First Steps in the Algorithmic Reconstruction of Digital Convex Sets

Paolo Dulio[1], Andrea Frosini[2], Simone Rinaldi[3], Lama Tarsissi[4(✉)], and Laurent Vuillon[4]

[1] Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
paolo.dulio@polimi.it

[2] Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 65, 50134 Firenze, Italy
andrea.frosini@unifi.it

[3] Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Via Roma, 56, 53100 Siena, Italy
rinaldi@unisi.it

[4] Laboratoire de Mathématiques, Université de Savoie Mont Blanc, CNRS UMR 5127, 73376 Le Bourget du Lac, France
lama.tarsissi@uiv-smb.fr, laurent.vuillon@univ-smb.fr

Abstract. Digital convex (DC) sets plays a prominent role in the framework of digital geometry providing a natural generalization to the concept of Euclidean convexity when we are dealing with polyominoes, i.e., finite and connected sets of points. A result by Brlek, Lachaud, Provençal and Reutenauer (see [4]) on this topic sets a bridge between digital convexity and combinatorics on words: the boundary word of a *DC* polyomino can be divided in four monotone paths, each of them having a Lyndon factorization that contains only Christoffel words.

The intent of this paper is to provide some local properties that a boundary words has to fulfill in order to allow a single point modifications that preserves the convexity of the polyomino.

Keywords: Digital convexity · Discrete geometry · Discrete tomography · Reconstruction problem

1 Introduction

Digital convex sets play a prominent role in the framework of digital geometry providing a natural generalization to the concept of Euclidean convexity. It is not so easy to define digital convex sets because for example in the papers of Sklansky [9] and Minsky and Papert [9] digital convex sets may contain many connected components. In order to have exactly one connected component, Chaudhuri and Rosenfeld [5] impose implicitly that a digital convex set must be a polyomino. Recall that a *polyomino* P is a simply connected union of unit squares, that is a union of unit squares without holes. In fact the two authors propose the notion

S. Brlek et al. (Eds.): WORDS 2017, LNCS 10432, pp. 164–176, 2017.
DOI: 10.1007/978-3-319-66396-8_16

of DL-convexity (where DL means digital line) and by definition a region is DL-convex if, for any two squares belonging to it, there exists a digital straight line between them all of whose squares belong to the region. Thus for Chaudhuri and Rosenfeld the region must be a polyomino. Debled-Rennesson, Rémy and Rouyer-Degli have worked on the arithmetical properties of discrete segments in order to detect the convexity of polyominoes (see [8]). Another nice result on this topic using a bridge between digitally convex notion and combinatorics on words is stated by Brlek, Lachaud, Provençal and Reutenauer (see [4]). Indeed, a polyomino P is described by its boundary word b. The boundary word b can be divided in 4 monotone paths and we compute the Lyndon factorization of each path. If each of these factorizations contains only Christoffel words then we have a digitally convex polyomino. We will recall these definitions and use technics to address the following problem: how to give a sequence of single square modifications of a digitally convex polyomino in order to remain at each step in the set of digitally convex polyominoes? This approach is usual in discrete tomography where we would like to reconstruct a polyomino from the horizontal and vertical projections. For example, if we consider HV-convex polyominoes which are polyominoes formed by horizontal and vertical bars of squares, Barcucci et al. [1] shown that the enumeration of all solutions with given projections could be done using switching components. Switching components are local modifications on the boundary of the polyomino which preserve the horizontal and vertical projections. In this paper, we point out the positions where by a single modification of a square the digital convexity of the whole polyomino is maintained. These local decompositions allow us to detect the possible positions of a switching component. And this is crucial in order to make only modifications in the set of digitally convex polyominoes.

2 Preliminaries

Let A be a finite alphabet and w be the word obtained by the concatenation of finite letters of A. We write $w = l_1 l_2 \dots l_n$ where n represents the length of w and denoted by $|w|$. For all $l \in A$, the number of occurrences of this letter in a word w is denoted: $|w|_l$. The set of finite words is denoted A^*, the empty word is denoted by ϵ and by convention $A^+ = A^* \backslash \{\epsilon\}$. The word $\tilde{w} = l_n \dots l_2 l_1$ is the reversal of $w = l_1 l_2 \dots l_n$, where w is called a palindrome if $\tilde{w} = w$. Let p represent the period of the word w such that $w_{i+p} = w_i$ for all $1 \leq i \leq |w| - p$. The following notation $w^k = w^{k-1}.w$ represents the k^{th} power of $w \in A^*$, where $w^0 = \epsilon$. A word w is said *primitive* if it is not the power of a nonempty word.

Two words w and w' in A^* are conjugate if there exists $u, v \in A^*$, such that: $w = uv$ and $w' = vu$. The set of all circular permutations of a word w of its letters is equivalent to the conjugacy class of the word that is defined as the set of all the conjugates of w.

We call $w' = w[i, j]$ a factor of w if it is a subword of w of length $j - i + 1$, starting from the i^{th} position of w to its j^{th} position, where $1 \leq i < j$. For brevity sake, we write $w[i]$ in place of $w[i, i]$. Respectively, if $w = uv$ where u and v are nonempty words in A^*, u is called prefix of w and v is its suffix.

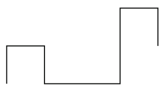
2.1 Digital Convexity and Convexity on Polyominoes

Kim and Rosenfeld introduced different characterizations of discrete convex sets, where a set in Euclidean geometry is convex if and only if for any pair of points p_1, p_2 in a region R, the line segment joining them is completely included in R. In discrete geometry on square grids, this notion refers to the convexity of unit squares.

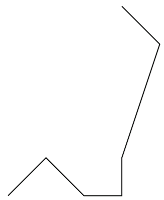
Let $x = (x_1, x_2)$ be an an element of $\mathbb{Z}^2$, we define two norms: $||x||_1 = \sum_{i=1}^{2} |x_i|$ and $||x||_\infty = \max\{|x_1|, |x_2|\}$ to study the connectedness in $\mathbb{Z}^2$.

Definition 1. *A sequence of points* $p_0, p_1, \ldots, p_n \in \mathbb{Z}^2$ *is* $k-$*connected ;* $k \in \mathbb{N}^*$ *if* $||p_i - p_{i-1}||_\infty \leq 1$ *and* $||p_i - p_{i-1}||_1 \leq k$.

Definition 2. *A path* P *is* $k-$*connected if* $\forall\ x, y \in P$, $\exists\ p_0, p_1, \ldots, p_n$ *points that are* $k-$*connected such that* $p_0 = x$, $p_n = y$ *and* $p_i \in P\ \forall i$.



Example of 1−connected path,



Example of 2−connected path.

In this paper, we deal with polyominoes that are defined as following:

Definition 3. *We call* $P \in \mathbb{Z}^2$ *a polyomino if* P *is a* $1-$*connected and finite with* $\mathbb{Z}^2/P$ *also* $1-$*connected.*

Given a finite subset S of $\mathbb{Z}^2$, its *convex hull* is defined as the intersection of all Euclidean convex sets containing S. We say that a polyomino P is *digitally convex* if the convex hull denoted $conv(P)$ and $\mathbb{Z}^2$ are in P. In other words: $conv(P) \cap \mathbb{Z}^2 \in P$.

In Sect. 3, we define the boundary of a polyomino and we give a second equivalent definition for the convexity.

2.2 Christoffel and Lyndon Words

Now we provide the basic definitions and some known results about Christoffel and Lyndon words:

Christoffel Words. In 1771, Jean Bernoulli [2] was the first to give the definition of Christoffel words in the discrete plane and in 1990 Jean Berstel gave them this name with respect to Elwin. B. Christoffel (1829–1900).
Let a, b be two co-prime numbers, i.e. $\gcd(a, b) = 1$. The *Lower* Christoffel path of slope a/b is the $1-$connected path joining the origin $O(0, 0)$ to the point (b, a) and respecting the following characteristics:

1. it is the nearest path below the Euclidean line segment joining these two points;
2. there are no points of $\mathbb{Z} \times \mathbb{Z}$ between the path and line segment.

Analogously, the *Upper* Christoffel path is the path that lies above the line segment. By convention, the Christoffel path is exactly the *Lower* Christoffel path.

The Christoffel word of slope a/b, denoted $C(\frac{a}{b})$, is a word defined on a binary alphabet $A = \{0, 1\}$ that codes the path. We obtain the Christoffel word by assigning a 0 for each horizontal step and a 1 for each vertical one for the Christoffel path of slope a/b. We get that the fraction $\frac{a}{b}$ is exactly: $\frac{|w|_1}{|w|_0}$. The following result is known.

Property 1. Let $w = C\left(\frac{a}{b}\right)$ be the Christoffel word of slope a/b, we write $w = 0w'1$, where w' is a palindrome.

We name w' the central part of w. Note that the lower and upper Christoffel words have the same central part.
We define the morphism $\rho : A^* \longrightarrow \mathbb{Q} \cup \{\infty\}$ by:

$$\rho(\epsilon) = 1 \text{ and } \rho(w) = \frac{|w|_1}{|w|_0} \ \forall\ w \neq \epsilon \in A^*;$$

where $\frac{1}{0} = \infty$. This morphism determines the slope for each given word in A^*.

Example 1. Consider the line segment joining the origin $O(0, 0)$ to the point $(8, 5)$. We have $a = 5, b = 8$ and $n = a + b = 13$. The Christoffel word of slope $5/8$ is: $C(\frac{5}{8}) = 00100101001010$.

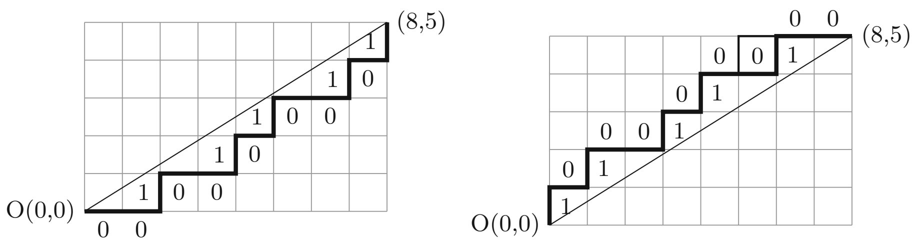


Fig. 1. The Lower and Upper Christoffel word of slope 5/8 are 0010010100101 and 1010010100100, respectively.

The word $w = C(\frac{5}{8}) = 0\boxed{01001010010}1$, where the central part 01001010010 is a palindrome.

Another definition for the Christoffel word of slope a/b was introduced by Christoffel [7].

Definition 4. *Suppose a and b are relatively prime and* $(b, a) \neq (0, 1)$*. The label of a point* (i, j) *on the Christoffel path of slope* $\frac{a}{b}$ *is the number* $\frac{ia-jb}{b}$*. That is, the label of* (i, j) *is the vertical distance from the point* (i, j) *to the line segment from* $(0, 0)$ *to* (b, a)*.*

Lemma 1. *Suppose w is a Christoffel word of slope a/b with a and b relatively prime. If* $\frac{s}{b}$ *and* $\frac{t}{b}$ *are two consecutive labels, as defined in Definition 4, on the Christoffel path from* $(0, 0)$ *to* (b, a)*, then* $t \equiv s + a \mod (a + b)$*. Moreover, t takes as values each integer* $0, 1, 2, \dots, a + b - 1$ *exactly once.*

This part shows that every Christoffel word can be expressed as the product of two Christoffel words in a unique way. This factorization is called the *standard factorization* and was introduced by Jean-Pierre Borel and Laubie [3] (Fig. 2).

Definition 5. *The* maximal point *of a given Christoffel word w, is the point P of w closest to the line segment. Analogously, the* minimal point *Q of w is the furthest from the line segment.*

By Lemma 1, we obtain that the maximal and minimal points of a Christoffel word are obtained when t takes the values 1 and -1, respectively.

Definition 6. *The* standard factorization *of the Christoffel word* $w = C(\frac{a}{b})$ *of slope a/b is the factorization* $w = (w_1, w_2)$*, where* w_1 *encodes the portion of the Christoffel path from (0, 0) to P and* w_2 *encodes the portion from P to* (b, a)*.*

Example 2. The standard factorization of the Christoffel word of slope 5/8 is: $C(\frac{5}{8}) = (00100101, 00101)$ and is shown in the Fig. 2.

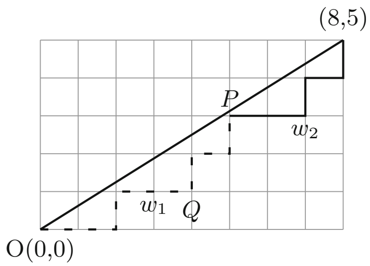


Fig. 2. The standard factorization of $C(\frac{5}{8}) = (00100101, 00101)$.

Theorem 1 *(Borel, Laubie [3]). A nontrivial Christoffel word w (i.e. different from* $(0)^n$ *or* $(1)^n$*) has a unique factorization* $w = (w_1, w_2)$ *with* w_1 *and* w_2 *Christoffel words.*

Lyndon Words. In 1954, Roger Lyndon introduced the *Lyndon words* by defining an order relation over all the words in A^*. The *Standard lexicographic sequence* order denoted $<_l$ is defined as the alphabetic order defined in a dictionary. Hence for the two words $w = 00101$ and $w' = 01001$, we have $w <_l w'$. The order relation between w and w' can be defined as follows:

$$w <_l w' \; if \; w' = w.u \; where \; u \in A^*, \; or$$
$$w = v0z \; and \; w' = v1z' \; where \; v, z \; and \; z' \in A^*.$$

Definition 7. *Let $w = uv$ with $u, v \in A^+$, w is a Lyndon word if it is the smallest between all its conjugates with respect to the lexicographic order.*

Note that Lyndon words are always primitive.

Example 3. The word $w = 00101$ of length 5 is a Lyndon word since it is the minimal element of the set of all conjugates of w. Hence, we can write: $00101 = \min\{00101, 01010, 01001, 10100, 10010\}$. While 0010100101 is not a Lyndon word since it is not primitive.

An important factorization is deduced from the definition of Lyndon words and given by the following theorem by Chen, Fox and Lyndon in [6]

Theorem 2. *Every non-empty word w admits a unique factorization as a lexicographically decreasing sequence of Lyndon words. We write $w = w_1^{n_1} w_2^{n_2} \ldots w_k^{n_k}$, such that $w_k <_l \cdots <_l w_2 <_l w_1$, $n_i \geq 1$ and w_i are Lyndon words for all $1 \leq i \leq k$.*

3 Theoretical Results

The definition of (digital) convexity of a connected set S implies that the set can be described by a word on a four letters alphabet Σ that codes its border, say $Bd(S)$. Each of the four letters in $\Sigma = \{1, \overline{1}, 0, \overline{0}\}$ provides a step along the border in one of the four different directions North, South, East and West, respectively.

To reach a standard coding of the border of $Bd(S)$, it can be noticed that a convex set touches the border of its minimal bounding rectangle in four bars, called *(N)orth, (S)outh, (E)ast* and *(W)est foot*. Moving counterclockwise on the border of the set, let us denote the ending corner of each foot by N, W, S and E according to the correspondent foot, as shown in Fig. 3. The word $Bd(S)$ starts from W and runs clockwise along the border of S in a closed path: $Bd(S)$ can be factorized into four non-void sub-paths WN, NE, ES and SW each using only two of the four steps in Σ to connect the related points; such a factorization is called *standard*. We say that a word in $\{0, 1\}$ is WN*-convex* if it is the NW path of a convex set. The NE, ES, and SW *convexity* can be defined similarly. Obviously, a path is convex if its standard factorization is made by four paths that are NW, NE, ES, and SW convex.

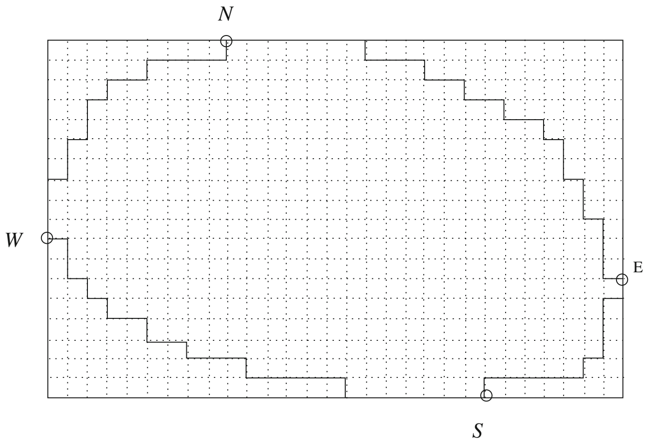


Fig. 3. A convex polyomino and its standard factorization. The word $w \in \{0,1\}^*$ coding the WN path is $w = 1110110110100100001$.

3.1 Perturbations on the WN Paths

From now on, we will consider the WN path only, assuming that all the properties hold for the other three paths up to rotations. In [4], the authors characterized the words that are border of a convex connected set by means of Lyndon and Christoffel words:

Property 2. A word w is WN-convex iff its unique Lyndon factorization $w_1^{n_1} w_2^{n_2} \ldots w_k^{n_k}$ is such that all w_i are primitive Christoffel words.

Such a result highlights the fact that a WN convex path is composed by line segments, i.e. Christoffel words, having a decreasing slope, so that they respect the lexicographical order and produce a Lyndon factorization. Furthermore, in the same paper, the authors pointed out that such a decomposition can be obtained in linear time.

In order to define a procedure that reconstructs convex sets from projections, we are interested in finding a set of loci of a WN convex path where it is possible to add one single point without loosing the convexity.

Let us consider a primitive Christoffel word w and define $min(w)$ to be the length of the prefix that reaches its minimal point as in Definition 5.

As an example, let $w = 00100101$ be a primitive Christoffel word; its minimal point is at position $(4, 1)$ reached by the prefix 00100 of w, so $min(w) = |00100| = 5$. Since we assume w to be primitive, then $min(w)$ is unique. Figure 4 shows a WN path and the minimal points of the Christoffel words.

By definition, we remark that if $k = min(w)$, then $w[k] = 0$, and $w[k+1] = 1$. The following property states that if we flip the elements of w at positions k and $k+1$ the obtained word w' is not a Christoffel word; on the other hand, it can be split into two words $w'[1, k]$ and $w'[k + 1, n]$, with $n = |w'|$, that are Christoffel words. Furthermore, position k is the only one allowing such a decomposition:

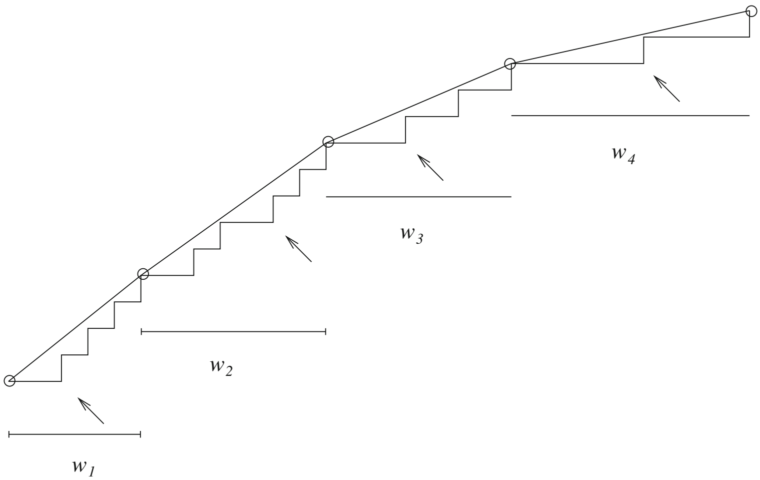


Fig. 4. A WN path and its decomposition into four Christoffel words w_1, w_2, w_3, and w_4 related to four line segments. The four minimal points of each segment are highlighted.

Proposition 1. *Let w be a primitive Christoffel word of length n and $k = min(w)$.*

(i) The words $u = w[1, k-1]\,1$ and $v = 0\,w[k+2, n]$, are two Christoffel words;
(ii) for each k' different from k, the words $u' = w[1, k'-1]1$ and $v' = 0w[k'+2, n]$ are not both Christoffel words.

The proof is a direct consequence of Property 1 and Lemma 1. We point out the following immediate and useful consequence:

Corollary 1. *Let w, u and v be as in Property 1. It holds*

(i) $\rho(u) > \rho(v)$;
(ii) $\rho(w[1, k']1) > \rho(u)$, for each $k' \neq k(= min(w))$, and $w[k'] = 0$. A symmetric result holds for v.

3.2 Definition of the *split* Operator

Proposition 1 allows us to define a *split operator*, that acts on the Christoffel word w and provides as output the concatenation of the two words u and v, by simply flipping the sequence 01 at position k and $k+1$ of w into the sequence 10, i.e., $split(w) = u\,v$. From now on, we consider the extension of the operator to *sequences* of Christoffel words, and we index it with the (index of the) sub-word where the split takes place, i.e., if $w = w_1\,w_2 \ldots w_n$ is a sequence of primitive Christoffel words, then $split_k(w) = w_1\,w_2 \ldots split(w_k) \ldots w_n$. Consecutive applications of the split operator to the word w will be indexed by the sequence of the indexes of the involved sub-words.

Given two words p_1 and p_2 of the same length l, we say that p_1 is *greater* than or equal to p_2 ($p_1 \geq p_2$), if for each $k \leq l$ it holds $|p_1[1 \cdots k]|_1 \geq |p_2[1 \cdots k]|_1$. The "$\geq$" relation is a natural partial ordering on words.

As an immediate consequence we have:

Property 3. Let $w = w_1\, w_2 \ldots w_n$ be a sequence of Christoffel words. It holds:

(i) $split_k(w) \geq w$, with $k \leq n$;
(ii) the split operator commutes with respect to successive applications, i.e., $split_{k,h}(w) = split_{h,k}(w)$.

Attention must be paid when we are dealing with sequences of Christoffel words that are paths of a convex polyomino: the split operator provides an

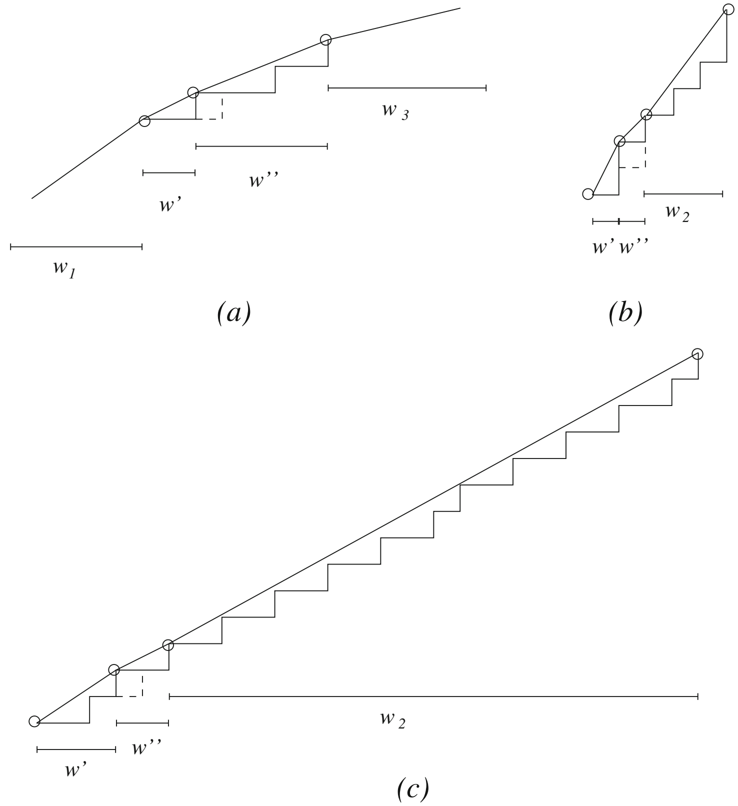


Fig. 5. Three WN paths of a convex polyomino: (a) the Lyndon factorization and the global convexity are preserved; (b), the global convexity is preserved, but not the Lyndon factorization that has to be modified including w'' in w_2, i.e., $w_2' = w''\, w_2 = (01)(0101011)$; ($c$) the global WN convexity is lost. Observe that the word $w''w_2 = (001)(0010010010010010100100100100101)$ is not a Christoffel word. So, we need a second split in w_2 and the addition of a further point to obtain a WN path back.

efficient way to add one point on a line segment of the border of the polyomino without loosing the convexity on that segment, but it does not guarantee either to preserve the Lyndon factorization of the related word or its convexity.

We can classify the perturbations performed by the split operator on the factor w_i (i.e. $split(w_i) = u_i\, v_i$) of the Lyndon decomposition of a convex path into three different types, according to the values of the slopes of the consecutive Lyndon factors after the perturbation:

(a) the Lyndon factorization and the global convexity are preserved (see Fig. 5 (a)), i.e., the two new factors u_i and v_i globally preserve the slope decreasing of the line segments of the path;
(b) the Lyndon factorization is not preserved but the obtained path is still convex (see Fig. 5 (b)), i.e., $w_{i-1}u_iv_iw_{i+1}$, with w_{i+1} eventually void, is not a Lyndon factorization, so it does not preserve the slope decreasing of the line segments of the path, while the new Lyndon factorization does;
(c) neither the Lyndon factorization nor the convexity are preserved (see Fig. 5 (c)), i.e., $w_{i-1}u_iv_iw_{i+1}$, with w_{i+1} eventually void, is not a Lyndon factorization. Furthermore, the new Lyndon factorization is not composed by Christoffel words only.

3.3 Commutativity of the *split* Operator

In what follow, we are interested in showing under which assumptions the commutative behavior of the split operator is preserved in a WN path (as already underlined, by symmetry the found results hold for the remaining three kinds of paths). In particular, we are going to show that, if the split operator produces perturbations of type (a) on two consecutive Christoffel words in a same octant of a WN path, then the result of the two successive applications is independent from their appliance order.

Theorem 3. *Let w_1 and w_2 be two consecutive Christoffel words in the same octant of a WN path of a polyomino, and let $split(w_1) = u_1\, v_1$ and $split(w_2) = u_2\, v_2$. If $\rho(v_1) > \rho(w_2)$ and $\rho(w_1) > \rho(u_2)$ (i.e. the split operator provides two perturbations of type (a) on w_1 and w_2), then it holds $\rho(v_1) > \rho(u_2)$.*

Proof. Let $\rho(u_2) = \frac{b}{a}$ and $\rho(v_1) = \frac{b'}{a'}$, and assume without loss of generality that w_1 and w_2 lie in the upper octant, i.e., $a > b$, and $a' > b'$. Since the split operator acts on $min(w_1)$ and $min(w_2)$, then it holds

$$\frac{b-1}{a-1} < \rho(w_2) < \frac{b}{a}, \text{ and } \frac{b'-1}{a'-1} < \rho(w_1) < \frac{b'}{a'}. \tag{1}$$

Let us proceed by contradiction, assuming that $\rho(v_1) < \rho(u_2)$, i.e.,$ab' < ba'$. We first prove that the inequality

$$\frac{b'-1}{a'-1} < \frac{b-1}{a-1} \tag{2}$$

is always satisfied: several cases have to be considered according to the mutual dimensions of the four parameters a, a', b, and b':

Case (1) $a = a'$. From $a = a'$ it follows $b' < b$ and consequently $b' - 1 < b - 1$, so Inequality 2 holds. The case $b = b'$ is symmetrical.

Case (2) $a < a'$. Two subcases arise: if $b' < b$, then we have $b' - 1 < b - 1$. Since we also have $a - 1 < a' - 1$, then Inequality 2 again is a direct consequence. On the other hand, let $b' > b$. In this case we show that a contradiction arises, i.e., $\rho(w_1) < \rho(u_2)$, against the hypothesis.
The Christoffel tree is isomorphic to the Stern-Brocot tree that contains all the irreducible fractions. The fractions are distributed all over the tree using the Farey addition, which is:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

Let the two Christoffel words w_1 and w_2, of slopes respectively $\rho(w_1)$ and $\rho(w_2)$, be split into Christoffel sub-words as $w_1 = u_1\, v_1$ and $w_2 = u_2\, v_2$, as shown in Fig. 6.

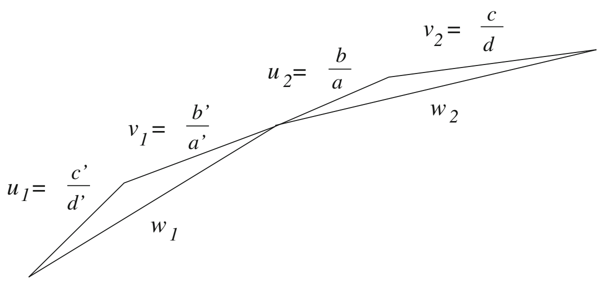


Fig. 6. The split of the Christoffel words w_1 and w_2 into (u_1, v_1) and (u_2, v_2).

We let, $\rho(u_1) = \frac{c'}{d'}$, $\rho(v_1) = \frac{b'}{a'}$, $\rho(u_2) = \frac{b}{a}$, and $\rho(v_2) = \frac{c}{d}$.

Using the construction of the Stern-Brocot tree, we know that there exist $k, t \in \mathbb{N}$ such that $c' = b' - k$ and $d' = a' - t$.

Since, by assumption, $b' > b$ and $a' > a$, there also exist $k', t' \in \mathbb{N}$ such that $b = b' - k'$ and $a = a' - t'$.

The following inequalities hold:

$$\begin{aligned} \frac{b'}{a'} &< \frac{b}{a} \\ ab' &< a'b \\ a'b' - a'b &< a'b' - ab' \\ \frac{b'-b}{a'-a} &< \frac{b'}{a'} \\ \frac{k'}{t'} &< \frac{b'}{a'} \\ 2a'k' - 2b't' &< 0. \end{aligned}$$

Reminding that we assumed to be confined in the first octant where the slopes of the Christoffel words are less than 1, then $b < a$ implies $b' - k' < a' - t'$. Therefore: $-(a' - t') < -(b' - k')$ and, since by the Stern-Brocot tree we have $\frac{c'}{d'} \oplus \frac{k}{t} = \frac{b'}{a'}$, then $\frac{k}{t} < 1$ and consequently $k < t$, then we get

$$-k(a' - t') < -t(b' - k') \text{ i.e. } -k(a' - t') + t(b' - k') < 0.$$

The inequalities gathered up to now lead to the following ones: $\rho(w_1) = \frac{b'+c'}{a'+d'} = \frac{2b'-k}{2a'-t}$ and $\rho(u_2) = \frac{b}{a} = \frac{b'-k'}{a'-t'}$ that are enough to prove that $\rho(w_1) < \rho(u_2)$ always holds against the hypothesis. In fact

$$2b'a' - 2b't' - ka' + kt' < 2a'b' - 2a'k' - b't + k't$$
$$2a'k' - 2b't' - k(a' - t') + t(b' - k') < 0$$

which is always true since $2a'k' - 2b't' < 0$ and $-k(a' - t') + t(b' - k') < 0$.

Case (3) $a > a'$. As above, two subcases arise:

- if $b < b'$, then since $\frac{1}{a} < \frac{1}{a'}$ we get $\frac{b}{a} < \frac{b'}{a'}$, i.e., a contradiction to the initial hypothesis.
- The other case concerns $b > b'$. Let us consider $a = a' + h$ and $b = b' + k$ (see Fig. 7), and consequently Inequality 2 can be written as

$$\frac{b' + k - 1}{a' + h - 1} - \frac{b' - 1}{a' - 1} > 0. \quad (3)$$

Since by hypothesis, we have $\frac{b'}{a'} = \frac{b-k}{a-h} < \frac{b}{a}$, hence $ab - ak - ab + hb < 0$ and $\frac{b}{a} < \frac{k}{h}$ as shown in Fig. 7. Therefore the following relations hold:

$$\frac{k}{h} > \frac{b}{a} > \frac{b'}{a'} > \frac{b' - 1}{a' - 1}.$$

By the last one, it holds $k(a'-1) > h(b'-1)$ and consequently $ka'-hb'-k+h > 0$ that is equivalent to Inequality 3.

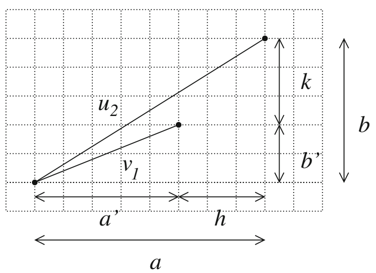


Fig. 7. An example of the case $a > a'$ and $b > b'$ in the proof of Theorem 3.

In the 3 cases, and assuming that $\rho(v_1) > \rho(u_2)$, we obtain Inequality 2. The following inequalities are deduced:

$$\frac{b'-1}{a'-1} < \frac{b-1}{a-1} < \rho(w_2) < \frac{b'}{a'} < \frac{b}{a} < \rho(w_1)$$

and, comparing with the first inequality of the second chain in Inequality 1, we get a contradiction. □

Corollary 2. *After performing a sequence of perturbations of type* (a) *in a* WN *path, then the obtained path is still a* WN *path.*

This last result can be rephrased by saying that the split operator commutes in case of perturbations of type (a) inside the same octant. A further analysis has to be carried on in presence of perturbations of type (b), both in the same octant and in the whole quadrant.

Acknowledgment. This study has been partially supported by INDAM - GNCS Project 2017.

References

1. Barcucci, E., Del Lungo, A., Nivat, M., Pinzani, R.: Reconstructing convex polyominoes from horizontal and vertical projections. Theoret. Comput. Sci. **155**(2), 321–347 (1996). http://dx.doi.org/10.1016/0304-3975(94)00293–2
2. Bernoulli, J.: Sur une nouvelle espèce de calcul, pp. 255–284. Recueil pour les astronomes
3. Borel, J.P., Laubie, F.: Quelques mots sur la droite projective réelle. J. Théor. Nombres Bordeaux **5**(1), 23–51 (1993). http://jtnb.cedram.org/item?id=JTNB_1993_5_1_23_0
4. Brlek, S., Lachaud, J.O., ProvenÃSS, X., Reutenauer, C.: Lyndon+christoffel = digitally convex. Pattern Recogn. **42**(10), 2239–2246 (2009). Selected papers from the 14th IAPR International Conference on Discrete Geometry for Computer Imagery 2008, http://www.sciencedirect.com/science/article/pii/S0031320308004706
5. Chaudhuri, B.B., Rosenfeld, A.: On the computation of the digital convex hull and circular hull of a digital region. Pattern Recogn. **31**(12), 2007–2016 (1998)
6. Chen, K.T., Fox, R.H., Lyndon, R.C.: Free differential calculus, IV: the quotient groups of the lower central series. Ann. Math. **68**(2), 81–95 (1958). http://dx.doi.org/10.2307/1970044
7. Christoffel, E.: Observatio arithmetica. Annali di Matematica **6**, 145–152 (1875)
8. Debled-Rennesson, I., Jean-Luc, R., Rouyer-Degli, J.: Detection of the discrete convexity of polyominoes. In: Borgefors, G., Nyström, I., Baja, G.S. (eds.) DGCI 2000. LNCS, vol. 1953, pp. 491–504. Springer, Heidelberg (2000). doi:10.1007/3-540-44438-6_40
9. Sklansky, J.: Recognition of convex blobs. Pattern Recogn. **2**(1), 3–10 (1970)