Matching Lexicographic and Conjugation Orders on the Conjugation Class of a Special Sturmian Morphism

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Abstract. The conjugation class of a special Sturmian morphism carries a natural linear order by virtue of the two elementary conjugations $conj_a$ and $conj_b$ with the single letters a and b, with the standard morphism of the class as the smallest element in this order. We show that a lexicographic order on the morphisms of the given conjugation class can be defined that matches the conjugation order.

Keywords: Sturmian morphisms · Sturmian involution · Christoffel words · Standard words and their conjugates

1 Motivation

Conjugation classes of special Sturmian morphisms carry a natural linear order by virtue of the two elementary conjugations $conj_a$ and $conj_b$ with the single letters a and b (see [6]). For every morphism f in the class—except for the anti-standard morphism—either $conj_a \circ f$ or $conj_b \circ f$ belongs to the class and can be identified as the successor of f. Starting from the standard morphism in the class as the smallest element all the others can be iteratively reached in this way. The largest element in the order is the anti-standard morphism in the class. Figure 1 shows a directed graph, containing five such conjugation classes—including the trivial class of the identity morphism. The four non-trivial classes are aligned along concentric circular arcs around the identity morphism.

In addition to these linear graphs, whose counter-clockwise circular arrows are labeled with either $conj_a$ or $conj_b$, there are outward reaching arrows, which are labeled with the four generators $G, \tilde{G}, D, \tilde{D}$ of the special Sturmian monoid St_0. From every inner node of the graph depart two arrows, labeled either G and $\tilde{G}$ or D and $\tilde{D}$. Hence, there are $2^4 = 16$ paths leading from the central node to the nodes on the outermost arc. Intuitively these pathways can also be ordered in a counter-clockwise manner. The intuition will be made precise in Sect. 4.

Our initial consideration is the following: Each conjugation class is ordered counter-clockwise along the corresponding arc. Each node along this arc can also

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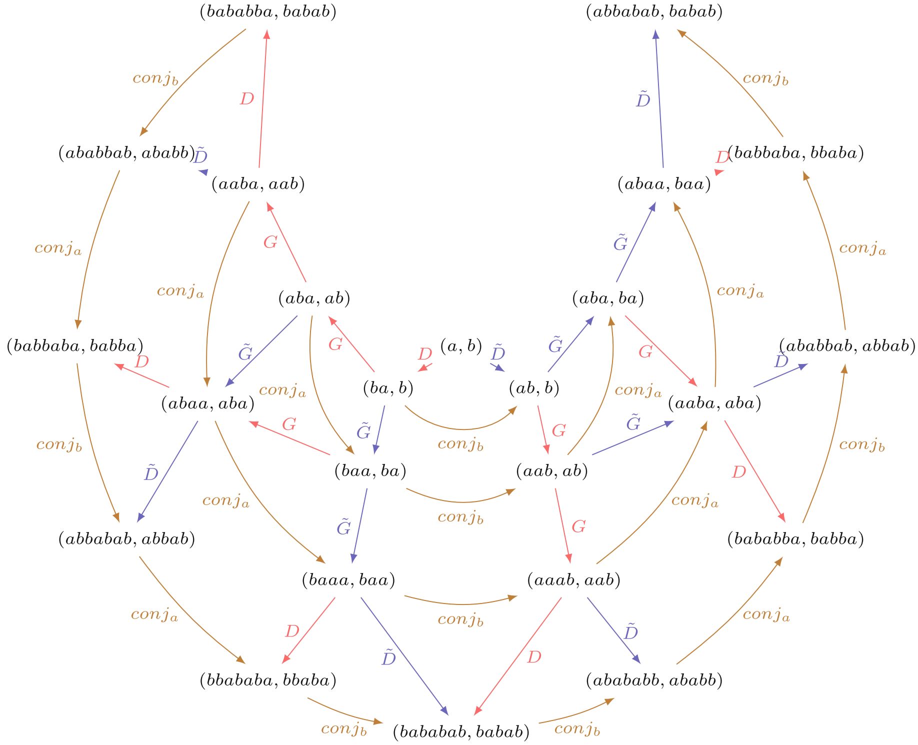


Fig. 1. The nodes along each of the concentric circular arcs form a complete conjugation class of special Sturmian morphisms. Each morphism f is represented by the pair $(f(a), f(b))$ of images of the letters a and b. The node (a, b) in the center represents the Identity map. Then from inside outwards the conjugation classes of D, GD, GGD and $DGGD$ are displayed. The graph forms a subgraph of the Cayley graph of the group $Aut(F_2)$ with respect to the generators $G, \tilde{G}, D, \tilde{D}, conj_a, conj_b$ (and E). Each single conjugation class forms a linear graph, whose arrows are all labeled with one of the conjugations $conj_a$ or $conj_b$. The outward reaching arrows, connecting nodes on successive arcs, are labeled with the generators $G, \tilde{G}, D, \tilde{D}$ of the special Sturmian monoid (= monoid of special positive automorphisms).

be reached along one or more pathways from the center. All the pathways from the center to the nodes on the same arc can also be ordered in a counterclockwise-outward right-to-left lexicographic manner as follows: For each node we postulate that $\overset{G}{\longleftarrow}$ precedes $\overset{\tilde{G}}{\longleftarrow}$ or that $\overset{D}{\longleftarrow}$ precedes $\overset{\tilde{D}}{\longleftarrow}$, in accordance with the counterclockwise arrangement of these arrows. Paths can be ordered lexicographically from right to left (= from the center outward). $\overset{D}{\longleftarrow}\overset{G}{\longleftarrow}\overset{G}{\longleftarrow}\overset{D}{\longleftarrow}$ precedes $\overset{\tilde{D}}{\longleftarrow}\overset{G}{\longleftarrow}\overset{G}{\longleftarrow}\overset{D}{\longleftarrow}$ precedes $\overset{D}{\longleftarrow}\overset{\tilde{G}}{\longleftarrow}\overset{G}{\longleftarrow}\overset{D}{\longleftarrow}$ precedes $\overset{\tilde{D}}{\longleftarrow}\overset{\tilde{G}}{\longleftarrow}\overset{G}{\longleftarrow}\overset{D}{\longleftarrow}$, etc. The last path is $\overset{\tilde{D}}{\longleftarrow}\overset{\tilde{G}}{\longleftarrow}\overset{\tilde{G}}{\longleftarrow}\overset{\tilde{D}}{\longleftarrow}$. Hence the question arises, whether the two orders match.

In the strictest sense the orders do not match: The path $\xleftarrow{\tilde{D}}\xleftarrow{\tilde{G}}\xleftarrow{G}\xleftarrow{D}$ precedes the path $\xleftarrow{D}\xleftarrow{G}\xleftarrow{\tilde{G}}\xleftarrow{D}$. Yet the morphism $DG\tilde{G}D$ with the node label $(babbaba, babba)$ precedes the morphism $\tilde{D}\tilde{G}GD$ with the node label $(abbabab, abbab)$ in the conjugation order: $conj_b \circ DG\tilde{G}D = \tilde{D}\tilde{G}GD$. There is a weaker sense, though, according to which the two orders match. In the concrete example, one may bring to bear that the morphisms G and $\tilde{G}$ commute. Hence with $\xleftarrow{D}\xleftarrow{\tilde{G}}\xleftarrow{G}\xleftarrow{D}$ there is an equivalent path to the node $(babbaba, babba)$ which is lexicographically smaller than $\xleftarrow{\tilde{D}}\xleftarrow{\tilde{G}}\xleftarrow{G}\xleftarrow{D}$, which is the smallest path to the node $(abbabab, abbab)$. Section 4 establishes a general result to that effect, proving a conjecture made in [3].

The following section is of a preparatory nature and—among other things—it inspects various commutative triangles and squares in Fig. 1, such as:

$$\begin{array}{ll} conj_a \circ G = G \circ conj_a = \tilde{G} & conj_b \circ D = D \circ conj_b \;= \tilde{D} \\ conj_a \circ \tilde{G} = \tilde{G} \circ conj_a & conj_b \circ \tilde{G} = G \circ conj_b \\ conj_a \circ \tilde{D} = D \circ conj_a & conj_b \circ \tilde{D} = \tilde{D} \circ conj_b. \end{array}$$

2 Special Sturmian Morphisms and Conjugation

Let F_2 denote the free group generated by the two letters a and b. Following [4] we consider the special Sturmian monoid St_0 as a submonoid of the automorphism group $Aut(F_2)$. It is generated by the four positive automorphisms $G, \tilde{G}, D, \tilde{D}$. On the letters a and b they are defined as follows:

$$\begin{array}{l|l|l|l} G(a) = a & \tilde{G}(a) = a & D(a) = ba & \tilde{D}(a) = ab \\ G(b) = ab & \tilde{G}(b) = ba & D(b) = b & \tilde{D}(b) = b. \end{array}$$

Lemma 1. *On the inverted letters a^{-1} and b^{-1} the morphisms $G, \tilde{G}, D, \tilde{G}$ have the following images:*

$$\begin{array}{l|l|l|l} G(a^{-1}) = a^{-1} & \tilde{G}(a^{-1}) = a^{-1} & D(a^{-1}) = a^{-1}b^{-1} & \tilde{D}(a^{-1}) = b^{-1}a^{-1} \\ G(b^{-1}) = b^{-1}a^{-1} & \tilde{G}(b^{-1}) = a^{-1}b^{-1} & D(b^{-1}) = b^{-1} & \tilde{D}(b^{-1}) = b^{-1} \end{array}$$

Proof. All four morphisms f have to satisfy $f(a^{-1}) = f(a)^{-1}$ as $\epsilon = f(aa^{-1}) = f(a)f(a^{-1})$, analogously for b. Thus, we obtain $G(a^{-1}) = G(a)^{-1} = a^{-1}$, $G(b^{-1}) = G(b)^{-1} = (ab)^{-1} = b^{-1}a^{-1}$ etc.

For some purposes it is useful to know the inverses of $G, \tilde{G}, D, \tilde{D}$ within the group $Aut(F_2)$:

Lemma 2. *The inverses of the special Sturmian morphisms $G, \tilde{G}, D, \tilde{D}$ within the automorphism group $Aut(F_2)$ are given as follows on the letters and the inverted letters:*

$$\begin{array}{l|l|l|l} G^{-1}(a) = a & \tilde{G}^{-1}(a) = a & D^{-1}(a) = b^{-1}a & \tilde{D}^{-1}(a) = ab^{-1} \\ G^{-1}(b) = a^{-1}b & \tilde{G}^{-1}(b) = ba^{-1} & D^{-1}(b) = b & \tilde{D}^{-1}(b) = b \\ G^{-1}(a^{-1}) = a^{-1} & \tilde{G}^{-1}(a^{-1}) = a^{-1} & D^{-1}(a^{-1}) = a^{-1}b & \tilde{D}^{-1}(a^{-1}) = ba^{-1} \\ G^{-1}(b^{-1}) = b^{-1}a & \tilde{G}^{-1}(b^{-1}) = ab^{-1} & D^{-1}(b^{-1}) = b^{-1} & \tilde{D}^{-1}(b^{-1}) = b^{-1} \end{array}$$

Proof. All four inverses have to satisfy $f^{-1}(f(a)) = f(f^{-1}(a)) = a$ and $f^{-1}(f(b)) = f(f^{-1}(b)) = b$. So we check $G^{-1}(G(a)) = G^{-1}(a) = a$, $G^{-1}(G(b)) = G^{-1}(ab) = aa^{-1}b = b$, etc.

As indicated in Sect. 1 we verify now the equations behind the commutative triangles and squares in Fig. 1, which we need later in Sect. 4.

Proposition 1. *Let* $conj_w : F_2 \to F_2$ *denote the conjugation automorphism with the element* w*, i.e.,* $conj_w(u) = w^{-1}uw$*. Then the following equalities hold:* $\tilde{G}G^{-1} = G^{-1}\tilde{G} = conj_a$ *and* $\tilde{D}D^{-1} = D^{-1}\tilde{D} = conj_b$*.*

Proof. It suffices to verify $\tilde{G}G^{-1} = conj_a$ and $\tilde{D}D^{-1} = conj_b$ on the letters a and b and to take into consideration that $G^{-1}\tilde{G} = \tilde{G}G^{-1}$ and $\tilde{D}D^{-1} = D^{-1}\tilde{D}$. Thus we verify $\tilde{G}G^{-1}(a) = \tilde{G}(a) = a = a^{-1}aa = conj_a(a)$, $\tilde{G}G^{-1}(b) = \tilde{G}(a^{-1}b) = a^{-1}ba = conj_a(b)$ and $\tilde{D}D^{-1}(a) = \tilde{D}(b^{-1}a) = b^{-1}ab = conj_b(a)$, $\tilde{D}D^{-1}(b) = \tilde{D}(b) = b = b^{-1}bb = conj_b(b)$.

Corollary 1. $conj_a \circ \tilde{G} = \tilde{G} \circ conj_a$ *and* $conj_b \circ \tilde{D} = \tilde{D} \circ conj_b$*.*

Proof. Substituting $conj_a = \tilde{G}G^{-1}$ and $conj_b = \tilde{D}D^{-1}$ we obtain:

$$\begin{aligned} conj_a \circ \tilde{G} &= (\tilde{G}G^{-1}) \circ \tilde{G} = \tilde{G} \circ (G^{-1}\tilde{G}) = \tilde{G} \circ conj_a \\ conj_b \circ \tilde{D} &= (\tilde{D}D^{-1}) \circ \tilde{D} = \tilde{D} \circ (D^{-1}\tilde{D}) = \tilde{D} \circ conj_b \end{aligned}$$

Proposition 2. $conj_b \circ \tilde{G} = G \circ conj_b$ *and* $conj_a \circ \tilde{D} = D \circ conj_a$*.*

Proof. We apply the morphisms to the letters a and b and compare both sides:

$$\begin{aligned} conj_b(\tilde{G}(a)) &= b^{-1}\tilde{G}(a)b = b^{-1}ab = G(b^{-1}aaa^{-1}b) = G(b^{-1}ab) = G(conj_b(a)) \\ conj_b(\tilde{G}(b)) &= b^{-1}\tilde{G}(b)b = b^{-1}bab = G(b) = G(b^{-1}bb) = G(conj_b(b)) \\ conj_a(\tilde{D}(a)) &= a^{-1}\tilde{D}(a)a = a^{-1}aba = D(a) = D(a^{-1}aa) = D(conj_a(a)) \\ conj_a(\tilde{D}(b)) &= a^{-1}\tilde{D}(b)a = a^{-1}ba = D(a^{-1}bbb^{-1}a) = D(a^{-1}ba) = D(conj_a(b)). \end{aligned}$$

To define the linear *conjugation order* on the conjugation class of a special Sturmian morphism, we recall the following known facts (e.g., see [2,4]).

1. The outer group $Out(F_2) = Aut(F_2)/Inn(F_2)$ of automorphisms of the free group F_2 modulo conjugations is isomorphic to the automorphism group $Aut(\mathbb{Z}^2) = GL_2(\mathbb{Z})$ of the commutative image $\mathbb{Z}^2$ of F_2. This implies that all conjugates of a given special Sturmian morphism f are characterized by the fact that they share the same incidence matrix $M_f = \begin{pmatrix} |f(a)|_a & |f(b)|_a \\ |f(a)|_b & |f(b)|_b \end{pmatrix}$.
2. The incidence matrices of special Sturmian morphisms belong to the monoid $SL_2(\mathbb{N})$, freely generated by the matrices $M_G = M_{\tilde{G}} = R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M_D = M_{\tilde{D}} = L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. This implies that the representations of conjugate morphisms share the same sequence of basic letters G and D and differ at most in the distribution of diacritic $\sim$ marks attached to these letters.

3. With every special Sturmian morphism $f \in St_0$ we may associate the word $w = f(ab) = w_1 \dots w_n \in \{a,b\}^*$. By conjugating the word w with its first letter w_1 we obtain the word $w_1^{-1} w w_1 = w_2 \dots w_n w_1$. By iterating these conjugations with the respective first letter we obtain a full cycle of n conjugated words, namely $w_1 \dots w_n$, $w_2 \dots w_n w_1$, $\dots$, $w_n w_1, \dots w_{n-1}$.
4. All but one of these conjugated words (the bad conjugate) are images of the type $g(ab)$ of a special Sturmian morphism from the same conjugation class as f, and these $n-1$ morphisms also exhaust the conjugation class. Removing the bad conjugate from the full cycle of single-letter conjugations thereby induces a linear order $f_1 < f_2 < \dots < f_{n-1}$ on the conjugation class of f. Its initial element f_1 is a special standard morphism, i.e., $f_1 \in \langle G, D \rangle$ and its terminal element f_{n-1} is a special anti-standard morphism, i.e., $f_{n-1} \in \left\langle \tilde{G}, \tilde{D} \right\rangle$.

Definition 1. *Consider a special standard morphism $f_1 \in \langle G, D \rangle \subset St_0$ and let $n = |f_1(ab)|$ denote the length of the image of the word ab. The linear order $f_1 < f_2 < \dots < f_{n-1}$ on the conjugation class of f_1 (as described above) is called* conjugation order*.*

3 The Path Monoid and the Abacus Relations

In addition to the special Sturmian monoid $St_0 = \left\langle G, \tilde{G}, D, \tilde{D} \right\rangle \subset Aut(F_2)$ we consider the *path monoid* $\Sigma^* = \{\mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D}, \tilde{\mathcal{D}}\}^*$, freely generated over the set $\Sigma = \{\mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D}, \tilde{\mathcal{D}}\}$ of four formal symbols, which we distinguish from the four generating Sturmian morphisms $G, \tilde{G}, D, \tilde{D}$ themselves. The projection $\mu : \Sigma^* \to St_0$ with

$$\mu(\mathcal{G}) = G, \mu(\tilde{\mathcal{G}}) = \tilde{G}, \mu(\mathcal{D}) = D, \mu(\tilde{\mathcal{D}}) = \tilde{D},$$

mediating between the path monoid and the special Sturmian monoid, is well-understood by virtue of the following result from [4] (proposition 2.1):

Proposition 3 *(Kassel and Reutenauer). The special Surmian monoid has a presentation of the form*

$$St_0 \cong \left\langle \mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D}, \tilde{\mathcal{D}} \quad | \quad \mathcal{G}\mathcal{D}^k\tilde{\mathcal{G}} = \tilde{\mathcal{G}}\tilde{\mathcal{D}}^k\mathcal{G}, \mathcal{D}\mathcal{G}^k\tilde{\mathcal{D}} = \tilde{\mathcal{D}}\tilde{\mathcal{G}}^k\mathcal{D} \; for\; all\; k \in \mathbb{N} \right\rangle.$$

We will refer to these relations on paths as the *abacus relations.*

Definition 2. *On the set $\Sigma = \{\mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D}, \tilde{\mathcal{D}}\}$ we introduce the total order $\mathcal{G} < \tilde{\mathcal{G}} < \mathcal{D} < \tilde{\mathcal{D}}$.*[1] *This order induces a natural right-to-left* lexicographic order *on the free monoid Σ^*. For any two words $U, V \in \Sigma^*$ consider the longest common suffix Y, such that $U = X_1 L_1 Y$ and $V = X_2 L_2 Y$ for $X_1, X_2 \in \Sigma^*$ and $L_1, L_2 \in \Sigma$. The two symbols L_1 and L_2 necessarily differ from each other. So we say $U < V$ iff $L_1 < L_2$.*

[1] The setting $\tilde{\mathcal{G}} < \mathcal{D}$ is arbitrary here. It could be likewise $\mathcal{D} < \tilde{\mathcal{G}}$ without consequences for the content of the article.

Lemma 3. *The following permutations $\tau, \tau_{\mathcal{G}}, \tau_{\mathcal{D}} : \Sigma \to \Sigma$ generate monoid automorphisms on Σ^*:*

$$\begin{array}{ll} \tau(\mathcal{G}) := \tilde{\mathcal{G}},\ \ \tau(\tilde{\mathcal{G}}) := \mathcal{G}, & \tau(\mathcal{D}) := \tilde{\mathcal{D}},\ \ \tau(\tilde{\mathcal{D}}) := \mathcal{D}, \\ \tau_{\mathcal{G}}(\mathcal{G}) := \tilde{\mathcal{G}},\ \tau_{\mathcal{G}}(\tilde{\mathcal{G}}) := \mathcal{G}, & \tau_{\mathcal{G}}(\mathcal{D}) := \mathcal{D},\ \tau_{\mathcal{G}}(\tilde{\mathcal{D}}) := \tilde{\mathcal{D}}, \\ \tau_{\mathcal{D}}(\mathcal{G}) := \mathcal{G},\ \tau_{\mathcal{D}}(\tilde{\mathcal{G}}) := \tilde{\mathcal{G}}, & \tau_{\mathcal{D}}(\mathcal{D}) := \tilde{\mathcal{D}},\ \tau_{\mathcal{D}}(\tilde{D}) := \mathcal{D}. \end{array}$$

Proof. This is true for any permutation of Σ.

Proposition 4. *Consider a special standard morphism $f \in \langle G, D \rangle$ and its conjugation class $\mathfrak{F} \subset St_0$. Let $\mathfrak{W} = \mu^{-1}(\mathfrak{F}) \subset \Sigma^*$ denote the set of all words representing these Sturmian morphisms. With respect to lexicographic order for any two words $U, V \in \mathfrak{W}$ the following holds:*

$$U < V \quad iff \quad \tau(V) < \tau(U).$$

Proof. Consider the evaluation $ev : \Sigma \to \{0,1\}$ with $ev(\mathcal{G}) = ev(\mathcal{D}) := 0$ and $ev(\tilde{\mathcal{G}}) = ev(\tilde{\mathcal{D}}) := 1$. Let m denote the common length of all words $W \in \mathfrak{W}$. We define $ev^* : \mathfrak{W} \to \{0, 1, \dots 2^m - 1\}$ with $ev^*(W_1, ..., W_m) = \sum_{k=1}^{m} ev(W_k)2^{k-1}$. The map ev^* is an order-preserving bijection, i.e., we have $U < V$ in lexicographic order if and only if $ev^*(U) < ev^*(V)$ in the order of natural numbers. Furthermore, we have $ev^*(\tau(W)) = 2^m - ev^*(W)$ for any word W. Hence, for any two words $U, V \in \mathfrak{W}$ we have $U < V$ iff $ev^*(U) < ev^*(V)$ iff $2^m - ev^*(V) < 2^m - ev^*(U)$ iff $ev^*(\tau(V)) < ev^*(\tau(U))$ iff $\tau(V) < \tau(U)$.

4 Matching Lexicographic and Conjugation Order

Proposition 5. *Consider a word $W \in \{\tilde{\mathcal{G}}, \tilde{\mathcal{D}}\}^*$ and the associated anti-standard morphism $f = \mu(W)$. The following equations hold:*

$$conj_a \circ f \circ G = \mu(\tau_{\mathcal{D}}(W)) \circ \tilde{G}, \quad conj_b \circ f \circ D = \mu(\tau_{\mathcal{G}}(W)) \circ \tilde{D}.$$

Proof. Any anti-standard morphism f can be expressed in the form:

$$f = \tilde{G}^{n_k} \tilde{D}^{m_k} \dots \tilde{G}^{n_1} \tilde{D}^{m_1}$$

where $k \geq 0$, $n_k, m_1 \geq 0$ and $n_1, \dots, n_{k-1}, m_2, \dots, m_k > 0$. Iteratively applying equations from Proposition 2 we obtain:

$$\begin{aligned} conj_a \circ f \circ G &= conj_a \circ \tilde{G}^{n_k} \tilde{D}^{m_k} \dots \tilde{G}^{n_1} \tilde{D}^{m_1} G \\ &= \tilde{G}^{n_k} \circ conj_a \circ \tilde{D}^{m_k} \dots \tilde{G}^{n_1} \tilde{D}^{m_1} G \\ &= \tilde{G}^{n_k} D^{m_k} \circ conj_a \dots \tilde{G}^{n_1} \tilde{D}^{m_1} G \\ &= \tilde{G}^{n_k} D^{m_k} \dots \tilde{G}^{n_1} D^{m_1} \circ conj_a \circ G \\ &= \tilde{G}^{n_k} D^{m_k} \dots \tilde{G}^{n_1} D^{m_1} \tilde{G} \\ &= \mu(\tau_{\mathcal{D}}(W)) \circ \tilde{G} \end{aligned}$$

The proof for the second equation is analogous.

Proposition 6. *Consider a word $W \in \{\mathcal{G}, \mathcal{D}\}^*$ and the associated standard morphism $f = \mu(W)$. The following equations hold:*

$$f \circ \tilde{G} = conj_a \circ \mu(\tau_{\mathcal{D}}(W)) \circ G, \quad f \circ \tilde{D} = conj_b \circ \mu(\tau_{\mathcal{G}}(W)) \circ D.$$

Proof. Any standard morphism f can be expressed in the form:

$$f = G^{n_k} D^{m_k} \dots G^{n_1} D^{m_1}$$

where $k \geq 0$, $n_k, m_1 \geq 0$ and $n_1, \dots, n_{k-1}, m_2, \dots, m_k > 0$. Then applying the lemma once, and iteratively applying the abacus relation and commutativity of G and $\tilde{G}$:

$$\begin{aligned} conj_a \circ \mu(\tau_{\mathcal{D}}(W)) \circ G &= conj_a \circ G^{n_k} \tilde{D}^{m_k} \dots G^{n_1} \tilde{D}^{m_1} \circ G \\ &= \tilde{G} G^{n_k - 1} \tilde{D}^{m_k} \dots G^{n_1} \tilde{D}^{m_1} G \\ &= G^{n_k - 1} (\tilde{G} \tilde{D}^{m_k} G) G^{n_{k-1} - 1} \dots G^{n_1} \tilde{D}^{m_1} G \\ &= G^{n_k - 1} (G D^{m_k} \tilde{G}) G^{n_{k-1} - 1} \dots G^{n_1} \tilde{D}^{m_1} G \\ &= G^{n_k} D^{m_k} G^{n_{k-1}} D^{m_{k-1}} \dots \tilde{G} G^{n_1 - 1} \tilde{D}^{m_1} G \\ &= G^{n_k} D^{m_k} G^{n_{k-1}} D^{m_{k-1}} \dots G^{n_1 - 1} (\tilde{G} \tilde{D}^{m_1} G) \\ &= G^{n_k} D^{m_k} G^{n_{k-1}} D^{m_{k-1}} G^{n_k - 2} \dots G^{n_1} D^{m_1} \tilde{G} \\ &= f \circ \tilde{G} \end{aligned}$$

The proof for the second equation is analogous.

Proposition 7. *Consider a non-anti-standard special Sturmian morphism $f \in St_0$ and let $\mathfrak{F} \subset St_0$ denote the conjugation class of f. Consider the smallest representative $U \in \mu^{-1}(f)$ of f in lexicographic order. Let $W \in \{\tilde{\mathcal{G}}, \tilde{\mathcal{D}}\}^*$ denote the maximal anti-standard prefix of U such that $U = W\mathcal{L}X$ with a letter $\mathcal{L} \in \{\mathcal{G}, \mathcal{D}\}$ and some suffix $X \in \Sigma^*$. Then the word $U' = \tau_{\mathcal{L}}(W)\tilde{\mathcal{L}}X$ is the smallest representative of the successor of f' of f in conjugation order.*

Proof. For a moment we consider the special case where X is empty. We then have to show that every word $V\tilde{\mathcal{L}} \in \{\mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D}, \tilde{\mathcal{D}}\}^*$, which is lexicographically larger then $W\mathcal{L}$ and smaller than $\tau_{\mathcal{L}}(W)\tilde{\mathcal{L}}$ represents a Sturmian morphism, which—in conjugation order—either precedes or coincides with f.

We look at the case where $\mathcal{L} = \mathcal{G}$. The proof for $\mathcal{L} = \mathcal{D}$ is completely analogous. For $n_k, m_1 \geq 0$ and $n_1, \dots, n_{k-1}, m_2, \dots, m_k > 0$ we obtain the following general form for $W\mathcal{L}$ and $\tau_{\mathcal{G}}(W)\tilde{\mathcal{L}}$:

$$W\mathcal{L} = \tilde{\mathcal{G}}^{n_k} \tilde{\mathcal{D}}^{m_k} \dots \tilde{\mathcal{G}}^{n_1} \tilde{\mathcal{D}}^{m_1} \mathcal{G}, \quad \tau_{\mathcal{G}}(W)\tilde{\mathcal{L}} = \tilde{\mathcal{G}}^{n_k} \mathcal{D}^{m_k} \dots \tilde{\mathcal{G}}^{n_1} \mathcal{D}^{m_1} \tilde{\mathcal{G}}$$

Now we consider a word V satisfying $W\mathcal{L} < V\tilde{\mathcal{L}} < \tau_{\mathcal{L}}(W)\tilde{\mathcal{L}}$. It is specified by the exponents $l_j > 0$ and $l_{j+1}, \dots, l_k, h_{j+1}, \dots, h_k \geq 0$ as follows:

$$\begin{aligned} V &= (\tilde{\mathcal{G}}^{n_k - l_k} \mathcal{G}^{l_k})(\tilde{\mathcal{D}}^{m_k - h_k} \mathcal{D}^{h_k}) \dots \\ &\dots (\tilde{\mathcal{G}}^{n_{j+1} - l_{j+1}} \mathcal{G}^{l_{j+1}})(\tilde{\mathcal{D}}^{m_{j+1} - h_{j+1}} \mathcal{D}^{h_{j+1}})(\tilde{\mathcal{G}}^{n_j - l_j} \mathcal{G}^{l_j}) \mathcal{D}^{m_j} \tilde{\mathcal{G}}^{n_{j-1}} \mathcal{D}^{m_{j-1}} \dots \tilde{\mathcal{G}}^{n_1} \mathcal{D}^{m_1}. \end{aligned}$$

The index j marks the right-most factor in V, where there is a nontrivial power of $\mathcal{G}$. Powers of $\tilde{\mathcal{D}}$ can only occur on the left side of the factor with index j, where they have no influence on the following calculation. If they were to be to the right of that index, the resulting word would be lexicographically larger than $\tau_{\mathcal{L}}(W)\tilde{\mathcal{L}}$, contrary to the assumption.

We substitute $V\tilde{\mathcal{L}}$ with the help of suitable abacus relations until the final $\tilde{\mathcal{L}}$ is replaced by $\mathcal{L}$. This implies that the substitution is smaller or equal to $W\mathcal{L}$.

$$\begin{aligned}
V\tilde{\mathcal{L}} &= \ldots(\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^l)\mathcal{D}^{m_j}\tilde{\mathcal{G}}^{n_{j-1}}\mathcal{D}^{m_{j-1}}\ldots\tilde{\mathcal{G}}^{n_1}\mathcal{D}^{m_1}\tilde{\mathcal{G}} \\
&= \ldots\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^{l-1}(\mathcal{G}\mathcal{D}^{m_j}\tilde{\mathcal{G}})\tilde{\mathcal{G}}^{n_{j-1}-1}\mathcal{D}^{m_{j-1}}\ldots\tilde{\mathcal{G}}^{n_1}\mathcal{D}^{m_1}\tilde{\mathcal{G}} \\
&\cong \ldots\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^{l-1}(\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_j}\mathcal{G})\tilde{\mathcal{G}}^{n_{j-1}-1}\mathcal{D}^{m_{j-1}}\ldots\tilde{\mathcal{G}}^{n_1}\mathcal{D}^{m_1}\tilde{\mathcal{G}} \\
&\cong \ldots\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^{l-1}\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_j}\tilde{\mathcal{G}}^{n_{j-1}-1}(\mathcal{G}\mathcal{D}^{m_{j-1}}\tilde{\mathcal{G}})\tilde{\mathcal{G}}^{n_{j-2}-1}\ldots\tilde{\mathcal{G}}^{n_1}\mathcal{D}^{m_1}\tilde{\mathcal{G}} \\
&\cong \ldots\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^{l-1}\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_j}\tilde{\mathcal{G}}^{n_{j-1}-1}(\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_{j-1}}\mathcal{G})\tilde{\mathcal{G}}^{n_{j-2}-1}\ldots\tilde{\mathcal{G}}^{n_1}\mathcal{D}^{m_1}\tilde{\mathcal{G}} \\
&\;\;\vdots \\
&\cong \ldots\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^{l-1}\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_j}\tilde{\mathcal{G}}^{n_{j-1}}\tilde{\mathcal{D}}^{m_{j-1}}\tilde{\mathcal{G}}^{n_{j-2}}\ldots\tilde{\mathcal{G}}^{n_1-1}(\mathcal{G}\mathcal{D}^{m_1}\tilde{\mathcal{G}}) \\
&\cong \ldots\tilde{\mathcal{G}}^{n_j-l}\mathcal{G}^{l-1}\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_j}\tilde{\mathcal{G}}^{n_{j-1}}\tilde{\mathcal{D}}^{m_{j-1}}\tilde{\mathcal{G}}^{n_{j-2}}\ldots\tilde{\mathcal{G}}^{n_1-1}(\tilde{\mathcal{G}}\tilde{\mathcal{D}}^{m_1}\mathcal{G})
\end{aligned}$$

The calculation remains valid if the exponent m_1 of the rightmost power of $\tilde{\mathcal{D}}$ vanishes. In this case the abacus relation reduces to $\tilde{\mathcal{G}}\mathcal{G} \cong \mathcal{G}\tilde{\mathcal{G}}$. If the suffix X is not empty, nothing in the above argument changes.

Corollary 2. *Consider a non-standard special Sturmian morphism $f \in St_0$ and let $\mathfrak{F} \subset St_0$ denote the conjugation class of f. Consider the largest representative $U \in \mu^{-1}(f)$ of f in lexicographic order. Let $W \in \{\mathcal{G}, \mathcal{D}\}^*$ denote the maximal standard prefix of U such that $U = W\tilde{\mathcal{L}}X$ with a letter $\tilde{\mathcal{L}} \in \{\tilde{\mathcal{G}}, \tilde{\mathcal{D}}\}$ and some suffix $X \in \{\mathcal{G}, \tilde{\mathcal{G}}, \mathcal{D}, \tilde{\mathcal{D}}\}^*$. Then the word $U' = \tau_{\mathcal{L}}(W)\mathcal{L}X$ is the largest representative of the predecessor of f' of f in conjugation order.*

Proof. This follows from the application of Proposition 4 to Proposition 7.

Theorem 1. *Consider a conjugation class $\mathfrak{F} = \{f_1 < \cdots < f_{n-1}\} \subset St_0$ of special Sturmian morphisms in conjugation order. Let $\mathfrak{W} = \mu^{-1}(\mathfrak{F}) \subset \Sigma^*$ denote the set of all their representing words. $\mathfrak{W} = \mathfrak{W}_1 \sqcup \cdots \sqcup \mathfrak{W}_{n-1}$, where $\mathfrak{W}_k = \mu^{-1}(f_k)$, for $k = 1, \ldots, n-1$. Let $U_k, V_k \in \mathfrak{W}_k$ denote the lexicographically smallest and largest elements of $\mathfrak{W}_k$, respectively. Then the following holds:*

1. *$\tau(\mathfrak{W}_k) = \mathfrak{W}_{n-k}$ and $\tau(U_k) = V_{n-k}$ for $k = 1, \ldots, n-1$.*
2. *$U_1 < U_2 < \cdots < U_{n-1}$ and $V_1 < V_2 < \cdots < V_{n-1}$ in lexicographic order.*

Figure 2 illustrates the theorem with an example.

$GDGDG$	$(abaababa, abaababaabaab)$
$\tilde{G}DGDG$	$(baababaa, baababaabaaba)$
$G\tilde{D}GDG$	$(aababaab, aababaabaabab)$
$\tilde{G}\tilde{D}GDG \cong GD\tilde{G}DG$	$(ababaaba, ababaabaababa)$
$\tilde{G}D\tilde{G}DG$	$(babaabaa, babaabaababaa)$
$G\tilde{D}\tilde{G}DG \cong GDG\tilde{D}G$	$(abaabaab, abaabaababaab)$
$\tilde{G}\tilde{D}\tilde{G}DG \cong \tilde{G}DG\tilde{D}G$	$(baabaaba, baabaababaaba)$
$G\tilde{D}G\tilde{D}G$	$(aabaabab, aabaababaabab)$
$\tilde{G}\tilde{D}G\tilde{D}G \cong GD\tilde{G}\tilde{D}G \cong GDGD\tilde{G}$	$(abaababa, abaababaababa)$
$\tilde{G}D\tilde{G}\tilde{D}G \cong \tilde{G}DGD\tilde{G}$	$(baababaa, baababaababaa)$
$G\tilde{D}\tilde{G}\tilde{D}G \cong G\tilde{D}GD\tilde{G}$	$(aababaab, aababaababaab)$
$\tilde{G}\tilde{D}\tilde{G}\tilde{D}G \cong \tilde{G}\tilde{D}GD\tilde{G} \cong GD\tilde{G}D\tilde{G}$	$(ababaaba, ababaababaaba)$
$\tilde{G}D\tilde{G}D\tilde{G}$	$(babaabaa, babaababaabaa)$
$G\tilde{D}\tilde{G}D\tilde{G} \cong GDG\tilde{D}\tilde{G}$	$(abaabaab, abaababaabaab)$
$\tilde{G}\tilde{D}\tilde{G}D\tilde{G} \cong \tilde{G}DG\tilde{D}\tilde{G}$	$(baabaaba, baababaabaaba)$
$G\tilde{D}G\tilde{D}\tilde{G}$	$(aabaabab, aababaabaabab)$
$\tilde{G}\tilde{D}G\tilde{D}\tilde{G} \cong GD\tilde{G}\tilde{D}\tilde{G}$	$(abaababa, ababaabaababa)$
$\tilde{G}D\tilde{G}\tilde{D}\tilde{G}$	$(baababaa, babaabaababaa)$
$G\tilde{D}\tilde{G}\tilde{D}\tilde{G}$	$(aababaab, abaabaababaab)$
$\tilde{G}\tilde{D}\tilde{G}\tilde{D}\tilde{G}$	$(ababaaba, baabaababaaba)$

Fig. 2. The complete 20-element conjugation class of the special standard morphism $GDGDG$ is listed row by row in terms of all representing words (left side) and in terms of the pairs $(f(a), f(b))$ of images of the letters a and b (right side). The rows are ordered according to the conjugation order. Within each row the equivalent words are lexicographically ordered. Smallest representatives are placed to the left, largest words are placed to the right. The thick polygon traverses the $32 = 2^5$ words in lexicographic order. On this trajectory the smallest words are traversed in conjugation order. The same is true for the largest words.

5 Dualizing the Network

Berthé et al. [2] introduce Sturmian involution, an anti-automorphism of the monoid St_0 that sends f in St_0 to f^* by fixing G and $\tilde{G}$ while exchanging D and $\tilde{D}$. They relate conjugation order on the morphisms f_i to the lexicographic order on words $f_i^*(ab)$, where f_1 is a standard morphism and f_1^* is a Christoffel morphism. They show that with $a < b$, $f_1^*(ab) < f_2^*(ab) < \cdots < f_{n-1}^*(ab)$. Another perspective on the lexicographic ordering, via the Burrows-Wheeler Transform, is available in [5]. In this section we revisit and illustrate the finding of [2] by constructing an isography between the diagram in Fig. 1 and a dualized diagram to be fed from the former by applying Sturmian involution to all its components.

The Sturmian monoid St_0 generates the subgroup $ST_0 = M^{-1}(SL_2(\mathbb{Z}))$ of index 2 within $Aut(F_2)$, of all group automorphisms with incidence matrices of determinant 1. ST_0 acts on itself from the left via $\lambda : ST_0 \times ST_0 \to ST_0$ with $\lambda_g(f) = g \circ f$ and from the right via $\rho : ST_0 \times ST_0 \to ST_0$ where $\rho_g(f) = f \circ g$.

In order to manage the right action ρ in terms of conventional function application we consider the generating transformations and the conjugations separately:

$$\rho_G = \Gamma, \rho_{\tilde{G}} = \tilde{\Gamma}, \rho_D = \Delta, \rho_{\tilde{D}} = \tilde{\Delta}, \rho_{conj_a} = \chi_a, \rho_{conj_b} = \chi_b : ST_0 \to ST_0 \text{ with:}$$

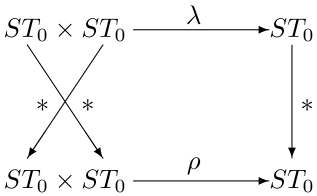
$$\begin{array}{r|r|l} \Gamma(f) = f \circ G & \Delta(f) = f \circ D & \chi_a(f) = conj_{f(a)}, \\ \tilde{\Gamma}(f) = f \circ \tilde{G} & \tilde{\Delta}(f) = f \circ \tilde{D} & \chi_b(f) = conj_{f(b)}. \end{array}$$

Lemma 4. *The transformations $\Gamma, \tilde{\Gamma}, \Delta, \tilde{\Delta}$ satisfy the equations:*

$$\tilde{\Gamma}\Delta^k\Gamma = \Gamma\tilde{\Delta}^k\tilde{\Gamma} \quad \text{and} \quad \tilde{\Delta}\Gamma^k\Delta = \Delta\tilde{\Gamma}^k\tilde{\Delta}$$

Proof. $\tilde{\Gamma}\Delta^k\Gamma = \rho_{GD^k\tilde{G}} \cong \rho_{\tilde{G}\tilde{D}^kG} = \Gamma\tilde{\Delta}^k\tilde{\Gamma}$. Analogously for $\tilde{\Delta}\Gamma^k\Delta$.

Here we regard Sturmian Involution as an anti-automorphism $* : ST_0 \to ST_0$ generated by $G^* = G$, $\tilde{G}^* = \tilde{G}$, $D^* = \tilde{D}$, $\tilde{D}^* = D$. Applying the anti-automorphism $*$ to all components of the left action λ naturally yields a transformation into the right action ρ (see diagram below):



The diagram in Fig. 3 is the result of a thorough application of Sturmian involution to all components (nodes and arrows) of the diagram in Fig. 1. Thereby we may revisit the relations between the generators and the conjugations: from Sect. 2.

Proposition 8.

$$\begin{array}{lll} \chi_a \circ \Gamma = \Gamma \circ \chi_a = \tilde{\Gamma} & \chi_b \circ \Delta = \Delta \circ \chi_b = \tilde{\Delta} & \Leftrightarrow \chi_{b^{-1}} \circ \tilde{\Delta} = \tilde{\Delta} \circ \chi_{b^{-1}} = \Delta \\ \chi_a \circ \tilde{\Gamma} = \tilde{\Gamma} \circ \chi_a & \chi_b \circ \Gamma = \tilde{\Gamma} \circ \chi_b & \Leftrightarrow \chi_{b^{-1}} \circ \tilde{\Gamma} = \Gamma \circ \chi_{b^{-1}} \\ \chi_a \circ \Delta = \tilde{\Delta} \circ \chi_a & \chi_b \circ \tilde{\Delta} = \tilde{\Delta} \circ \chi_b & \Leftrightarrow \chi_{b^{-1}} \circ \tilde{\Delta} = \tilde{\Delta} \circ \chi_{b^{-1}} \end{array}$$

Proof. These relations arise from translating the analogous relations (Proposition 1, Corollary 1, Proposition 2) from the left action λ to the right action ρ:

$$\begin{array}{lllll} f \circ conj_a \circ G = [\Gamma \circ \chi_a](f) & \text{and} & f \circ G \circ conj_a = [\chi_a \circ \Gamma](f) & \text{and} & f \circ \tilde{G} = \tilde{\Gamma}(f) \\ f \circ conj_b \circ D = [\Delta \circ \chi_b](f) & \text{and} & f \circ D \circ conj_b = [\chi_b \circ \Delta](f) & \text{and} & f \circ \tilde{D} = \tilde{\Delta}(f) \\ f \circ conj_b \circ \tilde{G} = [\tilde{\Gamma} \circ \chi_b](f) & \text{and} & f \circ G \circ conj_b = [\chi_b \circ \Gamma](f) & & \\ f \circ conj_a \circ \tilde{G} = [\tilde{\Gamma} \circ \chi_a](f) & \text{and} & f \circ \tilde{G} \circ conj_a = [\chi_a \circ \tilde{\Gamma}](f) & & \\ f \circ conj_a \circ \tilde{D} = [\tilde{\Delta} \circ \chi_a](f) & \text{and} & f \circ D \circ conj_a = [\chi_a \circ \Delta](f) & & \\ f \circ conj_b \circ \tilde{D} = [\tilde{\Delta} \circ \chi_b](f) & \text{and} & f \circ \tilde{D} \circ conj_b = [\chi_b \circ \tilde{\Delta}](f) & & \end{array}$$

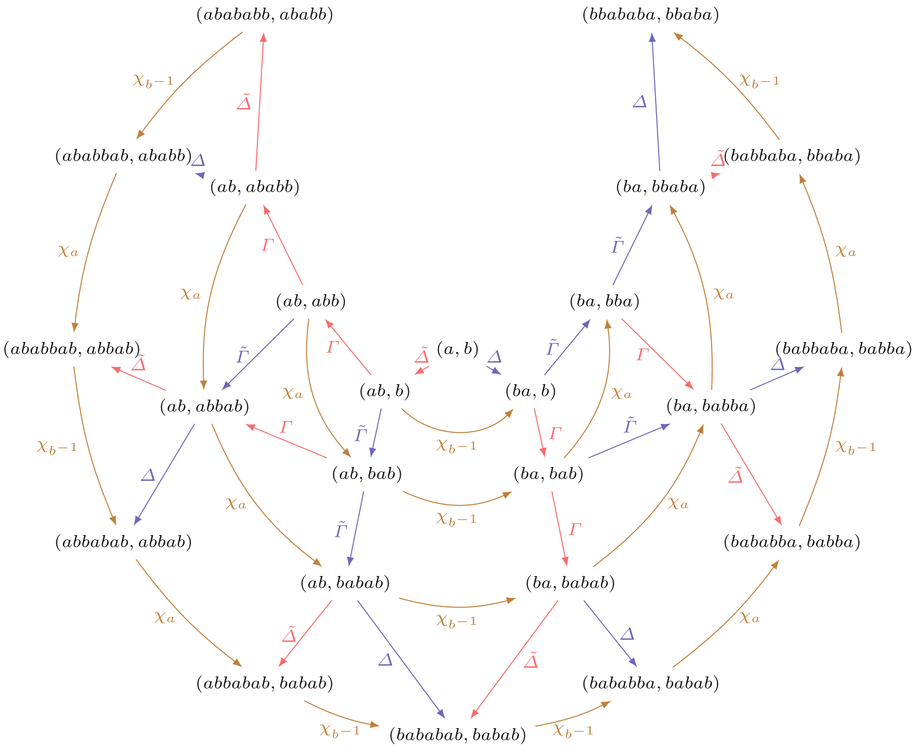


Fig. 3. We obtain the diagram in this Figure from the isographic diagram in Fig. 1 by replacing their node and arrow labels as follows: (1) Each node label $(f(a), f(b))$ is replaced by $(f^*(a), f^*(b))$; (2) each arrow label $conj_a$ or $conj_b$ is replaced by χ_a or $\chi_{b^{-1}} = {\chi_b}^{-1}$, respectively; (3) each arrow label G or $\tilde{G}$ is replaced by Γ or $\tilde{\Gamma}$, respectively; (4) each arrow label D or $\tilde{D}$ is replaced by $\tilde{\Delta}$ or Δ, respectively.

Our insight about the incidence of the lexicographic order of the smallest (or largest) paths with the conjugation order for each conjugation class is faithfully transferred along the duality. The conjugation order has another meaning though, in so far as the conjugating elements are no longer the letters a and b. Also the lexicographic order of the paths has to be modified in view of the Sturmian involution. After involution it is induced by the order $\mathcal{G} < \tilde{\mathcal{G}} < \tilde{\mathcal{D}} < \mathcal{D}$ on Σ. In Fig. 3 we observe that the node labels of each conjugation class $(f^*(a), f^*(b))$ are lexicographically ordered. They are aligned along the corresponding arc in the left-to-right ordering of the images $f^*(ab)$ which is induced by the ordered alphabet $\{a < b\}$. With the final considerations we intend to relate this order to the right-to-left lexicographic order of the words encoding paths. To that end we need to define a suitable lexicographic order on a given conjugation class:

Definition 3. *Consider a conjugation class $\mathfrak{F} \subset St_0$ of special Sturmian morphisms. We say that its elements are in left-to-right lexicographic order:* $\{f_1 \prec$

$\cdots \prec f_{n-1}\}$ *iff their images of the word* $ab \in \{a < b\}^*$ *are in left-to-right lexicographic order:* $\{f_1(ab) < \cdots < f_{n-1}(ab)\}$.

Lemma 5. *For any special Sturmian morphism* $f \in St_0$ *one has* $f(ab) < f(ba)$ *with respect to the left-to-right lexicographic order in* $\{a < b\}^*$.

Proof. For special standard words one has $f(a)f(b) = cab$ and $f(b)f(a) = cba$, where c is the associated central word (see [1]). Hence $f(ab) < f(ba)$. Conjugation with the prefixes w of c preserves the order relation: $conj_w f(ab) < conj_w f(ba)$. And this exhausts the conjugation class of f.

Corollary 3. *For all* $f \in St_0$ *one has*

$$\Gamma(f(a), f(b)) \prec \tilde{\Gamma}(f(a), f(b)) \text{ and } \tilde{\Delta}(f(a), f(b)) \prec \Delta(f(b), f(a))$$

Proof. For any $f \in St_0$ we have

$$\Gamma(f(a), f(b)) \prec \tilde{\Gamma}(f(a), f(b)) \text{ iff } (f(a), f(ab)) \prec (f(a), f(ba)) \text{ iff } f(ab) < f(ba)$$
$$\tilde{\Delta}(f(a), f(b)) \prec \Delta(f(a), f(b)) \text{ iff } (f(ab), f(b)) \prec (f(ba), f(b)) \text{ iff } f(ab) < f(ba)$$

The condition $f(ab) < f(ba)$ is always satisfied by virtue of Lemma 5.

From this result we may finally conclude, that the inherited conjugation order after the application of Sturmian involution coincides with the lexicographic order $\prec$ from Definition 3.

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