fully fuzzy monomial geometric programming problem with trapezoidal fuzzy numbers by keep fuzzy numbers in whole solution. We show that if the exponents is negative number, the method holds. The special thing is that if the objective function is posynomial, the FFGP can solved. the condition of being monomial is only for constraints function and objective function which can be posynomial geometric programming.

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Note on Max-Łukasiewicz Bipolar Fuzzy Relation Equation

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Abstract. System of max-Łukasiewicz bipolar fuzzy relation equations is considered in this paper. A modified concept of characteristic matrix and the corresponding necessary and sufficient condition is given to check a solution in such system. Based on the necessary and sufficient condition, novel method for checking the consistency of system of max-Łukasiewicz bipolar fuzzy relation equations is also proposed, with some illustrative examples.

Keywords: Fuzzy relation equation · Max-Łukasiewicz composition · Consistency · Solution · Characteristic matrix

1 Introduction

In this paper we consider the following system of bipolar fuzzy relation equations with max-Łukasiewicz composition

$$\max_{j\in J}\max\{T_L(a_{ij}^+, x_j), T_L(a_{ij}^-, \bar{x}_j)\} = b_i, \quad i = 1, 2, \cdots, m. \tag{1}$$

where $a_{ij}^+, a_{ij}^-, x_j, b_i \in [0, 1]$, $\bar{x}_j = 1 - x_j$ $i \in I = \{1, 2, \cdots, m\}$, $j \in J = \{1, 2, \cdots, n\}$, and T_L represents the max-Łukasiewicz composition.

The max-Łukasiewicz bipolar fuzzy relation equation system (1) was recently introduced by Li [2] and Liu et al. [3]. The authors provided some effective methods for minimizing a linear objective function subject to system (1). As we know, concept of bipolar fuzzy relation equation was proposed by Freson et al. [1] for the first time. In [1], max-min composition was considered. System of max-min bipolar fuzzy relation equations was applied to describe the public awareness of some products. For maximizing the benefits of the suppliers, the authors investigated the corresponding linear optimization problem.

As pointed out in [2], there exists three commonly used compositions in unipolar or bipolar fuzzy relation equation, i.e. *minimum* $T_M(x_0, y_0) = \min\{x_0, y_0\}$,

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product $T_P(x_0, y_0) = x_0 \cdot y_0$ and *Łukasiewicz t-norm* $T_L(x_0, y_0) = \max\{x_0 + y_0 - 1, 0\}$, where $x_0, y_0 \in [0, 1]$. According to the operation of Łukasiewicz t-norm, system (1) can also be written as

$$\max_{j \in J}\{\max\{a_{ij}^+ + x_j - 1, 0\} \vee \max\{a_{ij}^- + \bar{x}_j - 1, 0\}\} = b_i, \quad i = 1, 2, \cdots, m, \quad (2)$$

or in its matrix form

$$A^+ \circ x \vee A^- \circ \bar{x} = b, \quad (3)$$

where $A^+ = (a_{ij}^+)_{m\times n}$, $A^- = (a_{ij}^-)_{m\times n}$, $x = (x_1, x_2, \cdots, x_n)^T$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^T$, $b = (b_1, b_2, \cdots, b_m)^T$, and $\circ$ represents the max-Łukasiewicz composition.

Denote $X = [0, 1]^n$

This short note is motivated by the following considerations based on the existing works [2,3].

- How to check the feasibility of a vector in system (1)? In Ref. [2], after calculation of the value of the lower bound $\check{x}$ and upper bound $\hat{x}$, the equations with right side $b_i = 0$ were deleted. This process is necessary before continuing to check the feasibility of a vector in system (1). Otherwise, the checking condition will be invalid. However, it is found that when the characteristic matrix is modified, then the necessary and sufficient condition will be effective without deleting the equations with $b_i = 0$. In this paper we will give the modified concept of characteristic matrix and the corresponding condition to check a solution in system (1).
- How to check the consistency of system (1)? Consistency of a system of unipolar or bipolar fuzzy relation equation is usually an important issue the relevant investigation. Such as in Refs. [1–3], before solving the linear optimization problem with fuzzy relation constraint, one should check the consistency of the constraint first. If the constraint is inconsistent, then it is unnecessary to continue the solution procedure. There exists no optimal solution to the problem. Unfortunately, we didn't find a necessary and sufficient condition for checking the consistency of the bipolar fuzzy relation equations in [1,3]. The authors just given a necessary condition for the consistency. In this paper, an effective method is proposed to check the consistency of system (1).

We try to give answers to the above-mentioned questions. The rest of the paper is organized as follows. Section 2 is preliminaries, in which some basic concepts and relevant results are presented. In Sect. 3 we give a new definition to characteristic matrix, based on which a necessary and sufficient condition is developed for checking a solution in system (1). In Sect. 4 a necessary and sufficient condition is proposed to check the consistency of system (1) with illustrative examples. Section 5 is the conclusions.

2 Preliminaries

For convenience, we denote

$$f_{ij}(x_j) = \max\{a_{ij}^+ + x_j - 1, 0\} \vee \max\{a_{ij}^- + \bar{x}_j - 1, 0\}, \quad (4)$$

for any $i \in I$, $j \in J$. Then system (1) can be written as

$$\max_{j \in J} f_{ij}(x_j) = b_i, \quad i = 1, 2, \cdots, m, \tag{5}$$

or

$$\begin{cases} f_{11}(x_1) \vee f_{12}(x_2) \vee \cdots \vee f_{1n}(x_n) = b_1, \\ f_{21}(x_1) \vee f_{22}(x_2) \vee \cdots \vee f_{2n}(x_n) = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m1}(x_1) \vee f_{m2}(x_2) \vee \cdots \vee f_{mn}(x_n) = b_m, \end{cases} \tag{6}$$

System (1) is called consistent (inconsistent) its solution set is nonempty (empty). Its solution set is always denoted by $X(A^+, A^-, b)$ in this paper. That is,

$$X(A^+, A^-, b) = \{x \in X | A^+ \circ x \vee A^- \circ \bar{x} = b\}. \tag{7}$$

Lemma 1 *(See Lemma 1 in [2]). For any $a^+, a^-, b \in [0, 1]$, the inequality*

$$\max\{T_L(a^+, x_0),\ T_L(a^-, 1 - x_0)\} \le b$$

holds if and only if

$$T_L(a^-, 1 - b) \le x_0 \le S_L(1 - a^+, b),$$

where $S_L(x_0, y_0) = \min\{x_0 + y_0, 1\}$.

Lemma 2 *(See Lemma 3 in [2]). A vector $x \in [0, 1]^n$ is a solution to system of bipolar max-T_L equations $A^+ \circ x \vee A^- \circ \bar{x} = b$ if and only if $\max\{T_L(a^+_{ij}, x_j), T_L(a^-_{ij}, 1 - x_j)\} \le b_i$ for every $i \in I$ and $j \in J$, and there exists an index $j_i \in J$ for each $i \in I$ such that $\max\{T_L(a^+_{ij_i}, x_{j_i}), T_L(a^-_{ij_i}, 1 - x_{j_i})\} = b_i$.*

Based on the above-mentioned Lemmas 1 and 2, the following Lemma 3 was developed in Ref. [3].

Lemma 3 *(See Lemma 2 in [3]). If $x = (x_j)_{j \in J} \in X(A^+, A^-, b) \ne \emptyset$ is a feasible solution for (1), then*

$$\max_{i \in I}\{a^-_{ij} - b_i, 0\} \le x_j \le \min_{i \in I}\{1 - a^+_{ij} + b_i, 1\}, \quad \forall j \in J.$$

Denote

$$\check{x} = (\check{x}_1, \check{x}_2, \cdots, \check{x}_n) = (\max_{i \in I}\{a^-_{ij} - b_i, 0\})_{j \in J} \tag{8}$$

and

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) = (\min_{i \in I}\{1 - a^+_{ij} + b_i, 1\})_{j \in J} \tag{9}$$

Lemma 3 indicates that a solution of system (1), denoted by x, should be bigger than or equal to $\check{x}$, and meanwhile less than or equal to $\hat{x}$. That is to say, $\check{x} \le x \le \hat{x}$ holds for any $x \in X(A^+, A^-, b)$. In general, $\check{x}$ and $\hat{x}$ are called the *lower bound* and *upper bound* of the solution in system (1).

Lemma 4 *(See Lemma 3 in [3]).* *If $x = (x_j)_{j\in J} \in X(A^+, A^-, b) \neq \emptyset$ is a feasible solution for (1), then*

$$\max_{j\in J}\{\frac{1}{2}(a_{ij}^+ + a_{ij}^- - 1), 0\} \leq b_i \leq \max_{j\in J}\{a_{ij}^+, a_{ij}^-\}, \quad \forall i \in I.$$

In [2], the characteristic matrix is denoted by $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$, where

$$\tilde{q}_{ij} = \begin{cases} \{\check{x}_j\}, & \text{if } T_L(a_{ij}^-, 1-\check{x}_j) = b_i \neq T_L(a_{ij}^+, \hat{x}_j), \\ \{\hat{x}_j\}, & \text{if } T_L(a_{ij}^-, 1-\check{x}_j) \neq b_i = T_L(a_{ij}^+, \hat{x}_j), \\ \{\check{x}_j, \hat{x}_j\}, & \text{if } T_L(a_{ij}^-, 1-\check{x}_j) = b_i = T_L(a_{ij}^+, \hat{x}_j), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{10}$$

Based on the above concept of characteristic matrix, necessary and sufficient condition was given to check whether a vector $x \in [0,1]^n$ is a solution of system (1), as shown in the following Theorem 1.

Theorem 1 [2]**.** *Let $A^+ \circ x \vee A^- \circ \bar{x} = b$ be a system of bipolar max-T_L equations. A vector $x \in [0,1]^n$ is a solution of $A^+ \circ x \vee A^- \circ \bar{x} = b$ if and only if $\check{x} \leq x \leq \hat{x}$ and the induced binary matrix $Q^x = (q_{ij}^x)_{m\times n}$ has non zero rows where*

$$q_{ij}^x = \begin{cases} 1, & if\ x_j \in \tilde{q}_{ij}, \\ 0, & otherwise. \end{cases} \tag{11}$$

3 New Definition of Characteristic Matrix and the Corresponding Necessary and Sufficient Condition for Checking a Solution

In this section we modify the concept of characteristic matrix, based on which a necessary and sufficient condition is developed for checking a solution in system (1). The necessary and sufficient condition lies in Theorem 2. Two lemmas are presented first, which are helpful to the proof of Theorem 2.

Lemma 5. *Let $\check{x}$ and $\hat{x}$ be the lower and upper bounds of system (1), as defined by (8) and (9). Then for any $i \in I$, $j \in J$ and $x_j \in [\check{x}_j, \hat{x}_j]$, we have $f_{ij}(x_j) \leq b_i$. Especially it holds that*

$$f_{ij}(\check{x}_j) \leq b_i, \quad f_{ij}(\hat{x}_j) \leq b_i. \tag{12}$$

Proof. According to (8) and (9), we get

$$\check{x}_j = \max_{i\in I}\{a_{ij}^- - b_i, 0\} \geq \max\{a_{ij}^- - b_i, 0\} = T_L(a_{ij}^-, 1-b_i),$$

and

$$\check{x}_j \leq \hat{x}_j = \min_{i\in I}\{1 - a_{ij}^+ + b_i, 1\} \leq \min\{1 - a_{ij}^+ + b_i, 1\} = S_L(1-a_{ij}^+, b_i).$$

That is $\check{x}_j \in [T_L(a_{ij}^-, 1-b_i), S_L(1-a_{ij}^+, b_i)]$. In the similar way we may check $\hat{x}_j \in [T_L(a_{ij}^-, 1-b_i), S_L(1-a_{ij}^+, b_i)]$. Hence $[\check{x}_j, \hat{x}_j] \subseteq [T_L(a_{ij}^-, 1-b_i), S_L(1-a_{ij}^+, b_i)]$. Considering $x_j \in [\check{x}_j, \hat{x}_j]$, it follows from Lemma 1 that $f_{ij}(x_j) = \max\{T_L(a_{ij}^+, x_j),\ T_L(a_{ij}^-, 1-x_j)\} \leq b_i$.

Lemma 6. *Let $y \in X(A^+, A^-, b) \neq \emptyset$ be a solution of system (1), with lower bound $\check{x}$ and upper bound $\hat{x}$. Suppose there exists some $i \in I$ and $j \in J$ such that*

$$f_{ij}(y_j) = \max\{a_{ij}^+ + y_j - 1, 0\} \vee \max\{a_{ij}^- + \bar{y}_j - 1, 0\} = b_i. \tag{13}$$

Then we have

(i) If $b_i \neq 0$, then $y_j \in \{\check{x}_j, \hat{x}_j\}$.
(ii) If $b_i = 0$, then $y_j \in [\check{x}_j, \hat{x}_j]$.

Proof. (i) $b_i \neq 0$, i.e. $b_i > 0$

Equation (13) indicates either

$$\max\{a_{ij}^+ + y_j - 1, 0\} = b_i \tag{14}$$

or

$$\max\{a_{ij}^- - y_j, 0\} = b_i \tag{15}$$

holds. Since $b_i > 0$, we have either

$$a_{ij}^+ + y_j - 1 = b_i, \tag{16}$$

or

$$a_{ij}^- - y_j = b_i. \tag{17}$$

In order to complete the proof of (i), next we will verify that "$\check{x}_j \leq y_j \leq \hat{x}_j$" doesn't hold, according to Eqs. (16) and (17).

(By contradiction) Assume that $\check{x}_j \leq y_j \leq \hat{x}_j$.

Case 1. If Eq. (16) holds, then

$$\begin{aligned} f_{ij}(\hat{x}_j) &= \max\{a_{ij}^+ + \hat{x}_j - 1, 0\} \vee \max\{a_{ij}^- - \hat{x}_j, 0\} \\ &\geq a_{ij}^+ + \hat{x}_j - 1 \\ &> a_{ij}^+ + y_j - 1 \\ &= b_i. \end{aligned} \tag{18}$$

This is conflict with Lemma 5.

Case 2. If Eq. (17) holds, then

$$\begin{aligned} f_{ij}(\check{x}_j) &= \max\{a_{ij}^+ + \check{x}_j - 1, 0\} \vee \max\{a_{ij}^- - \check{x}_j, 0\} \\ &\geq a_{ij}^- - \check{x}_j \\ &> a_{ij}^- - y_j \\ &= b_i. \end{aligned} \tag{19}$$

This is conflict with Lemma 5.

(ii) This is trivial according to Lemma 3.

Definition 1. For system (1), $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ is said to be the characteristic matrix, where

$$\tilde{q}_{ij} = \begin{cases} \{\check{x}_j\}, & \text{if } T_L(a^-_{ij}, 1-\check{x}_j) = b_i \neq T_L(a^+_{ij}, \hat{x}_j), \\ \{\hat{x}_j\}, & \text{if } T_L(a^-_{ij}, 1-\check{x}_j) \neq b_i = T_L(a^+_{ij}, \hat{x}_j), \\ \{\check{x}_j, \hat{x}_j\}, & \text{if } T_L(a^-_{ij}, 1-\check{x}_j) = b_i = T_L(a^+_{ij}, \hat{x}_j) \neq 0, \\ [\check{x}_j, \hat{x}_j], & \text{if } T_L(a^-_{ij}, 1-\check{x}_j) = b_i = T_L(a^+_{ij}, \hat{x}_j) = 0, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{20}$$

In Definition 1, it is obvious that the element of the characteristic matrix $\tilde{Q}$ takes value in four situations, i.e. (i) single-element set, (ii) two-element set, (iii) closed interval, (iv) empty set.

Theorem 2. *In system (1), let $x \in [\check{x}, \hat{x}]$ be a fuzzy vector. Define a binary matrix $Q^x = (q^x_{ij})_{m\times n}$ with respect to x as follows:*

$$q^x_{ij} = \begin{cases} 1, & if\ x_j \in \tilde{q}_{ij}, \\ 0, & otherwise. \end{cases} \tag{21}$$

Then $x \in X(A^+, A^-, b)$ if and only if each row in Q^x has at least one nonzero element.

Proof. According to the proof of Theorem 1 (See Theorem 1 in [2]) and the above-proved Lemma 6, it is not difficult to verify the result presented in this theorem.

4 Necessary and Sufficient Condition for Checking the Consistency of System (1)

In Ref. [3] the authors just provided a necessary condition to check the (in)consistency of system (1). In the following a numerical example is given to illustrate that such necessary condition may be insufficient in some cases.

Example 1. Consider the following bipolar fuzzy relation equations with max-Łukasiewicz composition:

$$A^+ \circ x \vee A^- \circ \bar{x} = b, \tag{22}$$

where

$$A^+ = (a^+_{ij})_{3\times 3} = \begin{bmatrix} 0.9 & 0.8 & 0.7 \\ 0.6 & 0.5 & 0.3 \\ 0.4 & 0.9 & 0.7 \end{bmatrix}, \quad A^- = (a^-_{ij})_{3\times 3} = \begin{bmatrix} 0.7 & 0.6 & 0.7 \\ 0.6 & 0.3 & 0.4 \\ 0.6 & 0.6 & 0.8 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.4 \end{bmatrix},$$

$x = (x_1, x_2, x_3)^T$ and $\bar{x} = 1 - x = (1-x_1, 1-x_2, 1-x_3)^T$.

Lemma 4 was used to check the case of the empty solution set in [3]. Now we apply Lemma 4 to system (22). Compute $\max_{j=1,2,3}\{\frac{1}{2}(a^+_{ij}+a^-_{ij}-1),0\}$ and $\max_{j=1,2,3}\{a^+_{ij},a^-_{ij}\}$ for $i=1,2,3$ as follows.

$$\begin{aligned}&\max_{j=1,2,3}\{\frac{1}{2}(a^+_{1j}+a^-_{1j}-1),0\}\\ =&\max_{j=1,2,3}\{\frac{1}{2}(0.9+0.7-1),\frac{1}{2}(0.8+0.6-1),\frac{1}{2}(0.7+0.7-1),0\}=0.3.\end{aligned}$$

Similarly, after calculation we get $\max_{j=1,2,3}\{\frac{1}{2}(a^+_{2j}+a^-_{2j}-1),0\}=0.1$, $\max_{j=1,2,3}\{\frac{1}{2}(a^+_{3j}+a^-_{3j}-1),0\}=0.25$, $\max_{j=1,2,3}\{a^+_{1j},a^-_{1j}\}=0.9$, $\max_{j=1,2,3}\{a^+_{2j},a^-_{2j}\}=0.6$, $\max_{j=1,2,3}\{a^+_{3j},a^-_{3j}\}=0.9$.

It is clear that the vector $b^T=(0.5,0.3,0.4)$ satisfies

$$\max_{j=1,2,3}\{\frac{1}{2}(a^+_{ij}+a^-_{ij}-1),0\}\le b_i\le\max_{j=1,2,3}\{a^+_{ij},a^-_{ij}\},\quad\forall i\in\{1,2,3\}.$$

However, system (22) is inconsistent (See Example 2 below), i.e., its solution set is empty set.

Next we propose a necessary and sufficient condition to avoid the case demonstrated in Example 1.

Construct the bound vector (or terminal vector) set as follows:

$$V^B=\{x=(x_1,x_2,\cdots,x_n)|x_j\in\{\check{x}_j,\hat{x}_j\},j=1,2,\cdots,n\}.\tag{23}$$

Here $\check{x}$ and $\hat{x}$ are the lower and upper bounds defined by (8) and (9). Each element in the set V^B is called a bound vector or terminal vector. Obviously V^B is a finite set and has 2^n elements.

Definition 2. A bound vector $x\in V^B$ is said to be a bound solution, if x is also a solution of system (1).

Proposition 1. *If $\check{x}$ and $\hat{x}$ are the lower and upper bounds of system* (1) *and there exists $i'\in I$ such that $b_{i'}=0$, then for any $\check{x}\le x\le\hat{x}$ it holds that*

$$f_{i'j}(x_j)=b_{i'},\quad\forall j\in J.\tag{24}$$

Proof. It is obvious that $f_{i'j}(x_j)=\max\{a^+_{i'j}+x_j-1,0\}\vee\max\{a^-_{i'j}+\bar{x}_j-1,0\}\ge 0=b_{i'}$. On the other hand, it follows from Lemma 5 that $f_{i'j}(x_j)\le b_{i'}$. So we get $f_{i'j}(x_j)=b_{i'}$ for arbitrary $j\in J$.

Theorem 3. *System* (1) *is consistent if and only if there exists at least one bound solution.*

Proof. If system (1) is consistent take arbitrary $y \in X(A^+, A^-, b)$. It follows from Lemma 3 that $\check{x}_j \leq y_j \leq \hat{x}_j, \forall j \in J$. Based on the solution y, construct vectors $\check{x}^y = (\check{x}^y_1, \check{x}^y_2, \cdots, \check{x}^y_n)$ and $\hat{x}^y = (\hat{x}^y_1, \hat{x}^y_2, \cdots, \hat{x}^y_n)$ as follow:

$$\check{x}^y_j = \begin{cases} y_j, & \text{if } y_j \in \{\check{x}_j, \hat{x}_j\}, \\ \check{x}_j, & \text{if } y_j \notin \{\check{x}_j, \hat{x}_j\}. \end{cases} \tag{25}$$

and

$$\hat{x}^y_j = \begin{cases} y_j, & \text{if } y_j \in \{\check{x}_j, \hat{x}_j\}, \\ \hat{x}_j, & \text{if } y_j \notin \{\check{x}_j, \hat{x}_j\}. \end{cases} \tag{26}$$

In fact both $\check{x}^y$ and $\hat{x}^y$ are bound solution. However, in order to complete the proof, we just need to check one of them. Next we verify that $\check{x}^y$ is a bound solution. Obviously it is a bound vector. Hence only $\check{x}^y \in X(A^+, A^-, b)$ need to be checked.

$\check{x}_j \leq y_j \leq \hat{x}_j$ indicates $\check{x}_j \leq \check{x}^y_j \leq \hat{x}_j$, $\forall j \in J$. Take arbitrary $i \in I$. Following Lemma 5, it is holds that

$$f_{ij}(\check{x}^y_j) \leq b_i, \quad \forall j \in J. \tag{27}$$

Moreover, considering $y \in X(A^+, A^-, b)$ and Lemma 2, that there exists $j_i \in J$ such that

$$f_{ij_i}(y_{j_i}) = b_i. \tag{28}$$

Case 1. If $b_i \neq 0$, then $y_{j_i} \in \{\check{x}_j, \hat{x}_j\}$, according to Lemma 6. By (25) and (28) we get

$$f_{ij_i}(\check{x}^y_{j_i}) = f_{ij_i}(y_{j_i}) = b_i. \tag{29}$$

Case 2. If $b_i = 0$, then $y_{j_i} \in [\check{x}_j, \hat{x}_j]$, again by Lemma 6. It follows from Lemma (5) that $f_{ij_i}(\check{x}^y_{j_i}) \leq b_i = 0$. On the other hand, $f_{ij_i}(\check{x}^y_{j_i}) = \max\{a^+_{ij_i} + \check{x}^y_{j_i} - 1, 0\} \vee \max\{a^-_{i_ij} - \check{x}^y_{j_i}, 0\} \geq 0$ holds naturally. Combining these two aspects we get

$$f_{ij_i}(\check{x}^y_{j_i}) = f_{ij_i}(y_{j_i}) = 0 = b_i. \tag{30}$$

According to Lemma 2 and Eqs. (27), (29) and (30), $\check{x}^y$ is a solution of system (1) and the proof is complete.

Denote the set of all bound solution by $X^B = V^B \cap X(A^+, A^-, b)$. Then the following Theorem 4 is direct corollary of Theorem 3.

Theorem 4. *System* (1) *is consistent if and only if* $X^B \neq \emptyset$.

Example 2. Consider the bipolar fuzzy relation equations given in Example 1, i.e. system (22). Now we check the consistency of (22) according to Theorems 2 and 3.

After calculation, the lower and upper bound vectors turn out to be $\check{x} = (0.3, 0.2, 0.4)$ and $\hat{x} = (0.6, 0.5, 0.7)$ respectively. Thus the set of all bound vectors is $V^B = \{x^1, x^2, \cdots, x^8\}$, where

$$\begin{aligned} x^1 &= (0.3, 0.2, 0.4), \quad x^2 = (0.3, 0.2, 0.7), \\ x^3 &= (0.3, 0.5, 0.4), \quad x^4 = (0.3, 0.5, 0.7), \\ x^5 &= (0.6, 0.2, 0.4), \quad x^6 = (0.6, 0.2, 0.7), \\ x^7 &= (0.6, 0.5, 0.4), \quad x^8 = (0.6, 0.5, 0.7). \end{aligned} \tag{31}$$

Compute the characteristic matrix by (20) in Definition 1:

$$\tilde{Q} = \begin{bmatrix} \{0.6\} & \emptyset & \emptyset \\ \{0.3\} & \emptyset & \emptyset \\ \emptyset & \{0.2, 0.5\} & \{0.4, 0.7\} \end{bmatrix}.$$

Following Theorem 2, it is found that the induced binary matrices with respect to x^1, x^2, x^3 and x^4 are identical and all equal to

$$Q^{'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

while the induced binary matrices with respect to x^5, x^6, x^7 and x^8 are all equal to

$$Q^{''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

It is obvious that both of $Q^{'}$ and $Q^{''}$ have at least one zero row. According to Theorem 2, none of the bound vectors (i.e. $x^1, x^2, \cdots, x^8$) is a solution to system (22). That is to say, there doesn't exist any bound solution to (22). It follows from Theorem 3 that system (22) is inconsistent.

Example 3 [3]**.** Check the consistency of the following system of max-Łukasiewicz bipolar fuzzy relation equations, which is picked out from [3],

$$\begin{cases} \max\{g_1(0.9, 0.9, x_1), g_2(0.8, 0.7, x_2), g_3(0.9, 0.8, x_3), g_1(0.6, 0.9, x_4)\} = 0.8, \\ \max\{g_1(0.8, 0.7, x_1), g_2(0.9, 0.9, x_2), g_3(0.7, 0.7, x_3), g_1(1, 0.8, x_4)\} = 0.8, \\ \max\{g_1(0.8, 0.6, x_1), g_2(0.6, 0.8, x_2), g_3(0.8, 0.9, x_3), g_1(0.4, 0.9, x_4)\} = 0.7, \\ \max\{g_1(0.5, 0.6, x_1), g_2(0.6, 0.9, x_2), g_3(0.4, 0.8, x_3), g_1(0.4, 0.4, x_4)\} = 0.6, \end{cases} \tag{32}$$

where $g_j(a_0^+, a_0^-, x_j) = T_L(a_0^+, x_j) \vee T_L(a_0^-, \bar{x}_j) = \max\{a_0^+ + x_j - 1, 0\} \vee \max\{a_0^- - x_j, 0\}$, $j = 1, 2, 3, 4$, T_L is the max-Łukasiewicz composition.

The lower bound and upper bound are

$$\check{x} = (0.1, 0.3, 0.2, 0.2) \text{ and } \hat{x} = (0.9, 0.9, 0.9, 0.8).$$

After calculation, we get the characteristic matrix of system (32), i.e.,

$$\tilde{Q} = \begin{bmatrix} \{0.1, 0.9\} & \emptyset & \{0.9\} & \emptyset \\ \emptyset & \{0.9\} & \emptyset & \{0.8\} \\ \{0.9\} & \emptyset & \{0.2, 0.9\} & \{0.2\} \\ \emptyset & \{0.3\} & \{0.2\} & \emptyset \end{bmatrix}.$$

Take the bound vector $x^B = (0.1, 0.9, 0.2, 0.2)$. According to Theorem 2, the induced binary matrix with respect to x^B is

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Obviously in the matrix Q, each row has at least one nonzero element. Hence x^B is a bound solution and consequently system (32) is consistent.

5 Conclusion

Bipolar fuzzy relation equations is a new kind of fuzzy relation system. It was proposed by Freson in 2013 for the first time. Optimization problems with linear objective function and bipolar fuzzy relation equations constraint were investigated in [1–3]. In this paper, the max-Łukasiewicz composition is considered, as the same to that in [2,3]. We give a modified condition for checking a solution in system (1). Checking by this new condition, the equations with $b_i = 0$ needn't to be deleted in the computing processes. Besides, we propose a necessary and sufficient condition to check the consistency of system (1), which is important in solving the corresponding fuzzy relation optimization problems.

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Recommender: Academic Conference on 30th anniversary of fuzzy geometric programming and 40th education year by and of Professor Cao Bingyuan.

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Multi-level Linear Programming Subject to Max-product Fuzzy Relation Equalities

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Abstract. In this paper, we consider the multi-level linear programming subject to max-product fuzzy relation equations which is used to characterize a kind of wireless communication EBS model. Based on the theorem of lexicography order, we developed a algorithm to find the unique optimal solution. And a numerical example is given to illustrate the feasibility and efficiency of the algorithm.

Keywords: Fuzzy relation equations · Multi-level linear programming · Lexicography order

1 Introduction

The notion of fuzzy relational equations has been investigated both from a theoretical standpoint and in a view of applications since they were first introduced by Sanchez [1,2]. Various effective methods were proposed by many scholars.

The optimization problems withe fuzzy relation equations or inequalities constraint is said to be a fuzzy relation optimization problem. Because of the special structure of the feasible domain i.e. the solution set of fuzzy relation equations or inequalities, the solution method is much different from the ordinary optimization problem. Many different kinds fuzzy relation optimization problems were investigated such as: P.Z. Wang et al. proposed the fuzzy relation latticized linear programming with max-composition [11], P. Li and Fang focused on the optimization of a system of fuzzy relational equations with sup-T composition [9,10].

Recently X.-P. Yang and B.-Y. Cao et al. investigated a latticized linear programming subject to max-product fuzzy relation inequalities [5]. In this paper a kind of wireless communication optimization management models into a system of max-product fuzzy relation inequalities. And in another article [4], the author propose a new order relation during solving a multi-level linear programming subject to addition-min fuzzy relation inequalities.

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Based on the application of wireless communication, we want to reduce the damage of electromagnetic radiation by limiting power of electromagnetic. So we give a multi-level linear programming subject to system of the max-product equations and solve it this paper. In Sect. 2 we introduce some basic theorem of max-product fuzzy relation equations and corresponding multi-level linear programming problem. In Sect. 3, resolution of this problem is studied with a step-by-step algorithm based some theorems of lexicography order. Section 4 provides a numerical example to illustrate the feasibility and efficiency of the algorithm. At last we give a sample conclusion.

2 Preliminaries

In this section we present some basic definitions and theory about the system of max-product fuzzy relation equations.

A system of fuzzy relation equalities is formulated as follow:

$$\begin{cases} a_{11}x_1 \vee a_{12}x_2 \vee \cdots \vee a_{1n}x_n = b_1, \\ a_{21}x_1 \vee a_{22}x_2 \vee \cdots \vee a_{2n}x_n = b_2, \\ \cdots \\ a_{m1}x_1 \vee a_{m2}x_2 \vee \cdots \vee a_{mn}x_n = b_m, \end{cases} \tag{1}$$

or

$$a_{i1}x_1 \vee a_{i2}x_2 \vee \cdots \vee a_{in}x_n = b_i, i = 1, 2, \ldots \tag{2}$$

or

$$A \circ x^T = b^T, \tag{3}$$

where $\circ$ represents the max-product composition, $A = (a_{ij})_{m\times n}, x = (x_1, x_2, \cdots, x_n)$, $b = (b_1, b_2, \cdots, b_n), a_{ij}, x_j, b_j \in [0, 1], b_j > 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$.

In this paper we denote $X = [0, 1]^n, I = \{1, 2, \cdots, m\}, J = \{1, 2, \cdots, n\}$. The nature order relation $\leq$ on the set X is defined below.

Definition 1. *Denote* $X = [0; 1]^n$ *Let* $x^1 = (x_1^1, x_2^1, \cdots, x_n^1), x^2 = (x_1^2, x_2^2, \cdots, x_n^2) \in X$, *we define:*

(i) $x^1 \leq x^2$, *if* $x_j^1 \leq x_j^2, \forall j \in J$;

(ii) $x^1 < x^2$, *if* $x^1 \leq x^2$ *and there some* $j \in J$ *such that* $x_j^1 < x_j^2$.

The dual order relations of $\leq$ and $<$ are denote by $\geq$ and $>$, obviously $(X, \leq)$ forms a partial order set.

Let $X(A, b) = \{x|A \circ x = b\}$ denotes the solution set of system (1). If $X(A, b) = \{x|A \circ x = b\} \neq \emptyset$, we say the system is consistent, otherwise it is inconsistent.

Definition 2. *Define*

$$a_{ij} \odot^{-1} b_i = \begin{cases} \frac{b_i}{a_{ij}}, & a_{ij} > b_i, \\ 1, & a_{ij} \leq b_i, \end{cases}$$

where $\odot^{-1}$ is an operator defined on $[0,1], i = 1,2,\ldots,m, j = 1,2,\ldots,n$. Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$, and $\hat{x}_j = \bigwedge_{i=1}^{m} a_{ij} \odot^{-1} b_i \; j = 1,2,\ldots,n$.

Lemma 1 [9,10]. *The system (1) is consistent if and only if $\hat{x} \in X(a,b)$.*

Definition 3. *A solution $\hat{x} \in X(A,b)$ is said to be a maximum or greatest solution if and only if $x \leq \hat{x}$ for all $x \in X(A,b)$. A solution $\check{x} \in X(A,b)$ is said to be a minimal solution if and only if $x \leq \check{x}$ implies $x = \check{x}$ for any $x \in X(A,b)$.*

Obviously the maximum solution of the system (1), if it exists, is unique. While if the system has a minimal solution, the minimal solutions are usually not unique. If we denote the set of all the minimal solutions by $\check{X}(A,b)$, then we have [2]:

$$X(A,b) = \bigcup_{\check{x} \in \check{X}(A,b)} \{x | \check{x} \leq x \leq \hat{x}\}$$

As shown in [6], a wireless communication EBS system can be reduced to a system of max-product fuzzy relation equations as (1). In this system the major optimization is to minimize $\{x_1, x_2, \ldots, x_n\}$, in which the every x_i is in same level. However in real practical situation, the importance of every x_i is different. In this paper we consider the optimization objective with priority rank: $x_{j_1} \to x_{j_2} \to \cdots \to x_{j_n}$, where $\{j_1, j_2, \ldots, j_n\} = J$ and the notation $x_{j_1} \to x_{j_2} \to \cdots \to x_{j_n}$ means x_{j_p} is more important than x_{j_q} if $p < q$. Based on this consideration we reduce a wireless communication EBS system into a multi-level programming as follow:

$$\begin{aligned} min \quad & x_{j_1} \\ min \quad & x_{j_2} \\ & \cdots \\ \text{s.t.} \quad & A \circ x^T = b^T. \end{aligned} \tag{4}$$

Definition 4. *Denote $X = [0;1]^n$, let $x^1 = (x_1^1, x_2^1, \cdots, x_n^1)$, $x^2 = (x_1^2, x_2^2, \cdots, x_n^2) \in X$, and $\{j_1, j_2, \ldots, j_n\} = J$ is an ranking set of J, which means for any $p > q \in [1,n], j_p$ is more important than j_q in the order relation. And we denote:*

(i) $x^1 = x^2$, if $x_j^1 \leq x_j^2$, for all $j \in J$;

(ii) $x \prec y$, if there exists a $k \in J$ such that $x_{j_1}^1 = x_{j_1}^1, \ldots, x_{j_{k-1}}^1 = x_{j_{k-1}}^2$ and $x_{j_k}^1 < x_{j_k}^2$.

$x \preceq y$ represent $x \prec y$ or $x = y$. The dual notations of "$\prec$" and "$\preceq$" are "$\succ$" and "$\succeq$" respectively.

The order relation defined in Definition 4 is said to be lexicography order (based a certain ranking set). Obviously it is a total order on the X.

Theorem 1. *Let* $x, y, z \in X$. *Then:*

(i) $x \preceq x$,
(ii) $x \preceq y$ *and* $y \preceq x$ *imply* $x = y$,
(iii) $x \preceq y$ *and* $y \preceq z$ *imply* $x \preceq z$.

Definition 5. *A solution* $x^* \in X(A, b)$ *is said to be a lexicography minimum solution if and only if* $x \preceq x^*$ *impliesx* $= x^*$ *for any* $x \in X(A, b)$

Absolutely the lexicography minimum solution of (1) is the optimal solution of (4)

3 Resolution of Problem (4)

Theorem 2. *If the system (1) is consistent, the lexicography minimum solution must exist and be unique, then problem (4) has an unique optimal solution.*

Theorem 3. *If the system (1) is consistent, any minimal solution* $\check{x} \in X(A, b)$ *must be a lexicography minimum solution based on a certain ranking set.*

Because lexicography order is a total order on the X, so Theorems 2 and 3 are obviously right.

Theorem 4. *If* $\check{x}$ *is minimal solution*$\in X(A, b)$, $\hat{x}$ *is the maximum solution, then* $\check{x} \leq \hat{x}$ *and there must exist at least one* $k \in J$ *such that* $\check{x}_k = \hat{x}_k$.

Proof. Absolutely, $\check{x} \leq \hat{x}$, if for all $k \in J$ we have $\check{x}_k < \hat{x}_k$, then $a_{ik}\check{x}_k < a_{ik}\hat{x}_k = b_i$ hold for each $i \in I$, $\check{x} \notin X(A, b)$, so there must exist at least one $k \in J$ such that $\check{x}_k = \hat{x}_k$

The optimal solution of (2) can be choose from the minimal solution set of (1) by compare every element, but it is very difficult, so we need to find a new algorithm to compute the optimal solution of (2).

Algorithm

Step 1. Compute $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ by Definition 3.

Step 2. Check the feasibility of (1) by Lemma 1. If $X(A, b) \neq \emptyset$, Let $k = 1$ and go to the Step 3, otherwise the system is not consistent, and there is no lexicography minimum solution.

Step 3. Denote $y^k = (y_1^k, y_2^k, \ldots, y_n^k), x^* = (x_1^*, x_2^*, \ldots, x_n^*)$, and,

$$y_j^k = \begin{cases} x_j^*, \, j < k, \\ \;0, \;\, j = k, \\ \hat{x}_j, \, j > k. \end{cases}$$

Step 4. If the $y^k \in X(A, b)$, let $x_k^* = 0$, otherwise let $x_k^* = \hat{x}_k$.

Step 5. If $k < n$, let $k = k + 1$ and return to the Step 3. If $k = n$, the solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is the lexicography minimum solution.

Theorem 5. *The vector x^* obtained by the Algorithm mentioned above is the lexicography minimum solution of system 1.*

Proof. (i) Firstly, we prove that $x^* \in X(A,b)$, denote $p^l = (p_1^l, p_2^l, \ldots, p_n^l)$, and

$$p^l = \begin{cases} x_j^* \ j \leq l, \\ \hat{x}_j \ j > l, \end{cases}$$

$x^* = p^n$, when $l = 1$, $y^1 = (0, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_n)$, if $y^1 \in X(A,b)$, then $x_1^* = 0$ and $p^1 = y^1 \in X(A,b)$.

If $y^1 \notin X(A,b)$, then $x_1^* = \hat{x}_1$ and $p^1 = \hat{x} \in X(A,b)$.

When $l = k$, we assume that $p^k \in X(A,b)$, if $y^{k+1} \in X(A,b)$, then $x_{k+1}^* = 0$ and $p^{k+1} = y^{k+1} \in X(A,b)$.

If $y^{k+1} \notin X(A,b)$, then $x_{k+1}^* = \hat{x}_{k+1}$ and $p^{k+1} = p^k \in X(A,b)$

So from the all $x^* = p^n \in X(A,b)$.

(ii) Secondly we prove $x^*$is the lexicography minimum solution. Assume $x = (x_1, x_2, \ldots, x_n) \in X(A,b)$ is an arbitrary solution. Then we prove $x^* \preceq x$. If $x_1^* = 0, x_1^* \leq x_1$ obviously.

If $x_1^* = \hat{x}_1$, then $y^1 = (0, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_n) \notin X(A,b)$, so exist an $i_1 \in I$ such that:

$$a_{i_1 2}\hat{x}_2 \vee a_{i_1 3}\hat{x}_3 \vee, \cdots, \vee a_{i_1 n}\hat{x}_n \neq b_{i_1}.$$

At the same time $\hat{x} \in X(A,b)$, so we have:

$$a_{i_1 1}\hat{x}_1 \vee a_{i_1 2}\hat{x}_2 \vee, \cdots, \vee a_{i_1 n}\hat{x}_n \neq b_{i_1}.$$

So we have $a_{i_1 1}\hat{x}_1 = b_{i_1}$ and

$$a_{i_1 j}\hat{x}_j < b_{i_1}, j = 2, 3 \cdots$$

For the x,

$$a_{i_1 1}x_1 \vee a_{i_1 2}x_2 \vee, \cdots, \vee a_{i_1 n}x_n = b_{i_1}.$$

Because $\hat{x}$ is the maximum solution, $x_j \leq \hat{x}_j$, and

$$a_{i_1 j}x_j \leq a_{i_1 j}\hat{x}_j < b_{i_1}, \ for \ j = 2, 3 \cdots$$

So $a_{i_1 1}x_1 = b_{i_1}$ and then $x_1 = \hat{x}_1 = x_1^*$.

Then prove that if $x_k^* = x_k$, then $x_{k+1}^* = x_{k+1}$ for $k = 1, 2, 3, \ldots$ by the same way.

So $x^* \preceq x$ holds.

From the (i)(ii) is the lexicography minimum solution of system (1).

4 Numerical Example

$$\begin{aligned} \min \quad & x_1 \\ \min \quad & x_2 \\ & \cdots \\ \min \quad & x_8 \\ \text{s.t.} \quad & A \circ x^T = b^T. \end{aligned}$$

In this problem we consider the optimization objective with priority rank: $x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_6 \to x_7 \to x_8$ and

$$A = \begin{pmatrix} 0.8 & 0.6 & 0.2 & 0.4 & 0.2 & 0.7 & 0.7 & 0.5 \\ 0.6 & 0.3 & 0.7 & 0.6 & 0.1 & 0.3 & 0.5 & 0.3 \\ 0.5 & 0.8 & 0.7 & 0.4 & 0.7 & 0.8 & 0.3 & 0.8 \\ 0.2 & 0.4 & 0.5 & 0.1 & 0.3 & 0.5 & 0.8 & 0.4 \\ 0.6 & 0.2 & 0.5 & 0.5 & 0.1 & 0.4 & 0.7 & 0.2 \\ 0.9 & 0.9 & 0.8 & 0.2 & 0.8 & 0.6 & 0.1 & 0.4 \end{pmatrix},$$

$$b = (0.56, 0.42, 0.64, 0.4, 0.42, 0.72), x = (x_1, x_2, \cdots, x_8).$$

Step 1. By the Definition 3, we obtain that $\hat{x} = (0.7, 0.8, 0.6, 0.7, 0.9, 0.8, 0.5, 0.8)$
Step 2. Since

$$A \circ \hat{x} = \begin{pmatrix} 0.8 & 0.6 & 0.2 & 0.4 & 0.2 & 0.7 & 0.7 & 0.5 \\ 0.6 & 0.3 & 0.7 & 0.6 & 0.1 & 0.3 & 0.5 & 0.3 \\ 0.5 & 0.8 & 0.7 & 0.4 & 0.7 & 0.8 & 0.3 & 0.8 \\ 0.2 & 0.4 & 0.5 & 0.1 & 0.3 & 0.5 & 0.8 & 0.4 \\ 0.6 & 0.2 & 0.5 & 0.5 & 0.1 & 0.4 & 0.7 & 0.2 \\ 0.9 & 0.9 & 0.8 & 0.2 & 0.8 & 0.6 & 0.1 & 0.4 \end{pmatrix} \circ \begin{pmatrix} 0.7 \\ 0.8 \\ 0.6 \\ 0.7 \\ 0.9 \\ 0.8 \\ 0.5 \\ 0.8 \end{pmatrix} = b = \begin{pmatrix} 0.56 \\ 0.42 \\ 0.64 \\ 0.4 \\ 0.42 \\ 0.72 \end{pmatrix},$$

so $A \circ x^T = b^T$ is consistent and go to step 3.
Step 3–5. $y^1 = (0, 0.8, 0.6, 0.7, 0.9, 0.8, 0.5, 0.8)$ and it is easy to check that $y^1 \notin X(A, b)$, so $x_1^* = \hat{x}_1 = 0.7$.

$y^2 = (0.7, 0, 0.6, 0.7, 0.9, 0.8, 0.5, 0.8)$ and it is easy to check that $y^2 \in X(A, b)$, so $x_2^* = 0$.

$y^3 = (0.7, 0, 0, 0.7, 0.9, 0.8, 0.5, 0.8)$ and it is easy to check that $y^3 \in X(A, b)$, so $x_3^* = 0$.

$y^4 = (0.7, 0, 0, 0, 0.9, 0.8, 0.5, 0.8)$ and it is easy to check that $y^4 \in X(A, b)$, so $x_1^* = 0$.

$y^5 = (0.7, 0, 0, 0, 0, 0.8, 0.5, 0.8)$ and it is easy to check that $y^5 \notin X(A, b)$, so $x_5^* = \hat{x}_5 = 0.9$.

$y^6 = (0.7, 0, 0, 0, 0.9, 0, 0.5, 0.8)$ and it is easy to check that $y^6 \in X(A, b)$, so $x_1^* = 0$.

$y^7 = (0.7, 0, 0, 0, 0.9, 0, 0, 0.8)$ and it is easy to check that $y^7 \notin X(A, b)$, so $x_7^* = \hat{x}_7 = 0.5$.

$y^8 = (0.7, 0, 0, 0, 0.9, 0, 0.5, 0)$ and it is easy to check that $y^8 \notin X(A, b)$, so $x_8^* = \hat{x}_8 = 0.8$, the lexicography minimum solution is $x^* = (0.7, 0, 0, 0, 0.9, 0, 0.5, 0)$.

5 Conclusion

In this paper, we introduce a optimization model, i.e. multi-level linear programming subject to max-product fuzzy relation equations which can be used to describe a wireless communication EBS system. And the optimal solution is one of the minimal solutions of the max-product fuzzy relation equations. We introduce a method to find it without finding all the minimal solutions.

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Quadratic Programming with Max-product Fuzzy Relation Inequality Constraints

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Abstract. In this paper, a new method for quadratic programming with max-product fuzzy relation inequality constraints is proposed. First, the properties of the optimal solution are analyzed in several special cases of fuzzy relation quadratic programming. Simultaneously, some rules are presented to simplify the original fuzzy relation quadratic programming problem. Then, an algorithm is presented, based on rules, the branch and bound method and numerical algorithm for solving traditional quadratic programming problems with interval constraints. The proposed algorithm does not need to find all feasible minimal solutions. Hence, the amount of calculation is reduced. Some numerical examples are given to illustrate the feasibility and effectiveness of the proposed algorithm.

Keywords: Quadratic programming · Max-product fuzzy relation inequality · Fuzzy relation quadratic programming · Optimal solution

1 Introduction

In this paper, the following quadratic programming problem with max-product fuzzy relation inequality constraints is studied:

$$\begin{aligned} \min\ & f(x) = \frac{1}{2}x^TQx + c^Tx \\ \text{s.t. } & A \circ x \geqslant b, \\ & D \circ x \leqslant e, \\ & x \in [0,1]^n, \end{aligned} \tag{1}$$

where $Q = (q_{ij})_{n\times n}$ is the n order symmetric matrix, $A = (a_{kj})_{m\times n}, D = (d_{lj})_{h\times n}$, $x = (x_1,\cdots,x_n)^T$, $b = (b_1,\cdots,b_m)^T, e = (e_1,\cdots,e_h)^T$, $c = (c_1,\cdots,c_n)^T, k \in K, j \in J, l \in L$, here, $K = \{1,2,\cdots,m\}, J = \{1,2,\cdots,n\}, L = \{1,2,\cdots,h\}$, "$\circ$" denotes the max-product operator.

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As is well known, Quadratic Programming (QP) can be viewed as a generalization of linear programming [1]. It arises in a wide variety of scientific and engineering applications including regression analysis and function approximation [2], pattern recognition [3], portfolio selection [4], and so on. Ref. [5] has given an algorithm for finding the global optimal solution of fuzzy relation quadratic programming with a max-min fuzzy relation equation constraint. Ref. [6] discusses quadratic programming with a max-product fuzzy relation inequality constraint. Some sufficient conditions are presented to determine its optimal solution in terms of the maximal solution or the minimum solutions of its feasible domain, and some simplification operations have been given to accelerate the resolution of the problem by removing the components that have no effect on the solution process. Ref. [7] use some properties of $n \times n$ real symmetric indefinite matrices, Choleskys decomposition, and the least square technique, and convert the problem to a separable programming problem. Furthermore, a relation in terms of a closed form is presented to solve it. Finally, an algorithm is proposed to solve the original problem. Ref. [8] propose a new definition of FRI path of max-product fuzzy relation inequality. In this paper, we revise the definition of the FRI path of the max-product fuzzy relation inequality, and put forward some new properties of problem (1). Several new simplified constrained quadratic programming methods are obtained based on the new properties.

This paper is organized as follows: In Sect. 2, the solution set structure of the max-product fuzzy relation inequality and some properties of the FRI path are introduced. In Sect. 3, the properties of the optimal solution for the quadratic programming with max-product fuzzy relation inequality constraints is introduced, and the process of simplified constrained quadratic programming is described. Then the new algorithm for the quadratic programming with max-product fuzzy relation inequality constraints is proposed. In Sect. 4, some numerical examples are presented to illustrate the effectiveness of the algorithm. Finally, the conclusions are presented in Sect. 5.

2 Max-product Fuzzy Relation Inequality

In this section, some concepts and properties of the max-product fuzzy relation inequality will be introduced based on the following form:

$$\begin{aligned} A \circ x &\geqslant b, \\ D \circ x &\leqslant e, \end{aligned} \tag{2}$$

where $0 \leqslant x_j \leqslant 1 (j \in J)$. Suppose that $X(A, b, D, e)$ is the solution set of (2).

Definition 2.1 [9,10]**.** $\hat{x} \in X(A, b, D, e)$ is called maximal solution of (2), if $x \leqslant \hat{x}$, for all $x \in X(A, b, D, e)$. $\check{x} \in X(A, b, D, e)$ is called a minimum solution of (2), if $x \leqslant \check{x}$ for any $x \in X(A, b, D, e)$, we have $x = \check{x}$.

If $X(A, b, D, e) \neq \emptyset$, then it can be completely determined by a unique maximal solution and a finite number of minimum solutions. Denote the set of minimum solutions of (2) by $\check{X}$, then we have $X(A,b,D,e) = \bigcup_{\check{x}\in\check{X}}\{x \in [0,1]^n | \check{x} \leqslant x \leqslant \hat{x}\}$. The maximal solution $\hat{x}$ can be solved by the following formula (see [9,10]):

$$\hat{x}_j = \begin{cases} \min\{\frac{e_l}{d_{lj}} | d_{lj} > e_l\}, & \text{if } \{l \in L | d_{lj} > e_l\} \neq \emptyset, \\ 1, & \text{else.} \end{cases} \tag{3}$$

Let $J_k = \{j \in J | a_{kj}\hat{x}_j \geqslant b_k\}(k \in K), K_j = \{k \in K | a_{kj}\hat{x}_j \geqslant b_k\}(j \in J)$ and $\Lambda = J_1 \times J_2 \times \cdots \times J_m$. The vector $p = (p_1, p_2, \cdots, p_m) \in \Lambda$ if and only if $p_k \in J_k, \forall k \in K$. For all $p \in \Lambda$, we can calculate the index set

$$K_p^j = \{k \in K \mid p_k = j\}, j \in J, \tag{4}$$

and define

$$x_j^p = \begin{cases} \max\limits_{k \in K_p^j} \frac{b_k}{a_{kj}} & \text{if } K_p^j \neq \emptyset, \\ 0 & \text{if } K_p^j = \emptyset, \end{cases} \quad \forall j \in J. \tag{5}$$

The vector $p \in \Lambda$ is called the general path or G-path of (2). We denote the set of all G-paths of (2) by GP.

Theorem 2.1 [9]**.** Suppose that $X(A,b,D,e) \neq \emptyset$, the following conclusions are evident: (1) If $p \in \Lambda$, then $x^p \in X(A,b,D,e)$. (2) For all $\check{x} \in \check{X}$, there exists some G-paths p, such that $x^p = \check{x}$.

Definition 2.2 [8]**.** A vector p is called an FRI path of (2) if for any $k \in K$,

$$p_k \begin{cases} = 0, & \begin{array}{l}\text{if there exists some } k' \in \{1, \cdots, k-1\}, \text{ such that} \\ j_0 = p_{k'} \in J_k \cap \{p_1, \cdots, p_{k-1}\} \text{ and } \dfrac{b_{k'}}{a_{k'j_0}} \geqslant \dfrac{b_k}{a_{kj_0}},\end{array} \\ \in J_k, & \text{otherwise.} \end{cases}$$

Definition 2.3. For all $k_1, k_2 \in K$, $k_1 < k_2$ and $j_0 \in J_{k_1} \cap J_{k_2}$, suppose that $\dfrac{b_{k_1}}{a_{k_1,j_0}} \geqslant \dfrac{b_{k_2}}{a_{k_2,j_0}}$, if the vector $p \in \Lambda$ satisfying the following conditions

$$p_k \begin{cases} \in J_k, & \text{if } J_k \cap \{p_1, \cdots, p_{k-1}\} = \emptyset, \\ = 0, & \text{else,} \end{cases} \quad \forall k \in K,$$

then the vector p is called an FRI path of (2).

Definition 2.4. For all $k_1, k_2 \in K$, $k_1 < k_2$ and $j \in J$, suppose that $\dfrac{b_{k_1}}{a_{k_1,j}} \geqslant \dfrac{b_{k_2}}{a_{k_2,j}}$, for any $k \in K$, if the vector $p \in \Lambda$ satisfying the following conditions

$$p_k \begin{cases} \in J_k, & \text{if } J_k \cap \{p_1, \cdots, p_{k-1}\} = \emptyset, \\ = 0, & \text{else,} \end{cases} \quad \forall k \in K,$$

then the vector p is called an FRI path of (2).

Suppose that the set of all FRI paths(which can be calculated based on the above definition) is $FRIP$. We can obtain the following conclusions.

Theorem 2.2 [8]**.** Suppose that $b_k > 0$ for all $k \in K$, p and q are FRI paths of (2), then we have the following results:

(1) For all $k_1, k_2 \in K, k_1 < k_2$, if $p_{k_1} = p_{k_2}$, and one of the following two conditions holds ① $J_{k_1} \bigcap J_{k_2} = \emptyset$, ② $\dfrac{b_{k_1}}{a_{k_1 j_0}} \geqslant \dfrac{b_{k_2}}{a_{k_2, j_0}}$ for all $j_0 \in J_{k_1} \bigcap J_{k_2}$, then $p_{k_1} = p_{k_2} = 0$.

(2) Suppose that $\dfrac{b_{k_1}}{a_{k_1,j}} > \dfrac{b_{k_2}}{a_{k_2,j}}$ for all $k_1, k_2 \in K$, $k_1 < k_2$, $j \in J_{k_1} \cap J_{k_2}$, p and q is the path of (2) and $p \neq q$, then $x^p \neq x^q$.

(3) Suppose $\check{x}$ is a minimum solution of system (2). Then there must exist an FRI path p such that $\check{x} = x^p$, where x^p is defined by (5).

(4) For any FRI path p of (2), suppose that $\dfrac{b_{k_1}}{a_{k_1,j}} > \dfrac{b_{k_2}}{a_{k_2,j}}$ for all $k_1, k_2 \in K$, $k_1 < k_2$ and $j \in J$, then x^p is a minimum solution of (2), where x^p is defined by (5).

Theorem 2.3 [8]**.** Suppose that $\check{x}$ is a minimum solution of a fuzzy relation inequality (2), then there exists an FRI path $p \in FRIP$ satisfying $\check{x} = x^p$. If $\dfrac{b_{k_1}}{a_{k_1,j}} > \dfrac{b_{k_2}}{a_{k_2,j}}$ for all $k_1, k_2 \in K$, $k_1 < k_2$ and $j \in J$, then for all FRI paths p, x^p is a minimum solution of (2).

Definition 2.5 [10]**.** Suppose that $p \in GP$. A solution $x^p = (x_1^p, x_2^p, \cdots, x_m^p)^T$ is called the quasi-minimum solution corresponding to the G-path and p is called a corresponding G-path of x^q.

Theorem 2.4 [8]**.** If $X(A, b, D, e) \neq \emptyset$, then $X(A, b, D, e) = \{x \in X \mid x^q \leqslant x \leqslant \hat{x}, q \in GP\} = \{x \in X \mid x^q \leqslant x \leqslant \hat{x}, q \in FRIP\}$.

3 Properties and Algorithms

This section first analyzes several special cases of problem (1) and then some rules are proposed to simplify the problem for the general case of (1). Finally, the new global optimal solution algorithm for solving problem (1) is constructed based on the branch and bound method in a special case and numerical algorithms for solving a classical quadratic programming problem[23,31-33].

For all $i_0 \in J$, set $J_{i_0}^+ = \{j \in N | q_{i_0 j} \geqslant 0\}, J_{i_0}^- = J \backslash J_{i_0}^+$.

Lemma 3.1. If there exists some $i_0 \in J$ satisfying $c_{i_0} \leqslant 0, q_{i_0 i_0} \leqslant 0$, and $c_{i_0} + 0.5 q_{i_0 i_0} \hat{x}_{i_0} + \sum\limits_{j \in J_{i_0}^+} q_{i_0 j} < 0$, then we can get $x_{i_0}^* = \hat{x}_{i_0}$ for any optimal solution x^*.

Proof. Let x^* be the optimal solution of problem (1) and $x^*_{i_0} < \hat{x}_{i_0}$. Suppose that $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ satisfies

$$\bar{x}_j = \begin{cases} x^*_j, \text{ if } j \neq i_0, \\ \hat{x}_j, \text{ if } j = i_0. \end{cases}$$

Obviously, $\bar{x}$ is a solution of (2). If $j \neq i_0$, we have $0 \leqslant x^* = \bar{x} \leqslant \hat{x}$, otherwise, $0 \leqslant x^*_j < \bar{x}_j = \hat{x}_j$. Therefore, we can obtain

$$\begin{aligned} f(x^*) &= \tfrac{1}{2} x^{*T} Q x^* + c^T x^* \\ &= \tfrac{1}{2} \sum_{i=1, i\neq i_0}^{n} \sum_{j=1, j \neq i_0}^{n} q_{ij} x^*_i x^*_j + \sum_{j \in J^-_{i_0}, j\neq i_0} q_{i_0,j} x^*_{i_0} x^*_j + \sum_{i=1, i \neq i_0}^{n} c_i x^*_i \\ &\quad + \Big(\sum_{j \in J^+_{i_0}, j \neq i_0} q_{i_0,j} x^*_j + \tfrac{1}{2} q_{i_0 i_0} x^*_{i_0} + c_{i_0} \Big) x^*_{i_0} \\ &> \tfrac{1}{2} \sum_{i=1, i\neq i_0}^{n} \sum_{j=1, j \neq i_0}^{n} q_{ij} x^*_i x^*_j + \sum_{j \in J^-_{i_0}, j\neq i_0} q_{i_0,j} \hat{x}_{i_0} x^*_j + \sum_{i=1, i \neq i_0}^{n} c_i x^*_i \\ &\quad + \Big(\sum_{j \in J^+_{i_0}, j \neq i_0} q_{i_0,j} x^*_j + \tfrac{1}{2} q_{i_0 i_0} \hat{x}_{i_0} + c_{i_0} \Big) \hat{x}_{i_0} \\ &= \tfrac{1}{2} \bar{x}^T Q \bar{x} + c^T \bar{x}, \end{aligned}$$

where the inequality follows from $c_{i_0} \leqslant 0, q_{i_0 i_0} \leqslant 0$, and $c_{i_0} + 0.5 q_{i_0 i_0} \hat{x}_{i_0} + \sum_{j \in J^+_{i_0}} q_{i_0 j} < 0$. It is a contradiction that x^* is an optimal solution of problem (1). Therefore, $x^*_{i_0} = \hat{x}_{i_0}$ for all optimal solutions x^*.

The following conclusions can be directly obtained using Lemma 3.1.

Corollary 3.1. If $c_i \leqslant 0, q_{ii} \leqslant 0$, and $c_i + 0.5 q_{ii} \hat{x}_i + \sum_{j \in J^+_i} q_{ij} < 0$ for all $i, j \in J$, then the maximal solution of (2) is an optimal solution of (1).

Due to Lemma 3.1, we propose the following rule to simplify problem (1).

Rule 3.1. Suppose that $J^0 = \{ j \in J | c_j \leqslant 0, q_{jj} \leqslant 0, c_j + 0.5 q_{jj} \hat{x}_j + \sum_{i \in J^+_j} q_{ij} < 0\}$. For all $j \in J^0$, let $x^*_j = \hat{x}_j$, and remove the j^{th} column of matrix A. Suppose that $K^0 = \{ k \in K | a_{kj} \hat{x}_j \geqslant b_k, \forall j \in J^0 \}$. For any $k \in K^0$, remove the k^{th} row of matrix A and the k^{th} component of vector b.

By using Rule 3.1, we can reduce the objective function of problem (1) into the following function

$$\begin{aligned} \min\ f(x) &= \frac{1}{2} x^T Q x + c^T x \\ &= \frac{1}{2} \sum_{i \in J \setminus J^0} \sum_{j \in J \setminus J^0} q_{ij} x_i x_j + \sum_{i \in J \setminus J^0} c_i x_i + \frac{1}{2} \sum_{i \in J^0} \sum_{j \in J \setminus J^0} q_{ij} \hat{x}_i x_j \\ &\quad + \frac{1}{2} \sum_{i \in J \setminus J^0} \sum_{j \in J^0} q_{ij} x_i \hat{x}_j + \frac{1}{2} \sum_{i \in J^0} \sum_{j \in J^0} q_{ij} \hat{x}_i \hat{x}_j + \sum_{i \in J^0} c_i \hat{x}_i. \end{aligned}$$

Suppose that matrix A' and vector b' are created by deleting rows or columns of the original matrix A and vector b based on Rule 3.1, $x^1 = (x_j)_{j\in J\setminus J^0}$, $Q^1 = Q_{(J-J^0)\times(J-J^0)}$ denote the matrix Q obtained by deleting the $i^{th}(i \in J^0)$ row and the $j^{th}(j \in J^0)$ column, and let $c_j^1 = c_j + \sum_{i\in J^0} q_{ij}\hat{x}_i$ for all $j \in J - J^0$, $\alpha = \frac{1}{2}\sum_{i\in J^0}\sum_{j\in J^0} q_{ij}\hat{x}_i\hat{x}_j + \sum_{i\in J^0} c_i\hat{x}_i$. Thus, we can reduce problem (1) into the following equivalent problem

$$\begin{aligned} \min\ & f(x) = \frac{1}{2}{x^1}^T Q^1 x^1 + {c^1}^T x^1 + \alpha \\ \text{s.t. } & A' \circ x^1 \geqslant b', \\ & 0 \leqslant x_j \leqslant \hat{x}_j,\ j \in J\setminus J^0. \end{aligned} \tag{6}$$

It follows from Note 2.5 that we can construct Rule 3.2 to reduce problem (6).

Rule 3.2. According to the constraint conditions of problem (6), for all $k \in K\setminus K^0$, calculate $J_k = \{j \in J\setminus J^0 | a_{kj}\hat{x}_j \geqslant b_k\}, \Lambda = \prod_{k\in K\setminus K^0} J_k$. If there exists $k_1 \geqslant k_2$, and $J_{k_1} \supseteq J_{k_2}$ such that $\dfrac{b_{k_1}}{a_{k_1 j}} \leqslant \dfrac{b_{k_2}}{a_{k_2 j}}(\forall j \in J_{k_2})$, then deleting J_{k_1} from Λ will not affect the minimum solution set of problem (6), that is, the k_1^{th} inequality of $A' \circ x^1 \geqslant b'$ can be deleted.

Lemma 3.2. If there exists some $i_0 \in J$ such that $c_{i_0} \geqslant 0, q_{i_0 i_0} \geqslant 0$, and $c_{i_0} + \sum_{j\in J_{i_0}^-} q_{i_0 j} > 0$, then there exists a minimum solution $\check{x}$ satisfying $x_{i_0}^* = \check{x}_{i_0}$ for any optimal solution x^*.

Proof. Since x^* is a feasible solution of problem (1), then there exists a minimum solution $\check{x}$ satisfying $0 \leqslant \check{x} \leqslant x^* \leqslant \hat{x}$, that is, for all $j \in J$,we must have $0 \leqslant \check{x}_j \leqslant x_j^* \leqslant \hat{x}_j$. To prove the conclusion, equivalent to proving $\check{x}_{i_0} = x_{i_0}^*$, suppose that $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ satisfies

$$\bar{x}_j = \begin{cases} x_j^*, \text{ if } j \neq i_0, \\ \check{x}_j, \text{ if } j = i_0. \end{cases}$$

Obviously, $\bar{x}$ is a solution of (2) since $\check{x} \leqslant \bar{x} \leqslant x^*$. If $j \neq i_0$, we have $0 \leqslant x^* = \bar{x} \leqslant \hat{x}$, otherwise, $0 \leqslant x_j^* < \bar{x}_j = \hat{x}_j$. Thus, we can get

$$\begin{aligned} f(x^*) &= \tfrac{1}{2}{x^*}^T Q x^* + c^T x^* \\ &= \tfrac{1}{2}\sum_{i=1,i\neq i_0}^{n}\sum_{j=1,j\neq i_0}^{n} q_{ij}x_i^*x_j^* + \sum_{j\in J_{i_0}^-, j\neq i_0} q_{i_0,j}x_{i_0}^*x_j^* + \sum_{i=1,i\neq i_0}^{n} c_i x_i^* \\ &\qquad + (\sum_{j\in J_{i_0}^+, j\neq i_0} q_{i_0,j}x_j^* + \tfrac{1}{2}q_{i_0 i_0}x_{i_0}^* + c_{i_0})x_{i_0}^* \\ &> \tfrac{1}{2}\sum_{i=1,i\neq i_0}^{n}\sum_{j=1,j\neq i_0}^{n} q_{ij}x_i^*x_j^* + \sum_{j\in J_{i_0}^-, j\neq i_0} q_{i_0,j}\check{x}_{i_0}x_j^* + \sum_{i=1,i\neq i_0}^{n} c_i x_i^* \\ &\qquad + (\sum_{j\in J_{i_0}^+, j\neq i_0} q_{i_0,j}x_j^* + \tfrac{1}{2}q_{i_0 i_0}\check{x}_{i_0} + c_{i_0})\check{x}_{i_0} \\ &= \tfrac{1}{2}\bar{x}^T Q\bar{x} + c^T\bar{x}, \end{aligned}$$

where the inequality follows from $c_{i_0} \geqslant 0, q_{i_0 i_0} \geqslant 0$, and $c_{i_0} + \sum_{j \in J_{i_0}^-} q_{i_0 j} > 0$. It is a contradiction that x^* is an optimal solution of problem (1). So, there exists a minimum solution $\check{x}$ satisfying $x_{i_0}^* = \check{x}_{i_0}$ for any optimal solution x^*.

The following corollary can be directly obtained by using Lemma 3.2.

Corollary 3.2. If $c_i \geqslant 0, q_{ii} \geqslant 0$, and $c_i + \sum_{j \in J_i^-} q_{ij} > 0$ for all $i \in J$, then there exists a minimum solution $\check{x}$ of (2) such that $x^* = \check{x}$, where x^* is an optimal solution of problem (1).

Lemma 3.3. Suppose that $K_j = \{k \in K | a_{kj} \wedge \hat{x}_j \geqslant b_k\} (j \in J)$. If there exists some $i_0 \in J$ satisfying the following conditions: (a)$K_{i_0} = \emptyset$; (b)$c_{i_0} \geqslant 0, q_{i_0 i_0} \geqslant 0$, and $c_{i_0} + \sum_{j \in J_{i_0}^-} q_{i_0 j} > 0$, then we have $x_{i_0}^* = 0$ for any optimal solution x^*.

Proof. Suppose that x^* is an optimal solution of problem (1). It follows from Lemma 3.2 that there exists a minimum solution $\check{x}$ such that $x_{i_0}^* = \check{x}_{i_0}$. According to Theorem 2.3, there exists an FRI path p such that $\check{x} = x^p$. Because $K_{i_0} = \emptyset$, it implies that $i_0 \notin J_k$ for all $k \in K$, that is, $p_k \neq i_0$ for any FRI path p of (2). It implies that $\check{x}_{i_0} = x_{i_0}^p = 0$ based on (5).

Based on Lemma 3.3, we present the following rule to simplify problem (6).

Rule 3.3. Let $J^1 = \{j \in J \backslash J^0 | K_j = \emptyset, c_j \geqslant 0, q_{jj} \geqslant 0, c_j + \sum_{i \in J_j^-} q_{ij} > 0\}$. For all $j \in J^1$, set $x_j^* = 0$.

Lemma 3.4. If there exists $k_0 \in K$ and $j_0 \in J$ satisfying: (a) $J_{k_0} = \{j_0\}$, (b) $\frac{b_{k_0}}{a_{k_0 j_0}} \geqslant \frac{b_k}{a_{k j_0}}$ for all $k \neq k_0$ and $j_0 \in J_k \cap J_{k_0}$, (c) $c_{j_0} \geqslant 0, q_{j_0 j_0} \geqslant 0$, and $c_{j_0} + \sum_{j \in J_{j_0}^-} q_{j_0 j} > 0$, then any optimal solution x^* of problem (1) must meet $x_{j_0}^* = \frac{b_{k_0}}{a_{k_0 j_0}}$.

Proof. It follows from conditions (a) and (b) that any FRI path $p = (p_1, \cdots, p_m)$ of (2) must satisfy $p_{k_0} = j_0$ and $p_k \neq j_0 (k > k_0)$. This implies that $x_{j_0}^p = \frac{b_{k_0}}{a_{k_0 j_0}}$. From condition (3) and Lemma 3.2, we have $x_{j_0}^* = \frac{b_{k_0}}{a_{k_0 j_0}}$ for any optimal solution x^* of problem (1).

Based on Lemma 3.4, we present the following rule to simplify problem (6).

Rule 3.4. Let $J^2 = \{j_0 \in J \backslash J^0 | (a) \exists k_0 \in K \backslash K^0, J_{k_0} = \{j_0\}, (b) \forall k \neq k_0, j_0 \in J_k \cap J_{k_0}, \text{have} \frac{b_{k_0}}{a_{k_0 j_0}} \geqslant \frac{b_k}{a_{k j_0}}, (c) c_{j_0} \geqslant 0, q_{j_0 j_0} \geqslant 0, c_{j_0} + \sum_{j \in J_{j_0}^-} q_{j_0 j} > 0\}$. For all $j \in J^2$, set $x_j^* = \frac{b_{k_0}}{a_{k_0 j_0}}$, and when $j_0 \in J_k$, deleting J_k, that is, deleting the k^{th} inequality of $A' \circ x^1 \geqslant b'$.

Lemma 3.5. If there exists some $k_0 \in K, j_0 \in J$ satisfying (a)$J_{k_0} = \{j_0\}$, (b) $\hat{x}_{j_0} = \dfrac{b_{k_0}}{a_{k_0 j_0}}$, then for any optimal solution x^*, we can get $x^*_{j_0} = \hat{x}_{j_0}$.

Proof. If $j_0 \in J_k (k \in K, k \neq k_0)$, then, from the definition of J_k, we have $\dfrac{b_{k_0}}{a_{k_0 j_0}} = \hat{x}_{j_0} \geqslant \dfrac{b_k}{a_{kj_0}}$. Suppose that $p = (p_1, \cdots, p_m)$ is any FRI path of (2).

If $p_k \neq j_0$ for all $k < k_0$, then we can get $p_{k_0} = j_0$. Thus we have $x^p_{j_0} = \dfrac{b_{k_0}}{a_{k_0 j_0}} = \hat{x}_{j_0}$.

If there exists some $k < k_0$ satisfying $p_k = j_0$, then must have $\dfrac{b_{k_0}}{a_{k_0 j_0}} = \hat{x}_{j_0} \geqslant \dfrac{b_k}{a_{kj_0}}$. Now, we consider the following two cases:

(1) $\dfrac{b_{k_0}}{a_{k_0 j_0}} = \hat{x}_{j_0} > \dfrac{b_k}{a_{kj_0}}$. According to Definition 2.2, we have $p_{k_0} = j_0$. This implies $x^p_{j_0} = \dfrac{b_{k_0}}{a_{k_0 j_0}} = \hat{x}_{j_0}$.

(2) $\dfrac{b_{k_0}}{a_{k_0 j_0}} = \hat{x}_{j_0} = \dfrac{b_k}{a_{kj_0}}$. By using Definition 2.2, we can obtain $p_{k_0} = 0$. It follows that we have $x^p_{j_0} = \dfrac{b_k}{a_{kj_0}} = \hat{x}_{j_0}$.

Thus, there is always $x^p_{j_0} = \hat{x}_{j_0}$ for any FRI path p. By Theorems 2.3 and 2.4, for any one feasible solution x of problem (1), we can get $x_{j_0} = \hat{x}_{j_0}$. So, for any one optimal solution x^*, we must have $x^*_{j_0} = \hat{x}_{j_0}$.

The following rule for simplifying problem (6) is proposed based on Lemma 3.5.

Rule 3.5. Let $J^3 = \{j_0 \in J | \exists k_0 \in K \backslash K^0, J_{k_0} = \{j_0\}, \hat{x}_{j_0} = \dfrac{b_{k_0}}{a_{k_0 j_0}}\}$. For all $j \in J^3$, let $x^*_j = \hat{x}_{j_0}$, and when $j_0 \in J_k$, deleting J_k, that is, deleting the k^{th} inequality of $A' \circ x^1 \geqslant b'$.

Corollary 3.3. If there exists some $i \in J$ satisfying: (a) $K_i \neq \emptyset$, and $K_i \cap K_j = \emptyset$ for all $j \in J \backslash \{i\}$, (b) $c_i \geqslant 0, q_{ii} \geqslant 0$, and $c_i + \sum\limits_{j \in J_i^-} q_{ij} > 0$, then any optimal solution x^* must have $x^*_i = \max\{\dfrac{b_k}{a_{ki}} | k \in K_i\}$.

Proof. Suppose that an optimal solution of problem (1) is x^*. From Lemma 3.2, there exists a minimum solution $\check{x}$ satisfying $x^*_i = \check{x}_i$. Let $k_1, \cdots, k_r \in K_i$, $k_1 < k_2 < \cdots < k_r$ and $\dfrac{b_{k_l}}{a_{k_l i}} = \max\{\dfrac{b_k}{a_{ki}} | k \in K_i\} (1 \leqslant l \leqslant r)$. Since $K_i \cap K_j = \emptyset$ for all $j \in J \backslash \{i\}$, then we have $J_{k_1} = J_{k_2} = \cdots = J_{k_l} = \{i\}$, and $i \notin J_k (k \in K, k \notin \{k_1, \cdots, k_r\})$. From the definition of the FRI path, any FRI path $p = (p_1, \cdots, p_m)$ of (2) satisfies $p_{k_l} = i$. It then follows from Theorem 3.3 that for any minimum solution $\check{x}$, there exists some FRI path p meeting $\check{x} = x^p$. According to (5), we can get $\check{x}_i = x_i^p = \max\{\frac{b_k}{a_{ki}}|p_k = i\} = \frac{b_{k_l}}{a_{k_l i}} = \max\{\frac{b_k}{a_{ki}}|k \in K_i\}$.

The following conclusion can be obtained easily from Corollaries 3.2 and 3.3:

Corollary 3.4. If problem (1) satisfies: (a) for all $i \in J$, then wehave $c_i \geqslant 0, q_{ii} \geqslant 0$, and $c_i + \sum\limits_{j \in J_i^-}^{n} q_{ij} > 0$; (b) for all $t, j \in J$ we have $K_t \cap K_j = \emptyset$, then problem (1) has a unique optimal solution $x^* = (x_1^*, \cdots, x_n^*)^T$, and

$$x_j^* = \begin{cases} \max\{\frac{b_k}{a_{kj}}|k \in K_j\}, \text{ if } K_j \neq \emptyset, \\ 0, \qquad\qquad\qquad\qquad \text{if } K_j = \emptyset, \end{cases} \quad j \in N.$$

Theorem 3.1. If there exists some $j_1 \in J$ such that: (a) $\bigcup\limits_{j=1, j \neq j_1}^{n} K_j \subset K_{j_1}$, (b)$\exists k_0 \in K_{j_1} - \bigcup\limits_{j=1, j \neq j_1}^{n} K_j$ satisfies $\frac{b_{k_0}}{a_{k_0 j_1}} \geqslant \frac{b_k}{a_{kj_1}} (\forall k \in K_{j_1})$; (c) $c_i \geqslant 0, q_{ii} \geqslant 0$, and $c_i + \sum\limits_{j \in J_i^-}^{n} q_{ij} > 0$ for all $i \in J$; then problem (1) has a unique optimal solution $x^* = (x_1^*, \cdots, x_n^*)^T$, where

$$x_j^* = \begin{cases} \frac{b_{k_0}}{a_{k_0 j_1}}, \text{ if } j = j_1, \\ 0, \qquad \text{if } j \neq j_1, \end{cases} \quad j \in J.$$

Proof. When $X(A, b, D, e) \neq \emptyset$, we can obtain $K_{j_1} = K$. Otherwise, there exists some $k \in K$ such that $k \notin K_{j_1}$. This implies $k \notin K_j (\forall j \in J)$, that is, for all $j \in J, a_{kj}\hat{x}_j \ngeqslant b_k$. So, the k^{th} inequality of $A \circ x \geqslant b$ has no solution. It is a contradiction to $X(A, b, D, e) \neq \emptyset$. Thus $a_{kj}\hat{x}_j \geqslant b_k$ for all $k \in K$, that is, $x = (0, \cdots, 0, \hat{x}_j, 0, \cdots, 0)^T$ is a solution of $A \circ x \geqslant b$. Since $k_0 \in K_{j_1} - \bigcup\limits_{j=1, j \neq j_1}^{n} K_j$, we have $J_{k_0} = \{j_1\}$. Based on the proof of Corollary 3.3, we have $\check{x}_{j_1} = \frac{b_{k_0}}{a_{k_0 j_1}}$ for any minimum solution $\check{x}$ of (2). Since $c_i \geqslant 0, q_{ii} \geqslant 0$ and $c_i + \sum\limits_{j \in J_i^-}^{n} q_{ij} > 0$ for all $i \in j$, then any optimal solution x^* satisfies $x_{j_1}^* = \frac{b_{k_0}}{a_{k_0 j_1}}$. Thus $(0, \cdots, 0, \frac{b_{k_0}}{a_{k_0 j_1}}, 0, \cdots, 0)^T$ is an optimal solution of problem (1).

Theorem 3.2. If there exists some $j_1, \cdots, j_r \in J$ satisfying: (1) for all $l \in \{1, 2, \cdots, r\}$, $\bigcup\limits_{j=1, j \notin \{j_1, \cdots, j_r\}}^{n} K_j \subset K_{j_l}$ and $K_{j_l} = K$; (2) for all $l \in \{1, 2, \cdots, r\}$, there exists some $k_l \in K_{j_l} - \bigcup\limits_{j=1, j \neq k}^{n} K_j$ such that $\frac{b_{k_l}}{a_{k_l j_l}} \geqslant \frac{b_k}{a_{kj_l}} (\forall k \in K_{j_l})$; (3)

for all $i, j \in N$, $c_i \geqslant 0, q_{ij} \geqslant 0$, and $c_i + \sum_{j \in J_i^-}^{n} q_{ij} > 0$; then problem (1) has a unique optimal solution $x^* = (x_1^*, \cdots, x_n^*)^T$, where

$$x_j^* = \begin{cases} \dfrac{b_{k_l}}{a_{k_l j_l}}, & \text{if } j = j_l, \\ 0, & \text{if } j \neq j_l, \end{cases} \quad j \in J.$$

The following rule to simplify problem (1) is proposed based on Corollary 3.3.

Rule 3.6. Let $J^4 = \{j \in J \backslash J^0 | (a) K_j \neq \emptyset$, and for all $i \in J \backslash \{j\}$ have $K_i \cap K_j = \emptyset, (b) c_j \geqslant 0, q_{jj} \geqslant 0$, and $c_j + \sum_{i \in J_j^-}^{n} q_{ij} > 0\}$. For all $j \in J^4$, set $x_j^* = \max\{\dfrac{b_k}{a_{kj}} | k \in K_j\}$, and deleting the j^{th} column of matrix A, that is, deleting all j from J_k.

By Rules 3.1–3.6, problem (1) can be reduced to the following questions

$$\begin{aligned} \min\ f(x) = & \frac{1}{2} \sum_{i \in J^5} \sum_{j \in J^5} q_{ij} x_i x_j + \sum_{i \in J^5} c_i x_i + \frac{1}{2} \sum_{i \in J^0} \sum_{j \in J^5} q_{ij} \hat{x}_i x_j \\ & + \frac{1}{2} \sum_{i \in J^5} \sum_{j \in J^0} q_{ij} x_i \hat{x}_j + \frac{1}{2} \sum_{i \in J^6} \sum_{j \in J^5} q_{ij} x_i^* x_j + \frac{1}{2} \sum_{i \in J^5} \sum_{j \in J^6} q_{ij} x_i x_j^* + \alpha' \\ \text{s.t. } & A'' \circ x \geqslant b'', \\ & 0 \leqslant x_j \leqslant \hat{x}_j, \ j \in J^5, \end{aligned} \tag{7}$$

where, A'', b'' are new matrix and vector that are made by deleting rows and columns from the original matrix and vector based on Rules 3.1–3.6, $J^5 = J \backslash J^0 \backslash J^1 \backslash J^2 \backslash J^3 \backslash J^4, J^6 = J^2 \cup J^3 \cup J^4$,

$$\alpha' = \frac{1}{2} \sum_{i \in J^0} \sum_{j \in J^0} q_{ij} \hat{x}_i \hat{x}_j + \sum_{i \in J^0} c_i \hat{x}_i + \frac{1}{2} \sum_{i \in J^6} \sum_{j \in J^6} q_{ij} x_i^* x_j^* + \sum_{i \in J^6} c_i x_i^*.$$

Suppose that the entire FRI path set of problem (7) is $FRIP$, and $|FRIP| = h$. In order to solve problem (7), we must solve the following h quadratic programming problems with interval constraints by numerical algorithms as in Bazaraa et al. [1]

$$\begin{aligned} & \min\ f(x) \\ & \text{s.t. } x_j^q \leqslant x_j \leqslant \hat{x}_j, \quad j \in J', \end{aligned} \tag{8}$$

where $q \in FRIP$. Suppose that the optimal solution of problem (8) is $x^l (l = 1, 2, \cdots, h)$, then the optimal solution x^* of problem (7) is $f(x^*) = \min_{l=1,2,\cdots,h} \{f(x^l)\}$.

If the conditions of Corollary 3.2 are met, suppose that the entire FRI path set of problem (7) is $FRIP$, and $|FRIP| = h$, then we just solve the following problems

$$\begin{aligned} &\min \ f(x) \\ &\text{s.t. } x = x^q, \quad q \in FRIP. \end{aligned} \tag{9}$$

We can then solve problem (9) by using the branch and bound method [1].

Now we build algorithms to solve problem (1) based on the above discussion.

Step 1. Calculate the maximal solution of the quadratic programming with max-product fuzzy relation inequality constraints by Formula (3). If $A \circ \hat{x} \geqslant b$ holds, then go to Step 2, otherwise, there is no feasible solution to the original problem (1), stop.

Step 2. Compute $J^0 = \{j \in J | c_j \leqslant 0, q_{jj} \leqslant 0, c_j + 0.5 q_{jj}\hat{x}_j + \sum_{i \in J_j^+} q_{ij} < 0\}$, $J' = \{j \in J | c_j \geqslant 0, q_{jj} \geqslant 0, c_j + \sum_{i \in J_j^-} q_{ij} > 0\}$, $J'' = J \backslash J^0 \backslash J'$, go to Step 3.

Step 3. If $J^0 = J$, the maximal solution $\hat{x}$ is the optimal solution of problem (1), stop, otherwise, go to Step 4.

Step 4. If $J' = J$, go to step 6, otherwise, go to Step 5.

Step 5. Calculate $K^0 = \{k \in K | a_{kj}\hat{x}_j \geqslant b_k, \forall j \in J^0\}$, and simplify problem (1) based on Rule 3.1, that is, for all $j \in J^0$, set $x_j^* = \hat{x}_j$, deleting the j^{th} column of matrix A, and deleting the k^{th} row of matrix A and the k^{th} component of vector b for any $k \in K^0$. Set $J = J \backslash J^0$. Go to Step 6.

Step 6. Calculate $J_k = \{j \in J' \cup J'' | a_{kj}\hat{x}_j \geqslant b_k\}$ for all $k \in K \backslash K^0$, and $\Lambda = \prod_{k \in K \backslash K^0} J_k$. Simplify problem (1) or (7) based on Rule 3.2, and update index set Λ, go to Step 7.

Step 7. Calculate $K_j = \{k \in K \backslash K^0 | a_{kj} \wedge \hat{x}_j \geqslant b_k\} (j \in J \backslash J^0)$ and $J^1 = \{j \in J \backslash J^0 | K_j = \emptyset, c_j \geqslant 0, q_{jj} \geqslant 0, c_j + \sum_{i \in J_j^-} q_{ij} > 0\}$, and simplify problem (1) or (7) based on Rule 3.3. Set $J' = J' \backslash J^1$. If $J' = \emptyset$ and $J'' = \emptyset$, stop, and calculate the optimal solution and the optimal value. Otherwise, update index set Λ, go to Step 8.

Step 8. Calculate J^2, and simplify problem (1) or (7) based on Rule 3.4. Set $J' = J' \backslash J^2$. If $J' = \emptyset$ and $J'' = \emptyset$ or $\Lambda = \emptyset$, stop. Set $x_j^* = 0 (j \in J \backslash J^0 \backslash J^1 \backslash J^2)$, and calculate the optimal solution and the optimal value. Otherwise, update index set Λ, go to Step 9.

Step 9. Calculate J^3, and simplify problem (1) or (7) based on Rule 3.5. Set $J' = J' \backslash J^3$. If $J' = \emptyset$ and $J'' = \emptyset$ or $\Lambda = \emptyset$, stop. Set $x_j^* = 0 (j \in J \backslash J^0 \backslash J^1 \backslash J^2 \backslash J^3)$, and calculate the optimal solution and the optimal value. If $J' \neq \emptyset$ and $J'' = \emptyset$, update index set Λ, and go to Step 10. If $J'' \neq \emptyset$, update index set Λ, and go to Step 11.

Step 10. Calculate J^4, and simplify problem (1) or (7) based on Rule 3.6. Set $J' = J' \backslash J^1$. If $J' = \emptyset$ and $J'' = \emptyset$ or $\Lambda = \emptyset$, stop. Set $x_j^* = 0 (j \in J \backslash J^0 \backslash J^1 \backslash J^2 \backslash J^3 \backslash J^4)$, and calculate the optimal solution and the optimal value.

If $J' \neq \emptyset$ and $J'' = \emptyset$, update index set Λ, and go to Step 11. If $J'' \neq \emptyset$, update index set Λ, and go to Step 12.

Step 11. Solve the optimal solution for problem (9) based on the FRI path and branch and bound method, and the optimal solution of (1) will be obtained.

Step 12. Find all FRI paths of problem (8) and corresponding solutions x^p based on the definition of the FRI path. Look for the optimal solutions of problem (8) based on numerical optimization methods for solving traditional quadratic programming with interval constraints. The optimal solution of problem (1) can then be obtained.

4 Numerical Examples

Example 4.1. Consider the following fuzzy relation quadratic programming

$$\begin{aligned} \min\ & f(x) = \frac{1}{2}x^T Q x + cx \\ \text{s.t. } & A \circ x \geqslant b, \\ & D \circ x \leqslant e, \\ & x \in [0,1]^4, \end{aligned} \tag{10}$$

where $c = (-2.415, -1.947, -1.772, -3.522)^T$, $b = (0.2, 0.25, 0.45, 0.4)^T$, $e = (0.141, 0.2, 0.435, 0.333)^T$,

$$Q = \begin{bmatrix} -1.94 & 1.38 & 0.55 & 0.33 \\ 1.38 & -1.21 & 0.31 & -1.47 \\ -0.55 & 0.31 & -1.2 & 1.23 \\ 0.33 & -1.47 & 1.23 & -1.31 \end{bmatrix}, A = \begin{bmatrix} 0.9 & 0.8 & 0.75 & 0.68 \\ 0.75 & 0.88 & 0.6 & 0.8 \\ 0.66 & 0.89 & 1 & 0.56 \\ 0.7 & 0.25 & 0.9 & 0.4 \end{bmatrix}, D = \begin{bmatrix} 0.25 & 0.4 & 0.1 & 0.4 \\ 0.3 & 0.6 & 0.24 & 0.8 \\ 0.71 & 0.5 & 0.23 & 0.3 \\ 0.6 & 0.8 & 0.5 & 0.6 \end{bmatrix}.$$

Solution. Step 1. Calculate the maximal solution of problem (10) based on formula (3). The maximal solution is $\hat{x} = (0.555, 0.333, 0.666, 0.25)^T$ and $A \circ \hat{x} \geqslant b$, and go to Step 2.

Step 2. $J = \{1, 2, 3, 4\}$, $J^0 = \{1, 2, 3, 4\}$, $J' = \emptyset$, $J'' = \emptyset$, go to Step 3.

Step 3. Since $J^0 = J$, the maximal solution $\hat{x} = (0.555, 0.333, 0.666, 0.25)^T$ is the optimal solution of problem (10), the optimal value is $f(\hat{x}) = -4.2702$.

Example 4.2

$$\begin{aligned} \min\ & f(x) = \frac{1}{2}x^T Q x + cx \\ \text{s.t. } & A \circ x \geqslant b, \\ & D \circ x \leqslant e, \\ & x \in [0,1]^6, \end{aligned} \tag{11}$$

where $c = (1.8551, 2.506, 1.6991, 3.8909, 1.9593, 4.5472)^T$,
$b = (0.3, 0.3, 0.4, 0.5, 0.6, 0.6)^T$,
$e = (0.3, 0.6, 0.6, 0.6, 0.4, 0.7, 0.8, 0.9)^T$,

$$Q = \begin{bmatrix} 0.7577 & 0.7431 & -0.3922 & 0.6555 & -1.1712 & 0.7060 \\ 0.7431 & 0.2769 & 0.0462 & 0.0971 & -0.8235 & 0.6948 \\ -0.3922 & -1.0462 & 0.0344 & -0.4387 & 0.3816 & 0.7655 \\ 0.6555 & 0.0971 & -0.4387 & 0.4456 & 0.6463 & -2.7094 \\ 1.1712 & -0.8235 & 0.3816 & 0.6463 & 0.1626 & -0.1190 \\ 0.7060 & 0.6948 & 0.7655 & -2.7094 & -0.1190 & 0.7513 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.1 & 0.8 & 0.3 & 0.6 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 1 & 0.2 & 0.5 & 0.5 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.4 & 0.5 & 0.5 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.8 & 0.6 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{bmatrix}, D^T = \begin{bmatrix} 0.3 & 0.5 & 0.1 & 0.6 & 0.4 & 0.5 & 0.8 & 0.3 \\ 0.2 & 0.6 & 0.2 & 0.5 & 0.4 & 0.5 & 0.8 & 0.9 \\ 0.4 & 0.3 & 0.3 & 0.4 & 0.4 & 0.6 & 1 & 0.3 \\ 0.5 & 0.3 & 0.4 & 0.3 & 0.3 & 0.6 & 1 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.2 & 0.3 & 0.7 & 1 & 0.3 \\ 0.3 & 0.4 & 0.6 & 0.1 & 0.3 & 0.7 & 0.8 & 0.4 \end{bmatrix}.$$

Solution. Step 1. The maximal solution is $\hat{x} = (1, 1, 0.75, 0.6, 0.6, 1)^T$ and $A \circ \hat{x} \geqslant b$. Go to Step 2.

Step 2. $J = \{1,2,3,4,5,6\}, J^0 = \emptyset, J' = \{1,2,3,4,5,6\}, J'' = \emptyset$, go to Step 3.

Step 3. $J^0 \neq J$, go to Step 4.

Step 4. $J' = J$, go to Step 6.

Step 6. For all $k \in K = \{1,2,3,4,5,6\}$, we calculate index set J_k:

$$J_1 = \{2,4,5\}, J_2 = \{1,3,4\}, J_3 = \{1\}, J_4 = \{6\}, J_5 = \{6\}, J_6 = \{2\}, \Lambda = \prod_{k=1}^{6} J_k.$$

Since $J_4 = J_5 = \{6\}$, and $\frac{b_4}{a_{46}} \geqslant \frac{b_5}{a_{56}}$, delete J_5 from Λ based on Rule 3.2, that is, delete the 5^{th} inequality of $A \circ x \geqslant b$. Update $\Lambda : \Lambda = J_1 \times J_2 \times J_3 \times J_4 \times J_6$, go to Step 7.

Step 7. For all $j \in J' = \{1,2,3,4,5,6\}$, calculate index set K_j $K_1 = \{2,3\}, K_2 = \{1,6\}, K_3 = \{2\}, K_4 = \{1,2\}, K_5 = \{1\}, K_6 = \{4\}$ and $J^1 = \emptyset$. Therefore, Rule 3.3 can not be taken advantage of simplification problem (11), go to Step 8.

Step 8. It is clear that $J^2 = \{1,2,6\}, J_3 = \{1\}, J_4 = \{6\}, J_6 = \{2\}$. By using Rule 3.4, set $x_1 = \frac{b_3}{a_{31}} = 0.4, x_2 = \frac{b_6}{a_{62}} = 1, x_6 = \frac{b_4}{a_{46}} = 1$, and delete J_1, J_2, J_3, J_4, J_6. Since $J = J \backslash J^2 = \{3,4,5\}$ and $\Lambda = \emptyset$, stop. Set $x_3^* = x_4^* = x_5^* = 0$. Thus, the optimal solution and the optimal value are $x^* = (0.4, 1, 0, 0, 0, 1)$ and $f(x^*) = 9.6444$, respectively.

Example 4.3. Solving fuzzy relation quadratic programming [6]:

$$\begin{array}{ll} \min & f(x) = \frac{1}{2}x^T Q x + c^T x \\ \text{s.t} & A \circ x \geqslant b, \\ & D \circ x \leqslant e, \\ & 0 \leq x_j \leq 1, j \in J \end{array} \tag{12}$$

$c = (2,-1,-1,-3,5,1,1)^T$, $b = (0.3, 0.24, 0.3, 0.15, 0.35, 0.24, 0.23)^T$, $e = (0.2, 0.5, 0.4, 0.8, 0.4, 0.6)^T$,

$$Q = \begin{bmatrix} 1 & -1.5 & 2 & 2 & 3 & 1 & 1 \\ -1.5 & -3 & -1 & -2.5 & -3.5 & -5.5 & -8 \\ 2 & -1 & 3 & 4 & 7 & 5 & 1 \\ 2 & -2.5 & 4 & 6.6 & 2 & 2.2 & 1 \\ 3 & -3.5 & 7 & 2 & 3.5 & 3 & 3 \\ 1 & -5.5 & 5 & 2.2 & 3 & 4 & 5 \\ 1 & -8 & 1 & 1 & 3 & 5 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.9 & 0.3 & 0.3 & 0.56 & 0.4 & 0.34 & 0.99 \\ 0.8 & 0.95 & 0.7 & 0.6 & 0.3 & 0.25 & 0.43 \\ 0.3 & 0.5 & 0.6 & 0.44 & 0.35 & 0.65 & 1 \\ 0.8 & 0.4 & 0.56 & 0.66 & 0.34 & 0.23 & 0.12 \\ 0.24 & 0.34 & 0.46 & 0.87 & 0.94 & 0.34 & 0.27 \\ 0.45 & 0.44 & 0.56 & 0.87 & 0.65 & 0.44 & 0.23 \\ 0.55 & 0.45 & 1 & 0.42 & 0.41 & 0.24 & 0.35 \end{bmatrix}.$$

$$D = \begin{bmatrix} 0.94 & 0.6 & 0.45 & 0.9 & 0.34 & 0.33 & 0.44 \\ 0.65 & 0.24 & 0.5 & 0.87 & 0.42 & 0.24 & 0.2 \\ 0.52 & 0.35 & 0.65 & 0.76 & 0.27 & 0.15 & 0.15 \\ 0.6 & 0.44 & 0.8 & 0.65 & 0.66 & 0.23 & 0.4 \\ 0.3 & 0.5 & 0.26 & 0.8 & 0.44 & 0.7 & 0.76 \\ 0.64 & 0.4 & 0.43 & 0.25 & 0.34 & 0.66 & 0.1 \end{bmatrix}.$$

Solution. Step 1. The maximal solution is $\hat{x} = (0.213, 0.333, 0.444, 0.222, 0.588, 0.571, 0.454)^T$. Obviously, $A \circ \hat{x} \geqslant b$, go to Step 2.

Step 2. $J^0 = \{2\}, J' = \{1, 5\}, J'' = \{3, 4, 6, 7\}$, go to Step 3.

Step 3. $J^0 \neq J$, go to Step 4.

Step 4. $J' \neq J$, go to Step 5.

Step 5. Since $J^0 = \{2\}, K^0 = \{2\}$, according to Rule 3.1, set $x_2 = 0.333$, and delete the 2^{th} column, the 2^{th} row of matrix A and the 2^{th} component of vector b. Then problem (12) can be reduced as follows:

$$\begin{aligned} &\min\ f(x) = \frac{1}{2}{x^1}^T Q^1 x^1 + c^1 x^1 + \alpha \\ &\text{s.t. } A' \circ x^1 \geqslant b', \\ &0 \leqslant x_i \leqslant \hat{x}_i, i = 1, 3, 4, 5, 6, 7, \end{aligned} \tag{13}$$

where $c^1 = (1.5005, -1.333, -3.8325, 3.8345, -0.8315, -1.664), x^1 = (x_1, x_3, x_4, x_5, x_6, x_7)^T, b' = (0.3, 0.3, 0.15, 0.35, 0.24, 0.23)^T, \alpha = -0.4993$ and

$$Q^1 = \begin{array}{c} i\backslash j \\ 1 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 1 \\ 2 & 3 & 4 & 7 & 5 & 1 \\ 2 & 4 & 6.6 & 2 & 2.2 & 1 \\ 3 & 7 & 2 & 3.5 & 3 & 3 \\ 1 & 5 & 2.2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 3 & 5 & 1 \end{bmatrix} \end{array}, \quad A' = \begin{array}{c} k\backslash j \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.9 & 0.3 & 0.56 & 0.4 & 0.34 & 0.99 \\ 0.3 & 0.6 & 0.44 & 0.35 & 0.65 & 1 \\ 0.8 & 0.56 & 0.66 & 0.34 & 0.23 & 0.12 \\ 0.24 & 0.46 & 0.87 & 0.94 & 0.34 & 0.27 \\ 0.45 & 0.56 & 0.87 & 0.65 & 0.44 & 0.23 \\ 0.55 & 1 & 0.42 & 0.41 & 0.24 & 0.35 \end{bmatrix} \end{array}.$$

So,$J = \{1, 3, 4, 5, 6, 7\}, J^0 = \emptyset, J' = \{1, 5\}, J'' = \{3, 4, 6, 7\}$. Go to Step 6.

Step 6. For all $k = 1, 2, 4, 5, 6, 7$, and $J_k = \{j \in J' \cup J'' | a_{kj}\hat{x}_j \geqslant b_k\}$, calculate index set J_k : $J_1 = \{7\}, J_3 = \{6, 7\}, J_4 = \{1, 3, 5\}, J_5 = \{5\}, J_6 = \{3, 5, 6\}, J_7 = \{3, 5\}, \Lambda = J_1 \times J_3 \times J_4 \times J_5 \times J_6 \times J_7$. Because $7 \in J_1 \subseteq J_3, 5 \in J_5 \subseteq J_6, \dfrac{b_1}{a_{17}} \geqslant \dfrac{b_3}{a_{37}}, \dfrac{b_5}{a_{55}} \geqslant \dfrac{b_6}{a_{65}}$, delete J_3 and J_6 based on Rule 3.2, that is, delete the 3^{th} and the 6^{th} inequalities of $A \circ x \geqslant b$. Λ is updated to $\Lambda = J_1 \times J_4 \times J_5 \times J_7 = \{7\} \times \{1, 3, 5\} \times \{5\} \times \{3, 5\}$. Go to Step 7.

Step 7. For all $j \in \{1, 3, 4, 5, 6, 7\}$, calculate index set K_j $K_1 = \{4\}, K_3 = \{5, 7\}, K_4 = \emptyset, K_5 = \{4, 5, 7\}, K_6 = \emptyset, K_7 = \{1\}$ and $J^1 = \emptyset$. Therefore, Rule 3.3 can not be taken advantage of simplification problem (13), go to Step 8.

Step 8. Because $J^2 = \emptyset$, Rule 3.4 cannot be applied, go to Step 9.

Step 9. Because $J^3 = \emptyset$, Rule 3.5 cannot be applied, go to Step 10.

Step 10. Because $J^4 = \emptyset$, Rule 3.6 cannot be applied. Since $J'' \neq \emptyset$, go to Step 12.

Step 12. By using $\Lambda = J_1 \times J_4 \times J_5 \times J_7 = \{7\} \times \{1, 3, 5\} \times \{5\} \times \{3, 5\}$ and Definition 2.2, we find all FRI paths p of (14) are p_1, p_2, p_3, p_4, p_5, where

$$p_1 = (7, 1, 5, 3), p_1 = (7, 1, 5, 5), p_3 = (7, 3, 5, 0), p_4 = (7, 5, 0, 3), p_5 = (7, 5, 0, 5).$$

The corresponding solutions are

$$\begin{aligned} &x^{p_1} = (0.1875, 0.23, 0, 0.3723, 0, 0.303), \quad x^{p_2} = (0.1875, 0, 0, 0.561, 0, 0.303), \\ &x^{p_3} = (0, 0.2678, 0, 0.3723, 0, 0.303), \quad x^{p_4} = (0, 0.23, 0, 0.4412, 0, 0.303), \\ &x^{p_5} = (0, 0, 0, 0.561, 0, 0.303). \end{aligned}$$

So, all minimum solutions are:

$$\begin{aligned} &x^{p_1} = (0.1875, 0.23, 0, 0.3723, 0, 0.303), x^{p_3} = (0, 0.2678, 0, 0.3723, 0, 0.303), \\ &x^{p_4} = (0, 0.23, 0, 0.4412, 0, 0.303), x^{p_5} = (0, 0, 0, 0.561, 0, 0.303). \end{aligned}$$

We solve the following four quadratic programming problems by numerical algorithms [1]

$$\begin{aligned} &\min \ f(x) = \frac{1}{2} {x^1}^T Q^1 x^1 + c^1 x^1 + \alpha \\ &\text{s.t. } x^{p_l} \leqslant x^1 \leqslant \hat{x}^1, \end{aligned} \tag{14}$$

where $l = 1, 3, 4, 5, x^1 = (x_1, x_3, x_4, x_5, x_6, x_7)^T, \hat{x}^1 = (0.213, 0.444, 0.222, 0.588, 0.571, 0.454)^T$. The optimal solution and optimal value of problem (14) can be seen in Table 1

Table 1. The optimal solution and optimal value of (14) for any x^{p_l}

l	x^*	$f(x^*)$
1	(0.1875,0.23,0.222,0.3723,0,0.303)	1.976082
3	(0,0.2678,0.222,0.3723,0,0.303)	1.362797
4	(0,0.23,0.222,0.4412,0,0.303)	1.807865
5	(0,0,0.222,0.561,0,0.303)	1.882416

So the optimal solution of problem (12) is $x^* = (0, 0.333, 0.2678, 0.222, 0.3723, 0, 0.303)$, the optimal value is $f(x^*) = 1.362797$.

5 Conclusion

We have presented a new algorithm for solving quadratic programming problems with max-product fuzzy relation inequality constraints, based on FRI paths, branch and bound methods and a numerical algorithm for solving traditional quadratic programming with interval constraints. The proposed algorithm has avoided a rather large amount of work that has nothing to do with finding the optimal solution within the feasible region. As the new algorithm does not need to find all minimum solutions of the fuzzy relation equations, the efficiency of the new algorithm has been demonstrated. Numerical examples have proved that the new algorithm can smoothly reach the optimal point when the variable scale of the fuzzy relation inequality (2) is not very large. However, when the size of fuzzy relation inequality (2) is very large, how to effectively solve problem (1) is still a problem to be studied.

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Recommender: Academic Conference on 30th anniversary of fuzzy geometric programming and 40th education year by and of Professor Cao Bingyuan.

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A New Algorithm to Shortest Path Problem with Fuzzy Arc Lengths

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Abstract. Network flow problem is prevalent in engineering and management. In real life the parameters between nodes are not certain. Many authors try to proposed the solution method in different types of this problem. This paper proposes the simple method to compute the fuzzy shortest path (fuzzy shortest distance) between source node and destination node in network flow problem. This algorithm only considered the nodes that need to reach destination node and doesn't involve all nodes, so it has expended less time. The goal is to reach destination node and find the shortest path, so it's not important which nodes will be past through.

Keywords: Ranking function · Trapezoidal fuzzy numbers · Fuzzy shortest path

1 Introduction

Network is a graph that contains finite set of nodes and arcs, which arcs length have numerical value. Zadeh [5] presented the parameters by fuzzy numbers. Lin and Chern [6] proposed the algorithm for the type of network flow that arc lengths have fuzzy numbers to finding the most vital arcs. In [11], Okada presented the conception of "degree of possibility" for the arcs that used in shortest path of network.

Chuang and Kung [2] proposed the new algorithm for computing the fuzzy shortest path with triangular fuzzy set on arc lengths. Sujatha and Elizabeth [12] investigated all the paths between source node and destination node and then find the fuzzy shortest path by fuzzy shortest length heuristic procedure.

Hernandes et al. [3] considered a generic algorithm by using any fuzzy numbers ranking index on the decision-maker. In [9] Mahdavi et al. first proposed ranking order between fuzzy numbers, then improved a dynamic programming

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approach for the fuzzy shortest chain problem. Amit Kumar [4] proposed a new method that decision maker obtain the fuzzy shortest path between each node and source node and use ranking function for comparing paths. We focus on shortest path between source node and destination node and only consider paths and nodes utilized in shortest path by using ranking function.

The rest of this paper is organized as follows: Sect. 2 review some basic definitions of fuzzy numbers and trapezoidal fuzzy number's arithmetic and introduce ranking function. In Sect. 3 proposed the algorithm and illustrate the method by helping an example. The conclusions are discussed in Sect. 4.

2 Preliminaries

In this section we review some basic and necessary definitions and notices.

Definition 1 [1]. *The subset $\tilde{A}$ in set X defined as $\tilde{A} = \{(\mu_{\tilde{A}}(x), x) | x \in X\}$, where $\mu_{\tilde{A}}(x)$ is a real number belong to interval $[0, 1]$. $\mu_{\tilde{A}}(x)$ is degree of membership x in $\tilde{A}$ and call*

$$\mu_{\tilde{A}} : X \to [0, 1],$$
$$x \to \mu_{\tilde{A}}(x)$$

a membership function in fuzzy set $\tilde{A}$.

Definition 2. *We denote the trapezoidal fuzzy number as $\tilde{A} = (a_1, a_2, a_3, a_4)$ and show the set of all trapezoidal fuzzy numbers with $F(R)$.*

Definition 3 [1]. *Fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be a trapezoidal fuzzy number if its membership function defined as follows*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0\ , & x < a_1, x > a_4, \\ \frac{x-a_1}{a_2-a_1}\ , & a_1 x < a_2, \\ 1\ , & a_2 x a_3, \\ \frac{a_4-x}{a_4-a_3}\ , & a_3 < x a_4. \end{cases}$$

Definition 4 [1]. *Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. The arithmetic operations properties on trapezoidal fuzzy numbers denote as follows:*

(1) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$,
(2) $c \geqslant 0, c \in R; \quad c\tilde{A} = (ca_1, ca_2, ca_3, ca_4)$,
(3) $c < 0, c \in R; \quad c\tilde{A} = (ca_4, ca_3, ca_2, ca_1)$,
(4) $\tilde{A} - \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$.

Ranking Function
The accessible method for comparing fuzzy numbers is to use ranking function (see [7,13]).

Definition 5 *(Ranking Function). We call $\mathcal{R} : F(R) \rightarrow R$ a ranking function that effects on elements of $F(R)$by natural ordering and maps this fuzzy numbers into real number in R. Suppose $\tilde{A}_1$, $\tilde{A}_2 \in F(R)$, we define the orders respect to ranking function $\mathcal{R}$ as follows:*

$$\tilde{A}_1 \overset{\mathcal{R}}{>} \tilde{A}_2 \quad iff \quad \mathcal{R}(\tilde{A}_1) > \mathcal{R}(\tilde{A}_2),$$

$$\tilde{A}_1 \overset{\mathcal{R}}{<} \tilde{A}_2 \quad iff \quad \mathcal{R}(\tilde{A}_1) < \mathcal{R}(\tilde{A}_2),$$

$$\tilde{A}_1 \overset{\mathcal{R}}{=} \tilde{A}_2 \quad iff \quad \mathcal{R}(\tilde{A}_1) = \mathcal{R}(\tilde{A}_2).$$

Note. It is obvious that $\mathcal{R}$ is linear ranking function such that $\mathcal{R}(c\tilde{A}_1 + \tilde{A}_2) = c\mathcal{R}(\tilde{A}_1) + \mathcal{R}(\tilde{A}_2)$, where $c \in R$.

Remark 1 [4,14]. Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number. Ranking function defined on $\tilde{A}$ as follows:

$$\mathcal{R}(\tilde{A}) = \frac{a_1 + a_2 + a_3 + a_4}{4}.$$

3 Process of the Algorithm and Numerical Example

Here by using the definitions and theorem mentioned in Sect. 2, we want to explain the algorithm step by step.

Suppose that $N = \{1, 2, ..., n\}$ be the set of all nodes in network and node 1 be the source node and node n be the destination node. $\tilde{P}_{ij}$ denotes the fuzzy distance between node i and node j. $Np(i)$ is the set of all nodes that have relationship with node i.

Algorithm

(1) $\tilde{P}_{11} = \tilde{0} = (0, 0, 0, 0)$ because the distance between each node and itself (loop) is zero.

(2) Put $i = 1$, find j for $\tilde{P}_{1j} = \tilde{P}_{11} + \bigwedge_{j \in Np(1)} \tilde{P}_{1j}$, then compute $\tilde{P}_{1j}$.

(3) Replace j instead of i, i.e., put $i = j$.

(4) Now find the new j by $\tilde{P}_{1j} = \tilde{P}_{1i} + \bigwedge_{j \in Np(i)} \tilde{P}_{ij}$.
Note. If minimum of d_{ij} appears for more than one value of j, it denote that there exist more than one optimal fuzzy path, so continue the algorithm by all of them. Consider that which path is minimal in next step, and then choose the best.

(5) Continue the process since j become equal to n, i.e., arrive to destination node n, so the algorithm will stops when $j = n$ appears.

Numerical Examples

Here we illustrate the algorithm by helping numerical example. The example presented in [4,8] will solving by new method and then you can compare the results that approached by Liu and Kao in [8] and Kumar and Kaur in [4] and this method.

Example
In this example we want to find the shortest path between source node and the destination node with the fuzzy distances in following graph.

Solution
The graph shows the source node is node 1 and the destination node is node 6, so $n = 6$ and for the first step $\tilde{P}_{11} = (0, 0, 0, 0)$.

At first $i = 1$, so it needs to find j by the equation

$$\tilde{P}_{1j} = \tilde{P}_{11} + \bigwedge_{j \in Np(1)} \tilde{P}_{1j} = \tilde{P}_{11} + \bigwedge_{j \in \{2,3\}} \tilde{P}_{1j} = \tilde{P}_{11} + (\tilde{P}_{12} \wedge \tilde{P}_{13})$$

$$= (0, 0, 0, 0) + ((10, 20, 20, 30) \wedge (52, 62, 65, 70)),$$

$$\mathcal{R}(10, 20, 20, 30) = \frac{10 + 20 + 20 + 30}{4} = 20,$$

$$\mathcal{R}(52, 62, 65, 70) = \frac{52 + 62 + 65 + 70}{4} = 62.25.$$

Because $\mathcal{R}(10, 20, 20, 30) < \mathcal{R}(52, 62, 65, 70)$, so

$$\tilde{P}_{1j} = (0, 0, 0, 0) + ((10, 20, 20, 30) \wedge (52, 62, 65, 70)) = (0, 0, 0, 0) + (10, 20, 20, 30) = (10, 20, 20, 30).$$

It implies that $j = 2$, therefore $\tilde{P}_{12} = (10, 20, 20, 30)$.

For the next step substitute i by $j = 2$, i.e., $i = 2$ and again it needs to computing the new j to find the shortest path of $\tilde{P}_{1j}$ by using the equation $\tilde{P}_{1j} = \tilde{P}_{1i} + \bigwedge_{j \in Np(i)} \tilde{P}_{ij}$.

$$\tilde{P}_{1j} = \tilde{P}_{12} + \bigwedge_{j \in Np(2) \tilde{P}_{2j}} = \tilde{P}_{12} + \bigwedge_{j \in \{3,5\}} \tilde{P}_{1j} = \tilde{P}_{12} + (\tilde{P}_{23} \wedge \tilde{P}_{25}) =$$

$$(10, 20, 20, 30) + ((35, 38, 40, 45) \wedge (52, 55, 60, 65)),$$

$$\mathcal{R}(35, 38, 40, 45) = \frac{35 + 38 + 40 + 45}{4} = 39.5,$$

$$\mathcal{R}(52, 55, 60, 65) = \frac{52 + 55 + 60 + 65}{4} = 58,$$

so $\tilde{P}_{1j} = (10, 20, 20, 30) + (35, 38, 40, 45) = (45, 58, 60, 75)$.

Here $j = 3$, it implies that $\tilde{P}_{13} = (45, 58, 60, 75)$.
Put $i = j = 3$, compute following equation to find new j.

$$\tilde{P}_{1j} = \tilde{P}_{13} + \bigwedge_{j \in Np(3)} \tilde{P}_{3j} = \tilde{P}_{13} + \bigwedge_{j \in \{4,5\}} \tilde{P}_{1j} = \tilde{P}_{13} + (\tilde{P}_{34} \wedge \tilde{P}_{35}) =$$

$$(45, 58, 60, 75) + ((10, 13, 17, 20) \wedge (8, 9, 9, 10)),$$

$$\mathcal{R}(10, 13, 17, 20) = \frac{10 + 13 + 17 + 20}{4} = 15,$$

$$\mathcal{R}(8, 9, 9, 10) = \frac{8 + 9 + 9 + 10}{4} = 9$$

and

$$\tilde{P}_{1j} = (45, 58, 60, 75) + (8, 9, 9, 10) = (53, 67, 69, 85).$$

So $j = 5$ and $\tilde{P}_{15} = (53, 67, 69, 85)$.

Since now we find the shortest path from the source node to node 5. By the graph, there is only one path from node 5 to node 6. To finding an optimal path we will solve the last equation as follows. $i = 5$

$$\tilde{P}_{1j} = \tilde{P}_{15} + \bigwedge_{j \in Np(5)} \tilde{P}_{5j} = \tilde{P}_{15} + \bigwedge_{j \in \{6\}} \tilde{P}_{1j} = \tilde{P}_{15} + \tilde{P}_{56}$$

$$= (53, 67, 69, 85) + (50, 70, 80, 100) = (103, 137, 149, 185).$$

The process stops because here $j = n = 6$. It shows that we find optimal path, so the algorithm finished. Therefore $\tilde{P}_{16} = (103, 137, 149, 185)$ is the value of fuzzy shortest path between source node (node 1) to destination node (node 6).

Note. Nodes can labeled by their distance from the source node and the previous node.

In this example, node 1 can labeled as $[(0, 0, 0, 0), -]$, then arrive to node 2 from node 1, i.e., $1 \to 2$ so node 2 labels as $[(10, 20, 20, 30), 1]$, it means that you come from node 1 to node 2 by value distance $(10, 20, 20, 30)$.

Then $1 \to 2 \to 3$, so node 3 can labeled as $[(45, 58, 60, 75), 2]$.

Next node is node 5, $1 \to 2 \to 3 \to 5$, so node 5 can labeled as $[(53, 67, 69, 85), 3]$. And for the destination node, the path is $1 \to 2 \to 3 \to 5 \to 6$, and node 6 can labeled as $[(103, 137, 149, 185), 5]$.

4 Conclusion

In this paper we proposed a new method to solve fuzzy shortest path network problem with trapezoidal fuzzy numbers. There exist different algorithms to find the fuzzy shortest path. In [4] the algorithm for computing fuzzy shortest path between source node and the destination node, involved all the nodes in network and it needs to find the fuzzy shortest path between source node and each other nodes. In this paper we try to improve this method that it doesn't need to involve all node to find the shortest path, only the nodes that utilized to approach destination node, are enough. As you see, the results are the same.

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Application Research of Improved Classification Recognition Algorithm Based on Causality Analysis

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Abstract. When the causality-relationship is incomplete, it's easy to have problem on sample classification. For the sake of solving this problem, this paper proposes an improved classification recognition algorithm based on causality analysis. This algorithm has improved the process of classification and recognition which is proposed in Causality Analysis in Factor Spaces [1], and it's based on the nearest-neighbor rule and maximum subordination principle. In addition, aiming at the case that can be only applied in the discrete groups in Pei-Zhuang Wang's paper, this article has transformed the continuous data into discrete data by segmentation method. Therefore, this algorithm expands on its original application into the case involving continuous data. Experimental results indicate that this improved classification recognition algorithm can successfully identify all the samples, and it also significantly improves the overall recognition rate. Simultaneously, when continuous data is centralizing, this algorithm is better than most common classification algorithms, and it can be effectively applied to image classification areas.

Keywords: Factor space · Causality analysis · Nearest-neighbor rule · Maximum subordination principle · Improved classification recognition algorithm

1 Introduction

Pei-Zhuang Wang proposed factor space theory [2] in 1982, and built other mathematical theories of knowledge representation based upon it factor space theory [3]. During the first stage of its development, factor space theory focused on concept representation and inferential decision, which has provided mathematical basis for China's artificial intelligence [4]. With the arrival of the big data era, in order to use factor space theory to solve the problems that generated by this era, Prof. Wang came up with a new dataset–factor data-bases in 2013 [5], and then proposed causality analysis in 2014. This method has been widely used in data mining, especially on data classification problems. This indicates that the development of factor space theory has entered the second stage.

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2 Propaedeutic

There are two main parts in this article: one is improving the algorithm in Causality Analysis in Factor Spaces; the other is expanding the application of this algorithm. For the following study, we introduce the relevant definitions from references and the factor relation as well as its expression.

2.1 Definition

Definition 1. In factor space theory, the factor being concerned is the resulted factor, that have impact on them are called conditional factors.

Definition 2. A state of conditional factor f_j denotes by s, and the union of all the objects of s states in f_j denotes as $[s]$. If all the objects in $[s]$ has the same results g, then $[s]$ is one determining class in f_j.

Definition 3. The union of all the determining classes of f_j is the determining region from f_j to result g.

Definition 4. The ratio of the number of factors in determining region to the total number of objects m is the determining degree from f_j to result g.

Definition 5. In the causality analysis table (Table 1), if factor $f_{j_1} = s_{j_1}$ and $f_{j_2} = s_{j_2} .. f_{j_{end}} = s_{j_{end}}$, then the resulted factor g is t ($f_{j_1}, \ldots, f_{j_{end}} \in F$). The causality-relationship between these kinds of factors are called inference sentences, and its general notation:

$$T : f_{j1}\_s_{j_2} {}^\wedge f_{j2}\_s_{j_2} {}^\wedge \ldots {}^\wedge f_{jk}\_s_{j_{end}} \to g\_t$$

2.2 Causality Analysis Table

According to factor space theory, there is causality relationship of mutual influence and mutual restriction between factors. Investigating m objects $u_1, \ldots, u_m$, then there will be a m × n table, when rows are objects, columns are conditional factors, and the last column is the resulted factor. Table 1 is an example of a causality analysis table.

Table 1. Causality analysis table

U	$F \to g$			
	f_1	$\cdots$	f_n	g
u_1		$f_j(u_i)$		$g(u_i)$
u_2				
$\cdots$				
u_m				

2.3 Causality Analysis in Factor Space

Focusing on the causality analysis table like Table 1, the calculation steps of causality analysis table in factor space can be concluded as follows:

Step 1. Set the causality analysis table of training as the training domain discourse U_{train}.

Step 2. Select the determined class from the state of all the unclassified factors $f_j \in F_{not\_finish}$, and find the determining region, then calculate the determining degree of result g in each conditional factor.

Step 3. Transform all the determining classes of conditional factors, which possesses the maximum determining degree, into inference sentence. For instance, the determining class having max determining degree in conditional factor f_j is $[s]$, all the objects in $[s]$ have an unique result t, then the inference sentence will be $T : f_j\_s \rightarrow g\_t$.

Step 4. Delete the max determining degree factors from the unclassified factor set, and group them into the classified factor set F_{finish}.

Step 5. Update the state of the unclassified factor set into the union of the state of classified factor set: the state of unclassified factor $f_p \in F_{not\_finish}$ of object u_i is $f_p(u_i) = s_p$, the state of the new classified factor $f_j \in F_{finish}$ is $f_j(u_i) = s$, then the updating state of unclassified is $f_p(u_i) = \{s_p, s\}$.

Step 6. Deleting the max determining degree conditional factor from the training domain discourse, we get one new domain discourse. If the new domain discourse is an empty set, then move onto step 7; otherwise, return to step2.

Step 7. Set the causality analysis table of the testing set as the testing domain discourse U_{test}, let $i = 1$.

Step 8. For $u_i \in U_{test}, f_{j1}(u_i) = s_{j1}, f_{j2}(u_i) = s_{j2}, \ldots, f_{j\max}(u_i) = s_{j\max}$. If the inference sentence $T_k : f_{j1}\_s_{j1}\hat{\ } f_{j2}\_s_{j2}\hat{\ } \ldots \hat{\ } f_{j\max}\_s_{j\max} \rightarrow g_k$ exists, then $g(u_i) = g_k$. Let $i = i + 1$.

Step 9. If $i > N(U_{test})$, then $N(U_{test})$ is the row number of testing domain discourse, and the algorithm comes to an end. Otherwise, return to step 8.

In summary, Step 1 to Step 6 concludes the casual rule of factors in the training set, and extract inference sentences; Step 7 to Step 9 is the classification recognition process in the testing set.

3 Improved Classification Recognition Algorithm Based on Causality Analysis

3.1 The Problems Exist in Factor Analysis Algorithm

From the naissance to maturation of new algorithms, tests need to be continuously done to evaluate and find possible problems, and then improve the algorithm to gradually perfect development. Causality analysis is a new algorithm which arose in the last two years. It's inevitable to have some problems: 1. Paper 1 only discusses the application of factor analysis algorithm in discrete data. There is lack of literature referring to the continuous data; 2. Since the process of factor analysis algorithm is conducting causal analysis on training set to get the causality rule, the testing set does the result estimation based on these causality rules. If the causality-relationship from the training set is incomplete, then some testing objects cannot be recognized.

For problem 1, this article tries to divide the continuous data into discrete intervals, so as to apply it to the factor analysis algorithm. For problem 2, Yanke Bao improved the algorithm by increasing the utilization of sample information in training set, and proposed a new knowledge mining algorithm [6]; Haitao Liu built the inference model of causality analysis by using "knowledge base and an inference engine" principle [7]. This paper attempts to start from a new perspective. It improves the factor analysis algorithm by utilizing the nearest-neighbor rule and maximum subordination principle, so where the sample cannot be recognized in the original factor analysis algorithm can be recognized here, improving the overall recognition rate.

For the sake of visually representing which conditional factors will affect the resulted factors in the inference sentence to make the algorithm expression more clear, this paper denotes the inference sentence by a 1 × (n + 1) vector:

$$T = (\ \varnothing\ \ \varnothing\ \ldots s_{j_1}\ \ldots s_{j_2}\ \ldots s_{j_{end}}\ \ldots \varnothing\ \ t\ ).$$

The $j_1^{th}, j_2^{th}, \ldots, j_{end}^{th}$ columns respectively are denoted as $s_{j_1}, s_{j_2}, \ldots, s_{j_{end}}$, representing the state of $j_1^{th}, j_2^{th}, \ldots, j_{end}^{th}$ conditional factor is $s_{j_1}, s_{j_2}, \ldots, s_{j_{end}}$, respectively; other columns are denoted as $\varnothing$, representing conditional factors in inference sentence T which have no impact on resulted factor; all the not empty factors called non-empty factors; The $(n+1)^{th}$ column is the confirmable resulted factor t in this inference sentence.

3.2 Preprocess of Continuous Data

The recognition rate of classification and recognition in factor analysis algorithm depends on the reflection degree of inference sentence set between conditional factors and resulted factors, that is, whether the inference sentence set is complete. There is generally less possibility of the state in discrete data. Few training samples may already include most of even all the possibility state, making it easier for the inference sentence set to be complete. Compared with discrete data, there are too many possible state in continuous data. It's really hard to contain all possibilities of state in the training set, leading to the inference sentence being hard to be complete, resulting in the low recognition rate of the algorithm. Meanwhile, due to a plethora of state possibility, the amount of inference sentences will increase and reduce the algorithm efficiency. Hence, transforming the continuous data set into discrete data will be beneficial to the application of causality analysis in continuous data.

For a finite continuous data set S, $\forall s_i \in S$, always has $\min S \le s_i \le \max S$. Thus, data set S can be the union of n disjoint sets with $Length = \frac{\max S - \min S}{n}$, that is $S = [\min S, \min S + Length) \cup \ldots \cup [\min S + (n-1)Length, \max S]$. Then S can be divided into n intervals: $I_1 = [\min S, \min S + Length)$, $I_2 = [\min S + Length, \min S + 2Length), \ldots, I_n = [\min S + (n-1)Length, \max S]$.

For $\forall s_i \in S$, there always exist $I_j \in I_1, \ldots, I_n$ such that $s_i \in I_j$,that is all the data from the finite continuous data set S can be divided into one of n discrete intervals. The n divided intervals reflect the n level of original data set S from small to large. The process of segment can be regarded as the map from data set S to different degree set, denoted as $f(s_i) = j, \ \ if \ s_i \in I_j$ (Fig. 1).

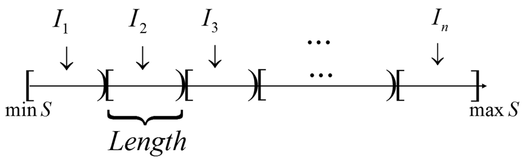

Fig. 1. Interval segmentation of continuous data

3.3 Design of Improved Classification Recognition Algorithm Based on Causality Analysis

The factor analysis algorithm in factor space depends on the complete degree of causal rule training by the training data set. If the causal rule is incomplete, it will cause some testing objects to not be recognized, and reduce the overall recognition rate of the algorithm. If we can do correct identification on the non-recognized object, it will efficiently increase the overall recognition rate of the algorithm. Learning from the concept of nearest-neighbor rule and maximum subordination principle, this paper ameliorates the classified recognition process of factor analysis algorithm in factor space, and proposes the improved classification recognition algorithm based on causality analysis, enabling all the test samples to be successfully classified and recognized.

3.3.1 Nearest-Neighbor Rule

Nearest-neighbor rule is the core principle of K nearest neighbor categorization algorithm. Its core idea is that if most of K samples closest to one sample in characteristic space belongs to one class, then this sample also belongs to this class [8]. Based on this principle, this article calculates the Euclidean distance between the testing sample and all inference sentences, and the resulting closest distance of inference sentence becomes the classified result of this testing sample. Thus, all the testing sample can be classified:

Assume the conditional factors of testing sample u_i are $f_1(u_i), f_2(u_i), \ldots, f_n(u_i)$, respectively, $\exists T_k = (\ \varnothing \ \ \varnothing \ \ldots \ s_{j_1} \ \ldots \ s_{j_2} \ \ldots \ s_{j_{end}} \ \ldots \ \varnothing \ \ t_k\ )$, the Euclidean distance between testing sample u_i and inference sentence T_k defined as:

$$D_k = \sqrt{(s_{j_1} - f_{j_1}(u_i))^2 + (s_{j_2} - f_{j_2}(u_i))^2 + \ldots + (s_{jn} - f_{jn}(u_i))^2}. \tag{1}$$

In formula (1), the empty conditional factors in inference sentence and resulted factors are not involved in the calculation. If the data is discrete, then it needs to do the data standardization in order to eliminate the dimensional impact on the calculation;

doing the segmentation on continuous data has the same effect so standardization for continuous data isn't required.

When doing classified recognition on testing sample u_i, calculate the distance D between u_i and all inference sentences T, then the resulted factor of the closest distance inference sentence becomes the testing classified result of u_i. If $D_k = 0$, then the distance between u_i and T_k will be the minimum distance, and the result of u_i is identified as t_k. This means that the testing sample is in full agreement with the state of non-empty conditional factor of T_k, and it's equivalent to the classified recognition principle of causality analysis in factor space. If $\forall D_k \in \{D\}$, $D_k \neq 0$, this indicates that u_i doesn't match with any inference sentence. This is the case that this sample cannot be recognized in the causality analysis, and according to the nearest-neighbor rule, the result of u_i matches the closest distance result of inference sentence.

3.3.2 Membership Degree and Maximum Subordination Principle

Doing classified recognition based on the nearest-neighbor rule, thought it can classify and recognize all the non-recognized samples, if u_i has non-unique nearest-neighbors, that is, if there are two or more inference sentences which coincide with the distance of u_i and are minimum, it necessary to properly classify and determine which nearest neighbor is most reliable. This paper introduces the membership degree and maximum subordination principle from fuzzy mathematics theory [9], in order to figures out the most reliable inference sentence. Membership degree is the concept that measures the degree of object belonging to one factor. If there is one object u and one factor $\underset{\sim}{A}$, then $\underset{\sim}{A}(u)$ is the membership degree from u to $\underset{\sim}{A}$. The range of membership degree is $[0, 1]$. The closer it is to 1, the higher degree of u belonging to $\underset{\sim}{A}$; the closer it is to 0,the lower the degree. If there is one object u , and factors $\underset{\sim 1}{A}, \underset{\sim 2}{A}, \ldots, \underset{\sim n}{A}$, and if $\underset{\sim i}{A} = \max\{\underset{\sim 1}{A}, \underset{\sim 2}{A}, \ldots, \underset{\sim n}{A}\}$, then u belongs to $\underset{\sim i}{A}$. This is called maximum subordination principle.

Focusing on causality analysis tables like Table 1, assume that the resulted factors are divided into p categories: $g = \{g_1, g_2, \ldots, g_p\}$, conditional factors f_j has q different states: $f_j = \{s_1, s_2, \ldots, s_q\}$, denotes the row number of $g = g_k (k \leq p)$ in the table as m_k, denotes the row number of the result is g_k and conditional factors $f_j = s_l (l \leq q)$ as $n_{k,j,l}$, then the calculation formula of membership degree is:

$$L(k, j, l) = \frac{n_{k,j,l}}{m_k}. \tag{2}$$

In formula (2), $L(k, j, l)$ represents the result of g_k, the membership degree of factor $f_j = s_l$ indicates that the degree of the state s_l from factor f_j can represent the g_k category. The higher the membership degree, the higher the degree of this sample belonging to the g_k category when $f_j = s_l$. After calculating membership degree of all categories, factors and states, we will obtain $p \times n \times q$ membership degree table L.

For sample u_i, if $\exists T_{k_1}, T_{k_2}, \ldots, T_{k_{end}}$, $D_{k_1} = D_{k_2} = \ldots = D_{k_{end}} \neq 0$ and all of them are the minimum, for any $T_k = (\ \varnothing\ \ \varnothing\ \ldots\ s_{j_1}\ \ldots\ s_{j_2}\ \ldots\ s_{j_N}\ \ldots\ \varnothing\ \ t_k\ )$ $(T_k \in \{T_{k_1}, T_{k_2}, \ldots, T_{k_{end}}\})$ having $N (N \leq n)$ non-empty factors. This defines the average membership degree L_k as the membership degree between sample u_i and inference sentence T_k, which represents the matching degree of u_i with T_k. The larger the average membership degree, the more u_i match with T_k.

In order to calculate average membership degree L_k, define function $H(v)$ to determine whether sample u_i is consistent with the state of non-empty factor in inference sentence T_k:

$$H(v) = \begin{cases} 1 & if\ s_{j_v} = f_{j_v}(u_i), \\ 0 & if\ s_{j_v} \neq f_{j_v}(u_i). \end{cases} \tag{3}$$

Then calculate the average membership degree:

$$L_k = \frac{\sum_{v=1}^{N} H(v) L(t_k, j_v, s_{j_v})}{N}. \tag{4}$$

Based on formulas (3) and (4), we can calculate that the average membership degree of $T_{k_1}, T_{k_2}, \ldots, T_{k_{end}}$ are $L_{k_1}, L_{k_2}, \ldots, L_{k_{end}}$, respectively. If $L_{k\max} = \max\{L_{k_1}, L_{k_2}, \ldots, L_{k_{end}}\}$, according to the maximum subordination principle, then sample u_i belongs to inference sentence $T_{k\max}$.

4 Result Analysis

This paper uses three data sets from UCI machine learning database to test the improved classification recognition algorithm based on causality analysis, and the overall recognition rate serves as the evaluation index to detect the effect of practical applications of this algorithm.

4.1 Dataset Description

4.1.1 The United States Wisconsin Breast Cancer Database [10, 11]

This dataset is a discrete dataset, having 699 samples. There are 9 conditional factors in this dataset and each factor includes 10 states, marked as 1–10 respectively; the resulted factors are divided into 2 classes, $g = 2$ represents benign, and $g = 4$ represents malignant. There are some abnormal state '?' existing in some factors in this dataset, after deleting the abnormal samples, there are 683 samples left, selecting the former 533 samples as the training set, and the latter 150 samples as testing set. Using the original causality analysis, 140 samples are correctly recognized, 1 sample is misidentified, 9 samples cannot be recognized, resulting in an overall recognition rate of 93.33%.

4.1.2 Image Segmentation Database [12]

This dataset has 210 training samples and 2100 testing samples. There are 9 conditional factors, and the resulted factors are divided into 7 categories: brickface, sky, foliage, cement, window, path, and grass, respectively. This dataset is continuous data, using the method of segmentation to divide each conditional factor into 4 types.

4.1.3 Letter Recognition Database [13]

This dataset has 20000 samples, selecting the first 16000 samples into the training set, and 4000 samples into the testing set. There are 16 conditional factors in this dataset. The original data of these conditional factors is continuous data and is divided into 16 categories after segmentation. The resulted factors have 26 categories, denoted by letter $A \sim Z$.

4.2 Testing Results and Analysis

The United States Wisconsin breast cancer database is a discrete dataset, and is tested by the improved classification recognition algorithm based on causality analysis. The testing result: 148 samples are correctly classified, with 2 identification sample errors, resulting in an overall recognition rate of 98.67%. This test result indicates that the improved algorithm can classify and recognize the 9 non-recognized samples, and 8 of them are classified properly with 1 misidentified, resulting in an increase of overall recognition rate by 5.34%.

Image segmentation database and letter recognition database are both continuous figure classification datasets. After segmentation, they can be tested by the improved classification recognition algorithm based on causality analysis. The test results: the overall recognition rate in image segmentation database is 86.95%, and is 77.65% in the letter recognition database. Amund Tveit used MIPSVM, C4.5, Navie Bayes, C_SVM, which are common classified algorithms to get these results. [14] Compared with classified results conducted by Amund Tveit, we can find only C4.5 has better performance in two dataset than the improved algorithm; though C-SVM has a higher overall recognition rate in letter recognition database than the improved algorithm, its performance in the image segmentation database is far beneath the improved algorithm. The overall recognition rate of MIPSVM and Navie Bayes are both smaller than the improved algorithm. The comparative result indicates that the improved classification recognition algorithm based on causality analysis has a good performance in continuous dataset, coming in the second position amongst the four common classified algorithms discussed above.

5 Conclusion

In situations where some objects cannot be recognized in the practical application of causality analysis, the paper improves the classified process based on the nearest-neighbor principle, and builds the membership degree table, making further selections among the non-unique nearest cases based on the maximum subordination principle. The result of testing the breast cancer database shows that the improved algorithm can recognize all the testing sample, and the recognition rate has significantly increased comparing to the causality analysis in factor space. Meanwhile, to apply the improved algorithm to continuous cases, this paper divides the continuous data into several disjoint intervals, and arranges the original data into the intervals, then transforms the continuous data into discrete data by the method of segmentation. This paper tests the performance of the improved algorithm by selecting the image segmentation database and letter recognition to be the testing dataset. The results indicate that the improved classification recognition algorithm based on causality analysis can effectively apply to continuous cases. Making a comparison with MIPSVM, C4.5, Navie Bayes, and C_SVM, its overall recognition rate is only inferior to C4.5, and superior to the other classified algorithms.

Though the improved classification recognition algorithm based on causality analysis has explored the continuous case, it hasn't address the mixed case. In the process of improving the classified recognition, in order to address the not-recognized problem and pursue a higher overall recognition rate, it's inevitable to increase the complexity of the algorithm. The future direction of study will be to even further improve the algorithm effectiveness and efficiency at the same time.

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On Intuitionistic Fuzzy Filters of Filteristic Soft BE-algebras

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Abstract. In this study, we define notion of intuitionistic fuzzy filters (IFF) in BE -algebras. We introduced intuitionistic fuzzy filters of filteristic soft BE-algebras and discussed related properties by means of $\in$-soft sets and q-soft sets.

Keywords: BE-algebra · Soft filters · Intuitionistic fuzzy filters · $\in$-soft sets and q-soft sets

1 Introduction

Molodtsov [1], popularized the soft sets as a new soft tool in mathematics for dealing with uncertainties. Subsequently, this theory has been applied in many research areas such as data analysis, approximate reasoning and decision-making. Maji et al. [2], proposed some useful results in soft sets theory. In 2009 Ali et al. [3], presented some new operations in soft set theory. Nowadays, research in this area is advancing rapidly with remarkable applications. Soft sets have been applied in various algebraic structures, such as in group theory, rings theory and semi-rings theory etc. In 2007 Kim and Kim [4], presented concept of BE-algebras as a generalization of a dual BCK-algebra. They examined some properties of BE-algebras by utilizing upper sets in BE-algebras. In [5], Rezaei and Saeid investigated commutative ideals of BE −algebras and see [6]. Ahn and So [7], characterized BE -algebras by using ideals. Ahn, Kim, So [8], presented the notion of fuzzy BE-algebras, and Jun, Lee, Song [9], investigated the concept of fuzzy ideals in BE-algebras. Recently, Abdullah et al. [11] presented N-structures in implicative filters of BE-algebras. Applications of soft sets in BE-algebras presented in [10]. In this work, we introduce intuitionistic fuzzy filters of filteristic soft BE-algebras using notions presnted in [13].

In this study, we define notion of intuitionistic fuzzy filters (IFF) in BE -algebras. We introduced intuitionistic fuzzy filters of filteristic soft BE-algebras and discussed related properties by means of $\in$-soft sets and q-soft sets.

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2 Preliminaries

In what follows, let X be a BE-algebra unless otherwise specified.

Definition 2.1 [4]**.** *An algebra* $(X, \star, 1)$ *of type* $(2, 0)$ *is called a BE -algebra if the following axioms holds:*

(i) $a \star a = 1$,
(ii) $a \star 1 = 1$,
(iii) $1 \star a = a$,
(iv) $a_1 \star (a_2 \star a_3) = a_2 \star (a_1 \star a_3)$, *for all* $a_1, a_2, a_3 \in X$.

Example 1. Let $X = \{1, p, q, r, s\}$ be a set with table:

$\star$	1	p	q	r	s
1	1	p	q	r	s
p	1	1	p	s	s
q	1	p	1	s	s
r	1	1	p	1	p
s	1	1	1	1	1

Clearly, $(X, \star, 1)$ is a BE-algebra.
We consider the relation, $\preceq$ on $(X, \star, 1)$ by $a \preceq b$ if and only if $a \star b = 1$.
In rest of paper, X is BE-algebra, unless else we particularized.

Definition 2.2 [7]**.** *A subset* $L \neq \emptyset$ *of a BE-algebra* X *is called an filter of* X *if it holds:*

(i) $1 \in L$,
(ii) $(y \star x) \in L$, $y \in X \implies x \in X$, $\forall x, y \in L$.

Definition 2.3. *A fuzzy set* λ *in a BE-algebra* X *is called a fuzzy filter of* X *if it satisfies for all* $x, y \in X$*:*

(1) $\lambda(1) \geq \lambda(x)$,
(2) $\lambda(x) \geq \lambda(y \star x) \wedge \lambda(y)$.

Let X *be an universe. Let* $P(X)$ *be a set of power set of* X *and* E *be non-empty subsets of* X*. Then we define following definitions:*

Definition 2.4 [1]**.** *A soft set* (S*-set*) $(\widetilde{\gamma}, E)$ *over* X *is defined as* $\widetilde{\gamma} : E \to P(X)$ *such that* $\widetilde{\gamma}(a) = \emptyset$ *if* $a \notin E$*. It can be represented by* $(\widetilde{\gamma}, E) = \{(a, \widetilde{\gamma}(a)) \mid a \in X, \widetilde{\gamma}(a) \in P(X)\}$*. The set of all soft sets over* X *is denoted by* $S(X)$*.*

Definition 2.5. *Let* $(\widetilde{\gamma}_1, X), (\widetilde{\gamma}_2, X)$ *and* $(\widetilde{\gamma}, X) \in S(X)$*. Then,*

(i) *A soft set is called null soft set if* $\widetilde{\gamma}(a) = \emptyset$*, for* $a \in X$*. It denoted by* $\widetilde{\Phi}_L$*.*
(ii) *A soft set is called whole soft set if* $\widetilde{\gamma}(a) = X$*, for* $a \in X$*. It denoted by* $\widetilde{\mu}_L$*.*
(iii) *Soft set* $\widetilde{\gamma}_1$ *is called subset of* $\widetilde{\gamma}_2$*, denoted by* $(\widetilde{\gamma}_1, X) \sqsubseteq (\widetilde{\gamma}_2, X)$ *and defined by* $\widetilde{\gamma}_1(a) \subseteq \widetilde{\gamma}_2(a)$ *for all* $a \in X$*.*
(iv) *Intersection of* $\widetilde{\gamma}_1$ *and* $\widetilde{\gamma}_2$ *denoted by* $(\widetilde{\gamma}_1, X) \sqcap (\widetilde{\gamma}_2, X)$ *and defined by* $(\widetilde{\gamma}_1 \sqcap \widetilde{\gamma}_2)(a) = \widetilde{\gamma}_1(a) \cap \widetilde{\gamma}_2(a)$ *for all* $a \in X$*.*
(v) *Union of* $\widetilde{\gamma}_1$ *and* $\widetilde{\gamma}_2$ *denoted by* $(\widetilde{\gamma}_1, X) \sqcup (\widetilde{\gamma}_2, X)$ *and defined by* $(\widetilde{\gamma}_1 \sqcup \widetilde{\gamma}_2)(a) = \widetilde{\gamma}_1(a) \cup \widetilde{\gamma}_2(a)$ *for all* $a \in X$*.*

3 Filteristic Soft *BE*-algebras

In [12], an intuitionistic fuzzy set is defined as in following form:

Definition 3.1. *An intuitionistic fuzzy subset (IFS in short) A is a non-empty set X is an object having the form* $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, *where the functions* $\mu_A : X \longrightarrow [0,1]$ *and* $\nu_A : X \longrightarrow [0,1]$ *denote the degree of membership and the degree of non membership, respectively, and* $0 \le \mu_A(x) + \nu_A(x) \le 1$ *for all* $x \in X$.

Definition 3.2. *An intuitionistic fuzzy set A in a BE-algebra X is called an intuitionistic fuzzy filter of X* ($IFF(X)$) *if it satisfies:*

(IFF$_1$) $\mu_A(1) \ge \mu_A(x), \nu_A(1) \le \nu_A(x)$ $\forall x, y \in X$,
(IFF$_2$) $\mu_A(x) \ge \min\{\mu_A(y \star x), \mu_A(y)\}, \nu_A(x) \le \max\{\nu_A(y \star x), \nu_A(y)\}$ $\forall x, y \in X$.

Definition 3.3. *Let x be a fixed point of a non-empty set X. If* $t \in (0,1]$ *and* $s \in [0,1)$ *be two real numbers such that* $0 \le s+t \le 1$ *then the intuitionistic fuzzy set of the form* $\langle x, (t,s)\rangle = \langle x, x_t, x_{1-s}\rangle$, *is called an intuitionistic fuzzy point. We shall use the notation* $x_{(t,s)}$ *instead of* $\langle x, x_t, x_{1-s}\rangle$. *For an IFP* $x_{(t,s)}$ *in X and an IFS* $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, *we define* $x_{(t,s)} \in A$ *as follows:*

$x_{(t,s)} \in A$ *resp.* $x_{(t,s)}qA$ *means that* $\mu_A(x) \ge t$ *and* $\nu_A(x) \le s$ *(resp.* $\mu_A(x) + t > 1$ *and* $\nu_A(x) + s < 1$ *and in this case we say that* $x_{(t,s)}$ *belong to (resp. quasi-coinsident with) an intuitionistic fuzzy set* $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$.

Definition 3.4. *Let* $(\widetilde{\gamma}, E)$ *be a soft set over X, then* $(\widetilde{\gamma}, E)$ *is called a filteristic soft BE-algebra over X if* $\widetilde{\gamma}(x)$ *is a filter of X for all* $x \in E$.

For an intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ *of X and* $E = E_1 \cup E_2$ *where* $E_1 = (0,1]$ *and* $E_2 = [0,1)$ *consider two set valued function* $\widetilde{\gamma} : E \longrightarrow S(X)$, $(t,s) \longmapsto \{x \in X \mid x_{(t,s)} \in A\} = \{x \in X \mid \mu_A(x) \ge t, \nu_A(x) \le s\}$. *Then* $(\widetilde{\gamma}, E)$ *is called* $\in$*-soft set.*

$\widetilde{\gamma}_q : E \longrightarrow S(X)$, $(t,s) \longmapsto \{x \in X \mid x_{(t,s)}qA\} = \{x \in X \mid \mu_A(x) + t \ge 1, \nu_A(x) + s \le 1\}$. *Then* $(\widetilde{\gamma}_q, E)$ *is called q-soft set.*

Theorem 3.1. *Let* $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ *be a IFS of X and* $(\widetilde{\gamma}, E)$, *be an* $\in$*-soft set over X with* $E = E_1 \cup E_2$ *where* $E_1 = (0,1]$ *and* $E_2 = [0,1)$. *Then the following are equivalent:*

(i) $(\widetilde{\gamma}, E)$ *is a filteristic soft BE-algebra over X,*
(ii) *A is an intuitionistic fuzzy filter of X.*

Proof. (i) $\Longrightarrow$(ii). Assume that $(\widetilde{\gamma}, E)$ is a filteristic soft BE-algebra over X, then $\widetilde{\gamma}(t,s)$ is a filter of X for all $t \in (0,1]$ and $s \in [1,0)$. If there exist some $x, y \in X$ such that $\mu_A(1) < \mu_A(x)$ and $\nu_A(1) > \nu_A(x)$, then for some $t \in (0,1]$ and $s \in [1,0)$ we have $\mu_A(1) < t \le \mu_A(x)$ and $\nu_A(1) > s \ge \nu_A(x)$. Thus $x \in \widetilde{\gamma}(t,s)$ but $1 \overline{\in} \widetilde{\gamma}(t,s)$, which is contradiction, therefore $\mu_A(1) \ge t$ and $\nu_A(1) \le s$. Now let $x, y \in X$ such that $\mu_A(x) < \min\{\mu_A(y \star x), \mu_A(y)\}$ and

$\nu_A(x) > \max\{\nu_A(y \star x), \nu_A(x)\}$, then for some $t \in (0,1]$ and $s \in [1,0)$ we have $\mu_A(x) < t \leq \min\{\mu_A(y \star x), \mu_A(y)\}$ and $\nu_A(x) > s > \max\{\nu_A(y \star x), \nu_A(x)\}$. Thus $y \star x \in \widetilde{\gamma}(t,s), y \in \widetilde{\gamma}(t,s)$ but $x \overline{\in} \widetilde{\gamma}(t,s)$, which is contradiction, therefore $\mu_A(x) \geq t$ and $\nu_A(x) \leq s$. Hence A is IFF of X.

(ii) $\Longrightarrow$(i). Assume that A is an intuitionistic fuzzy filter of X. Let $x \in X$, $t \in (0,1]$ and $s \in [1,0)$ such that $x \in \widetilde{\gamma}(t,s)$ that is $x_{(t,s)} \in A$ or $\mu_A(x) \geq t$ and $\nu_A(x) \leq s$. Since A is an intuitionistic fuzzy filter of X, so we have $\mu_A(1) \geq \mu_A(x) \geq t$ and $\nu_A(1) \leq \nu_A(x) \leq s$. Then $\mu_A(1) \geq t$ and $\nu_A(1) \leq s$, it follows that, $1 \in \widetilde{\gamma}(t,s)$. Now, let $y \in \widetilde{\gamma}(t,s)$ and $y \star x \in \widetilde{\gamma}(t,s)$ that is $\mu_A(y) \geq t$, $\nu_A(y) \leq s$ and $\mu_A(y \star x) \geq t$, $\nu_A(y \star x) \leq s$. Since A is an intuitionistic fuzzy filter of X, so we have $\mu_A(x) \geq \min\{\mu_A(y \star x), \mu_A(y)\} \geq t$ and $\nu_A(x) \leq \max\{\nu_A(y \star x), \nu_A(y)\} \leq s$. Then $\mu_A(x) \geq t$ and $\nu_A(x) \leq s$, it follows that, $x \in \widetilde{\gamma}(t,s)$.

Theorem 3.2. *Let $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ be a IFS of X and $(\widetilde{\gamma}, E)$ a q-soft set over X with $E = E_1 \cup E_2$, where $E_1 = (0,1]$ and $E_2 = [0,1)$. Then the following are equivalent:*

(i) *A is an intuitionistic fuzzy filter of X,*
(ii) *(For all $t \in E_1$ and $s \in E_2$) $(\widetilde{\gamma}_q(t,s) \neq \emptyset \Longrightarrow \widetilde{\gamma}_q(t,s)$ filter of X.*

Proof. (i) $\Longrightarrow$(ii). Assume that A is an intuitionistic fuzzy filter of X. Let $x \in X$, $t \in (0,1]$ and $s \in [1,0)$ such that $x \in \widetilde{\gamma}_q(t,s)$ that is $x_{(t,s)} qA$ or $\mu_A(x) + t \geq 1$ and $\nu_A(x) + s \leq 1$. Since A is an intuitionistic fuzzy filter of X, so we have $\mu_A(1) + t \geq \mu_A(x) + t \geq 1$ and $\nu_A(1) + s \leq \nu_A(x) + s \leq 1$. Then $\mu_A(1) + t \geq 1$ and $\nu_A(1) + s \leq 1$, it follows that, $1 \in \widetilde{\gamma}_q(t,s)$. Now, let $y \in \widetilde{\gamma}_q(t,s)$ and $y \star x \in \widetilde{\gamma}_q(t,s)$ that is $\mu_A(y) + t \geq 1$, $\nu_A(y) + s \leq 1$ and $\mu_A(y \star x) + t \geq 1$, $\nu_A(y \star x) + s \leq 1$. Since A is an intuitionistic fuzzy filter of X, so we have $\mu_A(x) + t \geq \min\{\mu_A(y \star x), \mu_A(y)\} + t \geq 1$ and $\nu_A(x) + s \leq \max\{\nu_A(y \star x), \nu_A(y)\} + s \leq 1$. Then $\mu_A(x) + t \geq 1$ and $\nu_A(x) + s \leq 1$, it follows that, $x \in \widetilde{\gamma}_q(t,s)$.

(ii) $\Longrightarrow$(i). Assume that $\widetilde{\gamma}_q(t,s) \neq \emptyset$ is a filter of X for all $t \in (0,1]$ and $s \in [1,0)$. If there exist some $x, y \in X$ such that $\mu_A(1) < \mu_A(x)$ and $\nu_A(1) > \nu_A(x)$, then for some $t \in (0,1]$ and $s \in [1,0)$ we have $\mu_A(1) + t < 1 \leq \mu_A(x) + t$ and $\nu_A(1) + t > 1 \geq \nu_A(x) + s$. Thus $x_{(t,s)} \overline{q} A$ but$1_{(t,s)} qA$, which is contradiction, therefore $\mu_A(1) + t \geq 1$ and $\nu_A(1) + s \leq 1$ and $\mu_A(1) \geq \mu_A(x)$ and $\nu_A(1) \leq \nu_A(x)$. Now let $x, y \in X$ such that $\mu_A(x) < \min\{\mu_A(y \star x), \mu_A(y)\}$ and $\nu_A(x) > \max\{\nu_A(y \star x), \nu_A(x)\}$, then for some $t \in (0,1]$ and $s \in [1,0)$ we have $\mu_A(x) + t < 1 \leq \min\{\mu_A(y \star x), \mu_A(y)\} + t$ and $\nu_A(x) + s > 1 > \max\{\nu_A(y \star x), \nu_A(x)\} + s$. Thus $y \star x \in \widetilde{\gamma}_q(t,s), y \in \widetilde{\gamma}_q(t,s)$ but $x \overline{\in} \widetilde{\gamma}_q(t,s)$, which is contradiction, therefore $\mu_A(x) + t \geq 1$ and $\nu_A(x) + s \leq 1$. Hence A is IFF of X.

Example 2. Consider BE-algebra $X = \{1, r_1, r_2\}$ defined as follow:

$\star$	1	r_1	r_2	r_3
1	1	r_1	r_2	r_3
r_1	1	1	r_2	r_2
r_2	1	r_1	1	r_1
r_3	1	1	1	1

We define IFS $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ in X as follows:

$$\mu_A(x) = \begin{cases} 0.7, & x = 1, \\ 0.5, & x = r_1, \\ 0.3, & x = r_2, r_3, \end{cases} \qquad \nu_A(x) = \begin{cases} 0, & x = 1, \\ 0, & x = r_1, \\ 0.6, & x = r_2, r_3. \end{cases}$$

Let $(\widetilde{\gamma}_q, E)$ a q-soft set over X with $E = E_1 \cup E_2$ where $E_1 = (0, 1]$ and $E_2 = [0, 1)$. And $\widetilde{\gamma}_q(t, s) = \{x \in X \mid \mu_A(x) + t \geq 1, \nu_A(x) + s \leq 1\} = \{1, r_1\}$. One can easily check that $A \in IFF(X) \iff \widetilde{\gamma}_q(t, s)$ is a filter of X.

4 Conclusion

In above study, we investigated IFF-filters in BE-algebras and discussed some basic results of IFF-filters. Our approach provides a new insights into BE-algebras using properties of the $\in$-soft sets and q-soft sets. Hopefully, these notions and essential results may lead to significant and new results in related fields.

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Recommender: Academic Conference on 30th anniversary of fuzzy geometric programming and 40th education year by and of Professor Cao Bingyuan.

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Part IV:
Fuzzy Systems and Operations Research and Management

Extremal Graphs of Chemical Trees with Minimal Atom-Bond Connectivity Index

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Abstract. The atom-bond connectivity (ABC) index is a degree-based molecular structure descriptor that found chemical applications. A chemical tree is a tree whose maximum degree is no greater than 4. The lower bound of ABC index for chemical trees has been obtained by Furtula et al. [Discrete Applied Mathematics, 157(2009)2828-2835]. In this paper, we sharpened the lower bound of atom-bond connectivity index of chemical trees of some particular cases. In addition, a way to obtain extremal graphs is obtained, which shows the complexity of the structure of extremal graphs.

Keywords: Chemical trees · Atom-bond connectivity index · Extremal graphs · Minimal values

1 Introduction

Topological molecular index, which is widely studied in QSPR/QSAR of chemical theory, is one of the most active research fields in modern chemical graph theory [1].

If G = (V,E) is a molecular graph, and $\mathbf{d_u}$, $\mathbf{d_v}$ are the degree of the its terminal vertices **u** and **v** of edge **uv**, then the atom-bond connectivity (ABC) index of G is defined by $\mathbf{ABC(G)} = \sum_{\mathbf{uv \in E(T)}} \sqrt{\frac{\mathbf{d_u + d_v - 2}}{\mathbf{d_u d_v}}}$. This molecular structure descriptor, introduced by Estrada et al. in 1998, can serve as a descriptor of thermodynamic stability of acyclic saturated hydrocarbons and the strain energy of their cyclic congeners.

In 2009, it has been proven by Furtula et al. [2] that, among all trees, the star tree $\mathbf{S_n}$ hasthe maximal ABC-index. But, finding the minimal ABC values of trees and its graphs remain to be a difficult problem. Gutman et al. [3] have proven some structural features of the trees with minimal ABC-index. As a particular case, chemical trees also have the same structural features.

Lemma 1.1 [3]. If n $\geq$ 10, then the n-vertex tree with minimal ABC-index does not contain internal paths of length k $\geq$ 2.

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Lemma 1.2 [4]. If n $\geq$ 10 and T is an n-vertex tree with minimal ABC-index, then each pendent path of T is of length 2or 3, and T contains at most one pendent path of length 3.

Lemma 1.3 [5]. Let T be a tree with minimal ABC-index. For every positive integer d the vertices with degrees at least d induce a subtree of T.

Boris Furtula et al. [2] reached a conclusion about value ranges of ABC-index of chemical trees.

Lemma 1.4 [2]. Let T be a chemical tree with n vertices. Then,

$$\left.\begin{cases} 0, & n \leq 2 \\ \frac{1}{\sqrt{2}}(n-1), & 3 \leq n \leq 9 \\ \frac{8}{\sqrt{2}} + \frac{2}{3}, & n = 10 \\ \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132}n + \frac{-63\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132}, & n > 10 \end{cases}\right\} \leq ABC(T) \leq \frac{n+3}{4}\sqrt{3} + \frac{n-5}{4}\sqrt{2}.$$

At the same time, extremal graph $\mathbf{T}_1$ (see Fig. 1) which reaches the lower bound is given. This type of extremal graph meets the situation that n = 11 k + 10 (k $\geq$ 1). Its ABC index is $\mathbf{ABC}(\mathbf{T}_1) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132}\mathbf{n} + \frac{-63\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132}$.

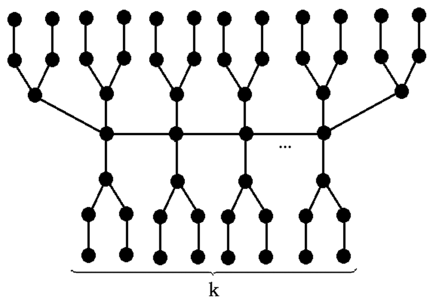

Fig. 1. Chemical tree T_1 (n = 11 k + 10)

Based on the theory of Furtula et al., this paper improved the lower bond of Lemma 1.4, found out the minimal ABC index of n = 11 k + x (x = 0, 1, 2,…, 9) and a way to construct various kinds of extremal graphs with a same minimal ABC index. The conclusion of paper [2] is a special case for x = 10.

2 Minimal ABC Index of Chemical Tree

Definition 2.1. Chemical tree T_n with minimal ABC index has vertex of degree 4, called basic chemical tree.

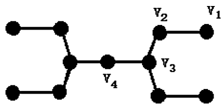

Fig. 2. Chemical tree T_2

Lemma 2.1. If basic chemical tree T_n contains chemical tree T_2 (see Fig. 2) for $n > 16$, then it doesn't contain structure B_1 (see Fig. 3).

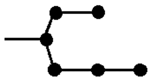

Fig. 3. Structure B_1

Proof. We proof it by way of contradiction for two cases.

Case 1: If both T_2 and B_1 exist and are adjacent, it can lead to structure B_2 (see Fig. 4). Its ABC index is $ABC(B_2) = \frac{\sqrt{15}+13\sqrt{2}}{2}$.

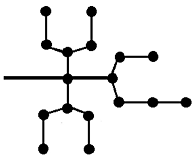

Fig. 4. Structure B_2

But B_2 can be transformed into structure B_3 (see Fig. 5) whose ABC index is $ABC(B_3) = \frac{\sqrt{6}+14\sqrt{2}}{2}$. Obviously, ABC index of B_3 is smaller than B_2.

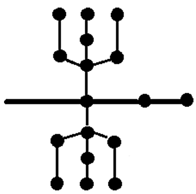

Fig. 5. Structure B_3

Case 2: If both T_2 and B_1 exist but not adjacent, it can lead to structure B_4 (see Fig. 6) where circles represent trees or forests.

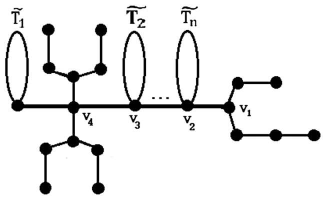

Fig. 6. Structure B_4

From Lemma 1.3, V_2 of B_4 must be a vertex of degree 4 or degree 3. Hence, we have to discuss the following two cases:

Case 2.1: V_2 is a vertex of degree 3.

The ABC index of B_4 is $ABC(B_4) = K + \frac{4+2\sqrt{15}+39\sqrt{2}}{6}.\frac{4+2\sqrt{15}+39\sqrt{2}}{6}$ where K is the sum of ABC index of edges in B_4 which is neither V_1V_2 nor in B_1 or T_2.

But B_4 can be transformed into structure B_5 (See Fig. 7) whose ABC index is $ABC(B_5) = K + \frac{\sqrt{6}+14\sqrt{2}}{2}$ which is obviously smaller than B_4. A contradiction is therefore obtained.

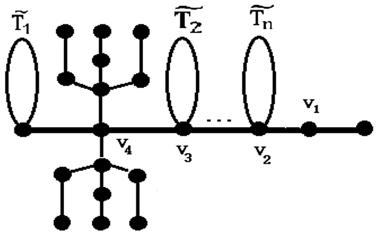

Fig. 7. Structure B_5

Case 2.2: V_2 is a vertex of degree 4.

The ABC index of B_4 is $ABC(B_4) = K + \frac{\sqrt{15}+13\sqrt{2}}{2}$. But the ABC index of B_5 is $ABC(B_5) = K + \frac{\sqrt{6}+14\sqrt{2}}{2}$. Hence, ABC index of B_5 is smaller than B_4. Again, a contradiction.

In summary, if T_n contains chemical tree T_2, T_n will not contain structure B_1.

As for n = 11 k + 10, when k = 1, T_3 (see Fig. 8) obviously is the extremal graph of chemical tree with minimal ABC index, which is called the basic chemical tree of T_1. The ABC index of T_3 is $ABC(T_3) = \frac{4\sqrt{15}+48\sqrt{2}}{6}$.

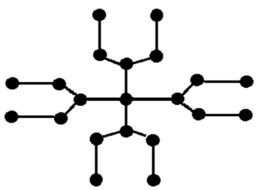

Fig. 8. Chemical tree T_3

When k > 1, T_1 is a type of extremal graphs of chemical trees with minimal ABC index. When k increases by 1(while n increases by 11 at the same time), we split an edges in T_3 which is adjacent to a pair of vertices with degree 4 and degree 3 respectively with a new vertex v, and attach T_2 to the splitted graph by identifying v and V_4. Thus,

$$\begin{aligned}ABC(T_{11k+10}) &= \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-1)+\frac{4\sqrt{15}+48\sqrt{2}}{6} \\ &= \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}k+\frac{-3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12} \\ &= \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132}n+\frac{-63\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132}.\end{aligned}$$

From 1.4, the ABC index of T_1 consists of the ABC index of chemical tree $T_3 ABC(T_3) = \frac{4\sqrt{15}+48\sqrt{2}}{6}$ and the ABC index of $T_2' ABC(T_2') = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}$ which is the sum of the ABC index of newly attached k (k > 1) chemical tree T_2.

Combining Lemma 1.4, we obtain the extremal graphs of chemical trees with minimal ABC index and its ABC index formula below.

Proposition 2.1. Assuming n = 11 k (k is a positive integer), then

(1) For k = 1, T_4 (see Fig. 9) is the chemical trees with minimal ABC index. Its ABC index is $ABC(T_4) = \frac{4+27\sqrt{2}}{6}$.
(2) For k $\geq$ 2, the minimal ABC index is given by the following formula

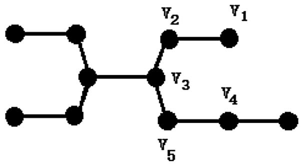

Fig. 9. Chemical tree T_4 (n = 11)

$$ABC(T_{11k}) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-2)+\frac{3\sqrt{6}+\sqrt{15}+54\sqrt{2}}{6} \quad (1)$$

T_5 (see Fig. 10) is a type of extremal graphs of chemical trees with minimal ABC index.

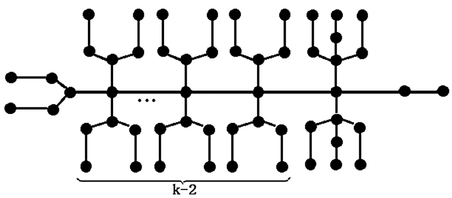

Fig. 10. Chemical tree T_5 (n = 11 k,k > 1)

Proof. When k = 1, we list all possible graphs out by using enumeration method. Restrained by Lemmas 1.1, 1.2 and 1.3, T_3 is the only one chemical tree with ABC index.

In a similar way, when k = 2, T_6 (see Fig. 11), which is called basic chemical tree of T_5, is the chemical tree with minimal ABC index. Its ABC index is $ABC(T_6) = \frac{3\sqrt{6}+\sqrt{15}+54\sqrt{2}}{6}$.

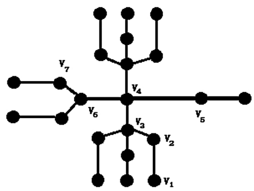

Fig. 11. Chemical tree T_6 (n = 22)

We prove the following by way of contradiction.

When k $\geq$ 2, we assume that formula (1) does not hold. That is to say, there exists a chemical tree $\tilde{T}_5$ which is not isomorphic to T_5 but has the minimal ABC index such that $ABC(\tilde{T}_5) = L + \frac{3\sqrt{6}+\sqrt{15}+54\sqrt{2}}{6}$ where $L < \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}$.

Hence, we can obtain another chemical tree $\tilde{T}_1$ from T_2 by using the same way. We constructed chemical tree T_1. And $ABC(\tilde{T}_1) = L + \frac{4\sqrt{15}+48\sqrt{2}}{6}$ < ABC (T_1).

This is a contradiction to Lemma 1.4. So formula (1) must hold.

The following Propositions 2.2–2.10 can be proved by a similar way.

Proposition 2.2. Suppose n = 11 k + 1 (k is a positive integer), for k $\geq$ 1, the minimal ABC index of T_{11k+1} can be calculated by the following formula

$$ABC(T_{11k+1}) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-1) + \frac{\sqrt{15}+30\sqrt{2}}{6}$$

T_7 (see Fig. 12) is a kind of extremal graphs of chemical trees with minimal ABC index.

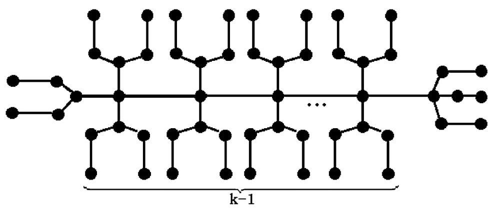

Fig. 12. Chemical tree T_7 (n = 11 k + 1)

Proposition 2.3. Suppose n = 11 k + 2 (k is a positive integer), for k $\geq$ 1, the minimal ABC index of T_{11k+2} can be calculated by the following formula

$$ABC(T_{11k+2}) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-1)+\frac{4+15\sqrt{2}}{3}$$

T_8 (see Fig. 13) is a kind of extremal graphs of chemical trees with minimal ABC index.

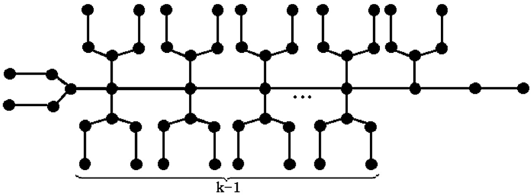

Fig. 13. Chemical tree T_8 (n = 11 k + 2)

Proposition 2.4. Suppose n = 11 k + 3 (k is a positive integer), for k $\geq$ 1, the minimal ABC index of T_{11k+3} can be calculated by the following formula

$$ABC(T_{11k+3}) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-1)+\frac{\sqrt{6}+24\sqrt{2}}{4}$$

T_9 (see Fig. 14) is a kind of extremal graphs of chemical trees with minimal ABC index.

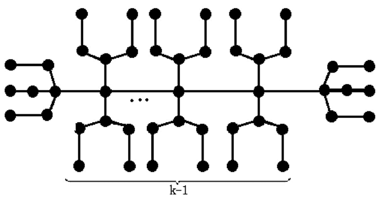

Fig. 14. Chemical tree T_9 (n = 11 k + 3)

Proposition 2.5. Suppose n = 11 k + 4 (k is a positive integer), for k $\geq$ 1, the minimal ABC index of T_{11k+4} can be calculated by the following formula

$$ABC(T_{11k+4}) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-1)+\frac{\sqrt{15}+18\sqrt{2}}{3}$$

T_{10} (see Fig. 15) is a kind of extremal graphs of chemical trees with minimal ABC index.

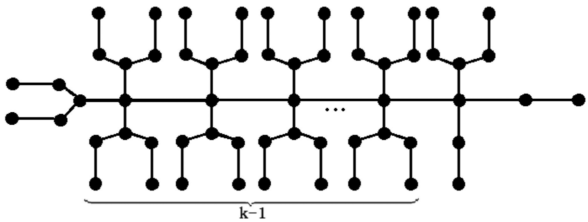

Fig. 15. Chemical tree T_{10} (n = 11 k + 4)

Proposition 2.6. Suppose n = 11 k + 5 (k is a positive integer), then

(1) When k = 1, T_{11} (see Fig. 16) is the chemical trees with minimal ABC index. Its ABC index is

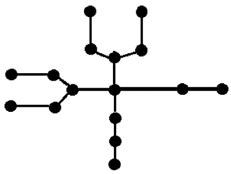

Fig. 16. Chemical tree T_{11} (n = 16)

$$ABC(T_{11}) = \frac{\sqrt{15} + 39\sqrt{2}}{6}$$

(2) When k $\geq$ 2, the minimal ABC index of T_{11k+5} can be calculated by the following formula

$$ABC(T_{11k+5}) = \frac{3\sqrt{6} + 4\sqrt{15} + 48\sqrt{2}}{12}(k - 2) + \frac{9\sqrt{6} + 2\sqrt{15} + 132\sqrt{2}}{12}$$

T_{12} (see Fig. 17) is a kind of extremal graphs of chemical trees with minimal ABC index.

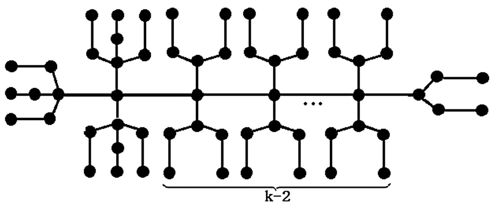

Fig. 17. Chemical tree T_{12} (n = 11 k + 5,k > 1)

Proposition 2.7. Suppose n = 11 k + 6 (k is a positive integer), for $k \geq 1$, the minimal ABC index of T_{11k+6} can be calculated by the following formula

$$ABC(T_{11k+6}) = \frac{3\sqrt{6} + 4\sqrt{15} + 48\sqrt{2}}{12}(k-1) + \frac{3\sqrt{6} + 2\sqrt{15} + 84\sqrt{2}}{12}$$

T_{13} (see Fig. 18) is a kind of extremal graphs of chemical trees with minimal ABC index.

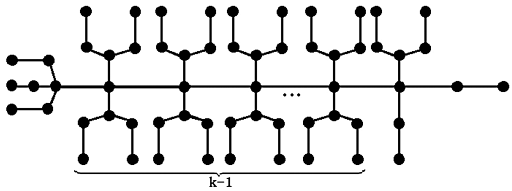

Fig. 18. Chemical tree T_{13} (n = 11 k + 6)

Proposition 2.8. Suppose n = 11 k + 7 (k is a positive integer), for $k \geq 1$, the minimal ABC index of T_{11k+7} can be calculated by the following formula

$$ABC(T_{11k+7}) = \frac{3\sqrt{6} + 4\sqrt{15} + 48\sqrt{2}}{12}(k-1) + \frac{9\sqrt{6} + 2\sqrt{15} + 84\sqrt{2}}{12}$$

T_{14} (see Fig. 19) is a kind of extremal graphs of chemical trees with minimal ABC index.

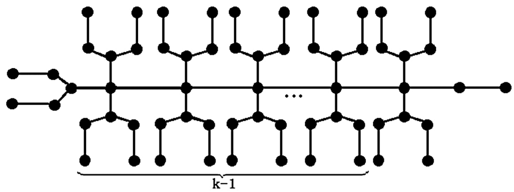

Fig. 19. Chemical tree T_{14} (n = 11 k + 7)

Proposition 2.9. Suppose n = 11 k + 8 (k is a positive integer), for $k \geq 1$, the minimal ABC index of T_{11k+8} can be calculated by the following formula

$$ABC(T_{11k+8}) = \frac{3\sqrt{6} + 4\sqrt{15} + 48\sqrt{2}}{12}(k-1) + \frac{\sqrt{6} + 16\sqrt{2}}{2}$$

T_{15} (see Fig. 20) is a kind of extremal graphs of chemical trees with minimal ABC index.

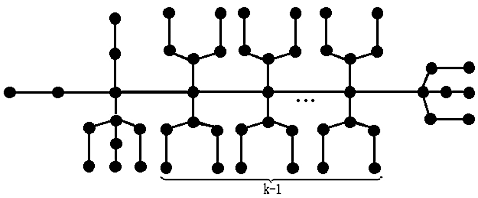

Fig. 20. Chemical tree T_{15} (n = 11 k + 8)

Proposition 2.10. Suppose n = 11 k + 9 (k is a positive integer), for k $\geq$ 1, the minimal ABC index of T_{11k+9} can be calculated by the following formula

$$ABC(T_{11k+9}) = \frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(k-1) + \frac{3\sqrt{6}+4\sqrt{15}+96\sqrt{2}}{12}$$

T_{16} (see Fig. 21) is a kind of extremal graphs of chemical trees with minimal ABC index.

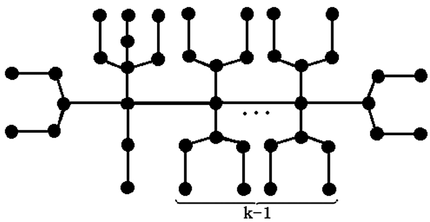

Fig. 21. Chemical tree T_{16} (n = 11 k + 9)

Based on Lemma 1.4 and combining Propositions 2.1–2.10, we can obtain the following theorem (Fig. 22).

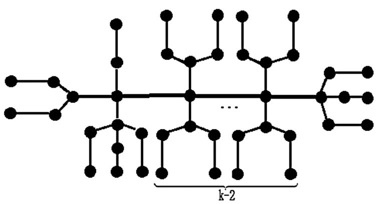

Fig. 22. Chemical tree T_{17} (n = 11 k, k > 1)

Theorem 2.1. Suppose T_n is an n-vertex chemical tree, then

$$\left\{\begin{array}{ll}
0, & n \le 2 \\
\frac{1}{\sqrt{2}}(n-1), & 3 \le n \le 9 \\
\frac{8}{\sqrt{2}} + \frac{2}{3}, & n = 10 \\
\frac{9}{\sqrt{2}} + \frac{2}{3}, & n = 11 \\
\frac{11}{\sqrt{2}} + \frac{\sqrt{15}}{3}, & n = 16 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-2) + \frac{3\sqrt{6}+\sqrt{15}+54\sqrt{2}}{6}, & n = 11k (k > 1) \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{\sqrt{15}+30\sqrt{2}}{6}, & n = 11k+1 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{4+15\sqrt{2}}{3}, & n = 11k+2 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{\sqrt{6}+24\sqrt{2}}{4}, & n = 11k+3 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{\sqrt{15}+18\sqrt{2}}{3}, & n = 11k+4 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-2) + \frac{9\sqrt{6}+2\sqrt{15}+132\sqrt{2}}{12}, & n = 11\mathrm{k}+5 (k > 1) \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{3\sqrt{6}+2\sqrt{15}+84\sqrt{2}}{12}, & n = 11k+6 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{9\sqrt{6}+2\sqrt{15}+84\sqrt{2}}{12}, & n = 11k+7 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) + \frac{\sqrt{6}+16\sqrt{2}}{2}, & n = 11k+8 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{12}(\mathrm{k}-1) \frac{3\sqrt{6}+4\sqrt{15}+96\sqrt{2}}{12}, & n = 11k+9 \\
\frac{3\sqrt{6}+4\sqrt{15}+48\sqrt{2}}{132} n + \frac{-63\sqrt{6}+4\sqrt{15}+46\sqrt{2}}{132}, & n = 11k+10
\end{array}\right\} \le ABC(T_n) \le \frac{n+3}{4}\sqrt{3} + \frac{n-5}{4}\sqrt{2}.(k \ge 1)$$

3 The Diversity of Extremal Graphs of Chemical Trees with Minimal ABC Index

From Proposition 2.1, for n = 11 k, T_5 is a type of extremal graphs of chemical trees with minimal ABC index, when k > 1. We prove the diversity of extremal graphs (Fig. 23).

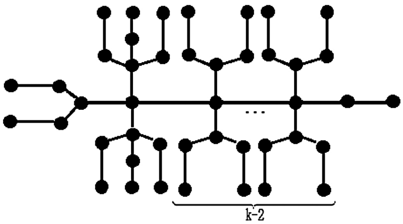

Fig. 23. Chemical tree T_{18} (n = 11 k, k > 1)

When k increases by m, we attach m T_2 to T_6.

Let $a \sim b$ be an edge which is adjacent to a pair of vertices with degree a and degree b respectively. So ABC $(a \sim b) = \sqrt{\frac{a+b-2}{ab}}$.

Just an observation, we know that there are 5 places despite of isomorphism that T_2 can be inserted into. These places are $4 \sim 4$, $4 \sim 3$, $4 \sim 2$, $3 \sim 2$, $2 \sim 1$.

If chemical tree T_2 is inserted to a ∼ b and V_4 is the attaching point, then it will destroy an a ∼ b edge and create an a ∼ 4 edge and a 4 ∼ b edge. At the same time, the structure of T_2 is changed. V_4 will become a vertex of degree 4 rather than degree 2. So we need to consider the changing of the ABC index of a ∼ 4 edges and 4 ∼ b edges during the process.

① If a = b = 4, then a ∼ 4 and 4 ∼ b become 4 ∼ 4 and 4 ∼ 4 respectively. That is to say, it increases the ABC index by $\sqrt{\frac{4+4-2}{4*4}}$.

② If a = 4 and b = 3, then a ∼ 4 and 4 ∼ b become 4 ∼ 4 and 4 ∼ 3 respectively. That is to say, it increases the ABC index by $\sqrt{\frac{4+4-2}{4*4}}$.

③ If a = 4 and b = 2, then a ∼ 4 and 4 ∼ b become 4 ∼ 4 and 4 ∼ 2 respectively. That is to say, it increases the ABC index by $\sqrt{\frac{4+4-2}{4*4}}$.

④ If a = 3 and b = 2, then a ∼ 4 and 4 ∼ b become 3 ∼ 4 and 4 ∼ 2 respectively. That is to say, it increases the ABC index by $\sqrt{\frac{3+4-2}{3*4}}$.

⑤ If a = 2 and b = 1, then a ∼ 4 and 4 ∼ b become 2 ∼ 4 and 4 ∼ 1 respectively. That is to say, it increases the ABC index by $\sqrt{\frac{4+1-2}{4*1}}$.

Because ABC (4 ∼ 4) < ABC (4 ∼ 3) < ABC (4 ∼ 1) < ABC (2 ∼ x) = $\frac{\sqrt{2}}{2}$ (x = 1,2,3,4), cases ① ② ③ are more desired than ④ ⑤.

If we only consider case ①, it will get the following graph.

If we only consider case ②, it will get chemical tree T_5 (see Fig. 10).

If we only consider case ③, it will get the following graph.

Extremal graph T_5 is obtained from chemical tree T_6 by inserting k–2 T_2 as in case ②. We can approach different graphs which are not isomorphic by taking different inserting strategy as shown in different cases. In addition, all of them meet the characteristics of chemical trees with minimal ABC index and their ABC indices are in fact minimized. In another word, ABC (T_{18}) = ABC (T_5) = ABC (T_{19}). We can also take multiple inserting strategies picked from those different cases we have shown which will create even more desired extremal graphs. We are not going to linger around more details here.

Thus, for n = 11 k + x (x = 0,1,2,3,4,5,6,7,8,9), there are many kinds of extremal graphs with minimal ABC index.

4 Conclusion

The problem of finding chemical trees with minimal ABC index is solved. This paper provided more accurate lower bond of ABC index of chemical trees and sorted out the extremal graphs of chemical trees with minimal ABC index and showed its diversity. But, finding the minimal ABC index of trees remains an open and attractive problem.

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Recommender: Muhuo Liu, South China Agricultural University, Associate Professor.

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A Soft Approach to Evaluate the Customer Satisfaction in E-retailing

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Abstract. Online shopping behavior has received much attention. This paper investigates the customer satisfaction in electronic retailing. The satisfaction is surveyed by questionnaires and the evaluation criteria are measured by using some simple symbols. The symbols in questionnaires are collected and aggregated into an intuitionistic fuzzy information in a group decision-making environment. The examined online retail companies are ranked by using an extended TOPSIS (technique for order preference by similarity to ideal solution) technique.

Keywords: Customer satisfaction · Electronic retailing · Group decision-making · Intuitionistic fuzzy information · Symbol information

1 Introduction

Electronic commerce, commonly known as e-commerce, is trading in products or services using computer networks, for example, the Internet. E-commerce is generally considered to be the sale aspects of e-business.

China Internet Network Information Center (CNNIC) released a statistical report [1]. According to the report, up to December 2016, China had 731 million Internet users, with a yearly increase of 42.99 million. The Internet penetration rate reached 53.2%, up 2.9% points from the end of 2015. As of December 2016, the number of mobile Internet users in China reached 695 million, an increase of 75.5 million from the end of 2015. Mobile netizens accounted for 95.1% of the total netizen population, while this percentage was 90.1% in 2015. According to the report, by December 2016, China had 466.7 million online shopping customers, a yearly increase 63.8%. The Chinese online shopping market still maintains robust growth. Meanwhile, the number of mobile online commerce (or m-commerce) customers is growing rapidly to 440.93 million, an increase of 63.4%.

In the current business world, there are many online retail companies in China. For example, Taobao Mall, Jingdong Mall (360buy), Amazon.cn and Dangdang. The competition among companies has become increasingly fierce, companies need to differentiate themselves from other companies in order to keep their relationships with their customers. Customer satisfaction is an important

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object that all companies are seeking. To facilitate business growth in retail sector, the assessment of customer's satisfaction and loyalty levels is very important work, which needs in-depth investigation.

As we all know, questionnaire is an important survey tool. Consider that participants hope a simple questionnaire. This paper plans to employ some simple symbols $\{\checkmark, \times, \bigcirc\}$ to answer questionnaire, where the symbols $\checkmark, \times, \bigcirc$ denote, respectively, satisfaction, dissatisfaction, and hesitation or abstention. Consider that an intuitionistic fuzzy number is composed of three parameters, one is the membership degree, which can measure the customer satisfaction ($\checkmark$); another is the non-membership degree, which can measure the customer dissatisfaction($\times$); and third is a hesitation or indeterminacy index, which can measure the uncertain information. The nonresponse and $\bigcirc$ are uncertain information. That is to say, the symbols $\checkmark, \times, \bigcirc$ are consistent with an intuitionistic fuzzy number. How is the symbol information aggregated into an intuitionistic fuzzy information? And how is a new evaluation approach to customer satisfaction based on intuitionistic fuzzy theory developed? This is a pressing problem. To solve this problem, the rest of the paper is structured as follows. Section 2 reviews the related work and introduces the research motivation. Section 3 briefly reviews the interval-valued intuitionistic fuzzy information and some related decision tools. Section 4 presents an evaluation methodology and algorithm based on the above idea, including a real example. And Sect. 5 gives our conclusions and future research.

2 Related Work and Research Motivation

Customer satisfaction has been defined in various ways. Oliver [2] regarded customer satisfaction as a customer's response to consumption experiences. Armstrong et al. [3] defined customer satisfaction as the level of a person's felt state resulting from comparing a product's perceived performance or outcome with his/her own expectations. Numerous studies have treated satisfaction as the essential principle for the retention of customers, and customer satisfaction has moved to the head of relationship marketing approaches. For example, Terpstra and Verbeeten [4] investigated the relation between customer satisfaction, customer servicing costs, and customer value in a financial services firm. Chow [5] focused on the relationship between customer satisfaction, measured by customer complaints, and the service quality of Chinese carriers. Saeidi et al. [6] examined the mediating role of competitive advantage, reputation, and customer satisfaction. Homburg et al. [7] focused on understanding how price importance links to customer's price search and satisfaction. Shi et al. [8] compared casino service quality evaluations, customer satisfaction and loyalty between casino members and nonmembers. Kang and Park [9] developed a new framework for measurement of customer satisfaction for mobile services. Demirci Orel and Kara [10] examined the service quality of supermarket/grocery store and its impact on customer satisfaction and loyalty in an emerging market. Chen et al. [11] studied the customer satisfaction in homebuilding industry.

Customer satisfaction is a key driver of loyalty in the retail context [12], and it is considered an antecedent of repurchase intention [13]. This relationship has been confirmed in the online context [14]. So customer satisfaction has become a key factor for company's survival, development and competitiveness. How to quantify the customer satisfaction is an important research topic, which has been widely considered in above-mentioned literature. However, there are only a few papers evaluating the customer satisfaction in e-retailing industry. For example, based on multivariate analysis of covariance techniques, Thirumalai and Sinha [15] investigated the customization of the online purchase process in electronic retailing. Ihtiyar et al. [16] examined the relationships of the conceptual model among intercultural competence, reliability, and customer satisfaction; and Rose et al. [14] developed and empirically tested a model of the relationship between antecedents and outcomes of online customer experience; and Srivastava and Kaul [17] investigated the impact of both convenience and social interaction on customer satisfaction and the mediating role of customer experience. Endo et al. [18] investigated the e-satisfaction for online shoes retailing by using statistical analysis method. Low et al. [19] examined the relationship between customer satisfaction and price sensitivity by using hierarchical regression analysis.

The above-mentioned methods, including statistical methods, have made great contributions to examine the customer satisfaction. However, there are some research gaps, which need in-depth investigation:

1. Previous studies pay little attention to the ranking of evaluation objects.
2. It is often the case that there is still the lack of answers to questions in tested questionnaires, on which the respondents may be hesitation or negligence. The nonresponse is also an information resource. Current statistical methods, as far as we know, have failed to consider it.
3. Some participants might be pressed for time, they always complain that the tested questionnaire has too many terms and options. They more prefer the questionnaire in which there are fewer terms and options, and the options can be answered by some simple symbols, such as $\checkmark, \times, \bigcirc$, where the symbols $\checkmark, \times, \bigcirc$ denote, respectively, satisfaction, dissatisfaction, and hesitation or abstention.

To solve these problems, this paper employs a decision-making method to deal with the evaluation of customer satisfaction in retail industry. Multi-criteria decision-making is one of the most complex administrative processes in management, which is the procedure to find the best alternative among a set of feasible alternatives. Owing to the increasing complexity of the socio-economic environment, a single decision maker (DM) or expert may be impossible to consider all relevant aspects of a problem. In this case, some decision-making problems require to be further extended to group decision-making (GDM) [20]. Recently, the GDM has drawn great attention from researchers [21–24].

Fuzzy logic system is more suitable for dealing with the lack of answers to questions in tested questionnaire. Fuzzy logic, emerged from the theory of fuzzy set [25], is one of the techniques of soft computing which can deal with the inherent subjectivity, imprecision and vagueness in the articulation of opinions. As a generalization of Zadeh's fuzzy set, the intuitionistic fuzzy theory [26] has been extensively applied to various areas in the recent decade. The intuitionistic fuzzy number [27] and interval-valued intuitionistic fuzzy number (IVIFN) [28] were developed and have been applied to many multi-criteria decision-making [29] and GDM problems [30]. In this model, we will employ the intuitionistic fuzzy theory to handle the symbol-based evaluation problem.

We consider that some evaluations in questionnaires might be very positive; while others might be very negative. If the evaluations are aggregated into a collective decision of DMs, the positive information and the negative information will be offset. To avoid this, this paper attempts to propose a direct GDM method for evaluating the customer satisfaction based on an extended TOPSIS (technique for order preference by similarity to ideal solution) technique [31]. The TOPSIS technique is a compromise method [32], which can highlight the positive and the negative information, and compromises them by a relative closeness.

The main research motivation and contributions of this work are as follows.

1. This paper intendeds to contribute a new evaluation method to customer satisfaction. The evaluation information only includes three simple symbols $\{\checkmark, \times, \bigcirc\}$.
2. The evaluation information will be aggregated into an intuitionistic fuzzy information in group decision-making setting.

3 Preliminaries

As preliminaries, this section intends to prepare two contents: interval-valued intuitionistic fuzzy information and GDM.

3.1 Interval-Valued Intuitionistic Fuzzy Information

Zadeh [25] given the concept of fuzzy set. As a generalization of fuzzy set, Atanassov [26] introduced the intuitionistic fuzzy set. Later, Atanassov et al. [33] extended the intuitionistic fuzzy set to interval-valued intuitionistic fuzzy set as follows.

Let X be a universe of discourse. An interval-valued intuitionistic fuzzy set $\tilde{A}$ in X is an object of the form:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in X\}, \tag{1}$$

where $\mu_{\tilde{A}}(x) = [\mu^l_{\tilde{A}}(x), \mu^u_{\tilde{A}}(x)] \subseteq [0,1]$ and $\nu_{\tilde{A}}(x) = [\nu^l_{\tilde{A}}(x),\ \nu^u_{\tilde{A}}(x)] \subseteq [0,1]$ are intervals, $\mu^l_{\tilde{A}}(x) = \text{inf}\mu_{\tilde{A}}(x)$, $\mu^u_{\tilde{A}}(x) = \text{sup}\mu_{\tilde{A}}(x)$, $\nu^l_{\tilde{A}}(x) = \text{inf}\nu_{\tilde{A}}(x)$, $\nu^u_{\tilde{A}}(x) = \text{sup}\nu_{\tilde{A}}(x)$, and $\mu^u_{\tilde{A}}(x) + \nu^u_{\tilde{A}}(x) \leq 1$, for all $x \in X$, and $\pi_{\tilde{A}}(x) = [\pi^l_{\tilde{A}}(x), \pi^u_{\tilde{A}}(x)]$, where $\pi^l_{\tilde{A}}(x) = 1 - \mu^u_{\tilde{A}}(x_i) - \nu^u_{\tilde{A}}(x), \pi^u_{\tilde{A}}(x) = 1 - \mu^l_{\tilde{A}}(x) - \nu^l_{\tilde{A}}(x)$, for all $x \in X$.

Especially, if $\mu_{\tilde{A}}(x) = \mu^l_{\tilde{A}}(x) = \mu^u_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x) = \nu^l_{\tilde{A}}(x) = \nu^u_{\tilde{A}}(x)$, then, $\tilde{A}$ is reduced to an intuitionistic fuzzy set.

Xu and Chen [28] called the pair $\tilde{\alpha} = (\mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$ an interval-valued intuitionistic fuzzy number (IVIFN), and denoted an IVIFN by

$$\tilde{\alpha} = ([\mu^l_{\tilde{\alpha}}, \mu^u_{\tilde{\alpha}}], [\nu^l_{\tilde{\alpha}}, \nu^u_{\tilde{\alpha}}]), \tag{2}$$

where $[\mu^l_{\tilde{\alpha}}, \mu^u_{\tilde{\alpha}}], [\nu^l_{\tilde{\alpha}}, \nu^u_{\tilde{\alpha}}], [\pi^l_{\tilde{\alpha}}, \pi^u_{\tilde{\alpha}}] \subseteq [0, 1], \mu^u_{\tilde{\alpha}} + \nu^u_{\tilde{\alpha}} \leq 1, \pi^l_{\tilde{\alpha}} = 1 - \mu^u_{\tilde{\alpha}} - \nu^u_{\tilde{\alpha}}, \pi^u_{\tilde{\alpha}} = 1 - \mu^l_{\tilde{\alpha}} - \nu^l_{\tilde{\alpha}}$.

And Xu and Chen [28] introduced the following operations:

Definition 1. *Let* $\tilde{\alpha} = ([\mu^l_{\tilde{\alpha}}, \mu^u_{\tilde{\alpha}}], [\nu^l_{\tilde{\alpha}}, \nu^u_{\tilde{\alpha}}])$ *and* $\tilde{\beta} = ([\mu^l_{\tilde{\beta}}, \mu^u_{\tilde{\beta}}], [\nu^l_{\tilde{\beta}}, \nu^u_{\tilde{\beta}}])$ *be two IVIFNs and* λ *be a real number. Then*

1. $\tilde{\alpha} + \tilde{\beta} = ([\mu^l_{\tilde{\alpha}} + \mu^l_{\tilde{\beta}} - \mu^l_{\tilde{\alpha}}\mu^l_{\tilde{\beta}}, \mu^u_{\tilde{\alpha}} + \mu^u_{\tilde{\beta}} - \mu^u_{\tilde{\alpha}}\mu^u_{\tilde{\beta}}], [\nu^l_{\tilde{\alpha}}\nu^l_{\tilde{\beta}}, \nu^u_{\tilde{\alpha}}\nu^u_{\tilde{\beta}}])$;
2. $\lambda\tilde{\alpha} = ([1 - (1 - \mu^l_{\tilde{\alpha}})^\lambda, 1 - (1 - \mu^u_{\tilde{\alpha}})^\lambda], [(\nu^l_{\tilde{\alpha}})^\lambda, (\nu^u_{\tilde{\alpha}})^\lambda]), \lambda > 0$;
3. $\tilde{\alpha}^c = ([\nu^l_{\tilde{\alpha}}, \nu^u_{\tilde{\alpha}}], [\mu^l_{\tilde{\alpha}}, \mu^u_{\tilde{\alpha}}])$, *where the* $\tilde{\alpha}^c$ *is the complement of* $\tilde{\alpha}$.

Definition 2. *Let* $\tilde{\alpha} = ([\mu^l_{\tilde{\alpha}}, \mu^u_{\tilde{\alpha}}], [\nu^l_{\tilde{\alpha}}, \nu^u_{\tilde{\alpha}}])$ *and* $\tilde{\beta} = ([\mu^l_{\tilde{\beta}}, \mu^u_{\tilde{\beta}}], [\nu^l_{\tilde{\beta}}, \nu^u_{\tilde{\beta}}])$ *be two IVIFNs. Similar to literature [34], the Euclidean distance between* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *is given as follows:*

$$s(\tilde{\alpha}, \tilde{\beta}) = \sqrt{(\mu^l_{\tilde{\alpha}} - \mu^l_{\tilde{\beta}})^2 + (\mu^u_{\tilde{\alpha}} - \mu^u_{\tilde{\beta}})^2 + (\nu^l_{\tilde{\alpha}} - \nu^l_{\tilde{\beta}})^2 + (\nu^u_{\tilde{\alpha}} - \nu^u_{\tilde{\beta}})^2 + (\pi^l_{\tilde{\alpha}} - \pi^l_{\tilde{\beta}})^2 + (\pi^u_{\tilde{\alpha}} - \pi^u_{\tilde{\beta}})^2}, \tag{3}$$

where, by Eq. (2), the $\pi^l_{\tilde{\alpha}} = 1 - \mu^u_{\tilde{\alpha}} - \nu^u_{\tilde{\alpha}}, \pi^u_{\tilde{\alpha}} = 1 - \mu^l_{\tilde{\alpha}} - \nu^l_{\tilde{\alpha}}, \pi^l_{\tilde{\beta}} = 1 - \mu^u_{\tilde{\beta}} - \nu^u_{\tilde{\beta}}$ *and* $\pi^u_{\tilde{\beta}} = 1 - \mu^l_{\tilde{\beta}} - \nu^l_{\tilde{\beta}}$.

Definition 3. *Let* $X = (\tilde{x}_{ij})_{m\times n}$ *be a matrix. If all elements* $\tilde{x}_{ij}$ *are IVIFNs, then we call* X *an interval-valued intuitionistic fuzzy matrix.*

Similar to Eq. (3), we define the Euclidean measure between two interval-valued intuitionistic fuzzy matrices $X_1 = (([\mu^{1l}_{ij}, \mu^{1u}_{ij}], [\nu^{1l}_{ij}, \nu^{1u}_{ij}]))_{mn}$ and $X_2 = (([\mu^{2l}_{ij}, \mu^{2u}_{ij}], [\nu^{2l}_{ij}, \nu^{2u}_{ij}]))_{mn}$ as:

$$S(X_1, X_2) = \sqrt{\sum_{i=1}^m \sum_{j=1}^n ((\mu^{1l}_{ij} - \mu^{2l}_{ij})^2 + (\mu^{1u}_{ij} - \mu^{2u}_{ij})^2 + (\nu^{1l}_{ij} - \nu^{2l}_{ij})^2 + (\nu^{1u}_{ij} - \nu^{2u}_{ij})^2 + (\pi^{1l}_{ij} - \pi^{2l}_{ij})^2 + (\pi^{1u}_{ij} - \pi^{2u}_{ij})^2)}, \tag{4}$$

where, by Eq. (2), the $\pi^{1l}_{ij} = 1 - \mu^{1u}_{ij} - \nu^{1u}_{ij}, \pi^{1u}_{ij} = 1 - \mu^{1l}_{ij} - \nu^{1l}_{ij}, \pi^{2l}_{ij} = 1 - \mu^{2u}_{ij} - \nu^{2u}_{ij}$ and $\pi^{2u}_{ij} = 1 - \mu^{2l}_{ij} - \nu^{2l}_{ij} (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$.

3.2 Group Decision-Making

For convenience, the following indexes and sets of key elements are considered to represent the GDM problem in this paper.

1. A set of m feasible alternatives written as $A = \{A_i | i \in M\}$, where $M = \{1, 2, \ldots, m\}$;
2. A set of criteria written as $U = \{u_j | j \in N\}$, where $N = \{1, 2, \ldots, n\}$;

3. A weight set of criteria written as $w = \{w_j | j \in N\}$, with $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j = 1$;
4. A set of DMs written as $D = \{d_k | k \in T\}$, where $T = \{1, 2, \ldots, t\}$.

A GDM problem with t DMs, m alternatives and n criteria can be characterized by the following matrix:

$$X_i = \begin{array}{c} \\ d_1 \\ d_2 \\ \vdots \\ d_t \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & \cdots & u_n \end{array} \\ \begin{pmatrix} x_{11}^i & x_{12}^i & \cdots & x_{1n}^i \\ x_{21}^i & x_{22}^i & \cdots & x_{2n}^i \\ \vdots & \vdots & \vdots & \vdots \\ x_{t1}^i & x_{t2}^i & \cdots & x_{tn}^i \end{pmatrix} \end{array}, i \in M, \tag{5}$$

where the $X_i = (x_{kj}^i)_{t \times n}$ and the x_{kj}^i, provided by kth DM, is the score of the ith alternative with respect to jth criterion.

The aim of GDM is to rank the alternatives $A_i (i \in M)$ according to the $X_i (i \in M)$.

4 Methodology

This section is a framework this research, which will be illustrated by the following model in detail.

4.1 Alternatives, Criteria and Decision Makers

Three online retail companies, located in China, are considered as alternatives in this paper. That is to say, the evaluation set is $A = \{A_1, A_2, A_3\} = \{$company 1, company 2, company 3$\}$. The customer satisfaction are evaluated and compared among three companies. Evaluation criteria in this paper are based on the recommendation of evaluators, which are $U = \{u_1, u_2, u_3\} = \{$product quality, product price, after-sales service$\}$. The DMs are respondents (or evaluators, or raters) in this model. A DM is a group of respondents in a district. All respondents had purchased products from websites of electronic retailers ensuring familiarity with the online purchase process in this setting. Specifically, the set of DMs is $D = \{d_1, d_2, d_3\} = \{$respondents in district 1, respondents in district 2, respondents in district 3$\}$, each of which is further divided into four age grades: respondents aged younger than 20; respondents between the ages of 21 and 35; respondents between the ages of 36 and 50; respondents aged older than 50.

4.2 Data Collection and Measurement

The customer's opinions are collected by questionnaires. This paper selected a random sample of 8788 participants, who had successfully participated to the evaluation. The data are collected by postmen, who asks for his/her customer to complete a questionnaire after every delivery. The questionnaire includes five main content: age and three evaluation criteria. The age term is divided into the above-mentioned four stages. The customer is asked for marking by symbol ✓ on own age stage. The three criteria are evaluated by customer, where each criterion is marked by using only one symbol from {✓, ×, ◯}. As described above, the symbol ✓ denotes satisfaction; the symbol × denotes dissatisfaction; and the symbol ◯ denotes hesitation or abstention. That is to say, each respondent marks at most thirteen symbols in a questionnaire.

These symbols in questionnaires will be collected, quantified, and aggregated by the following steps 1–3, then the elements of $X_i (i = 1, 2, 3)$ in Eq. (7) are formed, which are characterized by IVIFNs. The complete evaluation procedure is as follows.

Step 1. Data collection and statistics.
For each examined company $A_i (i = 1, 2, 3)$, the questionnaires are divided into three class $d_k (k = 1, 2, 3)$ by the districts of respondents, and each class is further divided into four subclasses according to the ages of respondents. For each subclass of questionnaires, the total number of questionnaires of A_i with respect to d_k at hth $(h = 1, 2, 3, 4)$ ages is written as s_k^{ih}, the symbols ✓ and × are collected respectively from the three criteria $\{u_j | j = 1, 2, 3\}$ in all the questionnaires. The total number of symbols ✓ is written as n_{kj}^{ih}, and the total number of symbols × is written as m_{kj}^{ih}. The statistics is shown in Table 1.

The statistics in Table 1 involve only two symbols ✓ and ×. However, the symbol ◯ is not neglected. It is considered as the same as the nonresponse in statistics. In fact, the symbol ◯ can show that the respondent is not negligence this evaluation of criterion, but hesitation or abstention. For convenience, the information ◯ along with the nonresponse will be quantified and aggregated into a hesitancy degree π_{kj}^{il} and π_{kj}^{iu} in Eqs. (11) and (12) below, although the missing values are not equivalent complete hesitation.

Step 2. Data normalization.
All data are normalized by the following formulas:

$$\xi_{kj}^{ih} = \frac{n_{kj}^{ih}}{s_k^{ih}}, \eta_{kj}^{ih} = \frac{m_{kj}^{ih}}{s_k^{ih}}, i \in M, j \in N, k \in T, \tag{6}$$

where the s_{kj}^{ih}, n_{kj}^{ih} and $m_{kj}^{ih} (i, j, k = 1, 2, 3, h = 1, 2, 3, 4)$ are the same as in Table 1.
The normalized data are shown in Table 2.

Step 3. Determine the interval-valued intuitionistic fuzzy information of evaluation criteria.

Table 1. Statistics of assessment information

Alternative	DM	Age	s_k^{ih}	u_1		u_2		u_3	
				$n_{k1}^{ih}(\checkmark)$	$m_{k1}^{ih}(\times)$	$n_{k2}^{ih}(\checkmark)$	$m_{k2}^{ih}(\times)$	$n_{k3}^{ih}(\checkmark)$	$m_{k3}^{ih}(\times)$
A_1	d_1	<25	250	86	155	66	96	27	58
		26–40	220	80	79	83	135	63	82
		41–55	307	129	132	77	138	99	79
		>56	202	73	87	65	80	38	91
	d_2	<25	190	56	85	86	76	97	80
		26–40	230	122	79	89	75	93	81
		41–55	258	87	102	129	97	122	118
		>56	202	66	67	68	70	73	76
	d_3	<25	260	81	79	84	70	88	84
		26–40	250	78	73	94	76	132	65
		41–55	300	125	119	132	85	84	131
		>56	172	54	87	65	72	66	83
A_2	d_1	<25	260	90	60	86	120	110	125
		26–40	280	81	177	93	69	125	133
		41–55	260	36	120	137	99	25	108
		>56	182	41	59	65	93	37	72
	d_2	<25	260	85	62	80	122	104	127
		26–40	300	78	180	92	72	120	135
		41–55	250	40	125	132	97	30	110
		>56	200	41	61	61	95	42	74
	d_3	<25	270	89	152	67	93	30	55
		26–40	250	83	76	86	132	66	79
		41–55	292	132	130	80	135	102	76
		>56	200	76	84	68	78	43	94
A_3	d_1	<25	220	58	87	88	78	99	83
		26–40	250	124	80	87	77	95	84
		41–55	290	89	105	127	99	120	120
		>56	218	68	68	70	73	75	77
	d_2	<25	250	81	80	84	74	88	85
		26–40	248	78	75	94	77	132	69
		41–55	290	125	124	132	88	84	130
		>56	208	54	88	65	72	66	89
	d_3	<25	243	85	66	80	122	104	129
		26–40	285	78	176	92	82	120	145
		41–55	260	40	115	132	107	30	110
		>56	208	41	76	61	98	42	78

Table 2. Normalization of assessment information

Alternative	DM	Age and [min,max]	u_1		u_2		u_3	
			$\xi_{k1}^{ih}(\checkmark)$	$\eta_{k1}^{ih}(\times)$	$\xi_{k2}^{ih}(\checkmark)$	$\eta_{k2}^{ih}(\times)$	$\xi_{k3}^{ih}(\checkmark)$	$\eta_{k3}^{ih}(\times)$
A_1	d_1	<25	0.344	0.620	0.264	0.384	0.108	0.232
		26–40	0.364	0.359	0.377	0.614	0.286	0.373
		41–55	0.420	0.430	0.251	0.450	0.323	0.257
		>56	0.361	0.431	0.322	0.396	0.188	0.451
		[min, max]	[0.344,0.420]	[0.359,0.620]	[0.251,0.377]	[0.384,0.614]	[0.108,0.323]	[0.232,0.451]
	d_2	<25	0.295	0.447	0.453	0.400	0.511	0.421
		26–40	0.530	0.344	0.387	0.326	0.404	0.352
		41–55	0.337	0.395	0.500	0.376	0.473	0.457
		>56	0.327	0.332	0.337	0.347	0.361	0.376
		[min, max]	[0.295,0.530]	[0.332,0.447]	[0.337,0.500]	[0.326,0.400]	[0.361,0.511]	[0.352,0.457]
	d_3	<25	0.312	0.304	0.323	0.269	0.339	0.323
		26–40	0.312	0.292	0.376	0.304	0.528	0.260
		41–55	0.417	0.397	0.440	0.283	0.280	0.437
		>56	0.314	0.506	0.378	0.419	0.384	0.483
		[min, max]	[0.312,0.417]	[0.292,0.506]	[0.323,0.440]	[0.269,0.419]	[0.280,0.528]	[0.260,0.483]
A_2	d_1	<25	0.346	0.231	0.331	0.462	0.423	0.481
		26–40	0.289	0.632	0.332	0.246	0.446	0.475
		41–55	0.139	0.462	0.527	0.381	0.096	0.415
		>56	0.225	0.324	0.357	0.511	0.203	0.396
		[min, max]	[0.139,0.346]	[0.231,0.632]	[0.331,0.527]	[0.246,0.511]	[0.096,0.446]	[0.396,0.481]
	d_2	<25	0.327	0.239	0.308	0.469	0.400	0.489
		26–40	0.260	0.600	0.307	0.240	0.400	0.450
		41–55	0.160	0.500	0.528	0.388	0.120	0.440
		>56	0.205	0.305	0.305	0.475	0.210	0.370
		[min, max]	[0.160,0.327]	[0.239,0.600]	[0.305,0.528]	[0.240,0.475]	[0.120,0.400]	[0.370,0.489]
	d_3	<25	0.330	0.563	0.248	0.344	0.111	0.204
		26–40	0.332	0.304	0.344	0.528	0.264	0.316
		41–55	0.452	0.445	0.274	0.462	0.349	0.260
		>56	0.380	0.420	0.340	0.390	0.215	0.470
		[min, max]	[0.330,0.452]	[0.304,0.563]	[0.248,0.344]	[0.344,0.528]	[0.111,0.349]	[0.204,0.470]
A_3	d_1	<25	0.264	0.396	0.400	0.355	0.450	0.377
		26–40	0.496	0.320	0.348	0.308	0.380	0.336
		41–55	0.307	0.362	0.438	0.341	0.414	0.414
		>56	0.312	0.312	0.321	0.335	0.344	0.353
		[min, max]	[0.264,0.496]	[0.312,0.396]	[0.321,0.438]	[0.308,0.355]	[0.344,0.450]	[0.336,0.414]
	d_2	<25	0.324	0.320	0.336	0.296	0.352	0.340
		26–40	0.315	0.302	0.379	0.311	0.532	0.278
		41–55	0.431	0.428	0.455	0.303	0.290	0.448
		>56	0.260	0.423	0.313	0.346	0.317	0.428
		[min, max]	[0.260,0.431]	[0.302,0.428]	[0.313,0.455]	[0.296,0.346]	[0.290,0.532]	[0.278,0.448]
	d_3	<25	0.350	0.272	0.329	0.502	0.428	0.531
		26–40	0.274	0.618	0.323	0.288	0.421	0.509
		41–55	0.154	0.442	0.508	0.412	0.115	0.423
		>56	0.197	0.365	0.293	0.471	0.202	0.375
		[min, max]	[0.154,0.350]	[0.272,0.618]	[0.293,0.508]	[0.288,0.502]	[0.115,0.428]	[0.375,0.531]

For each evaluation criterion u_j of alternative A_i, the evaluation value can be characterized by an IVIFN as follows:

$$x^i_{kj} = ([\mu^{il}_{kj}, \mu^{iu}_{kj}], [\nu^{il}_{kj}, \nu^{iu}_{kj}]), i \in M, j \in N, k \in T, \tag{7}$$

where the $\mu^{il}_{kj} = \xi^{il}_{kj}/\sigma^{iu}_{kj}, \mu^{iu}_{kj} = \xi^{iu}_{kj}/\sigma^{iu}_{kj}, \nu^{il}_{kj} = \eta^{il}_{kj}/\sigma^{iu}_{kj}, \nu^{iu}_{kj} = \eta^{iu}_{kj}/\sigma^{iu}_{kj}$, and the $\sigma^{iu}_{kj} = \xi^{iu}_{kj} + \eta^{iu}_{kj}, \xi^{il}_{kj} = \min_{1\leq h\leq 4}\{\xi^{ih}_{kj}\}$, $\xi^{iu}_{kj} = \max_{1\leq h\leq 4}\{\xi^{ih}_{kj}\}, \eta^{il}_{kj} = \min_{1\leq h\leq 4}\{\eta^{ih}_{kj}\}, \eta^{iu}_{kj} = \max_{1\leq h\leq 4}\{\eta^{ih}_{kj}\}$, and $M = N = T = \{1, 2, 3\}$.

It is clear that the μ^{iu}_{kj} and ν^{iu}_{kj} satisfy the condition $\mu^{iu}_{kj} + \nu^{iu}_{kj} \leq 1 (i \in M, j \in N, k \in T)$ in Eq. (2).

The assessment values based on interval-valued intuitionistic fuzzy information are shown in Table 3.

Table 3. Assessment based on interval-valued intuitionistic fuzzy information

Decision	DM	u_1	u_2	u_3
X_1	d_1	([0.331,0.404],[0.345,0.596])	([0.253,0.381],[0.388,0.619])	([0.140,0.417],[0.300,0.583])
	d_2	([0.301,0.542],[0.339,0.458])	([0.374,0.556],[0.362,0.444])	([0.373,0.527],[0.364,0.473])
	d_3	([0.338,0.452],[0.317,0.548])	([0.376,0.512],[0.314,0.488])	([0.277,0.522],[0.257,0.478])
X_2	d_1	([0.142,0.354],[0.236,0.646])	([0.319,0.508],[0.237,0.492])	([0.104,0.481],[0.427,0.519])
	d_2	([0.173,0.353],[0.257,0.647])	([0.304,0.526],[0.239,0.474])	([0.135,0.450],[0.416,0.550])
	d_3	([0.325,0.445],[0.300,0.555])	([0.285,0.394],[0.395,0.606])	([0.136,0.426],[0.249,0.574])
X_3	d_1	([0.296,0.556],[0.350,0.444])	([0.405,0.553],[0.389,0.447])	([0.398,0.521],[0.389,0.479])
	d_2	([0.302,0.502],[0.352,0.498])	([0.390,0.568],[0.369,0.432])	([0.295,0.543],[0.284,0.457])
	d_3	([0.159,0.362],[0.281,0.638])	([0.290,0.503],[0.285,0.497])	([0.120,0.446],[0.391,0.554])

Step 4. Construct the weighted decision.

For the criteria' weight vector $w = (w_1, w_2, \ldots, w_n)$, the weighted decision is constructed by the following matrix:

$$Y_i = (y^i_{kj})_{tn}, i \in M, k \in T, j \in N, \tag{8}$$

where the $y^i_{kj} = w_j x^i_{kj} = ([\tau^{il}_{kj}, \tau^{iu}_{kj}], [\upsilon^{il}_{kj}, \upsilon^{iu}_{kj}])$, the $\tau^{il}_{kj} = 1-(1-\mu^{il}_{kj})^{w_j}, \tau^{iu}_{kj} = 1 - (1 - \mu^{iu}_{kj})^{w_j}, \upsilon^{il}_{kj} = (\nu^{il}_{kj})^{w_j}, \upsilon^{iu}_{kj} = (\nu^{iu}_{kj})^{w_j} (i \in M, k \in T, j \in N)$ by Definition 1, and the $x^i_{kj} \in X_i$ are the same as in Eq. (7).

The weight vector of criteria is $(w_1, w_2, w_3) = (0.3, 0.3, 0.4)$, which is negotiated and determined by some representative DMs. The weighted decisions Y_i are shown in Table 4, where the $M = N = T = \{1, 2, 3\}$.

Step 5. Determine the ideal decisions.

For the $Y_i (i \in M)$ in Eq. (8), according to the framework of TOPSIS technique, we let

$$Y_+ = (y^+_{kj})_{tn}, k \in T, j \in N, \tag{9}$$

Table 4. Weighted assessment based on interval-valued intuitionistic fuzzy information

Decision	DM	u_1	u_2	u_3
Y_1	d_1	([0.113,0.144],[0.727,0.856])	([0.084,0.134],[0.752,0.866])	([0.058,0.194],[0.618,0.806])
	d_2	([0.102,0.209],[0.723,0.791])	([0.131,0.216],[0.737,0.784])	([0.171,0.259],[0.667,0.741])
	d_3	([0.116,0.165],[0.708,0.835])	([0.132,0.194],[0.706,0.806])	([0.122,0.256],[0.581,0.744])
Y_2	d_1	([0.045,0.123],[0.648,0.877])	([0.109,0.192],[0.650,0.808])	([0.043,0.231],[0.711,0.769])
	d_2	([0.055,0.122],[0.665,0.878])	([0.103,0.201],[0.651,0.799])	([0.056,0.213],[0.704,0.787])
	d_3	([0.111,0.162],[0.696,0.838])	([0.096,0.140],[0.757,0.860])	([0.057,0.199],[0.573,0.801])
Y_3	d_1	([0.100,0.216],[0.730,0.784])	([0.144,0.214],[0.753,0.786])	([0.184,0.255],[0.685,0.745])
	d_2	([0.102,0.189],[0.731,0.811])	([0.138,0.223],[0.742,0.777])	([0.131,0.269],[0.604,0.731])
	d_3	([0.051,0.126],[0.683,0.874])	([0.098,0.189],[0.686,0.811])	([0.050,0.211],[0.687,0.789])

be the positive ideal decision (PID) of all $Y_i (i \in M)$, where, the $y^+_{kj} = ([\tau^{+l}_{kj}, \tau^{+u}_{kj}], [\upsilon^{+l}_{kj}, \upsilon^{+u}_{kj}])$, and $\tau^{+l}_{kj} = \max_{i \in M}\{\mu^{il}_{kj}\}, \tau^{+u}_{kj} = \max_{i \in M}\{\mu^{iu}_{kj}\}, \upsilon^{+l}_{kj} = \min_{i \in M}\{\nu^{il}_{kj}\}$ and $\upsilon^{+u}_{kj} = \min_{i \in M}\{\nu^{iu}_{kj}\} (k \in T, j \in N)$.

A negative ideal decision (NID) should have the maximum separation from the PID, so we let

$$Y_- = (y^-_{kj})_{tn}, k \in T, j \in N, \tag{10}$$

be a NID of all $Y_i (i \in M)$, where, the $y^-_{kj} = ([\tau^{-l}_{kj}, \tau^{-u}_{kj}], [\upsilon^{-l}_{kj}, \upsilon^{-u}_{kj}])$, and $\tau^{-l}_{kj} = \min_{i \in M}\{\mu^{il}_{kj}\}, \tau^{-u}_{kj} = \min_{i \in M}\{\mu^{iu}_{kj}\}, \upsilon^{-l}_{kj} = \max_{i \in M}\{\nu^{il}_{kj}\}$ and $\upsilon^{-u}_{kj} = \max_{i \in M}\{\nu^{iu}_{kj}\} (k \in T, j \in N)$.
The ideal decisions are shown in Table 5.

Table 5. Ideal decisions of all decisions

Decision	DM	u_1	u_2	u_3
Y_+	d_1	([0.113,0.216],[0.648,0.784])	([0.144,0.214],[0.650,0.786])	([0.184,0.255],[0.618,0.745])
	d_2	([0.102,0.209],[0.665,0.791])	([0.138,0.223],[0.651,0.777])	([0.171,0.269],[0.604,0.731])
	d_3	([0.116,0.165],[0.683,0.835])	([0.132,0.194],[0.686,0.806])	([0.122,0.256],[0.573,0.744])
Y_-	d_1	([0.045,0.123],[0.730,0.877])	([0.084,0.134],[0.753,0.866])	([0.043,0.194],[0.711,0.806])
	d_2	([0.055,0.122],[0.731,0.878])	([0.103,0.201],[0.742,0.799])	([0.056,0.213],[0.704,0.787])
	d_3	([0.051,0.126],[0.708,0.874])	([0.096,0.140],[0.757,0.860])	([0.050,0.199],[0.687,0.801])

Step 6. Calculate the separations of each decision from its ideal decisions.
The separation of each decision Y_i from its PID Y_+, S^+_i, is given by the Euclidean distance between Y_i and Y_+ (see Eq. (4)) as follows:

$$S^+_i = \sqrt{\sum_{k=1}^{t}\sum_{j=1}^{n}((\tau^{il}_{kj} - \tau^{+l}_{kj})^2 + (\tau^{iu}_{kj} - \tau^{+u}_{kj})^2 + (\upsilon^{il}_{kj} - \upsilon^{+l}_{kj})^2 + (\upsilon^{iu}_{kj} - \upsilon^{+u}_{kj})^2 + (\pi^{il}_{kj} - \pi^{+l}_{kj})^2 + (\pi^{iu}_{kj} - \pi^{+u}_{kj})^2)}, i \in M, \tag{11}$$

where the $\tau_{kj}^{+l}, \tau_{kj}^{+u}, \upsilon_{kj}^{+l}$ and υ_{kj}^{+u} are the same as in Eq. (9), the $\pi_{kj}^{il} = 1 - \tau_{kj}^{iu} - \upsilon_{kj}^{iu}, \pi_{kj}^{iu} = 1 - \tau_{kj}^{il} - \upsilon_{kj}^{il}, \pi_{kj}^{+l} = 1 - \tau_{kj}^{+u} - \upsilon_{kj}^{+u}$ and $\pi_{kj}^{+u} = 1 - \tau_{kj}^{+l} - \upsilon_{kj}^{+l} (i \in M, k \in T, j \in N)$ by Eq. (2).

Similarly, the separation of each Y_i from its NID Y_-, S_i^-, is given by:

$$S_i^- = \sqrt{\sum_{k=1}^{t} \sum_{j=1}^{n} ((\tau_{kj}^{il} - \tau_{kj}^{-l})^2 + (\tau_{kj}^{iu} - \tau_{kj}^{-u})^2 + (\upsilon_{kj}^{il} - \upsilon_{kj}^{-l})^2 + (\upsilon_{kj}^{iu} - \upsilon_{kj}^{-u})^2 + (\pi_{kj}^{il} - \pi_{kj}^{-l})^2 + (\pi_{kj}^{iu} - \pi_{kj}^{-u})^2)}, i \in M, \quad (12)$$

where the $\tau_{kj}^{-l}, \tau_{kj}^{-u}, \upsilon_{kj}^{-l}$ and υ_{kj}^{-u} are the same as in Eq. (10), the π_{kj}^{il} and π_{kj}^{iu} are the same as in Eq. (11), the $\pi_{kj}^{-l} = 1 - \tau_{kj}^{-u} - \upsilon_{kj}^{-u}$ and $\pi_{kj}^{-u} = 1 - \tau_{kj}^{-l} - \upsilon_{kj}^{-l} (i \in M, k \in T, j \in N)$ by Eq. (2).

Step 7. Calculate the relative closeness.

For each decision Y_i, an extended relative closeness in TOPSIS technique is calculated by [35]:

$$RC_i = \frac{S_i^-}{S_i^+ + S_i^-}, i \in M. \quad (13)$$

Step 8. Rank the preference order of alternatives.

All alternatives are ranked in descending order in accordance with their relative closeness. The greater the relative closeness RC_i, the better the alternative $A_i (i \in M)$ is.

The separations, relative closeness, ranking of alternatives are summarized in Table 6.

Table 6. Separations, relative closeness and ranking of alternatives

Alternative	S_i^+	S_i^-	RC_i	Ranking
A_1	0.3487	0.3403	0.4939	2
A_2	0.3788	0.3138	0.4531	3
A_3	0.3302	0.3745	0.5314	1

Table 6 shows that the order of customer satisfaction of three examined online retail companies is as follows:

$$A_3 \succ A_1 \succ A_2.$$

Specifically, the A_3 is the best online retail company, followed by the A_1 and A_2.

5 Conclusion

The aim of this study is to develop a comprehensive GDM model to evaluate the customer satisfaction in the e-retailing. We now offer insights into the findings of the study and discuss implications for academics and practitioners, and conclusion and future research.

This study makes four contributions to existing evaluation methods. The first contribution is that the questionnaire is answered by using some simple symbols, which makes respondents easy to complete the questionnaires. The second contribution is that this study provides a data aggregation method. These data are characterized by symbols, and they are aggregated into an intuitionistic fuzzy information in a GDM setting. The third contribution is to solve an information fusion problem for the lack of answers to questions in questionnaires. The study identifies the nonresponse as an information regarding the customer satisfaction, and fuses it into an intuitionistic fuzzy information. The fourth contribution is made by extending existing evaluation methodology into a new context. Specifically, the current method extends the previous research on the relationship between factors and customer satisfaction into a new research on the ranking of evaluation objects.

This study helps three retail companies to distinguish their ranking and to further improve their work based on customer satisfaction. As in all commercial contexts, a key managerial objective should be high customer satisfaction ratings for an online site. However, a high level of online customer experience in any one transaction does not ensure repeat purchase. Rather e-retailers must provide a compelling online customer experience continuously over time in order to build levels of cumulative satisfaction which drives trust in the e-retailer [14]. The future work should build the cumulative satisfaction over time.

The findings and contributions of our study are to some extent constrained by certain limitations, some of which provide opportunities for further research. First, if the ages of respondents in Tables 1 and 2 are no longer distinguished, then it will simplify the procedure of survey and the calculation of model. The future work might simplify this procedure for assessing customer satisfaction. Second, the data are collected by postmen of express/delivery companies in this study, which is a limitation. The proposed approach can and should be improved by using multi-form questionnaires for survey. For example, the online retail company can pay a return visit to its customers by telephone; the customer can pay a feedback on the online evaluation system, etc.

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An M/G/1 Queue with Second Optional Service and General Randomized Vacation Policy

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Abstract. This paper studies a continuous time queue system with second optional service where all the arriving customers demand the first "essential" service while only some of them demand the second "optional" service with probability α. The service time of the first essential service and the second optional service both are independent and arbitrarily random variables. Whenever a busy period is completed, the server takes a vacation. If there is at least one customer waiting at a vacation, the server immediately serves the customer. Otherwise, the server takes another vacation with probability p, or remains idle with probability $1 - p$. We give some performances analysis of this model. Finally, it gives some numerical examples to illustrate the effect of the probabilities λ and p on the mean system size, waiting time, the probabilities when the server is idle and is on a vacation.

Keywords: Continuous time queue · Second optional service · General randomized vacation policy · Supplementary variable method

1 Introduction

As soon as the first essential service of a customer is completed, he or she immediately leaves the system with probability α or accepts the second optional service with probability $1-\alpha$. This service policy is called second optional service policy and was firstly studied by Madan [1]. The literature discussed an M/G/1 queue with the second optional service in which the first essential service time follows a general distribution, but the second optional service is assumed to be exponentially distributed. Medhi [2] extended Madan's model by considering that the second optional service follows a general distribution. Wang [3] examined an M/G/1 queue with second optional service and breakdowns in which the first

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essential service time follows a general distribution, but the second optional service is assumed to be an exponential distribution. In addition, there are many other queue models concerned second optional service which have been studied in recent years, details of which may be seen [4–18].

When a busy period is completed, the server immediately takes a vacation. The server will serve the customers if there are customers waiting in the queue at the end of a vacation. Otherwise, the server either remains idle with probability p or takes another vacation with probability $1-p$. This pattern continues until the server has taken J vacations. The server keeps idle, if there are not customers in the system at J^{th} vacation. This vacation policy is called randomize vacation policy and was studied by Ke [10]. However, some more complex queue systems with this policy are hard to analysis, as in a queue system with working vacations. Therefore, we cancel the limit of randomized vacation policy, namely the server remains idle with probability p or takes another vacation with probability $1-p$ if no customers are waiting for service at the end of any vacation, and then let the pattern continue forever. Here we define this vacation policy as general randomized vacation policy. The policy eliminates a parameter J so that it is easy to be widely applied to some more complex queue systems. Moreover it is not a stand alone vacation policy but also summarizes multiple and single vacation policy. That is our motivation to put forward the general randomized vacation policy.

The remainder of this paper is organized as follows. A full description of the model and analysis of the system embedded with the Markov chain are given in Sect. 2. In Sect. 3, some important measures performance of the system are obtained. In Sect. 4, we give two special cases of the model. Finally in Sect. 5, we present some numerical results to illustrate the effect of α and p on the performance of the system. Section 6 concludes the paper.

2 Description and Analysis of Model

In the section, we describe our model with following assumptions. Customers arrive the system according to a Poisson process with rate λ. When the first "essential" service of a customer is completed by the server, he or she will demand the second "optional" service with probability α. We assume that the first "essential" service and the second "optional" service both follow general distributions, with probability distribution functions $G_1(x)$ and $G_2(x)$, respectively. In addition, let $g_k(x)$, $\frac{1}{u_k}$, and $u_k(x)dx = \frac{dG_k(x)}{1-G_k(x)}, k = 1, 2$, denote the corresponding probability density functions, means and hazard rate functions. When an busy period is completed, the server immediately takes a vacation with general distribution $V(x)$. Let $v(x), v$ and $w(x)dx = \frac{dV(x)}{1-V(x)}$ be the corresponding probability density function, mean and hazard rate function. If there is at least one customer in the system at the end of the vacation, the server will immediately serve the customer. Otherwise, the server will either take another vacation with probability p or remain idle waiting for the arrival of customers with probability $1-p$. Obviously, if $p = 1$, our model can be simplified to the

M/G/1 queue with second optional service and multiple vacations; if $p = 0$, the model can be also simplified to the M/G/1 queue with second optional service and single vacation.

We assume, throughout this paper, that various stochastic processes involved in the system are mutual independence and obey first-come first-served (FCFS) service discipline. For a given function $F(x)$, its Laplace-Stieltjes transform (LST) denotes by $F^*(s) = \int_0^\infty e^{-sx} dF(x)$. And then, we define $\rho = \frac{\lambda}{u_1} + \alpha\frac{\lambda}{u_2}$. Obviously, $\rho < 1$ is the necessary and sufficient condition when a steady state solution exists.

Let $N(t)$ be the system size including the one being served (if any) at time t, and denote by $G_1^-(x), G_2^-(x)$ and $V^-(x)$ the elapsed first "essential" service, elapsed second "optional" service and elapsed vacation at time t, respectively. In addition, we introduce the following random variable

$$J(t) = \begin{cases} 0, & \text{if the server is idle at time t,} \\ 1, & \text{if the server is busy providing a essential service at time t,} \\ 2, & \text{if the server is busy providing a second optional service at time t,} \\ 3, & \text{if the server is taking a vacation at time t} \end{cases}$$

At time t, the system can be described by the process $(N(t), c(t))$ where $c(t) = 0$ if $J(t) = 0$; $c(t) = G_1^-(x)$ if $J(t) = 1$; $c(t) = G_2^-(x)$ if $J(t) = 2$ and $c(t) = V^-(x)$ if $J(t) = 3$. For further studying the model, we define the following limiting probabilities:

$$\begin{aligned} p_{0,0} &= \lim_{t\to\infty} p(N(t) = 0, c(t) = 0), \\ p_{1,n} &= \lim_{t\to\infty} p(N(t) = n, c(t) = G_1^-(x); x \le G_1^-(x) \le x + dx),\ n \ge 1,\ x \ge 0, \\ p_{2,n} &= \lim_{t\to\infty} p(N(t) = n, c(t) = G_2^-(x); x \le G_2^-(x) \le x + dx),\ n \ge 1,\ x \ge 0, \\ p_{3,n} &= \lim_{t\to\infty} p(N(t) = n, c(t) = V^-(x); x \le V^-(x) \le x + dx),\ n \ge 0,\ x \ge 0 \end{aligned}$$

Then in steady-state condition, the Kolmogorov forward equations to govern the model can be written as follows:

$$\lambda p_{0,0} = (1 - p)\int_0^\infty p_{3,0}(x)w(x)dx \tag{1}$$

$$\frac{dp_{1,1}(x)}{dx} + [\lambda + u_1(x)]p_{1,1}(x) = 0 \tag{2}$$

$$\frac{dp_{1,n}(x)}{dx} + [\lambda + u_1(x)]p_{1,n}(x) = \lambda p_{1,n-1}(x),\ n \ge 2 \tag{3}$$

$$\frac{dp_{2,1}(x)}{dx} + [\lambda + u_2(x)]p_{2,1}(x) = 0 \tag{4}$$

$$\frac{dp_{2,n}(x)}{dx} + [\lambda + u_2(x)]p_{2,n}(x) = \lambda p_{2,n-1}(x),\ n \ge 2 \tag{5}$$

$$\frac{dp_{3,0}(x)}{dx} + [\lambda + w(x)]p_{3,0}(x) = 0 \tag{6}$$

$$\frac{dp_{3,n}(x)}{dx} + [\lambda + w(x)]p_{3,n}(x) = \lambda p_{3,n-1}(x),\ n \geq 1 \tag{7}$$

Equations (1)–(7) will be solved under the following boundary conditions at time $x = 0$

$$p_{1,1}(0) = \lambda p_{0,0} + (1-\alpha)\int_0^\infty p_{1,2}u_1(x)dx + \int_0^\infty p_{2,2}u_2(x)dx + \int_0^\infty p_{3,1}w(x)dx \tag{8}$$

$$p_{1,n}(0) = (1-\alpha)\int_0^\infty p_{1,n+1}u_1(x)dx + \int_0^\infty p_{2,n+1}u_2(x)dx + \int_0^\infty p_{3,n}w(x)dx,\ n \geq 2 \tag{9}$$

$$p_{2,n}(0) = \alpha\int_0^\infty p_{1,n}u_1(x)dx,\ n \geq 1 \tag{10}$$

$$p_{3,0}(0) = (1-\alpha)\int_0^\infty p_{1,1}u_1(x)dx + \int_0^\infty p_{2,1}u_2(x)dx + p\int_0^\infty p_{3,0}w(x)dx \tag{11}$$

In order to solve the above Equations, we define some probability generating functions as follows:

$$P_i(x,z) = \sum_{n=1}^{\infty} p_{i,n}(x)z^n,\ P_3(x,z) = \sum_{n=0}^{\infty} p_{3,n}(x)z^n, P_i(z) = \int_0^\infty P_k(x,z)dx$$

where $i = 1, 2; k = 1, 2, 3$.

Multiplying both sides of Eqs. (2) and (3) by z^n $(n = 1, 2, \cdots)$ and summing over n, then we have

$$P_1(x,z) = P_1(0,z)[1 - G_1(x)]e^{-\lambda(1-z)x} \tag{12}$$

Similar proceeding on the Eqs. (4)–(7), then we obtain

$$P_2(x,z) = P_2(0,z)[1 - G_2(x)]e^{-\lambda(1-z)x} \tag{13}$$

and

$$P_3(x,z) = P_3(0,z)[1 - V(x)]e^{-\lambda(1-z)x} \tag{14}$$

In the same way, we can get the following equation from Eqs. (8) and (9)

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - p_{3,0}(0) + \frac{1-\alpha}{z}P_1(0,z)G_1^*(\lambda(1-z)) \\ +\tfrac{1}{z}P_2(0,z)G_2^*(\lambda(1-z)) + P_3(0,z)V^*(\lambda(1-z)) \end{aligned} \tag{15}$$

For convenience, let $r(z) = \lambda(1-z)$. From Eq.(15), we have

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - p_{3,0}(0) + \frac{1-\alpha}{z}P_1(0,z)G_1^*(r(z)) \\ +\tfrac{1}{z}P_2(0,z)G_2^*(r(z)) + P_3(0,z)V^*(r(z)) \end{aligned}$$

Solving the differential Eq.(6) yields

$$p_{3,0}(x) = p_{3,0}(0)(1 - V(x))e^{-\lambda x} \tag{16}$$

Then multiplying both sides of Eq.(16) by $w(x)$ and integrating with x from 0 to ∞, together with Eq.(1), we have

$$p_{3,0} = \frac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} \tag{17}$$

Substituting Eq.(17) into Eq.(15), we obtain

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - \tfrac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} + \frac{1-\alpha}{z} P_1(0,z) G_1^*(r(z)) \\ + \frac{1}{z} P_2(0,z) G_2^*(r(z)) + P_3(0,z) V^*(r(z)) \end{aligned} \tag{18}$$

Since $P_3(0,z) = p_{3,0}(0)$, Eq.(18) can be written as follows:

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - \tfrac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} + \frac{1-\alpha}{z} P_1(0,z) G_1^*(r(z)) \\ + \frac{1}{z} P_2(0,z) G_2^*(r(z)) + \frac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} V^*(r(z)) \end{aligned} \tag{19}$$

Multiplying both sides of Eq. (10) by z^n $(n = 1, 2, \cdots)$ and summing over n, then we have

$$P_2(0,z) = \alpha P_1(0,z) G_1(r(z)). \tag{20}$$

Substituting Eq.(20) into Eq.(19), we obtain

$$P_1(0,z) = \frac{\lambda z p_{0,0}[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]}{(1-p)V^*(\lambda)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{21}$$

Integrating both sides of Eq.(12) with x from 0 to ∞, then we get

$$P_1(z) = P_1(0,z)\frac{1 - G_1^*(r(z))}{\lambda(1-z)} \tag{22}$$

Substituting Eq.(21) into Eq.(22), we have

$$P_1(z) = \frac{z p_{0,0}[1 - G_1^*(r(z))][1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]}{(1-p)V^*(\lambda)(1-z)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{23}$$

Performing similar operations on Eqs. (13) and (14), then we get

$$P_2(z) = P_2(0,z)\frac{1 - G_2^*(r(z))}{\lambda(1-z)} \tag{24}$$

and

$$P_3(z) = P_3(0,z)\frac{1 - V^*(\lambda(1-z))}{\lambda(1-z)} \tag{25}$$

Then, substituting Eqs. (20) and (17) into (24) and (25), respectively, we have

$$P_2(z) = \frac{p_{0,0} z \alpha G_1^*(r(z))[1 - G_2^*(r(z))][1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]}{(1-p)V^*(\lambda)(1-z)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{26}$$

and

$$P_3(z) = \frac{p_{0,0}[1 - V^*(r(z))]}{(1-p)V^*(\lambda)(1-z)} \tag{27}$$

From Eqs. (23), (26) and (27), we get the probability generating function for steady-state system size

$$\begin{aligned} P(z) &= P_1(z) + P_2(z) + P_3(z) + p_{0,0} \\ &= \frac{p_{0,0}[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{(1-p)V^*(\lambda)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \end{aligned} \tag{28}$$

Using the normalization condition $P_1(z) + P_2(z) + P_3(z) + p_{0,0} = 1$, thus we have

$$p_{0,0} = \frac{(1-p)(1-\rho)V^*(\lambda)}{\lambda v + (1-p)V^*(\lambda)} \tag{29}$$

Substituting $p_{0,0}$ into Eq. (28), it is given as

$$P(z) = \frac{(1-\rho)[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{[\lambda v + (1-p)V^*(\lambda)][(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]}$$

Based on the above analysis, we will give some performance analysis for the system in the next section.

3 Performance Analysis

In the section, we will obtain the probability generating function of the steady state system size at a departure epoch, and the mean values for the steady state system size, waiting time, sojourn time. In addition, we will obtain the probability for each state of the server.

We denote by $\pi_n, n = 0, 1, \cdots$ the probabilities that there are n customers in the system at a departure point (no including the one just departing from the system). Then, we can obtain the forward equations as follows:

$$\pi_n = M(1-\alpha)\int_0^\infty p_{1,n+1}u_1(x)dx + M\int_0^\infty p_{2,n+1}u_2(x)dx,\ n = 0, 1, \cdots \tag{29}$$

where M is the normalizing constant.

Multiplying Eq.(29) by z^n $(n = 1, 2, \cdots)$ and summing over n, then together with Eqs. (12) and (13), we obtain the probability generating function of the system size $\Pi(z)$ at a departure epoch as follows:

$$\Pi(z) = \frac{M\lambda p_{0,0}[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{(1-p)V^*(\lambda)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{30}$$

by utilizing the normalizing condition $\Pi(1) = 1$, from Eq.(30), we have

$$M = \frac{(1-\rho)(1-p)V^*(\lambda)}{\lambda p_{0,0}[\lambda v + (1-p)V^*(\lambda)]} \tag{31}$$

Substituting Eq. (31) into Eq. (30), we obtain

$$\Pi(z) = \frac{(1-\rho)[1+(1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha)+\alpha G_2^*(r(z))]}{[\lambda v + (1-p)V^*(\lambda)][(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{32}$$

Thus, the probability generating function of the steady state system size at a departure epoch is same as the one of the system size at a random epoch. From the Eq. (32), we can have a theorem as follows:

Theorem 1. *If $\rho < 1$, the steady-state system size L can be decomposed into the sums of two stochastic variables, i.e., $L = L_0 + L_d$, where L_0 denotes the steady-state system size at departure epoch of M/G/1 queue with second optional service whose generating function has been given in* [1]*, L_d is the steady-state additional system size due to the general randomized vacations with the probability generating function as follows*

$$L_d(z) = \frac{1+(1-p)V^*(\lambda)(1-z) - V^*(r(z))}{(1-z)[\lambda v + (1-p)V^*(\lambda)]} \tag{33}$$

Proof. From Eq. (32), it is very easy to obtain the theorem.

Utilizing Theorem 1, we can obtain a remark as follows.

Remark 1. If $\rho < 1$, the mean system size can be written as $E[L] = E[L_0] + E[L_d]$, where $E[L_0]$ denotes the mean system size at departure epoch of M/G/1 queue with second optional service whose detailed expression has been given in [1], $E[L_d]$ is the additional mean system size due to the general randomized vacations with the detailed expression as follows

$$E[L_d] = \frac{\lambda^2 v^{(2)}}{2[\lambda v + (1-p)V^*(\lambda)]} \tag{34}$$

where $v^{(2)}$ stands for the two moment of the general distribution $V(x)$.

Utilizing Remark 1 and Little formula, we can obtain the other two remarks as follows.

Remark 2. If $\rho < 1$, the expected value for the sojourn time of a customer in the system is given by

$$E[W] = \frac{E[L_0]}{\lambda} + \frac{\lambda v^{(2)}}{2[\lambda v + (1-p)V^*(\lambda)]} \tag{35}$$

Remark 3. If $\rho < 1$, the expected value for the waiting time of a customer in the system is given by

$$E[W_q] = \frac{E[L_0]}{\lambda} + \frac{\lambda v^{(2)}}{2[\lambda v + (1-p)V^*(\lambda)]} - \frac{1}{u_1} - \frac{\alpha}{u_2} \tag{36}$$

From the expressions of $P_1(z), P_2(z), P_3(z)$ and $p_{0,0}$, we can determined the probability for each state of the server, as in the following Corollary 1.

Corollary 1. *If $\rho < 1$, then*

(1) the probability when the server is idle is

$$p_{0,0} = \frac{(1-p)(1-\rho)V^*(\lambda)}{\lambda v + (1-p)V^*(\lambda)}$$

(2) the probability when the server is busy with supplying the first essential service is

$$P_1 = \rho_1$$

(3) the probability when the server is busy with supplying the second optional service is

$$P_2 = \rho_2$$

(4) the probability when the server is taking a vacation is

$$P_3 = \frac{(1-\rho)\lambda v}{\lambda v + (1-p)V^*(\lambda)}$$

where $\rho_1 = \dfrac{\lambda}{u_1}$, $\rho_2 = \dfrac{\alpha\lambda}{u_2}$.

4 Special Cases of the Model

In the section, we will give two special cases of our model by choosing the different value of p. We will only study $\Pi(z)$ for the two cases of the model, and the other parameters can be studied similarly.

Case 1. Let $p = 1$. Then our model can be simplified to the M/G/1 queue with second optional service and multiple vacations. Let $p = 1$ in $\Pi(z)$. We have the probability generating function of system size at a departure epoch as follows

$$\Pi(z) = \frac{(1-\rho)[1 - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{\lambda v[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]}$$

Case 2. Let $p = 0$. Then our model can be simplified to the M/G/1 queue with second optional service and single vacation. In addition, let $p = 0$ in $\Pi(z)$. We have the probability generating function of system size at a departure epoch as follows

$$\Pi(z) = \frac{(1-\rho)[1 + V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{[\lambda v + V^*(\lambda)][(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]}$$

5 Numerical Results

In the section, our first purpose is to study the effects of parameters p and λ on the expected system size of messages and the expected waiting time of messages in the system. We assume that the length of a first essential service, a second optional service and a vacation all follow exponential distributions with parameters μ_1, μ_2 and ν, respectively.

For convenience, we choose $\mu_1 = 2.5, \mu_2 = 2.0, \nu = 1.5, \alpha = 0.5$ and $p = 0, 0.2, 0.5, 0.7, 1$, and then vary the value of λ from 0 to 1.0.

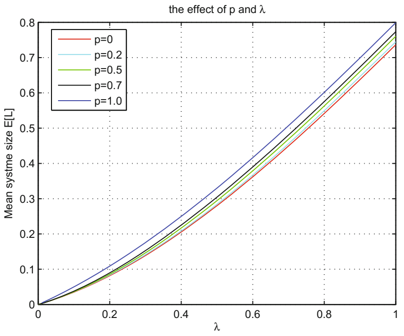

Fig. 1. The expected system size

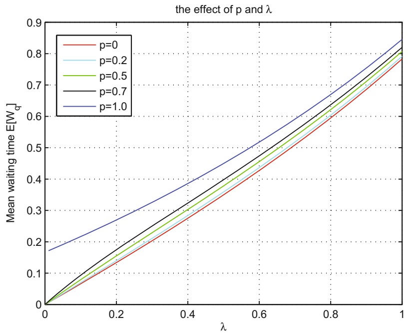

Fig. 2. The expected waiting time

Figures 1 and 2 show that the expected system size and the expected waiting time are functions of the arrival rate λ and p. We find that whenever λ increases, the expected system size and expected waiting time increase at a higher level with a fixed p, so the both are increasing functions of λ. Similarly the both are also increasing functions of p with a fixed λ.

The second purpose is to study the effects of parameters p and λ on probabilities $p_{0,0}$ and P_3. We make some assumptions as above.

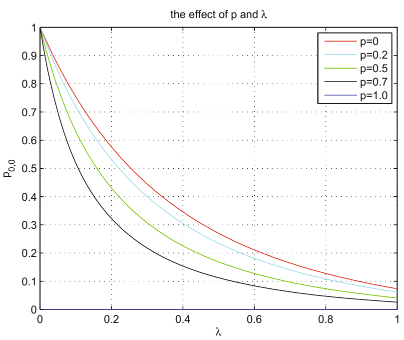

Fig. 3. The expected system size

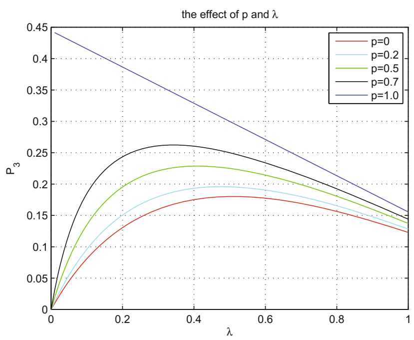

Fig. 4. The expected waiting time

Figures 3 and 4 show that $p_{0,0}$ is a function of the arrival rate λ and p. We find that λ increases, $p_{0,0}$ decreases at a lower level with a fixed p, so it is a decreasing function of λ. Furthermore, P_3 is increasing function about p with a fixed λ, but not of the monotonicity, of λ with a fixed p.

6 Conclusions

In this paper, we study the general randomized vacation policy for the M/G/1 queueing system with second optional service. By the Kolmogorov forward equations and supplementary variable method, we obtain the probability generating functions for the steady state system size and expected values for the steady state system size, waiting time and sojourn time. Additionally, utilizing numerical illustration, we study the effects of parameters p and λ on the expected system size of messages, the expected waiting time of messages and the probabilities when the server is idle and is on vacation.

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The Matrix Representation of Fuzzy Error Logic Conjunction and Applied Research

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Abstract. Based on the paper [1–11], this paper studied matrix representation of six basic transformations in error logic and their application in eliminating or avoiding errors, including the following: Similar conjunctions corresponds to T_x^{-1} (reverse similarity); The replacement conjunction corresponds to T_z^{-1} (inverse permutation); The addition of the conjunction corresponds to T_{zn}^{-1} (reducing the transforming word); The decomposed conjunction corresponds to T_f^{-1} (combination transformation words); Destroyed conjunction corresponds to T_h^{-1} (produce a transformative word); Unit conversion Conjunction word system Td (unit); corresponds to T_d^{-1} (inverse unit conversion word); Error logical quantifier system.

Keywords: Error logic · Convert conjunctions · Matrix representation · Eliminate the wrong

1 The Concept of Fuzzy Error Matrix

Definition 1.1

$$\text{Set A} = \begin{pmatrix} ((u_{111}\,u_{112}\ldots u_{11k}),\,x_{11}) & ((u_{121}\,u_{122}\ldots u_{12k}),\,x_{12}) & \ldots & ((u_{1n1}\,u_{1n2}\ldots u_{1nk}),\,x_{1n}) \\ ((u_{211}\,u_{212}\ldots u_{21k}),\,x_{21}) & ((u_{221}\,u_{222}\ldots u_{22k}),\,x_{22}) & \ldots & ((u_{2n1}\,u_{2n2}\ldots u_{2nk}),\,x_{2n}) \\ \ldots & \ldots & \ldots & \ldots \\ ((u_{m11}\,u_{m12}\ldots u_{m1k}),\,x_{m1}) & ((u_{m21}\,u_{m22}\ldots u_{m2k}),\,x_{m2}) & \ldots & ((u_{mn1}\,u_{mn2}\ldots u_{mnk}),\,x_{mn}) \end{pmatrix}$$

Then A is called an m × n-order fuzzy error matrix, in which $x_{ij} \in [0, 1]$, I = 1,2,…, m; j = 1,2,…,n.

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Definition 1.2

$$\text{Set A} = \begin{pmatrix} U_{10}\,S_{10}(t) & \vec{p}_{10}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_1(t) & L_1(t) & x_{10}(t)=f_{10}((u(t),\ \vec{p}_1),G_{U10}(t)) & G_{U10}(t) \\ U_{11},S_{11}(t) & \vec{p}_{11}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{11}(t) & L_{11}(t) & x_{11}(t)=f_{11}((u(t),\ \vec{p}_1),G_{U11}(t)) & G_{U11}(t) \\ \cdots & & \cdots & & \cdots & \cdots \\ U_{1t}\,S_{1t}(t) & \vec{p}_{1t}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{1t}(t)L_{1t}(t) & & x_{1t}(t)=f_{1t}((u(t),\ \vec{p}_1),G_{U1t}(t)) & G_{U1t}(t) \end{pmatrix}$$

Then A is called (t + 1) × 7 fuzzy error matrix, (t + 1) × 7 fuzzy error matrix elements are the collection, in which $x_{1i} \in[0,1]$, I = 0,1,2,…,t.

2 Matrix Representation of Fuzzy Error Logic Transformation

By the definition of fuzzy matrix multiplication, each column of the matrix can be defined as a decomposition transformation. It can act on every element of the right matrix.

$$(\mathrm{u},\mathrm{x}) = (\mathrm{U1},\mathrm{S1(t)},\vec{p}_1,T_1(t),L_1(t))\,,x(t)=f((u(t),\vec{p}_1),G_\mathrm{U}(t)) =$$
$$\begin{pmatrix} U_{10}S_{10}(t) & \vec{p}_{10}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_1(t) & L_1(t) & x_{10}(t)=f_{10}((u(t),\ \vec{p}_1),G_{U10}(t)) & G_{U10}(t) \\ U_{11},S_{11}(t) & \vec{p}_{11}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{11}(t) & L_{11}(t) & x_{11}(t)=f_{11}((u(t),\ \vec{p}_1),G_{U11}(t)) & G_{U11}(t) \\ \cdots & & \cdots & & \cdots & \cdots \\ U_{1\mathrm{t}}S_{1\mathrm{t}}(t) & \vec{p}_{1t}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{1\mathrm{t}}(t)L_{1\mathrm{t}}(t) & & x_{1\mathrm{t}}(t)=f_{1\mathrm{t}}((u(t),\ \vec{p}_1),G_{U1\mathrm{t}}(t)) & G_{U1\mathrm{t}}(t) \end{pmatrix}$$

$X_{1i} \in[0,1]$, I = 0,1,2,…,t;.

Definition 2.1

$$\text{Set }(\mathrm{u},\mathrm{x}) = (U_1,S_1(\mathrm{t}),\vec{p}_1,T_1(t),L_1(t))\,,x(t)=f((u(t),\vec{p}_1),G_\mathrm{A}(t)) =$$
$$\begin{pmatrix} U_{10}S_{10}(t) & \vec{p}_{10}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_1(t) & L_1(t) & x_{10}(t)=f_{10}((u(t),\ \vec{p}_1),G_{U10}(t)) & G_{U10}(t) \\ U_{11},S_{11}(t) & \vec{p}_{11}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{11}(t) & L_{11}(t) & x_{11}(t)=f_{11}((u(t),\ \vec{p}_1),G_{U11}(t)) & G_{U11}(t) \\ \cdots & & \cdots & & \cdots & \cdots \\ U_{1\mathrm{t}}S_{1\mathrm{t}}(t) & \vec{p}_{1t}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{1\mathrm{t}}(t)L_{1\mathrm{t}}(t) & & x_{1\mathrm{t}}(t)=f_{1\mathrm{t}}((u(t),\ \vec{p}_1),G_{U1\mathrm{t}}(t)) & G_{U1\mathrm{t}}(t) \end{pmatrix}$$

$X_{1i} \in[0,1]$, I = 0,1,2,…,t;.

$$(\mathrm{V},\mathrm{y}) = (V_2,S_2(t),\vec{p}_2,T_2(t),L_2(t)),y(t)=f((v(t),\vec{p}_2),G_\mathrm{V}(t)) =$$
$$\begin{pmatrix} (V_{20},S_{20}(t)) & \vec{p}_{20}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_2(t) & L_2(t) & y_{20}(t)=f_{20}((u(t),\ \vec{p}_2),G_\mathrm{V}(t)) & G_{V10}(t) \\ (V_{21},S_{21}(t)) & \vec{p}_{21}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{21}(t) & L_{21}(t) & y_{21}(t)=f_{21}((u(t),\ \vec{p}_2),G_\mathrm{V}(t)) & G_{V11}(t) \\ \cdots & & \cdots & & \cdots & \cdots \\ (V_{2\mathrm{t}},S_{\mathrm{tt}}(t)) & \vec{p}_{2t}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{2\mathrm{t}}(t) & ,L_{2\mathrm{t}}(t) & y_{2\mathrm{t}}(t)=f_{2\mathrm{t}}((u(t),\ \vec{p}_2),G_\mathrm{V}(t)) & G_{V1\mathrm{t}}(t) \end{pmatrix}$$

$Y_{2i} \in[0,1]$, I = 0,1,2,…,t;

$$A = (u, x)\begin{pmatrix} ((u_{11} & u_{12} & \dots & u_{1k}), & x_1) \\ ((u_{21} & u_{22} & \dots & u_{2k}), & x_2) \\ \dots & \dots & \dots & \dots & \dots \\ ((u_{m1} & u_{m2} & \dots & u_{mk}), & x_m) \end{pmatrix}$$

$x_i \in [0,1]$, I = 1,2,…,m.

$$B = (v,y) = [((v_{11} v_{12} \quad \dots \quad v_{1k}),\ y)]$$

$$AB = \begin{pmatrix} ((w_{11} w_{12} & \dots & w_{1k}),\ x_1 \wedge y) \\ ((w_{21} w_{22} & \dots & w_{2k}),\ x_2 \wedge y) \\ \dots & \dots & \dots \\ ((wu_{m1} w_{m2} & \dots & w_{mk}),\ x_m \wedge y) \end{pmatrix}$$

If $x_i \geq y,\ x_i = 1, 2, \cdots\cdots, m, y \in [0, 1]$, then this element is empty, or this element $((u_{i1} u_{i2} \cdots u_{ik}), x_{i1} \wedge y) = ((u_{i1} u_{i2} \cdots u_{ik}), x_i)$, then v = ($u_1$ hu_1 h$_{\dots}$ h $u_{m\circ}$. Then called

$$A = \begin{pmatrix} ((u_{11} & u_{12} & \dots & u_{1k}), & x_1) \\ ((u_{21} & u_{22} & \dots & u_{2k}), & x_2) \\ \dots & \dots & \dots & \dots & \dots \\ ((u_{m1} & u_{m2} & \dots & u_{mk}), & x_m) \end{pmatrix}$$

is decomposition transformation of ((v_{11} v_{12} … v_{1k}), y).

$x_i \in [0,1]$, I = 1,2,…,m;.

Definition 2.2

$$A = \begin{pmatrix} ((u_{11} & u_{11} & \dots & u_{1k}), & x_1) \\ ((u_{21} & u_{21} & \dots & u_{2k}), & x_2) \\ \dots & \dots & \dots & \dots & \dots \\ ((u_{m1} & u_{m1} & \dots & u_{mk}), & x_m) \end{pmatrix}$$

$x_i \in [0,1]$, I = 1,2,…,m;

$$B = [((v_{11} v_{11} \cdots v_{1k}), y_1) \quad ((v_{21} v_{22} \cdots v_{2k}), y_2) \quad \cdots \quad ((v_{m1} v_{m1} \cdots v_{mk}), y_m)]$$

$$BA = (w, z)[((w_{11} w_2 \cdots w_k), \quad (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \quad \cdots \quad \vee (x_m \wedge y_m))]$$

W_J = u_{1J} h u_{2J} h … h u_{mJ} = v_{1J} h v_{2J} h … h v_{mJ}; z is the combined target error value, and $z = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_m \wedge y_m)$.

(For example, if $y_i \geq x_i, x_i, y_i = 1, 2, \cdots, m$; and z = max($x_1$, x_2, … , x_m). Then $z = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_m \wedge y_m)$ .Can be satisfied.)

Then called

$$A = \begin{pmatrix} ((u_{11} & u_{12} & \ldots & u_{1k}), & x_{11}) \\ ((u_{21} & u_{22} & \ldots & u_{2k}), & x_{21}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((u_{m1} & u_{m2} & \ldots & u_{mk}), & x_{m1}) \end{pmatrix}$$

is combination transformation of $(((v_{11}\ v_{11} \ldots v_{1k}), y_1)\ ((v_{21}\ v_{22} \ldots v_{2k}), y_2) \ldots ((v_{m1}\ v_{m1} \ldots v_{mk}), y_m)$.

From this we can see that the decomposition transformation and the combined transformation (i.e., its inverse transformation) are the matrix multiplications of right and left. Therefore, only six basic transformations are discussed below, and its inverse is not discussed.

Definition 2.3

$$\text{Set B} = [((v_1, y_1)\quad ((v_2, y_2)\quad \cdots \quad ((v_m, y_m), y_i \in [0,1], I = 1,2,\ldots,m;]$$

$$A = \begin{pmatrix} 1 & 0 & & \ldots & & 0 & & 0 \\ 0 & 1 & & \ldots & & 0 & & 0 \\ & & & \ldots & & & & \\ 0 & 0 & \ldots & a & \ldots & 0 & 0 & i \\ & & & \ldots & & & & \\ 0 & 0 & & \ldots & & 0 & 1 & mn \\ & & & j & & & & \end{pmatrix}$$

$AB = ((v_1, y_1)\ ((v_2, y_2) \ldots\ldots a\ ((v_j, y_j) \ldots\ldots ((v_m, y_m)$, then called

$$A = \begin{pmatrix} 1 & 0 & & \ldots & & 0 & & 0 \\ 0 & 1 & & \ldots & & 0 & & 0 \\ & & & \ldots & & & & \\ 0 & 0 & \ldots & a & \ldots & 0 & 0 & i \\ & & & \ldots & & & & \\ 0 & 0 & & \ldots & & 0 & 1 & mm; \end{pmatrix}$$

is similarity transformation of $B = ((v_1, y_1)\ ((v_2, y_2) \ldots a\ ((v_j, y_j) \ldots ((v_m, y_m). y_i \in [0,1]$, $I = 1,2,\ldots,m$;

Definition 2.4

$$\text{Set } (u, x) = (U_1, S_1(t), \vec{p}_1, T_1(t), L_1(t)), x(t) = f((u(t), \vec{p}_1), G_A(t)) =$$

$$\begin{pmatrix} U_{10}S_{10}(t) & \vec{p}_{10}(x_1, x_2, \cdots, x_n) & T_1(t) & L_1(t) & x_{10}(t) = f_{10}((u(t),\ \vec{p}_1), G_{U10}(t)) & G_{U10}(t) \\ U_{11}, S_{11}(t) & \vec{p}_{11}(x_1, x_2, \cdots, x_n) & T_{11}(t) & L_{11}(t) & x_{11}(t) = f_{11}((u(t),\ \vec{p}_1), G_{U11}(t)) & G_{U11}(t) \\ \ldots & & \ldots & & \ldots & \ldots \\ U_{1t}S_{1t}(t) & \vec{p}_{1t}(x_1, x_2, \cdots, x_n) & T_{1t}(t)L_{1t}(t) & & x_{1t}(t) = f_{1t}((u(t),\ \vec{p}_1), G_{U1t}(t)) & G_{U1t}(t) \end{pmatrix}$$

$X_{1i} \in [0,1]$, $I = 0,1,2,\ldots,t$;.

$$(\mathrm{V},\mathrm{y})=(V_2,S_2(t),\ \vec{p}_2,T_2(t),L_2(t)),y(t)=f((v(t),\ \vec{p}_2),G_{\mathrm{B}}(t))=$$

$$\begin{pmatrix} (V_{20},S_{20}(t)) & \vec{p}_{20}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_2(t) & L_2(t) & y_{20}(t)=f_{20}((u(t),\ \vec{p}_2),G_{\mathrm{V}}(t)) & G_{V10}(t) \\ (V_{21},S_{21}(t)) & \vec{p}_{21}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{21}(t) & L_{21}(t) & y_{21}(t)=f_{21}((u(t),\ \vec{p}_2),G_{\mathrm{V}}(t)) & G_{V11}(t) \\ & \cdots & & \cdots & \cdots & \cdots \\ (V_{2t},S_{tt}(t)) & \vec{p}_{2t}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{2t}(t) & ,L_{2t}(t) & y_{2t}(t)=f_{2t}((u(t),\ \vec{p}_2),G_{\mathrm{V}}(t)) & G_{V1t}(t) \end{pmatrix}$$

$Y_{2i}\in[0,1]$, I = 0,1,2,…,t;.

$$\mathrm{A}=(\mathrm{u},\mathrm{x})\begin{pmatrix} ((\mathrm{u}_{01} & \mathrm{u}_{02} & \ldots & \mathrm{u}_{0k}), & \mathrm{x}_0) \\ ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \mathrm{x}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \mathrm{x}_m) \end{pmatrix}$$

$x_i\in[0,1]$, I = 0,1,2,…,m.

$$\mathrm{B}\ =(\mathrm{v},\mathrm{y})=[((\mathrm{v}_{11}\quad \mathrm{v}_{12}\quad \cdots \mathrm{v}_{1k}),\mathrm{y})]$$

$$\mathrm{AB}\ =\begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}), & \mathrm{x}_0\wedge\mathrm{y}) \\ ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \mathrm{x}_1\wedge\mathrm{y}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \mathrm{x}_m\wedge\mathrm{y}) \end{pmatrix}$$

If $\mathrm{x}_0\geq\mathrm{y},\quad \mathrm{x}_i\leq\mathrm{y},\mathrm{i}=1,2,\cdots,\mathrm{m}$, then

$$\begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}), & \mathrm{x}_0\wedge\mathrm{y}) \\ ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \mathrm{x}_1\wedge\mathrm{y}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \mathrm{x}_m\wedge\mathrm{y}) \end{pmatrix}=\begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}),\mathrm{y}) \\ ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}),\mathrm{x}_1) \\ \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}),\mathrm{x}_m) \end{pmatrix}$$

then called

$$\mathrm{A}=\begin{pmatrix} ((\mathrm{u}_{01} & \mathrm{u}_{02} & \ldots & \mathrm{u}_{0k}), & \mathrm{x}_0) \\ ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \mathrm{x}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \mathrm{x}_m) \end{pmatrix}$$

is the increasing transformation of $((\mathrm{v}_{11}\ \mathrm{v}_{12}\ \ldots\ \mathrm{v}_{1k}),\ \mathrm{y})$.

Definition 2.5

$$\text{Set }(\mathrm{u},\mathrm{x})=(U_1,S_1(\mathrm{t}),\vec{p}_1,T_1(t),L_1(t))\,,x(t)=f((u(t),\vec{p}_1),G_{\mathrm{A}}(t))\ =$$

$$\begin{pmatrix} U_{10}S_{10}(t) & \vec{p}_{10}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_1(t) & L_1(t) & x_{10}(t)=f_{10}((u(t),\ \vec{p}_1),G_{U10}(t)) & G_{U10}(t) \\ U_{11},S_{11}(t) & \vec{p}_{11}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{11}(t) & L_{11}(t) & x_{11}(t)=f_{11}((u(t),\ \vec{p}_1),G_{U11}(t)) & G_{U11}(t) \\ & \cdots & & \cdots & \cdots & \cdots \\ U_{1t}S_{1t}(t) & \vec{p}_{1t}(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\mathrm{n}) & T_{1t}(t) & L_{1t}(t) & x_{1t}(t)=f_{1t}((u(t),\ \vec{p}_1),G_{U1t}(t)) & G_{U1t}(t) \end{pmatrix}$$

$X_{1i} \in [0,1]$, I = 0,1,2,…,t;.

$$(\mathrm{V}, \mathrm{y}) = (V_2, S_2(t), \vec{p}_2, T_2(t), L_2(t)), y(t) = f((v(t), \vec{p}_2), G_{\mathrm{B}}(t)) =$$
$$\begin{pmatrix} (V_{20}, S_{20}(t)) & \vec{p}_{20}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_2(t) & L_2(t) & y_{20}(t) = f_{20}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & G_{V10}(t) \\ (V_{21}, S_{21}(t)) & \vec{p}_{21}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_{21}(t) & L_{21}(t) & y_{21}(t) = f_{21}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & G_{V11}(t) \\ & \cdots & & \cdots & \cdots & \cdots \\ (V_{2\mathrm{t}}, S_{\mathrm{tt}}(t)) & \vec{p}_{2t}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_{2\mathrm{t}}(t) & , L_{2\mathrm{t}}(t) & y_{2\mathrm{t}}(t) = f_{2\mathrm{t}}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & G_{V1\mathrm{t}}(t) \end{pmatrix}$$

$Y_{2i} \in [0,1]$, I = 0,1,2,…,t;.

$$\mathrm{A} = (\mathrm{u}, \mathrm{x}) \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1\mathrm{k}}), & \mathrm{x}_0) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2\mathrm{k}}), & \mathrm{x}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{\mathrm{m}1} & \mathrm{u}_{\mathrm{m}2} & \ldots & \mathrm{u}_{\mathrm{mk}}), & \mathrm{x}_\mathrm{m}) \end{pmatrix}$$

$x_i \in [0,1]$, I = 0,1,2,…,m;.

$$\begin{aligned} \mathrm{B} &= (\mathrm{v}, \mathrm{y}) \\ &= [\, ((\mathrm{v}_{11}\mathrm{v}_{12}\ldots\mathrm{v}_{1\mathrm{k}}), \mathrm{y}_0) \quad ((\mathrm{v}_{21}\mathrm{v}_{22}\ldots\mathrm{v}_{2\mathrm{k}}), \mathrm{y}_1) \quad \cdots \quad ((\mathrm{v}_{\mathrm{m}1}\mathrm{v}_{\mathrm{m}2}\ldots\mathrm{v}_{\mathrm{mk}}), \mathrm{y}_\mathrm{m}) \,] \end{aligned}$$

$y_i \in [0,1]$, I = 0,l,2,…,m.

$$\mathrm{AB} = \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1\mathrm{k}}), & \mathrm{x}_0 \wedge \mathrm{y}_1) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2\mathrm{k}}), & \mathrm{x}_1 \wedge \mathrm{y}_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{\mathrm{m}1} & \mathrm{u}_{\mathrm{m}2} & \ldots & \mathrm{u}_{\mathrm{mk}}), & \mathrm{x}_\mathrm{m} \wedge \mathrm{y}_\mathrm{m}) \end{pmatrix}$$

If $\mathrm{x}_\mathrm{i} \leq \mathrm{y}_\mathrm{i}$, i = 1,2, $\cdots\cdots$, m. Then

$$\mathrm{AB} = \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1\mathrm{k}}), & \mathrm{x}_1) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2\mathrm{k}}), & \mathrm{x}_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{\mathrm{m}1} & \mathrm{u}_{\mathrm{m}2} & \ldots & \mathrm{u}_{\mathrm{mk}}), & \mathrm{x}_\mathrm{m}) \end{pmatrix}$$

then called

$$\mathrm{A} \;= \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1\mathrm{k}}), & \mathrm{x}_0) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2\mathrm{k}}), & \mathrm{x}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{\mathrm{m}1} & \mathrm{u}_{\mathrm{m}2} & \ldots & \mathrm{u}_{\mathrm{mk}}), & \mathrm{x}_\mathrm{m}) \end{pmatrix}$$

is permutation transformation of $((v_{11}\ v_{12}\ \ldots\ v_{1k}), y)$.

Or

$$A = (u, x)\begin{pmatrix} ((u_{11} & u_{12} & \ldots & u_{1k}), & x_0) \\ ((u_{21} & u_{22} & \ldots & u_{2k}), & x_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((u_{m1} & u_{m2} & \ldots & u_{mk}), & x_m) \end{pmatrix}$$

$x_i \in[0,1]$, I = 0,1,2,…,m;.

$$B \;= (v, y) = \begin{pmatrix} ((v_{11} & v_{12} & \ldots & v_{1k}), & y_0) \\ ((v_{21} & v_{22} & \ldots & v_{2k}), & y_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((v_{m1} & v_{m2} & \ldots & v_{mk}), & y_m) \end{pmatrix}$$

$y_i \in[0,1]$, I = 0,1,2,…,m;.

$$A + B = \begin{pmatrix} ((u_{11} & u_{12} & \ldots & u_{1k}), & x_0 \vee y_1) \\ ((u_{21} & u_{22} & \ldots & u_{2k}), & x_1 \vee y_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((u_{m1} & u_{m2} & \ldots & u_{mk}), & x_m \vee y_m) \end{pmatrix}$$

If $x_i \geq y_i, \quad i = 1,2, \cdots, m$. Then

$$A + B = \begin{pmatrix} ((u_{11} & u_{12} & \ldots & u_{1k}), & x_1) \\ ((u_{21} & u_{22} & \ldots & u_{2k}), & x_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((u_{m1} & u_{m2} & \ldots & u_{mk}), & x_m) \end{pmatrix}$$

Called

$$A = \begin{pmatrix} ((u_{11} & u_{12} & \ldots & u_{1k}), & x_0) \\ ((u_{21} & u_{22} & \ldots & u_{2k}), & x_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((u_{m1} & u_{m2} & \ldots & u_{mk}), & x_m) \end{pmatrix}$$

is the second kind of permutation transformation of ((v11 v12 … v1 k), y).

Definition 2.6

$$\text{Set } (u, x) = (U_1, S_1(t), \vec{p}_1, T_1(t), L_1(t)), x(t) = f((u(t), \vec{p}_1), G_A(t)) =$$

$$\begin{pmatrix} U_{10}\, S_{10}(t) & \vec{p}_{10}(x_1, x_2, \cdots, x_n) & T_1(t) & L_1(t) & x_{10}(t) = f_{10}((u(t), & \vec{p}_1), G_{U10}(t)) & G_{U10}(t) \\ U_{11}, S_{11}(t) & \vec{p}_{11}(x_1, x_2, \cdots, x_n) & T_{11}(t) & L_{11}(t) & x_{11}(t) = f_{11}((u(t), & \vec{p}_1), G_{U11}(t)) & G_{U11}(t) \\ \ldots & & \ldots & & \ldots & & \ldots \\ U_{1t}\, S_{1t}(t) & \vec{p}_{1t}(x_1, x_2, \cdots, x_n) & T_{1t}(t) L_{1t}(t) & & x_{1t}(t) = f_{1t}((u(t), & \vec{p}_1), G_{U1t}(t)) & G_{U1t}(t) \end{pmatrix}$$

$X_{1i} \in[0,1]$, I = 0,1,2,…,t;.

$$(\mathrm{V},\mathrm{y}) = (V_2, S_2(t), \vec{p}_2, T_2(t), L_2(t)), y(t) = f((v(t), \vec{p}_2), G_{\mathrm{B}}(t)) =$$

$$\begin{pmatrix} (V_{20}, S_{20}(t)) & \vec{p}_{20}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_2(t) & L_2(t) & y_{20}(t) = f_{20}((u(t), \vec{p}_2), G_\mathrm{V}(t)) & G_{V10}(t) \\ (V_{21}, S_{21}(t)) & \vec{p}_{21}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_{21}(t) & L_{21}(t) & y_{21}(t) = f_{21}((u(t), \vec{p}_2), G_\mathrm{V}(t)) & G_{V11}(t) \\ \ldots\ldots & & \ldots\ldots & & \ldots\ldots & \ldots\ldots \\ (V_{2t}(t), S_{tt}(t)) & \vec{p}_{2t}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_{2t}(t) & , L_{2t}(t) & y_{2t}(t) = f_{2t}((u(t), \vec{p}_2), G_\mathrm{V}(t)) & G_{V1t}(t) \end{pmatrix}$$

$Y_{2i} \in [0,1]$, I = 0,1,2,…,t;.

$$\mathrm{A} = (\mathrm{u},\mathrm{x})\left[\,((\mathrm{u}_{11}\mathrm{u}_{12}\ldots\mathrm{u}_{1k}),\Phi) \quad ((\mathrm{u}_{21}\mathrm{u}_{22}\ldots\mathrm{u}_{2k}),\Phi) \quad \cdots \quad ((\mathrm{u}_{m1}\mathrm{u}_{m2}\ldots\mathrm{u}_{mk}),\Phi)\,\right]$$

$$\mathrm{B} = (\mathrm{v},\mathrm{y}) = \begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}), & \mathrm{y}_0) \\ ((\mathrm{v}_{21} & \mathrm{v}_{22} & \ldots & \mathrm{v}_{2k}), & \mathrm{y}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{v}_{m1} & \mathrm{v}_{m2} & \ldots & \mathrm{v}_{mk}), & \mathrm{y}_m) \end{pmatrix}$$

$$\mathrm{AB} = \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \Phi \wedge \mathrm{y}_1) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2k}), & \Phi \wedge \mathrm{y}_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \Phi \wedge \mathrm{y}_m) \end{pmatrix}$$

Then

$$\mathrm{B} = (\mathrm{v},\mathrm{y}) = \begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}), & \mathrm{y}_0) \\ ((\mathrm{v}_{21} & \mathrm{v}_{22} & \ldots & \mathrm{v}_{2k}), & \mathrm{y}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{v}_{m1} & \mathrm{v}_{m2} & \ldots & \mathrm{v}_{mk}), & \mathrm{y}_m) \end{pmatrix}$$

Called

$$\mathrm{A} = \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \Phi) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2k}), & \Phi) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \Phi) \end{pmatrix}$$

is destruction transformation of quantities of ((v_{11} v_{12} … v_{1k}), y).

Or

$$\mathrm{A} = (\mathrm{u},\mathrm{x})\begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \Phi) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2k}), & \Phi) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \Phi) \end{pmatrix}$$

$$B = (\mathrm{v}, \mathrm{y}) = \begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}), & \mathrm{y}_0) \\ ((\mathrm{v}_{21} & \mathrm{v}_{22} & \ldots & \mathrm{v}_{2k}), & \mathrm{y}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{v}_{m1} & \mathrm{v}_{m2} & \ldots & \mathrm{v}_{mk}), & \mathrm{y}_m) \end{pmatrix}$$

$$A①B = \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \Phi \wedge \mathrm{y}_1) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2k}), & \Phi \wedge \mathrm{y}_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \Phi \wedge \mathrm{y}_m) \end{pmatrix}$$

Then

$$B = (\mathrm{v}, \mathrm{y}) = \begin{pmatrix} ((\mathrm{v}_{11} & \mathrm{v}_{12} & \ldots & \mathrm{v}_{1k}), & \mathrm{y}_0) \\ ((\mathrm{v}_{21} & \mathrm{v}_{22} & \ldots & \mathrm{v}_{2k}), & \mathrm{y}_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{v}_{m1} & \mathrm{v}_{m2} & \ldots & \mathrm{v}_{mk}), & \mathrm{y}_m) \end{pmatrix}$$

Called

$$A = \begin{pmatrix} ((\mathrm{u}_{11} & \mathrm{u}_{12} & \ldots & \mathrm{u}_{1k}), & \Phi) \\ ((\mathrm{u}_{21} & \mathrm{u}_{22} & \ldots & \mathrm{u}_{2k}), & \Phi) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ((\mathrm{u}_{m1} & \mathrm{u}_{m2} & \ldots & \mathrm{u}_{mk}), & \Phi) \end{pmatrix}$$

is destruction transformation of quantities of $((\mathrm{v}_{11}\ \mathrm{v}_{12} \ldots \mathrm{v}_{1k}), \mathrm{y})$.

Particularly, if

$$A = (\mathrm{u}, \mathrm{x}) \begin{pmatrix} \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \\ \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \\ & & & \ldots & & & \\ \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \end{pmatrix}$$

$$B = \begin{pmatrix} (V_{20}, S_{20}(t)) & \vec{p}_{20}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) & T_2(t) & L_2(t) & y_{20}(t) = f_{20}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & G_{V10}(t) \\ (V_{21}, S_{21}(t)) & \vec{p}_{21}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) & T_{21}(t) & L_{21}(t) & y_{21}(t) = f_{21}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & G_{V11}(t) \\ & \ldots & & \ldots & \ldots & \ldots \\ (V_{2t}, S_{tt}(t)) & \vec{p}_{2t}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) & T_{2t}(t) & , L_{2t}(t) & y_{2t}(t) = f_{2t}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & G_{V1t}(t) \end{pmatrix}$$

$Y_{2i} \in [0,1]$, I = 0,1,2,…,t;.

$$A①B = \begin{pmatrix} \Phi \wedge (V_{20}, & \Phi \wedge S_{20}(t)) & \Phi \wedge \vec{p}_{20}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) & \Phi \wedge T_2(t) & \Phi \wedge L_2(t) & \Phi \wedge y_{20}(t) = f_{20}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & \Phi \wedge G_{V10}(t) \\ \Phi \wedge (V_{21}, & \Phi \wedge S_{21}(t)) & \Phi \wedge \vec{p}_{21}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) & \Phi \wedge T_{21}(t) & \Phi \wedge L_{21}(t) & \Phi \wedge y_{21}(t) = f_{21}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & \Phi \wedge G_{V11}(t) \\ \ldots & & & \ldots & & \ldots & \ldots \\ \Phi \wedge (V_{2t}, \Phi \wedge S_{tt}(t)) & & \Phi \wedge \vec{p}_{2t}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) & \Phi \wedge T_{2t}(t) & , \Phi \wedge L_{2t}(t) & \Phi \wedge y_{2t}(t) = f_{2t}((u(t), \vec{p}_2), G_{\mathrm{V}}(t)) & \Phi \wedge G_{V1t}(t) \end{pmatrix}$$

$$= \begin{pmatrix} \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \\ \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \\ & & & \ldots & & & \\ \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \end{pmatrix}$$

Called

$$A = \begin{pmatrix} \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \\ \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \\ & & & \cdots & & & \\ \Phi & \Phi & \Phi & \Phi_{\ldots} & \Phi & \Phi & \Phi \end{pmatrix}$$

is the second kind of complete destruction transformation of $((v_{11}\ v_{12} \ldots v_{1k}), y)$.

Definition 2.7

$$\text{Set } (u, x) = (U_1, S_1(t), \vec{p}_1, T_1(t), L_1(t)), x(t) = f((u(t), \vec{p}_1), G_A(t)) =$$
$$\begin{pmatrix} U_{10}\ S_{10}(t) & \vec{p}_{10}(x_1, x_2, \cdots, x_n) & T_1(t) & L_1(t) & x_{10}(t) = f_{10}((u(t),\ \vec{p}_1), G_{U10}(t)) & G_{U10}(t) \\ U_{11},\ S_{11}(t) & \vec{p}_{11}(x_1, x_2, \cdots, x_n) & T_{11}(t) & L_{11}(t) & x_{11}(t) = f_{11}((u(t),\ \vec{p}_1), G_{U11}(t)) & G_{U11}(t) \\ \cdots & & \cdots & & \cdots & \cdots \\ U_{1t}\ S_{1t}(t) & \vec{p}_{1t}(x_1, x_2, \cdots, x_n) & T_{1t}(t) L_{1t}(t) & & x_{1t}(t) = f_{1t}((u(t),\ \vec{p}_1), G_{U1t}(t)) & G_{U1t}(t) \end{pmatrix}$$

$x_{i1} \in [0,1]$, I = 0,1,2,…,t;.

$$(V, y) = (V_2, S_2(t), \vec{p}_2, T_2(t), L_2(t)), y(t) = f((v(t), \vec{p}_2), G_B(t)) =$$
$$\begin{pmatrix} (V_{20}, S_{20}(t)) & \vec{p}_{20}(x_1, x_2, \cdots, x_n) & T_2(t) & L_2(t) & y_{20}(t) = f_{20}((u(t), \vec{p}_2), G_V(t)) & G_{V10}(t) \\ (V_{21}, S_{21}(t)) & \vec{p}_{21}(x_1, x_2, \cdots, x_n) & T_{21}(t) & L_{21}(t) & y_{21}(t) = f_{21}((u(t), \vec{p}_2), G_V(t)) & G_{V11}(t) \\ \cdots & & \cdots & & \cdots & \cdots \\ (V_{2t}, S_{tt}(t)) & \vec{p}_{2t}(x_1, x_2, \cdots, x_n) & T_{2t}(t) & L_{2t}(t) & y_{2t}(t) = f_{2t}((u(t), \vec{p}_2), G_V(t)) & G_{V1t}(t) \end{pmatrix}$$

$Y_{2i} \in [0,1]$, I = 0,1,2,…,t;.

$$A = (u, x)[((u_1 \quad u_2 \quad \ldots \quad u_k), \infty)]$$

$$B = (v, y) = [((v_{11} \quad v_{12} \quad \ldots \quad v_{1k}), y)]$$

$$\begin{aligned} AB &= [((w_1 \quad w_2 \quad \ldots \quad w_k), \infty \wedge y)] \\ &= ((v_{11} \quad v_{12} \quad \ldots \quad v_{1k}), y) \end{aligned}$$

Then

$A = (u, x)[((u_1 \quad u_2 \quad \ldots \quad u_k), \infty)]$ Is called unit transformation of $B = (v, y) = [((v_{11} \quad v_{12} \quad \ldots \quad v_{1k}), y)]$.

3 Application Examples for the Matrix Representation of Fuzzy Error Logic Conjunction

The set equation for fuzzy error logic matrix

$$\text{Set A} = \begin{pmatrix} U_1 & S_1(t) & \vec{p}_1(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_1(t) & L_1(t) & y_1(t) = f_1((u(t), & \vec{p}_1), G_{\mathrm{U1}}(t)) & G_{\mathrm{U1}}(t) \\ U_2 & S_2(t) & \vec{p}_2(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_2(t) & L_2(t) & y_2(t) = f_2((u(t), & \vec{p}_2), G_{\mathrm{U2}}(t)) & G_{\mathrm{U2}}(t) \\ U_3 & S_3(t) & \vec{p}_3(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_3(t) & L_3(t) & y_3(t) = f_3((u(t), & \vec{p}_3), G_{\mathrm{U3}}(t)) & G_{\mathrm{U3}}(t) \\ U_4 & S_4(t) & \vec{p}_4(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) & T_4(t) & L_4(t) & y_4(t) = f_4((u(t), & \vec{p}_4), G_{\mathrm{U4}}(t)) & G_{U4}(t) \end{pmatrix}$$

$y_i \in [0,1]$, I = 1,2,3,4;.

$$\begin{aligned}
&U_1 = \{\mathrm{u}_{12}, \mathrm{u}_{13}, \cdots, \mathrm{u}_{15}\},\\
&S_{\mathrm{U1}}(t) = \{\mathrm{s}_{11}, \mathrm{s}_{12}, \cdots, \mathrm{s}_{15}\},\\
&\vec{p}_{U1}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) = \{\vec{p}_{11}, \vec{p}_{12}, \cdots, \vec{p}_{15}\},\\
&T_{\mathrm{U1}}(t) = \{\mathrm{t}_{11}, \mathrm{t}_{12}, \cdots, \mathrm{t}_{15}\},\\
&L_{\mathrm{U1}}(t) = \{\mathrm{l}_{11}, \mathrm{l}_{12}, \cdots, \mathrm{l}_{15}\},\\
&Y_{\mathrm{U1}}(t) = f_1((u(t), \vec{p}_1) = \{\mathrm{y}_{11}, \mathrm{y}_{12}, \cdots, \mathrm{y}_{15}\},\\
&G_{U1}(t) = \{\mathrm{g}_{11}, \mathrm{g}_{12}, \cdots, \mathrm{g}_{15}\}.
\end{aligned}$$

$$\begin{aligned}
&U_{\mathrm{U2}}(t) = \{\mathrm{u}_{21}, \mathrm{u}_{22}, \cdots, \mathrm{u}_{26}\},\\
&S_{\mathrm{U2}}(t) = \{\mathrm{s}_{21}, \mathrm{s}_{22}, \cdots, \mathrm{s}_{26}\},\\
&\vec{p}_{U2}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) = \{\vec{p}_{21}, \vec{p}_{22}, \cdots, \vec{p}_{26}\},\\
&T_{\mathrm{U2}}(t) = \{\mathrm{t}_{21}, \mathrm{t}_{22}, \cdots, \mathrm{t}_{26}\},\\
&L_{\mathrm{U2}}(t) = \{\mathrm{l}_{21}, \mathrm{l}_{22}, \cdots, \mathrm{l}_{26}\},\\
&y_{\mathrm{U2}}(t) = f_2((u(t), \vec{p}_2) = \{\mathrm{y}_{21}, \mathrm{y}_{22}, \cdots, \mathrm{y}_{26}\},\\
&G_{U2}(t) = \{\mathrm{g}_{21}, \mathrm{g}_{22}, \cdots, \mathrm{g}_{26}\}.
\end{aligned}$$

$$\begin{aligned}
&U_{\mathrm{U3}}(t) = \{\mathrm{u}_{31}, \mathrm{u}_{32}, \cdots, \mathrm{u}_{36}\},\\
&S_{\mathrm{U3}}(t) = \{\mathrm{s}_{31}, \mathrm{s}_{32}, \cdots, \mathrm{s}_{36}\},\\
&\vec{p}_{U3}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) = \{\vec{p}_{31}, \vec{p}_{32}, \cdots, \vec{p}_{26}\},\\
&T_{\mathrm{U3}}(t) = \{\mathrm{t}_{31}, \mathrm{t}_{32}, \cdots, \mathrm{t}_{36}\},\\
&L_{\mathrm{U3}}(t) = \{\mathrm{l}_{31}, \mathrm{l}_{32}, \cdots, \mathrm{l}_{36}\},\\
&Y_{\mathrm{U3}}(t) = f_3((u(t), \vec{p}_3) = \{\mathrm{y}_{31}, \mathrm{y}_{32}, \cdots, \mathrm{y}_{36}\},\\
&G_{U3}(t) = \{\mathrm{g}_{31}, \mathrm{g}_{32}, \cdots, \mathrm{g}_{36}\}.
\end{aligned}$$

$$\begin{aligned}
&U_4 = \{\mathrm{u}_{41}, \mathrm{u}_{42}, \cdots, \mathrm{u}_{47}\},\\
&S_{\mathrm{U4}}(t) = \{\mathrm{s}_{41}, \mathrm{s}_{42}, \cdots, \mathrm{s}_{47}\},\\
&\vec{p}_{U4}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) = \{\vec{p}_{41}, \vec{p}_{42}, \cdots, \vec{p}_{47}\},\\
&T_{\mathrm{U4}}(t) = \{\mathrm{t}_{41}, \mathrm{t}_{42}, \cdots, \mathrm{t}_{47}\},\\
&L_{\mathrm{U4}}(t) = \{\mathrm{l}_{41}, \mathrm{l}_{42}, \cdots, \mathrm{l}_{47}\},\\
&Y_{\mathrm{U4}}(t) = f_4((u(t), \vec{p}_4) = \{\mathrm{y}_{41}, \mathrm{y}_{42}, \cdots, \mathrm{y}_{4\mathrm{n}}\},\\
&G_{U4}(t) = \{\mathrm{g}_{41}, \mathrm{g}_{42}, \cdots, \mathrm{g}_{47}\}.
\end{aligned}$$

$$X = \begin{pmatrix} U_{1x} & S_{1x}(t) & \vec{p}_{1x}(x_1, x_2, \cdots, x_n) & T_{1x}(t) & L_{11x}(t) & x_{1x}(t) = f_{1x}((u(t),\ \vec{p}_{1x}), G_{U1x}(t)) & G_{U1x}(t)) \\ U_{2x} & S_{2x}(t) & \vec{p}_{2x}(x_1, x_2, \cdots, x_n) & T_{2x}(t) & L_{2x}(t) & x_{2x}(t) = f_{2x}((u(t),\ \vec{p}_{2x}), G_{U2x}(t)) & G_{U2x}(t)) \\ U_{3x} & S_{3x}(t) & \vec{p}_{3x}(x_1, x_2, \cdots, x_n) & T_{3x}(t) & L_{3x}(t) & x_{3x}(t) = f_{3x}((u(t),\ \vec{p}_{3x}), G_{U3x}(t)) & G_{U3x}(t)) \\ U_{4x} & S_{4x}(t) & \vec{p}_{4x}(x_1, x_2, \cdots, x_n) & T_{4x}(t) & L_{4x}(t) & x_{4x}(t) = f_{4x}((u(t),\ \vec{p}_{4x}), G_{U4x}(t)) & G_{U4x}(t)) \end{pmatrix}$$

$x_{ix} \in [0,1]$, I = 1,2,…,4;.

$$B = \begin{pmatrix} ((V_1\ S_{v1}(t)) & \vec{p}_{v1}(x_1, x_2, \cdots, x_n) & T_{v1}(t) & L_{v1}(t) & y_{v1}(t) = f_{v1}((v(t),\ \vec{p}_{v1}), G_{V1}(t)) & G_{V1}(t)) \\ ((V_2,\ S_{v2}(t)) & \vec{p}_{v2}(x_1, x_2, \cdots, x_n) & T_{v2}(t) & L_2(t) & y_{v2}(t) = f_{v2}((v(t),\ \vec{p}_{v2}), G_{V2}(t)) & G_{V2}(t)) \\ ((V_3,\ S_{v3}(t)) & \vec{p}_{v3}(x_1, x_2, \cdots, x_n) & T_{v3}(t) & L_3(t) & y_{v3}(t) = f_{v3}((v(t),\ \vec{p}_{v3}), G_{V3}(t)) & G_{V3}(t)) \\ ((V_4,\ S_{v4}(t)) & \vec{p}_{v4}(x_1, x_2, \cdots, x_n) & T_{v4}(t) & L_4(t) & y_{v4}(t) = f_{v4}((v(t),\ \vec{p}_{v4}), G_{V4}(t)) & G_{V4}(t)) \end{pmatrix}$$

$Y_{vi} \in [0,1]$, I = 1,2,…,4;.

$$\begin{aligned} &V_1 = \{u_{12}, u_{13}, \cdots, u_{15}\}, \\ &S_{V1}(t) = \{s_{11}, s_{13}, \cdots, s_{15}\}, \\ &\vec{p}_1(x_1, x_2, \cdots, x_n) = \{\vec{p}_{11}, \vec{p}_{12}, \cdots, \vec{p}_{15}\}, \\ &T_{V1}(t) = \{t_{12}, t_{13}, \cdots, t_{15}\}, \\ &L_{V1}(t) = \{l_{12}, l_{13}, \cdots, l_{15}\}, \\ &Y_{V1}(t) = f_1((u(t), \vec{p}_1) = \{y_{11}, y_{12}, \cdots, y_{15}\}, \\ &G_{V1}(t) = \{g_{11}, g_{12}, \cdots, g_{15}\}. \end{aligned}$$

$$\begin{aligned} &U_{V2} = \{u_{21}, u_{22}, \cdots, u_{25}\}, \\ &S_{V2}(t) = \{s_{21}, s_{22}, \cdots, s_{25}\}, \\ &\vec{p}_{V2}(x_1, x_2, \cdots, x_n) = \{\vec{p}_{21}, \vec{p}_{22}, \cdots, \vec{p}_{25}\}, \\ &T_{V2}(t) = \{t_{21}, t_{22}, \cdots, t_{25}\}, \\ &L_{V2}(t) = \{l_{21}, l_{22}, \cdots, l_{25}\}, \\ &y_{V2}(t) = f_2((u(t), \vec{p}_2) = \{y_{21}, y_{22}, \cdots, y_{25}\}, \\ &G_{V2}(t) = \{g_{21}, g_{22}, \cdots, g_{25}\}. \end{aligned}$$

$$\begin{aligned} &V_3 = \{u_{31}, u_{32}, \cdots, u_{36}\}, \\ &S_{V3}(t) = \{s_{31}, s_{32}, \cdots, s_{36}\}, \\ &\vec{p}_{V3}(x_1, x_2, \cdots, x_n) = \{\vec{p}_{31}, \vec{p}_{32}, \cdots, \vec{p}_{26}\}, \\ &T_{V3}(t) = \{t_{31}, t_{32}, \cdots, t_{36}\}, \\ &L_{V3}(t) = \{l_{31}, l_{32}, \cdots, l_{36}\}, \\ &Y_{V3}(t) = f_3((u(t), \vec{p}_3) = \{y_{31}, y_{32}, \cdots, y_{36}\}, \\ &G_{V3}(t) = \{g_{31}, g_{32}, \cdots, g_{36}\}. \end{aligned}$$

$$V_4 = \{\mathrm{u}_{41}, \mathrm{u}_{42}, \cdots, \mathrm{u}_{45}\},$$
$$S_{\mathrm{V4}}(t) = \{\mathrm{s}_{41}, \mathrm{s}_{42}, \cdots, \mathrm{s}_{45}\},$$
$$\vec{p}_{V4}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) = \{\vec{p}_{41}, \vec{p}_{42}, \cdots, \vec{p}_{45}\},$$
$$T_{\mathrm{V4}}(t) = \{\mathrm{t}_{41}, \mathrm{t}_{42}, \cdots, \mathrm{t}_{45}\},$$
$$L_{\mathrm{V4}}(t) = \{\mathrm{l}_{41}, \mathrm{l}_{42}, \cdots, \mathrm{l}_{45}\},$$
$$Y_{\mathrm{V4}}(t) = f_4((u(t), \vec{p}_4) = \{\mathrm{y}_{41}, \mathrm{y}_{42}, \cdots, \mathrm{y}_{45}\},$$
$$G_{V4}(t) = \{\mathrm{g}_{41}, \mathrm{g}_{42}, \cdots, \mathrm{g}_{45}\}.$$

A $\supseteq$ B is given by the given condition, and the theorem is solved by the fuzzy error matrix, If X = B, then X is the solution of equation $X \wedge A = B$, and it can be obtained by X transform to be B, here is X = B.

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Variational Iteration Method for Solving an Inverse Parabolic Problem

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Abstract. In this paper, the variational iteration method is applied to solving an inverse problem of determining more than one unknown parameters in a linear parabolic equation with Neumann boundary conditions. If one of boundary conditions is considered as unknown, it is desirable to be able to determine more than one parameter from the given data. This method is based on the use of Lagrange multipliers for identification of optimal valuse of parameters in a functional. We get a rapid convergent sequence tending to the exact solution of the inverse problem. To show the efficiency of the present method, one interesting example is presented.

Keywords: Variational iteration method · Inverse parabolic equation · Neumann boundary conditions · Lagrange multipliers

1 Introduction

In this work, we will consider the following inverse problem of simultaneously finding unknown coefficients $p(t)$, one boundary condition $q(t)$ and $u(x,t)$ from the following parabolic equation

$$u_t = u_{xx} + p(t)u + f(x,t), x \in (0,1), t \in (0,T], \tag{1}$$

with the initial-boundary conditions

$$u(x,0) = \varphi(x), x \in (0,1), t \in (0,T], \tag{2}$$

$$u_x(0,t) = q(t), t \in (0,T], \tag{3}$$

$$u_x(1,t) = \mu_1(t), t \in (0,T], \tag{4}$$

$$u(1,t) = \mu_2(t), t \in (0,T], \tag{5}$$

and the additional specification

$$u(x^*,t) = E(t), x^* \in (0,1), t \in (0,T], \tag{6}$$

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where $f(x,t), \varphi(x), \mu_1(t), \mu_2(t)$ and $E(t) \neq 0$ are known functions, $p(t)$ and $q(t)$ are unknown function, x^* is a fixed prescribed interior point in $(0,1)$.

The determination of unknown coefficients in partial differential equations of parabolic type from additional boundary conditions (i.e., measured data taken on the boundary) is well known in literature as inverse coefficient problems (ICP). Physically, the ICP is the reconstruction of an intra property of a medium in some bounded region by using state measurements taken on the boundary. ICP for semi-linear parabolic equations have been studied by many people, for example, by Cannon and Lin [1], Emine [2], Hasanov and Liu [3], Liu [4–6], Odibat [7], Varedi, Hosseini, Rahimi, et al. [8].

The variational iteration method is introduced by He [9–11] as a modification of a general Lagrange multiplier method [12], which has been proved by many authors to be a powerful mathematical tool for various types of nonlinear problems. It was successfully applied to burger's equation and coupled equation [13], a biochemical reaction model [14], singular perturbation initial value problems [15], strongly nonlinear problems [16,17], nonlinear differential equations of fractional order [18,26], generalized nonlinear Boussinesq equation [19] and generalized KdV [20], Dehghan, Liu Jinbo, Huang Dejian and Ma Yunjie have studied the inverse problems by use of the variational iteration method [7,21,22,25,27].

In this paper, we will apply the variational iteration method to find the exact solution of a control parameter $p(t)$, a boundary condition $q(t)$ in parabolic equation.

2 The Variational Iteration Method

In this section the application of variational iteration method is discussed for solving problem (1)–(5) with over specification (6). Applying a pair of transformations [2] as follows:

$$r(t) = \exp(-\int_0^t p(s)ds), \tag{7}$$

$$w(x,t) = u(x,t)r(t). \tag{8}$$

We reduce the original inverse problem (1)–(6) to the following auxiliary problem:

$$w_t = w_{xx} + r(t)f(x,t), x \in (0,1), t \in (0,T]. \tag{9}$$

$$w(x,0) = \varphi(x), x \in (0,1), t \in (0,T], \tag{10}$$

$$w_x(0,t) = r(t)q(t), t \in (0,T], \tag{11}$$

$$w_x(1,t) = r(t)\mu_1(t), t \in (0,T], \tag{12}$$

$$w(1,t) = r(t)\mu_2(t), t \in (0,T], \tag{13}$$

subject to

$$r(t) = \frac{w(x^*,t)}{E(t)}, t \in (0,T]. \tag{14}$$

It is easy to show that the original inverse problem (1)–(6) is equivalent to the auxiliary problem (9)–(14). Obviously, Eq. (9) has only one unknown function $w(x,t)$ [23,24] and has suitable form to apply the variational iteration method.

According to the variational iteration method, we consider the correction functional in $t-$ direction in the following form

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda(s)\{\frac{\partial w_n(x,s)}{\partial s} - \frac{\partial^2 \tilde{w}_n(x,s)}{\partial x^2} - \frac{\tilde{w}_n(x^*,s)}{E(s)} f(x,s)\}ds, \quad (15)$$

where $\lambda(t)$ is the general Lagrange multiplier, its optimal value is found by using variational theory, $w_0(x,t)$ is an initial approximation which must be chosen suitably and $\tilde{w}_n$ is the restricted variation i.e. $\delta\tilde{w}_n = 0$ [9].

To find the optimal value of $\lambda(t)$, we have

$$\delta w_{n+1}(x,t) = \delta w_n(x,t) + \delta \int_0^t \lambda(s)\{\frac{\partial w_n(x,s)}{\partial s} - \frac{\partial^2 \tilde{w}_n(x,s)}{\partial x^2} - \frac{\tilde{w}_n(x^*,s)}{E(s)} f(x,s)\}ds, \quad (16)$$

or

$$\delta w_{n+1}(x,t) = \delta w_n(x,t) + \delta \int_0^t \lambda(s)\{\frac{\partial w_n(x,s)}{\partial s}\}ds. \quad (17)$$

Using integration by parts, we have

$$\delta w_{n+1}(x,t) = \delta w_n(x,t)(1+\lambda(t)) - \int_0^t \delta w_n(x,s)\lambda'(s)ds = 0, \quad (18)$$

which yields

$$\lambda'(s) = 0|_{s=t}, \quad (19)$$

$$1+\lambda(s) = 0|_{s=t}. \quad (20)$$

Thus we have

$$\lambda(t) = -1. \quad (21)$$

and we obtain the following iteration formula

$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t \{\frac{\partial w_n(x,s)}{\partial s} - \frac{\partial^2 w_n(x,s)}{\partial x^2} - \frac{w_n(x^*,s)}{E(s)} f(x,s)\}ds. \quad (22)$$

Now using (22) we can find the solution of Eq. (9). Then we get the solutions of the original inverse problem from the following

$$u(x,t) = \frac{w(x,t)}{E(t)}, \quad (23)$$

and

$$p(t) = -\frac{r'(t)}{r(t)}, \quad (24)$$

then

$$q(t) = u_x(0,t), \tag{25}$$

where $r(t)$ is given in (7).

Also we can consider w_n as an approximation of the exact solution for sufficiently large values of n.

3 The Test Example

To show the efficiency of the present method, we consider the following example, which can be solved iteratively by using the variational iteration method.

Consider Eqs. (1)–(6) with the following conditions:

$$u(x,0) = \cos(\pi x) + x, \tag{26}$$
$$u_x(1,t) = \exp(t), \tag{27}$$
$$u(1,t) = 0, \tag{28}$$
$$f(x,t) = \pi^2 \exp(t)\cos(\pi x) - t^2 \exp(t)[\cos(\pi x) + x], \tag{29}$$
$$E(t) = (\frac{\sqrt{2}}{2} + \frac{1}{4})\exp(t), \tag{30}$$

with $x^* = 0.25$. The exact solution of this problem is [26]

$$u(x,t) = \exp(t)[\cos(\pi x) + x], \tag{31}$$

and

$$p(t) = 1 + t^2, \tag{32}$$
$$q(t) = \exp(t). \tag{33}$$

We set from (10)

$$w_0 = \varphi(x) = \cos(\pi x) + x. \tag{34}$$

Using Eq. (22), we obtain

$$\begin{aligned} w_1(x,t) &= w_0(x,t) - \int_0^t \{\tfrac{\partial w_0(x,s)}{\partial s} - \tfrac{\partial^2 w_0(x,s)}{\partial x^2} - \tfrac{w_0(x^*,s)}{E(s)} f(x,s)\} ds \\ &= \cos(\pi x) + x - \int_0^t \{\pi^2 \cos(\pi x) - [\pi^2 \cos(\pi x) - s^2(\cos(\pi x) + x)]\} ds \\ &= [\cos(\pi x) + x](1 - \tfrac{t^3}{3}) \\ &= \sum_{j=0}^{1} \tfrac{(-\frac{t^3}{3})^j}{j!} [\cos(\pi x) + x], \end{aligned} \tag{35}$$

$$\begin{aligned} w_2(x,t) &= w_1(x,t) - \int_0^t \{\tfrac{\partial w_1(x,s)}{\partial s} - \tfrac{\partial^2 w_1(x,s)}{\partial x^2} - \tfrac{w_1(x^*,s)}{E(s)} f(x,s)\} ds \\ &= [\cos(\pi x) + x](1 - \tfrac{t^3}{3}) - \int_0^t \{-[\cos(\pi x) + x]s^2 + (s^2 - \tfrac{s^5}{3})[\cos(\pi x) + x]\} ds \\ &= [\cos(\pi x) + x](1 - \tfrac{t^3}{3} + \tfrac{t^6}{18}) \\ &= \sum_{j=0}^{2} \tfrac{(-\frac{t^3}{3})^j}{j!} [\cos(\pi x) + x], \end{aligned} \tag{36}$$

$$
\begin{aligned}
w_3(x,t) &= w_2(x,t) - \int_0^t \{ \tfrac{\partial w_2(x,s)}{\partial s} - \tfrac{\partial^2 w_2(x,s)}{\partial x^2} - \tfrac{w_2(x^*,s)}{E(s)} f(x,s) \} ds \\
&= [\cos(\pi x) + x](1 - \tfrac{t^3}{3} + \tfrac{t^6}{18}) - \int_0^t \{ [\cos(\pi x) + x] \tfrac{s^8}{18} \} ds \\
&= [\cos(\pi x) + x](1 - \tfrac{t^3}{3} + \tfrac{t^6}{18} - \tfrac{t^9}{162}) \\
&= \textstyle\sum_{j=0}^{3} \tfrac{(-\frac{t^3}{3})^j}{j!} [\cos(\pi x) + x],
\end{aligned} \tag{37}
$$

and so on.

Generally we obtain

$$
w_n(x,t) = \sum_{j=0}^{n} \frac{(-\frac{t^3}{3})^j}{j!} [\cos(\pi x) + x]. \tag{38}
$$

Thus the exact value of w in a closed form is

$$
w(x,t) = \exp(-\frac{t^3}{3})[\cos(\pi x) + x] \quad (n \to \infty), \tag{39}
$$

which results the exact solution of the problem. It can be seen that the same results are obtained using Finite difference method [25], Comparing with Finite difference method, it is easy to know that the approximation obtained by the variational iteration method converges to its exact solution faster than those of Finite difference without calculating implicit difference scheme. The results show the computation efficiency of the variational iteration method for the studied model.

4 Conclusion

In this work, the variational iteration method has been successfully applied to inverse parabolic equation with Neumann boundary conditions. Since this method solves the problem without any need to discretization of the variables, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. The example shows that this method provides the solution of the problem in a closed form without calculating implicit difference scheme, which is an advantage of the variational iteration method over Finite difference method. Thus we can say the proposed method is very simple and straightforward.

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Using Coloring Function to Partition Vertices in a Fuzzy Graph

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Abstract. Motivated by some problems in real life, we consider the problem how to partition vertices of a fuzzy graph $\xi = (V, \sigma, \mu)$. We define a *coloring function* (or coloring for short) of a fuzzy graph ξ to be a mapping $f : V \rightarrow R$ such that $|\sigma(v)f(v) - \sigma(u)f(u)| \geq \mu(vu)$ for any $v, u \in V$. If $|\{f(v) : v \in V\}| \leq |\{g(v) : v \in V\}|$ for any coloring g, then f is a *minimum coloring* and the cardinality $|\{f(v) : v \in V\}|$ is called the *chromatic number* of ξ, denoted $\chi(\xi)$.

The topic is interesting because a series of results show that the chromatic number problem for fuzzy graphs is essential a new combinatorial optimization problem different from, but having some relations with, the chromatic number problem for crisp graphs.

Keywords: Coloring function · Chromatic number · Combinatorial optimization · Fuzzy graph

1 Introduction

Background. Graph models have been widely used in many real world phenomena, such as social networks, biological networks and finance. A graph is a convenient way of representing information with relationships between objects. In recent years, the current complicated ubiquitous information enables the relationship between objects to become more vague [6]. Therefore, fuzzy graph model has attracted a lot of attention from the communities of data science and fuzzy logic [14].

The concept of a fuzzy set was introduced by L. Zadeh in [22]. A *fuzzy set* in a referential (universe of discourse) X is characterized by a membership function A which associates with each element $x \in X$ a real number $A(x) \in [0, 1]$, having the interpretation that $A(x)$ is the membership degree of x in the fuzzy set A. A *fuzzy graph* $\xi = (V, \sigma, \mu)$ is an algebraic structure of non-empty set V together with a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$, where μ is a symmetric relation, $\sigma(x)$ and $\mu(x, y)$ (or $\mu(xy)$) represent the membership values of the vertex x and of the edge xy in ξ respectively.

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Rosenfeld introduced the concept of a fuzzy graph in 1975 [13]. Since then, the topic on fuzzy graph and its applications has been extensively studied in, for example [2,4,7,10,11,18,21]. Among them, some versions of fuzzy domination is studied in [11,15,16], and a version of fuzzy coloring is studied in [12]. Motivated by the following problem in real life, we introduce a type of coloring of a fuzzy graph.

Motivating Example. In a fuzzy graph $\xi = (V, \sigma, \mu)$ we consider the vertices in ξ as signals. The membership value $\sigma(v)$ is interpreted as characteristic value of signal v. The relation value $\mu(v_i, v_j)$ is considered as a threshold such that if only the difference of the characteristic values are more than this threshold, putting these two signals in a same information channel would not cause confusion. Specifically, two adjacent signals with a same characteristic value should be put into different information channels in order to avoid confusion. For two non-adjacent signals or two adjacent signals with big difference in their characteristic values, we can clearly put them into a same information channel. For two adjacent signals with small difference in their characteristic values, we can first assign each of them a coloring (or function value) to enlarge the difference so that the difference can be recognized, and then put them in a same information channel. The standard if two signals u, v can be put into a same information channel is $|\sigma(v)f(v) - \sigma(u)f(u)| \geq \mu(vu)$, which is a natural generalization of coloring in crisp graph. Then at least how many information channels are needed for this purpose? This problem can be mathematically stated as follows.

Definition 1. A *coloring* of a fuzzy graph $\xi = (V, \sigma, \mu)$ is a mapping $f : V \to R$ such that $|\sigma(v)f(v) - \sigma(u)f(u)| \geq \mu(vu)$ for any two vertices $v, u \in V$. If $|\{f(v) : v \in V\}| \leq |\{g(v) : v \in V\}|$ for any colorings g, then f is a *minimum coloring* and the cardinality $|\{f(v) : v \in V\}|$ is called the *chromatic number* of ξ, denoted $\chi(\xi)$.

One can easily see that a crisp graph $G = (V, E)$ can be seen as a fuzzy graph $\xi = (V, \sigma, \mu)$ such that $\sigma(v) = 1$ for every $v \in V$ and $\sigma(e) = 1$ for every $e \in E$ and $\sigma(e) = 0$ for every $e \notin E$. This is why we use the same coloring terms in fuzzy graphs as in crisp graphs. In the coloring of a crisp graph, minimum number of colors are assigned to the vertices of this graph such that any two adjacent vertices don't have a same color.

The topic is interesting because the chromatic number of a fuzzy graph has some relations with that of a crisp graph, but it is also shown that these two parameters are not necessarily related to each other. Specifically, some relations between chromatic number of a fuzzy graph and that of its corresponding crisp graph are provided. Based on these relations, we design an Heuristic to produce a coloring of a fuzzy graph from that of its corresponding crisp graph. On the other hand, it is also shown that the chromatic number problem of a fuzzy graph is essentially a new combinatorial problem different from the chromatic number problem of a crisp graph.

The *base graph* $G_\xi = (V, E)$ of a fuzzy graph $\xi = (V, \sigma, \mu)$ is a crisp graph such that $E = \{uv : \mu(u, v) > 0\}$. If two vertices are adjacent in G_ξ, so too are them in ξ. ξ is a *fuzzy complete graph* (or other type of graph) if its base graph G_ξ is a complete graph (or other type of graph). For two crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, G_1 is called a *subgraph* of G_2, if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$, denoted by $G_1 \subseteq G_2$. Let ξ_1 and ξ_2 be two fuzzy graphs. If $G_{\xi_1} \subseteq G_{\xi_2}$, then we say that ξ_1 is a *fuzzy subgraph* of ξ_2, denoted $\xi_1 \subseteq \xi_2$. The *induced subgraph of* G by $S \subseteq V$ is the graph $G[S]$ with vertex set S and edge set $\{uv \in E : u, v \in S\}$. A *clique* is a maximal set of vertices which induces a complete subgraph. The base graph G_ξ is usually written by G for short if there is no ambiguousness occurs. The notions not defined here can be seen in text books [3,13].

2 Coloring Fuzzy Graphs

To generalize the coloring of a crisp graph to that of a fuzzy graph, we first give a new equivalent version of the coloring of a crisp graph.

Definition 2.1. A *coloring* of a graph $G = (V, E)$ is a mapping $f : V \to N$ such that for any edge $vu \in E$, $|f(v) - f(u)| \geq 1$, where N is the set of natural numbers. The cardinality $|\{f(v) : v \in V\}|$ is called the *coloring number* of G. The minimum coloring number of G is called the *chromatic number* of G, denoted $\chi(G)$.

Since a crisp graph $G = (V, E)$ can be considered as a fuzzy graph $\xi = (V, \sigma, \mu)$ such that $\sigma(v) = 1$ for any $v \in V$, $\mu(e) = 1$ if $e \in E$ and $\mu(e) = 0$ if $e \notin E$, it is easy to see that Definition 1 is a natural generalization of Definition 2.1.

Lemma 1. *Let f be a fuzzy coloring of $\xi = (V, \sigma, \mu)$ and let M be a positive number. Denote by $g = M + f$ such that $g(v) = f(v) + M$ for every $v \in V$. Denote by $h = Mf$ such that $h(v) = Mf(v)$ for every $v \in V$. Then both $M + f$ and Mf are also colorings of ξ, as long as M is large enough.*

Proof. The proof is easy to be observed, and thus we omit. □

By Lemma 1, it is easy to see that we can assign M a proper value to make all the function values of a coloring Mf being natural numbers. So Definition 1 has the following equivalent version.

Definition $1'$**.** A *coloring* of a fuzzy graph $\xi = (V, \sigma, \mu)$ is a mapping $f : V \to N$ such that $|\sigma(v)f(v) - \sigma(u)f(u)| \geq \mu(v, u)$ for any two vertices $v, u \in V$.

In what follows, we will always use Definition $1'$ to continue our studies. We further have the following property.

Lemma 2. *Let f be a coloring of $\xi = (V, \sigma, \mu)$ with $v_0 \in V$. Then there exists a large enough number M such that the function g such that $g(v_0) = f(v_0) + M$ and $g(v) = f(v)$ for any $v \neq v_0$ is also a coloring of ξ.*

Proof. Since f is a coloring of ξ, $|\sigma(v)f(v) - \sigma(u)f(u)| \geq \mu(v,u)$ for all $v,u \in V$. If $\sigma(v_0) = 0$, then for $\forall\ v \neq v_0$, $|\sigma(v_0)g(v_0) - \sigma(v)g(v)| = |\sigma(v_0)f(v_0) - \sigma(v)f(v)| \geq \mu(v_0,v)$. If $\sigma(v_0) \neq 0$, then choose M such that $M \geq (|\sigma(v_0)f(v_0) - \sigma(v)f(v)| + |\mu(v_0,v)|)/\sigma(v_0)$. It follows that $|\sigma(v_0)g(v_0) - \sigma(v)g(v)| = |\sigma(v_0)f(v_0) - \sigma(v)f(v) + \sigma(v_0)M| \geq \sigma(v_0)M - |\sigma(v_0)f(v_0) - \sigma(v)f(v)| \geq \mu(v_0,v)$. $\square$

It is well known that two vertices u,v in a crisp graph G can take a same color in a coloring of G if and only if they are not adjacent. For coloring of a fuzzy graph, by combining Lemmas 1 and 2 and its proof, we have the further conclusion as follows.

Theorem 1. *Two vertices u,v of a fuzzy graph ξ can take a same color (function value) in a coloring of ξ if and only if they are not adjacent or $\sigma(u) \neq \sigma(v)$.*

Proof. If u,v are adjacent and $\sigma(u) = \sigma(v)$, then it is clear that they cannot take a same color (function value) by Definition $1'$. It remains to prove the converse direction. Let f be a coloring of ξ. Assume $\sigma(u) \neq \sigma(v)$ and $f(v) - f(u) = k > 0$. If $\sigma(u) = 0$, then from the proof of Lemma 2 we know that the function g such that $g(u) = f(u) + k$ and $g(x) = f(x)$ for every $x \neq u$ is also a colorings of ξ, in which $g(u) = g(v)$. Suppose $\sigma(u) \neq 0$. We first choose a large enough number M to enlarge $f(v) - f(u)$ such that $Mk = Mf(v) - Mf(u) \geq (|\sigma(u)Mf(u) - \sigma(v)Mf(v)| + |\mu(u,v)|)/\sigma(u)$. From Lemma 1 we know that Mf is also a colorings of ξ. Then by Lemma 2 we know that the function g such that $g(u) = Mf(u) + Mk$ and $g(x) = f(x)$ for all $x \neq u$ is also a colorings of ξ, in which $g(u) = g(v)$. $\square$

A *color class* (under some coloring of ξ) is a set of vertices having the same color. Suppose f is a coloring of ξ such that $f(u) = f(v)$ for two vertices u,v. By Lemmas 1 and 2, the functions $Mf, M+f$ and $M + f(v_0)$ in the two lemmas are still coloring of ξ. So we can use Theorem 1 once again to add another vertex to the set $\{u,v\}$ to form a larger color class (under a new coloring). Starting from all possible sets and repeating this step can result in a minimum coloring of ξ. However, the complexity for running these all possible steps is NP-hard. In fact, the chromatic number problem for a crisp graph $G = (V,E)$ is a special case of the chromatic number problem for a fuzzy graph $\xi = (V,\sigma,\mu)$ such that $\sigma(v) = 1$ for each $v \in V$ and $\mu(e) = 1$ for each $e \in E$ and $\mu(e) = 0$ for each $e \notin E$. On the other hand, the chromatic number problem for a general crisp graph is well-known to be NP-hard. By the above analysis, Definitions 1 and $1'$ have also the following equivalent form, which can help us to better understanding the NP-hard property of this problem.

Definition $1''$**.** A *coloring* of a fuzzy graph $\xi = (V,\sigma,\mu)$ is a partition of V such that any two vertices from a same part are not adjacent or have different membership values. The minimum partition number is called the *chromatic number* of ξ.

By Definition 1″, it is easy to see that a coloring of the base graph G_ξ is also a coloring of the fuzzy graph ξ. So we have the following relation between the chromatic number of a fuzzy graph and that of its base graph.

Theorem 2. *For any fuzzy graph* ξ, $\chi(\xi) \leq \chi(G_\xi)$.

The chromatic number of a fuzzy complete graph can be determined as follows.

Theorem 3. *Let* $\xi = (V, \sigma, \mu)$ *be a fuzzy complete graph.* $\{\sigma(v) : v \in V\} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. $V(\sigma_i) = \{v \in V : \sigma(v) = \sigma_i\}$, $i = 1, 2, \ldots, k$. *Then* $\chi(\xi) = M = \max_{1 \leq i \leq k}\{|V(\sigma_i)|\}$.

Proof. Since every pair of vertices is adjacent, then noting Theorem 1, we only need to consider a partition such that any two vertices from a same part have different membership degrees. We first choose exact one vertex from each set $V(\sigma_i)$, and put these chosen vertices in a set C_1. Then We choose exact one vertex from each non-empty set $V(\sigma_i)$, and put these chosen vertices in a set C_2. Repeat this step until all vertices are chosen. Obviously, the resulted sets correspond a minimum coloring of ξ. On the other hand, we note that there are all together $M = \max_{1 \leq i \leq k}\{|V(\sigma_i)|\}$ steps are needed. So the desired result. □

By Theorem 3 we know that the chromatic number of a fuzzy complete graph ξ may be any integer from 1 to n, where n is the vertex number of ξ. By contrast, a complete graph of n vertices has the chromatic number n.

Based on Theorem 3 and its proof, we can also easily obtain the chromatic number of a fuzzy split graph. In crisp graph theory, a *split graph* is a graph whose vertex set can be partitioned into a clique and a stable set, where, a *clique* is a set of vertices each pair of vertices in which is adjacent, a *stable set* consists of vertices not adjacent each other.

Corollary 1. *Let* $\xi = (V, \sigma, \mu)$ *be a fuzzy split graph with clique* K. $\{\sigma(v)|\ v \in K\} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. $V(\sigma_i) = \{v \in K|\ \sigma(v) = \sigma_i\}$, $i = 1, 2, \ldots, k$. *Then* $\chi(\xi) = \max_{1 \leq i \leq k}\{|V(\sigma_i)|\}$.

By above conclusions one can easily see that the chromatic number problem for fuzzy graphs is essential anther combinatorial optimization problem different from, but having some relations with, the chromatic number problem for crisp graphs.

Computing the chromatic number of a fuzzy graph is NP-hard. In practical situations, one must therefore be content with efficient heuristic procedures which perform reasonably well.

Based on the relations between the chromatic number of a fuzzy graph and that of its base graph, we show a heuristic which derives a coloring of a fuzzy graph from a coloring of its base graph.

FuzzyColorHeuristicI.
Input: a fuzzy graph ξ, and a coloring of G_ξ with color classes $c_1, c_2, \ldots, c_k$;
Output: color classes of ξ.
for each $c_i \neq \emptyset$, i from 1 to k **do**
  for each $v \in c_i$ **do**
    while there exists some $j \neq i$ such that $\mu(vu) = 0$ or $\sigma(v) \neq \sigma(u)$ for every $u \in c_j$ **do**
      $c_j \leftarrow c_j \cup \{v\}$; $c_i \leftarrow c_i \setminus \{v\}$;
    end while
    if $c_i = \emptyset$ **then**
      turn to the next c_i;
    else
      set back each color class to its status before c_i is considered;
      turn to the next c_i;
end for

When heuristic FuzzyColorHeuristicI stops, the output non-empty sets $c_i, i \leq k$ are color classes of ξ.

We further provide a heuristic to produce a coloring of a fuzzy graph from the fuzzy graph itself, which may help us to better understand the relationship between a coloring of a fuzzy graph and that of its base graph.

FuzzyColorHeuristicII.
Input: a fuzzy graph $\xi = (V, \sigma, \mu)$ and its base graph G;
Output: color classes $C_1, C_2, \ldots$ of ξ.
let $\{\sigma(v) : v \in V\} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$;
initially $V(\sigma_i) = \{v \in V : \sigma(v) = \sigma_i\}$, $i = 1, 2, \ldots, k$;
for each C_s, $s = 1, 2 \ldots,$ **do**
initially $C_s = \emptyset$;
  for each $V(\sigma_i) \neq \emptyset$, i from 1 to k **do**
    find color classes $c_{i1}, c_{i2}, \ldots, c_{it}$ of $G[V(\sigma_i)]$
    such that the number of color is as small as possible;
    $V(\sigma_i) \leftarrow \{c_{i1}, c_{i2}, \ldots, c_{it}\}$;
    choose exact one element c_{ij} from $V(\sigma_i)$;
    $C_s \leftarrow C_s \cup c_{ij}$;
    $V(\sigma_i) \leftarrow V(\sigma_i) \setminus \{c_{ij}\}$;
  end for;
end for

When heuristic FuzzyColorHeuristicII stops, the output sets C_s are color classes of ξ. Further, from this Heuristic and the arguments in Theorem 3 we easily have the following sharp upper bound for the chromatic number of a fuzzy graph.

Theorem 4. *Let $\xi = (V, \sigma, \mu)$ be a fuzzy complete graph. $\{\sigma(v) : v \in V\} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. $V(\sigma_i) = \{v \in V : \sigma(v) = \sigma_i\}$, $i = 1, 2, \ldots, k$. Then $\chi(\xi) \leq \max_{1 \leq i \leq k}\{\chi(G[V(\sigma_i)])\}$.*

3 An Example

Coloring fuzzy graphs has many applications in real life. Here we provide another example different from the motivated example in Introduction: Chemical storage problem.

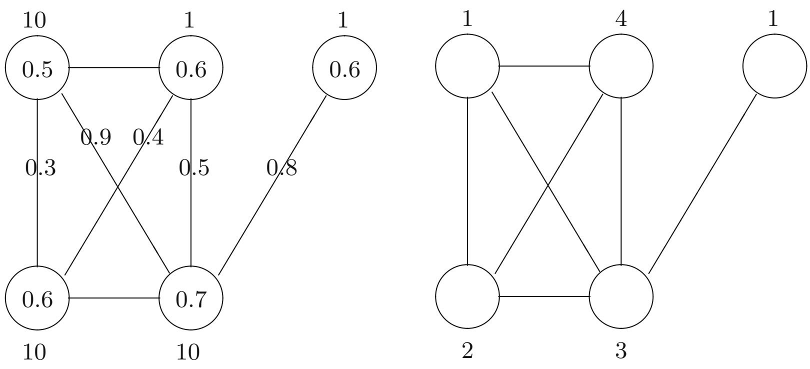

Fig. 1. Minimum colorings of a fuzzy graph and its base graph

A company manufactures n chemicals $C_1, C_2, \ldots, C_n$. Each chemical C_i has a characteristic value $\sigma(C_i)$. Each pair of chemicals C_i and C_j have a threshold $\mu(C_i, C_j)$ such that if only the difference of the characteristic values are more than this threshold, putting these two chemicals in a same warehouse would not cause explosions. As a precautionary measure, what is the least number of warehouses such that the chemicals in a same warehouse is safety? We first use a fuzzy graph $\xi = (V, \sigma, \mu)$ to denote the n chemicals and their thresholds. It is easy to see that the least number of warehouses is equal to the chromatic number of ξ.

We consider the Chemical storage problem in Fig. 1. For the crisp graph on the right side, four warehouses (four colors 1, 2, 3 and 4) are needed to store the chemicals. By contrast, for the fuzzy graph on the left side, only two warehouses (two colors 1 and 10) are needed to store these chemicals, where, the colors 1 and 10 can be considered as the strengths of the packages of the chemicals.

4 Further Study

In this paper, we concentrate our attention mainly on the existence of a minimum coloring of a fuzzy graph, and not on what such a coloring function is. For further studies on the coloring of fuzzy graphs, finding more properties of the coloring of a fuzzy graph and determining a coloring function f such that $\max\{f(v) : v \in V\}$ or $\sum_{v \in V} f(v)$ is minimum among all minimum colorings of a fuzzy graph are also interesting topics.

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Recommender: Lianying Miao, China University of Mining and Technology, Professor.

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A New Approach in Geometric Brownian Motion Model

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Abstract. Geometric Brownian Motion is One of the basic and useful models applicable in different regions such as Mathematical biology, Financial Mathematics and etc. Its differential is $dS = \alpha Sdt + \sigma Sdw_t$. Where α and σ are constant and w_t is Wiener process. αSdt is deterministic part and σSdw_t is stochastic part. α and σ could be estimated with data about amount of S in past. In this paper, we estimate α and σ in each time of past, we use numerical method to prepare quadratic function based on time, and we set them on last constant amount. Finally, we can find $dS = \alpha(t)Sdt + \sigma(t)Sdw_t$. Eventually, we express some path with real data and MATLAB.

Keywords: Geometric Brownian Motion · Implied volatility · Historic volatility · Curve fitting

1 Introduction

Geometric Brownian Motion in physics is a kind of particles motion in fluids. This motion is obtained from impact of particle to atoms or molecules in fluid. In 1827, Robert Brown, botanist, saw on microscope that pollen in water has special motion, but he couldn't find justification for this movement. In 1905, in his paper, Albert Einstein described that the motion observed by Brown in water is result of water's molecules motion. Then, he tried to model this structure [1]. However it should be noted that Thiele 1880 and Bachelier 1900 worked on Brownian Motion [2, 3]; Unaware of each other's works.

The importance and necessity of this model in some sciences like Financial Mathematics is undeniable. Bachelier was the first who studied Brownian Motion in price behavior of Paris stock market, in his PhD thesis although it wasn't welcomed. In 1965 Paul Samuelson used it in economic system and showed application of this thesis. After Samuelson, Fischer Black, Myron Scholes, and Robert K. Merton extended this model [1, 4].

In this paper we present a new model for Geometric Brownian Motion. In this structure we use numerical method, and introduce elements based on time.

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We suppose $B(t)$ whose dynamic follows formula [5]:

$$dB(t) = \mu(B(t), t)dt, \tag{1}$$

Where μ depends on the value of $B(t)$ and time.

If we rewrite model (1) in free risk market we would have:

$$dB(t) = r(t)\,B(t)dt, \tag{2}$$

$r(t)$ is interest rate.

Now we want introduce the variable that has a stochastic part or in other words has risk in market. For better understanding, suppose that some factors like weather, policy or other things have effect on the value of variable and make volatility.

In model (1), suppose variable has stochastic change. We know

$$\frac{dS}{dt} = \mu(B(t), t). \tag{3}$$

Clearly stochastic model is seen, we must have a stochastic parameter, so with constant α we can find

$$\mu(B(t), t) = \alpha(B(t), t) + \text{``}noise\text{''}, \tag{4}$$

$$\frac{dS}{dt} = \alpha(B(t), t) + \text{``}noise\text{''}. \tag{5}$$

In next section we introduce the Geometric Brownian Motion model and its properties and explain the concept of volatility. In section three, we present our new Brownian Motion model, and numerical result.

2 Geometric Brownian Motion (GBM)

We consider this noise in form of stochastic process that has following features, that it is definition of Wiener process [6, 7]:

(π, F, p) is probability space.

1. For all $W_t \in \pi, t \to W_t$ be continuous;
2. For every group $(W_{t_0}, W_{t_1} - W_{t_0}, \ldots W_{t_{k-1}} - W_{t_{k-2}}, W_{t_k} - W_{t_{k-1}})$; $t_0 < t_1 < \ldots < t_k$, each of them is independent of other one;
3. For every $0 < s < t$ the difference $(W_t - W_s)$ has Normal distribution with zero mean and $(t - s)$ variance:$(W_t - W_s) \sim N(0, t - s)$.

With this property, we define dynamic of Geometric Brownian Motion that stochastic source follows above process:

$$dS = \alpha S dt + \sigma S dw_t. \tag{6}$$

In the following, we present concept volatility.

2.1 Volatility in Geometric Brownian Motion Model

If we ignore "noise", we have this formula [4]:

$$dS = \alpha S dt. \tag{7}$$

Solving (7), we find:

$$S = S_0 e^{\alpha \Delta t}. \tag{8}$$

This formula is smooth curve. Now, we import "noise" and have volatility near formula (8).

We can find:

$$S = S_0 e^{(\alpha - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta w_t}. \tag{9}$$

To estimate value of variable in Geometric Brownian Motion model, we need α, σ, S_0 and t. To find these parameters there are tow way Implied Volatility and Historic Volatility (Fig. 1).

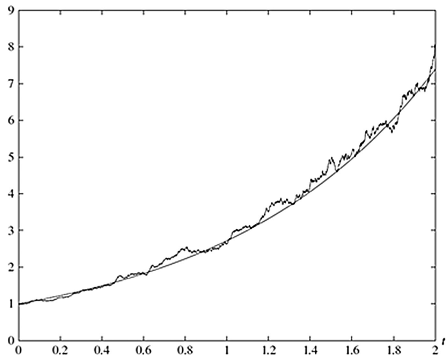

Fig. 1. Geometric Brownian Motion with $\alpha = 1$ and $\sigma = 0.4$

2.1.1 Implied Volatility

In this method, we consider expectation of decision makers who use this variable in their program and we can estimate future of variable. In this method we have some special phrase like scowl and smile of variable; and decision maker makes the best decision in this situation [4].

2.1.2 Historic Volatility

In this section with historical data, we find α and σ which are constant.

$$dS = \alpha S dt + \sigma S dw_t. \tag{10}$$

Now we divide the interval (t, T) in n sections $t_0 = t < t_1 < \ldots < t_n = T$. We use property of Ito integral so we can find [4]:

$$\begin{aligned} &Z_i = Ln\left(\frac{S_{i+1}}{S_i}\right), \\ &E[Z] = (\alpha - \tfrac{1}{2}\sigma^2)\Delta t, \\ &Var[Z] = \sigma^2 \times \Delta t, \\ &E[Z] = \tfrac{1}{n-1}\sum Z_i, \\ &Var[Z] = \tfrac{1}{n-1}\sum (Z_i - E[Z])^2. \end{aligned} \tag{11}$$

To find numerical result we use property of Brownian Motion, and we assume:

$$\Delta w_t = \varepsilon \times \sqrt{\Delta t}, \quad \varepsilon \in N(0, 1). \tag{12}$$

3 The New Geometric Brownian Motion (NGBM)

Suppose we have amount of variable S in times D $= \{t_0, t_1, \ldots, t_n\}$. Now for each T_i, $i > 0$ from set D we consider the data from t_0 to T_i and obtain α_i and σ_i with historical volatility method (11).

With this method, we will have $\{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{n-1}\}$ and $\{\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{n-1}\}$ (Fig. 2).

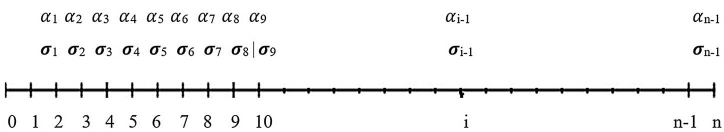

Fig. 2. Amount of $\alpha_i \& \sigma_i$ on each time unit interval

With Curve fitting, we can find quadric equation $\alpha(t)$ and $\sigma(t)$. α and σ have positive amount, so in formula (6), we consider absolute value of $\alpha(t)$ and $\sigma(t)$.

So we rewrite formula (6):

$$dS = |\alpha(t)| S dt + |\sigma(t)| S dw_t. \tag{13}$$

With numerical method, we have:

$$S_{t_{i+1}} = S_{t_i} + |\alpha(t_i)| S_{t_i} \Delta t + |\sigma(t_i)| S_{t_i} \Delta w_t. \tag{14}$$

3.1 MATLAB Code

```
function GBM(A)
B=log(A(2:end)./A(1:end-1));
sigma=zeros(1,length(B));
moy=zeros(1,length(B));
for i=1:length(B)
sigma(i)=sqrt(var(B(1:i))/i);
moy(i)=mean(B)/i+1/2*sigma(i)^2;
end
SIG=polyfit(1:length(B),sigma,2);
MO=polyfit(1:length(B),moy,2);
m=input('How many days do you like
predicte : ');
s(1)=A(end);
s1(1)=A(end);
dw=sqrt(1/m)*randn(1,m);
for i=2:m
s1(i)=s1(i-1)+s1(i-1)*moy(end)...
*1/m+sigma(end)*s1(i-1)*dw(i);
s(i)=s(i-1)+s(i-1)...
*abs(polyval(MO,i+length(B)-1))...
*1/m+abs(polyval(SIG,i+length(B)-
1))...
*s(i-1)*dw(i);
end
figure(1)
plot(s)
figure(2)
plot(s1)
figure(3)
plot(s-s1)
end
```

3.2 Numerical Result

We have data of Stock Index of country X for 101 days in vector Data; A = [61700.1; 61710.2; 61728.1; 62055.9; 62632.8; 62655; 62461; 62788.7; 62841.2; 63516.9; 64860.9; 65424.1; 65119.5; 66118.1; 66562.5; 66960.4; 67740.1; 68857.5; 70999.5; 71011; 71393.9; 71118.8; 72912.8; 73725.5; 73684; 74103.6; 74569.4; 75783.8; 78199.1; 77497.9; 76594.2; 76709.5; 77377.5; 77141.6; 77888; 77560.9; 77234.9; 77587.7; 77704; 77840.2; 78228.3; 77733.8; 77539.2; 77648.8; 77697.8; 78022.2; 78093; 78158.4; 78220; 78312.1; 79366.5; 80236.7; 80037.5; 80219.4; 81200.3; 81261; 80935.7; 80561.3; 81480.4; 81536.9; 80852.7; 80872.1; 80965.8; 80752.7; 80654; 80280.7; 80262.4; 80109.3; 7588.3; 77984.8; 77516.2; 78435.4; 78430.9; 78269; 78281.7; 78448.3; 78688.4; 78394.4; 78404.7; 78414.6; 78384.4; 78285.4; 78044.4; 78033.8; 77423.9; 77045.3; 77106; 76630.4; 75982.6; 76138.6; 75863.2; 75980.5; 76292.9; 76413.3; 76448.3; 76613.9; 76692.8; 76853; 76690.6; 76387; 76431.5].

There is a path that we estimate with programming. The Fig. 3 is obtained from GBM and the Fig. 4 is our new model (NGBM) with same stochastic source. Figure 5 shows the difference between GBM and NGBM.

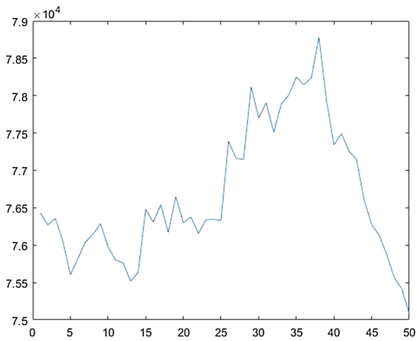

Fig. 3. The GBM estimate

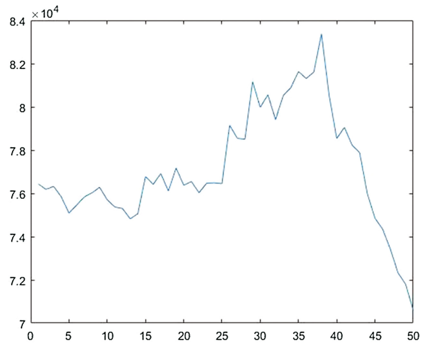

Fig. 4. The NGBM estimate

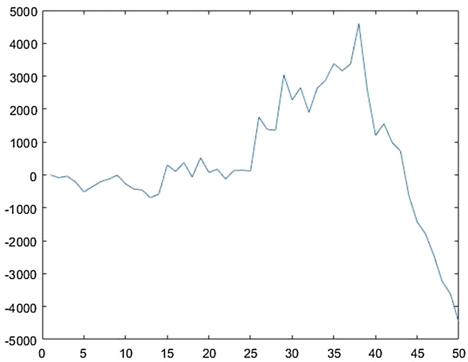

Fig. 5. The difference between NGBM and GBM

4 Conclusion

In this paper we studied Geometric Brownian Motion, and presented more dynamism for Geometric Brownian Motion. In fact, we prepared α and σ by using historical data, repeating the estimate that we used in Geometric Brownian Motion, and numerical method (Curve fitting) that they depend on time and put them in the last model. At last, with programming in MATLAB, we produced some path.

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A Study on Comprehensive Traffic Capacity of Urban Roads

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Abstract. In this paper, aimed at the congestion of urban roads, the author analyses the possibility of building an evaluation index system that is suitable for the traffic capacity of urban roads. The author proposes that urban traffic capacity should be evaluated in a dynamic comprehensive evaluation method according to the real time and dynamics of traffic condition. The roads can be divided into five types and their traffic capacity should be worked out in a respective calculation formula. On this basis, the comprehensive calculation model of the traffic capacity is set up. In order to improve the urban traffic condition, the author utilizes MATLAB7.0 to calculate and compare the traffic capacity of urban roads with three different structures. The results show that three types of open residential areas can raise the traffic capacity of the whole system and improve the road traffic conditions.

Keywords: Traffic capacity · Saturation · Community structure · MATLAB

1 Basic Assumptions

(1) Neglect the traffic jams caused by traffic accident, natural disaster, the pedestrian, etc.
(2) The management of communities, urban traffic roads and its intersection is reasonable.
(3) No traffic violation records.
(4) The traffic control system operates normally.

2 The Establishment and Solution of the Model

2.1 The Selection and Calculation of Evaluation Indexes

In order to select the appropriate evaluation index system to reflect the complicated and changing road condition, using comprehensive analysis method,indexes reflecting the traffic condition are selected. According to the traffic condition, roads are divided into common segments and intersections, as shown in Figs. 1 and 2.

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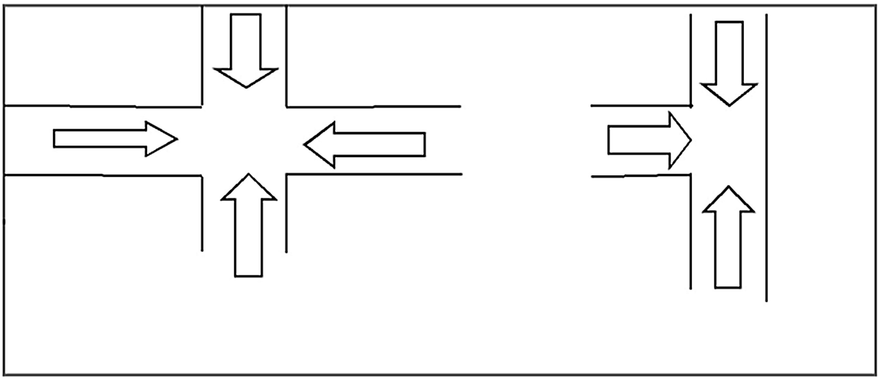

Fig. 1. The intersection

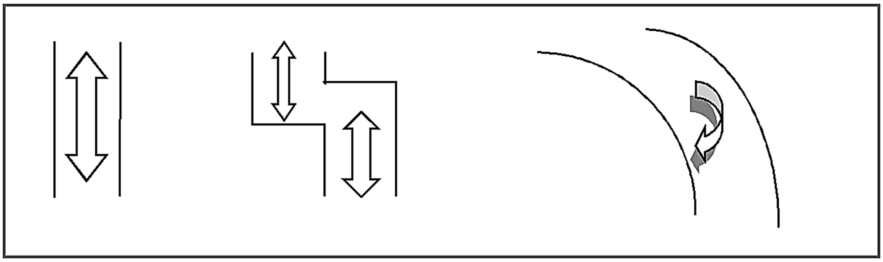

Fig. 2. Ordinary road

According to the characters of road traffic and the functions of urban road intersections, indicators include: traffic density *E*, time share *T*, saturation *B*, the average delay time J, queue length *K*, signal intersection queue rate *R*.

(1) Traffic density *E*: At one point, the number of vehicles in a unit is the indicator of road congestion, computation expression is: $E = \frac{N}{L}$
Where, *E* is the traffic density, *N* is the number of vehicles (number/km),
(2) Time share T: refers to the product of observation time and number of the traffic detectors per occupancy time in a fixed period of time, which reflects the overall road congestion [1] the formula is: $T = \frac{t_c}{L_1}$
(3) Saturation *B*: the ratio of traffic flow and road capacity at traffic crossing, which is the measure of intersection congestion. Which is: $B = \frac{S}{E}$
Where, *S* stands for actual traffic volume, which reflects the traffic capacity of entrance lane (car equivalent/hour).
(4) Average delay time J: is the average delay time of each car entering the intersection, the formula is $J = \frac{0.5T\left(1-\frac{t_b}{T}\right)}{1-\left[\min(1,d)\frac{t_b}{T}\right]}$, which reflects traffic stuck in the intersection and the queuing situation [3].

Where, T is signal cycle length, t_b is green time, d is the saturation of one car lane, saturation is, under ideal conditions, the ratio of maximum service traffic volume and basic capacity.

(5) Queue length k: is the vehicle queue length in the intersection, average queue length can reflect the intersection congestion directly and visually. Generally speaking, the more serious traffic congestion is, the longer the queue length will be. In urban road, when the vehicles reach the intersection, because of the traffic control signal, they usually need to wait or slow down, the vehicles reaching the intersection when the lights are red are bound to wait in line, we normally think the situation that the vehicles can pass in 1–2 signal doesn't belong to the category of traffic jams [4].

(6) Secondary line rate of signal intersection r: is the ratio of the number of vehicles parked twice or more in one cycle to that of vehicles passing through the green light, which is $r = \frac{H}{C}$, $H = D + G - Z$. Secondary line rate can intuitively reflect whether the intersection is crowded, which is belong to the traffic management indicators that must be strictly controlled, secondary parking can be expressed as: the number of vehicles parked twice in this cycle equals to the number of vehicles stranded last cycle plus the number of vehicles arriving at the red time this cycle minus that of vehicles leaving at the green time this cycle.

Where D is the number of stranded vehicles in the previous cycle, G is the number of vehicles arriving at the red time this cycle, Z is the number of vehicles leaving at the green time this cycle, r is the secondary line rate.

Through literature, the structure of the community could be roughly divided into the following three types (Fig. 3).

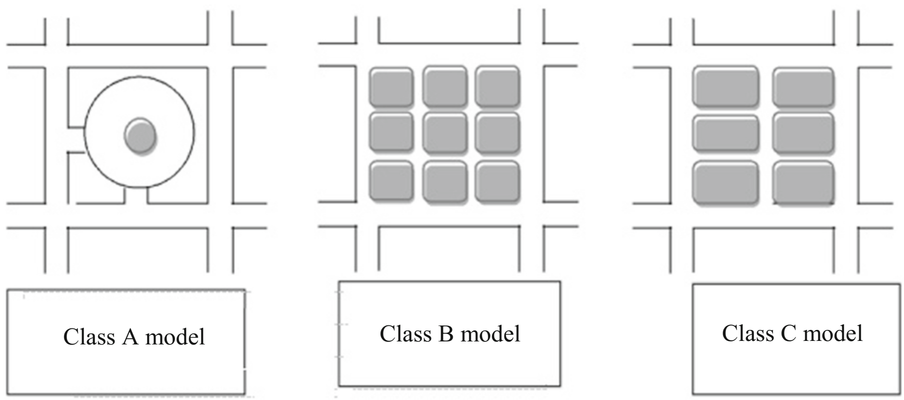

Fig. 3. Neighborhood structure type

Where:

Class A model:	Enclosed plot diagram—Ring road model
Class B model:	The traffic open plot diagram—Little closed, great open
Class C model:	Enclosed plot diagram—The number of road system

This article employs brittleness indexes based on the urban road network to reflect the influences from opening different structures of village.

Brittleness of urban road network: refers to the possibility of the function loss of the whole system for the whole or partial system getting damaged under external disturbance. Vulnerability of urban road network is mainly manifested in the level of road network service degree. Urban road network is a complex network system and brittleness is inevitable. The study found that vulnerability of urban road network is related to the topology of the urban road network. This paper mainly studies the impact of the level of openness of the community, residential location and scale of residential area on the topology structure of road network, we may analyze the brittleness of urban road network according to the change of its topology structure.

Brittleness: urban road network evaluation indexes include node degree and betweenness centrality, etc. Edge node degree refers to the number of sides adjacent to the node, in the network v_i, the number of adjacent side k_i is called the degree of node v_i. Average the degree of all the nodes in the network then we get the average degrees of the network $k = 1/\sum_{i=1}^{N} k_i$.

Betweenness could be divided into node betweenness and edge betweenness, it reflects the importance of the nodes and edges in the network. Node (side) betweenness refers to the ratio of the number of the shortest paths through the node (edge) to the number of all the shortest paths, and its calculation formula is:

$$B_i = \sum_{i,j,k \in (k \neq j)} \frac{n_{jk}(i)}{n_{jk}}$$

Where, n_{jk} is the number of the shortest paths between v_j and v_k. $n_{jk}(i)$ is the number of shortest paths which pass through node v_i between v_j and v_k.

The betweenness centrality $C_B(v_i)$ of node v_i is the normalized betweenness and its formula is

$$C_B(v_i) = \frac{2B_i}{[(N-2)(N-1)]}$$

Degree and Betweenness are to reflect the degree of importance of a structure unit in the network, studies have shown that the fewer the important structural units are, the brittler the urban road network is. Based on the above ideas, this paper takes the variances of the node degrees as the evaluation indexes of the vulnerability of urban road network.

Variance reflects the degree of deviation from the average. The smaller the variance of node degrees in urban road network is, the better the uniformity of urban road network will be. The smaller the difference of important level of the nodes, the stronger the capacity of resisting damage of the road network, i.e., the smaller the brittleness of urban road network. Therefore it is reasonable to take the variances of the node degrees as evaluation indexes of the brittleness of urban road network [7] (Tables 1 and 2).

Variance calculation formula is: $D(k) = \frac{1}{N}\sum_{i=1}^{N}(k_i - \bar{k})^2$.

Table 1. Indicators than choose standard

Standard of comparison	Detailed instructions
Clear	Unambiguous professional work
Simple	Easy to understand, easy to analyze and apply
Description and prediction ability	Describe and evaluate the present situation, find out the problem, predict the traffic growth in the future, reflect the traffic fluctuation
Analytical ability	Few data requirements, low cost of collecting data, easy collection and statistical analysis, congestion degree can be described reasonably
Feasibility	Suitable for all kinds of transportation, can reflect road type, congestion and continuously to collect and congestion area; Applied to different areas; Can reflect the degree of congestion based on the urban size; Can reflect the air quality and energy consumption

Table 2. Applied analysis of Primary road congestion index

Importance of indicators	Road indicators						
	Traffic density	The time to share	Saturation	Evaluation of delay time	Queue length	Rate of signal intersection line	Community structure
Primary	✓		✓	✓			
Secondary		✓			✓	✓	
General							✓

2.2 Pretreatment of Indicators

Multi-index comprehensive evaluation system includes different evaluation indexes, therefore, the comprehensive indexes should be pretreated before there are assessed.

2.2.1 Uniformization of Evaluation Indexes

In the multi-index comprehensive evaluation system, some indicators are as small as possible, these indicators are called inverse indicators and some of them are the bigger the better, these indicators are called positive indicators. Before the evaluation appraisal, the indicators have to be standardized, the reverse indicators should be converted into positive indicators, and vice versa. With said into positive indicators, the transform is: $z_i' = \frac{c}{z_i}, i = 1, 2, 3, \cdots, n$ (generally, c equals to 1), That is $z_i' = \frac{1}{z_i}$.

2.2.2 Standardization of Appraisal Indicators

In the multi-index comprehensive evaluation system, there are differences between these indicators in units of measurement, the intrinsic attributes, orders of magnitude, etc. Therefore, it is inappropriate to weigh average the data directly, the results will be meaningless, hence the data need to be unified, namely, they should be converted into proper dimensionless indexes. Conventional methods: "(1) standardization law, (2) extreme value method, (3) efficacy function method".

(1) Formula of Standardization method (method of data standardization) is:

$$y_i = \frac{x_i - s}{\bar{x}}$$

In the expression, y_i is the standardized value of x_i, $\bar{x}$ and s_i indicate he sample mean and the mean square error of x_i, respectively.

(2) The extremum method:
If we set: $W_j = Max\{x_{ij}\}, w_j = \min\{x_{ij}\}$, then: $x_{ij}' = \frac{x_{ij} - w_j}{W_j - w_j}$ is a dimensionless, and x_{ij}' ∈[0,1].
即 $X_{ij} \in [60, 100]$

(3) Efficacy function method:
Let $X_{ij} = c + \frac{x_{ij} - m_j}{M_j - m_j} d$

Where M_j, m_j are the satisfied and not permitted values of x_{ij}, and c, d are positive constant. c is used to translate the value after transformation, d is used to enlarge or narrow the value after transformation. Generally, take c = 60, d = 40, namely $X_{ij} \in [40, 60]$.

2.3 The Choice of Comprehensive Evaluation Method

After the urban road traffic evaluation system was determined, each indicator will be endowed with weight, we also need to select an appropriate evaluation method of the evaluation system. There are many multi-index comprehensive evaluation methods such as the data envelopment analysis (DEA) model, the fuzzy synthetic method, the dynamic comprehensive evaluation method, etc. They are frequently applied.

Each kind of comprehensive evaluation method has its characteristics and limitations, among the multi-index evaluation methods, lateral summery of static scheduling has been widely used. As the application objects in this paper are the urban road traffic conditions, where traffic conditions are real-time dynamic, we should consider the dynamicity and real time when selecting the corresponding evaluation indexes. In order to fully consider the characteristics of demand, this article will use the dynamic comprehensive evaluation method as the best evaluation index.

3 The Establishment of the Comprehensive Urban Transportation Capacity Model

3.1 Basic Capacity of Road Traffic

Basic capacity refers to under the ideal conditions, the maximum capacity of traffic flow of each road per unit time. Ideal road conditions refer to that the lane width should be greater than 3.65 m (assuming the provisions of 3.75 m), roadside lateral remaining is not less than 1.75 m, longitudinal slope is gentle, and with wide field of vision, good road conditions. Ideal traffic conditions, mainly refer to cars of standard size, composed of single vehicles in the same lanes at different speeds, drive continuously, each vehicle keeps the minimum distance appropriate to the speed, meanwhile without any interference from each direction. In such ideal road and traffic conditions, this paper establishes traffic flow calculation model, then figures out the maximum traffic flow, that is, the basic capacity. The calculation formula is: $C_B = \frac{3600}{t_0} = \frac{3600}{\frac{h_0}{\frac{v_0}{3.6}}} = \frac{1000 v_0}{h_0}$.

Where: C_B is the basic capacity, v_0 is the vehicle speed (km/h); and t_0 is the average headway (s);

$$h_0 = h_1 + h_2 + h_3 + h_4 = \frac{v_0}{3.6} t + \frac{v_0^2}{254\varphi} + h_3 + h_4$$

Where h_0 is the minimum vehicle distance (m); t is the driver's reaction time (s), where preferable time is about 1.2 s; h_1 is the distance of vehicle within the driver's reaction time; h_2 is the braking distance; h_3 is the safe distance between two cars (m), where desirable distance is 5 m; h_4 is the average length of the vehicles (m), where desired length is 5 m, the expected value of large passenger cars is 9 m, and that of buses and trackless vehicles is 14 m.

3.2 Possible Capacity

The calculation of possible capacity is based on the basic capacity, taking the actual road and traffic conditions into account, then we may determine its correction coefficient. And then multiply the correction coefficient by basic traffic capacity, we could get the possible capacity in the actual road, traffic and certain environment conditions. The correction coefficients of factors that affect the traffic capacity, are:

(1) Correction coefficient of lane width ε_1;
(2) Correction coefficient of longitudinal slope width ε_2;
(3) Correction coefficient of traffic condition ε_3;
(4) Correction coefficient of design speed ε_4;
(5) Correction coefficient of direction distribution ε_5;
(6) Correction coefficient of lateral width ε_6;
(7) Correction coefficient of transverse interference ε_7.

Where correction coefficient of traffic condition ε_3 mainly refers to the composition of different types of vehicles, the determination of reduction factor of longitudinal slope

ε_2 is related to the conversion coefficient between different vehicles, usually, it is based on the percentage of trucks, its calculation is: $v_3 = \frac{100}{100 - d_r + E_r d_r}$.

Where: d_r represents for the percentage of trucks; Er is the equivalent value of trucks in terms of vehicles, it could be calculated according to the slope gradient and slope length from the table. Other reduction factors are determined from the size of the difference between actual conditions and ideal conditions which could be achieved by the corresponding form. The possible capacity of the road is: $W_k = N_{\max}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7$.

3.3 The Actual Capacity

The actual capacity, usually as the basis of road planning and design, could be calculated by multiplying the possible traffic capacity by the ratio of the service traffic volume to the traffic capacity. Namely:

$$R_k = W_k \times \text{serving traffic} \div \text{traffic capacity} \; ((\text{m/h}\;))$$

3.4 The Traffic Capacity of Plane Intersection

Level crossing refers to two or more roads intersecting in the same plane, the traffic capacity of plane intersection refers to the maximum traffic flow when traffic moving in two different directions through the intersection.

Plane intersection could be divided into three forms: intersection with traffic lights, round about of the central island and intersection without traffic lights.

3.5 The Traffic Capacity of Intersection Without Traffic Lights

Its computational principle is to see the traffic flow on the main road as continuous traffic flow, then suppose the vehicles arrive at a Poisson process, therefore the spacing interval of the vehicles obeys negative exponential distribution [7]. But not all intervals might be used for vehicle passing, only when this interval is greater than the critical threshold. When there is a time interval, traffic flow in the non-primary route could pass in order the accessory interval as the following formula: $Q_r = \frac{Q_g \cdot e^{-q\delta}}{1 - e^{-q\delta}}$.

Where Q_r is the traffic flow in the non-primary route without priority (m/h); Q_g represents for the traffic flow on the main road with priority (m/h); and q is: $\frac{Q_g}{3600}$ (m/h).

3.6 The Traffic Capacity of Roundabout

Roundabout is the center of intersection where several streets intersect, it would be set to round island or arc shape, thus vehicles entering the intersection would pass the island along the same direction. Formula expressed as [9]:

$$Q_n = \left[354v\left(Q_1 + \frac{h}{v}\right)\left(1 - \frac{v}{l}\right)\right]$$

Where, Q_n is the maximum capacity of the weaving segment (m/h) and l is for the weaving segment length (m), v is the weaving section width (m), h is the width of approach road of the intersection (m), and r is the ratio of weaving vehicles to all vehicles, expressed as a percentage.

3.7 The Traffic Capacity of Signal Intersection

In the signal intersection, red, yellow and green light are used to command vehicles pass, stop, turn left and right. The traffic lights change, according to the length of the signal cycle and the time occupancy of each kind of signal, could be used to calculate the traffic capacity of signal intersection. Taking the stop line of the approach as standard, as long as through the line of vehicles to have passed through the intersection, traffic capacity of each inlet: go straight, And turn right, turn left for three kinds of situations, and at the same time each inlet lane is divided into lanes and mixed with lane, so the intersection traffic capacity is the sum of the intersection traffic capacity at the road entrance. According to three different lanes, introduce three kinds of formulas.

(1) A single special straight lane capacity $K_S = \frac{3600}{N_q} \times \frac{t_i - t_g}{t_j}$
Where: Nq signal cycle time, t_j for both before and after the car through the mean time between you, t_i is green time of one cycle, and t_g is to green light loss of one cycle time, contains the car starting acceleration time.

(2) A single right turn lanes of traffic capacity formula is as follows: $K_Y = \frac{3600}{t_k}$, (m/h)
Where: t_k is both before and after the time interval of right turn vehicles successively

(3) A single left turn lanes of traffic capacity $K_L = n\frac{3600}{N_q}$
Where: n is a cycle number of vehicles allowed to turn left.

(4) Straight, left a driveway when driving with mixed traffic capacity (M_{SL})

At the same path straight, left, on the basis of different directional interference, even parking phenomenon, therefore, should be multiplied by the appropriate reduction factor K. At the same time, because the left car is often greater than straight driving through time by time, generally is about 1.75 times that of straight driving through time we left the car should be multiplied by the proportion of 1.75 times, and set to left the vehicle percentage accounted for, is: $M_{SL} = K_S\left(1 - \frac{3}{4}n_A\right)K$, unit: m/h

$$Q_x = K_s + K_Y + K_L + M_{SL} + M_{SR}$$

(5) Straight, right human traffic capacity of a road motorised (M_{SR})

General principle ditto, but what right transfer on time is 1.5 times that of the straight road. Which is expressed in n_B right transfer accounts for a percentage: $M_{SR} = K_S\left(1 - \frac{1}{2}n_B\right)K$, unit: m/h. The entire signal intersection traffic capacity for each imported straight, turn right, turn left the sum of all the ability to influence [10,11].

$$Q_x = K_s + K_Y + K_L + M_{SL} + M_{SR}$$

Where, Q_x for the whole of the traffic capacity of signal intersection.

To sum up, the actual capacity of the whole traffic system can be drawn from computation formula is as follows:

$$R_sQ_\gamma = W_k \times \text{service volume} \div Q_\gamma$$
$$R_sQ_n = W_k \times \text{service volume} \div Q_n$$
$$R_sQ_x = W_k \times \text{service volume} \div Q_x$$

Where, R_sQ_γ no traffic lights to control the actual capacity of intersection, R_sQ_n is the actual capacity of intersection, the R_sQ_x is the actual capacity of the whole signal intersection. Q_x for the whole signal intersection traffic capacity, and the H_L for traffic capacity of the whole transportation system.

3.8 The Establishment of the Comprehensive Urban Transportation Capacity Model

According to the above analysis, the comprehensive capacity of urban road computation formula is as follows:

$$S_A = (f_1 \times N_sQ_\gamma + f_2 \times N_sQ_n + f_3 \times N_sQ_x + f_4 \times C_B) \div f_1 + f_2 + f_3 + f_4$$

Among them: S_A said integrated transportation capacity.

4 The Model Application and Based on the Analysis of Traffic Capacity

In accordance with the established model, 3.3 will be opened area before the relevant data input MATLAB7.0, concluded that the actual capacity of different road in front of the village is opened. As shown in Table 3:

Table 3. Before opening the actual capacity of different roads

	The actual capacity (pcu/h)						
Base this section	396	363.64	357.14	348.3	336.7	320.51	229.4
No intersection modes of pipe system	12142	17543	17544	18903	40044	57449	65159
Ring shape intersection	94.649	95.724	109.4	147.34	218.5	358.25	406.33
Signal intersection	60.62	89.157	90.7136	98.0599	211.105	307.64	352.66

According to the above data, again by MATLAB, the application of the model of the formula:

$$S_A = (f_1 \times N_s Q_y + f_2 \times N_s Q_n + f_3 \times N_s Q_x + f_4 \times C_B) \div (f_1 + f_2 + f_3 + f_4)$$

It is concluded that village before opening the whole transportation system capacity (Table 4).

Table 4. The village before opening the whole transport capacity

	The actual capacity (pcu/h)
The whole transport system	2620

To solve by the model under the three different structure of the village after the opening of the whole traffic capacity.

Hypothesis: village road intersection with the outside world in light, and the internal of the village, no lights, and that roads are standard in the community.

When open cell types for type *A*, we learn that the added two signal intersection, the traffic network basic sections increased n_1, so will be modified road data, using the model using the MATLAB software to calculate the results as shown in Table 5:

Table 5. Type A village after opening traffic capacity

	The actual capacity (pcu/h)
The whole transport system	3390

When the open cell type is type B, according to the plot structure, the whole added eight signal intersection, the traffic network basic sections increased article n_2, so will be modified road data, using the model using the MATLAB software to calculate the results as shown in Table 6:

Table 6. The class B district traffic capacity after opening

	The actual capacity (pcu/h)
The whole transport system	3690

When open cell types for type C, according to the plot structure, the signal of the traffic network intersection increased by six basic sections increased n_3, so will be modified road data, using the model using the MATLAB software to calculate the results as shown in Table 7:

Table 7. The class C district traffic capacity after opening

	The actual capacity (pcu/h)
The whole transport system	3796

Will be opened area before and after comparing the actual capacity of the whole traffic system as shown in Table 8:

Table 8. The village before and after open the actual capacity of the whole transportation system

	Village before opening	After the opening area		
		Type A village	Type B village	Type C village
The actual capacity (pcu/h)	2620	3390	3690	3796

Through the comparison and analysis can be concluded that: open area can effectively improve the actual capacity of the whole traffic system, is conducive to the improvement of road capacity.

5 Conclusion

Through rational filtration, analysis and selection, this article selects traffic density, time share, saturation, average delay time, queue length, the rate of signal intersection line, community structure on traffic impact as significant indicators to form the evaluation index system. Considering the real time and dynamic of traffic situation, dynamic comprehensive evaluation method is applied: the road could be divided into several types: basic road section, plane intersection, intersection without traffic lights, roundabout intersection and signal intersection, then the calculation formula of their traffic capacity is achieved. On this basis, we may establish the model to calculate the comprehensive capacity of the whole traffic system. Based on the important influence of community open on improving urban traffic conditions, combined with three different plot structure, the model is solved by using MATLAB7.0. We may find that: before the village open, the comprehensive capacity of the whole transportation system was 2620 (pcu/h), after the opening of community A, B, C, the comprehensive capacity of the whole traffic system becomes 3390 (pcu/h), 3690 (pcu/h), 3796 (pcu/h), respectively. By comparing the comprehensive capacity of the whole traffic system before and after the opening of the community, it could be concluded the opening of the community would improve the comprehensive capacity of the whole traffic system and the road traffic conditions and provide hard reference data for the urban planning and development.

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Solving First Order Fuzzy Initial Value Problem by Fourth Order Runge-Kutta Method Based on Different Means

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Abstract. In this paper, we compare exact solution and four approximate solutions obtained from the R-K method by four different means. We illustrate related results and investigate better and closer solutions to the exact solutions. The accuracy of presented method is showed by solving examples from the fuzzy differential equations with initial values.

Keywords: Fuzzy differential equations · Runge-Kutta method · Initial value problems

1 Introduction

Fuzzy differential equation (FDE) has been studied in the recent past. This topic has much application in many fields such as mathematics, engineering and in the field of medicines. At the first the concept of fuzzy derivative introduced by Chang and Zade [2]. Dubois et al. [5] discussed differentiation with fuzzy features by using extension principle in their approach. The fuzzy differential equations and initial value problems were studied by Kaleva [10, 11] and Seikkala [18], respectively. Also, some related methods are discussed by Ouyang and Wu [15], Kloden [12] and Wu [19]. Puri et al. [17] and Goescjel et al. [9] contributed toward the differential of fuzzy functions. Runge-Kutta methods have been studied by Abbasbandy [1] and Palligkinis et al. [16]. Numerical solution of FDE by Runge-Kutta method of order two with new parameters has been by Nirmala, Savetha, N and S Chenthur Pandian [13] and three order have been studied by Duraisamy and Usha [6] and by Runge-Kutta method of order four with new parameters by Nirmala et al. [14].

In this paper, Runge-Kutta method applied in several means to obtain the approximate solutions and compare it with exact solution. The present concepts extended the previously introduced results of centroidal mean [8], harmonic mean [3, 4] and contraharmonic mean [7].

This paper is organized as follows: in Sect. 2, some basic definition of fuzzy numbers and fuzzy derivative are given. Section 3, contains definitions of fuzzy

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Runge-Kutta method by different means. The numerical example and proposed method and the result of compared different means with the exact solutions are in Sect. 5 and conclusions are in Sect. 6.

2 Preliminaries

In this paper, we present Fuzzy Runge-Kutta method by using centroidal mean, harmonic mean and contraharmonic mean.

First, we present some preliminary from the fuzzy number, fuzzy function and etc.

Definition 2.1 [3]. A fuzzy number u is satisfied by an ordered of functions $(\underline{u}(r), \bar{u}(r))$. $r \in [0, 1]$ by following conditions:

(i) as bounded left continuous non-decreasing function over [0, 1] with respect to any r,
(ii) as a bounded right continuous non-decreasing function over [0, 1] with respect to any r,
(iii) $\underline{u}(r) \leq \bar{u}(r)\ 0 \leq r \leq 1$,
(iv) $\forall u \in R_F$ u is upper semi continuous on R.

r-level set $\{[u]_r = \{x|u(x) \geq r\}\ 0 \leq r \leq 1$ is closed and bounded interval denoted by $[u]_r = \left[\underline{u(r)}, \overline{u(r)}\right]$ and $[u]_0 = \{x|u(x) \geq 0\}$ is compact.

Definition 2.2 [4] ($\propto -$ level set). Let I be the real interval. A mapping y: I $\rightarrow$ E is called a fuzzy process and its $\propto -$ level set is denoted by $[y]_\propto = [\underline{y}(t, \propto), \bar{y}(t, \propto)] = t \in I\ 0 \leq \propto \leq 1$.

Definition 2.3 [3]. A trapezoidal fuzzy number u is a fuzzy set in E, defined by for real numbers $k < l < m < n$ where our membership function is:

$$u(x) = \begin{cases} \frac{x-k}{l-k}; & k \leq x \leq l \\ 1; & l \leq x \leq m \\ \frac{x-k}{l-k}; & m \leq x \leq n \end{cases},$$

we have:

1. $u > 0$ if $k > 0$;
2. $u > 0$ if $l > 0$;
3. $u > 0$ if $m > 0$;
4. $u > 0$ if $n > 0$.

Lemma 2.1. If the sequence of non-negative numbers $\{W_n\}_{n=0}^{N}$ satisfy $|W_n + 1| \leq A\ |W_n| + B, 0 \leq n \leq N - 1$.

For the given positive constant A and B, then

$$|W_n| \leq A^n |W_0| + B \frac{A^n - 1}{A - 1}, 0 \leq n \leq N.$$

Lemma 2.2. If the sequence of non-negative numbers $\{W_n\}_{n=0}^{N}, \{V_n\}_{n=0}^{N}$ satisfy

$$|W_n + 1| \leq |W_n| + A \ \max\{|W_n|, |V_n|\} + B,$$
$$|W_n + 1| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B.$$

For the given positive constant A and B, then

$$U_n = |W_n| + |V_n|, 0 \leq n \leq N,$$

we have $U_n \leq \overline{A^n} U_0 + B \frac{\overline{A^n} - 1}{A - 1}, 0 \leq n \leq N$ where $\bar{A} = I + 2A$ and $\bar{B} = 2B$.

Lemma 2.3. Let F(t, u, v) and G(t, u, v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$,

$$D(y(t_{n+1}), y^0(t_{n+1})) \leq h^2 L(1 + 2C),$$

where L is a bounded of partial derivatives of F and G, and

$$C = \max \{|G[t_N, (y(t_N, r), \bar{y}(t_{N-1})|, r \in [0, 1]\} < \infty.$$

Lemma 2.4. Let F(t, u, v) and G(t, u, v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0 \leq r \leq 1$, the numerical solution of $\underline{y}(t_{n+1}; r)$ and $\bar{y}(t_{N+1}; r)$ converge the exact solutions $\underline{Y}(t; r)$ and $\bar{Y}(t; r)$ uniformly in t.

Theorem 2.1. Let F(t, u, v) and G(t, u, v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed r, $0 \leq r \leq 1$, the iterative numerical solution of $\underline{y^{(j)}}(t_n; r)$ and $\bar{y}(t_{N+1}; r)$ converge to the numerical solutions $\underline{y}(t; r)$ and $\bar{y}(t; r)$ uniformly in $t_0 \leq t_n \leq t_N$, when $j \to \infty$.

3 Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), t \in [t_0, T] \\ y(t_0) = y_0 \end{cases}, \tag{3.1}$$

where y is a fuzzy function of t, f(t, y) is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a trapezoidal or a trapezoidal shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \bar{y}]$. It means that the r-level set of y(t) for $t \in [t_0, T]$ is

$$[y(t)]_r = \left[\underline{y}(t;r), \bar{y}(t;r)\right],$$
$$[y(t_0)]_r = \left[\underline{y}(t_0;r), \bar{y}(t_0;r)\right], r \in (0,1].$$

We write f(t, y) = $[\underline{f}(t,y), \bar{f}(t,y)]$ and

$$\underline{f}(t,y) = F\left[t, \underline{y}, \bar{y}\right],$$
$$\bar{f}(t,y) = G\left[t, \underline{y}, \bar{y}\right].$$

Because of y' = f(t, y), we have

$$\bar{f}(t, y(t); r) = G\left[t, \underline{y}(t;r), \bar{y}(t;r)\right].$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\},$$

so fuzzy number f(t, y(t)). From this, it follows that

$$[f(t, y(t))]_r = \left[\underline{f}(t, y(t); r), \bar{f}(t, y(t); r)\right], r \in (0,1],$$

where

$$\underline{f}(t, y(t); r) = \min\{f(t,u) | u \in [y(t)]_r\},$$
$$\bar{f}(t, y(t); r) = \max\{f(t,u) | u \in [y(t)]_r\}.$$

Definition 3.1. A function $f : R \to R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, $\delta > 0$ such that $|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \varepsilon$ exists.

Throughout this paper, we also consider fuzzy functions which are continuous in metric D. Then the continuity of f(t, y(t); r) guarantees the existence of the definition of f(t, y(t); r) for $t \in [t_0, T]$ and $r \in [0, 1]$. Therefore, the functions G and F can be definite too.

4 The Forth Order Runge-Kutta Method

The basis of all Runge-Kutta method is to express the difference between the value of *y*,

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i, \tag{4.1}$$

where for i = 1, 2, …, m, w_i's are constants and

$$k_i = hf\left(t_n + c_i h, y_n h \sum_{i=1}^{i-1} a_{ij} k_j\right) \tag{4.2}$$

Equation (4.1) is to be exact for powers of h through h^m because it is to be coincident with Taylor series of order m.

Therefore, the truncation error T_m, can be written as

$$T_m = \gamma_m h^{m+1} + o\left(h^{m+2}\right).$$

4.1 The Fourth Order Runge-Kutta Method Based on Contraharmonic Mean

We consider fuzzy initial value problem (3.1) with the grid points (3.2). Let the exact solution $[\mathrm{Y(t)}]_r = [Y_1(\mathrm{t;r}), Y_r(\mathrm{t;r})]$ is approximated by some $[y(t)]_r = [y_1(\mathrm{t;r}), y_2(\mathrm{t;r})]$. From (2.6), (2.7) we define

$$y_1(t_{n+1};r) - y_1(t_n) = \sum_{i=1}^{4} w_i k_{i,1}(t_n, y(t_n;r)),$$
$$y_2(t_{n+1};r) - y_2(t_n) = \sum_{i=1}^{4} w_i k_{i,2}(t_n, y(t_n;r)),$$

where the w_i's are constants and

$$[k_1(t_n, y(t_n;r))]_r = \left[k_{i,1}(t_n, y(t_n;r)), k_{i,2}(t_n, y(t_n;r))\right],$$
$$k_{i,1}(t_n, y(t_n;r)) = hf(t_n + c_i h, y_1(t_n) + \sum_{j=1}^{i-1} a_{ij} k_{j,1}(t_n, y(t_n;r)),$$
$$k_{i,2}(t_n, y(t_n;r)) = hf(t_n + c_i h, y_2(t_n) + \sum_{j=1}^{i-1} a_{ij} k_{j,2}(t_n, y(t_n;r)),$$

and

$$k_{11}(t_n, y(t_n;r)) = \min\{hf(t,u)|u \in [y_1(t;r), y_2(t;r)]\},$$
$$k_{12}(t_n, y(t_n;r)) = \max\{hf(t,u)|u \in [y_1(t;r), y_2(t;r)]\},$$

$$k_{21}(t_n, y(t_n;r)) = \min\{hf(t+h/2,u)|u \in \left[z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))\right]\},$$
$$k_{22}(t_n, y(t_n;r)) = \max\{hf(t+h/2,u)|u \in \left[z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))\right]\},$$

$$k_{31}(t_n, y(t_n;r)) = \min\left\{hf(t+h/2,u)|u \in \left[z_{2,1}(t, y(t;r)), z_{2,2}(t, y(t;r))\right]\right\},$$
$$k_{32}(t_n, y(t_n;r)) = \max\left\{hf(t+h/2,u)|u \in \left[z_{2,1}(t, y(t;r)), z_{2,2}(t, y(t;r))\right]\right\},$$

$$k_{41}(t_n, y(t_n;r)) = \min\{hf(t+((1/4)+(-3/4)+(3/2))h,u)|u \in \left[z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))\right]\},$$
$$k_{42}(t_n, y(t_n;r)) = \max\{hf(t+((1/4)+(-3/4)+(3/2))h,u)|u \in \left[z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))\right]\},$$

where the fourth order Runge-Kutta method based on Contraharmonic Mean

$$z_{1,1}(t, y(t; r)) = y_1(t, r) + 1/2k_{1,1}(t, y(t; r)),$$
$$z_{1,2}(t, y(t; r)) = y_2(t, r) + 1/2\, k_{1,2}(t, y(t; r)),$$

$$z_{2,1}(t, y(t; r)) = y_1(t, r) + 1/8\, k_{1,1}(t, y(t; r)) + 3/8\, k_{2,1}(t, y(t; r)),$$
$$z_{2,2}(t, y(t; r)) = y_2(t, r) + 1/8\, k_{1,2}(t, y(t; r)) + 3/8\, k_{2,2}(t, y(t; r))$$

$$z_{3,1}(t, y(t; r)) = y_1(t, r) + 1/8\, k_{1,1}(t, y(t; r)) - 3/8\, k_{2,1}(t, y(t; r)) + 3/8\, k_{2,2}(t, y(t; r)),$$
$$z_{3,2}(t, y(t; r)) = y_2(t, r) + 1/4\, k_{1,2}(t, y(t; r)) - 3/8\, k_{2,2}(t, y(t; r)) + 3/2\, k_{2,2}(t, y(t; r)).$$

Define

$$\mathrm{F}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))] = \frac{k_{11}^2(t, y(t; r)) + k_{21}^2(t, y(t; r))}{k_{11}(t, y(t; r)) + k_{21}(t, y(t; r))} + \frac{k_{21}^2(t, y(t; r)) + k_{31}^2(t, y(t; r))}{k_{21}(t, y(t; r)) + k_{31}(t, y(t; r))} + \frac{k_{31}^2(t, y(t; r)) + k_{41}^2(t, y(t; r))}{k_{31}(t, y(t; r)) + k_{41}(t, y(t; r))},$$

$$\mathrm{G}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))] = \frac{k_{12}^2(t, y(t; r)) + k_{21}^2(t, y(t; r))}{k_{12}(t, y(t; r)) + k_{21}(t, y(t; r))} + \frac{k_{22}^2(t, y(t; r)) + k_{31}^2(t, y(t; r))}{k_{22}(t, y(t; r)) + k_{31}(t, y(t; r))} + \frac{k_{32k}^2(t, y(t; r)) + k_{42}^2(t, y(t; r))}{k_{32}(t, y(t; r)) + k_{42}(t, y(t; r))}.$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by

$$[Y(t_n)]_r = [Y_1(t_n; r), Y_2(t_n; r)] \text{ and } [y(t_n)]_r = [y_1(t_n; r), y_2(t_n; r)]$$

respectively. The solution is calculated by grid points at (2.13). By (4.1) and (4.5), we have

$$Y_1(t_{n+1}; r) \approx Y_1(t_n; r) + 1/3\, \mathrm{F}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))],$$
$$Y_2(t_{n+1}; r) \approx Y_2(t_n; r) + 1/3\, \mathrm{G}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))].$$

We define

$$y_1(t_{n+1}; r) \approx y_1(t_n; r) + 1/3\, \mathrm{F}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))],$$
$$y_2(t_{n+1}; r) \approx y_2(t_n; r) + 1/3\, \mathrm{G}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))].$$

4.2 The Fourth Order Runge-Kutta Method Based on Centroidal Mean

The same that last method, we have four k_is, but by different exponent and different mean, then

$$y_1(t_{n+1}; r) - y_1(t_n) = \sum_{i=1}^{4} w_i s_{i,1}(t_n, y(t_n; r)),$$
$$y_2(t_{n+1}; r) - y_2(t_n) = \sum_{i=1}^{4} w_i s_{i,2}(t_n, y(t_n; r)),$$

where the w_i's are constants

$$
\begin{aligned}
\left[s_1(t_n, y(t_n; r))\right]_r &= \left[s_{i,1}(t_n, y(t_n; r)), s_{i,2}(t_n, y(t_n; r))\right],\\
s_{i,1}(t_n, y(t_n; r)) &= hf(t_n + d_i h, y_1(t_n) + \sum\nolimits_{j=1}^{i-1} b_{ij} s_{j,1}(t_n, y(t_n; r)),\\
s_{i,2}(t_n, y(t_n; r)) &= hf(t_n + d_i h, y_2(t_n) + \sum_{j=1}^{i-1} b_{ij} s_{j,2}(t_n, y(t_n; r))
\end{aligned}
$$

and

$$
\begin{aligned}
s_{11}(t_n, y(t_n; r)) &= \min\{hf(t, u) | u \in [y_1(t; r), y_2(t; r)]\},\\
s_{12}(t_n, y(t_n; r)) &= \max\{hf(t, u) | u \in [y_1(t; r), y_2(t; r)]\},
\end{aligned}
$$

$$
\begin{aligned}
s_{21}(t_n, y(t_n; r)) &= \min\{hf(t + h/2, u) | u \in \left[v_{1,1}(t, y(t; r)), v_{1,2}(t, y(t; r))\right]\},\\
s_{22}(t_n, y(t_n; r)) &= \max\{hf(t + h/2, u) | u \in \left[v_{1,1}(t, y(t; r)), v_{1,2}(t, y(t; r))\right]\},
\end{aligned}
$$

$$
\begin{aligned}
s_{31}(t_n, y(t_n; r)) &= \min\{hf(t + h/2, u) | u \in \left[v_{2,1}(t, y(t; r)), v_{2,2}(t, y(t; r))\right]\},\\
s_{32}(t_n, y(t_n; r)) &= \max\{hf(t + h/2, u) | u \in \left[v_{2,1}(t, y(t; r)), v_{2,2}(t, y(t; r))\right]\},
\end{aligned}
$$

$$
\begin{aligned}
s_{41}(t_n, y(t_n; r)) &= \min\{hf(t + ((1/4) + (-3/4) + (3/2))h, u) | u \in \left[v_{1,1}(t, y(t; r)), v_{1,2}(t, y(t; r))\right]\},\\
s_{42}(t_n, y(t_n; r)) &= \max\{hf(t + ((1/4) + (-3/4) + (3/2))h, u) | u \in \left[v_{1,1}(t, y(t; r)), v_{1,2}(t, y(t; r))\right]\},
\end{aligned}
$$

where in the fourth order Runge-Kutta method based on Centroidal Mean

$$
\begin{aligned}
v_{1,1}(t, y(t; r)) &= y_1(t, r) + (1/2) s_{1,1}(t, y(t; r)),\\
v_{1,2}(t, y(t; r)) &= y_2(t, r) + (1/2) s_{1,2}(t, y(t; r)),
\end{aligned}
$$

$$
\begin{aligned}
v_{2,1}(t, y(t; r)) &= y_1(t, r) + (1/2) s_{1,1}(t, y(t; r)) + (11/24) s_{2,1}(t, y(t; r)),\\
v_{2,2}(t, y(t; r)) &= y_2(t, r) + (1/2) s_{1,2}(t, y(t; r)) + (11/24) s_{2,2}(t, y(t; r)),
\end{aligned}
$$

$$
\begin{aligned}
v_{3,1}(t, y(t; r)) &= y_1(t, r) + (1/2) s_{1,1}(t, y(t; r)) + (11/24) s_{2,1}(t, y(t; r)) + (73/66) s_{2,2}(t, y(t; r)),\\
v_{3,2}(t, y(t; r)) &= y_2(t, r) + 1/2\, s_{1,2}(t, y(t; r)) + 11/24\, s_{2,2}(t, y(t; r)) + 73/662\, s_{2,2}(t, y(t; r))
\end{aligned}
$$

Define

$$
\begin{aligned}
\mathrm{F}[(\mathrm{t}, \mathrm{y}(\mathrm{t}; \mathrm{r}))] = {} & \frac{s_{11}^2(t, y(t; r)) + s_{11}(t, y(t; r)) s_{21}(t, y(t; r)) + s_{21}^2(t, y(t; r))}{s_{11}(t, y(t; r)) + s_{21}(t, y(t; r))}\\
& + \frac{s_{21}^2(t, y(t; r)) + s_{21}(t, y(t; r)) s_{31}(t, y(t; r)) + s_{31}^2(t, y(t; r))}{s_{21}(t, y(t; r)) + s_{31}(t, y(t; r))}\\
& + \frac{s_{31}^2(t, y(t; r)) + s_{31}(t, y(t; r)) s_{41}(t, y(t; r)) + s_{41}^2(t, y(t; r))}{s_{31}(t, y(t; r)) + s_{41}(t, y(t; r))},
\end{aligned}
$$

$$\begin{aligned}\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))=&\frac{s_{12}^2(t,y(t;r))+s_{12}(t,y(t;r))s_{21}(t,y(t;r))+s_{21}^2(t,y(t;r))}{s_{12}(t,y(t;r))+s_{21}(t,y(t;r))}\\&+\frac{s_{22}^2(t,y(t;r))+s_{22}(t,y(t;r))s_{31}(t,y(t;r))+s_{31}^2(t,y(t;r))}{s_{22}(t,y(t;r))+s_{31}(t,y(t;r))}\\&+\frac{s_{32}^2(t,y(t;r))+s_{32}(t,y(t;r))s_{41}(t,y(t;r))+s_{42}^2(t,y(t;r))}{s_{32}(t,y(t;r))+s_{42}(t,y(t;r))}.\end{aligned}$$

The exact and approximate solutions at $t_n, 0 \le n \le N$ are denoted by

$$[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)] \text{ and } [y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$$

respectively. The solution is calculated by grid points at (2.13). By (4.1) and (4.5), we have

$$\begin{aligned}Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + 2/9\,\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],\\ Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + 2/9\,\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],\end{aligned}$$

We define

$$\begin{aligned}y_1(t_{n+1};r) &\approx y_1(t_n;r) + 2/9\,\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],\\ y_2(t_{n+1};r) &\approx y_2(t_n;r) + 2/9\,\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))].\end{aligned}$$

4.3 The Fourth Order Runge-Kutta Method Based on Harmonic Mean

Consider

$$\begin{aligned}y_1(t_{n+1};r) - y_1(t_n) &= \sum\nolimits_{i=1}^{4} p_i g_{i,1}(t_n, y(t_n;r)),\\ y_2(t_{n+1};r) - y_2(t_n) &= \sum\nolimits_{i=1}^{4} p_i g_{i,2}(t_n, y(t_n;r)),\end{aligned}$$

where the w_i's are constants and

$$[g_1(t_n, y(t_n;r))]_r = \left[g_{i,1}(t_n, y(t_n;r)), g_{i,2}(t_n, y(t_n;r))\right],$$

$$\begin{aligned}g_{i,1}(t_n, y(t_n;r)) &= hf(t_n + c_i h, y_1(t_n) + \sum\nolimits_{j=1}^{i-1} a_{ij} g_{j,1}(t_n, y(t_n;r)),\\ g_{i,2}(t_n, y(t_n;r)) &= hf(t_n + c_i h, y_2(t_n) + \sum\nolimits_{j=1}^{i-1} a_{ij} g_{j,2}(t_n, y(t_n;r)),\end{aligned}$$

and

$$\begin{aligned}g_{11}(t_n, y(t_n;r)) &= \min\{hf(t,u) | u \in [y_1(t;r), y_2(t;r)]\},\\ g_{12}(t_n, y(t_n;r)) &= \max\{hf(t,u) | u \in [y_1(t;r), y_2(t;r)]\},\end{aligned}$$

$$g_{21}(t_n, y(t_n;r)) = \min\{hf(t+h/2,u)|u \in [z_{1,1}(t,y(t;r)), z_{1,2}(t,y(t;r))]\},$$
$$g_{22}(t_n, y(t_n;r)) = \max\{hf(t+h/2,u)|u \in [z_{1,1}(t,y(t;r)), z_{1,2}(t,y(t;r))]\},$$

$$g_{31}(t_n, y(t_n;r)) = \min\{hf(t+h/2,u)|u \in [z_{2,1}(t,y(t;r)), z_{2,2}(t,y(t;r))]\},$$
$$g_{32}(t_n, y(t_n;r)) = \max\{hf(t+h/2,u)|u \in [z_{2,1}(t,y(t;r)), z_{2,2}(t,y(t;r))]\},$$

$$g_{41}(t_n, y(t_n;r)) = \min\{hf(t+h,u)|u \in [z_{1,1}(t,y(t;r)), z_{1,2}(t,y(t;r))]\},$$
$$g_{42}(t_n, y(t_n;r)) = \max\{hf(t+h,u)|u \in [z_{1,1}(t,y(t;r)), z_{1,2}(t,y(t;r))]\},$$

where in the fourth order Runge-Kutta method based on Contraharmonic Mean

$$z_{1,1}(t,y(t;r)) = y_1(t,r) + 1/2\, g_{1,1}(t,y(t;r)),$$
$$z_{1,2}(t,y(t;r)) = y_2(t,r) + 1/2\, g_{1,2}(t,y(t;r)),$$

$$z_{2,1}(t,y(t;r)) = y_1(t,r) - 1/8\, g_{1,1}(t,y(t;r)) + 5/8\, g_{2,1}(t,y(t;r)),$$
$$z_{2,2}(t,y(t;r)) = y_2(t,r) - 1/8\, g_{1,2}(t,y(t;r)) + 5/8\, g_{2,2}(t,y(t;r)),$$

$$z_{3,1}(t,y(t;r)) = y_1(t,r) - 1/4\, g_{1,1}(t,y(t;r)) + 7/20\, g_{2,1}(t,y(t;r)) + 9/10\, g_{2,2}(t,y(t;r)),$$
$$z_{3,2}(t,y(t;r)) = y_2(t,r) - 1/4\, g_{1,2}(t,y(t;r)) + 7/20\, g_{2,2}(t,y(t;r)) + 9/10\, g_{2,2}(t,y(t;r)).$$

Define

$$\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))] = \frac{g_{11}(t,y(t;r))g_{11}(t,y(t;r))}{g_{11}(t,y(t;r)) + g_{21}(t,y(t;r))} + \frac{g_{21}(t,y(t;r))g_{31}(t,y(t;r))}{g_{21}(t,y(t;r)) + g_{31}(t,y(t;r))} + \frac{g_{31}(t,y(t;r))g_{41}(t,y(t;r))}{g_{41}(t,y(t;r)) + g_{41}(t,y(t;r))},$$

$$\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))] = \frac{g(t,y(t;r))g_{21}(t,y(t;r))}{g(t,y(t;r)) + g_{21}(t,y(t;r))} + \frac{g_{21}(t,y(t;r))g_{31}(t,y(t;r))}{g_{21}(t,y(t;r)) + g_{31}(t,y(t;r))} + \frac{g_{32}(t,y(t;r)) + g_{42}(t,y(t;r))}{g_{32}(t,y(t;r)) + g_{42}(t,y(t;r))}.$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by

$$[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)] \text{ and } [y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$$

respectively. The solution is calculated by grid points at (2.13). By (4.1) and (4.5), we have

$$Y_1(t_{n+1};r) \approx Y_1(t_n;r) + 2/3\,\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],$$
$$Y_2(t_{n+1};r) \approx Y_2(t_n;r) + 2/3\,\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))].$$

We define

$$y_1(t_{n+1};r) \approx y_1(t_n;r) + 2/3\,\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],$$
$$y_2(t_{n+1};r) \approx y_2(t_n;r) + 2/3\,\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))].$$

4.4 The Fourth Order Runge-Kutta Method Based on General Mean

The same that last method, we have four k_is, but by different exponent and different mean, then

$$y_1(t_{n+1};r) - y_1(t_n) = \sum_{i=1}^{4} w_i l_{i,1}(t_n, y(t_n;r)),$$
$$y_2(t_{n+1};r) - y_2(t_n) = \sum_{i=1}^{4} w_i l_{i,2}(t_n, y(t_n;r)),$$

where the w_i's are constants and

$$[l_1(t_n, y(t_n;r))]_r = \left[l_{i,1}(t_n, y(t_n;r)), l_{i,2}(t_n, y(t_n;r))\right],$$
$$l_{i,1}(t_n, y(t_n;r)) = hf(t_n + d_i h, y_1(t_n) + \sum_{j=1}^{i-1} b_{ij} l_{j,1}(t_n, y(t_n;r)),$$
$$l_{i,2}(t_n, y(t_n;r)) = hf(t_n + d_i h, y_2(t_n) + \sum_{j=1}^{i-1} b_{ij} l_{j,2}(t_n, y(t_n;r)),$$

and

$$l_{11}(t_n, y(t_n;r)) = \min\{hf(t,u)|u \in [y_1(t;r), y_2(t;r)]\},$$
$$l_{12}(t_n, y(t_n;r)) = \max\{hf(t,u)|u \in [y_1(t;r), y_2(t;r)]\},$$

$$l_{21}(t_n, y(t_n;r)) = \min\left\{hf(t+h/2,u)|u \in \left[v_{1,1}(t, y(t;r)), v_{1,2}(t, y(t;r))\right]\right\},$$
$$l_{22}(t_n, y(t_n;r)) = \max\left\{hf(t+h/2,u)|u \in \left[v_{1,1}(t, y(t;r)), v_{1,2}(t, y(t;r))\right]\right\},$$

$$l_{31}(t_n, y(t_n;r)) = \min\left\{hf(t+h/2,u)|u \in \left[v_{2,1}(t, y(t;r)), v_{2,2}(t, y(t;r))\right]\right\},$$
$$l_{32}(t_n, y(t_n;r)) = \max\left\{hf(t+h/2,u)|u \in \left[v_{2,1}(t, y(t;r)), v_{2,2}(t, y(t;r))\right]\right\},$$

$$l_{41}(t_n, y(t_n;r)) = \min\left\{hf(t + ((1/4) + (-3/4) + (3/2))h, u)|u \in \left[v_{1,1}(t, y(t;r)), v_{1,2}(t, y(t;r))\right]\right\},$$
$$l_{42}(t_n, y(t_n;r)) = \max\left\{hf(t + ((1/4) + (-3/4) + (3/2))h, u)|u \in \left[v_{1,1}(t, y(t;r)), v_{1,2}(t, y(t;r))\right]\right\},$$

where in the fourth order Runge-Kutta method based on Centroidal Mean

$$v_{1,1}(t, y(t;r)) = y_1(t,r) + (1/2)l_{1,1}(t, y(t;r)),$$
$$v_{1,2}(t, y(t;r)) = y_2(t,r) + (1/2)l_{1,2}(t, y(t;r)),$$

$$v_{2,1}(t, y(t;r)) = y_1(t,r) + (1/2))l_{2,1}(t, y(t;r)),$$
$$v_{2,2}(t, y(t;r)) = y_2(t,r) + (1/2)l_{2,2}(t, y(t;r)),$$

$$v_{3,1}(t, y(t;r)) = y_1(t,r) + l_{3,2}(t, y(t;r)),$$
$$v_{3,2}(t, y(t;r)) = y_2(t,r) + l_{3,2}(t, y(t;r)).$$

Define

$$\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))] = s_{11}(t, y(t;r)) + 2s_{21}(t, y(t;r)) + 2s_{31}(t, y(t;r)) + s_{41}(t, y(t;r)),$$
$$\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))] = s_{12}(t, y(t;r)) + 2s_{22}(t, y(t;r)) + 2s_{32}(t, y(t;r)) + s_{42}(t, y(t;r)).$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by

$$[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)] \text{ and } [y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$$

respectively. The solution is calculated by grid points at (2.13). By (4.1) and (4.5), we have

$$Y_1(t_{n+1};r) \approx Y_1(t_n;r) + 1/6\,\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],$$
$$Y_2(t_{n+1};r) \approx Y_2(t_n;r) + 1/6\,\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))].$$

We define

$$y_1(t_{n+1};r) \approx y_1(t_n;r) + 1/6\,\mathrm{F}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))],$$
$$y_2(t_{n+1};r) \approx y_2(t_n;r) + 1/6\,\mathrm{G}[(\mathrm{t},\mathrm{y}(\mathrm{t};\mathrm{r}))].$$

5 Numerical Example

Fuzzy initial value problem

$$\begin{cases} \mathrm{y}' = \mathrm{y}(\mathrm{t}), & \mathrm{t} \geq 0 \\ \mathrm{y}(0) = (0.8 + 0.125\mathrm{r}, 1.1 - 0.1\mathrm{r}). \end{cases}$$

The exact solution is given by

$$\mathrm{Y}(\mathrm{t};\mathrm{r}) = [(0.8 + 0.125r)e^t, (1.1 - 0.1r)e^t].$$

At t = 1 we get

$$\mathrm{Y}(1;\mathrm{r}) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e], 0 \leq r \leq 1.$$

The value of exact solution and approximate solutions is given in Tables 1, 2, 3 and 4.

Table 1. .

r	t	R-k 4 CHmean		Exact		Error	
		y_1	y_2	Y_1	Y_2	y_1	y_1
0	1	2.1531	2.999	2.1746	2.9901	0.006484	0.0089174
0.2	1	2.2204	3.0143	2.2426	3.0445	0.022198	0.030136
0.4	1	2.3174	3.1081	2.3105	3.0988	0.006898	0.0092417
0.6	1	2.3856	3.1626	2.3785	3.1532	0.007094	0.0094038
0.8	1	2.4537	3.2171	2.4465	3.2076	0.007291	0.009566
1	1	2.5219	3.2717	2.5144	3.2619	0.007498	0.0097281

Table 2. .

r	t	R-k 4 method		Exact			
		y_1	y_2	Y_1	Y_2	y_1	y_1
0	1	2.1667	2.9792	2.1746	2.9901	0.0079588	0.010943
0.2	1	2.2344	3.0333	2.2426	3.0445	0.0082075	0.011142
0.4	1	2.3021	3.0875	2.3105	3.0988	0.0084562	0.011341
0.6	1	2.3698	3.1417	2.3785	3.1532	0.0087049	0.01154
0.8	1	2.4375	3.1958	2.4465	3.2076	0.0089536	0.011739
1	1	2.5052	3.25	2.5144	3.2619	0.0092024	0.011938

Table 3. .

r	t	R-k4 Cnt mean		Exact		Error	
		y_1	y_2	Y_1	Y_2	y_1	y_1
0	1	2.5794	3.5466	2.1746	2.9901	0.40474	0.55652
0.2	1	2.66	3.6111	2.2426	3.0445	0.41739	0.56664
0.4	1	2.7406	3.6756	2.3105	3.0988	0.43004	0.57676
0.6	1	2.8212	3.7401	2.3785	3.1532	0.44269	0.58688
0.8	1	2.9018	3.8046	2.4465	3.2076	0.45534	0.597
1	1	2.9824	3.8691	2.5144	3.2619	0.46798	0.60711

Table 4. .

r	t	R-k 4 H mean		Exact		Error	
		y_1	y_2	Y_1	Y_2	y_1	y_1
0	1	2.1531	2.9605	2.1746	2.9901	0.021525	0.029597
0.2	1	2.2204	3.0143	2.2426	3.0445	0.022198	0.030136
0.4	1	2.2877	3.0682	2.3105	3.0988	0.022871	0.030674
0.6	1	2.355	3.122	2.3785	3.1532	0.023543	0.031212
0.8	1	2.4222	3.1758	2.4465	3.2076	0.024216	0.03175
1	1	2.4895	3.2297	2.5144	3.2619	0.024889	0.03228

6 Conclusion

In this paper Runge-Kutta method by different means has been applied for finding the better and closed numerical solution and prepare by exact solution. The efficiency and the accuracy of the proposed method have been illustrated by a suitable example.

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Part V: Others

Non-traveling Wave Exact Solutions of (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama Equation

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Abstract. In this paper, the exact solutions for (3+1)-dimensional Yu-Toda-Sasa-Fukuyama equation have been investigated. By Lie group method and traveling wave transformation, we obtain two symmetry reduced equations of (3+1)-dimensional Yu-Toda-Sasa-Fukuyama equation. Then three classes of non-traveling wave exact solutions of (3+1)-dimensional Yu-Toda-Sasa-Fukuyama equation are constructed. At last, we achieve some computer simulations to illustrate our main results.

Keywords: (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama equation · Non-traveling exact wave solution · Meromorphic function · Elliptic function

1 Introduction and Main Results

One of the main topics in physics, chemistry, biology is to create mathematical models to describe the natural phenomenon and find exact solutions of nonlinear partial and ordinary differential equations. Exact solutions of differential equations can be used to help researchers to well understand the mechanism of the complicated natural phenomenon. Many effective methods being used to find exact solutions of differential equations have been developed, such as the exponential function method [1,2], F-expansion method [3], tanh-sech method [4], direct algebraic method [5], first integral method [6], sine-cosine method [7], transformed rational function method [8], inverse scattering transform method [9], Bäcklund transform method [10], Darboux transform method [11], $\frac{G'}{G}$ expansion method [12], Lie group method [13], and so on.

In 1998, Yu et al. [14] extended the Bogoyavlenskii Schiff equation

$$u_t + \Phi(u)u_z = 0, \Phi(u) = \partial_x^2 + 4u + 2u_x\partial_x^{-1} \tag{1}$$

to be the (3+1)-dimensional non-linear evolution equation

$$(-4u_t + \Phi(u)u_z)_x + 3u_{yy} = 0, \Phi(u) = \partial_x^2 + 4u + 2u_x\partial_x^{-1}. \tag{2}$$

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Setting $u := v_x$, Eq. (2) is changed into the (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama (YTSF) equation

$$v_{xxxz} - 4v_{xt} + 4v_x v_{xz} + 2v_{xx}v_z + 3v_{yy} = 0. \tag{3}$$

Substituting the traveling wave transformation

$$v = v(\xi), \xi = a_1x + a_2y + a_3z + a_4t \tag{4}$$

into the YTSF equation yields a non-linear ordinary differential equation, setting $w = v'$ and integrating it yields the ordinary differential equation

$$a_1^3a_3w'' + (3a_2^2 - 4a_1a_4)w + 3a_1^2a_3w^2 = \beta, \tag{5}$$

where a_1, a_2, a_3, a_4 are constants.

By Lie group method [13], we derive two symmetry reduced equations

$$\varphi_{xxx\eta} + 4\varphi_{x\rho} + 4\varphi_x\varphi_{x\eta} + 2\varphi_{xx}\varphi_\eta + 3\varphi_{\rho\rho} + 4 = 0, \tag{6}$$

$$\varphi_{xxx\eta} + 4\varphi_x\varphi_{x\eta} + 2\varphi_{xx}\varphi_\eta + 3\varphi_{yy} = 0. \tag{7}$$

Substituting the transformation

$$\varphi = v(x, \rho, \eta) - \frac{2}{3}\rho^2 + \epsilon\rho + \beta$$

into Eq. (6) yields the ordinary differential equation

$$v_{xxx\eta} + 4v_{x\rho} + 4v_xv_{x\eta} + 2v_{xx}v_\eta + 3v_{\rho\rho} = 0, \tag{8}$$

where ϵ, β are constants.

Next, substituting the traveling wave transformation

$$v(x, \rho, \eta) = v(\theta), \theta = kx + l\eta + r\rho \tag{9}$$

into Eq. (8), setting $w = v'$ and integrating it yields the differential equation

$$k^3lw'' + 3k^2lw^2 + (4kr + 3r^2)w = \beta, \tag{10}$$

where k, l, r are constants.

Substituting the traveling wave transformation

$$\varphi(x, \eta, y) = \varphi(\theta), \theta = kx + l\eta + ry \tag{11}$$

into Eq. (7), setting $w = \varphi'$ and integrating it yields the ordinary differential equation

$$k^3lw'' + 3k^2lw^2 + 3r^2w = \beta, \tag{12}$$

where k, l, r are constants.

Equations (5), (10) and (12) are special cases of the following non-linear ordinary differential equation

$$Aw'' + Bw + Cw^2 + D = 0, \tag{13}$$

where A, B, C, D are constants.

In Eq. (13), if $A = 0$, w must be a constant. If $C = 0, B \neq 0$, we have a linear ODE

$$Aw'' + Bw + D = 0. \tag{14}$$

Any meromorphic solution of Eq. (14) has no pole, because all the solutions of Eq. (14) are:

$$w(\xi) = \begin{cases} C_1 e^{\sqrt{-\frac{B}{A}}\xi} + C_2 e^{-\sqrt{-\frac{B}{A}}\xi} - \dfrac{D}{B}, & where - \dfrac{B}{A} \neq 0, \\ C_1 + C_2\xi - \dfrac{D}{2A}\xi^2, & where - \dfrac{B}{A} = 0. \end{cases} \tag{15}$$

In order to clarify our main results, we need some basic concepts. A meromorphic function is holomorphic in the complex plane $\mathbf{C}$ except for poles.

Let $\omega_1,\ \omega_2$ be two given complex constants such that $\mathrm{Im}\frac{\omega_1}{\omega_2} > 0,\ L = L[2\omega_1,\ 2\omega_2]$ be discrete subset $L[2\omega_1,\ 2\omega_2] = \{\omega \mid \omega = 2n\omega_1 + 2m\omega_2,\ n, m \in \mathbf{Z}\}$. The discriminant $\varDelta = \varDelta(c_1, c_2) := c_1^3 - 27c_2^2$ and

$$s_n = s_n(L) := \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}.$$

Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with two periods $2\omega_1,\ 2\omega_2$ and satisfying

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{16}$$

where $g_2 = 60s_4,\ g_3 = 140s_6$ and $\varDelta(g_2, g_3) \neq 0$.

Weierstrass zeta function $\zeta(z) := \frac{1}{z} + \sum\limits_{\omega \neq 0}(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2})$ is a meromorphic function and satisfying the differential equation

$$\wp(z) = -\zeta'(z). \tag{17}$$

Weierstrass elliptic function $\wp(z)$ satisfies the following addition formula

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}]^2. \tag{18}$$

Weierstrass zeta function $\zeta(z)$ satisfies the following addition formula

$$\zeta(z - z_0) = \zeta(z) - \zeta(z_0) + \frac{1}{2}[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}]. \tag{19}$$

In 2012, applying the complex method, for the first time, Yuan et al. achieved the general meromorphic solutions of Eq. (13) with $AC \neq 0$:

Theorem A [17]. *Suppose that $AC \neq 0$, then the general meromorphic solutions w of equation*

$$Aw'' + Bw + Cw^2 + D = 0$$

are of the following forms:

(I) The elliptic general solutions

$$w_d(z) = \tfrac{6A}{C}\wp(z) - \tfrac{3A}{2C}[\tfrac{\wp'(z)+F}{\wp(z)-E}]^2 + \tfrac{6AE}{C} - \tfrac{B}{2C}, \tag{20}$$

where $4DC = -12A^2 g_2 + B^2, F^2 = 4E^3 - g_2 E - g_3,\ g_2, g_3$ and E are arbitrary.

(II) The simply periodic solutions

$$w_s(z) = -\frac{3A}{2C}\alpha^2 \coth^2 \frac{\alpha}{2}(z - z_0) + \frac{A}{C}\alpha^2 - \frac{B}{2C}, \tag{21}$$

where $z_0 \in \mathbf{C}, \alpha \neq 0, 4DC = -A^2\alpha^4 + B^2$.

(III) The rational function solutions

$$w_r(z) = -\frac{6A}{C}\frac{1}{(z - z_0)^2} - \frac{B}{2C}, \tag{22}$$

where $z_0 \in \mathbf{C}$, $4CD = B^2$.

This paper is organized as follows. In the second section, we construct the traveling exact solutions of the YTSF equation. In the third section, we give the symmetry reduced equations and the non-traveling exact solutions of the YTSF equation. Our main results are the following Theorems.

2 Traveling Wave Exact Solutions

Theorem 1. *Suppose that $\xi = a_1 x + a_2 y + a_3 z + a_4 t$, $a_1 a_3 \neq 0$, then the traveling exact solutions v of Eq. (3) are the following forms:*

(I) The solutions with elliptic functions

$$v_d(\xi) = 2a_1[\zeta(\xi) - \zeta(\xi_0) + \tfrac{1}{2}\tfrac{\wp'(\xi)+F}{\wp(\xi)-E}] - \tfrac{3a_2^2 - 4a_1a_4}{6a_1^2 a_3}(\xi - \xi_0) + \gamma, \tag{23}$$

here $F^2 = 4E^3 - g_2 E - g_3,\ g_2, g_3,\ \xi_0$ and E are arbitrary, γ is integral constant.

(II) The solutions with some certain simply periodic functions

$$v_s(\xi) = a_1\alpha \coth\frac{\alpha}{2}(\xi - \xi_0) + \frac{4a_1a_4 - 3a_2^2 - a_1^3 a_3 \alpha^2}{6a_1^2 a_3}(\xi - \xi_0) + \gamma, \tag{24}$$

here $\alpha \neq 0$, ξ_0 is arbitrary, γ is integral constant.

(III) The solutions with rational functions

$$v_r(\xi) = \frac{2a_1}{\xi - \xi_0} + \frac{4a_1a_4 - 3a_2^2}{6a_1^2a_3}(\xi - \xi_0) + \gamma, \tag{25}$$

here ξ_0 is arbitrary, γ is integral constant.

Proof. Setting $A = a_1^3a_3, B = 3a_2^2 - 4a_1a_4, C = 3a_1^2a_3, D = -\beta$, by Theorem A and the addition formulas of Weierstrass η and ζ function, the solutions with elliptic functions of Eq. (3) are

$$\begin{aligned}
v_d(\xi) &= \int w_d(\xi)d\xi \\
&= \int \frac{6A}{C}\wp(\xi) - \frac{3A}{2C}[\frac{\wp'(\xi) + F}{\wp(\xi) - E}]^2 + \frac{6AE}{C} - \frac{B}{2C}d\xi \\
&= \int -\frac{6A}{C}\wp(\xi - \xi_0) - \frac{B}{2C}d\xi \\
&= \frac{6A}{C}\zeta(\xi - \xi_0) - \frac{B}{2C}(\xi - \xi_0) + \gamma \\
&= 2a_1[\zeta(\xi) - \zeta(\xi_0) + \frac{1}{2}\frac{\wp'(\xi) + F}{\wp(\xi) - E}] - \frac{3a_2^2 - 4a_1a_4}{6a_1^2a_3}(\xi - \xi_0) + \gamma.
\end{aligned}$$

here, γ is integral constant.

The solutions with some certain simply periodic functions of Eq. (3) are

$$\begin{aligned}
v_s(\xi) &= \int w_s(\xi)d\xi \\
&= \int -\frac{3A}{2C}\alpha^2 \coth^2\frac{\alpha}{2}(\xi - \xi_0) + \frac{A}{C}\alpha^2 - \frac{B}{2C}d\xi \\
&= \frac{3A}{C}\alpha \coth\frac{\alpha}{2}(\xi - \xi_0) - \frac{3A}{C}\frac{\alpha^2}{2}(\xi - \xi_0) + (\frac{A}{C}\alpha^2 - \frac{B}{2C})(\xi - \xi_0) + \gamma \\
&= a_1\alpha \coth\frac{\alpha}{2}(\xi - \xi_0) + \frac{4a_1a_4 - 3a_2^2 - a_1^3a_3\alpha^2}{6a_1^2a_3}(\xi - \xi_0) + \gamma.
\end{aligned}$$

The solutions with rational functions of Eq. (3) are

$$\begin{aligned}
v_r(\xi) &= \int w_r(\xi)d\xi \\
&= \int -\frac{6A}{C}\frac{1}{(\xi - \xi_0)^2} - \frac{B}{2C}d\xi \\
&= \frac{6A}{C}\frac{1}{\xi - \xi_0} - \frac{B}{2C}(\xi - \xi_0) + \gamma \\
&= \frac{2a_1}{\xi - \xi_0} + \frac{4a_1a_4 - 3a_2^2}{6a_1^2a_3}(\xi - \xi_0) + \gamma.
\end{aligned}$$

Remark 1. By Theorem 1, we give the computer simulations.

Setting $a_1 = a_2 = a_3 = a_4 = 1, x = 1, y = 1, \xi_0 = 0, \gamma = 0$, we have a solution with a rational function (See Fig. 1(a))

$$v_{r1}(\xi) = \frac{2}{z+t+2} + \frac{1}{6}(z+t) + \frac{1}{3}. \tag{26}$$

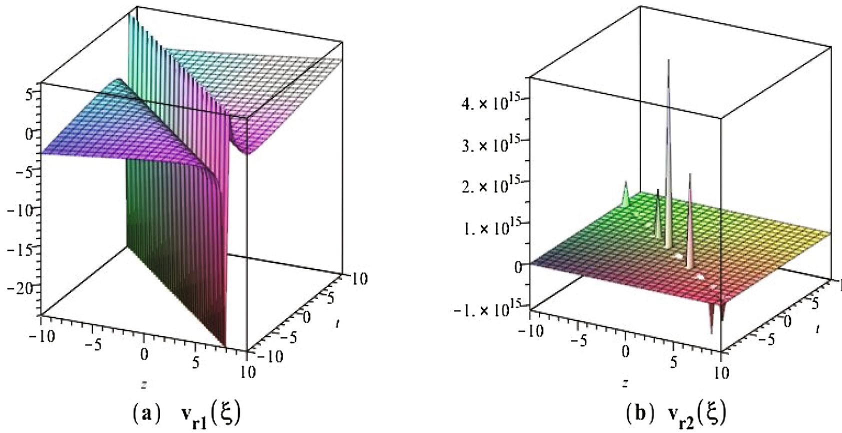

Fig. 1. Two solutions with rational functions of Eq. (3).

Setting $a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5, x = 1, y = 1, \xi_0 = 0, \gamma = 0$, we have a solution with a rational function (See Fig. 1(b))

$$v_{r2}(\xi) = \frac{4}{4z+5t+5} + \frac{13}{24}z + \frac{65}{96}t + \frac{65}{96}. \tag{27}$$

Setting $a_1 = a_2 = a_3 = a_4 = 1, \alpha = 2, x = 1, y = 1, \xi_0 = 0, \gamma = 0$, we have a solution with a simply periodic function (See Fig. 2(a))

$$v_{s1}(\xi) = 2\coth(z+t+2) - \frac{1}{2}(z+t) - 1. \tag{28}$$

Setting $a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5, \alpha = \frac{1}{2}, x = 1, y = 1, \xi_0 = 1, \gamma = 0$, we have a solution with a simply periodic function (See Fig. 2(b))

$$v_{s2}(\xi) = \coth(z + \frac{5}{4}t + 1) + \frac{5}{24}z + \frac{25}{96}t + \frac{5}{24}. \tag{29}$$

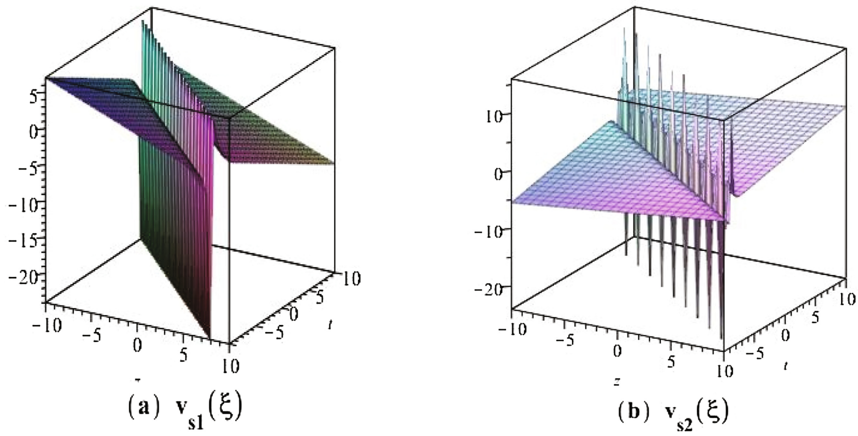

Fig. 2. Two solutions with simply periodic functions of (3).

3 Non-traveling Wave Exact Solutions and Symmetry Reduced Equations

3.1 Symmetry of YTSF Equation

In order to find out the symmetry $\sigma = \sigma(x, y, z, t, u)$ of Eq. (3), setting

$$\sigma = av_x + bv_y + cv_z + dv_t + ev + f. \tag{30}$$

Here u is the solution of Eq. (3), a, b, c, d, e, f are unknown functions of real variables x, y, z, t. According to Lie group analysis [13] and reference [12], σ satisfies

$$\sigma_{xxxz} - 4\sigma_{xt} + 4v_x\sigma_{xz} + 4v_{xz}\sigma_x + 2v_{xx}\sigma_z + 2v_z\sigma_{xx} + 3\sigma_{yy} = 0. \tag{31}$$

Substituting Eq. (30) into Eq. (31), we have a new differential equation, where

$$v_{xxxz} = 4v_{xt} - 4v_xv_{xz} - 2v_{xx}v_z - 3v_{yy}. \tag{32}$$

By Eqs. (30), (31) and (32), we have

$$\begin{gathered} a = c_1x + c_2, b = c_3y + c_4, c = (2c_3 - 3c_1)z + s(t), \\ d = (2c_3 - c_1)t + c_5, e = c_1, f = s'(t)x + \frac{2}{3}s''(t)y^2 + \tau(t)y + \mu(t), \end{gathered} \tag{33}$$

where c_i are real constants, $s(t), \tau(t), \mu(t)$ are arbitrary real functions of t. Substituting Eq. (33) into Eq. (30), we achieved the symmetry of YTSF equation

$$\begin{gathered} \sigma = (c_1x + c_2)v_x + (c_3y + c_4)v_y + ((2c_3 - 3c_1)z + s(t))v_z \\ +((2c_3 - c_1)t + c_5)v_t + c_1v + s'(t)x + \frac{2}{3}s''(t)y^2 + \tau(t)y + \mu(t). \end{gathered} \tag{34}$$

3.2 Symmetry Reduced Equations of YTSF Equation

By solving the characteristic equation of σ

$$\frac{dx}{c_1x+c_2}=\frac{dy}{c_3y+c_4}=\frac{dz}{(2c_3-3c_1)z+s(t)}$$
$$=\frac{dt}{(2c_3-c_1)t+c_5}=\frac{du}{c_1v+s'(t)x+\frac{2}{3}s''(t)y^2+\tau(t)y+\mu(t)}, \tag{35}$$

we find different symmetry reduced equations. Without loss of generality, we have two reduced equations as follows.

Setting $c_1=c_2=c_3=0, c_4=c_5=1, \tau(t)=0, s(t)=t$, solving $\sigma=0$, we have the first similarity solution of Eq. (3)

$$v=\varphi(x,\rho,\eta)-xt-\int\mu(t)dt, \tag{36}$$

where $\rho=y-t, \eta=z-\frac{1}{2}t^2$. Substituting Eq. (36) into Eq. (3), we have the first symmetry reduced equation Eq. (6)

$$\varphi_{xxx\eta}+4\varphi_{x\rho}+4\varphi_x\varphi_{x\eta}+2\varphi_{xx}\varphi_\eta+3\varphi_{\rho\rho}+4=0.$$

Setting $c_1=c_2=c_3=c_4=0, c_5=-1$, solving $\sigma=0$, we have the second similarity solution of Eq. (3)

$$v=\varphi(x,y,\eta)+s(t)x+\frac{2}{3}s'(t)y^2+\int(\tau(t)y+\mu(t))dt, \tag{37}$$

where $\eta=z+\int s(t)dt$. Substituting Eq. (37) into Eq. (3), we have the second symmetry reduced equation Eq. (7)

$$\varphi_{xxx\eta}+4\varphi_x\varphi_{x\eta}+2\varphi_{xx}\varphi_\eta+3\varphi_{yy}=0.$$

3.3 Non-traveling Exact Solutions of YTSF Equation

By Theorem A and Eq. (10), we derive the non-traveling exact solutions of Eq. (3).

Theorem 2. *Suppose that $\theta=kx+l\eta+r\rho, kl\neq 0, \rho=y-t, \eta=z-\frac{1}{2}t^2$, $\delta=-\frac{2}{3}\rho^2+\epsilon\rho+\beta-xt-\int\mu(t)dt$, here ϵ,β are arbitrary real numbers, $\mu(t)$ is arbitrary real function of t, then the non-traveling exact solutions v of Eq. (3) are the following forms:*

(I) The solutions with elliptic functions

$$v_d(x,y,z,t)=v_d(\theta)=2k[\zeta(\theta)-\zeta(\theta_0)+\tfrac{1}{2}\tfrac{\wp'(\theta)+F}{\wp(\theta)-E}]-\tfrac{4kr+3r^2}{6k^2r}(\theta-\theta_0)+\gamma+\delta, \tag{38}$$

here $F^2=4E^3-g_2E-g_3$, g_2,g_3,θ_0 and E are arbitrary, γ is integral constant.

(II) The solutions with some certain simply periodic functions

$$v_s(x,y,z,t) = v_s(\theta) = k\alpha \coth \frac{\alpha}{2}(\theta-\theta_0) - (\frac{k}{6}\alpha^2 + \frac{4kr+3r^2}{6k^2l})(\theta-\theta_0) + \gamma + \delta, \quad (39)$$

here θ_0 is arbitrary, $\alpha \neq 0$.

(III) The solutions with rational functions

$$v_r(x,y,z,t) = v_r(\theta) = \frac{2k}{\theta-\theta_0} - \frac{4kr+3r^2}{6k^2l}(\theta-\theta_0) + \gamma + \delta, \quad (40)$$

here θ_0 is arbitrary.

Remark 2. By Theorem 2, we give the computer simulations.

Setting $k=1, l=1, r=1, x=1, y=1, \epsilon=0, \beta=0, \mu(t)=0, \gamma=0, \theta_0=0$, we have a solution with a rational function (See Fig. 3(a))

$$v_{r3}(\theta) = \frac{2}{2+z-\frac{1}{2}t^2-t} - \frac{7}{3} - \frac{7}{6}z + \frac{7}{12}t^2 + \frac{1}{6}t - \frac{2}{3}(1-t)^2. \quad (41)$$

Setting $k=2, l=-1, r=1, \alpha=2, x=-1, y=1, \epsilon=1, \beta=1, \mu(t)=0, \gamma=0, \theta_0=0$, we have a solution with a simply periodic function (See Fig. 3(b))

$$v_{s3}(\theta) = 4\coth(-1-z+\frac{1}{2}t^2-t) - \frac{7}{16}t^2 + \frac{7}{8}t + \frac{7}{8}z + \frac{23}{8} - \frac{2}{3}(1-t)^2. \quad (42)$$

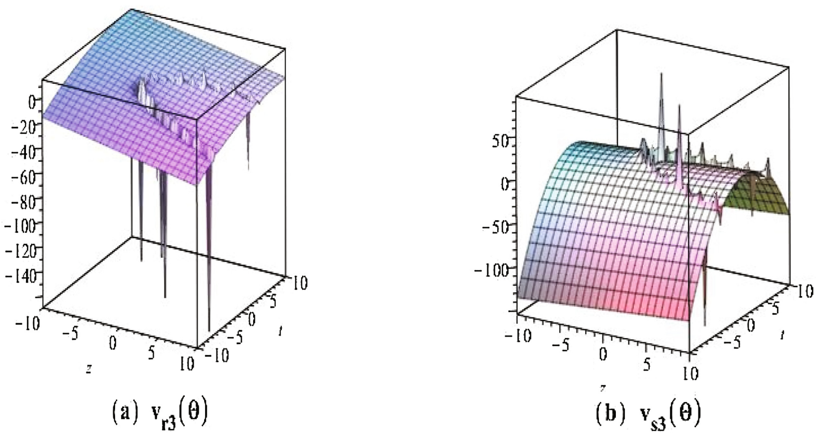

Fig. 3. Two solutions of Eq. (3).

Remark 3. By these Theorems, many non-traveling exact solutions can be constructed by selecting different initial value. Specially, setting $A = 4klr + 3lr^2 < 0$,

$\alpha = \frac{\sqrt{-kA}}{k^2 l}$, $\theta = kx + l\eta + r\rho$, $kl \neq 0$, $\rho = y - t, \eta = z - \frac{1}{2}t^2, \epsilon, \beta$ are arbitrary real numbers, $\mu(t)$ is arbitrary real function of t, by Theorem 2, we have

$$\begin{aligned} v_s(x,y,z,t) =& v_s(\theta) = \frac{1}{kl}\sqrt{-klr(4k+3r)}\coth\{\frac{1}{2}\frac{1}{k^2 l}\sqrt{-klr(4k+3r)}\theta - \theta_0\} + \gamma \\ & -\frac{2}{3}(y-t)^2 + \epsilon(y-t) + \beta - xt - \int \mu(t)dt, \end{aligned}$$

obviously, this solution is equivalent to the solution v_4 in reference [12].

Setting $A = -(4klr + 3lr^2) < 0, \alpha = \frac{\sqrt{kA}}{k^2 l}$, by Theorem 2, we have

$$\begin{aligned} v_s(x,y,z,t) \ =& v_s(\theta) = \frac{1}{kl}\sqrt{klr(4k+3r)}\coth\{\frac{1}{2}\frac{1}{k^2 l}\sqrt{klr(4k+3r)}\theta - \theta_0\} + \gamma \\ & -\frac{2}{3}(y-t)^2 + \epsilon(y-t) + \beta - xt - \int \mu(t)dt, \end{aligned}$$

obviously, this solution is equivalent to the solution v_5 in reference [12].

By Theorem 1 and Eq. (12), we derive the non-traveling solutions of Eq. (3).

Theorem 3. *Suppose that $\theta = kx + l\eta + ry$, $kl \neq 0$, $\eta = z + \int s(t)dt$, $\delta = s(t)x + \frac{2}{3}s'(t)y^2 + \int(\tau(t)y + \mu(t))dt$, then the non-traveling exact solutions v of Eq. (3) are the following forms:*

(I) The solutions with elliptic functions

$$v_d(x,y,z,t) = v_d(\theta) = 2k[\rho(\theta) - \rho(\theta_0) + \tfrac{1}{2}\tfrac{\wp'(\theta)+F}{\wp(\theta)-E}] - \tfrac{r^2}{2k^2 l}(\theta - \theta_0) + \gamma + \delta. \quad (43)$$

here $F^2 = 4E^3 - g_2 E - g_3$, g_2, g_3, θ_0 and E are arbitrary, γ is integral constant.

(II) The solutions with some certain simply periodic functions

$$v_s(x,y,z,t) = v_s(\theta) = k\alpha\coth\frac{\alpha}{2}(\theta-\theta_0) - (\frac{k}{6}\alpha^2 + \frac{r^2}{2k^2 l})(\theta-\theta_0) + \gamma + \delta, \quad (44)$$

here θ_0 is arbitrary, $\alpha \neq 0$.

(III) The solutions with rational functions

$$v_r(x,y,z,t) = v_r(\theta) = \frac{2k}{\theta-\theta_0} - \frac{r^2}{2k^2 l}(\theta-\theta_0) + \gamma + \delta, \quad (45)$$

here θ_0 is arbitrary.

Remark 4. By Theorem 3, we give the computer simulations.

Setting $k = 1, l = 2, r = 1, x = 1, y = 1, s(t) = t, \tau(t) = 0, \mu(t) = 0, \gamma = 0, \theta_0 = 0$, we have a solution with a rational function (See Fig. 4(a))

$$v_{r4}(\theta) = \frac{2}{t^2+2z+2} - \frac{1}{4}t^2 - \frac{1}{2}z + t + \frac{1}{6}. \quad (46)$$

Setting $k = 1, l = 2, r = 1, x = 1, y = 1, \alpha = 2, s(t) = t, \tau(t) = 0, \mu(t) = 0, \gamma = 0, \theta_0 = 0$, we have a solution with a simply periodic function (See Fig. 4(b))

$$v_{s4}(\theta) = 2\coth(t^2+2z+2) - \frac{11}{12}t^2 - \frac{11}{6}z + t - \frac{7}{6}. \quad (47)$$

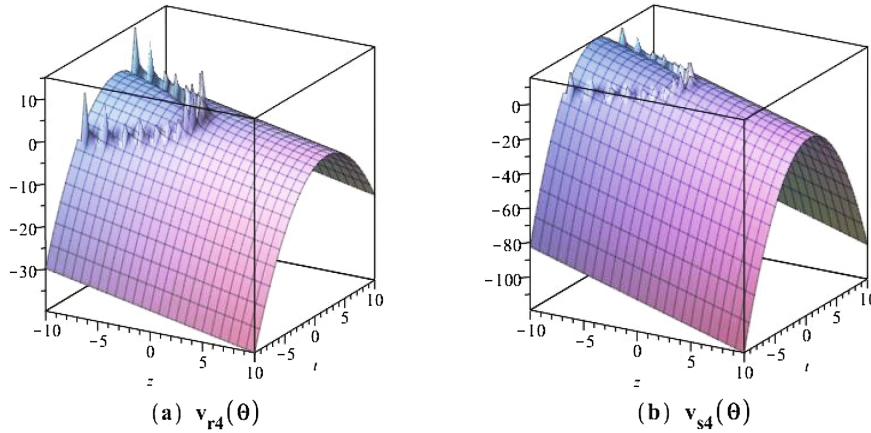

Fig. 4. Two solutions of Eq. (3).

4 Conclusion

By Theorem A and Lie group method, we achieved some new non-traveling exact solutions of YTSF equation. These solutions can be represented into three forms, containing the solutions with elliptic functions, some certain simply periodic functions or rational functions. This idea can be applied to other nonlinear partial differential equations.

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Recommender: Yadong Shang, Guangzhou University, Professor.

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Research of Solvability and Application of Fuzzy Errors Set Matrix 1 Equation of Type II

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Abstract. The concept of fuzzy sets and fuzzy error matrix is presented in this paper. On this basis, we studied the types of the set matrix of error matrix. It is especially researched that the elements of fuzzy error matrix are set, and each row of matrix is the decomposition of a fuzzy error logic proposition [1–5]. The error matrix equation is a general set of relations, not just the usual equation [6–12]. And the solvability of this error matrix, and solutions to it are presented. And an example of solving the fuzzy error set matrix would be given in the paper.

Keywords: Error logic · Convert conjunctions · Matrix representation · Eliminate the wrong

1 Fuzzy Set

1.1 Concept

1.1.1 Fuzzy Set

Suppose A is a mapping from set X to [0,1], A: X → [0,1], x → A (x), then A is the fuzzy set on X, A (x) is called the membership function of fuzzy set A, Or A (x) is the membership degree of x to fuzzy set A.

1.1.2 Representation of Fuzzy Set

1. When the elements in the fuzzy set are finite, the fuzzy set can be expressed as follows: suppose domain U = {u (1), u (2), …, u (n)}
   (1) Zadeh representation method: $\mathrm{A} = \frac{A(u1)}{u1} + \frac{A(u2)}{u2} + \frac{A(u3)}{u3} + \cdots$
   (2) Vector representation method: $\mathrm{A} = \{A(u_1), A(u_2), \cdots\}$
   (3) Sequence dual representation method: $\mathrm{A} = \big((u_1, \mathrm{A}(u_1)\big), \big((u_2, \mathrm{A}(u_2)\big), \cdots$
2. When there are infinitely elements in the fuzzy set, the fuzzy set can be expressed as Zadeh method as follows: $\mathrm{A} = \int \frac{A(u)}{u}$

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2 Fuzzy Error Set Matrix

2.1 Classification of Fuzzy Set Matrix

2.1.1 Fuzzy Set Matrix

A_{ij}, B_{ij}, i = 1, 2, …, m, j = 1, 2, …, n is a fuzzy set.

= A
= B are fuzzy set matrixs.

$$\begin{pmatrix} A_{12} & \dots & \dots & A_{1l} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2l} & \dots & A_{2n} \\ A_{i1} & A_{i2} & \dots & A_{il} & \dots & A_{in} \\ A_{m1} & A_{m} & \dots & A_{ml} & \dots & A_{mn} \end{pmatrix} = A$$

and

$$\begin{pmatrix} B_{11} & B_{12} & \dots & B_{1l} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2l} & \dots & B_{2n} \\ B_{i1} & B_{i2} & \dots & B_{il} & \dots & B_{in} \\ B_{m1} & B_{m} & \dots & B_{ml} & \dots & B_{mn} \end{pmatrix} = B$$

2.1.2 Fuzzy Error Set Atrix

This (t + 1) *7 matrix is an set matrix, which is also called a fuzzy error set matrix, since it is composed of seven (set) elements of the error logic proposition, where yij ∈ [0,1].

2.2 Fuzzy Error Set Matrix Operation

XA	X●A	X▲A	X∨A	X∧A
General	Excellent	Inferior	Or	Versus

3 Fuzzy Error Set Matrix Equation

3.1 Error Set Matrix Equation Type

1. Equality form

Type I	AX=B	A●X=B	A▲X=B	A∨X=B	A∧X=B
Type II	XA=B	X●A=B	X▲A=B	X∨A=B	X∧A=B
	General	Excellent	Inferior	Or	Versus

2. Connotative form

Type I	$AX \supseteq B$	$A \bullet X \supseteq B$	$A \blacktriangle X \supseteq B$	$A \vee X \supseteq B$	$A \wedge X \supseteq B$
Type II	$XA \supseteq B$	$X \bullet A \supseteq B$	$X \blacktriangle A \supseteq B$	$X \vee A \supseteq B$	$X \wedge A \supseteq B$
	General	Excellent	Inferior	Or	Versus

3.2 The Solution of Fuzzy Set Matrix Equation

3.2.1 The Solution Schemes of Fuzzy Matrices Set Equation

The equation of fuzzy matrices set has two types, equality form and connotative form. In general, the elements in a matrix are taken over a certain set. To solve fuzzy matrix equation, the solution is generally determined in a set. Here we first introduce the solving diagram of the fuzzy error matrix set equation (Figs. 1 and 2).

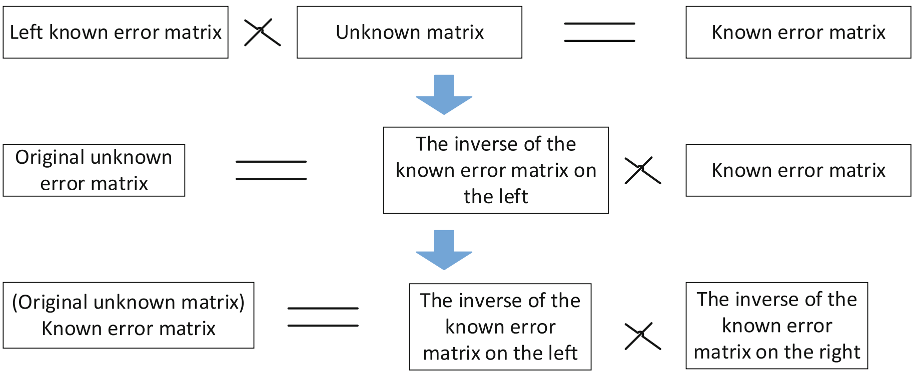

Fig. 1. An inverse solution to the left known matrix

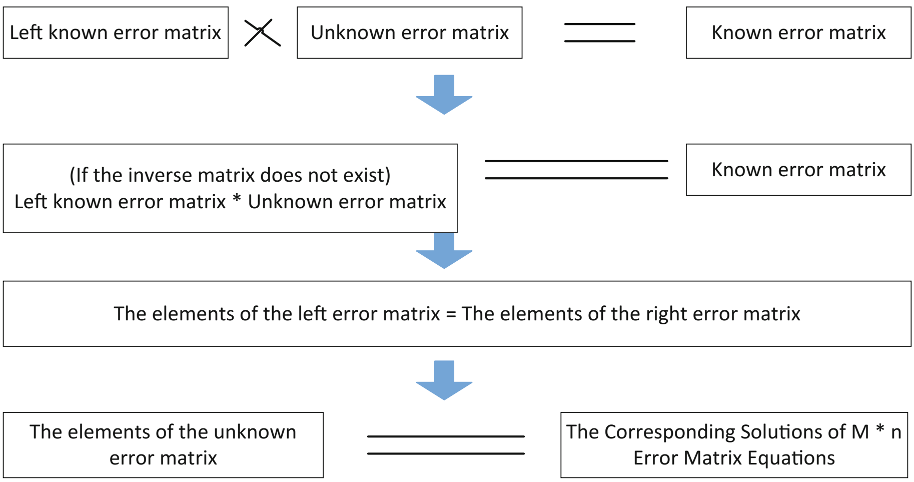

Fig. 2. No inverse solution for the left known matrix

3.2.2 The Solution of Fuzzy Error Matrix Equation

The solution of Fuzzy Error Matrix Equation of Type II 1

$$XA' \supseteq B = \begin{pmatrix} U_{10x} & S_{10x}(t) & \vec{p}_{10x} & (x_1, x_2, \ldots, x_n) & T_{10x}(t) & L_{10x}(t) & x_{10x}(t) = f_{10x}((u(t), & \vec{p}_{10x}), G_{U10x}(t)) & G_{U10x}(t) \\ U_{11x} & S_{11x}(t) & \vec{p}_{11x} & (x_1, x_2, \ldots, x_n) & T_{11x}(t) & L_{11x}(t) & x_{11x}(t) = f_{11x}((u(t), & \vec{p}_{11x}), G_{U11x}(t)) & G_{U11x}(t) \\ \ldots \\ U_{1tx} & S_{1tx}(t) & \vec{p}_{1tx} & (x_1, x_2, \ldots, x_n) & T_{1tx}(t) & L_{1tx}(t) & x_{1tx}(t) = f_{1tx}((u(t), & \vec{p}_{1tx}), G_{U1tx}(t)) & G_{U1tx}(t) \end{pmatrix}$$

$$A' = (U_2, S_2(t), \vec{P}_2, T_2(t), L_2(t), x_2(t) = f_2((u(t), \vec{P}_2), G_{U2}(t)))$$

$$= \begin{pmatrix} U_{20} & S_{20}(t) & \vec{p}_{20} & (x_1, x_2, \ldots, x_n) & T_{20}(t) & L_2(t) & y_{20}(t) = f_{20}((u(t), & \vec{p}_{20}), G_{U20}(t)) & G_{U20}(t) \\ U_{21} & S_{21}(t) & \vec{p}_{21} & (x_1, x_2, \ldots, x_n) & T_{21}(t) & L_{21}(t) & y_{21}(t) = f_{21}((u(t), & \vec{p}_{21}), G_{U21}(t)) & G_{U21}(t) \\ \ldots \\ U_{2t} & S_{tt}(t) & \vec{p}_{2t} & (x_1, x_2, \ldots, x_n) & T_{2t}(t) & L_{2t}(t) & y_{2t}(t) = f_{2t}((u(t), & \vec{p}_{2t}), G_{U2t}(t)) & G_{U2t}(t) \end{pmatrix}$$

$$B' = \begin{pmatrix} (b_{11}, y_{11}) & (b_{12}, y_{12}) & \ldots & (b_{1m1}, y_{1m1}) \\ (b_{21}, y_{21}) & (b_{22}, y_{22}) & \ldots & (b_{1m2}, y_{1m2}) \\ \ldots \\ (b_{m21}, y_{m21}) & (b_{m21}, y_{m21}) & \ldots & (b_{m2m1}, y_{m2m1}) \end{pmatrix}$$

$$= \begin{pmatrix} V_{201} \quad S_{v201}(t) \quad \vec{P}_{v201} \quad (x_1, x_2, \ldots, x_n) \quad T_{v201}(t) \quad L_{v201}(t) y_{v201}(t) = f_{v201}((v(t), \vec{P}_{v201}), G_{V201}(t)) \quad G_{V101}(t) \ldots \\ \ldots V_{21j} \quad S_{v21j}(t) \quad \vec{P}_{v21j} \quad (x_1, x_2, \ldots, x_n) \quad T_{v21j}(t) L_{21j}(t) y_{v21j}(t) = f_{v21j}((v(t), \vec{P}_{v21j}), G_{V21j}(t)) \quad G_{V21j}(t) \ldots \\ \ldots \\ \ldots V_{2m2m1} \quad S_{v2m2m1}(t) \quad \vec{P}_{v2m2m1} \quad (x_1, x_2, \ldots, x_n) \quad T_{v2m2m1}(t),\ L_{v2m2m1}(t) \quad y_{v2m2m1}(t) = f_{v2m2m1}((v(t), \\ \vec{P}_{v2m2m1}), G_{V2m2m1}(t)) G_{V1m2m1}(t) \end{pmatrix}$$

Definition 3.1 **let** $XA' \supseteq$

$$\begin{pmatrix} (w_{11}, z_{11}) \quad (w_{12}, z_{12}) \ldots (w_{1m1}, z_{1m1}) \\ (w_{21}, z_{21}) \quad (w_{22}, z_{22}) \ldots (w_{2m2}, z_{2m2}) \\ \ldots \\ (w_{m21}, z_{m21}) \, (w_{m21}, z_{m21}) \ldots (w_{m2m1}, z_{m2m1}) \end{pmatrix}$$

$$= \begin{pmatrix} (V_{201} S_{v201}(t) \quad \vec{P}_{v201} \quad (x_1, x_2, \ldots, x_n) \quad T_{v201}(t) \quad L_{v201}(t) y_{v201}(t) = f_{v201}((v(t), \vec{P}_{v201}),\ G_{V201}(t)) \quad G_{V101}(t) \ldots \\ \ldots \quad V_{21j} S_{v21j}(t) \vec{P}_{v21j} \quad (x_1, x_2, \ldots, x_n) \quad T_{v21j}(t) L_{21j}(t) y_{v21}(t) = f_{v21j}((v(t), \vec{P}_{v21j}),\ G_{V21j}(t)) \quad G_{V11j}(t) \ldots \\ \ldots \\ \ldots V_{2m2m1} S_{v2m2m1}(t) \vec{P}_{v2m2ml} \quad (x_1, x_2, \ldots, x_n) \quad T_{v2m2m1}(t),\ L_{v2m2m1}(t) y_{v2m2m1}(t) = f_{v2m2m1}((v(t), \quad \vec{P}_{v2m2ml}),\ G_{V2m2m1}(t)) G_{V1m2m1}(t) \end{pmatrix}$$

There into

$\left(w_{ij},\ z_{ij}\right) = U_{1ix} \wedge U_{2j},\ S_{1ix}(t) \wedge S_{2j}(t) \quad \vec{P}_{ijx} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2j} \quad T_{1ix}(t) \wedge T_{2j}(t) \quad L_{1ix}(t) \wedge L_{2j}$ $(t) \quad x_{1ix}(t) = f_{1i}((u(t), \vec{P}_{1ix}), \quad G_{U1i}(t)) \wedge y_{2j}(t) \quad G_{U1ix}(t) \wedge G_{U2j}(t)$

There into

$$\begin{pmatrix} U_{10x} \wedge U_{20} S_{10x}(t) \wedge S_{20}(t) \quad \vec{P}_{10x} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{20} T_{10x}(t) \wedge T_{20}(t) L_{10x}(t) \wedge L(t) x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{U10}(t)) \wedge y_{20}(t) \\ G_{U10x}(t) \wedge G_{U20}(t) \ldots \\ \ldots U_{11x} \wedge U_{21}, S_{11x}(t) \wedge S_{21}(t) \quad \vec{P}_{11x} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{21} T_{11x}(t) \wedge T_{21}(t) L_{11x}(t) \wedge L_{21}(t) x_{11x}(t) = f_{11x}((u(t), \vec{P}_{11x}), G_{U11}(t)) \wedge y_{21}(t) \\ G_{U11x}(t) \wedge G_{U21}(t) \ldots \\ \ldots \\ \ldots U_{1tx} \wedge U_{2t} S_{1tx}(t) \wedge S_{2t}(t) \vec{P}_{1tx \, (x_1, x_2, \ldots, x_n)} \wedge \vec{P}_{2t} T_{1tx}(t) \wedge T_{2t}(t) L_{1tx}(t) \wedge L_{2t}(t) x_{1tx}(t) = f_{1tx}((u(t), \vec{P}_{1tx}), G_{U1t}(t)) \wedge y_{2t}(t) G_{U1t}(t) \wedge G_{U2t}(t) \end{pmatrix}$$

$$= \begin{pmatrix} V_{201} S_{v201}(t) \vec{P}_{v201} \quad (x_1, x_2, \ldots, x_n) \quad T_{v201}(t) \quad L_{v201}(t) y_{v201}(t) = f_{v201}((v(t), \quad \vec{P}_{v201}), G_{V201}(t)) \quad G \\ {}_{V101}(t) \ldots \\ \ldots V_{21j} S_{v21j}(t) \vec{P}_{v21j} \quad (x_1, x_2, \ldots, x_n) \quad T_{v21j}(t) L_{21j}(t) y_{v21}(t) = f_{v21j}((v(t), \quad \vec{P}_{v21j}), G_{V21j}(t)) G_{V11j}(t) \ldots \\ \ldots \\ \ldots V_{2m2m1} S_{v2m2m1}(t) \vec{P}_{v2m2m1} \quad (x_1, x_2, \ldots, x_n) \quad T_{v2m2m1}(t), L_{v2m2m1}(t) y_{v2m2m1}(t) = f_{v2m2m1}((v(t), \vec{P}_{v2m2ml}), \\ G_{V2m2m1}(t)) G_{V1m2m1}(t) \end{pmatrix}$$

By the definition of equal matrices: if two matrices are contained, then the corresponding elements of the two matrices contain each other. Namely $\left(w_{ij}, z_{ij}\right) \supseteq \left(b_{ij}, y_{ij}\right)$, therefore,

$$U_{1ix} \wedge U_{2j}, S_{1ix}(t) \wedge S_{2j}(t) \quad \vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2j} \quad T_{1ix}(t) \wedge T_{2j}(t) \quad L_{1ix}(t) \wedge L_{2j}(t)$$
$$x_{1ix}(t) = f_{1i}((u(t), \vec{P}_{1ix}), G_{U1i}(t)) \wedge y_{2j}(t)\, G_{U1ix}(t) \wedge G_{U2j}(t) \quad \supseteq \left(b_{ij}, y_{ij}\right)$$
$$== (V_{2ij} S_{V2ij}(t) \vec{P}_{v2ij} \quad (x_1, x_2, \ldots, x_n) \quad T_{V2ij}(t) \quad L_{V2ij}(t) y_{V2ij}(t) = f_{V2ij}((v(t), \vec{P}_{v2ij}), G_{V2ij}(t)) G_{V2ij}(t))$$

Equals to the following set of equations:

$$U_{10x} \wedge U_{20} \supseteq V_{v20}$$
$$S_{10x}(t) \wedge S_{20}(t) \supseteq S_{v20}(t)$$
$$\vec{P}_{10x} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{20} \supseteq \vec{P}_{v20} \quad (x_1, x_2, \ldots, x_n)$$
$$T_{10x}(t) \wedge T_{20}(t) \supseteq T_{v20}(t)$$
$$L_{10x}(t) \wedge L_{20}(t) \supseteq L_{v20}(t)$$
$$x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{U10x}(t)) \wedge y_{20}(t) \supseteq y_{v20}(t) = f_{v20}((u(t), \vec{P}_{v20}), G_V(t))$$
$$G_{U10x}(t) \wedge G_{U20}(t) \supseteq G_{V20}(t)$$

$$U_{1ix} \wedge U_{2j} \supseteq V_{2j}$$
$$S_{1ix}(t) \wedge S_{2j}(t) \supseteq S_{V2j}(t)$$
$$\vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2j} \supseteq \vec{P}_{v2j} \quad (x_1, x_2, \ldots, x_n)$$
$$T_{1ix}(t) \wedge T_{2j}(t) \supseteq T_{V2j}(t)$$
$$L_{1ix}(t) \wedge L_{2j}(t) L_{V2j}(t)$$
$$x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) \wedge y_{2j}(t) \supseteq y_{V2j}(t) = f_{V2j}((u(t), \vec{P}_{v2j}), G_V(t))$$
$$G_{U1ix}(t) \wedge G_{U2j}(t) \supseteq G_{V2j}(t)$$

$$U_{ttx} \wedge U_{2t} \supseteq V_{V2t}$$
$$S_{ttx}(t) \wedge S_{2t}(t) \supseteq S_{V2t}(t)$$
$$\vec{P}_{ttx} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2t} \supseteq \vec{P}_{v2t} \quad (x_1, x_2, \ldots, x_n)$$
$$T_{ttx}(t) \wedge T_{2t}(t) \supseteq T_{V2t}(t)$$

$L_{\mathrm{ttx}}(t) \wedge L_{2\mathrm{t}}(t) \supseteq L_{V2\mathrm{t}}(t)$
$x_{\mathrm{ttx}}(t) = f_{\mathrm{ttx}}((u(t), \vec{P}_{ttx}), G_{U\mathrm{ttx}}(t)) \wedge y_{2\mathrm{t}}(t) \supseteq y_{\mathrm{v2t}}(t) = f_{\mathrm{v2t}}((u(t), \vec{P}_{v2t}), G_{\mathrm{V}}(t))$
$G_{U\mathrm{ttx}}(t) \wedge G_{\mathrm{U2t}}(t) \supseteq G_{V2\mathrm{t}}(t)$

About operations of "$\wedge$", if both sides of the equation are sets, then $\wedge$ means the "intersection" operation of the set. If $\wedge$ is the number on both sides of the equation, $\wedge$ means the "minimum" operation of the number.

As for $(U_{1\mathrm{ix}} \wedge U_{2\mathrm{j}})h_1(S_{1\mathrm{ix}}(t) \wedge S_{2\mathrm{j}}(t))h_2(\vec{P}_{1\mathrm{i}x}\quad(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n})\quad \wedge \vec{P}_{2j}) \vee h_3(T_{1\mathrm{ix}}(t) \wedge T_{2\mathrm{j}}(t))h_4(L_{1\mathrm{ix}}(t) \wedge L_{2\mathrm{j}}(t))h_5(x_{1\mathrm{ix}}(t) = f_{1\mathrm{ix}}((u(t), \vec{P}_{lix}), G_{\mathrm{U1ix}}(t)) \wedge y_{2\mathrm{j}}(t))h_6(G_{\mathrm{U1ix}}(t) \wedge G_{\mathrm{U2j}}(t)$ namely "h_i, i = 1,2, … 6" operations matrix elements, it mean that the elements have been computed are "combined" into a complete matrix element (proposition). The mode of combination depends on specific situations. One way is to use the parameters after the operation for the corresponding parameters constitute a new set of error elements or error logic proposition, which is called the multiplication of the m*7 error matrix.

Since we are not asking $\mathrm{X_iA'} = \mathrm{B}$ but $\mathrm{XiA'} = \mathrm{B}$ in the solvability of the real problem, we will discuss the more general error matrix equation model.

The solution of Fuzzy Error Matrix Equation of Type II 1 $\mathrm{XA'} \supseteq \mathrm{B}$

Theorem 1. The necessary and sufficient condition for the solvability of the fuzzy error matrix equation $\mathrm{XA'} \supseteq \mathrm{B}$ is the solvability of $\mathrm{x_iA'} \supseteq \mathrm{B_i}, \mathrm{i} = (1, 2, \ldots, \mathrm{m2})$.

Proof: If $\mathrm{XA'} \supseteq \mathrm{B}$ has solvability, it is can be known by the definition of $\mathrm{XA'} \supseteq \mathrm{B}$ and $\mathrm{x_iA'} \supseteq \mathrm{B_i}, \mathrm{i} = (1, 2, \ldots, \mathrm{m2})$ that are two equivalent equations, so it is necessary for $\mathrm{x_iA'} \supseteq \mathrm{B_i}, \mathrm{i} = (1, 2, \ldots, \mathrm{m2})$ has solvability; otherwise if $\mathrm{x_iA'} \supseteq \mathrm{B_i}, \mathrm{i} = (1, 2, \ldots, \mathrm{m2})$ has solutions, similarly it does for $\mathrm{XA'} \supseteq \mathrm{B}$.

Proved.

Therefore, we use the method of discussing the solvability of $\mathrm{x_i\,A'} = \mathrm{B_i}, \mathrm{i} = (0, 1, 2, \ldots, \mathrm{m2})$ to discuss the solution of $\mathrm{XA'} = \mathrm{B}$.

Then for in $\mathrm{X_iA'} \supseteq \mathrm{B_i}$

$$(U_{1\mathrm{ix}}\,S_{1\mathrm{ix}}(t)\vec{P}_{1\mathrm{i}x}\quad(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n})\quad T_{1\mathrm{ix}}(t)L_{1\mathrm{ix}}(t)x_{1\mathrm{ix}}(t) = f_{1\mathrm{ix}}((u(t), \vec{P}_{1ix}), G_{\mathrm{U1ix}}(t))G_{U1ix}(t))\,A'$$
$$\supseteq (U_{1\mathrm{ix}} \wedge U_{20}) \vee (S_{1\mathrm{ix}}(t) \wedge S_{20}(t)) \vee (\vec{P}_{1\mathrm{i}x}\quad(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n})\quad \wedge \vec{P}_{20}) \vee (T_{1\mathrm{ix}}(t) \wedge T_{20}(t))$$
$$\vee (L_{1\mathrm{ix}}(t) \wedge L_{20}(t)) \vee (x_{1\mathrm{ix}}(t) = f_{1\mathrm{ix}}((u(t), \vec{P}_{10x}), G_{U1\mathrm{ix}}(t)) \wedge x_{20}(t)) \vee (G_{U1\mathrm{ix}} \wedge G_{\mathrm{U20}}(t)$$

…

$$(U_{1\mathrm{ix}} \wedge U_{2\mathrm{j}}) \vee (S_{1\mathrm{ix}}(t) \wedge S_{2\mathrm{j}}(t)) \vee (\vec{P}_{1ix}\quad(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n})\quad \wedge \vec{P}_{2j}) \vee (T_{1\mathrm{ix}}(t) \wedge T_{2\mathrm{j}}(t))$$
$$\vee (L_{1\mathrm{ix}}(t) \wedge L_{2\mathrm{j}}(t)) \vee (x_{1\mathrm{ix}}(t) = f_{1\mathrm{ix}}((u(t), \vec{P}_{1ix}), G_{U1\mathrm{ix}}(t)) \wedge x_{2\mathrm{j}}(t)) \vee (G_{U1\mathrm{ix}}(t) \wedge G_{\mathrm{U2j}}(t)$$

…

$$(U_{1ix} \wedge U_{2m1}) \vee (S_{1ix}(t) \wedge S_{2m1}(t)) \vee (\vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2ml}) \vee (T_{iix}(t)$$
$$\wedge T_{2m1}(t)) \vee (L_{1ix}(t) \wedge L_{2m1}(t)) \vee (x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) \wedge x_{2m1}(t)) \vee (G_{U1ix}(t) \wedge G_{U2m1}(t)$$
$$\supseteq (b_{i1}, y_{i1})(b_{i2}, y_{i2}) \quad \ldots \quad (b_{im1}, y_{im1})$$

Which means

$$(U_{1ix} \wedge U_{20}) \vee (S_{1ix}(t) \wedge S_{20}(t)) \vee (\vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{20}) \vee (T_{1ix}(t) \wedge T_{20}(t))$$
$$\vee (L_{1ix}(t) \wedge L_{20}(t)) \vee (x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) \wedge x_{20}(t)) \vee (G_{U1ix}(t) \wedge G_{U20}(t)$$
$$\supseteq (V_{20} S_{v20}(t) \vec{P}_{v20} \quad (x_1, x_2, \ldots, x_n) \quad T_{v20j}(t) L_{v20}(t) y_{v20}(t) = f_{v20}((v(t), \vec{P}_{v20}), G_{2jV}(t)) \quad G_{V20}(t)$$

...

$$(U_{1ix} \wedge U_{2j}) \vee (S_{1ix}(t) \wedge S_{2j}(t)) \vee (\vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2j}) \vee (T_{1ix}(t) \wedge T_{2j}(t))$$
$$\vee (L_{1ix}(t) \wedge L_{2j}(t)) \vee (x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) \wedge x_{2i}(t)) \vee (G_{U1ix}(t) \wedge G_{U2j}(t)$$
$$\supseteq \left(V_{2j}, S_{v2j}(t)\right) \vec{P}_{v2j} \quad (x_1, x_2, \ldots, x_n) \quad T_{v2j}(t) \quad L_{v2j}(t) y_{v2j}(t) = f_{v2j}((v(t), \vec{P}_{v2j}), G_{2jV}(t)) G_{V2j}(t)$$

...

$$(U_{1tx} \wedge U_{2m1}) \vee (S_{1tx}(t) \wedge S_{2m1}(t)) \vee (\vec{P}_{1tx} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2ml}) \vee (T_{1tx}(t)$$
$$\wedge T_{2m1}(t)) \vee (L_{1tx}(t) \wedge L_{2m1}(t)) \vee (x_{1tx}(t) = f_{1tx}((u(t), \vec{P}_{1tx}), \quad G_{U1tx}(t)) \wedge x_{2m1}(t)) G_{U1tx}(t)$$
$$\wedge G_{U2m1}(t) \supseteq (V_{2t}, S_{v2t}(t)) \vec{P}_{v2t} \quad (x_1, x_2, \ldots, x_n) \quad T_{v2t}(t) L_{v2t}(t) y_{v2t}(t) = f_{v2t}((v(t), \vec{P}_{v2t}), G_V(t)) \quad G_{V2t}(t))$$

A series of equations is obtained,

$$(U_{1ix} \wedge U_{20}) \supseteq V_{20},$$
$$(S_{1ix}(t) \wedge S_{20}(t)) \supseteq S_{v20}(t),$$
$$(\vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{20}) \supseteq_{v20} \quad (x_1, x_2, \ldots, x_n) \quad ,$$
$$(T_{1ix}(t) \wedge T_{20}(t)) \supseteq T_{v20j}(t),$$
$$(L_{1ix}(t) \wedge L_{20}(t)) \supseteq L_{v20}(t),$$
$$(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) \wedge x_{20}(t)) \geq y_{v20}(t) = f_{v20}((v(t), \vec{P}_{v20}), G_{2jV}(t)),$$
$$(G_{U1ix}(t) \wedge G_{U20}(t) \supseteq G_{V20}(t);$$

...

$$(U_{1ix} \wedge U_{2j}) \supseteq V_{2j},$$
$$(S_{1ix}(t) \wedge S_{2j}(t)) \supseteq S_{v2j}(t),$$
$$(\vec{P}_{1ix} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2j}) \supseteq \vec{P}_{v2j} \quad (x_1, x_2, \ldots, x_n) \quad ,$$
$$(T_{1ix}(t) \wedge T_{2j}(t)) \supseteq T_{v2j}(t),$$
$$(L_{1ix}(t) \wedge L_{2j}(t)) \supseteq L_{v2j}(t),$$
$$(x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) \wedge x_{2i}(t)) \geq y_{v2j}(t) = f_{v2j}((v(t), \vec{P}_{v2j}), G_{2jV}(t)),$$
$$(G_{U1ix}(t) \wedge G_{U2j}(t) \supseteq G_{V2j}(t);$$

...

$U_{1tx} \wedge U_{2m1}) \supseteq V_{2t}$,
$(S_{1tx}(t) \wedge S_{2m1}(t)) \supseteq S_{v2t}(t)$,
$(\vec{P}_{1tx} \quad (x_1, x_2, \ldots, x_n) \quad \wedge \vec{P}_{2m1}) \supseteq \vec{P}_{v2t} \quad (x_1, x_2, \ldots, x_n) \quad ,$
$(T_{1tx}(t) \wedge T_{2m1}(t)) \supseteq T_{v2t}(t)$,
$(L_{1tx}(t) \wedge L_{2m1}(t)) \supseteq L_{v2t}(t)$,
$(x_{1tx}(t) = f_{1tx}((u(t), \vec{P}_{1tx}), G_{U1tx}(t)) \wedge x_{2m1}(t)) \geq y_{v2t}(t) = f_{v2t}((v(t), \vec{P}_{v2t}), G_V(t))$,
$G_{U1tx}(t) \wedge G_{U2m1}(t) \supseteq G_{V2t}(t))$

Theorem 2: The sufficient and necessary condition for the solvability of Fuzzy Error Matrix Equation $X_i A' \supseteq B_i$ is as follows:

$U_{20} \supseteq V_{20}$,
$S_{20}(t) \supseteq S_{v20}(t)$
$\vec{P}_{20} \supseteq \vec{P}_{v20} \quad (x_1, x_2, \ldots, x_n) \quad ,$
$T_{20}(t) \supseteq T_{v20j}(t)$,
$L_{20}(t) \geqslant L_{v20}(t)$,
$x_{20}(t) \geqslant y_{v20}(t) = f_{v20}((v(t), \vec{P}_{v20}), G_{2jV}(t))$,
$G_{U20}(t) \supseteq G_{V20}(t)$;

...

$U_{2j} \supseteq V_{2j}$,
$S_{2j}(t) \supseteq S_{v2j}(t)$,
$\vec{P}_{2j} \supseteq \vec{P}_{v2j} \quad (x_1, x_2, \ldots, x_n) \quad ,$
$T_{2j}(t) \supseteq T_{v2j}(t)$,
$L_{2j}(t) \geqslant L_{v2j}(t)$,
$x_{2i}(t) \geqslant y_{v2j}(t) = f_{v2j}((v(t), \vec{P}_{v2j}), G_{2jV}(t))$,
$G_{U2j}(t) \supseteq G_{V2j}(t)$;

...

$(U_{2m1}) \supseteq V_{2t}$
$S_{2m1}(t) \supseteq S_{v2t}(t)$,
$\vec{P}_{2ml} \supseteq \vec{P}_{v2t} \quad (x_1, x_2, \ldots, x_n) \quad ,$
$T_{2m1}(t) \supseteq T_{v2t}(t)$,
$L_{2m1}(t)) \geqslant L_{v2t}(t)$,
$x_{2m1}(t) \geqslant y_{v2t}(t) = f_{v2t}((v(t), \vec{P}_{v2t}), G_V(t))$,
$G_{U2m1}(t) \supseteq G_{V2t}(t))$

Firstly we discuss the necessity.

Proof: If one of the conditions above is not meet, without loss of generality, suppose that $S_{2j}(t) \supseteq S_{v2j}(t)$ is not meet, then in the $(S_{1ix}(t) \wedge S_{2j}(t)) = S_{v2j}(t)$, no matter what the value of $S_{1ix}(t)$ is, we can not get $(S_{1ix}(t) \wedge S_{2j}(t)) = S_{v2j}(t)$.

Proved.

Then discuss the sufficiency.

Because in the error matrix equation $X_iA' \supseteq B_i, A \supseteq B_i$, we should only take union operation between the corresponding element of A in $X_{i.}$, that is

$$U_{1ix} = U_{20} \cup U_{21} \cup, \ldots, \cup U_{2j}, \cup \ldots, \cup U_{2t}.$$
$$S_{1ix}(t) = S_{20}(t) \cup S_{21} \cup, \ldots, \cup S_{2j}, \cup \ldots, S_{2t}.$$
$$\vec{P}_{1ix\,(x_1,x_2,\ldots,x_n)} = \vec{P}_{20} \cup \vec{P}_{21} \cup, \ldots, \cup \vec{P}_{2j}, \cup \ldots, \cup \vec{P}_{2t}.$$
$$T_{1ix}(t) = T_{20}(t) \cup T_{21} \cup, \ldots, \cup T_{2j}, \cup \ldots, \cup T_{2t}.$$
$$L_{1ix}(t) = L_{20}(t)) \cup L_{21} \cup, \ldots, \cup L_{2j}, \cup \ldots, \cup L_{2t}.$$
$$x_{1ix}(t) = f_{1ix}((u(t), \vec{P}_{1ix}), G_{U1ix}(t)) = x_{20}(t) \cup x_{21} \cup, \ldots, \cup x_{2j}, \cup \ldots, \cup x_{2t}.$$
$$G_{U1ix}(t) = G_{U20}(t) \cup G_{U21}(t) \cup, \ldots, \cup G_{U2j}(t) \cup \ldots, \cup G_{U2t}(t).$$

Proved.

Then we discuss all the solution of $X_iA' \supseteq B_i$ and $XA' \supseteq B$:

After computing the solution $X\,(x_1, x_2, x_n)$, $X'(x'_1, x'_2, \ldots\ldots, x'_n) \in X'$ is obtained by the intersection operation between X and Kg, rw, xq.

3.3 The Example of Application of Error Matrix Equation

Let

$$A' = [a_1 a_2]$$
$$a_1 = (U_{201} S_{201}(t)\ \vec{P}_{201} \quad (x_1, x_2, \ldots, x_n) \quad T_{201}(t) L_2(t) \quad (u_1, 0, 6) G_{U201}(t))$$
$$a_2 = (U_{202} S_{202}(t)\ \vec{P}_{202} \quad (x_1, x_2, \ldots, x_n) \quad T_{202}(t) L_{202}(t)(u_2, 0, 8) G_{U202}(t))$$
$$U_{01} = \{u_1,\ u_2, u_3, u_4\},$$
$$S_{01}(t) = \{s_{011}, s_{012},\ s_{013}, s_{014}\},$$
$$\vec{P}_{201} \quad (x_1, x_2, \ldots, x_n) \quad = \{\vec{P}_{201}, \vec{P}_{202}, \ldots, \vec{P}_{20n}\}, n = 4$$
$$T_{01}(t) = \{t_{011},\ t_{012},\ t_{013},\ t_{014}\},$$
$$L_{01}(t) = \{\ l_{011},\ l_{012}, l_{013}, l_{014}\},$$
$$G_{U01}(t) = \{\ g_{011},\ g_{012}, g_{013},\ g_{014},\}.$$
$$U_{02} = \{\ u_1,\ u_2, u_3,\ u_4,\ u_{015}\},$$
$$S_{02}(t) = \{\ s_{011},\ s_{012},\ s_{013},\ s_{014},\ s_{015}\},$$
$$\vec{P}_{202} \quad (x_1, x_2, \ldots, x_n) \quad = \{\vec{P}_{201},\ \vec{P}_{202},\ \vec{P}_{20n} \ldots, \},\ n = 4$$
$$T_{02}(t) = \{\ t_{011},\ t_{012},\ t_{013},\ t_{014},\ t_{015}\},$$
$$L_{02}(t) = \{\ l_{011},\ l_{012}, l_{013}, l_{014}, l_{015}\},$$
$$G_{U02}(t) = \{\ g_{011},\ g_{012}, g_{013},\ g_{014}, g_{015}\}.$$

$$X = x$$

$$x = (U_{10x} S_{10x}(t) \vec{P}_{10x} \quad (x_1, x_2, \ldots, x_n) \quad T_{10x}(t) L_{10x}(t) x_{10x}(t) = f_{10x}((u(t), \vec{P}_{10x}), G_{U10x}(t)) \quad G_{U10x}(t))$$
$$B' = [b_{11}\ b_{12}]$$

$b_{11} = (V_{201}S_{v201}(t)\ \vec{P}_{v201}\quad (x_1, x_2, \ldots, x_n)\quad T_{v201}(t)L_{v201}(t)(v_1, 0, 5)G_{V101}(t))$
$b_{21} = (V_{202}S_{v201}(t)\ \vec{P}_{v201}\quad (x_1, x_2, \ldots, x_n)\quad T_{v202}(t)L_{202}(t)(v_2, 0, 6)G_{V211}(t))$
$V_{01} = \{\ u_1,\ u_2\}$
$S_{v201}(t) = \{\ s_{011}, s_{012},\ s_{013},\},$
$\vec{P}_{201}\quad (x_1, x_2, \ldots, x_n)\quad = \{\vec{P}_{201}, \vec{P}_{202}, \ldots, \vec{P}_{20n}\},\ k = 3$
$T_{v01}(t) = \{\ t_{011},\ t_{012},\ t_{013}\},$
$L_{v01}(t) = \{\ l_{011},\ l_{012},\},$
$G_{v01}(t) = \{\ g_{011},\ g_{012}\}.$
$V_{202} = \{\ u_1,\ u_2, u_3\},$
$S_{v202}(t) = \{\ s_{011}, s_{012},\ s_{013},\},$
$\vec{P}_{202}\quad (x_1, x_2, \ldots, x_n)\quad = \{\vec{P}_{201},\ \vec{P}_{202},\ \ldots,\ \vec{P}_{20k}\},\ k = 3$
$T_{v2\,02}(t) = \{\ t_{011},\ t_{012},\ t_{013}\},$
$L_{v202}(t) = \{\ l_{011},\ l_{012}, l_{013},\},$
$G_{v202}(t) = \{\ g_{011},\ g_{012}, g_{013}\},$

By the theorem 2 under the assumptions of the erasure planning, the solution of $XA' \supseteq B'$ is:

$U_{10x} = \{u_1, u_2, u_3, \ldots\},$
$S_{10x}(t) = \{s_{011}, s_{012},\ s_{013}, \ldots\},$
$\vec{P}_{10x}\quad (x_1, x_2, \ldots, x_n)\quad = \{\vec{P}_{201},\ \vec{P}_{201},\ \ldots, \vec{P}_{20n}\}, n = 3, 4, \ldots$
$T_{10x}(t) = \{t_{011},\ t_{012},\ t_{013}, \ldots\},$
$L_{10x}(t) = \{\ l_{011},\ l_{012}, l_{013}, \ldots\},$
$x_{10x}(t) = \{(u_1, 0, 6), (u_2, 0, 8), \ldots\}.$
$G_{10x}(t) = \{\ g_{011},\ g_{012}, g_{013}, \ldots\}.$

4 Conclusion

Therefore, in order to study the occurrence and transformation of fuzzy errors in systems science and system management, economy and management, such as T (u) = u1. Known T and u to seek u1; known T and u1 to seek u; known u and u1 to find T; Among them, transformation of T, u and u1, etc, it is necessary to study a mathematical tool to describe quantitatively these errors and its laws - fuzzy error sets, fuzzy error matrix equations, fuzzy error matrix set equations and its solvability theory and solving method.

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The Relation Between Mathematical Constant and Stock Market Crash

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Abstract. Using the method of quantitative analysis, we present that movements of the financial markets are related to mathematical constant e, with the Shanghai Composite Index and the Dow Jones Industrial Average Index as the evidences.

Keywords: Dow Jones Industrial Average Index · Mathematical constant · Fibonacci Sequence · Shanghai Composite Index · Stock market crash

1 Introduction

As it is known to all that the markets are emotional, human emotion is affected most by the movements and the relative position of the moon, and secondly by the macula, magnetic storm and the relative position of other planets in solar system, and so on. The trading market is a place that full of greed for money and naked competitions for survival. In such circumstances, many decisions of the traders are instinctive, subjective or emotional, and often in the control of emotion. With further research we found that movements of the financial markets are related to the transcendental numbers π and e [1].

Because 2π represents a cycle in the nature, and the movements of the financial markets are periodic motions, so movements of the financial markets are related to. The growth and recession models in the nature are related to mathematical constant e, and the movements in the financial markets are the growth and recession phenomenon in the nature, so movements of the financial markets are related to the mathematical constant e. For example, the product of $e/10$ and 6124.04 which is the 2007's peak of Shanghai Composite Index, is 1664 which is the 2008's low of Shanghai Composite Index. Another example is the product of $e/13$ and 1558.95 which is the 1993's peak of Shanghai Composite Index, is 325 which is the 1994's low of Shanghai Composite Index.

2 Main Conclusions

As we know, there are different π sequences for different markets that included varieties and stocks. Which π sequences should be used in different markets?

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For different varieties and stocks, the π sequences should be decided by the stock market fluctuation. If one sequence can explain one markets form of fluctuations effectively, so we should use this π sequence, and the success rate of the farther market forecast will be improved. For example, cyclotomic sequences π/n, exponential sequences $(\pi/4)^n$ and so on. For the mathematical constant e, if we have one e sequence can explain one markets form of fluctuations, so we should use this sequence, and the success rate of the farther market forecast will be improved. In general, natural constant sequence consist of e/n,$(e/4)^n$,$(e/3)^n$ and so on. Next, we introduce natural constant sequence e/n.

The well known Fibonacci Sequence

$$\{Fm = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \cdots, m = 1, 2, 3, \cdots.\}$$

And Lucas sequence:

$$\{Gm : m \in N^+\} = \{1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, \cdots\}$$

Suppose H is the high of one stage, for the bearish market, we define the prices included index price, stock price, futures price, conversion rate and so on, as follow

$$P_1(H, \beta) = H\beta \tag{1}$$

$$\beta = e/k \tag{2}$$

Formulas (1) and (2) are still applicable to stock market crash, only need to change the condition to when k is integer n, suppose $n = n_1 n_2 n_1, n_2 \in Fm$ or $n_1 \in Fm$,$n_2 \in Gm$, so the bottoms of some varieties can be approximated by

$$L_{predict} = H \times e/n \tag{3}$$

Similarly, suppose L is the bottom of one stage, for the bull market, we define the prices included index price, stock price, futures price, conversion rate and so on, as follow

$$P_2(L, \beta) = L \div \beta \tag{4}$$

$$\beta = e/k \tag{5}$$

When k is integer n, suppose $n = n_1 n_2 n_1, n_2 \in \{Fm\}$ or $n_1 \in \{Fm\}, n_2 \in \{Gm\}$, so some tops of some varieties approximated by inversion transform

$$H_{predict} = L \div (e/n) \tag{6}$$

As shown in Fig. 1, star from the high 1429.01 of Shanghai Composite Index in May 26, 1992, the difference between the $1429.01e/10 = 388.44$ and the bottom 386.85 in November 17, 1992 is small, the denominator n add 1, then from the point 386.85 we do the inversion transform, the difference between the $386.85 \div (e/11) = 1565.45$ and the high 1558.95 in February 16, 1993 is small, the denominator n add 1 again, because $12 = 3 \times 4, 3 \in \{Fm\}$, but $4 \notin \{Fm\}$, so the denominator n continue to add 1 again, now $3 \in \{Fm\}$, the difference between $1558.95e/13 = 325.97$ and the bottom 325.89 points in July 29,1994

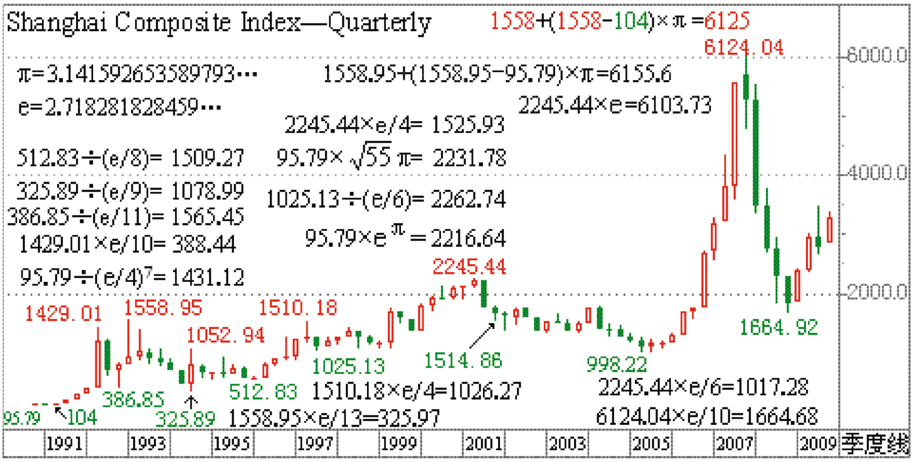

Fig. 1. Shanghai Composite Index from 1991 to 2009.

is just only 0.08 points. Star from $n = 10$, the denominator n minus 1, from the point 325.89 we do the inversion transform $325.89 \div (e/9) = 1078.99$ then we can estimate the high Shanghai Composite Index rebound to in 1994, the denominator n continue to minus 1, from the point 512.53 we do the inversion transform $512.83 \div (e/8) = 1509.27$, the difference between the 1509.27 and the high 1510.18 in May 12, 1997 is just only 0.91 points, and the difference between the $1510.18 \times e/4 = 1026.27$ and the bottom 1025.13 in September 23, 1997 is 1.14 points. From the point 1025.13 we do the inversion transform $1025.13 \div (e/6) = 2262.74$, and $95.79 \times e^{\pi} = 2216.64$, the mean value of 2262.74 and 2216.64 is 2239.69, this point is very close to the high 2254.44 in June 14, 2001. Then we use $2245.44 \times e/6 = 1017.28$ to estimate the ballpark drop target from 2245.44 points, and $2245.44 \times e = 6103.73$ is very close to the high 6124.04 in October 16, 2007.

Around the first callback target $1510.18 \times e/4 = 1026.27$, the second callback target $2245.44 \times e/6 = 1017.28$, from the additive property of the denominator, we can know the third callback target is around $6124.04 \times e/(4 + 6) = 1664.68$, and the difference between this point and the bottom 1664.92 in October 28, 2008 is just only 0.24 points.

3 Application

Figures 2, 3 and 4 show the relation between Dow Jones Industrial Average Index and the mathematical constant $e[= 2.718281828459 \cdots]$.

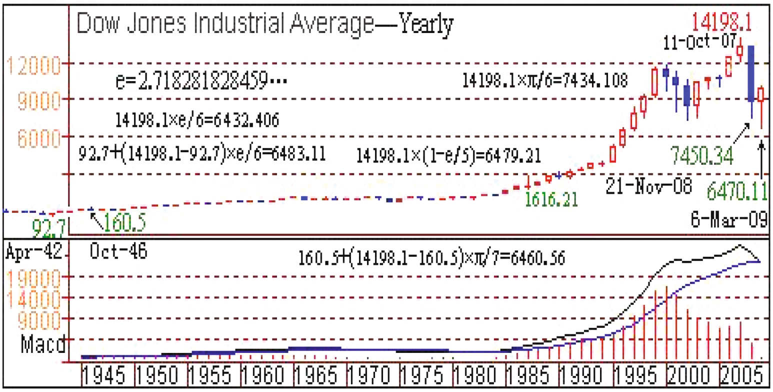

Fig. 2. Dow Jones industrial average yearly.

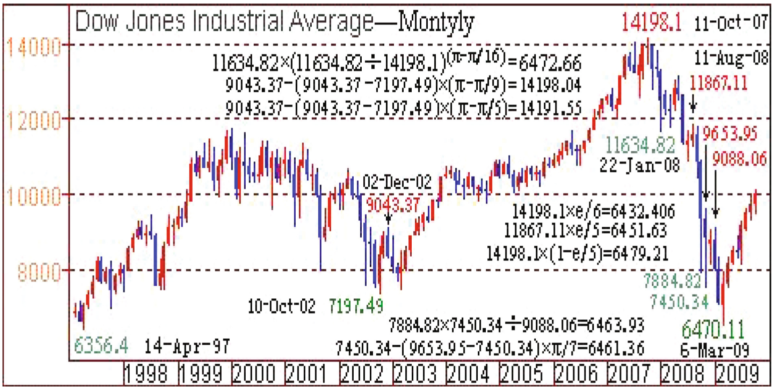

Fig. 3. Dow Jones industrial average monthly.

Fig. 4. Dow Jones industrial average weekly.

4 The Relation Between Stock Market Crash and the Mathematical Constant *e*

Figure 5 shows that the relation between the mathematical constant e and the Hang Seng Index which from 1774.96 points in March 9, 1973 fell to 150.11 points in December 10, 1974, is $1774.96 \times e/32 = 150.77$. And Fig. 6 show that the relation between the mathematical constant e and the Dow Jones Industrial Average Index which from 386.1 points in September 3, 1929 fell to 40.56 points in July 8, 1932, the stock market crash happened in New York October 1929, is $386.1 \times e/26 = 40.36$.

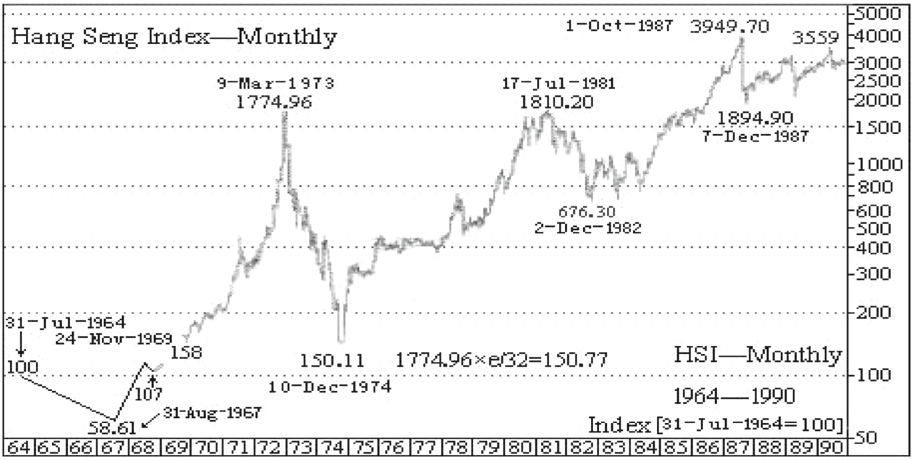

Fig. 5. Hang Seng Index monthly.

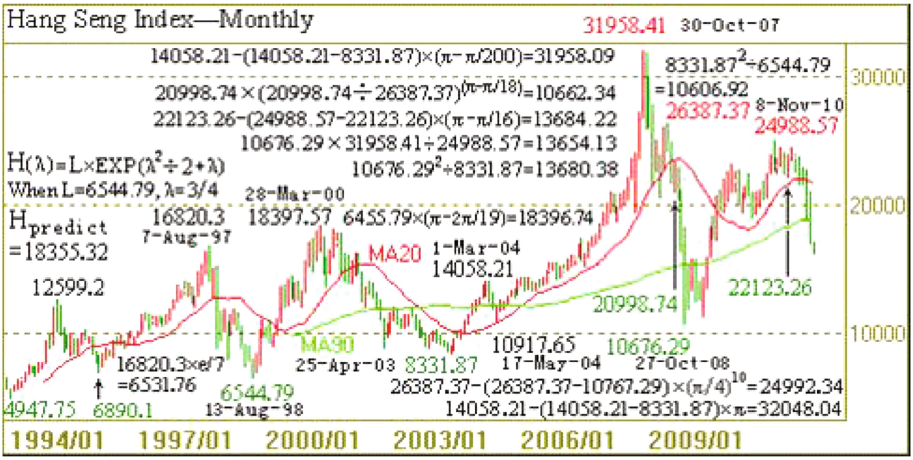

Fig. 6. Hang Seng Index monthly.

Figure 6 show that the relation between the mathematical constant e and the Hang Seng Index which from 16820.3 points in August 7, 1997 fell to 6544.79 points in August 13, 1998, is $16820.3 \times e/7 = 6531.76$ in the Asian Financial Crisis.

Figure 7 show the Dow Jones Industrial Average Index fell to 6470.11 points in 6-Mar-2009 from 14198.1 points in 11-Oct -2007 in the 2008's Subprime Lending Crisis, while $14198.1 \times e/6 = 6432.4$ and $14198.1 \times (1 - e/5) = 6479.21$.

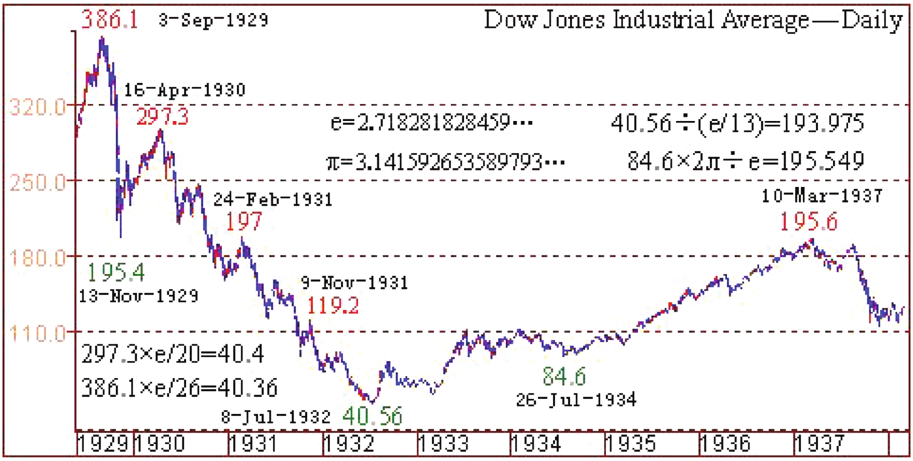

Fig. 7. Dow Jones Industrial Average Index daily.

5 The π-type Stock Market Crash

For the π type stock market crash, when n is integer, suppose $n = n_1 n_2, n_1, n_2 \in \{Fm\}$ or $n_1 \in \{Fm\}, n_2 \in \{Gm\}$, so some tops of some varieties approximated by inversion transform

$$L_{predict} = H \times \pi/n \tag{7}$$

Figure 8 show the Taiwan Weighted Stock Index fell to 2485.25 points in 12-Oct-1990 from 12682.41 points in 12-Feb-1990, while $14198.1 \times /16 = 2490.185$. This is the Taiwan area's stock market crash at the beginning of last 90's.

Figure 9 show the Nikkei Stock Average fell to 7603.76 points in 28-Apr-2003 from 38957.44 points in 29-Feb-1989, while $38957.44 \times /16 = 7649.27$ other $14556.11 \times /6 = 7621.56$. In 1989, the Japanese bubble economy burst.

The Vietnam stock market index fell to 367.46 points in June, 2008 from 1170.67 points in March, 2007, while $1770.67 \times /10 = 367.77$ in 2008's Vietnam stock market crash.

Besides, The Stock Market Crash of 1929 in USA [5], the price of United States Steel Corporation fell to 21 from 262, and we have $\frac{262}{4\pi} = 21$. And the price of General Motors Corporation felled to 7 from 92, and $\frac{92}{4\pi} = 7$.

Fig. 8. Taiwan Weighted Stock Index quarterly.

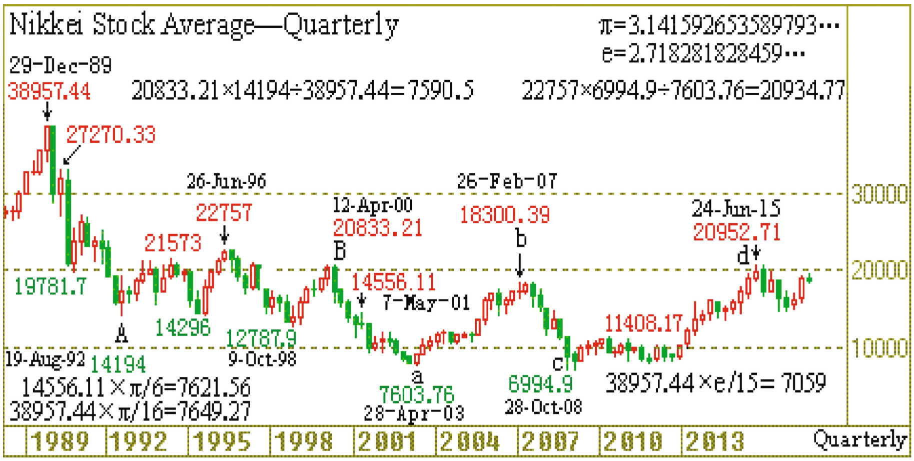

Fig. 9. Nikkei Stock Average quarterly.

6 Conclusion

The stock market crash analysis methods based on qualitative measures have some disadvantage. We found a method to do quantitative analysis of stock market crash. Just the same as the hydrogen energy level ($E = E_1/n^2, r_n = r_1n^2, E_1 = -13.6eV$) of Niels Henrik David Bohr, the ground level of stock market crash is discontinuous. A higher value of n indicates a bigger drops of the price($Lpredict = eH/n$).

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The Impact of Online Information to the Internet Reservations of Hotels in Guangzhou

With an Example of the eLong Network

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Abstract. The popularity of online hotel review websites has changed the way of customers' booking hotel rooms. Based on the review data of 1537 hotels in Guangzhou listed on the eLong network, we first adopt the stepwise linear regression, develop 4 categories of linear models for various classes of hotels, and analyze the fluctuately fuzzy information impacts of 4 sectors, including hotel features, customer reviews, platform booking recommendations and hotel services Index, to the online-booking effects of these hotels. Additionally, we have showed that the platform booking recommendations and customers' reviews impact highly on the online-booking of these hotels. Furthermore, hotel service information influences hotels' reservations except for four-star/upscale hotels, so do hotel features except for five-star/luxury ones. The study may provide online hotel booking platforms with guidance to providing better services for their customers.

Keywords: Online booking · Fluctuate data · Variables · Linear regression model

1 Introduction

The rapid development of information technology has changed travel industry, the Internet has become one of the most important ways for tourism enterprises to provide communication platform and services for visitors [1]. With the development of hotel online reservation, rather than relying on advantages of geography, hotels now rely on Internet platforms to attract guests. Meanwhile, more and more consumers release fluctuate evaluation on the Internet, the number of reviews is fast-growing on independent hotel review sites, such as ctrip, qnar, elong. Potential consumers not only search for product information such as price, type, brand, also concern about other consumers' post-purchase reviews [2]. After integrating information provided by third-party hotel review sites, potential consumers could reduce the information asymmetry, to lower the risk of purchase. Therefore, online information have a significant impact on consumers' purchase decisions, arising a wide attention from scholars now.

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2 Literature Review

As a typical experiential product, hotel industry is one of the earliest and most successful online channels for consumer to obtain information [3]. By investigating 1480 visitors, Gretzel and Yoo found that online reviews from other tourists affect significantly the made-up of decisions [4]. On the study of the influence factors of online hotel reservation, Dickinger and Mazanec also represented that online reviews can significantly influence consumers' purchase decision [5]. Vermeulen and Seegers found online reviews can improve consumer perception of the hotel, while the positive reviews can significantly improve the consumers' purchases willingness [6]. On the hotel industry study, Sparks and Browning (2011) developed that when the hotel has its own number of positive reviews, consumers also put marked high trust on it [7]. Through empirical research, Ye demonstrated that tourists comments has an important impact on hotel online sales, when increase on travel review reaches by 10%, increase on hotel online sales reaches by 5% [8]. By studying Paris's largest hotel reservation site data, Öğü t and Cezar (2012) found that higher ratings and lower prices will increase the number of comments, but extreme scores and star of hotels have no impact on comments' publish, in addition, room size may has negative effects on evaluation releasing [9]. Based on ctrip's data, Zhang Meng etc (2011) studied the effect of online information on online booking of 4 different ratings of hotels [10]. Xu Feng (2013) studied the impact of the different cities and different types on hotels online booking [11].

To sum up, the current research about online reviews of hotels focuses on impact of online reservation numbers, information of sentiment (positive and negative), hotel feature, and booking platform on hotels' online hotel bookings from large and medium cities. Given different features of each city and different potential consumer groups, factors that influence consumer hotels online booking also vary differently. Taking Guangzhou for example, this article attempts to take an in-depth study on influence of online information of elong on Guangzhou hotel booking online, and a comparative analysis of the difference between different stars of Guangzhou hotels'.

3 Research Method

3.1 Data Collection and Processing

First, through browsing ctrip (http://www.ctrip.com/), elong (http://www.elong.com/) and qunar (http://www.qunar.com/) the three most large and authoritative sites of customer reviews, we compare hotels reservation pages, comments and other information, ultimately we select elong as a sample for the study for the following three reasons as follows First, elong has a unique data-services index which can provide more detailed information for this study the data including promptly confirmation rate after booking and reservation success rate and user complaints rate, and the ranking of the hotel in the same city, and the data for a user who wants to reserve a hotel worth considering and should be paid attention by the hotel. Secondly, elong divides hotels into five-star hotels/Luxury, four star/upscale, three star/comfort, economic/Inn, having the same hotel classification with our study, which can be sampled directly. And thirdly, acquired by ctrip, elong can be represented to some extent ctrip. Therefore, we choose elong as the sample site.

Additionally as online information is dynamic, up-to-date, a time frame is needed. 2014 and 2015 newly-opened hotels are not included in the statistics, because of less number of reviews generally on new hotels. With the lack of time, the comparability between hotels of the same type is affected. Meanwhile, all included users' comments come from the date before August 30, 2015.

Finally, room numbers provided by different size hotels have an impact on the number of its customers' reviews, affecting the comparability between hotels of the same type. According to the data provided by the China National Tourism Administration, in 2014 the average occupancy rate for hotels is 54%. In principle, we select the number of online comments is greater than the total number of 54% room numbers.

After data collection and collation, we chose 1537 hotels as a sample of which 108 five-star/luxury hotels, 268 four star/upscale hotels, 510 three star/comfortable hotels, and 651 economic hotels.

3.2 Establishment of Indicator System

Variable selection in this study is based on online information provided by elong. Hotel online reservation volume and information of hotel features, hotel service, and recommendation of reservation platform can be obtained directly on the website, but recommended rate of customer comments instead of recommended rate need an calculation before obtaining. Recommended rate means recommended number dividing total review numbers, but non-recommended rate is not recommended number to divide total review numbers, and non- recommended number is made up of all reviews number minusing the recommended number and minusing the number to be improved.

Because ctrip online hotel's actual reservation data is not open to the public, we cannot obtain directly each hotel online reservation number on the site, thus this study adopts the alternative practice of online comments similar to online hotel bookings in the same period, proving that there exists a linear relationship between the two [12] (Table 1).

Table 1. eLong customer evaluation index system of Guangzhou Hotels

First class indicator	Second class indicator	Description
Num of hotel online booking	Y Num Reviews	All customer comment number of i Hotel before Oct, 2015
Hotel feature information	X_1 Room Types	Available room type showed on i Hotel website
	X_2 Lowest Price	The lowest price provided on i Hotel website
	X_3 Highest Price	The highest price provided on i Hotel website
	X_4 Room Pictures	Related hotel photos showed on i Hotel website

(*continued*)

Table 1. (*continued*)

First class indicator	Second class indicator	Description
Customer comment information	X_5 Applause Rate	Overall comment on i Hotel based on living experience
	X_6 Recommendation Rate	Recommendation of i Hotel based on living experience
	X_7 Not recommended Rates	No recommendation of i Hotel based on living experience
	X_8 Users upload Photos	Hotel photos by customers showed on i Hotel website
Hotel service information	X_9 Service Index	Promptly confirmation rate after booking and reservation success rate, user complaints rate, and the ranking of the hotel in the same city
	X_{10} Determine Rates	Promptly confirmation rate after booking
	X_{11} Success rate of booking	Success consumption after reservation rate
	X_{12} Customer complaint Rate	Complaint after consumption rate
Booking platform recommendation information	X_{13} Travelers' Rating	According to the evaluation of the hotel I, considering the various factors, booking platform rates for cooperated hotels
	X_{14} Browsing Index	According to the browse number of i Hotel, booking platform assign for the variables

(These factors include the assessed star of NTA, the hotel's facilities and equipment, supporting, service level, social reputation, brand visibility and integrity to the customer, and so on)

3.3 Research Process and Methods

In order to distinguish effects of different online information on different stars of hotels' online booking, the study takes the actual data from elong, using a linear regression model and 4 models to study the effect of online information on the five star/luxury, four star/luxury, three star/comfortable, economic hotels.

The model can be expressed as:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6 + \beta_7 X_7 + \beta_8 X_8 + \beta_9 X_9 + \beta_{10} X_{10} + \beta_{11} X_{11} + \beta_{12} X_{12} + \beta_{13} X_{13} + \beta_{14} X_{14}.$$

First of all, because of large number of independent variables, we study the bivariate correlation between the independent variables and the dependent ones, rejecting dependent variable irrelevant of independent one. Then, we use the linear regression to analyze dependent variables related to independent ones. And at last, as the model shows us, histogram does not assume a normal distribution of residuals, residuals scatter distribution regularly, indicating the existing of heteroscedasticity of the model. In order to solve this problem, in this study we use the log-linear regression model to analyze influence of online information on hotels online reservations.

$$\ln Y = \beta_0 + \beta_1 \ln X_1 + \beta_2 \ln X_2 + \beta_3 \ln X_3 + \beta_4 \ln X_4 + \beta_5 \ln X_5 + \beta_6 \ln X_6 + \beta_7 \ln X_7 + \beta_8 \ln X_8 + \beta_9 \ln X_9 + \beta_{10} \ln X_{10} + \beta_{11} \ln X_{11} + \beta_{12} \ln X_{12} + \beta_{13} \ln X_{13} + \beta_{14} \ln X_{14}.$$

4 Empirical Analysis

4.1 Related Analysis

Take the effect of online information on five-star/luxury hotels' online booking for example, we first take logarithm of all variables, using bivariate correlation analysis of IBM SPSS Statistics 20 to exclude independent variables irrelevant to dependent ones and keep those independent variable relevant to dependent variables. Significantly related independent variables are represented as follows (Table 2):

Table 2. Linear correlation of independent variable and dependent variables

	$\ln X_1$	$\ln X_4$	$\ln X_5$	$\ln X_6$	$\ln X_7$	$\ln X_8$	$\ln X_9$	$\ln X_{11}$	$\ln X_{13}$	$\ln X_{14}$
lnY	0.257**	0.300**	0.283**	0.192*	-0.181	0.553**	0.190*	0.235*	-0.227*	0.722**

**. significantly related to 0.1 (bilaterally) *. significantly related to 0.05 (bilaterally)

4.2 Linear Regression Analysis

Take the effect of online information from five-star/luxury hotels' online booking for example, we use linear regression analysis of IBM SPSS Statistics 20. Specifically, we use stepwise regression method to multicollinearity problems, DW testing way to test whether the model exists autocorrelationally. Thereafter, Its final result estimatation is shown in Table 3.

According to the analysis of model 1, log-linear model of five-star luxury hotel lnY = −15.192 + 0.112lnX8 + 3.851lnX11 − 0.401 lnX13 + 0.639 / lnX14. Among customer reviews, users upload photos X8 (β – 8 = 0.112, t = 2.563), hotel services reservation success rate X11 (β – 11 = 3.851, t = 4.868), users' rating of recommended information by reservation platform X13 (β – 13 = −0.401, t = −2.578) and browse and collection index X14 (β – 14 = 0.639, t = 8.170) significantly affected hotel online reservation. Since other variables did not pass the test, the hotel reservation service information was influenced significantly. Among reservation of five- star/luxury hotels, the impact of success rate of booking of hotel service information surpass reservation platform information and customer reviews as well. This shows that no matter what kind of reason leads to unsuccessful reservation would directly impact

Table 3. Summary of four models estimation results

Model	Five-star/luxury model 1	Four star/upscale model 2	Three star/comfortable model 3	Economic model 4
β0	−15.192 (−4.161)	1.102 (2.589)	−3.133 (−4.998)	−3.088 (−3.471)
$\ln X_1$	0.048 (0.430)	0.633 (7.862)	0.155 (1.966)	0.416 (6.153)
$\ln X_2$	−0.013 (−0.087)	0.198 (3.357)	0.112 (1.422)	0.353 (5.556)
$\ln X_3$	−0.002 (−0.017)	0.070 (1.074)	−0.007 (−0.093)	−0.243 (−3.624)
$\ln X_4$	−0.031 (−0.300)	0.047 (0.778)	−0.187 (−3.100)	−0.166 (−3.514)
$\ln X_5$	0.610 (0.237)	0.700 (0.682)	0.304 (4.136)	0.438 (2.366)
$\ln X_7$	0.098 (0.538)	−0.266 (−4.909)	−0.063 (−1.554)	0.038 (0.901)
$\ln X_8$	0.112 (2.563)	0.084 (2.317)	0.128 (2.980)	0.168 (5.034)
$\ln X_9$	0.281 (0.290)	−0.674 (−1.343)	0.088 (0.635)	0.830 (3.142)
$\ln X_{11}$	3.851 (4.868)	−0.180 (−1.678)	0.873 (6.735)	0.140 (1.591)
$\ln X_{13}$	−0.401 (−2.578)	0.798 (4.957)	−0.875 (−6.691)	0 (0)
$\ln X_{14}$	0.639 (8.170)	0.569 (14.904)	0.791 (27.342)	0.797 (29.608)
R-squared	0.645	0.651	0.761	0.723
Adjusted R-squared	0.632	0.646	0.758	0.720
F-statistic	46.852	115.564	228.619	238.846
Sum Squared Residual	23.083	137.623	230.767	266.335

Note: the numerical values in the brackets are t-statistics of parameter.

customer reservations, thus customers worry about their failing reservation. Booking platform information has greater impact on hotels online booking, customers rating demonstrates that customer think highly of the rating of a hotel whether it is five-star or luxury. Browse and collection index shows that customers tend to put into Favorites when they see hotels are interested for their quick access to the hotel, which also reflects the hotel's popularity and arise another customers' interest in their option of this hotel. Users upload photos in guests' comments also has certain impact on hotel online reservation, probably because customers believe that compared with hotel photos, users upload photos does not have a strong propaganda purposes, the authenticity and reliability of the photos is strong. Hotel feature information does not affect the five star/luxury hotel online reservation, possibly because the five star/luxury hotels are in higher grade, hardware and software services can reach a higher level, room types and photos in the customers mind have little difference. Customers of five star/luxury hotel have strong spending power, or charge on administrative unit or company, and also they are not very sensitive to the price.

According to the analysis of model 2, the log-linear model of four star/upscale hotel is $\ln Y = 1.102 + 0.633 \ln X1 + 0.198 \ln X2 - 0.266 \ln X7 + 0.084 \ln X8 + 0.798 \ln X13 + 0.569 \ln X14$. Room types in hotel feature information X1 ($\beta - 1 = 0.633$, $t = 7.862$), the lowest room rate X2 ($\beta - 2 = 0.198$, $t = 3.357$), not recommended rate in customer reviews information X7 ($\beta - 7 = -0.266$, $t = -4.909$), users uploading photos X8 ($\beta - 8 = 0.084$, $t = 2.317$), users rating in recommended information of booking platform X13 ($\beta - 13 = 0.798$, $t = 4.957$) and browse and collection index X14 ($\beta - 14 = 0.569$, $t = 14.904$) significantly affect hotel online reservation, other variables did not pass the test, of which recommended information on online booking platforms having a significant impact on online booking. This means customers of four star/upscale hotel are most concerned about the recommendations of reservation platform, and whether the hotel is four star or upscale. The same with customers of five-star/luxury hotels, customers of four star/upscale hotel, Hotel tend to put into Favorites when they seeknow that hotels are interested for their quick access to the hotel, which also reflects the hotel's popularity and arise another customers' interest in this hotel. On browsing other customer comments, not recommended rates reflects that customers are more concerned about negative feedback, thinking that users uploaded photos are more authentic. Last, as room types and the lowest rate of hotel feature information are concerned, we know the fact showing that different room types and room price have different impact on customers. Hotel service information did not pass the significance test, reflecting that hotel service information has no great impact at all.

According to the analysis of model 3, the log-linear model of three star/comfort hotel is $\ln Y = -3.133 + 0.155 \ln X1 - 0.187 \ln X4 + 0.304 \ln X5 + 0.128 \ln X8 + 0.873 \ln X11 - 0.875 \ln X13 + 0.791 \ln X14$. Room types in the hotel feature informationX1 ($\beta - 1 = 0.155$, $t = 1.966$), hotel photos X4 ($\beta - 4 = -0.187$, $t = -3.100$), praise rate of customer reviews X5 ($\beta - 5 = 0.304$, $t = 4.136$), users uploaded photos X8 ($\beta - 8 = 0.128$, $t = 2.980$), reservation success rate of hotel service information X11 ($\beta - 11 = 0.873$, $t = 6.735$), user rating in the recommendations of reservation platform X13 ($\beta - 13 = -0.875$, $t = -6.691$) and browse and collection index X14 ($\beta - 14 = 0.791$, $t = 27.342$) significantly affected hotel online reservation, other variables did not pass the test, of which reservation platform information and hotel reservation service information on the Internet have significant impact. This means customers are most concerned about the recommendations of reservation platform, and whether the hotel is three star or more comfortable. Customers tend to put into Favorites when they see hotels are interested. Secondly, the reservation success rate of hotel service information affects customers booking, customers worry about a failure reservation. The favorable rate of customer reviews indicates customer pays close attention to positive reviews. Finally room types in hotel feature information, hotel photos are given due consideration, but prices not significantly affect bookings.

According to the analysis of model 4, the log-linear models for economic hotels is lnY = −3.088 + 0.416lnX1 + 0.353lnX2 − 0.243lnX3 + 0.438lnX5 + 0.168lnX8 + 0.830lnX9 + 0.797lnX14. Room types in hotel feature information X1 (β − 1 = 0.416, t = 6.153), the lowest rate X2 (β − 2 = 0.353, t = 5.556), the highest rate X3 (β − 3 = −0.243, t = −3.624), favorable rate of customer reviews X5 (β − 5 = 0.438, t = 2.366), users upload photos X8 (β − 8 = 0.168, t = 5.034), hotel service information index X9 (beta 9 = 0.830, t = 3.142), browse and collection index of recommendations of booking platform X14 (β − 14 = 0.797, t = 29.608) significantly affect hotel online reservation, other variables did not pass the test. The impact of hotel service information, recommendations of reservation platform on online bookings is notable. Customers for economic hotels are most concerned about hotel service index, caring abour the comprehensive efficiency of interactive services provided by hotels. Second browse and collection in the booking platform recommended index, shows customers tend to put into Favorites when they see hotels are nterested in their access to it. And room type in hotel feature information shows the difference of different type of room, hotel reservation is positively correlated to the lowest rate and negatively correlated to the highest rate showing customers' sensitivity to room rate, prices significantly affect consumer choice. Finally favorable rate of guests' comments and users uploaded photos indicates that customer pays close attention to positive reviews. Economic hotels do not have users' rating information, the data thus is absent.

5 Research Conclusions

5.1 Conclusions

Comparing the analysis of four model parameter analysis, we come into a conclusion below:

(1) Booking platform recommendations have a significant effect on all hotel booking online. Except model 4 economic hotels having no users' rating information, its data being absent.
(2) Hotel service information significantly effects online reservation of five star/luxury, three star/comfort and economic hotels. Users' complaints data is zero, so there is no impact.
(3) The impact of customer reviews on all hotel booking online is notable. Significant effect of users' uploaded photos has shown on all hotel online booking.
(4) Hotel features information owns a significant effect on online reservation of four star/upscale, three star/comfort and economic hotels, and has no effect on five-star/luxury hotels.

5.2 Research Proposals

The conclusion above has certain implications for hotel to develop its online booking.

First of all, consumers focus on different information. Consumers of five-star/luxury hotels concern services index, namely, efficiency of the hotel itself offers online booking services. Consumers of four star/upscale and three star hotels concern about recommendation information of reservation platform, which is the ratings of cooperated hotels after several factors are considered.

Secondly, there are certain commonalities of management method to online booking platform to increase online reservation. Studies have shown that browse and collection index and customer reviews influence greatly all hotels in four models. Therefore, Internet booking platform should encourage all customers to browse and collect hotels, trying to make customers put cooperated hotels into their Favorites, for their first consideration of those hotels. Online booking platform should guide customers to upload photos. Because customer's photos are real enough with high reliability, with great influence on other potential customers. Online booking platform should also motivate customers to write down objective and real reviews. Customers of four star/upscale hotels concern more about negative feedback. Maybe they are least satisfied with hotel living experience, holding a pickier attitude to browse comments. Customers of three star/comfortable and economic hotel care more about positive feedback. Maybe they are satisfied with hotel living experience, holding an approval attitude to browse comments.

Finally, different types of hotels put different emphasis on online booking management. Hotel's feature information lies influence on online reservation of four star/luxury, three star/comfortable and economic hotels significantly. There is no effect on online reservation of five-star/luxury hotels. Four star/upscale, Three star/comfortable, economic hotels need to diversify products, and put forward a wider variety of room types for customer to choose from. Customers of four star/upscale hotels are sensitive to the lowest prices, they need to lower room rates or take some special rooms to attract customers. The room price of three star/comfortable hotels is moderate, thus no adjustment is needed, but more photos uploaded by customers for reference is in much need. Economic hotel reservation is positively correlated to the lowest rate and negatively correlated to the highest rate which shows customers' sensitivity to room rate, prices significantly affect consumer choice. Hotel service information has great effect on online reservation of five-star luxury, three star/comfortable and economic hotels. Among them, the reservation success rate has great influence on five-star/luxury and three star/comfortable hotel. The impact of reservation rate on five star/luxury hotels surpass other factors, and its impact on three star/comfortable hotels just inferior to impact of platform information. This shows that if the booking was not successful, it would affect customers' reservation psychology. Customers worry about a case that like other consumers, they suffer from failure reservations. Hotel reservations need to enhance the management of reservation at a successfulrate, and improve the consumption rate of customers' reservation. Economic hotels need to strengthen the management of service index, improve confirm rates, raise a reservation success rate and reduce the rate of customer complaints.

6 Limitations

Though the study rigorously builds model, standardizes data collection and control model evaluation, the study still has some limitations. When we select explanatory variables, we only consider the number index of online information, without considering the effect of comment texts, and the quality of hotel photos and users' uploaded photos. Therefore, data mining and text mining methods can be used in the future to research online information, the difference between hotel photos and users' uploaded photos is also worth studying.

7 Conclusion

Based on the review data of hotels in Guangzhou listed on the eLong network, we first adopt linear regression, develop 4 categories of linear models for various classes of hotels, and analyze the fluctuately fuzzy information impacts of 4 sectors. After we represented a high impact of platform booking recommendations and customers' reviews on the online-booking, we see clearly the important influences bwtween hotel service information and their reservations. And we also provide online hotel booking platforms with guidance to providing better services for their customers.

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Optimization of the Modified T Vacation Policy for a Discrete-Time $\mathrm{Geom}^{[X]}/\mathrm{G}/1$ Queueing System with Startup

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Abstract. In this paper, we discuss a discrete-time $\mathrm{Geom}^{[X]}/\mathrm{G}/1$ queueing system with modified T vacation policy and startup time. We derive the generating functions and the mean values for the steady state system size and the waiting time, and also get those of the busy period, the vacation period and the vacation cycle by using embedded Markov chain. Finally, we determine the optimal (T^*, J^*) to minimize the cost function with fixed cost elements by constructing a cost function.

Keywords: Queueing system model · Startup time · Stochastic decomposition · Modified T vacations policy · Embedded markov chain method

1 Introduction

The server leaves for a vacation with fixed length T slots when the system is empty. After a vacation, the server returns to the system. The server immediately begins to serve if there is at least one customer waiting for service in the system; otherwise, the server takes another vacation and so on until at least one customer waits for service. This vacation policy is called T vacation policy and was firstly studied by Levy and Yechiali [1] and Heyman [2]. Sen and Gupta [3] analyzed a time dependent M/M/1 queueing with T policy via a lattice path combinatoric technique. In recent years, some authors began to study the modified T policy queueing systems. Ke [4] considered modified T vacation policy M/G/1 with an unreliable server and startup, and obtained the expected number of customers, the expected waiting time and other performances. It followed that Ke [5] studied a batch arrival queueing system under modified T vacation

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policy with startup and closedown, and determined the optimal (T^*, J^*) by constructing a cost function. In addition, there are many other queueing models concerned T policy which have been studied in recent years, details of which may be seen [6–15].

In this paper, we consider a discrete time batch arrival queueing with modified T policy and startup, and derive the generating functions and the mean values for the steady state system size and the waiting time, and also get the generating functions and the expected values of the busy period, the vacation period and the vacation cycle. In addition, by constructing an cost function, we determine the optimal (T^*, J^*) to minimize the cost function. In fact, the modified T vacation policy is applied to many fields now. Take manufacturing systems for example, a machine will process a subproduct with fixed T slots after all ordinary products have been processed. And after finishing a processed subproduct while no ordinary products wait in queue, the machine continues to process another subproduct. This pattern continues cycle until at least one new ordinary product waits in the queue, otherwise it the server has already processed J subproducts. After that the machine stops to wait for arrival of the new ordinary products.

The remainder of this paper is organized as follows. A full description of the model and an embedded Markov chain are given in the Sect. 2. In Sect. 3, we obtain stochastic decomposition of the queue size and the expected values of waiting time. In Sect. 4, the expected values length of the vacation cycle, the vacation period and the busy period are obtained. We construct a cost function to introduce the optimal policy in Sect. 5. Finally in Sect. 6, we present some numerical results to illustrate the effect of λ on the expected queue size and the waiting time in the system, and obtain the optimal (T^*, J^*) with fixed cost elements.

2 Describing Model and Embedded Markov Chain

In the classical Geom$^{[X]}$/G/1 queueing system, we introduce the following vacation strategy: as soon as the system is empty, the server deactivates to take a vacation with fixed length of T. If no customers are found in the system when a vacation is finished, while the server takes another vacation with the same length T. This pattern continues cycle until a vacation is finished, the server finds at least one customer waiting in the queue or he will be already taken J vacations. If no customers are found at the end of the J-th vacation, the server stops in the system to wait for the arrival of one customer. If there is at least one customer waiting for service in the system when a vacation is finished or the server is idle in the system, he is immediately reactivated. But, the server will be need a startup time before supplying service for the waiting customers. As soon as the startup is finished, the server starts supplying service for the waiting customers until the system becomes empty again.

In the Geom$^{[X]}$/G/1 queueing model with T policy and startup time, we denote by Λ the number of customers who arrive in a single slot. The Λ is assumed to be an integral multiple, and its probability distribution and probability generating function of Λ are given by, respectively, $\lambda(k) = p(\Lambda = k), k = 0, 1, 2, \cdots; \Lambda(z) = \sum_{k=0}^{\infty} \lambda(k)z^k, |z| \leq 1$.

In addition, we denote by λ and $\lambda^{(i)}$ the mean and the i-th factorial moment of Λ, respectively, $\lambda = E[\Lambda], \lambda^{(i)} = E[\Lambda(\Lambda-1)\cdots(\Lambda-i+1)], i = 2, 3, \cdots$.

Let X be the service time of one customer and the length of the service time be an integral multiple of a slot duration, then its probability distribution and probability generating function are given by, respectively, $b(l) = p(X = l),\ l = 1, 2, \cdots; B(z) = \sum_{l=1}^{\infty} b(l)z^l, |z| \leq 1$.

Let b and $b^{(i)}$ be the mean and the i-th moment of the service time distribution, respectively, $b = E[X]; b^{(l)} = E[X^l],\ i = 2, 3, \cdots$.

Let S be the startup time and the length of the startup time be an integral multiple of a slot duration, then its probability distribution and probability generating are given by, respectively, $s(l) = p(S = l), l = 1, 2, \cdots; S(z) = \sum_{l=1}^{\infty} s(l)z^l, |z| \leq 1$.

Let s and $s^{(i)}$ be the mean and the ith factorial moment of the startup time distribution, respectively: $s = E[S]; s^{(i)} = E[S(S-1)\cdots(S-i+1)], i = 2, 3, \cdots$.

Now we consider a Markov chain $\{L_n; n = 1, 2, \cdots\}$, where L_n denotes the number of customers present in the system after the server has completed service for the n-th customer. And suppose that A_n is the number of arriving customers during the n-th customer's service and α is that of present customers in the system at the end of the startup time, thus we have

$$L_{n+1} = \begin{cases} L_n + A_{n+1} - 1, \ L_n \geq 1, \\ \alpha + A_{n+1} - 1, \quad L_n = 0 \end{cases}$$

Let $A(z)$ be the PGF for A_n, and $\alpha(z)$ for α. For the system, we imagine a Geom$^{[X]}$/G/1 queueing system with a vacation period that may terminate in one of the following two situations.

Case 1. If there is at least one customer waiting in the system at the end of the j-th vacation $(1 \leq j \leq J)$, the server immediately operates a startup. In this case, at the end of the startup time the PGF for the number of customers waiting in the system is given by $[1-\lambda^{JT}(0)][\Lambda^T(z)-\lambda^T(0)][1-\lambda^T(0)]^{-1}S[\Lambda(z)]$.

Case 2. If there is no customer waiting in the system at the end of the J-th vacation, the server stays idle in the system. Once a customer arrives, the server immediately operates a startup. Thus, in this case, at the end of the startup time the PGF for the number of customers found of in the system is given by $\lambda^{JT}(0)[\Lambda(z) - \lambda(0)][1 - \lambda(0)]^{-1}S[\Lambda(z)]$.

From the two cases above, at the end of the startup time the PGF $\alpha(z)$ for the number of customers waiting in the system is given by

$$\alpha(z) = [1 - \lambda^{JT}(0)]\frac{\Lambda^T(z) - \lambda^T(0)}{1 - \lambda^T(0)}S[\Lambda(z)] + \lambda^{JT}(0)\frac{\Lambda(z) - \lambda(0)}{1 - \lambda(0)}S[\Lambda(z)] \quad (1)$$

If we denote by $\{k_j, j = 0, 1, 2, \cdots\}$ and $\{b_j, j = 0, 1, 2, \cdots\}$ the probability distributions for A_n and $\alpha + A_n - 1$, respectively, then the PGFs for them are given by $A(z) = \sum_{j=0}^{\infty} k_j z_j = B[\Lambda(z)], \xi(z) = \sum_{j=0}^{\infty} b_j z_j = \frac{\alpha(z)B[\Lambda(z)]}{z}$.

Therefore, the transition probability matrix of Markov chain $\{L_n, n = 1, 2, \cdots\}$ is given by

$$\tilde{P} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ k_0 & k_1 & k_2 & k_3 & \cdots \\ & k_0 & k_1 & k_2 & \cdots \\ & & k_0 & k_1 & \cdots \\ & & & \vdots & \vdots \end{bmatrix}.$$

By the Foster rule, we can prove that the Markov chain $\{L_n, n = 1, 2, \cdots\}$ is positive recurrence if and only if $\rho = \lambda b < 1$.

3 Stochastic Decomposition of Queue Size and Expected Waiting Times in System

In the section, we will obtain the PGFs for the steady-state system size and the waiting time.

Theorem 1. *If $\rho < 1$, the steady-state system size L can be decomposed into the sum of two stochastic independent variables, i.e., $L = L_{Geom^{[X]}/G/1} + L_d$, where $L_{Geom^{[X]}/G/1}$ denotes the steady-state system size of classical $Geom^{[X]}/G/1$ model which generating function and expected value have been given in* [15]. *Then*

$$L_d(z) = \frac{\lambda[1 - \alpha(z)]}{E[\alpha][1 - \Lambda(z)]}$$

is the generating function of additional system L_d.

Proof. We assume that a steady-state distribution exists for the Markov chain $\{L_n; n = 1, 2, \cdots\}$ and that it is denoted by $\pi_k = \lim_{n\to\infty} p(L_n = k), k = 0, 1, 2, \cdots$.

Because the steady-state $\{\pi_k, k \geq 0\}$ satisfies $\Pi\widetilde{P} = \Pi$, we have

$$\pi_j = \pi_0 b_j + \sum_{i=1}^{j+1} \pi_i k_{j+1-i}, \ j \geq 0$$

where $\Pi = (\pi_0, \pi_1, \pi_2, \cdots)$.

Taking generating function, we obtain

$$\begin{aligned} L(z) &= \textstyle\sum_{j=0}^{\infty} \pi_j z^j = \pi_0 \sum_{j=0}^{\infty} b_j z^j + \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} \pi_i k_{j+1-i} z^j \\ &= \pi_0 \tfrac{\alpha(z)B[\Lambda(z)]}{z} + \tfrac{1}{z} B[\Lambda(z)][L(z) - \pi_0] \end{aligned} \quad (2)$$

Substituting Eq.(1) into Eq.(2), we get

$$L(z) = \frac{\pi_0 B[\Lambda(z)][\alpha(z) - 1]}{z - B[\Lambda(z)]}$$

By the normalization $L(1) = 1$ and the L'Hospital rule, we obtain

$$\pi_0 = \frac{1 - \rho}{E[\alpha]}$$

where

$$E[\alpha] = [1-\lambda^{JT}(0)]\frac{T\lambda + s\lambda[1-\lambda^T(0)]}{1-\lambda^T(0)} + \lambda^{JT}(0)\frac{\lambda + s\lambda[1-\lambda(0)]}{1-\lambda(0)}$$

is the mean number customers at the end of the startup time.

Substituting π_0 into Eq.(2), we obtain

$$L(z) = \frac{(1-\rho)B[\Lambda(z)][\alpha(z)-1]}{E[\alpha]\{z - B[\Lambda(z)]\}} = L_{Geom^{[X]}/G/1} \cdot \frac{\lambda[1-\alpha(z)]}{E[\alpha][1-\Lambda(z)]} \tag{3}$$

Thus, it yields

$$L_d(z) = \frac{\lambda[1-\alpha(z)]}{E[\alpha][1-\Lambda(z)]}$$

The proof is complete.

In addition, from the Theorem 1, we obtain the mean queue size in system given by

$$E[L] = E[L_{Geom^{[X]}/G/1}] + E[L_d] = E[L_{Geom^{[X]}/G/1}] + \frac{2\lambda E[\alpha(\alpha-1)] - \lambda^{(2)}E[\alpha]}{2\lambda E[\alpha]}$$

where

$$E[\alpha(\alpha-1)] = \frac{1-\lambda^{JT}(0)}{1-\lambda^T(0)}\{T(T-1)\lambda^2 + T\lambda^{(2)} + 2Ts\lambda^2 + [1-\lambda^T(0)](\lambda^2 s^{(2)} + \lambda^{(2)}s)\} + \frac{\lambda^{JT}(0)}{1-\lambda(0)}\{2\lambda^2 s + \lambda^{(2)} + [1-\lambda(0)](\lambda^2 s^{(2)} + \lambda^{(2)}s)\}$$

Theorem 2. *If $\rho < 1$, the steady-state waiting time W can be decomposed into the sum of two stochastic independent variables, i.e., $W = W_{Geom^{[X]}/G/1} + W_d$, where $W_{Geom^{[X]}/G/1}$ denotes the steady-state waiting time of classical $Geom^{[X]}/G/1$ model which generating function and expected value have been given in* [15]. *Then*

$$W_d(z) = \frac{[1-\lambda(0)][1-\beta(z)]}{E(\alpha)(1-z)}$$

is the generating function of additional system W_d.

Proof. We consider the waiting time of an arbitrary customer in FCFS systems and give explicit expressions for the PGF $W(z)$ of the waiting time for FCFS systems. The distribution of the waiting time can be easily obtained by assuming that a group of customers arrive in the same slot and they constitute one super-customer in a Geom/G/1 system. That is, the PGF $\Lambda_g(z)$ and the mean λ_g for the number of the super-customers who arrive in a slot in the Geom/G/1 system are given by, respectively,

$$\Lambda_g(z) = \lambda(0) + [1-\lambda(0)]z \tag{4}$$

$$\lambda_g = 1 - \lambda(0) \tag{5}$$

The PGF $B_g(z)$ for the service time of a super-customer is given by

$$B_g(z) = \widetilde{\Lambda}[B(z)] = \frac{\Lambda[B(z)] - \lambda(0)}{1 - \lambda(0)} \tag{6}$$

Therefore, the PGF for the number of present super-customer in the corresponding Geo$^{[X]}$/G/1 system at the end of super-customer's service is given by

$$L_g(z) = \frac{(1-\rho)B_g[\Lambda_g(z)][\alpha_g(z) - 1]}{E[\alpha]\{z - B_g[\Lambda_g(z)]\}} \tag{7}$$

where

$$\alpha_g(z) = [1 - \lambda^{JT}(0)]\frac{\Lambda_g^T(z) - \lambda^T(0)}{1 - \lambda^T(0)}S[\Lambda_g(z)] + \lambda^{JT}(0)\frac{\Lambda_g(z) - \lambda(0)}{1 - \lambda(0)}S[\Lambda_g(z)]$$

Let $\beta[\Lambda_g(z)] = \alpha_g(z)$, then we obtain

$$\beta(z) = [1 - \lambda^{JT}(0)]\frac{z^T - \lambda^T(0)}{1 - \lambda^T(0)}S(z) + \lambda^{JT}(0)\frac{z - \lambda(0)}{1 - \lambda(0)}S(z)$$

Since the number of present super-customer in the system at the end of super-customer's service equals just that arriving super-customer in the time interval that they have been in the system, by $W_g(z)$ denoting the PGF for the waiting time of the super-customer, we have the following expression

$$L_g(z) = W_g[\Lambda_g(z)]B[\Lambda_g(z)] \tag{8}$$

Note that the traffic intensity ρ is the same in classic Geom/G/1 and Geom$^{[X]}$/G/1 queue systems. Substituting Eqs.(4)–(7) into Eq.(8), we get the PGF $W_g(z)$ for the waiting time W_g of a supercustomer in an FCFS system as

$$W_g(z) = \frac{(1-\rho)[1 - \lambda(0)][1 - \beta(z)]}{E[\alpha]\{\Lambda[B(z)] - z\}} \tag{9}$$

The waiting time W of an arbitrary customer consists of two independent components. One is the waiting time W_g of a super-customer to who the arbitrary customer belongs; the other, denoted by J, is the sum of the service time for those customers within the same super-customer who are served in front of the arbitrary customer. Note that these components are independent. If $J(z)$ denotes the PGF for J, we have

$$W(z) = W_g(z)J(z) \tag{10}$$

In order to get the $J(u)$, we know that the number of customers within the super-customer that are served in front of the arbitrary customer is equivalent to the forward recurrence time in a discrete-time renewal process when the interrenewal time is given by the number of customers included in the super-customer. Hence we have

$$J(z) = \frac{1 - \Lambda[B(z)]}{\lambda[1 - B(z)]} \tag{11}$$

Substituting Eqs. (9) and (11) into Eq.(10), we get

$$W(z) = \frac{(1-\rho)[1-\lambda(0)][1-\beta(z)]\{1-\Lambda[B(z)]\}}{\lambda E[\alpha][1-B(z)]\{\Lambda[B(z)]-z\}} = W_{Geo^{[x]}/G/1}(z)W_d(z)$$

Thus, it yields

$$W_d(z) = \frac{[1-\lambda(0)][1-\beta(z)]}{E(\alpha)(1-z)}$$

The proof is complete.

In addition, from the Theorem 2, we obtain the mean waiting time in system given by

$$E[W] = E[W_{Geo^{[x]}/G/1}] + E[W_d] = E[W_{Geo^{[x]}/G/1}] + \frac{E[\beta(\beta-1)][1-\lambda(0)]}{2E(\alpha)}$$

where

$$E[\beta(\beta-1)] = \frac{1-\lambda^{JT}(0)}{1-\lambda^{T}(0)}\{T(T-1)+2Ts+[1-\lambda^{T}(0)]s^{(2)}\}+\frac{\lambda^{JT(0)}}{1-\lambda(0)}\{2s+[1-\lambda(0)]s^{(2)}\}$$

4 Expected Length of the Vacation Cycle, the Vacation Period and the Busy Period

We define a time interval as a vacation period that starts at the busy period and terminates at the beginning of the startup time, and denote it by I_v. It consists of a vacation and an idle period. Then we can obtain the probabilities and the PGF, respectively,

$$\begin{cases} P(I_v = kT) = [1-\lambda(0)]\lambda^{(k-1)T}(0)\sum_{j=0}^{T-1}\lambda^j(0),\ 1 \le k \le J, \\ P(I_v = JT+i) = \lambda^{JT+i-1}(0)[1-\lambda(0)],\ i \ge 1 \end{cases}$$

and

$$\begin{aligned} I_v(z) &= \textstyle\sum_{j=1}^{\infty} P(I_v = j)z^j \\ &= [1-\lambda(0)]\{\textstyle\sum_{k=1}^{J}\lambda^{(k-1)T}(0)\sum_{j=0}^{T-1}\lambda^j(0)z^{kT} + \sum_{i=1}^{\infty}\lambda^{JT+i-1}(0)z^{JT+i}\} \\ &= \tfrac{[1-\lambda(0)][1-\lambda^{JT}(0)z^{JT}]z^T}{1-\lambda^T(0)z^T} + \tfrac{\lambda^{JT}(0)[1-\lambda(0)]z^{JT+1}}{1-\lambda(0)z} \end{aligned}$$

Thus, it leads to the mean vacation period length

$$E(I_v) = \frac{-JT\lambda^{JT}(0)[1-\lambda(0)] + T[1-\lambda^{JT}(0)]}{1-\lambda^T(0)} + \frac{\lambda^{JT}(0)\{(JT+1)[1-\lambda(0)]+\lambda(0)\}}{1-\lambda(0)} \quad (12)$$

We denote by Θ_v a busy period defined as a time interval from the end of the startup time to the beginning of the next vacation. Since α is the number of customers in the system at the beginning of a busy period, the PGF $\Theta_v(z)$ and the mean $E(\Theta_v)$ for the length Θ_v of a busy period are given by

$$\begin{aligned}\Theta_v(z) &= \alpha[\Theta(z)] \\ &= [1-\lambda^{JT}(0)]\frac{\Lambda^T(\Theta(z))-\lambda^T(0)}{1-\lambda^T(0)}S[\Lambda(\Theta(z))]+\lambda^{JT}(0)\frac{\Lambda(\Theta(z))-\lambda(0)}{1-\lambda(0)}S[\Lambda(\Theta(z))]\end{aligned}$$

and

$$\begin{aligned}E(\Theta_v) &= \frac{E(\Theta)[1-\lambda^{JT}(0)]}{1-\lambda^T(0)}\{\lambda T+s\lambda[1-\lambda^T(0)]\}+\frac{\lambda E(\Theta)\lambda^{JT}(0)\{1+s[1-\lambda(0)]\}}{1-\lambda(0)} \\ &= \frac{\rho[1-\lambda^{JT}(0)]}{(1-\rho)[1-\lambda^T(0)]}\{T+s[1-\lambda^T(0)]\}+\frac{\rho\lambda^{JT}(0)\{1+s[1-\lambda(0)]\}}{(1-\rho)[1-\lambda(0)]}\end{aligned} \tag{13}$$

where Θ is the length of a busy period caused by the service time of a single customer in the system, $\Theta(z)$ and $E(\Theta)$ are the PGF and the mean of Θ, respectively.

A vacation cycle consists of a vacation period, startup time and the follow busy period. The PGF $C_v(z)$ and the mean $E[C_v]$ for the length C_v of the vacation cycle are given by

$$\begin{aligned}C_v(z) &= I_v(z)\cdot S(z)\cdot\Theta_v(z) \\ &= S(z)\times S[\Lambda(\Theta(z))]\times\{\frac{[1-\lambda^T(0)][1-\lambda^{JT}(0)z^{JT}]z^T}{1-\lambda^T(0)z^T}+\frac{\lambda^{JT}(0)[1-\lambda(0)]z^{JT+1}}{1-\lambda(0)z}\} \\ &\times\{[1-\lambda^{JT}(0)]\frac{\Lambda^T(\Theta(z))-\lambda^T(0)}{1-\lambda^T(0)}+\lambda^{JT}(0)\frac{\Lambda(\Theta(z))-\lambda(0)}{1-\lambda(0)}\}\end{aligned}$$

and

$$\begin{aligned}E[C_v] &= E(I_v)+s+E(\Theta_v) \\ &= s+\frac{-JT\lambda^{JT}(0)[1-\lambda^T(0)]+T[1-\lambda^{JT}(0)]}{1-\lambda^T(0)}+\frac{\lambda^{JT}(0)\{(JT+1)[1-\lambda(0)]+\lambda(0)\}}{1-\lambda(0)} \\ &+\frac{\rho[1-\lambda^{JT}(0)]}{(1-\rho)[1-\lambda^T(0)]}\{T+s[1-\lambda^T(0)]\}+\frac{\rho\lambda^{JT}(0)\{1+s[1-\lambda(0)]\}}{(1-\rho)[1-\lambda(0)]}\end{aligned} \tag{14}$$

5 Optimal Policy

In this section, we will construct a total long-run average cost function per customer per unit time for the system, in which T and J are all decision variables. Our purpose is to determine the optimal T and J to minimize this cost function. The following cost elements are considered: c_h is the holding cost per unit time for each present customer in the system; c_s is the setup cost for per busy cycle; c_i is the cost per unit time for keeping the server off; c_u is the startup cost per unit time for the preparatory work of the server before starting the service.

Employing the definition of each cost element and its corresponding system characteristics, the total long-run average cost per unit time is given by

$$\begin{aligned}F(J,T) &= c_hE[L]+c_s\frac{1}{E[C_v]}+c_i\frac{E[I_v]}{E[C_v]}+c_u\frac{s}{E[C_v]} \\ &= c_hE[L_{Geo^{[X]}/G/1}]+\frac{c_h\{2\lambda E[\alpha(\alpha-1)]-\lambda^{(2)}E[\alpha]\}}{2\lambda E[\alpha]}+\frac{c_s(1-\rho)(1-\lambda(0))[1-\lambda^T(0)]}{A} \\ &+\frac{c_i(1-\rho)(1-\lambda(0))[1-\lambda^T(0)]E[I_v]}{A}+\frac{c_ss(1-\rho)(1-\lambda(0))[1-\lambda^T(0)]}{A}\end{aligned} \tag{15}$$

where

$$E[L_{Geo^{[x]}/G/1}] = \rho + \frac{\lambda^2 b^{(2)} - \lambda\rho + b\lambda^{(2)}}{2(1-\rho)}$$

$$\begin{aligned} A = &s(1-\rho)[1-\lambda(0)][1-\lambda^T(0)] + (1-\rho)[1-\lambda(0)]\{-JT\lambda^{JT}(0)[1-\lambda^{JT}(0)]\} \\ &+T[1-\lambda^{JT}(0)] + (1-\rho)\lambda^{JT}(0)[1-\lambda^T(0)]\{(JT+1)[1-\lambda(0)]+\lambda(0)\} \\ &+\rho[1-\lambda(0)][1-\lambda^{JT}(0)]\{T+s[1-\lambda^T(0)]\} + \rho\lambda^{JT}(0)[1-\lambda^{JT}(0)]\{1+s[1-\lambda(0)]\} \end{aligned}$$

We consider the model with a minimum cost function. For fixed c_s, c_h, c_i and c_u, the optimization problem is described as follows:

$$\begin{aligned} &\min \quad F(J,T) = c_h E[L] + c_s \tfrac{1}{E[C_v]} + c_i \frac{E[I_v]}{E[C_v]} + c_u \tfrac{s}{E[C_v]}, \\ &\text{s.t.} \quad T \geq 1,\ J \geq 1, T, J \in N^+, \quad \text{and} \quad (c_h, c_s\ c_i,\ c_u > 0) \end{aligned} \tag{16}$$

We denote the solution by (J^*, T^*) that minimizes the cost function $F(J,T)$.

6 Numerical Illustration

In the section, the first purpose is to study the effects of some parameters on the expected values of the customers' number and waiting time in the system. We assume that the number of customers Λ in a single slot follows a poisson distribution with a parameter λ, and that service time X of a customer and setup time S follow geometric distributions with the parameters p_1 and p_2, respectively.

For convenience, we choose $T = 1, 10, 20, J = 5, p_1 = 0.8$ and $p_2 = 0.8$, vary the value of λ from 0.3 to 0.7.

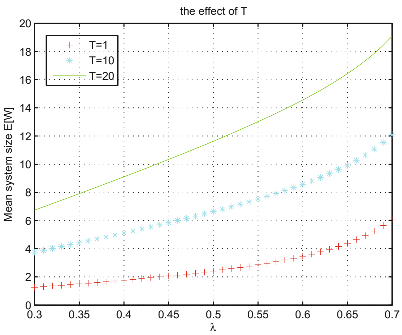

Fig. 1. The expected system size

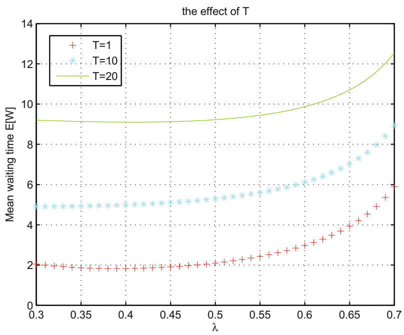

Fig. 2. The expected waiting time

Figures 1 and 2 show that the expected system size and the expected waiting time are all functions of the arrival rate λ. We find that whenever λ increases, the expected system size and the expected waiting time increase at a higher level. Meanwhile, the both increase faster with T increasing.

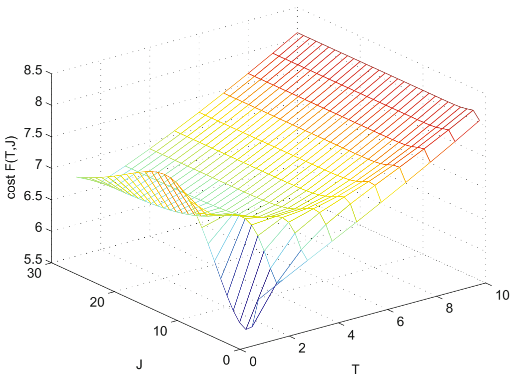

Fig. 3. Mean waiting time $E[W_v]$ versus traffic intensity ρ.

The second purpose is to study the effects of some parameters on the cost function. We assume that the number of customers Λ in a single slot follows a poisson distribution with a parameter λ, and that service time X of a customer and setup time S follow geometric distributions with parameters p_1 and p_2, respectively. We choose $\lambda = 0.15, p_1 = 0.3, p_2 = 0.5, c_h = 2, c_s = 20, c_i = 3$ and $c_u = 10$, vary the values of T and J from 1 to 10 and 1 to 30, respectively.

Figure 3 shows that the minimum cost value per unit time of 5.5731 is obtained at $(T^*, J^*) = (1, 3)$.

7 Conclusion

The paper introduces the optimal modified T vacation policy for the discrete-time Geom$^{[X]}$/G/1 queueing with startup. By using the embedded Markov chain method, we obtain the PGFs and the expected values for the steady state system size, waiting time, busy period and vacation cycle. Additionally, By constructing a cost function, we determine the optimal values of T and J to minimize the cost function. We will further try to study the N policy for the Geom$^{[X]}$/G/1 queueing system.

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Application of Fuzzy Comprehensive Evaluation Model in Mentality Adaptive Research of College Freshmen

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Abstract. This paper study the mentality adaption of college freshmen. Based on analysis and exception handling on the mentality adaption index data of freshmen from Guangdong province, fuzzy comprehensive evaluation (FCE) model to the mentality adaption index of college freshmen was established for evaluating mentality adaption capacity of college freshmen. It plays a positive role in evaluating mentality adaption capacity, early intervention to freshmen with mental crisis and preventing bad incident from happening. In further, it provides practical method and rational assessing for improving college students' comprehensive quality and let them grow up healthy.

Keywords: Fuzzy comprehensive evaluation · College fresh · Mentality adaption capacity

1 Introduction

College freshmen are the source of power in our developing country, they will achieve prosperity of the Chinese nation. However, something stunning was happened in such group, for example, poisoning cases from Tsinghua university, Pecking university and Fudan university. It is mostly because the lack of the mentality adaption that cause such mournful things. From the happening reasons point of view, we need to pay more attention to mentality adaption capacity and psychological status of college freshmen in order to ensure their physical and social well. Then make a contribution to cultivation of comprehensive quality talents for our nation. For the college freshmen, most of which are beginning their independent life journey, far away from their hometown and parents, with unfamiliar environment, strange roommates, different lift style, different cultures around, all elements are possible lead to going wrong in some freshmen mentally. The adaption problems are becoming more and more in the meantime.

In order to achieve early intervention to freshmen coming up mental crisis and further prevent and reduce adverse events recur. According to the diversity and the complex reasons of mentality problems of college freshmen, this paper analysis the mentality adaption related index data of college freshmen from Guangdong province and deal with the bad testing data, further establish fuzzy comprehensive evaluation

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(FCE) model based on mentality adaption quantitative index of college freshmen, which overall evaluate mental adaption status of college freshmen with multi factors and levels. Do evaluation of college freshman mentality adaption capacity through the fuzzy comprehensive evaluation model can not only avoid limitation and one-sidedness of results but also provide quantitative basis and method for mental adaption capacity evaluation of college freshmen. This paper take the college freshmen from one university from Guangdong province as an example, sample data are 200 students major in both art and science, of which 100 are boys and the others are girls, all students are testing by teenager self-appraisal table worked out by nations health education department, full score of 60.

2 Establishing of Index System and Data Processing

2.1 Establishing of Index System

In during the comprehensive evaluation of the college freshmen's mentality adaption, take three factors of mental adaption testing level (TL), academic performance (AP), conduct quality assessment (CQS) as index according to the characteristics of the students. Such index have 5 grades of excellent, very good, good, normal and poor. We can do the fuzzy comprehensive evaluation after the testing and colleting all the index.

2.2 Data Processing

All testing index results are shown in Tables 1, 2 and 3.

Table 1. Grades evaluation of TL

Sex (major)		Grades (scores)									
		51–60		41–50		31–40		21–30		0–20	
		No.	%	No.	%	No.	%	No.	%	No.	%
Science	Male	16	30.77	22	42.31	10	19.23	3	5.77	1	1.92
	Female	17	35.42	20	41.67	8	16.67	2	4.17	1	2.08
Art	Male	17	35.42	21	43.75	7	14.58	2	4.17	1	2.08
	Female	18	34.62	22	42.31	10	19.23	1	1.92	1	1.92

Table 2. Grades evaluation of AP

Sex (major)		Grades (scores)									
		Above 90		80–89		70–79		60–69		Below 60	
		No.	%	No.	%	No.	%	No.	%	No.	%
Science	Male	6	11.54	8	15.38	15	28.85	16	30.77	7	13.46
	Female	3	6.25	6	12.5	17	35.42	19	39.58	3	6.25
Art	Male	3	6.25	5	10.42	14	29.17	21	43.75	5	10.42
	Female	5	9.62	8	15.38	20	38.46	16	30.77	3	5.77

Table 3. Grades evaluation of CQS

Sex (major)		Grades (scores)									
		51–60		41–50		31–40		21–30		0–20	
		No.	%	No.	%	No.	%	No.	%	No.	%
Science	Male	10	19.23	16	30.77	8	15.38	15	28.85	3	5.77
	Female	22	45.83	12	25.00	7	14.58	5	10.42	2	4.17
Art	Male	12	25.00	15	31.25	10	20.83	9	18.75	2	4.17
	Female	22	42.31	13	25.00	10	19.23	6	11.54	1	1.92

3 Theory and Algorithm of FCE Model Based on Mentality Adaption of College Freshmen

3.1 Basic Theory [1, 2]

In recent years, theory of fuzzy comprehensive evaluation (FCE) has a widespread application with rapid expansion due to its fast development. The primary advantage of this approach is that it considers complexities of internal relationship between objective things and fizziness of value system.

In order to evaluate object O (one college freshman), considering m factors of u1, u2, …, um. We can construct mentality adaption evaluation index set of college freshmen as below:

$$U = \{u_1, u_2, \ldots, u_{\mathrm{m}}\}.$$

Take remark set of evaluation grades (excellent, very good, good, normal and poor) as:

$$V = \{v_1, v_2, \ldots, v_{\mathrm{n}}\}.$$

Fuzzy relationship between factor domain and remark domain can be evaluate by evaluation matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} (\text{in this paper,}\quad \mathrm{n} = 5, \mathrm{m} = 3). \tag{1}$$

where, r_{ij} = U(ui vj) represents membership degree that factor ui can be evaluate as v_j, in other word, membership of factor ui to Grade v_j. Row i of matrix R, R(i, :) = (r_{i1}, r_{i2}, …, r_{in}), is the single factor evaluation of ui, fuzzy subset of V.

Set fuzzy subset of factor domain as:

$$A = \frac{a_1}{u_1} + \frac{a_2}{u_2} + \cdots + \frac{a_m}{u_m} (0 < a_i < 1). \tag{2}$$

where, ai is membership degree of ui to A, measurement of role that single factor ui plays in total factor evaluation, A is usually called the fuzzy weighted vector of factor set U.

Set fuzzy subset of remark domain as:

$$B = \frac{b_1}{v_1} + \frac{b_2}{v_2} + \cdots + \frac{b_n}{v_n}(0 < b_j < 1). \tag{3}$$

where, bj is membership degree of object O to be rated as v_j and B is comprehensive evaluation result.

In fact, we can solve the comprehensive evaluation result B by operating the follow formula of B = A • R when A and R to be known, where • is a certain fuzzy product operation according to practical problem.

3.2 Algorithm

3.2.1 Determination of Fuzzy Weight

It is important and difficult issue to determine the weight in fuzzy comprehensive evaluation. In this particular case analytic hierarchy process [3] is carried to figure out the weighted vector A, which is A = (0.40 0.25 0.35), represent mental adaption testing level (TL), academic performance (AP), conduct quality assessment (CQS) of college freshmen respectively.

3.2.2 Determination of Fuzzy Matrix *R*

(1) Creation of evaluation set

Set evaluation factor set u = (TL, AP, CQS), set evaluation grade set v = (excellent, very good, good, normal, poor). So the range of the factor grade evaluation can be easily gotten, shown in Table 4.

Table 4. Range of the factor grade evaluation

Factors	Remarks				
	Excellent	Very good	Good	Normal	Poor
TL	51–60	41–50	31–40	21–30	0–20
AP	Above 90	80–89	70–79	60–69	Below 60
CQS	Excellent	Good	Middle	Qualified	Not qualified

(2) Distribution of factor grade evaluation of college freshmen major in science or art (Table 5).

Table 5. Percentage of factor grade evaluation of students majored in science and art

Major (factor)		Sex (remark)									
		Male					Female				
		Excellent	Very good	Good	Normal	Poor	Excellent	Very good	Good	Normal	Poor
Science	TL	30.77	42.31	19.23	5.77	7.14	35.42	41.67	16.67	4.17	2.08
	AP	11.54	15.38	28.85	30.77	13.46	6.250	12.50	35.42	39.58	6.25
	CQS	19.23	30.77	15.38	28.85	5.77	45.83	25.00	14.58	10.42	4.17
Art	TL	35.42	43.75	14.58	4.17	2.08	34.62	42.31	19.23	1.920	1.92
	AP	6.250	10.42	29.17	43.75	10.42	9.620	15.38	38.46	30.77	5.77
	CQS	25.00	31.25	20.83	18.75	4.17	42.31	25.00	19.23	11.54	1.92

(3) Determination of fuzzy matrix R of relationship between evaluation factor set and evaluation grade set (u × v)

Male majored in science: $R^{(MS)} = \begin{bmatrix} 0.3077 & 0.4231 & 0.1923 & 0.0577 & 0.0714 \\ 0.1154 & 0.1538 & 0.2885 & 0.3077 & 0.1346 \\ 0.1923 & 0.3077 & 0.1538 & 0.2885 & 0.0577 \end{bmatrix}$;

Male majored in art: $R^{(MA)} = \begin{bmatrix} 0.3542 & 0.4375 & 0.1458 & 0.0417 & 0.0208 \\ 0.0625 & 0.1042 & 0.2917 & 0.4375 & 0.1042 \\ 0.2500 & 0.3125 & 0.2083 & 0.1875 & 0.0417 \end{bmatrix}$;

Female majored in science: $R^{(FS)} = \begin{bmatrix} 0.3542 & 0.4167 & 0.1667 & 0.0417 & 0.0208 \\ 0.0625 & 0.1250 & 0.3542 & 0.3958 & 0.0625 \\ 0.4583 & 0.2500 & 0.1458 & 0.1042 & 0.0417 \end{bmatrix}$;

Female majored in art: $R^{(FA)} = \begin{bmatrix} 0.3462 & 0.4231 & 0.1923 & 0.0192 & 0.0192 \\ 0.0962 & 0.1538 & 0.3846 & 0.3077 & 0.0577 \\ 0.4231 & 0.2500 & 0.1923 & 0.1154 & 0.0192 \end{bmatrix}$.

4 Calculation Results and Data Processing

4.1 Calculation Results

4.1.1 Calculation of Fuzzy Comprehensive Evaluation Result *B*

Operating B = A • R to calculate comprehensive evaluation result B due to the fuzzy weight vector A = (0.40 0.25 0.35) and each fuzzy matrix R in the previous section. where • is a certain fuzzy product operation according to this practical problem.

Fuzzy comprehensive evaluation result of male majored in science is

$$\begin{aligned} B^{(MS)} &= A \bullet R^{(MS)} \\ &= (0.40 \quad 0.25 \quad 0.35) \bullet \begin{bmatrix} 0.3077 & 0.4231 & 0.1923 & 0.0577 & 0.0714 \\ 0.1154 & 0.1538 & 0.2885 & 0.3077 & 0.1346 \\ 0.1923 & 0.3077 & 0.1538 & 0.2885 & 0.0577 \end{bmatrix} \\ &= (0.2192 \quad 0.3154 \quad 0.2029 \quad 0.2010 \quad 0.0824). \end{aligned}$$

Similarly, we can get:

$$B^{(MA)} = A \bullet R^{(MA)} = (\,0.2448 \quad 0.3104 \quad 0.2042 \quad 0.1917 \quad 0.0490\,),$$
$$B^{(FS)} = A \bullet R^{(FS)} = (\,0.3177 \quad 0.2854 \quad 0.2063 \quad 0.1521 \quad 0.0385\,),$$
$$B^{(FA)} = A \bullet R^{(FA)} = (\,0.3106 \quad 0.2952 \quad 0.2404 \quad 0.1250 \quad 0.0288\,).$$

4.1.2 Calculation of Evaluation Total Score

If we give certain scores P(90, 80, 70, 60, 50) for evaluation grade set (excellent, very good, good, normal, poor) and compound fuzzy comprehensive evaluation result B and P, evaluation total score C can be obtained:

$$C^{(MS)} = B^{(MS)} \bullet P = (\,0.2192 \quad 0.3154 \quad 0.2029 \quad 0.2010 \quad 0.0824\,) \bullet \begin{bmatrix} 90 \\ 80 \\ 70 \\ 60 \\ 50 \end{bmatrix} = 75.3430.$$

Similarly, we can get:

$$C^{(MA)} = B^{(MA)} \bullet P = 75.1100,$$
$$C^{(FS)} = B^{(FS)} \bullet P = 76.9170,$$
$$C^{(FA)} = B^{(FA)} \bullet P = 77.3380.$$

4.2 Data Processing

From the evaluation total score C, female (76.9170, 77.3380) are obviously larger than male (75.3430, 75.1100), male majored in science 75.3430 are slightly larger than that majored in art 75.1100, same rules for female majored in science 77.3380 to that majored in art 76.9170. We get the final verdict only after Ridit analysis [4] to see whether the difference has a significant difference.

Here is the steps:

(1) Transform fuzzy evaluation membership into fuzzy evaluation frequency by multiplying the number of the male and female majored in science and art by fuzzy evaluation membership, then the fuzzy evaluation frequency set Z can be obtain:

$$Z^{MS} = (52 \times 0.2192 \quad 52 \times 0.3154 \quad 52 \times 0.2029 \quad 52 \times 0.2010 \quad 52 \times 0.0824) = (11 \quad 16 \quad 11 \quad 10 \quad 4);$$
$$Z^{MA} = (48 \times 0.2448 \quad 48 \times 0.3104 \quad 48 \times 0.2042 \quad 48 \times 0.1917 \quad 48 \times 0.0490) = (12 \quad 15 \quad 10 \quad 9 \quad 2);$$
$$Z^{FS} = (48 \times 0.3177 \quad 48 \times 0.2854 \quad 48 \times 0.2063 \quad 48 \times 0.1521 \quad 48 \times 0.0385) = (15 \quad 14 \quad 10 \quad 7 \quad 2);$$
$$Z^{FA} = (52 \times 0.3106 \quad 52 \times 0.2952 \quad 52 \times 0.2404 \quad 52 \times 0.1250 \quad 52 \times 0.0288) = (16 \quad 15 \quad 13 \quad 7 \quad 1).$$

(2) Take male majored in science as criterion group, calculation of standard value $\bar{R}$ [5] can be seen in Table 6.

Table 6. Calculation results of standard value

Definition and results	Calculation items				
	Number of people	Cumulative number down one line	Number of people / 2	Accumulation	(4) / total number of people
Cal. item no.	(1)	(2)	(3) = (1) / 2	(4) = (2) + (3)	
Poor	4	0	2	2	0.0385
Normal	10	4	5	9	0.1731
Good	11	14	5.5	19.5	0.3750
Very good	16	25	8	33	0.6346
Excellent	11	41	5.5	46.5	0.8942

$$\bar{R}^{(MS)} = \sum \mathrm{fR/n} = \frac{4 \times 0.0385 + 10 \times 0.1733 + 11 \times 0.3750 + 16 \times 0.6346 + 11 \times 0.8942}{52} = 0.5000;$$

$$\bar{R}^{(MA)} = \sum \mathrm{fR/n} = \frac{2 \times 0.0385 + 9 \times 0.1733 + 10 \times 0.3750 + 15 \times 0.6346 + 12 \times 0.8942}{48} = 0.5341;$$

$$\bar{R}^{(FS)} = \sum \mathrm{fR/n} = \frac{2 \times 0.0385 + 7 \times 0.1733 + 10 \times 0.3750 + 14 \times 0.6346 + 15 \times 0.8942}{48} = 0.5695;$$

$$\bar{R}^{(FA)} = \sum \mathrm{fR/n} = \frac{1 \times 0.0385 + 7 \times 0.1733 + 13 \times 0.3750 + 15 \times 0.6346 + 16 \times 0.8942}{52} = 0.5760.$$

(3) Calculation of standard error of $\bar{R}$ (S $\bar{R}$), and 95% confidence limit [6] of $\bar{R}$.

5 Analysis of Results

In Table 6, ranking the grade from poor to excellent, we calculate $\bar{R}$ by take male majored in science as criterion group. As for the data in Table 7, bigger $\bar{R}$ means the better students perform, vice is poorer. According to Table 7, considering horizontal comparison, obvious conclusion can be obtained:

(1) $\bar{R}$ value of female is larger than male and $\bar{R}$ value of student majored in art is larger than that majored in science. It reflects that mentality adaption of female is better than male overall, while for the same-sex student, that majored in art is better than that majored in science.
(2) In major, $\bar{R}$ value of female is larger than male as well whatever the major. It reflects that in same major, mentality adaption of female is better than male as well.
(3) In both male and female, majored in science and art. $\bar{R}$ value are overlapped together in their 95% confidence limit. It means it has no different difference in evaluation total score.

Table 7. Calculation of standard error S $\bar{R}$ and 95% confidence limit [6] of $\bar{R}$

Sex (major)		Calculation items		
		$\bar{R}$	$S_{\bar{R}}=1/\sqrt{12 \times N}$	95% confidence limit of $\bar{R}$ = $\bar{R} \pm 2S_{\bar{R}}$
Male	Science	0.5000	$1/\sqrt{12 \times 52}$	(0.4200, 0.5800)
	Art	0.5341	$1/\sqrt{12 \times 48}$	(0.4507, 0.6175)
Female	Science	0.5695	$1/\sqrt{12 \times 52}$	(0.4895, 0.6495)
	Art	0.5760	$1/\sqrt{12 \times 48}$	(0.4926, 0.6594)

6 Conclusion

This paper considers fuzziness while doing evaluation, total evaluates several factors that may affect mentality adaption of students. By utilizing time-point data, difference of different function of object was described and development level, advantages and disadvantages of functions of different object were compared. Results show that fuzzy comprehensive evaluation method can evaluate the mentality adaption capacity of college freshmen well.

By adapting fuzzy comprehensive evaluation method to evaluate mentality adaption of college freshmen, conclusion were obtained:

(1) Mentality adaption of the male and female students in the discussed college is in the same level, without significant difference.
(2) Mentality adaption of the sample students are in the grade of very good and excellent, it shows good mentality adaption of this college. It reflects that mentality adaption of female is better than male regardless of what they major in.

(3) Mentality adaption of the male and female students has no significant difference after Ridit analysis, this conclusion differ from our intuition of that female is better to mental adapt than male.

The reasons cause the differences of mentality adaption between male and female are multifactorial, but their adaption capacity has no significant difference. So as an educator, we should guild the students to form a vigorous mental state by tailoring teaching. For each individual, grassroots organizations such as class, dormitory should play its role to guild students to integrate into group quickly, clearing up their mental difficulty effectively.

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Assessing Holistic Tourism Resources Based on Fuzzy Evaluation Method: A Case Study of Hainan Tourism Island

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Abstract. Holistic tourism, as a new model of boosting regional competitiveness, can be applied in the concentration areas of tourism resources, led by the tourism industry, target for optimizing the resource and regional economic development. Hainan Island, with the pleasant tropical climate, has very rich tourism resources of tropical and coastal scenery, has developed into a large-scale tourist destination combining the natural scenery, folk customs, tropical rain forest, cultural monuments. Additionally, in 2016, Chinese authority carried out the work of developing first batch of holistic tourism demonstration area, which Hainan Province was identified as pilot province, would enjoy the National Tourism Administration launched a number of policy support. And assessment of tourism resource is an essential link in the development process and the foundation for the development of itself. Thus, this paper aim to build a comprehensive assessment model of holistic tourism resources, which based on fuzzy mathematical theory and comprehensive analytical methods, and analyses the tourism resources with the case study of Qionghai City, Hainan Province. The results confirm the potential and drawbacks of the development of holistic tourism in Hainan. Finally, based on our findings, the paper provides some suggestions on how to develop holistic tourism in Hainan.

Keywords: Tourism resources · Fuzzy comprehensive evaluation · Holistic tourism · Hainan island

1 Introduction

With the development of tourism, the traditional tourism forms have been far from meeting the needs of tourists, while the holistic tourism model is built to cater to the tourism market development demand. Holistic tourism refers to a new concept and model of tourism development in a certain area, with tourism as the dominant industry, through the comprehensive and systematic improving and optimizing the regional economic and social resources, especially tourism resources, related industries, ecological environment, public services, system and mechanism, policies and regulations, to achieve organic integration of regional resources, industrial integration and development, social

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co-construction and sharing [1–3]. In January 29, 2016, the National Tourism Work Conference held in Haikou stated the issue of development of holistic tourism, this indicates the development of holistic tourism officially started [4]. In May 26, 2016, the National Holistic Tourism Creation Conference held in Tonglu Zhejiang marked the development of China's tourism industry into the holistic tourism era [5]. The development of holistic tourism, is to solve the problems of imbalance on tourism supply, is conducive to promoting tourism transformation and upgrading, makes scenic spots, hotels, transportation to meet the needs of tourists. This paper will focusing on the tourism resources and build a model and method of comprehensive evaluation of holistic tourism resources, and take the Hainan International island as example for further analysis, then presents some thoughts on development of holistic tourism.

2 Evaluation Method and Model Construction of Holistic Tourism Resources

2.1 Theoretical Basis

Holistic tourism refers to a developing concept in specific administrative area, is to take full advantage of all the attractive elements in the destination for providing visitors with a full range of experience needs [3]. The aim of that is to promote the whole region, the whole factor, the whole process, the whole industry chain of cross-tourism integration in the specific context [4], and it has reality significance for the transformation of tourism development, optimizing the tourism industry structure, enhancing the tourism brand image and exploring tourism development potential.

While, in terms of academic researches, the concept of holistic tourism that is derived from a certain degree of economics and applied to tourism. The central idea of this concept has a high degree of fit with the theoretical system of Competitive Strategy [6] by Michael E. Porter. It is the innovation of the fusion development of tourism destination and industry under the framework of this theory. Its core is the combination of diamond theory and location theory to build the holistic tourism industrial cluster. Therefore, the local government should make efforts to eliminate the barriers that hinder the growth of regional tourism industry, and promote the promotion of industrial economic efficiency and innovation through effective competition. Holistic tourism emphasizes the controlment of development of the tourism industry and the whole socio-economic system, and applying with "point - axis - domain" space-time evolution system of space economics, through centralizing the major tourism projects, facilities and tourism resources, to drive the local relative industries and formats to spontaneously and cooperatively develop. And then promote the social and economic development of entire region.

In this paper, the author believes that the core of holistic tourism is, led on the tourism industry, reasonable allocate the productive elements, promote the regional economic development, strengthen the regional competitiveness, and clear tourism strategic position and social value.

2.2 Practical Basis

The phrase "Holistic tourism" recently has been mentioned often in conference of government, authorities and business sectors mentioned, and soon entered the relevant industry planning and other activities [4, 5]. In August 2015, the China National Tourism Administration issued "conducting national holistic tourism demonstration area", which indicated the government intention of developing holistic tourism, and also mentioned assessment indicators of holistic tourism [7].

In 2016, in the National Tourism Work Conference, Hainan, Ningxia, were entitled with the first two demonstration areas of holistic tourism, which has its development significance, no matter for these two provinces and for the country. Practically speaking, promoting the holistic tourism is conducive to separate from the highly dependence on the traditional model of tourism resources and environment, expand the space of tourism development, the enhance the attraction of different parts of the tourism characteristics. In the following, the author takes Qionghai city of Hainan as an example to try to find a comprehensive method and model for evaluating tourism resources of holistic tourism.

2.3 Building Assessment Criteria of Holistic Tourism Resources

For building a comprehensive assessment of holistic tourism, should have a reliable and accurate indexes system, which will directly influence the scientificity, reliability of the result.

We build the assessment criteria of holistic tourism resources from two aspect: tourism resource value and tourism development value, which is referenced to "Classification, investigation and evaluation of tourism resources" [8] and other

Table 1. Assessment indexes of tourism resource value

Object layer	First-order indexes		Second-order indexes	
Tourism resource value	Resource Element Value (**E1**)	Weight		Weight
		0.4	Ornamental & Recreational Value (**X01**)	0.35
			Historical, Scientific, Cultural, Artistic Value (**X02**)	0.3
			Uniqueness (**X03**)	0.2
			Scale, Abundance & Probability (**X04**)	0.1
			Completeness (**X05**)	0.05
	Environmental Value (**E2**)	0.4	Tourism Capacity (**X06**)	0.4
			Greenland Coverage Rate (**X07**)	0.2
			Safety And Stability (**X08**)	0.1
			Comfort (**X09**)	0.2
			Health Level (**X10**)	0.1
	Influential Value (**E3**)	0.2	Popularity & Influence (**X11**)	0.67
			Appropriate Visit Period (**X12**)	0.33

scholars' models [9, 10]. The assessment indexes of tourism resources value contains 3 first-order value (resource element value, environmental value and influential value), and 12 second-order indexes (as shown in Table 1), and the assessment indexes of tourism development value include 8 first-order indexes (as shown in Table 2).

Table 2. Assessment indexes of tourism development value

Object layer	First-order indexes	
Development value		Weight
	Market Location (**X15**)	0.15
	Economic Foundation (**X16**)	0.2
	Accessibility (**X17**)	0.15
	Scale, Abundance & Probability (**X18**)	0.1
	Completeness (**X19**)	0.05
	Distance To Destination (**X20**)	0.2
	Infrastructure (**X21**)	0.1
	Measures Of Dispersion (**X22**)	0.05

2.4 Assessment Method and Model

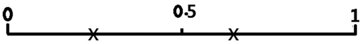
Using mathematical method to mature the assessment index has been recognized by many international scholars [11, 12]. Due to the massive factors could influence tourism evaluation and other practical problems, this paper focusing on quantify the metric data by using the fuzzy mathematics method and the hierarchical analysis method.

The attribute of the evaluation factor is divided into two categories: quantitative and qualitative. Therefore, the membership function is divided into two categories: continuous membership function and discrete membership function.

With regard to the method of obtaining the membership degree of the index, this article adopts the expert investigation method as follows:

- Step 1. Dividing and grading assessment factor index: based on the impact degree of each index.
- Step 2. Quantification each factors

μ represents the membership value of the each factor, and its flexible range can be set between real number interval of [0,1]. When it reaches highest level $\mu = 1$ and when it reaches the lowest level $\mu = 0$ the real number from this interval is corresponding to a certain level (see Tables 1 and 2), it can be scoring by 0 to 1 line method, specifically as follows:



Each member of the evaluation expert group evaluates each factor of each scenic spot according to the evaluation grade standard which has used the arithmetic mean to calculate the membership degree of each scenic spot index, shown as (1):

$$\mu_j = \frac{1}{n}\sum_{i=1}^{n}\mu_i, \tag{1}$$

where μ_i is the value of each evaluation index, and $i = 1, 2, \ldots, n$,
μ_j is the membership of each attraction index, and $j = 1, 2, \ldots, m$.

2.5 Development Assessment Index

2.5.1 Market Location

According to Losch's Location Theory, the market location is related with the population of source area (P_i) and target area (P_j), and the distance(D_{ij}) between them, the expression is shown as (2):

$$\mathrm{I}_{ij} = \mathrm{k} \cdot \frac{\mathrm{p}_i \cdot \mathrm{p}_j}{\mathrm{D}_{ij}}. \tag{2}$$

2.5.2 Distance to Destination

According to the characteristics of the elements, the lower half Cauchy function can be used as the membership function.

$$x \le a, \mu(x) = 1;\ x > a, \frac{1}{1 + \alpha(x - a)^{\beta}}; \tag{3}$$

(x refers to the distance to the source area, and a refers to reference distance).

2.5.3 Measures of Dispersion

The degree of resource concentration has practical significance. So we refer to the aggregation index.

$$R_i = \frac{\overline{r_0}}{\overline{r_e}} = 2\sqrt{D} \cdot \overline{r_0};\ \overline{r_e} = 1\Big/\left(2\sqrt{n/A}\right) = 1\Big/\left(2\sqrt{D}\right), \tag{4}$$

where $\overline{r_0}$ is the average value of the distance to the nearest spot; $\overline{r_e}$ is the theoretical closest distance; D is the point density; A is the attraction scenic area; n is a number of attraction.

2.6 Tourism Resource Evaluation

2.6.1 Assessing Set: $P = \{V, IV, III, II, I\}$

Assessing set is qualitative description set for elements or factors to evaluate its levels, which generally can be divided into five levels: V (excellent), IV (good), III (medium), II (poor), I (very poor); and the set is constructed as $P = \{V, IV, III, II, I\}$.

2.6.2 Factor Set: *U*

The evaluation factor set is the collection of influencing elements of tourism resources. This paper combines the resource value, development conditions and influence of tourism resources.

2.6.3 Weight Set: A

The weight set is used to represent the weight ratio between the factors.

2.6.4 Assessment Model of Holistic Tourism Resource

The comprehensive assessment model is expressed as (5):

$$B = A \circ R + \theta \tag{5}$$

$\mu : U \to P; u \mapsto \mu(u)$

and $u \in U, \mu(u) \in [0, 1], \mu(u)$ is a value corresponding to a level in the assessment set P. θ_i is a adjusted value.

And the assessing level is corresponds with Table 3.

Table 3. Assessing level

V	*IV*	*III*	*II*	*I*
$B \in [0.9, 1)$	$B \in [0.8, 0.9)$	$B \in [0.7, 0.8)$	$B \in [0.6, 0.7)$	$B \in [0, 0.6)$

The fuzzy evaluation model given above is graded and evaluated, and the weighted average calculation of the data is obtained. Using the following formula (6) to obtain the evaluation value b_i.

$$b_i = \sum_{i=1}^{n} a_i r_{ij} + \theta_i, \tag{6}$$

where, a_i is the weight coefficient of the i factor, b_i is the burden evaluation on i factors, r_{ij} is the evaluation i factors and j factors, θ_i is adjusted value.

It should be noted that the comprehensive assessment of tourism resources should be carried out according to the size of the evaluation value. The larger the evaluation value, the greater the value of tourism resources. If we mark the comprehensive score of tourism resources as b, we can divide the scenic spots or attractions into a certain level according to the range of the score and the level of the set.

3 Empirical Analysis of Holistic Tourism Resource Assessment Model

According to the theoretical basis and practical basis stated above, this part is based on the background of Hainan International Tourism Island, and specifically select Qionghai City in Hainan Province as a case to clarify the use of the model.

3.1 Basic Profile and Development Status of Hainan and Qionghai

Hainan Island, located in the southernmost tip of China, is a typical tropical monsoon climate zone, with fresh air, sunny, long period of summer and short period of winter, is suitable for entertaining tourists for most of the year. The central region has a relatively lower temperature, the southwest has relatively higher, the average temperature is 22 to 27 °C, annual sunshine for more than 300 days. Hainan had rich resources and products, the forest coverage rate of over 58%, has a large scale of tropical rainforest, the coastline of 1528 km, with clear water in the most area, white sand, shady trees, suitable water temperature, visitors can carry out various sea activities for most of the year Time. In addition to the typical tropical coastal and forest resources, Hainan is also a multi-ethnic cultural integration area, with exotic villages, where retains their original lifestyles and habits, which gives Hainan a unique attractiveness. From the distribution of tourism resources, tourism resources in Hainan Province are mainly concentrated in three regions Fig. 1: the provincial capital Haikou, world famous destination Sanya and the arising hot spot Qionghai. Hainan

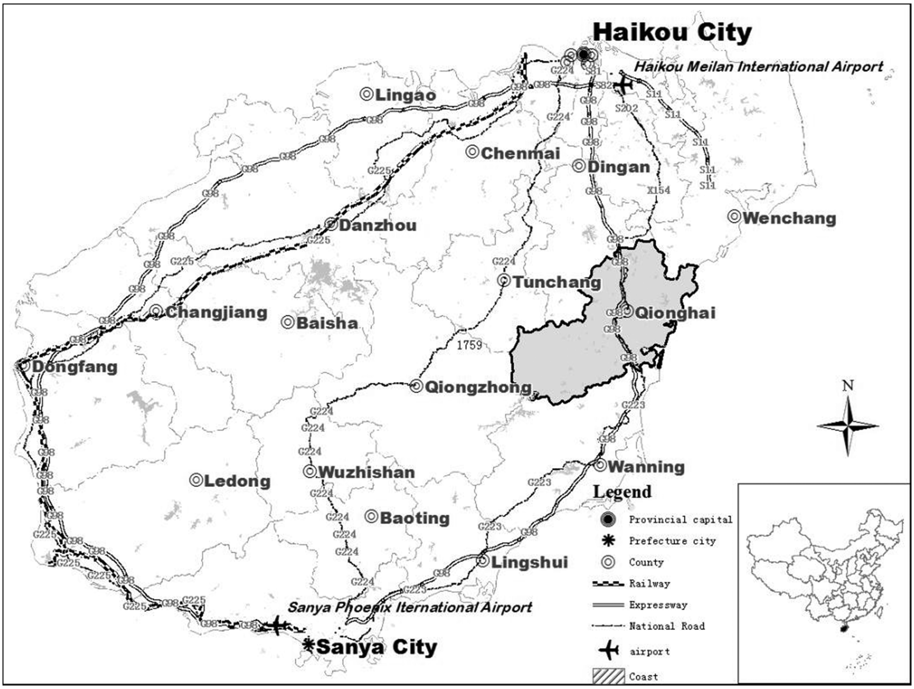


Fig. 1. Map of Hainan island.

relies on the unique natural environment, vigorously developed the tourism industry, real estate industry and tropical agriculture. In particular, the rapid growth of tourism, has been initially built chain industry of “eating, living, traveling, recreation, shopping and entertainment,” and a full range of resort tourism facilities and service system, with a capacity of 30 million visitors per year, attracts large numbers of domestic and international tourists with high quality and diverse resources [13].

Fig. 2. Map of Qionghai city.

Qionghai City is located in the east of Hainan Island, close the South China Sea. With 82.3 km coastline, Qionghai has 4 major ports Fig. 2: Tanmen Port, Qionghai Tourism Port, Longwan Port, Boao Port. The city is one favorite destination of overseas Chinese, with 55 million people from Hong Kong, Macao, and other 28 countries, is one of the famous hometown of overseas Chinese [14]. Qionghai City has a good economic foundation and great potentials, second only to Haikou and Sanya.

3.2 Fuzzy Comprehensive Assessment of Holistic Tourism Resources in Qionghai

In this part selects 17 representative tourism attractions (shown in Table 4) in Qionghai to evaluate their tourism resources value and tourism development value.

Table 4. Tourism attractions in qionghai

Hot destination				Non-hot destination			
Name	Rate	Attractions type	Code	Name	Rate	Attractions type	Code
Boao oriental cultural centre	AAA	Cultural	D01	Jiaji Town		Cultural	D09
Yudai beach		Natural	D02	Baishi Mountain	AAA	Natural	D10
BFA	AAAA	Cultural	D03	Yezhai Village	AA	Cultural	D11
BFA hote		Cultural	D04	Shazhou Island		Natural	D12
Red army memorial park	AAA	Cultural	D05	Jiuqu River		Natural	D13
Wanquan river	AAA	Natural	D06	Wanquan Lake	AA	Natural	D14
Wanquan floating	AAA	Natural	D07	Dongtainong		Cultural	D15
Duohe cultural village	AA	Cultural	D08	Tanmen Port		Natural	D16
				Guantan		Natural	D17

Based on the weight of each index from Table 1, the weighted average method can be used to obtain the evaluation results of the tourism resource value and tourism development value, the results calculated as follows:

3.2.1 Results of Assessment of Resource Element Value

E_1 = [0.92, 0.98, 0.95, 1, 0.95, 1, 0.98, 0.87, 0.96, 0.93, 0.81, 0.7, 0.9, 0.93, 0.82, 0.82, 1].

3.2.2 Results of Assessment of Environmental Value

E_2 = [0.97, 0.95, 0.95, 0.94, 0.87, 0.99, 0.98, 0.66, 0.87, 0.9, 0.81, 0.7, 0.94, 0.8, 0.92, 0.86, 0.79]

3.2.3 Results of Assessment of Influential Value

E_3 = [1, 1,, 1, 1, 1, 1, 0.99, 0.96, 0.91, 1, 1, 0.95, 0.93, 1, 0.85, 1, 1].

3.2.4 Results of Assessment of Tourism Resources Value

E = [0.96, 0.97, 0.96, 0.98, 0.93, 1, 0.98, 0.8, 0.92, 0.93, 0.85, 0.75, 0.92, 0.89, 0.87, 0.87, 0.91].

3.2.5 Results of Assessment of Tourism Development Value

D = [1, 0.99, 0.97, 0.73, 0.94, 0.99, 0.94, 0.87, 0.99, 0.79, 0.86, 0.78, 0.83, 0.83, 0.84, 0.85, 0.9]

3.2.6 Results of Assessment of Tourism Resource

B = [0.98, 0.98, 0.96, 0.88, 0.93, 1, 0.96, 0.83, 0.95, 0.87, 0.85, 0.76, 0.88, 0.87, 0.86, 0.86, 0.91].

According to the results and combing with the government report of ‘Classification, investigation and evaluation of tourism resources’ [8], the 17 attractions in Qionghai are classified into grades *V*, *IV* and *III* (see Appendix 1 and 2).

3.3 The Holistic Tourism Development Potentials and Limits in Qionghai

According to the above analysis, we can see that the tourism resources and development of Qionghai city have its potentials and limits..

3.3.1 Tourism Development Potentials

Qionghai City, its diverse natural resources such as the tropical coastal area, the ecological graceful Wanquan river, lush tropical rain forest, high grade hot springs and other forms of natural landscape; and identical and exotic ethnic minority villages and cultural landscape, internationally known reputation of BFA Conference, also the modern large-scale conference resorts, provides a solid foundation and support for tourism development. Additionally, Qionghai had rich hot springs and eco-tourism resources which are not fully developed yet, especially Wanquan River Tropical Ecosystem conservation, provides a great potential for the development.

3.3.2 Tourism Development Limits

Qionghai tourism heavily tilted in the eastern part of the Boao area, even most of the region has rich tourism resources, the development imbalance will hinder the tourism development in long-term. And due to the transportation-dependency from Haikou and Sanya, the tourists are dispelled from these two major destinations. This further limits its market scale. Moreover, Qionghai has large-scale of coastal area but limited suitable area for aquatic sports and activities on the beach. Compared with the Yalong Bay, etc., Qionghai coastal landscape is also inferior, which makes the local marine tourism at a relative disadvantage.

4 Holistic Tourism Development Suggestions in Hainan

Generally speaking, Hainan has unique advantages to develop holistic tourism: a good ecological environment, the country’s largest special economic zones, international tourism island and other strategic advantages; various and high quality tourism resources, the global reputation, etc. These advantages laid the basic conditions of holistic tourism development. Here is few thorough on Hainan holistic tourism development.

In the context of the holistic tourism concept, the development of Hainan tourism could consider with the concept of “Internet +” and “Whole Vision +”, deeply integrated the internet with the traditional industries, and accord with other industries as a whole vision, to strengthen the development, construction and protection of Hainan Eco-tourism park, Hainan cultural and ecological tourism park, and to speed up the improvement of urban transport, infrastructure and other public facilities, to achieve marketing integration and information computerization.

In additionally, Hainan holistic tourism must adopt an overall point of view, to create economic and industrial development zone, from the single spots to lines and forms the faces, for restructuring tourism elements, transforming the tourism resources advantages into industrial advantages, and forming the motive force of economic development.

Moreover, the modern transport network is direct influential elements to the tourism development, the railway (especially the east west line of high-speed rail), highways, scenic Expressway loop shortens the tourists psychological distance. The “Inner + outside” traffic system not only facilitate the local residents, but also make it much easier for be involved into tourism industry. The next step could start with the improvement of the road guidance mark, enhancing passageway system in the tourist area and public transport system in the urban area.

5 Conclusion

One of the prerequisites for the development of holistic tourism is the tourism industry as a dominant industry, that is, it need a higher resource abundance and characteristics. Hainan Province, with appropriate climate and temperature, the “sun, sea, sand” and other tropical coastal scenery, as well as the folk customs, tropical rain forest, cultural monuments, etc., has rich tourism resources. With the inherent advantages of resources, attracting a large number of tourists come to travel, making the rapid development of local tourism, and become a pillar industry of regional development, which laid the basis for the development of holistic tourism in Hainan. This paper adopts the mathematical basis of fuzzy evaluation, taking Hainan Qionghai as an example, trying to construct the Hainan tourism resources evaluation model, and obtain the assessment result and analysis present the development direction of holistic tourism.

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Appendix 1. Results of Tourism Resources Assessment

		Hot attraction								Non-hot attraction								
Indexes		D01	D02	D03	D04	D05	D06	D07	D08	D09	D10	D11	D12	D13	D14	D15	D16	D17
	X01	1.00	1.00	1.00	1.00	0.90	1.00	1.00	0.75	1.00	0.95	1.00	0.85	0.95	1.00	0.75	0.77	1.00

(*continued*)

(*continued*)

	Hot attraction								Non-hot attraction								
Indexes	D01	D02	D03	D04	D05	D06	D07	D08	D09	D10	D11	D12	D13	D14	D15	D16	D17
X02																	1.00
X03	1.00	1.00	1.00	1.00	0.97	1.00	1.00	0.94	0.90	0.83	0.85	0.75	0.96	0.85	0.65	0.72	1.00
X04	1.00	0.96	1.00	1.00	0.93	1.00	0.90	0.80	0.98	0.87	0.90	0.75	1.00	1.00	0.75	0.98	1.00
X05	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.97	1.00	1.00	1.00	1.00	0.95	0.92	0.96
X06	1.00	1.00	1.00	1.00	0.89	1.00	1.00	0.50	0.77	0.90	0.85	0.45	0.95	0.78	1.00	0.90	0.64
X07	1.00	1.00	1.00	0.80	0.84	1.00	0.95	0.75	0.86	0.83	0.88	0.85	0.85	0.75	0.87	0.80	0.76
X08	1.00	0.90	0.90	0.95	0.90	0.95	1.00	0.72	1.00	0.85	0.85	0.75	0.95	0.96	1.00	0.80	0.95
X09	0.87	0.80	0.78	0.94	0.79	0.98	1.00	0.67	1.00	0.98	0.53	0.89	0.97	0.77	0.75	0.87	0.96
X10	1.00	1.00	1.00	1.00	1.00	0.98	0.94	1.00	0.90	0.95	1.00	0.95	1.00	0.89	1.00	0.88	0.93
X11	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.96	0.95	1.00	1.00	0.92	0.89	1.00	0.78	1.00	1.00
X12	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.95	0.83	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Results E	0.96	0.97	0.96	0.98	0.93	1.00	0.98	0.80	0.92	0.93	0.85	0.75	0.92	0.89	0.87	0.87	0.91
levels	***V***	***V***	***V***	***V***	***V***	***V***	***V***	***V***	***V***	***V***	***IV***	***III***	***V***	***IV***	***IV***	***IV***	***V***

Appendix 2. Results of Tourism Development Assessment

	Hot attraction								Non-hot attraction								
Indexes	D01	D02	D03	D04	D05	D06	D07	D08	D09	D10	D11	D12	D13	D14	D15	D16	D17
X15	1.00	1.00	1.00	0.70	1.00	1.00	1.00	0.91	1.00	0.76	0.90	0.70	1.00	0.86	1.00	0.85	0.93
X16	1.00	0.98	0.99	0.80	0.85	0.95	0.94	0.83	1.00	0.67	0.88	0.75	0.75	0.82	0.82	0.91	0.80
X17	1.00	1.00	0.91	0.60	1.00	1.00	0.85	0.75	1.00	0.85	0.90	0.67	0.98	0.75	0.75	0.70	0.95
X18	0.97	0.97	0.93	0.64	1.00	1.00	0.95	0.80	0.97	0.64	0.97	1.00	0.70	0.80	0.69	0.71	0.93
X19	1.00	1.00	1.00	0.85	0.94	1.00	0.85	0.95	0.95	0.87	0.93	0.75	0.89	0.75	0.82	0.97	0.90
X20	1.00	1.00	0.97	0.80	1.00	1.00	0.95	1.00	1.00	1.00	0.85	0.82	0.68	0.91	0.89	0.85	1.00
X21	1.00	1.00	1.00	0.60	0.80	1.00	1.00	0.80	1.00	0.70	0.60	0.90	0.80	0.80	0.80	1.00	0.80
X22	1.00	1.00	1.00	0.91	0.87	0.93	1.00	0.90	0.91	0.67	0.77	0.71	0.98	0.90	0.90	1.00	0.81
Results D	1.00	0.99	0.97	0.73	0.94	0.99	0.94	0.87	0.99	0.79	0.86	0.78	0.83	0.83	0.84	0.85	0.90
Results	0.98	0.98	0.96	0.88	0.93	1.00	0.96	0.83	0.95	0.87	0.85	0.76	0.88	0.87	0.86	0.86	0.91
Levels	***V***	***V***	***V***	***IV***	***V***	***V***	***V***	***IV***	***V***	***IV***	***IV***	***III***	***IV***	***IV***	***IV***	***IV***	***V***

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Analysis of Flight Delays

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Abstract. This paper tries to analyze the problem D "The problems of flight delay" of 2015's Shenzhen Summer Camp College Students' Mathematical Contest in Modeling by establishing the relevant mathematical model to obtain the pertinent reasons responsible for the flight delays, also to apply the Markov chain model to forecast the delay of flight and to provide a theoretical basis for airline delay management.

Keywords: Flight delay · Delay rate · Hierarchy analysis method · Markov chain model

1 Background Information

This article discusses flight delays. With the development of China's economy, more and more people are choosing aircraft as a means of transport. However, the problem of flight delays is becoming more and more prominent not only in China, but also is ubiquitous in the global civil aviation industry. Hong Kong South China Morning Post claimed that according to flightstats.com statistics, China has the most serious flight delays, among the top 10 most frequent flight delays in the international airports, China accounted for seven, including Shanghai Hongqiao, Shanghai Pudong, Guangzhou Baiyun, Chengdu Shuangliu, Beijing International, Shenzhen Baoan, Hangzhou Xiaoshan and other airports, most of these airports are located in China's economically developed areas, with more population flow. This article will study the following questions based on the above content:

(1) Is the above conclusion correct?
(2) What is the main reason for our country's flight delays?
(3) What are the improvements?

2 Model Assumptions

Assumption 1: assuming that the collected data is true and reliable.
Assumption 2: assuming that the airport staffs' personal reasons are negligible for flight delays.
Assumption 3: assuming that the flights of national airlines are independent from each other and without mutual influence.
Assumption 4: assuming that the departure time of the aircraft is not affected by military activities.

Note: the following analysis and discussion are based on the above hypothesis.

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3 Problem Analysis

The main objective of this paper is to analyze the factors that contribute to flight delays and the corresponding measures to address it. Our overall research method is through the Internet to find relevant data for statistical analysis, to identify the various factors affecting flight delays and to make lists and draw pictures. Finally, for the third question, through inquiring date and literature to understand the impact of flight delays on passengers, combined with the model and method to have an early warning processing for flight delays and accordingly put forward the corresponding management measures. In view of this purpose, for the three questions in this paper, the following analysis can be carried out.

3.1 Analysis of the First Problem

Question 1 requires us to collect data on our own, to determine whether the conclusion is correct. First of all, we find the relevant flight information online, mainly focus on the following points:

1. Take international as the object of study, to collect domestic and foreign major aviation business data, including punctuality, delay rate and so on.
2. Take regions as the object of study, to collect large airlines' date in the Asia-Pacific region.
3. Take our country as the research object, to collect the data of our airlines.

So, with the above three points, we have done the work of analysis, statistics, and then integration, comparison and other operations to all the data collected.

The following Fig. 1 is the relevant data of the major international shipping enterprises in 2014. According to statistics, the international airline's average punctuality rate is 76.54%, while the three Chinese airlines' (Air China, China Eastern Airlines, China Southern Airlines) punctuality rate is lower than the average level [1].

May 2014 FLIGHTSTATS

Major International Airlines

On-time Arrival, Delay and Cancellation Data

Rank	Airline	Sample Size	On-time	Late (15-29 min)	Very Late (30-44 min)	Excessive (45+ min)	Cancelled	Diverted
1	(SU) Aeroflot	15,130	91.18%	4.94%	1.67%	2.22%	0.38%	0.00%
6	(AB) Air Berlin	19,607	87.23%	8.19%	2.08%	2.50%	0.28%	0.00%
23	(AC) Air Canada	16,865	81.97%	9.36%	3.71%	4.88%	0.02%	0.07%
36	(CA) Air China	33,091	65.77%	12.63%	6.59%	14.91%	2.02%	0.10%
3	(UX) Air Europa	4,415	89.64%	5.29%	2.09%	2.98%	0.29%	0.00%
21	(AF) Air France	23,921	82.16%	10.52%	3.63%	3.68%	0.74%	0.01%
37	(AI) Air India	10,998	64.83%	16.51%	8.09%	10.57%	0.21%	0.00%
13	(NZ) Air New Zealand	4,705	85.28%	9.51%	2.35%	2.86%	0.34%	0.00%
12	(AZ) Alitalia	12,160	85.50%	8.85%	2.99%	2.64%	0.81%	0.01%
27	(AA) American	59,521	78.49%	7.82%	4.01%	9.68%	2.06%	0.00%
16	(NH) ANA	19,042	83.99%	11.20%	2.75%	2.05%	0.25%	0.00%
30	(OZ) Asiana	9,657	76.08%	14.51%	4.39%	5.01%	0.88%	0.01%
10	(OS) Austrian Airlines	6,212	86.11%	8.88%	2.56%	2.46%	1.74%	0.00%
31	(AV) AVIANCA	13,364	74.37%	11.93%	4.88%	8.79%	0.11%	0.01%
32	(BA) British Airways	25,518	72.74%	13.77%	5.72%	7.77%	0.53%	0.00%
40	(CX) Cathay Pacific	8,590	58.04%	14.55%	8.95%	18.39%	1.13%	0.07%
43	(CI) China Airlines	5,706	51.33%	20.79%	11.50%	16.37%	0.04%	0.00%
38	(MU) China Eastern	48,917	61.98%	13.98%	7.97%	16.02%	3.05%	0.05%
39	(CZ) China Southern	53,274	61.67%	12.17%	6.77%	19.28%	2.95%	0.11%
15	(DL) Delta	79,942	84.21%	6.83%	3.01%	5.75%	0.12%	0.20%
34	(MS) Egyptair	6,781	68.89%	12.90%	5.28%	12.30%	1.45%	0.63%
24	(EK) Emirates	12,337	81.55%	11.28%	4.15%	3.01%	0.15%	0.01%

Fig. 1. Relevant data of the major international shipping enterprises in 2014

Figure 2 below is the data for the 2014 Asia Pacific Airline. According to the regional statistics, 15 Chinese airlines still do not have good performance, the average delay rate of flights in the Asia-Pacific region was 11.78%, while the delay rate of China's aviation enterprises is higher than the average level. And statistics found that Chinese airlines prefer long-time delays [2].

May 2014

Asia-Pacific Airlines

On-time Arrival, Delay and Cancellation Data

Rank	Airline	Sample Size	On-time	Late (15-29 min)	Very Late (30-44 min)	Excessive (45+ min)	Cancelled	Diverted
10	(BX) Air Busan	2,474	88.45%	6.88%	2.89%	1.79%	0.65%	0.00%
35	(CA) Air China	33,091	65.77%	12.63%	6.59%	14.91%	2.02%	0.10%
36	(AI) Air India	10,998	64.83%	16.51%	8.09%	10.57%	0.21%	0.00%
6	(RLK) Air Nelson	4,983	90.04%	6.36%	1.78%	1.82%	0.90%	0.00%
18	(NZ) Air New Zealand	4,705	85.28%	9.51%	2.35%	2.86%	0.34%	0.00%
34	(AK) Airasia	12,900	66.39%	15.86%	8.36%	9.39%	0.57%	0.00%
43	(D7) AirAsia X	1,109	60.26%	20.15%	7.78%	11.81%	1.53%	0.00%
22	(NH) ANA	19,042	83.99%	11.20%	2.75%	2.05%	0.25%	0.00%
4	(EH) ANA Wings	7,643	90.88%	5.90%	1.77%	1.45%	0.43%	0.00%
30	(OZ) Asiana	9,657	76.08%	14.51%	4.39%	5.01%	0.88%	0.01%
3	(PG) Bangkok Airways	4,893	90.90%	4.82%	1.73%	2.55%	0.78%	0.00%
44	(CX) Cathay Pacific	8,590	58.04%	14.55%	8.95%	18.39%	1.13%	0.07%
45	(EU) Chengdu Airlines	2,202	56.49%	14.08%	7.90%	21.48%	0.00%	0.05%
41	(MU) China Eastern	48,917	61.98%	13.98%	7.97%	16.02%	3.05%	0.05%
42	(CZ) China Southern	53,274	61.67%	12.17%	6.77%	19.28%	2.95%	0.11%
55	(KN) China United	4,171	40.68%	16.44%	10.70%	32.01%	4.34%	0.18%
8	(EAG) Eagle Airways	3,029	89.33%	5.93%	1.76%	2.98%	2.51%	0.00%
25	(QFA) Eastern Australia Airlines	3,001	82.28%	10.04%	3.40%	4.28%	1.07%	0.00%
48	(BR) EVA Airways	4,389	52.77%	18.23%	10.91%	18.09%	0.39%	0.00%
46	(HU) Hainan Airlines	16,645	53.94%	16.90%	8.69%	20.33%	1.19%	0.13%
52	(KA) Hong Kong Dragon	4,883	45.61%	15.22%	10.54%	28.63%	2.29%	0.00%
15	(6E) IndiGo	15,213	85.97%	7.74%	2.91%	3.39%	0.00%	0.00%
16	(JQ) Jetstar Airways	8,561	85.63%	6.95%	3.03%	4.38%	1.07%	0.00%
49	(HO) Juneyao Airlines	5,219	51.99%	16.34%	9.82%	21.70%	0.67%	0.15%
26	(KE) Korean Air Lines	12,402	82.16%	12.54%	3.38%	1.91%	0.65%	0.02%
28	(MH) Malaysia Airlines	15,429	80.16%	10.00%	3.88%	5.97%	1.84%	0.00%
11	(NZM) Mount Cook Airlines	2,808	86.95%	6.86%	2.78%	3.41%	3.92%	0.00%
20	(DD) Nok Air	4,232	85.03%	6.56%	3.56%	4.85%	1.68%	0.00%
56	(PK) Pakistan Intl Airlines	2,850	37.97%	16.05%	11.51%	34.32%	4.91%	0.15%
38	(PR) Philippine Airlines	2,609	63.87%	13.85%	8.71%	13.54%	1.46%	0.04%
12	(QF) Qantas	11,468	86.90%	7.96%	2.65%	2.48%	1.12%	0.01%
7	(ZL) Regional Express	5,390	90.02%	5.08%	1.60%	3.30%	0.39%	0.00%
54	(SC) Shandong Airlines	11,237	44.74%	20.33%	10.78%	23.99%	1.39%	0.15%
50	(FM) Shanghai Airlines	7,960	49.88%	16.62%	10.01%	23.39%	3.64%	0.09%

Asia-Pacific Airlines (Continued)

On-time Arrival, Delay and Cancellation Data

Rank	Airline	Sample Size	On-time	Late (15-29 min)	Very Late (30-44 min)	Excessive (45+ min)	Cancelled	Diverted
51	(ZH) Shenzhen	15,169	45.90%	15.63%	9.38%	28.91%	3.23%	0.19%
39	(3U) Sichuan	12,431	63.29%	12.91%	7.39%	16.26%	0.24%	0.15%
17	(SQ) Singapore Airlines	7,720	85.60%	8.90%	2.45%	3.06%	0.54%	0.00%
31	(BC) Skymark Airlines	3,949	71.82%	17.28%	5.72%	5.18%	0.35%	0.00%
47	(9C) Spring Airways	6,139	53.36%	12.74%	8.81%	25.01%	3.88%	0.07%
9	(SSQ) Sunstate Airlines	5,267	89.21%	5.58%	2.53%	2.66%	1.59%	0.02%
2	(FD) Thai AirAsia	6,491	92.39%	4.86%	1.39%	1.36%	1.39%	0.00%
23	(TG) Thai Airways	6,818	83.19%	10.72%	3.29%	2.79%	1.11%	0.00%
37	(GS) Tianjin Airlines	11,514	64.65%	14.94%	7.62%	12.65%	3.22%	0.13%
40	(GE) TransAsia Airways	3,630	62.54%	19.64%	7.66%	10.16%	1.82%	0.00%
19	(XR) Virgin Australia	2,605	85.22%	7.43%	2.57%	4.78%	1.31%	0.00%
32	(DJ) Virgin Australia	64	70.31%	20.31%	6.25%	3.12%	0.00%	0.00%
13	(VA) Virgin Australia Int'l	11,596	86.82%	7.33%	2.84%	3.01%	1.06%	0.00%
53	(MF) Xiamen	15,419	45.38%	16.54%	10.97%	27.02%	0.84%	0.08%
	Aggregate	521,137	68.72%	11.98%	5.86%	11.78%	1.61%	0.06%

Fig. 2. The data for the 2014 Asia Pacific Airline

To sum up, through data comparison with the international aviation enterprises and the Asia-Pacific region flight, we agree that the conclusion that China's flight delays are serious is correct.

According to the US air travel data provider aviation data network, among the world's 61 large airports, the seven worst performance airports are located in the Chinese mainland. Among them, the punctuality rate of Shanghai Hongqiao Airport, Shanghai Pudong Airport and Hangzhou Xiaoshan Airport is 37.17%, 37.26% and 37.74% respectively. Shenzhen Baoan Airport, Guangzhou Baiyun Airport, Beijing International Airport and Chengdu Shuangliu Airport are also among the most Severe ten airports. The 61 large airports, Tokyo, Japan Haneda airport punctuality rate is as high as 89.76%, Japan Osaka Yidan airport punctuality rate is as high as 94.56%.

According to the above analysis of the data, we believe that the conclusion in title of "the most serious flight delays problems in our Country occur in most of the airport that located in the economic development zone," is correct.

3.2 Analysis of Problem 2

Question 2 demands us to answer the main reason for flight delays. We first start statistics and processing work on the original data get the total number of flights per year, the number of normal flights, the number of abnormal flights, and on this basis, we have a statistical analysis on various factors that have influenced the flight delays, calculate the proportion of the distribution of factors, And then make a histogram of the scale distribution table. Through the chart, so as to clearly prove the several factors that have influenced the flight delays.

We first sort out all the data collected, and then get the total number of flights from 2009 to 2014, the total number of normal flights, the total number of abnormal flights and the number of all kinds of total number of flights affected by various factors, at last, we calculate the percentage [2]. as follows (Table 1):

Table 1. The total number of flights from 2009 to 2014

Year	Category							
	Flights number	Normal	Abnormal	Normal rate	Self-reasons	Flow	Weather	Other
2009	1759438	1437036	322601	81.68%	42.13%	22.49%	23.46%	11.92%
2010	2010652	1617150	393502	80.43%	41.63%	26.84%	20.03%	10.82%
2011	2204147	1701688	502459	77.20%	37.01%	26.00%	22.00%	14.99%
2012	2738472	2049198	689274	74.83%	38.50%	25.00%	21.60%	14.90%
2013	3442124	2413617	1028507	70.12%	37.00%	27.00%	21.60%	14.40%
2014	4013712	2626573	1387139	65.14%	38.00%	25.00%	21.60%	15.40%

We calculated the histogram of the ratio distribution table by the ratio distribution table of the above factors. as follows (Fig. 3):

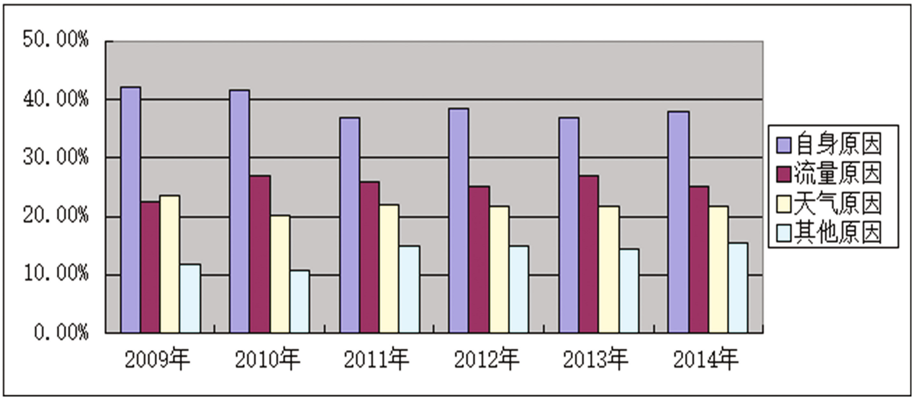

Fig. 3. The histogram of the ratio distribution table by the ratio distribution table of the above factors

Through the above steps to list the proportional distribution table and histograms that influence the flight delays, we conclude that the main reason for the impact of flight delays is the airline's own reasons. Therefore, we believe that we should start from the airline itself, to increase flight management efforts, to seize the key control points and weaknesses about flight operation, refine the safeguards, thereby reducing the aircraft delays.

3.3 Analysis of Problem Three

Question 3 requires us to propose a measure towards flight delays. We mainly target on the four elements, which are concluded in problem 2, that influence flight delays, use the Markov chain model to predict the delay of the flight, and then establish the simulation results to obtain the prediction results. The results will have some errors, and then the qualitative and quantitative analysis method is used to analyze the flight delay. (AHP), to have error analysis, and early warning treatment towards flight delays, we come to a way to help airlines management delays. At the same time, according to the law of flight delays, we also give passengers some reasonable travel recommendations [3].

Analytic Hierarchy Process (AHP) is a systematic analysis method proposed by Prof. A. L. Saaty of the University of Pittsburgh in the 1970s, which combines qualitative analysis and quantitative analysis. By clarifying the problems, it establishes the hierarchical analysis structure model, constructs the judgment matrix, the hierarchical single row and the hierarchical ranking. The five steps are used to calculate the combined weights of the constituent elements for the total target, so as to obtain the comprehensive evaluation value of different feasible schemes. To provide basis for the best solution.

According to the design of the above flight delay evaluation index, the AHP method is used to establish the flight delay model for the airline, as shown in the Fig. 4.

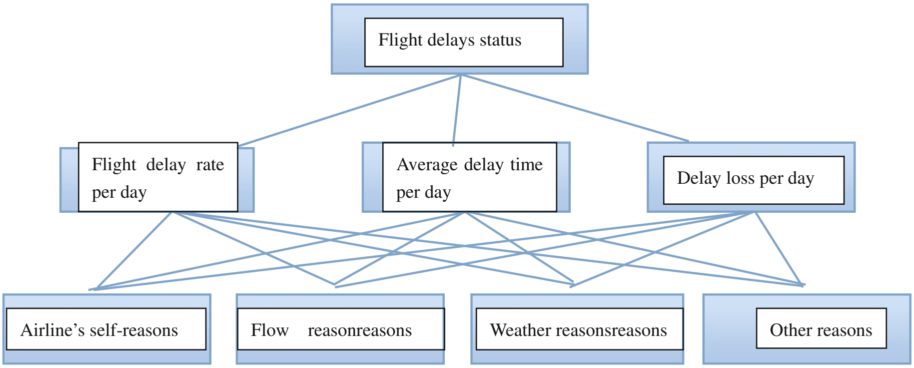

Fig. 4. The airline

According to the model, the occurrence and development of each evaluation index (delay rate, average delay time, delay loss) can be investigated from the program layer (four reasons for the flight delay), so as to provide the basis for the analysis.

According to the analytic hierarchy process model of flight delay state, combined with the status quo of the current domestic assessment flight delay problem, we can establish the warning level and basic index standard (as shown in the table below) for the airline flight delay state (Table 2):

Table 2. The warning level and basic index standard

Warning Level	Operation Indicator	Delay Status	Delay rate	The average delay time (min)
1	white warning	normal delay	<5%	<20
2	Green warning	mild delay	5%–10%	20–40
3	yellow warning	medium delay	10%–20%	40–60
4	orange warning	serious delay	20%–30%	60–120
5	red warning	dangerous delay	>30%	>120

As a result, we can provide operational control strategies for flight companies through early warning levels and index standard.

Table 3. The actual value of the flight delay rate in recent years

Year	Category			
	Self-reasons	Flow	Weather	Other
2009	42.13%	22.49%	23.46%	11.92%
2010	41.63%	26.84%	20.03%	10.82%
2011	37.01%	26.00%	22.00%	14.99%
2012	38.50%	25.00%	21.60%	14.90%
2013	37.00%	27.00%	21.60%	14.40%
2014	38.00%	25.00%	21.60%	15.40%
Average	39.05%	25.39%	21.71%	13.74%

Table 3 is the actual value of the flight delay rate in recent years. Based on these data, the transfer probability matrix is calculated using the Markov chain:

$$\begin{pmatrix} p_{11} & \cdots & p_{14} \\ \vdots & \ddots & \vdots \\ p_{41} & \cdots & p_{44} \end{pmatrix} = \begin{bmatrix} 0.00 & 1.00 & 0.00 & 0.00 \\ 0.23 & 0.16 & 0.14 & 0.40 \\ 0.35 & 0.51 & 0.14 & 0.00 \\ 0.35 & 0.31 & 0.00 & 0.00 \end{bmatrix}.$$

To use the data from the last year of the latest years as the initial state, that is $\pi(0) = (38.00\%, 25.00\%, 21.60\%, 15.40\%)$, with $\pi(m) = \pi(0)p^m$, we can calculate the delay rate in the next few years. The same method can be used to predict the average delay time [4].

The results of the simulation can be compared from the four types of factors that affect the flight delay, and also can analyze synthetically for one day's situation.

Error Self Cause Flow Reason Weather Reason Other Reason Average (Table 4).

Table 4. Error analysis of prediction results

Error	Self-cause	Flow reason	Weather reason	Other reason	Average value
Delay rate	0.30%	0.65%	0.33%	0.19%	0.37%
Average delay time/min	7	6	3	11	7

We can see from the forecast error value that the average error is very small to use airline's flight delay and the average delay time in the past few years to forecast the next few years', they are respectively 0.37% and 7 min.

The model is for the airline to deal with flight delays in the strategic model, and passengers how to face flight delays is still a problem worthy of attention, then we will provide some recommendations to passengers by analyzing the regular pattern of flight delay (Table 5).

Table 5. Average duration and average delay time for an airline within one week

Category	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Endurance	14847	12920	18843	13712	13859	36020	14706
Average delay duration/min	45	39	45	38	37	80	41

To table made according to the chart above (Fig. 5):

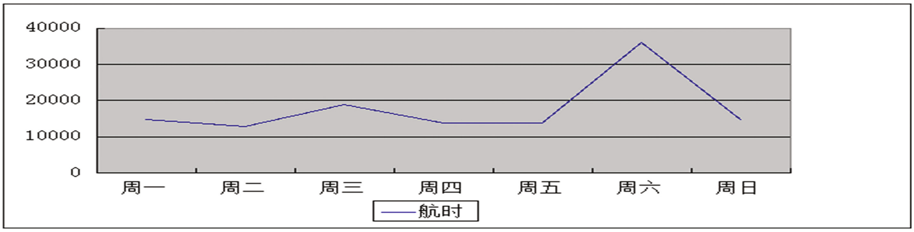

Fig. 5. Average duration for an airline within one week

From the above figure we can see the average daily flight peak is on Saturday, compared to other working days and Sundays, on Saturday, there are more passengers choose to travel by flight, while the seat supply is fixed, therefore, it, is bound to cause a certain the airline, and this pressure is just reflected in the following Fig. 6:

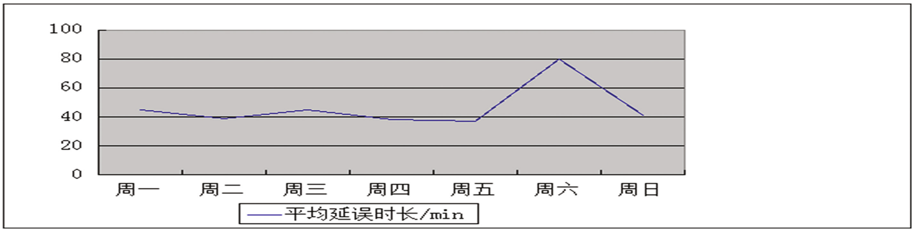

Fig. 6. Average delay time for an airline within one week

It can be seen from the figure that the daily delay time appears a large upward volatility on Saturday, which reflects the pressure caused by the increase in the number of passengers to the airline. Based on the above figure, we can come to the preliminary conclusion that if passengers choose to travel on Saturday, the possibility of flight delays will increase, and the length of time caused by delay will also accordingly be longer.

4 Conclusion

Due to the limited amount of data collected, the relevant conclusions of this article can only be given to airlines.

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Author Index

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