Constant Mean Curvature Hypersurfaces in the Steady State Space: A Survey

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Abstract In this survey, we review recent progress in the theory of spacelike hypersurfaces with constant mean curvature in the steady state space. Using the different models of this space, we outline the major concepts, techniques, and results with a special focus on Bernstein-type theorems, hypersurfaces with boundary in a slice, and the Dirichlet problem for the constant mean curvature equation.

Keywords Steady state space · Spacelike hypersurface · Mean curvature
Tangency principle · Omori-Yau maximum principle · Dirichlet problem

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1 Introduction

The *steady state space* $\mathcal{H}^{n+1}$ is the space $\mathbb{R}^{n+1}_{+} = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : x_{n+1} > 0\}$ endowed with the Lorentzian metric

$$g_{(x,x_{n+1})} = \frac{1}{x_{n+1}^2}(|dx|^2 - (dx_{n+1})^2).$$

Thus, $\mathcal{H}^{n+1} = (\mathbb{R}^{n+1}_{+}, g)$ is the Lorentzian analogue to the hyperbolic space $\mathbb{H}^{n+1}$. From the physical viewpoint, and for $n = 3$, $\mathcal{H}^4$ is a model of the universe proposed by Bondi and Gold [15] and Hoyle [34] under the belief in the "perfect cosmological principle", that is, the space looks the same not only at all points and in all directions (homogeneous and isotropic), but also at all times ([32, Sect. 5.2]). In particular, this model postulates a continuous creation of matter in order to be consistent with the idea of an unchanging universe, existing old and young galaxies in any large

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volume of space which are continuously forming by accretion of new matter. This cosmological model attracted the interest of physicists during part of the twentieth century but nowadays the steady state space has been discarded because it does not predict many physical observations (in contrast to the Big Bang model), as the abundance and the proportion of helium and hydrogen, or the evolution of stars and galaxies. Finally, it was the discovery of the cosmic microwave background (CMB) in late 1964, the most clear evidence against this model: in the initial stages, the universe was denser and hotter than now because it dilutes and cools as it expands. However, the steady state space forbids the existence of CMB because under this model, the density and the temperature are always the same; see [16] and especially [48, Chs. 14, 16].

If we come back to the steady state space viewed as a Lorentzian manifold, it opens up a wide variety of problems in the theory of submanifolds. Surprisingly, it has been until very recently that this space has gained the interest after the work of Montiel [42] in 2003 where, following ideas of [40], it is proved the existence of constant mean curvature spacelike hypersurfaces with boundary in the future infinity of $\mathscr{H}^{n+1}$. This pioneering article was the starting point which many geometers focused in the study of submanifolds in $\mathscr{H}^{n+1}$. Furthermore, this is accompanied by the property that $\mathscr{H}^{n+1}$ can be viewed equivalently in two different coordinates. First, as an open set of the de Sitter space $\mathbb{S}_1^{n+1}$ (such as it appeared in the original works of Bondi and Hoyle). The space $\mathbb{S}_1^{n+1}$ has a high relevance in general relativity that it deserves to study or, as in our case, an open set of $\mathbb{S}_1^{n+1}$. A second model of $\mathscr{H}^{n+1}$ is as a generalized Robertson–Walker (GRW) spacetime and thus forming part of a large family of cosmological models which made that authors working in these spacetimes put their focus in the steady state space.

In this survey, we will study spacelike hypersurfaces with constant mean curvature in $\mathscr{H}^{n+1}$. From a physical viewpoint, the hypersurfaces with constant mean curvature (cmc to abbreviate) are convenient initial data for the Cauchy problem corresponding to the Einstein equations. In spacetimes, there is also an interest to have foliations by means of cmc spacelike hypesurfaces because all points of each leaf of the foliation are instantaneous observers and the timelike unit normal vector of the hypersurfaces measures how the observers get away with respect to the next ones. Our aim in this chapter is twofold. We first want to summarize the most important results in this topic, and second, we are intended to give an overview of the main methods that lie behind the results, especially, the tangency principle, the Omori-Yau maximum principle, or the continuity method for the Dirichlet problem. In this chapter, we also provide a new approach in some results trying to unify, if possible, the techniques of the different models.

This chapter is organized as follows. In Sect. 2 we give a description of the spacelike umbilical hypersurfaces and we present two new models for $\mathscr{H}^{n+1}$, each one will be conveniently employed depending on the problems that we address. Section 3 is devoted to characterize the slices of $\mathscr{H}^{n+1}$ in the class of complete cmc spacelike hypersurfaces obtaining Bernstein-type results and extending some of these results to GRW spacetimes. In the last part of this exposition, we study the shape of a compact cmc hypersurface in relation with its boundary (Sect. 4) and we derive existence results of the Dirichlet problem for the cmc equation (Sect. 5). This chapter ends with a list of open problems.

2 The Steady State Space: A Space and Three Models

We have defined the steady state space as $\mathscr{H}^{n+1} = (\mathbb{R}^{n+1}_+, g)$, and we will say that this is the upper half-space model for $\mathscr{H}^{n+1}$. If $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle , \rangle)$ stands for the Lorentz–Minkowski space, where $\langle x, y\rangle = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$, then $\mathscr{H}^{n+1}$ is nothing but the open set $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}^+$ with the conformal metric $g = \langle , \rangle / x^2_{n+1}$. The time orientation is determined by $e_{n+1} = (0, \ldots, 0, 1)$. As a consequence, the isometries of $\mathscr{H}^{n+1}$ are the conformal transformations of $\mathbb{R}^{n+1}_1$ that preserve the upper half-space $\mathbb{R}^{n+1}_+$, as for example, the rotations about a vertical straight line, the horizontal translations, or the homotheties from a point of $\mathbb{R}^n \times \{0\}$.

We now introduce two equivalent models for $\mathscr{H}^{n+1}$, or to be more precise, we present two types of change of coordinates in $\mathscr{H}^{n+1}$.

1. The de Sitter model. Consider the de Sitter space, that is, the hyperquadric $\mathbb{S}^{n+1}_1 = \{x \in \mathbb{R}^{n+2}_1 : \langle x, x\rangle = 1\}$ of all unit spacelike vectors in $\mathbb{R}^{n+2}_1$, and take $a \in \mathbb{R}^{n+2}_1$ a nonzero null vector in the past half of the null cone. The steady state space is the open region $\mathbb{S}^{n+1}_{1,+} = \{x \in \mathbb{S}^{n+1}_1 : \langle x, a\rangle > 0\}$ with the induced metric. The time orientation is determined by $e_{n+2} = (0, \ldots, 0, 1)$.
2. A GRW spacetime. The steady state space is the vector space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^n\}$ with the Lorentzian metric $-dt^2 + e^{2t}|dx|^2$. In other words, $\mathscr{H}^{n+1}$ is the generalized Robertson–Walker spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^n$. Historically, the cosmological model $\mathscr{H}^4$ proposed by Hoyle is $-\mathbb{R} \times_{e^{Ht}} \mathbb{R}^3$ where H is the Hubble constant. The timelike orientation is determined by the vector field ∂_t.

The expressions of the change of coordinates between the three models are the following. The isometry $\Psi : \mathbb{S}^{n+1}_{1,+} \to \mathbb{R}^{n+1}_+$ is

$$\Psi(x) = \frac{1}{\langle x, a\rangle} (x - \langle x, a\rangle b - \langle x, b\rangle a, 1)$$

where $b \in \mathbb{R}^{n+2}_1$ is a null vector such that $\langle a, b\rangle = 1$. This isometry reverses the time orientation. The isometry $\Phi : \mathbb{S}^{n+1}_{1,+} \to -\mathbb{R} \times_{e^t} \mathbb{R}^n$ is

$$\Phi(x) = \left(\log(\langle x, a\rangle), \frac{x - \langle x, a\rangle b - \langle x, b\rangle a}{\langle x, a\rangle} \right)$$

where $b \in \mathbb{R}^{n+2}_1$ is as above. This isometry preserves the time orientation. Finally, the isometry between the GRW model and the upper half-space model is $\Xi : -\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+1}_+$, $\Xi(t, x) = (x, e^{-t})$ which reverses the time orientation.

Each one of the models has its advantages. For example, it is easy to visualize the isometries in the upper half-space model. In the de Sitter model, the analytic calculations are easier, as for example, when in Sect. 3 we compute the Laplacian of certain functions. Finally, the GRW model allows to see $\mathcal{H}^{n+1}$ as a product manifold with a distinguished role to the fibers of the space and again the analytic calculations in this model are simple (not necessarily easy).

The steady state space has two boundaries at the infinity. The *past infinity* of $\mathcal{H}^{n+1}$ is $\mathcal{J}^- \equiv \{x_{n+1} = \infty\}$ (the nullhypersurface $L_0 = \{x \in \mathbb{S}_1^{n+1} : \langle x, a\rangle = 0\}$ or the vertical hyperplane $\{-\infty\} \times \mathbb{R}^n$ in the GRW model). On the other hand, the *future infinity* $\mathcal{J}^+$ corresponds with the limit hyperplane $\{x_{n+1} = 0\}$ (or $L_\infty = \{x \in \mathbb{S}_1^{n+1} : \langle x, a\rangle = \infty\}$ or $\{\infty\} \times \mathbb{R}^n$ in the GRW model).

Since $\mathcal{H}^{n+1}$ is an open set of $\mathbb{S}_1^{n+1}$, then $\mathcal{H}^{n+1}$ is a non-complete Lorentzian manifold with constant sectional curvature equal to 1. For example, if $\{e_i\}$ is the usual basis of $\mathbb{R}^{n+2}$, and $a = e_1 - e_{n+2}$, the geodesic $\gamma(s) = \cosh(s)e_2 + \sinh(s)e_{n+2}$ is defined only if $s > 0$ with $\lim_{s\to 0} \gamma(s) \in \mathcal{J}^-$ and $\lim_{s\to\infty} \gamma(s) \in \mathcal{J}^+$. Motivated by this example, and following Hawking and Ellis in [32], in the steady state space any fundamental observer has a future event horizon but no past particle horizon. There also exist other geodesics in $\mathcal{H}^{n+1}$ defined for all s, for instance, $\gamma(s) = \cosh(s)e_1 - \sinh(s)e_{n+2}$.

We restrict our interest into *spacelike hypersurfaces* of $\mathcal{H}^{n+1}$. More generally, an immersion $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ of a n-dimensional (connected) manifold Σ is said to be a *spacelike* hypersurface if the induced metric on Σ via ψ is Riemannian. Because the orthogonal subspace $(T_p\Sigma)^\perp$ is timelike and there is a time orientation in the ambient space $\mathcal{H}^{n+1}$, we can define a timelike unit normal vector field N on Σ in such a way that N lies in the future half of the null cone: this concludes that any spacelike hypersurface is orientable. If $\bar{\nabla}$ and ∇ denote the Levi-Civita connections of $\mathcal{H}^{n+1}$ and Σ, respectively, the Gauss equation is $\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ for all $X, Y \in \mathfrak{X}(\Sigma)$. Then the mean curvature vector is $\mathbf{H} = \mathrm{tr}(\sigma)/n$. When we write $\mathbf{H} = HN$, then H is called the *mean curvature* of the immersion. In terms of the shape operator A, namely, $AX = -\bar{\nabla}_X N$ for $X \in \mathfrak{X}(M)$, the mean curvature is $H = -\mathrm{tr}(A)/n$. If H is constant, we say that Σ has constant mean curvature and we abbreviated by *H-hypersurface* if we want to emphasize the value of the mean curvature.

Remark 2.1 Throughout the rest of this chapter, all the spacelike hypersurfaces will be oriented with the choice of N pointing to the future. We also keep the convention that the mean curvature H is computed with this choice of N. We know that the isometry Ψ between the de Sitter model and the upper half-space model reverses the time orientation, and the same occurs with the isometry $\Psi \circ \Phi^{-1}$ between the GRW model and the upper half-space model. Thus, a spacelike hypersurface with mean curvature H in the upper half-space model has mean curvature $-H$ in the de Sitter and GRW models.

In the upper half-space model, the mean curvature of a hypersurface Σ can be calculated if we know the mean curvature of Σ viewed as submanifold of $\mathbb{R}_1^{n+1}$. Indeed, since both metrics are conformal, if H' is the mean curvature of $\Sigma \subset \mathbb{R}_1^{n+1}$, then

$$H = x_{n+1}H' - (x_{n+1} \circ N') \tag{1}$$

where $N' = N/x_{n+1}$ is the Gauss map of $\Sigma \subset \mathbb{R}_1^{n+1}$.

An important family of submanifolds in $\mathscr{H}^{n+1}$ are the totally umbilical ones. By the conformality between the metric g and the Lorentzian metric $\langle , \rangle$, these hypersurfaces are the intersection of the umbilical hypersurfaces of $\mathbb{R}_1^{n+1}$ (hyperbolic planes and spacelike planes) with the half-space $\mathbb{R}_+^{n+1}$. First, let us introduce the following notation: for $c \in \mathbb{R}^{n+1}$ and $r > 0$, let

$$\mathbb{H}^n(r; c) = \{x \in \mathbb{R}_+^{n+1} : \langle x - c, x - c\rangle = -r^2\}.$$

Depending on the relation between c and r, this hypersurface has one or two connected components, namely, the upper one $\mathbb{H}_+^n(r; c)$ and the lower one $\mathbb{H}_-^n(r; c)$. We describe the *spacelike umbilical hypersurfaces* of $\mathscr{H}^{n+1}$, including the value of H with respect to the future-directed orientation following our convention of Remark 2.1.

1. A *slice* is a horizontal hyperplane

$$L_\tau = \{x \in \mathbb{R}^{n+1} : x_{n+1} = \tau\}, \quad \tau > 0.$$

   A slice is complete with $H = -1$. After an isometry, a slice is $\mathbb{H}_+^n(r; c)$ where $c_{n+1} = -r$.
2. An *equidistant hypersurface* is a hypersurface of type $\mathbb{H}_-^n(r; c)$. For the existence of $\mathbb{H}_-^n(r; c)$, it is necessary that $c_{n+1} > r$. This hypersurface is not complete and its mean curvature is $H = -c_{n+1}/r$. After an isometry, they are also non-horizontal spacelike hyperplanes or the upper component $\mathbb{H}_+^n(r; c)$ where $c_{n+1} < -r$.
3. A *hyperbolic plane* of center $c \in \mathbb{R}^{n+1}$ and radius $r > 0$ is a hypersurface of type $\mathbb{H}_+^n(r; c)$ where $c_{n+1} > -r$. A hyperbolic plane is complete with $H = c_{n+1}/r$.

Equation (1) allows to write in local coordinates the mean curvature which reveals us the local nature of a cmc hypersurface. Indeed, since a spacelike hypersurface in $\mathscr{H}^{n+1}$ is locally a graph $x_{n+1} = u(x)$ of a function $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n \times \{0\}$, by the expression of the mean curvature in $\mathbb{R}_1^{n+1}$ and (1), we obtain

$$Q_H[u] := \operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) - \frac{n}{u}\left(H + \frac{1}{\sqrt{1 - |Du|^2}}\right) = 0. \tag{2}$$

The spacelike condition of the graph is equivalent to $|Du|^2 < 1$. Equation (2) is of elliptic type with the remarkable property that the difference of two solutions of (2) satisfies a linear elliptic PDE, and consequently, we can apply the strong maximum principle of Hopf [31]. This extends the usual tangency principle of Euclidean space for cmc hypersurfaces [37]:

Proposition 2.1 (Tangency principle) *Let Σ_1 and Σ_2 be two spacelike H-hypersurfaces which are tangent at a common interior point p and the unit* *normal vectors coincide at p. If one surface lies on one side of the other in a neighborhood of p, then Σ_1 and Σ_2 coincide in an open set around p. The same holds if $p \in \partial \Sigma_1 \cap \partial \Sigma_2$ provided that the tangent spaces $T_p \partial \Sigma_i$ coincide.*

Remark 2.2 The above notion of Euclidean graph coincides with the one of graph in $\mathscr{H}^{n+1}$. Indeed, if $\Omega \subset L_\tau$ is a (smooth) domain and $f \in C^\infty(\Omega)$, the *graph* of f is the hypersurface $\Sigma_f = \{\gamma(f(x); x) : x \in \Omega\}$, where $\gamma = \gamma(s; x)$ is the geodesic passing by x and orthogonal to L_τ. In the upper half-space model, this geodesic is a vertical line so Σ_f writes as $\{(x, u(x)) : x \in \Omega\}$ for a certain function u. A first example of an entire graph is a slice L_τ which is the graph of the constant function $f(x) = 0$ (or $u(x) = \tau$ in the upper half-space model). In the GRW model, a graph is $\{(u(x), x) : x \in \Omega\}$ where the spacelike condition reads as $|Du|^2 < e^{2u}$.

3 Bernstein-Type Characterizations of Slices

From the above section, we know that there do not exist complete umbilical hypersurfaces of $\mathscr{H}^{n+1}$ with $H < -1$, and that slices (for $H = -1$) are the first such examples. Notice that the steady state space is foliated by means of slices, indeed, $\mathbb{R}^{n+1}_+ = \cup_{\tau>0} L_\tau$ which it is of interest in the cosmological model. Slices also appear, via the tangency principle, as natural barriers for the existence of cmc hypersurfaces: see Sects. 4 and 5. Due to this distinguished role, in this section, we address with the following

> **Problem 1:** Under what geometric assumptions must a complete cmc spacelike hypersurface of $\mathscr{H}^{n+1}$ be a slice?

In this context, the remarkable chapter of Albujer and Alías [3] (part of the Ph. Doctoral Thesis of Albujer [2]) starts a series of works characterizing the slices under certain boundedness assumptions. The purpose of this section is to provide a general view of these results and the techniques employed in their proofs. Here, we use the de Sitter model of $\mathscr{H}^{n+1}$ where a slice corresponds with $L_\tau = \{x \in \mathbb{S}^{n+1}_1 : \langle x, a\rangle = \tau\}$, $\tau > 0$, and its mean curvature is $H = 1$ following the convention of Remark 2.1. First, we need the following definition.

Definition 3.1 A spacelike hypersurface $\psi : \Sigma \to \mathscr{H}^{n+1} = \mathbb{S}^{n+1}_{1,+}$ is said to be *bounded away from the future infinity* (resp. *from the past infinity*) if there exists $\tau > 0$ such that $\psi(\Sigma) \subset \{x \in \mathscr{H}^{n+1} : \langle x, a\rangle \leq \tau\}$ (resp. $\psi(\Sigma) \subset \{x \in \mathscr{H}^{n+1} : \langle x, a\rangle \geq \tau\}$). We say that Σ is *bounded away from the infinity*, or that Σ lies *between two slices*, if Σ is bounded away from the past and from the future infinity.

This definition is coherent with the future and the past infinity of $\mathscr{H}^{n+1}$: since $\mathscr{J}^+$ corresponds with $\langle x, a\rangle = \infty$, if Σ lies "away" from $\mathscr{J}^+$, then $\psi(\Sigma)$ is bounded from above. For a complete spacelike hypersurface $\Sigma \subset \mathscr{H}^{n+1}$, the boundedness of this definition imposes strong restrictions to its topology. First, notice that the spacelike property of Σ implies that the orthogonal projection of Σ on any slice is a local diffeomorphism. If we now suppose that Σ is a complete hypersurface bounded away from the future infinity, then this projection is a covering map on $\mathbb{R}^n$, and because $\mathbb{R}^n$ is simply connected, then it is a diffeomorphism onto $\mathbb{R}^n$, so Σ is an entire graph (in particular, Σ is not compact). This is the reason why results answering to the problem 1 are called of Bernstein-type because in Lorentz–Minkowski space, Cheng and Yau proved that hyperplanes are the only maximal entire graphs in $\mathbb{R}^{n+1}_1$ [23].

Theorem 3.1 ([3]) *If $\Sigma \subset \mathscr{H}^{n+1} = \mathbb{S}^{n+1}_{1,+}$ is a complete spacelike H-hypersurface between two slices, then $H = 1$. Furthermore, if $n = 2$, then Σ is a slice.*

The proof of this theorem involves two ingredients that are the keys in many other results that will appear in this section. First, it is the use of a large family of results known as "maximum principles" in the class of elliptic equations and where the tangency principle (Proposition 2.1) is a first example. For Theorem 3.1, we use he *Omori-Yau maximum principle* which is a type of maximum principle at infinity for complete Riemannian manifolds whose Ricci curvature is bounded from below [43, 49]; see also Remark 3.3.

Lemma 3.1 (Omori-Yau). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(M)$ is a function bounded from above, then there exists a sequence of points $\{p_k\} \subset M$ such that*

$$\lim_{k\to\infty} u(p_k) = \sup_{\Sigma} u, \quad |\nabla u(p_k)| < \frac{1}{k}, \quad \textit{and} \quad \Delta u(p_k) < \frac{1}{k}. \tag{3}$$

The second ingredient is the use of appropriate functions to which we apply the maximum principles. For a spacelike hypersurface $\Sigma \subset \mathscr{H}^{n+1}$, these functions are the height function $p \mapsto \langle \psi(p), a\rangle$ (abbreviated simply $\langle p, a\rangle$) and the Gauss map $p \mapsto \langle N(p), a\rangle$ (or simply $\langle N, a\rangle$). Following (3), we need to know their Laplacians. Here, the de Sitter model reveals very useful for these calculations, obtaining

$$\Delta\langle p, a\rangle = -n\langle p, a\rangle + nH\langle N, a\rangle, \quad \Delta\langle N, a\rangle = |A|^2\langle N, a\rangle - nH\langle p, a\rangle. \tag{4}$$

See [3, 42]. The expression of $\Delta\langle p, a\rangle$ holds when H is not constant, but for $\Delta\langle N, a\rangle$ is necessary that H is constant. We point out the difference of (4) with formula (8) in [42] by the reverse sign on H according to our convention in Remark 2.1.

Proof (*of Theorem* 3.1) The height function $\langle p, a\rangle$ defined in Σ is bounded because Σ lies between two slices. Since N is future-directed, then $\langle N, a\rangle > 0$. Because $a = \langle p, a\rangle p - \langle N, a\rangle N + a^T$, where a^T is the tangent part of a on Σ, then

$$\begin{aligned} 0 = \langle a, a\rangle &= \langle p, a\rangle^2 - \langle N, a\rangle^2 + |a^T|^2 \geq \langle p, a\rangle^2 - \langle N, a\rangle^2 \\ &= (\langle p, a\rangle - \langle N, a\rangle)(\langle p, a\rangle + \langle N, a\rangle). \end{aligned} \tag{5}$$

We point out that $|\nabla\langle p,a\rangle|^2=|a^T|^2=\langle N,a\rangle^2-\langle p,a\rangle^2$, so $\langle N(p_k),a\rangle\to\sup_\Sigma\langle p,a\rangle$. A computation of the Ricci curvature of Σ gives

$$\mathrm{Ric}_\Sigma(X,X)=n-1+nH\langle AX,X\rangle+\langle AX,AX\rangle\geq n-1-\frac{n^2H^2}{4} \tag{6}$$

for any $X\in\mathfrak{X}(\Sigma)$, in particular, Ric_Σ is bounded from below. Using (3) and (4), we have

$$H<\frac{\langle p_k,a\rangle}{\langle N(p_k),a\rangle}+\frac{1}{nk\langle N(p_k),a\rangle},$$

and letting $k\to\infty$, we conclude $H\leq 1$. The same argument holds with the function $-\langle p,a\rangle$ because the infimum of $\langle p,a\rangle$ is positive since Σ is bounded away from the past infinity. This yields $H\geq 1$, so $H=1$. When $n=2$, one can invoke a result of Akutagawa to conclude that Σ is a slice [1]. However, and such as it is rightly pointed in [3], it is better the following argument. From (4) and (5), we get $\Delta\langle p,a\rangle\geq 0$ showing that $\langle p,a\rangle$ is a subharmonic function. As $n=2$ and $H=1$, from (6) we deduce $K_\Sigma\geq 0$. Since Σ is complete, a result of Huber asserts that Σ is parabolic [35], so the subharmonic bounded function $\langle p,a\rangle$ is, indeed, constant, showing that Σ is a slice. □

Remark 3.1 If we only assume that Σ is bounded away from the future infinity, the proof yields $H\leq 1$. Since Σ is diffeomorphic to $\mathbb{R}^n$, Σ can not be compact. Taking into account the inequality (6), Bonnet-Myers's theorem implies that Ric_Σ is not bounded from below by 0 and this forces to $H^2\geq 4(n-1)/n^2$. Thus we have $2\sqrt{n-1}/n\leq H\leq 1$. In the particular case $n=2$, we conclude $H=1$, that is, *a complete cmc spacelike hypersurface in* $\mathscr{H}^3$ *bounded away from the future infinity must be a slice*.

Remark 3.2 We will prove in Remark 5.1 the existence of complete spacelike H-hypersurfaces with $H>1$ and bounded away from the past infinity.

Here, we present a new approach of Theorem 3.1 using a clever application of the tangency principle in the upper half-space model. Recall our convention on H in Remark 2.1.

Theorem 3.2 *Let* $\Sigma\subset\mathscr{H}^{n+1}$ *be a spacelike* H*-hypersurface in the upper half-space model.*

(i) If Σ *is an entire graph bounded away from the future infinity, then* $H\geq -1$.
(ii) If Σ *is complete and bounded away from the past infinity, then* $H\leq -1$.

Proof For (i), we know that there exists $\tau_1>0$ such that $\Sigma\subset\{x_{n+1}\geq\tau_1\}$. Let $m>1$, and consider the equidistant hypersurface $P_r=\mathbb{H}^n_-(r;(0,\ldots,0,mr))$ whose mean curvature is $H=-m$. Let us observe that the vertex of P_r is $V_r=(0,\ldots,0,(m-1)r)$. Take $r>0$ sufficiently small so $(m-1)r<\tau_1$. Then $P_r\cap\Sigma=\emptyset$. Let q be the intersection point of Σ with the x_{n+1}-axis ($q_{n+1}>\tau_1$): this point does

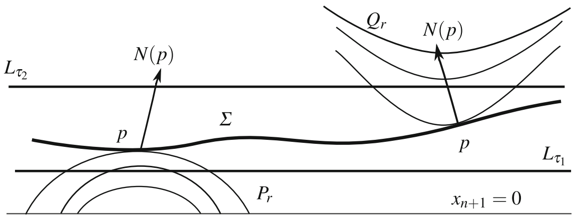

Fig. 1 Proof of Theorem 3.2

exist because Σ is a graph on $\mathbb{R}^n \times \{0\}$. Letting $r \to \infty$, we find a first value r_1, $(m-1)r_1 \leq q_{n+1}$, such that P_{r_1} meets the first time Σ at some point p; see Fig. 1, left. Since $\partial\Sigma = \emptyset$, then p is an interior common point of $\Sigma \cap P_{r_1}$, and the tangency principle says $H > -m$. Because this argument holds for any $m > 1$, we conclude $H \geq -1$.

For (ii), the completeness of Σ together the hypothesis on the boundedness says that Σ is a graph on $x_{n+1} = 0$ and $\Sigma \subset \{x_{n+1} \leq \tau_2\}$ for some $\tau_2 > 0$. The reasoning is similar replacing P_r by hyperbolic planes of type $\mathbb{H}^n_+(r; (0, \ldots, 0, mr))$ coming from $x_{n+1} = \infty$. The vertex of Q_r is $V_r = (0, \ldots, 0, (m+1)r)$ and $(m+1)r > 0$. If r is sufficiently big so $(m+1)r > \tau_2$, then $Q_r \cap \Sigma = \emptyset$. Letting $r \to 0$, we arrive until the first time $r_0 > 0$, $(m+1)r_0 \geq q_{n+1}$, such that Q_{r_0} meets Σ; see Fig. 1, right. The tangency principle gives $m > H$. Since this holds for any $m > -1$, then $H \leq -1$. □

Remark 3.3 (*A contact at "infinity"*) If in the above proof we use slices instead of equidistant hypersurfaces and hyperbolic planes we find with some troubles. For example, and for (i), suppose Σ lies above the slice L_{τ_1} but $\Sigma \not\subset L_t$ for all $t < \tau_1$. We can arrive from below with slices L_τ with $\tau < \tau_1$ without touching Σ, and it could occur that L_{τ_1} touches Σ at some point. Then, the tangency principle would say $\Sigma = L_{\tau_1}$ or $H > -1$ and this would prove the result. But it could happen that L_{τ_1} has a contact "at infinity" with Σ, that is, $L_{\tau_1} \cap \Sigma = \emptyset$ but $L_{\tau_1+\varepsilon} \cap \Sigma \neq \emptyset$ for all $\varepsilon > 0$. In such a case, we can not apply the tangency principle. This illustrates the difference between the tangency principle, which is utilized in local arguments, and the Omori-Yau maximum principle, which considers a touching point "at the infinity" by taking a sequence of points $\{p_k\}$ with the properties of (3).

Theorem 3.2 shows part of Theorem 3.1. It would remain to prove that a spacelike entire graph in $\mathscr{H}^3$ with $H = -1$ between two slices must be a slice. Suppose that Σ lies between the slices L_{τ_1} and L_{τ_2}, $\tau_1 < \tau_2$, and Σ does not lie in another smaller slab. By the tangency principle, it is not possible that Σ has a contact point with L_{τ_1} and L_{τ_2} (see Remark 3.3). Now the statement that we want to prove has the same flavor than the strong half-theorem in Euclidean space for minimal surfaces ([33]; see [44] in hyperbolic space). If in $\mathbb{R}^3$, the proof compares a minimal surface with a family of catenoids, in $\mathscr{H}^3$ a similar idea would be comparing Σ with rotational spacelike surfaces with $H = -1$. In the upper half-space model, and by Eq. (1), a rotational spacelike surface with respect to the x_3-axis has mean curvature H if the profile curve $\alpha(t) = (x(t), 0, z(t))$ satisfies

$$H = \frac{z(t)}{2}\left(\phi'(t) + \frac{\sinh(\phi(t))}{x(t)}\right) - \cosh(\phi(t))$$

where $\alpha'(t) = (\cosh\phi(t), 0, \sinh\phi(t))$. The initial conditions are $x(0) = 0$, $z(0) = z_0 > 0$, and $\phi(0) = \lambda$. Take $H = -1$, and let S_λ denote the rotational surface determined by the parameter λ. If $\lambda = 0$, the unique solution is the slice $x_3 = z_0$. If $\lambda < 0$ (resp. $\lambda > 0$), α is a graph on the x_1-axis with a singularity at $t = 0$, the function z is strictly decreasing (resp. increasing), and there exists $z_\lambda \geq 0$ such that $\lim_{t\to\infty} z(t) = z_\lambda$. This proves that S_λ is asymptotic to a horizontal hyperplane (a slice) and thus the use of S_λ is not adequate because we can not avoid a contact "at infinity" (such as it appeared in Remark 3.3 comparing Σ with slices).

The paper [3] motivated a series of works by different researchers that followed two directions: first, assuming other boundedness of the height function or with other r-mean curvatures, and second, extending to GRW spacetimes.

3.1 Other Assumptions on the Height Function

Caminha and de Lima replaced in [21] the assumption that Σ lies between two slices by some control on the growth of the height function. Let Σ denote a spacelike hypersurface in the GRW model $-\mathbb{R} \times_{e^t} \mathbb{R}^n$ which is oriented according to the future-directed orientation. The height function on Σ is $h = \pi_{\mathbb{R}} \circ \psi$, and since $\langle N, \partial_t\rangle \leq -1$, the *hyperbolic angle* is the function $\theta : \Sigma \to [0, \infty)$ given by $\cosh\theta = -\langle N, \partial_t\rangle$. Let us observe that e^h takes the same value that the function $\langle p, a\rangle$ in $\mathbb{S}^{n+1}_{1,+}$ by the isometry Φ. If $g = -\langle N, \partial_t\rangle$, then $|\nabla h|^2 = g^2 - 1$ and $\Delta h = 1 - n + nHg - g^2$. Hence, we obtain the Laplacian of the functions $v = e^h$ and $\eta = e^h\langle N, \partial_t\rangle$:

$$\Delta v = -nv + nH\eta, \quad \Delta\eta = nHv + |A|^2\eta, \tag{7}$$

where in the computation of $\Delta\eta$ we use that H is constant. Both equations are the analogous ones to (4) in the GRW model.

Theorem 3.3 ([21]) *Let $\Sigma \subset \mathscr{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n$ be a complete spacelike H-graph with $H \geq 1$. If $h \leq -\log(-\langle N, \partial_t\rangle - 1)$, then $H = 1$.*

Here, the growth of the height function h is bounded, in some sense, by the hyperbolic angle g. The proof uses (7) to conclude that the function $-v - \eta$ is subharmonic and the relation between h and g says exactly that this function is bounded, which allows to use the Omori-Yau maximum principle. In fact, we can replace the hypothesis on h by $h \leq -c\log(g - 1)$ for some $c > 0$. We give here the proof in the de Sitter model.

Proof Consider the function $\varphi = \langle N, a\rangle - \langle p, a\rangle$. The hypothesis says that φ is bounded from above so there exists $\sup_\Sigma \varphi$. Note that $\varphi \geq 0$ by (5). Then (4) yields

$$\begin{aligned}\Delta\varphi &= (|A|^2 - nH)\langle N, a\rangle + n(H-1)\langle p, a\rangle \\ &\geq (nH^2 - nH)\langle N, a\rangle + n(H-1)\langle p, a\rangle \\ &= n(H-1)(H\langle N, a\rangle - \langle p, a\rangle) \geq n(H-1)\varphi \geq 0\end{aligned}$$

where we have used $|A|^2 \geq nH^2$ and $H \geq 1$. This proves that φ is a subharmonic function, and taking the sequence $\{p_k\}$ of (3), the above inequality of $\Delta\varphi$ implies

$$0 \leq n(H-1)\varphi(p_k) \leq \Delta\varphi(p_k) < \frac{1}{k}.$$

By contradiction, suppose $H > 1$. Letting $k \to \infty$, we conclude $\sup_\Sigma \varphi = 0$, so $\varphi = 0$ on Σ. Then $\langle N, a\rangle = \langle p, a\rangle$ and $|\nabla\langle p, a\rangle|^2 = 0$, proving that Σ is a slice, that is, $H = 1$, a contradiction. This proves the theorem. □

In the following result, Camargo, Caminha, and de Lima replace the constancy of H by $H \geq 1$ and the integrability of the gradient of the height function. Now we do not conclude $H = 1$ (as in Theorems 3.1 and 3.3), but that Σ is a slice.

Theorem 3.4 ([18]) *Let $\Sigma \subset \mathscr{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n$ be a complete spacelike hypersurface between two slices with (not necessarily constant) mean curvature $H \geq 1$. If $|\nabla\langle p, a\rangle = |a^T|$ is Lebesgue integrable, then Σ is a slice.*

Proof By contradiction, suppose that $H > 1$ on Σ. From (4), $\Delta\langle p, a\rangle = n\langle HN - p, a\rangle)$. As $\langle HN - p, HN - p\rangle = 1 - H^2 < 0$, then $HN - p$ is a timelike vector on Σ, so $\langle HN - p, a\rangle$ is positive on Σ or its negative on Σ. Then $\Delta\langle p, a\rangle \geq 0$ or $\Delta\langle p, a\rangle \leq 0$. Up to a change of a sign, the function $\langle p, a\rangle$ is a subharmonic function which is bounded from above because Σ lies between two slices. Since Σ is complete and $|\nabla\langle p, a\rangle| \in \mathscr{L}^1(\Sigma)$, then $\langle p, a\rangle$ is harmonic by a result of Yau in [50]. Then $\Delta\langle p, a\rangle = 0$ and we derive that the timelike vector $HN - p$ satisfies $\langle HN - p, a\rangle = 0$, a contradiction. This proves that $H = 1$ on Σ. From (5), we have $\langle N, a\rangle - \langle p, a\rangle \geq 0$, and thus $\Delta\langle p, a\rangle \geq 0$ and $\langle p, a\rangle$ is bounded from above. The same argument as before proves that $\Delta\langle p, a\rangle = 0$ so $\langle N, a\rangle - \langle p, a\rangle = 0$, $|a^T| = 0$, and consequently, Σ is a slice. □

When $n = 2$, the same conclusion holds if we replace $|a^T| \in \mathscr{L}^1(\Sigma)$ by $K_\Sigma \geq 0$. This was proved by Aquino et al. in [14] and the argument is the same as in Theorem 3.1: now Σ is a parabolic surface, so the subharmonic function $\langle p, a\rangle$ must be constant, proving that Σ is a slice.

Finally, subsequent results in the literature replace the conditions on H by others about the higher order mean curvatures H_r. In order to use similar arguments, one needs, among other things, to extend the Laplacian operator and the Omori-Yau maximum principle. For the Laplacian, we take the so-called *r-th Newton transformations* P_r which are self-adjoint linear transformations on Σ involving H_r and the shape operator A. Then we define the second-order linear differential operator $L_r = \text{tr}(P_r \circ \text{Hess})$: for instance, $L_0 = \text{tr}(\text{Hess})$ is nothing that the Laplacian operator Δ. For $r = 1$, we have the known Yau's square operator $\Box = \text{tr}(P_1 \circ \text{Hess})$. It is also necessary to generalize the Omori-Yau maximum principle (this was done for the operator $\Box$ by Caminha and de Lima in [20, 22]). Besides that, the ellipticity of L_r is not assured and it depends on bounds on H_r. All these considerations make a bit difficult to give a clear statement of the results: we refer the interested reader to [4, 7, 12, 13, 18, 25], where much more of the results hold in GRW spacetimes. We refer also the reader to [9] for a recent account on Omori-Yau-type maximum principles for more general operators and its geometric applications.

3.2 Extension to GRW Spacetimes

A second scenario to extend Theorem 3.1 is by considering GRW spacetimes because of the third model for $\mathscr{H}^{n+1}$. We first review some basics of these spaces (we refer the article of Alías, Romero and Sánchez [11]). A *generalized Robertson–Walker* (GRW) spacetime is the product manifold $\bar{M} = I \times M$ endowed with the Lorentzian metric $-dt^2 + f^2\langle , \rangle_M$: here $(M^n, \langle , \rangle_M)$ is a n-dimensional Riemannian manifold, I is a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$), and $f : I \to \mathbb{R}$ is a positive smooth function, called the *warping function*. We denote this space as $-I \times_f M$. Among examples of GRW spacetimes, we have the Lorentz–Minkowski space $\mathbb{R}_1^{n+1} = -\mathbb{R} \times_1 \mathbb{R}^n$, de Sitter space $\mathbb{S}_1^{n+1} \equiv -\mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$, and the steady state space $\mathscr{H}^{n+1} \equiv -\mathbb{R} \times_{e^t} \mathbb{R}^n$. If $f(t) = e^t$, we say that $\bar{M}$ is a *steady state type space*, where besides $\mathscr{H}^{n+1}$, we point out the remarkable *de Sitter cusp space* $-\mathbb{R} \times_{e^t} \times \mathbb{T}^n$ in terminology of Galloway [29], where $\mathbb{T}^n$ is a flat n-torus.

A slice in $-I \times_f M$ is a hyperplane $L_\tau = \{\tau\} \times M$, for some $\tau \in I$. Then L_τ is an umbilical spacelike hypersurface with $H = f'(\tau)/f(\tau)$ computed with respect to $N = \partial_t$. Again, the question that we address is under what geometric assumption must a complete spacelike cmc hypersurface be a slice. The functions $\langle p, a\rangle$, and $\langle N, a\rangle$ used for the steady state space $\mathscr{H}^{n+1}$ correspond now with the height function $h = \pi_{|I} \circ \psi$ and the hyperbolic angle $\langle N, \partial_t\rangle$. If one wants to use the same techniques done in the previous results, it is necessary to consider the following remarks:

1. In $\mathbb{S}_{1,+}^{n+1}$, we computed the Laplacian of $\langle p, a\rangle$ in order to use the Omori-Yau maximum principle. In the GRW model, $\langle p, a\rangle$ corresponds with the function e^h which is nothing that $f(h)$ for the warping function $f(t) = e^t$. Thus, it is natural to compute the Laplacian of $f(h)$ in a GRW spacetime obtaining (see [26]):

$$\Delta f(h) = \left(\frac{f''f - f'^2}{f}(h)\right)|\nabla h|^2 - n\left(\frac{f'^2}{f}(h) + f'(h)H\langle N, \partial_t\rangle\right).$$

   If we want f to be subharmonic, then it is enough that both summands are not negative. For the first one is equivalent to say that $\log(f)$ is a convex function.

2. For the Omori-Yau maximum principle, we need that Ric_Σ is bounded from below. Here, we recall that a spacetime obeys the *null convergence condition* (NCC) if the Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{M}$ satisfies $\overline{\mathrm{Ric}}(Z, Z) \geq 0$ for any null vector Z. In a GRW spacetime, this inequality is expressed in terms of the warping function as

$$\mathrm{Ric}_M \geq (n-1)\sup_I (ff'' - f'^2)\langle , \rangle_M = (n-1)\sup_I f^2(\log f)''\langle , \rangle_M. \quad (8)$$

3. The condition $H \geq 1$ in Theorems 3.3 and 3.4 can be viewed as a comparison between H and the mean curvature of each slice. In a GRW spacetime, we would need to relate H with the quotient f'/f.

An example of generalization of Theorem 3.1 that indicates the type of results that we are referring is the following:

Theorem 3.5 *Let Σ be a complete spacelike hypersurface between two slices in a GRW spacetime. Suppose $|\nabla h| \in \mathscr{L}^1(\Sigma)$.*

1. *If $f'(h)H \geq f'^2/f(h) > 0$ ([26]), or*
2. *If H is bounded and $f(h)H_2 \geq f'(h)H \geq 0$ ([6]),*

then Σ is a slice.

In both statements we see again a comparison criterion between mean curvature quantities without being constant. For example, in item 2, we have $H_2^2/H^2 \leq f'^2/f^2$ where the right-hand side is the square of the mean curvature of the slice L_τ. The reader can see other Bernstein-type results for complete hypersurfaces in GRW spacetimes in [5, 17, 19, 27, 28, 30, 45].

4 Compact Spacelike Hypersurfaces with Boundary

In this section, we study how the boundary of a compact cmc hypersurface affects on the shape of the whole hypersurface. First, we precise our setting. Let $\psi : \Sigma \to \mathscr{H}^{n+1}$ be a spacelike immersion of a compact hypersurface Σ, in particular, $\partial\Sigma \neq \emptyset$ and let $\Gamma \subset \mathscr{H}^{n+1}$ be a $(n-1)$-submanifold. We say that Σ is a *hypersurface with boundary* Γ if the restriction of the immersion ψ to the boundary $\partial\Sigma$ is a diffeomorphism onto Γ. We abbreviate by saying that Γ is the boundary of Σ, or that $\partial\Sigma = \Gamma$, or that Σ spans Γ. We address the following

> **Problem 2:** Given a compact $(n-1)$-submanifold Γ included in a slice, does the geometry of Γ impose restrictions to the shape of a compact spacelike cmc hypersurface spanning Γ?

Related to the above problem, we study three specific questions:

(i) Whether the boundary Γ determines the position and the height of the hypersurface that spans with respect to the slice containing Γ.

(ii) Whether the geometry of Γ imposes restrictions to the possible values H of mean curvatures of H-hypersurfaces spanning Γ.
(iii) Whether the symmetries of Γ are inherited by the whole hypersurface. In other words, suppose that Γ is invariant by a rigid motion $M : \mathscr{H}^{n+1} \to \mathscr{H}^{n+1}$, that is, $M(\Gamma) = \Gamma$. If Σ is a cmc spacelike hypersurface with $\partial\Sigma = \Gamma$, we ask if Σ is also invariant by M. The simplest case is when Γ is a geodesic sphere and whether Σ is a hypersurface of revolution.

In this section, we will use the notation $\mathbb{H}^n_+(r; c)$ or $\mathbb{H}^n_-(r; c)$, assuming that $c = (0, \ldots, 0, c)$.

4.1 Height Estimates

Let Σ be a compact spacelike H-hypersurface in the upper half-space model with $\partial\Sigma \subset L_\tau$. Notice that if $\partial\Sigma$ is a simple closed curve, then Σ is a graph because the spacelike condition says that the orthogonal projection from Σ in L_τ is a covering map onto a simply connected domain. A first result gives us the position of Σ with respect to L_τ depending whether $H < -1$ or $H > -1$, that is, comparing H with the mean curvature of L_τ.

Proposition 4.1 *Let Σ be a compact spacelike hypersurface with $\partial\Sigma \subset L_\tau$. If $H < -1$ (resp. $H > -1$, $H = -1$), then $x_{n+1} \geq \tau$ in Σ (resp. $x_{n+1} \leq \tau$, $\Sigma \subset L_\tau$).*

Proof It is enough to consider the case $H < -1$. By contradiction, suppose that there are points strictly below L_τ. Let $q \in \Sigma$ be the lowest point with respect to L_τ, and let $\bar{\tau} = q_{n+1}$. We place the slice $L_{\bar{\tau}}$ at q. The orientation of Σ and $L_{\bar{\tau}}$ coincide at q (both ones are pointing to the future, so pointing upward). Since Σ lies above $L_{\bar{\tau}}$ around q, a comparison of the mean curvatures between Σ and $L_{\bar{\tau}}$ yields $H \geq -1$, a contradiction. This proves that $x_{n+1} \geq \tau$ in Σ. Following with the same argument, and if $\bar{\tau} < \tau$ and $H = -1$, the tangency principle would say that Σ lies contained in $L_{\bar{\tau}}$, a contradiction again because $\partial\Sigma \subset L_\tau$ and $\tau \neq \bar{\tau}$. □

Other approach to Proposition 4.1 is studying $\Delta\langle p, a\rangle$ in the de Sitter model. In these coordinates, Proposition 4.1 says that if $H > 1$ (resp. $H < 1$, $H = 1$), then $\langle p, a\rangle \leq \tau$ (resp. $\langle p, a\rangle \geq \tau$, $\Sigma \subset L_\tau$). We know from (5) that $\langle p, a\rangle - \langle N, a\rangle \leq 0$. Let $H > 1$. Since $\langle N, a\rangle > 0$, then (4) yields

$$\Delta\langle p, a\rangle \geq -n\langle p, a\rangle + n\langle N, a\rangle \geq 0$$

and the maximum principle asserts that $\langle p, a\rangle \leq \max_{\partial\Sigma}\langle p, a\rangle = \tau$, obtaining Proposition 4.1. When $H \leq 0$, then (4) gives directly $\Delta\langle p, a\rangle \leq 0$ and the maximum principle concludes $\langle p, a\rangle \geq \tau$. It would remain the case $0 < H < 1$ which is not deduced directly from $\Delta\langle p, a\rangle$, and the reasoning is the following. As $0 < H < 1$, $\langle HN - p, HN - p\rangle = H^2 - 1 < 0$, and consequently, the vector $HN - p$ is timelike so $\langle HN - p, a\rangle > 0$ on Σ or $\langle HN - p, a\rangle < 0$ on Σ. At the farest point $q \in$

int(Σ) from L_τ, $|\nabla\langle p,a\rangle|(q)=0$ and (5) gives $\langle N(q),a\rangle=\langle q,a\rangle$. Then $\langle HN-p,a\rangle(q)=(H-1)\langle q,a\rangle<0$. This proves definitively that $\langle HN-p,a\rangle<0$ on Σ yielding $\Delta\langle p,a\rangle\leq 0$ and consequently, $\langle p,a\rangle\geq\tau$ by the maximum principle.

Once we know that Σ lies on one side of L_τ, we want to estimate, if possible, how far Σ rises up from L_τ. In Euclidean space, this height is less than $1/|H|$ for H-graphs whose boundary Γ lies in a hyperplane, and thus this estimate is independent on the size of Γ (here and for a general reference in Euclidean space, we refer [37]). Following ideas of Meeks, the usual manner is by considering a linear combination of the height function and the Gauss map, then prove that this function is subharmonic, and finally apply the maximum principle to get the desired estimates. From (4) and because $|A|^2-nH^2\geq 0$, we have

$$\Delta(-H\langle p,a\rangle+\langle N,a\rangle)=(|A|^2-nH^2)\langle N,a\rangle\geq 0 \tag{9}$$

proving that $-H\langle p,a\rangle+\langle N,a\rangle$ is subharmonic. The maximum principle asserts

$$-H\langle p,a\rangle+\langle N,a\rangle\leq\max_{\partial\Sigma}(-H\langle p,a\rangle+\langle N,a\rangle)=-H\tau+\max_{\partial\Sigma}\langle N,a\rangle. \tag{10}$$

However, and in contrast to the Euclidean case, we can not get a similar estimate. For example, from (10) and for $H>0$, we have

$$\tau+\frac{\min_\Sigma\langle N,a\rangle-\max_{\partial\Sigma}\langle N,a\rangle}{H}\leq\langle p,a\rangle \tag{11}$$

but this estimate is not given only in terms of the boundary and H: inequality (11) is a bizarre estimate because involves the function $\langle N,a\rangle$ in Σ.

Proposition 4.1 generalizes to GRW spacetimes employing similar arguments. We replace $\Delta\langle p,a\rangle$ by the Laplacian of any primitive F of the warping function f. If h is the height function and $g=\langle N,\partial_t\rangle\leq -1$, the computation of ΔF in [30] yields $\Delta F=-nh((\log f)'(h)+Hg)$. Then it is immediate following the result proved by García-Martínez and Impera.

Theorem 4.1 ([30]) *Let Σ be a compact spacelike hypesurface in a GRW spacetime $-I\times_f M$ with $\partial\Sigma\subset L_\tau$.*

1. *If $H\geq\max\{0,\sup_I(\log f)'\}$, then $h\leq\tau$.*
2. *If $H\leq\min\{0,\inf_I(\log f)'\}$, then $h\geq\tau$.*

Let us observe that in $\mathscr{H}^{n+1}$, where $f(t)=e^t$, the above theorem says that if $H\geq 1$ (resp. $H\leq 0$), then $h\leq\tau$ (resp. $h\geq\tau$) but no information is obtained when $0<H<1$. If we proceed with the same arguments as in Eq. (9), and when H is constant, we consider the function $HF+fg$. Then

$$\Delta(HF+fg)=fg\left(|A|^2-nH^2+\mathrm{Ric}_M(N^*,N^*)-(n-1)(\log f)''|\nabla h|^2\right) \tag{12}$$

where $N^* = (\pi_M)_*(N)$. Because $\langle N^*, N^* \rangle_M = |\nabla h|^2/f^2$ and $|A|^2 \geq nH^2$, if we want to bound from below the parenthesis in (12), we need to estimate the Ricci curvature of M. Using (8) and the maximum principle, we have

Theorem 4.2 ([30]) *Let Σ be a compact spacelike H-hypersurface in a GRW spacetime satisfying NCC. Suppose $\partial\Sigma \subset L_\tau$. If f is nondecreasing function and $H \geq \max\{0, \sup_I (\log f)'\}$, then*

$$\tau - \alpha \leq h_{|\Sigma} \leq \tau, \quad \alpha = \frac{\frac{f(\tau)}{f(\min_\Sigma h)} \max_{\partial\Sigma}(-g) - 1}{H} \geq 0.$$

When we particularize to $\mathscr{H}^{n+1}$, then $H \geq 1$ and this estimate corresponds, up to a change of the models, with (11).

In the de Sitter model, and for $H > 1$, there exist compact H-hypersurfaces with boundary in a given slice and with arbitrary height as it is shown in the next example (we point out a gap in [24, Theorem 3.1] related with Remark 2.1).

Example 4.1 Consider the upper half-space model and H-hypersurfaces with $H < -1$. Fix the slice L_1. Consider the equidistant hypersurfaces $\mathbb{H}^{n+1}_-(r; c_r)$, $c_r = (0, \ldots, 0, -Hr)$, whose mean curvature is H. For $r > -1/(1+H)$, let $\Sigma_r = \mathbb{H}^n_-(r; c_r) \cap \{x_{n+1} \geq 1\}$, which is a compact H-hypersurface with $\partial\Sigma_r \subset L_1$. The height of Σ_r about L_1 is given by its vertex and this height is $\log|r(1+H)|$ which tends to ∞ as $r \to \infty$.

Following the above example, we observe that the boundary of Σ_r is a sphere of arbitrary large radius, namely, $\sqrt{(H^2-1)r^2 + 2Hr + 1}$. We now give an estimate of the height of a H-hypersurface with $H < -1$ depending only on H and the size of Γ.

Theorem 4.3 ([42]) *Let $\Gamma \subset L_\tau$ be a $(n-1)$-submanifold and let $\Omega \subset L_\tau$ denote the domain bounded by Γ. If $\Sigma \subset \mathscr{H}^{n+1}$ is a compact spacelike H-hypersurface with $\partial\Sigma = \Gamma$ and $H < -1$ in the upper half-space model, then the height h of Σ with respect to L_τ satisfies*

$$h \leq \log(h_0), \quad h_0 = \frac{-H + \sqrt{(H^2-1)R^2 + 1}}{1 - H}, \quad 2R = \operatorname{diam}\Omega. \tag{13}$$

Proof After an isometry of $\mathscr{H}^{n+1}$, suppose that $\tau = 1$ and let $B_R \subset L_1$ be the ball of radius $R > 0$ containing Ω inside: here R coincides with the Euclidean radius because the induced metric in L_1 is the Euclidean one in $\mathbb{R}^n \times \{1\}$. After a horizontal translation, if necessary, we assume the $(0, \ldots, 0, 1)$ is the center of B_R. Consider the equidistant hypersurfaces $\mathbb{H}^n_-(r; c_r)$ where $c_r = (0, \ldots, 0, -rH)$. For r sufficiently big, we can trap Σ in the convex domain of $\mathscr{H}^{n+1}$ determined between $\mathbb{H}^n_-(r; c_r)$ and $x_{n+1} \geq 1$. In particular, the disk determined by the sphere $L_1 \cap \mathbb{H}^n_-(r; c_r)$ contains B_R inside. Letting $t \searrow 0$, we arrive until the first value

$r = r_0$ such that $L_1 \cap \mathbb{H}^n_-(r_0; c_{r_0}) = \partial B_R$, just when $\mathbb{H}^n_-(r_0, c_{r_0})$ touches $\partial\Sigma$. During this process decreasing r, the tangency principle forbids the existence of an interior contact point between Σ and $\mathbb{H}^n_-(r; c_r)$ because both hypersurfaces have the same constant mean curvature. Thus, $r = r_0$ is the first time that $\mathbb{H}^n_-(r; c_r)$ meet Σ. This proves that the height of Σ is less than the height of $\mathbb{H}^n_-(r_0; c_{r_0})$, and this concludes (13). □

4.2 A Flux Formula

For the question (ii), we work in the de Sitter model following de Lima [36]. Let $\Gamma \subset L_\tau$ be a $(n-1)$-submanifold and let Ω denote the bounded domain that bounds Γ in L_τ. Let Σ be a compact spacelike H-hypersurface spanning Γ. The following argument follows the same steps as in Euclidean space. Consider the n-cycle $\Sigma \cup \Omega$ and define the Killing vector field in $\mathscr{H}^{n+1}$

$$Y_p = (\langle p, b\rangle a - \langle p, a\rangle b)/\langle a, b\rangle$$

where $b \in L_\tau$ with $\langle a, b\rangle \neq 0$. By the divergence theorem, we have

$$n|H||\Omega| = \left|\int_{\partial\Sigma} \langle Y_p, \nu\rangle\right| \tag{14}$$

where ν is the unit conormal vector field pointing to Σ and $|\Omega|$ is the volume of Ω. Identity (14) is usually called a *flux formula*. Let us observe that the left-hand side of (14) does not depend on Σ. Since $\langle p, a\rangle = \tau$ in L_τ, we have

$$n|H||\Omega| \leq \int_{\partial\Sigma} |\langle Y, \nu\rangle| = \int_{\partial\Sigma} \left|\frac{\langle p, b\rangle\langle \nu, a\rangle}{\tau} - \langle \nu, b\rangle\right|. \tag{15}$$

Although p, b, and ν are unit spacelike vectors, we cannot bound $\langle p, b\rangle$ and $\langle \nu, b\rangle$ by 1 (here we observe a gap in [36, p. 975]).

In Euclidean space $\mathbb{R}^{n+1}$, if $\partial\Sigma$ lies in a hyperplane, then (15) gives $|H| \leq |\Gamma|/((n-1)|\Omega|)$, obtaining an upper estimate for H depending *only on* Γ. For example, if Γ is a sphere of radius $r > 0$, then $|H| \leq 1/r$. However, in Lorentz–Minkowski space $\mathbb{R}^{n+1}_1$, and when the boundary lies in a spacelike hyperplane $< a >^\perp$, $|a| = 1$, the flux formula (14) gives $n|H||\Omega \leq \int_{\partial\Sigma} |\langle \nu, a\rangle|$ but we have the same problem than in (15) (see [8] for $n = 2$ and [10] for the general n-dimensional case).

We analyze the case when Γ is a sphere $\mathbb{S}^{n-1} \subset L_\tau$. Since L_τ is isometric to $\mathbb{R}^n$, in the upper half-space model, $\mathbb{S}^{n-1} \subset L_\tau$ is an Euclidean sphere in the hyperplane $x_{n+1} = \tau$. We have explicit examples of compact cmc hypersurfaces spanning a sphere obtained as follows. When we meet an umbilical hypersurface with a slice L_τ, we obtain a sphere $\mathbb{S}^{n-1}$ separating the umbilical hypersurface in two connected components. Depending on each case, there is at most one compact component, and called a *hyperbolic cap*. In fact, hyperbolic caps do exist always for equidistant hypersurfaces $\mathbb{H}^n_-(r;c)$, and only do exist for hyperbolic planes $\mathbb{H}^n_+(r;c)$ when $0 < c_{n+1} + r < \tau$. Furthermore, $\mathbb{S}^{n-1}$ determines a round disk called a *planar disk* in the very slice L_τ. Then we have:

Proposition 4.2 *Let $\Gamma = \mathbb{S}^{n-1}$ be a sphere of radius $\rho > 0$.*

1. *If $H < -1$, there exists a unique hyperbolic H-cap spanning Γ.*
2. *If $H = -1$, there exists a unique planar disk spanning Γ.*
3. *If $H \geq 0$, then there exists a hyperbolic H-cap spanning Γ if and only if $\rho < 1$. Moreover, this cap is unique.*
4. *Let $-1 < H < 0$. If $\rho \leq 1$, there exists a unique hyperbolic H-cap spanning Γ. If $\rho > 1$, then there exists a hyperbolic H-cap spanning Γ if and only if $H \leq H_0$, where $H_0 = -\sqrt{\rho^2 - 1}/\rho$. Moreover, the hyperbolic cap is unique if $H = H_0$ and there exactly two hyperbolic H-caps if $H < H_0$.*

Proof After an isometry of $\mathscr{H}^{n+1}$, suppose $\tau = 1$. Then the radius ρ of Γ coincides with the Euclidean radius.

1. For $H < -1$, we take the equidistant hypersurfaces $\mathbb{H}^n_-(r; -Hr)$ for all $r > 0$. When we meet with the slice L_1, we obtain spheres of arbitrary radius ρ.
2. Immediate.
3. Take a hyperbolic plane $\mathbb{H}^n_+(r; Hr)$. When we intersect with L_1, the sphere, if exists, has radius ρ such that $Hr + \sqrt{\rho^2 + r^2} = 1$. Thus, we study the solutions of $q(r) = 1$, where $q(r) = Hr + \sqrt{\rho^2 + r^2}$. Since $H \geq 0$, $q(r)$ is strictly increasing with $\lim_{r\to 0} q(r) = \rho$ and $\lim_{r\to\infty} q(r) = \infty$; see Fig. 2, left. This proves the result.
4. The above function $q(r)$ is now decreasing around $r = 0$, $q(r)$ has a unique minimum at $r_0 = -H\rho/\sqrt{1 - H^2}$ and $\lim_{r\to\infty} q(r) = \infty$; see Fig. 2, right. If $\rho \leq 1$, the result is immediate. If $\rho \geq 1$, it suffices to prove that $q(r_0) \leq 1$, which holds if and only if $H \leq H_0$. □

Example 4.2 Proposition 4.2 allows to give an example of two compact spacelike H-hypersurfaces spanning the same boundary. Let Γ be a sphere of radius $\rho > 1$. Let H be satisfying $-1 < H < -\sqrt{\rho^2 - 1}/\rho$, and denote by r_1 and r_2 the two roots of $Hr + \sqrt{\rho^2 + r^2} = 1$. Then the hyperbolic H-caps of radius ρ determined by $\mathbb{H}^n_+(r_i; Hr_i)$ and L_1 span Γ: see Fig. 3.

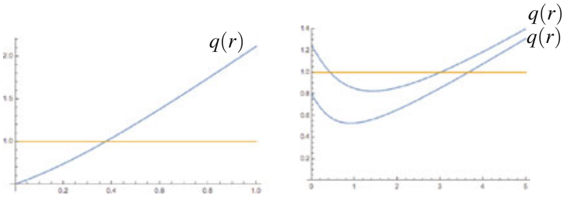

Fig. 2 Solutions of $q(r) = 1$ in the proof of Proposition 4.2. Left: $H = 1$ and $\rho = 1/2$; Right: $H = -3/4$ and $\rho = 0.8$ and $\rho = 1.25$

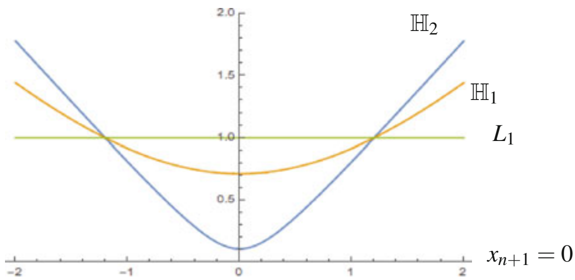

Fig. 3 Two hyperbolic H-caps $\mathbb{H}_1$ and $\mathbb{H}_2$ with the same boundary. Here, $H = -0.7$ and the boundary is a sphere of radius $\rho = 1.2$ in the slice L_1

Example 4.3 The above non-uniqueness result can be also proved as follows. For $H \in (-1, 0)$, let r be sufficiently small, so $\mathbb{H}^n_+(r; Hr)$ intersects L_1 in a sphere Γ of radius $\rho > 1$ and defining a hyperbolic cap C below L_1. The Euclidean cone with vertex the origin and containing Γ intersects $\mathbb{H}^n_+(r; Hr)$ again in other sphere of radius $\rho' \neq \rho$ and determining other hyperbolic cap C'. Without loss of generality, suppose $\rho' > \rho$, and thus, $C \subset C'$. If h denotes the homothety from the origin of $\mathbb{R}^{n+1}$ of ratio ρ/ρ', then $h(C')$ is a H-hypersurface spanning Γ and $h(C') \neq C$.

From Proposition 4.2, we deduce that the radius ρ of a sphere $\mathbb{S}^{n-1}$ imposes restrictions to the values H for hyperbolic H-caps. Indeed, we have:

1. If $\rho < 1$, then any real number H is a value of a hyperbolic H-cap spanning $\mathbb{S}^{n-1}$.
2. If $\rho \geq 1$, then $H \in (-\infty, -\sqrt{\rho^2 - 1}/\rho)$.

In general, we prove that the shape and the size of $\Gamma \subset L_\tau$ imposes restrictions to the value H for a compact H-hypersurface spanning Γ.

Theorem 4.4 ([38]) *Let $\Gamma \subset L_\tau$ be a closed submanifold and let $\Omega \subset L_\tau$ denote the domain bounded by Γ. Let $\Sigma \subset \mathscr{H}^{n+1}$ be a compact H-hypersurface spanning Γ.*

1. *If Ω contains a ball of radius 1, then $H < 0$.*
2. *If Ω contains a ball of radius*

$$\rho_0 = \sqrt{\frac{1-H}{1+H}}, \tag{16}$$

then $H \notin (-1, 0)$.

Proof We only prove the item 1. By contradiction, suppose $H \geq 0$. After an isometry, assume $\tau = 1$ and that Ω contains a sphere of radius 1 centered at $(0, \ldots, 0, 1)$. We know from Theorem 4.1 that Σ lies below L_1. Consider the hyperbolic H-caps of type $\mathbb{H}^n_+(r; Hr)$ with boundary in L_1. The radius of the boundaries of these caps is ρ, $0 < \rho < 1$, with $Hr + \sqrt{\rho^2 + r^2} = 1$, and its vertex is the point $V_r = (0, \ldots, 0, (H+1)r)$. When ρ is very small (and r is close to $1/(H+1)$), the cap lies below L_1 but it does not meet Σ. Letting $r \to 0$, the vertex goes to $(0, \ldots, 0)$ but $\rho < 1$. Thus, there will be a first value r_0 such that the hyperbolic plane $\mathbb{H}^n_+(r_0; Hr_0)$ intersects Σ at an interior contact point, a contradiction with the tangency principle. □

4.3 The Spherical Boundary Case

For the question (iii), we utilize the upper half-space model because we will work with the isometries of $\mathscr{H}^{n+1}$. As it was announced in Sect. 2, the foliation by slices of $\mathscr{H}^{n+1}$ allows a certain control on the position of a compact cmc hypersurface such as it appeared in Proposition 4.1 when $H = -1$. Assuming that the boundary of the surface lies in other type of umbilical hypersurface, Proposition 4.1 extends straightforward provided that we have a foliation of the ambient space by H-hypersurfaces for a given value of H (in Proposition 4.1 this value was $H = -1$).

Theorem 4.5 ([38]) *Let $H_0 \in (-\infty, -1] \cup [0, \infty)$. Let Σ be a compact spacelike H-hypersurface whose boundary is contained in an umbilical H_0-hypersurface P. Then we have one of the following three possibilities: either $H = H_0$ and $\Sigma \subset P$; or $H < H_0$ and Σ lies above P; or $H > H_0$ and Σ lies below P.*

Here, we revise a gap in [38] where it asserted that this result holds for any value H_0.

Proof 1. Case $H_0 < -1$. Then P is an equidistant hypersurface $\mathbb{H}^n_-(r; c)$ and take all Euclidean homotheties of P from a fix point of $x_{n+1} = 0$, obtaining the desired foliation of $\mathscr{H}^{n+1}$; see Fig. 4, left.

2. Case $H_0 \geq 0$. Without loss of generality, suppose that P is the hyperbolic plane $\mathbb{H}^n_+(r; c_r)$, where $c_r = (0, \ldots, 0, H_0 r)$. Let us observe that P is asymptotic to the upper light cone $\mathscr{C}^+ = \{x \in \mathbb{R}^{n+1}_+ : \langle x - c_r, x - c_r\rangle = 0, x_{n+1} \geq H_0 r\}$, and denote W the convex domain of $\mathbb{R}^{n+1}_+$ determined by $\mathscr{C}^+$. Let Σ be a compact H-hypersurface with $\partial\Sigma \subset P$. Take Q a lightlike hyperplane tangent to $\mathscr{C}^+$. Move parallely Q sufficiently far from Σ, and then come back to its initial position. By using that $\partial\Sigma \subset P$ and that P is asymptotic to $\mathscr{C}^+$, it is not possible that Q meets Σ because it should be at an interior point $p \in \Sigma$ and Q would be the tangent space of Σ at p violating the spacelike condition of Σ. If we do the same argument with all these hyperplanes Q, we have that Σ is included in W. Consider now $P_t = h_t(P)$ the homothety of P centered at the origin and ratio $t \in (0, \infty)$. Then $\{P_t\}$ is a foliation of the convex domain determined by the light cone $\langle x, x\rangle = 0$: note that this domain contains W which lies Σ. The argument follows the same steps as for Theorem 4.5. Suppose that $\mathrm{int}(\Sigma)$ has points in both sides of P. Coming from hyperbolic planes P_t with $t = \infty$ until the first contact point with Σ, the tangency principle implies that $H_0 > H$. A similar reasoning with planes P_t coming from $t = 0$ gives $H_0 < H$; see Fig. 4, right. This contradiction proves

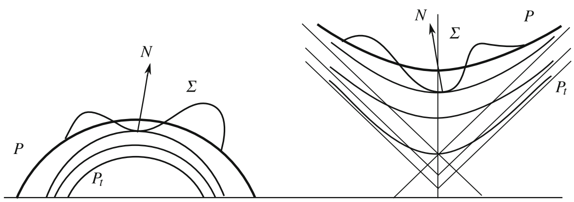

Fig. 4 Proof of Theorem 4.5: case $H_0 < -1$ (*left*) and case $H_0 \geq 0$ (*right*)

that Σ lies only on one side of P, but precisely the above argument gives that if $H < H_0$ (resp. $H > H_0$), then Σ lies above (resp. below) P. □

When the mean curvature H_0 lies in the range of the interval $(-1, 0)$, the family of umbilical H-hypersurfaces does not provide a foliation of the ambient space because any two members of this family meet each other: this appeared in Example 4.3.

Finally, we answer to the question (iii) when Γ is a sphere.

Theorem 4.6 ([38]) *Planar disks and hyperbolic caps are the only compact spacelike H-hypersurfaces in $\mathscr{H}^{n+1}$ spanning a sphere.*

Proof The proof uses the classical Alexandrov Reflection Method (see details [37] in the Euclidean space). The idea is to use the same hypersurface to compare with its reflection about a vertical hyperplane which preserves the constancy of the mean curvature, and finally, use the tangency principle. Let L_τ denote the slice containing Γ and $\Omega \subset L_\tau$ the bounded domain by Γ. We know by Theorem 4.5 that Σ is Ω when $H = -1$ or Σ lies on one side of L_τ if $H \neq -1$. In this case, $\Sigma \cup \Omega$ defines a domain in $\mathbb{R}^{n+1}_+$ and we can use the Alexandrov reflection method with reflections about vertical hyperplanes. This proves that Σ is a hypersurface of revolution with respect to a straight line orthogonal to L_τ. Finally, the hyperbolic caps are the only compact rotational cmc hypersurfaces spanning a sphere. □

Remark 4.1 When $H \in (-\infty, -1] \cup [0, \infty)$, we can do a proof of this result based on Theorem 4.5. For this argument, it is enough to prove that there exists an umbilical H-hypersurface containing Γ. If $H \leq -1$, the result follows from Proposition 4.2 (i, ii). If $H \geq 0$, then Theorem 4.4 says that $\rho < 1$, and now we use Proposition 4.2 (iii). Let us observe that we cannot complete the proof when $H \in (-1, 0)$.

We compare Theorem 4.6 to what happens in other ambient spaces. In Lorentz–Minkowski space $\mathbb{R}^{n+1}_1$ and in hyperbolic space when $|H| \leq 1$, the umbilical hypersurfaces are the only compact cmc hypersurfaces spanning a sphere [8, 10, 37]. However, it is an open question nowadays if spherical caps and planar disks are the only embedded compact H-hypersurfaces of $\mathbb{R}^{n+1}$ spanning a sphere (or in $\mathscr{H}^{n+1}$ when $|H| > 1$).

5 The Dirichlet Problem for the Mean Curvature Equation

In this section, we will prove the existence of complete H-hypersurfaces of $\mathscr{H}^{n+1}$, $H < -1$, whose boundary lies in the future infinity $\mathscr{J}^+$. This was proved by Montiel in [42] and was motivated by the Goddard conjecture: "the only complete spacelike cmc hypersurfaces in $\mathbb{S}^{n+1}_1$ must be umbilical." The hypersurfaces obtained in [42] (see Corollary 5.1 below) illustrate that this conjecture is not true in the steady state space. Recall that in $\mathbb{S}^{n+1}_1$, there was a great work of answering to this conjecture which is false, although in some cases is true, for example, when $|H| \leq 1$ and $n = 2$, when $0 \leq H^2 < 4(n-1)/n^2$ and $n \geq 3$, or when the hypersurface is compact [1, 41].

In order to prove the existence of Montiel's examples, and since the boundary lies in $\mathscr{J}^+$, the strategy is solving the Dirichlet problem in domains of slices, then take a sequence of such domains going to $\mathscr{J}^+$ and their corresponding solutions and having a suitable control of the solutions that ensures its convergence in the limiting process. Thus, in this section, we will study the existence of compact spacelike H-graphs on a domain of a slice.

The Dirichlet problem for the mean curvature equation is the following:

Problem 3: Given $\Omega \subset L_\tau$ a bounded domain, $H \in \mathbb{R}$ and $\tau > 0$, find a solution of

$$Q_H[u] = 0 \text{ on } \Omega, \quad u = \tau > 0 \text{ along } \partial\Omega. \tag{17}$$

The uniqueness of solutions of (17) is not assured by the standard theory because the term on u in the expression of $Q_H[u]$ in (2) is not necessarily nondecreasing. Recall that we showed in Example 4.2 two spacelike H-graphs on a disk of L_1 with the same boundary curve and $-1 < H < 0$.

The solvability of the Dirichlet problem (17) strongly depends whether $H < -1$ or $H > -1$, just the value of the mean curvature of a slice and, depending on each case, the hypersurface lies on one side of L_τ by Theorem 4.1.

Theorem 5.1 ([42]) *Let $\Omega \subset L_\tau$ be a compact bounded domain which has mean convex boundary. If $H < -1$, then there exists a unique solution of (17).*

We solve the Dirichlet problem using the method of continuity. For this technique, we refer the reader to [40] in the context of cmc hypersurfaces in hyperbolic space (the general reference for elliptic equations is Gilbarg and Trudinger [31]). Without loss of generality, suppose that $\tau = 1$. The method of continuity considers the uniparametric family of Dirichlet problems

$$\begin{cases} Q_{H(t)}[u] = 0 \text{ in} \Omega, \\ u = 1 \qquad\qquad \text{along} \partial\Omega \end{cases} \tag{18}$$

where $H(t) = t(1+H) - 1$, $t \in [0,1]$. Let us observe that for $t = 1$, a solution of (18) is the solution that we are looking for (17). We show that the subset of $[0,1]$ defined by

$$\mathscr{A} = \{t \in [0,1] : \exists u_t \in C^{2,\alpha}(\overline{\Omega}), \text{ such that } Q_{H(t)}(u_t) = 0 \text{ and } u_{t|\partial\Omega} = 1\}$$

is nonempty, open, and closed in $[0,1]$. In such a case, $1 \in \mathscr{A}$ proving that there exists a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (17). Finally, as H is constant and Ω is smooth, the regularity theory for the cmc equation proves that any $C^{2,\alpha}$ solution of (17) will be smooth, proving Theorem 5.1. We observe that $\mathscr{A} \neq \emptyset$ because $0 \in \mathscr{A}$ since $H(0) = -1$ and $u = 1$ are a solution in the domain Ω. The proof that $\mathscr{A}$ is open of [0, 1] is a consequence of the implicit function theorem in Banach spaces and it follows standard arguments.

The difficulties lie in proving that $\mathscr{A}$ is a closed set of [0, 1]. This follows once we establish a priori C^1 estimates of the prospective solutions of (18), that is, estimates of $|u|$ and $|Du|$ depending only on H and Ω.

The estimate for $|u|$ was proved in Theorem 4.3 where the estimate (13) depends only on H and Ω. For the estimate of $|Du|$, we need to work in the de Sitter model. The graph Σ_u of u corresponds with a graph $\Sigma_f \subset \mathbb{S}_1^{n+1}$ and the slice $L_1 \subset \mathbb{R}_+^{n+1}$ with the slice L_1 in $\mathbb{S}_1^{n+1}$. By the isometry Ψ, we have $u = e^f$ and

$$\langle p, a\rangle = \frac{1}{u} = e^{-f}, \quad \langle N, a\rangle = \frac{1}{u\sqrt{1-|Du|^2}}. \tag{19}$$

Thus, we will obtain bounds for $|Du|$ provided that we have a certain control of the functions $\langle p, a\rangle$ and $\langle N, a\rangle$. We proved in (10) that $-H\langle p, a\rangle + \langle N, a\rangle \leq -H + \langle N(q), a\rangle$ for some $q \in \partial\Sigma_u$. At q, the maximum principle gives again

$$H\langle \nu_q, a\rangle + \langle (dN)_q(\nu_q), a\rangle = \langle \nu_q, a\rangle(-H + \langle (dN)_q(\nu_q), \nu_q\rangle \leq 0$$

where ν is the unit conormal pointing along $\partial\Sigma_u$ toward Σ. Since $\langle \nu_q, a\rangle > 0$, then $-H + \langle (dN)_q(\nu_q), \nu_q\rangle \leq 0$. Using the mean convexity of Ω, we deduce $-H + \langle N(q), a\rangle \leq 0$, and thus

$$-H\langle p, a\rangle + \langle N, a\rangle \leq -H + \langle N(q), a\rangle \leq 0.$$

Hence, we deduce $\langle N, a\rangle \leq H\langle p, a\rangle \leq H$ because $\langle p, a\rangle \leq 1 (= \tau)$ by Proposition 4.1. Finally, using (19) we conclude

$$|Du|^2 \leq \frac{H^2 - 1}{H^2} \tag{20}$$

obtaining the desired uniform estimate for $|Du|$ and proving Theorem 5.1. The uniqueness is a consequence of the standard theory [31] and the inequality (20).

The estimates (13) and (20) are independent on the slice where lies Ω. This extends Theorem 5.1 allowing $u = 0$ along $\partial\Omega \subset \mathbb{R}^n \times \{0\}$. Indeed, we go by considering the solutions u_τ of (17) and by letting $\tau \to 0$, the domains $\Omega \times \{\tau\}$ converge to $\Omega \times \{0\} \subset \mathscr{J}^+$, and the funtions u_τ have a priori C^1-estimates independent on τ.

Theorem 5.2 ([39]) *Let $\Omega \subset \mathbb{R}^n$ be a compact domain which is mean convex boundary. If $H < -1$, then there exists a solution of (17) with $u = 0$ along $\partial\Omega$.*

If we introduce the concept of *asymptotic future boundary* of Σ as $\partial_\infty\Sigma = \overline{\Sigma} \cap \mathscr{J}^+$, the above result can be re-phrased as follows:

Corollary 5.1 ([39]) *Let $\Omega \subset \mathscr{J}^+$ be a compact domain which is mean convex boundary. If $H < -1$, then there exists a complete embedded spacelike H-hypersurface with $\partial_\infty\Sigma = \partial\Omega$.*

Proof It only remains to prove that the induced metric ds^2 on Σ_u is complete. From (20) we have

$$ds^2 = \frac{1}{u^2}(|dx|^2 - \langle Du, dx\rangle^2) \geq \frac{1}{H^2-1}\frac{\langle Du_x, dx\rangle^2}{u^2} \geq \frac{1}{H^2-1}|d\log u|^2.$$

This says that the length of any curve in Σ_u reaching $\partial_\infty \Sigma_u$ must be infinity. □

Remark 5.1 Let us observe that (13) says that $u \leq ct$ and this implies that the complete H-hypersurface obtained in Corollary 5.1 has $H < -1$ and it is bounded away from the past infinity of $\mathscr{H}^{n+1}$.

Consider now the Dirichlet problem (17) for values $H > -1$, so the graph lies below L_τ by Theorem 4.5. From Theorem 4.4, it is expectable some kind of smallness assumption on the domain Ω. If in Theorem 5.1 it was assumed to be mean convex, now we need strong convexity assumptions. For $\kappa > 0$, a domain $\Omega \subset L_\tau$ is said to be *κ-convex* if all the principal curvatures κ_i of $\partial\Omega$ with respect to the inward normal vector satisfy $\kappa_i \geq \kappa$.

Theorem 5.3 *Let $-1 \leq H < 0$ and let $\Omega \subset L_\tau$ be a κ-convex domain strictly contained in a ball of radius 1 in L_τ. If*

$$\kappa \geq \sqrt{1-H^2}, \tag{21}$$

then there exists a solution of the Dirichlet problem (17).

We utilize the method of continuity again. Suppose $\tau = 1$ and $(0, \ldots, 0, 1) \in \Omega$. Since the induced metric in L_1 is the Euclidean one, Ω is included in a ball Ω_1 of Euclidean radius 1. As $0 < u < 1$ in Ω and $\overline{\Omega} \subset \Omega_1$, then the spacelike condition, the convexity of Ω, and a similar argument as for Theorem 4.5, proves the existence of $0 < c < 1$ depending only on Σ_u such that $c < x_{n+1} \leq 1$ in Σ_u: this provides a priori estimates for $|u|$ in (17).

In Theorem 5.1, the gradient estimates were obtained using the subharmonicity of the function $-H\langle p, a\rangle + \langle N, a\rangle$. Now this is not enough and we will prove first the existence of an apriori estimate of $|Du|$ along $\partial\Omega$, and once obtained this estimate, we will get an estimate of $|Du|$ in the domain Ω.

Proposition 5.1 *Under the assumptions of Theorem 5.3, we have*

$$\sup_\Omega |Du| < \sqrt{1-H^2} \tag{22}$$

for every solution u of the Dirichlet problem (17).

Proof 1. Estimates along $\partial\Omega$. Because Ω is κ-convex and inequality (21), we can trap Σ_u in the domain determined by a hyperbolic H-hyperplane $\mathbb{H}^n_+(r; Hr)$ and the slice L_1 such the intersection $L_1 \cap \mathbb{H}^n_+(r; Hr)$ is the boundary of a round ball

B_ρ of radius $\rho = 1/\sqrt{1-H^2}$. Denoted by $\mathbb{H}^+_\rho \subset \mathbb{H}^n_+(r; Hr)$, the hyperbolic cap is determined by L_1. Then $\mathbb{H}^+_\rho$ lies below L_1, and Σ_u lies between $\mathbb{H}^+_\rho$ and B_ρ. The assumption on the κ-convexity of Ω says $\kappa \geq 1/\rho$, so the domain Ω has the following Blaschke's outer rolling sphere property: for every $p \in \partial\Omega$, there exists a ball in L_1 of radius ρ touching p and leaving the domain Ω inside the ball. By moving horizontally $\mathbb{H}^+_\rho$, we go touching every point of $\partial\Omega$ in such a way that during these translations (isometries of $\mathscr{H}^{n+1}$), the tangency principle forbids a contact between $\mathbb{H}^+_\rho$ and Σ_u. Therefore, we can reach any point of $\partial\Omega$ leaving Σ_u sandwiched by $\mathbb{H}^+_\rho$ and L_1, in particular, the slope of Σ_u along $\partial\Omega$ is bounded by the one of $\mathbb{H}^+_\rho$. This estimate is written precisely as $\sup_{\partial\Omega}|Du| < \sqrt{1-H^2}$, proving (22) for boundary points of Ω.

2. Estimates on Ω. In the de Sitter model, the function $-H\langle p, a\rangle + \langle N, a\rangle$ is subharmonic and the maximum principle gives

$$-H\langle p, a\rangle + \langle N, a\rangle \leq \sup_{\partial\Omega}\left(-H + \frac{1}{\sqrt{1-|Du|^2}}\right) < \frac{1-H^2}{H}$$

where we have used that $|Du| < \sqrt{1-H^2}$ along $\partial\Omega$. Hence, we obtain $\langle N, a\rangle$, and using (19), we conclude $\sup_\Omega |Du| < \sqrt{1-H^2}$. □

Finally, Proposition 5.1 proves definitively Theorem 5.3.

6 Oulook and Open Problems

The interest of the steady state space comes because it is the Lorentzian analogous of hyperbolic space. In this chapter, we have focused on spacelike hypersurfaces with constant mean curvature but the theory of submanifolds is much more: for example, one can study spacelike hypersurfaces with constant Gaussian curvature as in [46, 47]. The spacelike condition on the hypersurface is a strong difference between $\mathscr{H}^{n+1}$ and $\mathbb{H}^{n+1}$. The three models of $\mathscr{H}^{n+1}$ allow different approaches for a given problem, but we have observed in the literature similar results written in different coordinates. This is the reason why we have strived to make a common line of the progress in $\mathscr{H}^{n+1}$ after the paper of Montiel [42] and later of Albujer and Alías [3]. We have not fully studied the extension of results in other GRW spacetimes because we believe that a generalization in these spaces would bring long statements in the conditions on the warping function as well as the fiber manifold that could go away from our initial aim. Finally, we have emphasized on the techniques employed, where the maximum principles (tangency principle, Omori-Yau) play an important role.

In the literature, there has been a great interest for characterizing the slices, in connection with the Bernstein problem but we find missing more efforts in other directions. We have collected a list of open problems some of which could be tackled in the near future.

1. Obtain other types of characterizations for slices. The condition to be included between two slices in Theorem 3.1 is too strong. Extend the characterizations of Bernstein-type for hyperbolic planes, for example assuming that the hypersurface lies in the convex side of another hyperbolic plane.
2. Study spacelike H-hypersurfaces with $-1 < H < 0$ because this range of values for H appears remarkably in some results: Example 4.2 and Theorem 4.4 and 5.3.
3. Investigate the family of spacelike cmc hypersurfaces invariant by a uniparametric group of isometries. We are thinking not only in the rotational examples but also hypersurfaces invariant by a parabolic or a hyperbolic group of rotations.
4. Characterize cmc hypersurfaces whose asymptotic future boundary is known. In the upper half-space model, we can assume that $\partial_\infty \Sigma$ is one point, an Euclidean sphere or two concentric Euclidean spheres. In the literature, there are similar results in the hyperbolic space.
5. Study complete spacelike hypersurfaces in $\mathscr{H}^{n+1}$ with constant mean curvature $|H| = 1$. We know that if $n = 2$, then the hypersurface is a slice, but we do not know the existence of other examples in arbitrary dimension.
6. Solve the Dirichlet problem for the case $H > 0$. Following the discussion in Sect. 5, consider the case $H < -1$ but dropping the mean convex boundary assumption in Theorem 5.1.
7. We have not discussed on timelike cmc hypersurfaces in $\mathscr{H}^{n+1}$. Recall that there is a great activity in recent years in this topic in Lorentz–Minkowski space $\mathbb{R}^3_1$. The mean curvature equation for timelike surfaces is not elliptic so we cannot make use of the maximum principle. However, we have timelike slices which could be characterized with results of Bernstein-type.

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