Freeform Architecture and Discrete Differential Geometry

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Abstract. Freeform structures play an important role within contemporary architecture. While there is a wealth of excellent tools for the digital design of free-form geometry, the actual fabrication on the architectural scale is a big challenge. Key issues in this context are free-form surfaces composed of panels which can be manufactured at reasonable cost, and the geometry and statics of the support structure. The present article is an extended abstract of a talk on the close relation between geometric computing for free-form architecture and discrete differential geometry. It addresses topics such as skins from planar, in particular quadrilateral panels, geometry and statics of supporting structures, structures in force equilibrium.

1 Introduction

The mathematical and computational challenges posed by free-form shapes in architecture are twofold. One is *rationalization* which means approximating a given design surface by a collection of smaller parts which can be individually manufactured and put together. There is a great variety of constraints imposed on the individual parts, most having to do with manufacturing. The second challenge is *design* of free forms. The goal here is to develop tools which allow the user to interactively design free forms, such that key aspects of statics and fabrication are taken into account directly in the design phase. Meanwhile there is a wealth of results on these topics, and we want to point to the survey article [13].

2 Freeform Skins from Planar Panels and Associated Support Structures

Steel-glass constructions usually require a decomposition of freeform skins into *flat* panels, which leads us to the question of rationalization with polyhedral surfaces, and designing with polyhedral surfaces. The combinatorics of meshes plays an important role here: It is very easy to represent a given shape by a triangle mesh, and in fact the majority of freeform skins which exist are based on triangle meshes. However there are drawbacks: On average 6 edges meet in a vertex,

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so structures based on triangle meshes have complicated nodes. Further, they tend to be heavier than structures based on quad meshes. This has led to a new line of research into PQ (planar quad) meshes, which are meshes whose faces are planar quadrilaterals. Here an important link to discrete differential geometry is established: Combinatorially regular quad meshes decompose into two families of mesh polylines which can be seen as discrete versions of the parameter lines on a smooth surface. A quad mesh is then interpreted as a discrete version of a parametric surface. Properties of quad meshes relevant to architecture turn out to be equivalent to properties relevant in discrete differential geometry (in particular, the integrable systems viewpoint of discrete differential geometry, see [3]).

This connection between smooth surfaces and discrete surfaces is very important in investigating the degrees of freedom available for rationalization and design: E.g. a PQ mesh constitutes a discrete version of a so-called conjugate network of curves [9]. Meshes where the edge polylines appear smooth will need to approximate a conjugate network of curves. The conjugate networks are known and in theory there are many, but we nevertheless can draw the conclusion that in connection with practical considerations (e.g. angles between edges) there might be little flexibility or even no satisfactory network at all which serves as guidance for a PQ mesh (see Fig. 1).

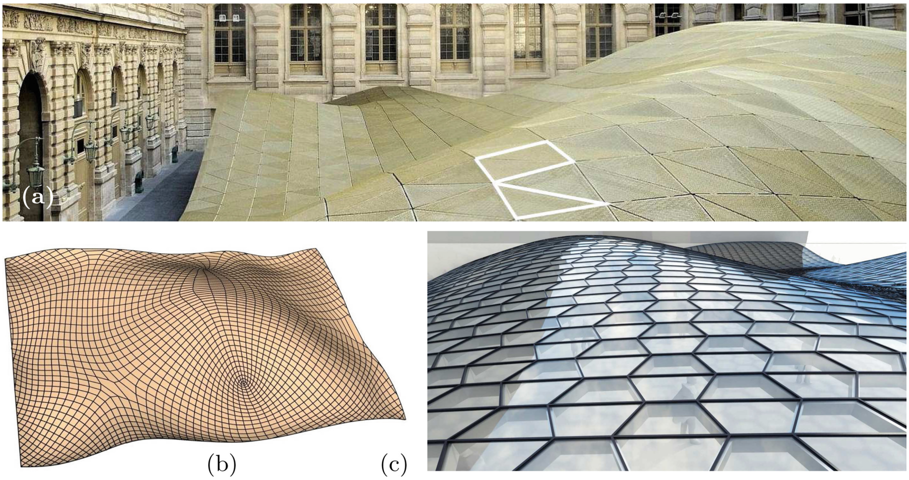

Fig. 1. Differential geometry informing rationalization. (a) The *Cour Visconti* roof in the Louvre (image courtesy Waagner-Biro Stahlbau). It was intended to be built in a lightweight way, possibly as a quad mesh. (b) A rationalization of this surface as a quad mesh with planar faces and smooth edge polylines must follow a conjugate network of curves, but these networks have unacceptable singularities. (c) If zigzags are allowed, rationalization as a PQ mesh with regular combinatorics is possible. For the actual roof, however, a different solution with both triangular and quadrilateral faces was found.

An important special case are *nearly rectangular panels.* Aside from aesthetics there are fundamental geometric reasons for constraining a PQ mesh to some form of orthogonality between edges. As it turns out, such meshes are discrete versions of principal curve networks, and the known nice behaviour of the surface normals along the curves of such a network translates to good properties of the *support structure* associated with the quad mesh, see Fig. 2. One is able to design so-called torsion-free nodes [9,11]. Research in this direction also led to progress in discrete differential geometry itself, in particular a new curvature theory for discrete surfaces [4]. Direct design of torsion-free support structures with quad combinatorics is related to special parametrizations of *congruences.* This word refers to a 2-dimensional system of straight lines and constitutes a classical topic of differential geometry. Its discrete incarnation turned out to be quite useful and has been explored systematically. We used it in connection with shading and guiding light by reflection, see [17] and Fig. 3.

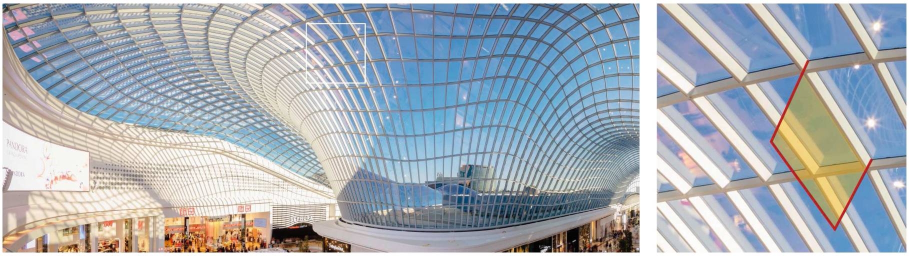

Fig. 2. Torsion-free support structures. The Chadstone shopping mall in Melbourne features a steel-glass roof in the shape of a planar quad mesh. The member corresponding to an edge is aligned along the *support plane* (yellow) of that edge, and the intersection of members in a node is defined by the *node axis* (red) where support planes meet. This behaviour of node axes is analogous to the behaviour of surface normals along principal curvature lines (original photo: T. Burgess, imageplay).

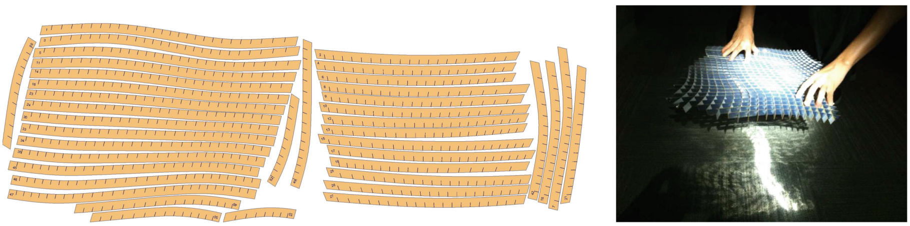

Fig. 3. Torsion-free support structures for shading and lighting. By cutting out and assembling the strips shown above one creates a torsion-free support structure capable of reflecting light into prescribed patterns. This arrangement of planes and lines discretizes the notion of *torsal parametrization of a line congruence.* The strips correspond to the two families of developable surfaces which make up the congruence (the system of normals of a surface along principal curvature lines is a special instance of this).

The previous paragraphs did not give an exhaustive list of the correspondences between discrete surfaces and smooth parametric surfaces which have already been used in the context of freeform architecture. In particular we did not mention *semidiscrete* surfaces relevant to structures with bent glass [12].

3 Structures in Static Equilibrium

Statics obviously is of paramount importance in architecture and building construction. It is therefore important that aspects of statics play a role already in the first stages of design. It is a long-term goal to create design tools which incorporate constraints relating to geometry, fabrication and statics while being still fast enough to allow interactive modeling. We are currently far from this goal, but partial results have been achieved. We start our summary by mentioning the *thrust network method* [1,2]: Maxwell's ideas on graphical statics are the basis of a method to treat systems of equilibrium forces which act in surface-like geometries. By separating vertical and horizontal components one is led to a discrete *Airy potential* polyhedron, which is a finite element discretization of the Airy stress potential well known in 2D elasticity theory. Compressive stresses are characterized by convexity of the stress potential.

A particularly nice application of this method is self-supporting masonry which is stable even without mortar, see Fig. 4. It is possible to interpret forces resp. stresses in differential-geometric terms, and we refer to [14,16] for this "geometrization" of the force balance condition, and for a treatment of the so-called isotropic differential geometry which occurs here. The direct interactive design of meshes (in particular polyhedral surfaces) with additional force balance conditions is a special case of constrained geometric modeling, see [15].

Recently we have worked on material-minimizing structures, see Fig. 4. This optimization problem was originally proposed in a groundbreaking paper by

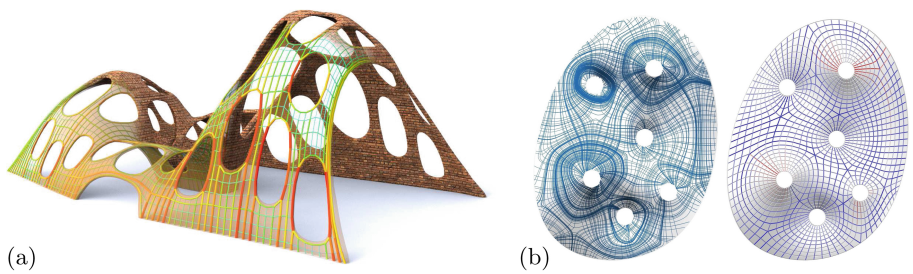

Fig. 4. Self-supporting and weight-optimal structures. (a) This masonry vault with holes contains a network of compressive forces which is in equilibrium with the deadload, implying the remarkable fact of stability of the structure when built of bricks even without mortar. Interactive design of such self-supporting surfaces is possible [15,16]. (b) The search for quad meshes with planar faces and minimal weight in the sense of M.G.M. Michell's *limit of economy* is converted into computing a variant of principal curves, by a suitable differential-geometric interpretation [8].

M.G.M. Michell [10] and is meanwhile formulated in modern language [18]. Our work, like others mentioned in this paper, is based on a differential-geometric interpretation of the subject of interest which, in this case, is the volume of members of a structure based on a mesh, and also the forces acting in these members. For example, 2-dimensional optimal trusses are characterized by Airy potential surfaces of minimal total curvature in the sense of isotropic geometry. This topic and its extension to shells is treated by [8] (Fig. 4).

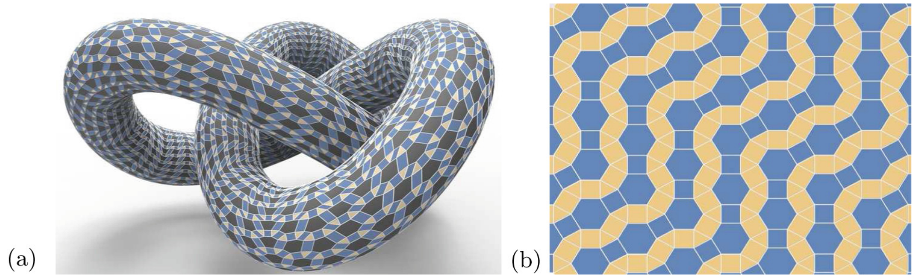

Fig. 5. Nonstandard notions of fairness. Here a given smooth surface is approximated by a polyhedral surface (a) of prescribed local combinatorics (b). The concept of fairness employed in the computation is based on existence of local approximate symmetries.

4 On Fairness, the Importance of Regularizers and Structures Beyond Discrete Differential Geometry

In all examples mentioned above, *fairness* plays an important role in identifying those discrete structures which meaningfully correspond to smooth objects. On a technical level, fairness functionals are used as regularizers in optimization and in iterative constraint solvers. There are, however, many different ways to express fairness computationally. The standard quadratic fairness energies composed of iterated differences might not be appropriate for meshes like the one shown by Fig. 1c. The zigzag polylines might be fully intentional, but they cause high (bad) values of such a fairness energy. Recently, alternative approaches to fairness have been successfully employed in creating polyhedral patterns [6]. They are based on existence of *local approximate symmetries*. An interpretation in terms of standard concepts of discrete differential geometry is still open. A difficult topic in general are fairness functionals of high nonlinearity, e.g. those involving kink angles. A fairness measure based on angles only [7] has led to a new concept of *smoothness of discrete surfaces* [5]. Recently we have investigated a functional defined as the sum of edge lengths times absolute value of kink angles. Its "isotropic" version surprisingly turns up in connection with material minimization (see previous paragraph). The shape of minimizers is a topic of current research; we conjecture that at least in negatively curved areas they are principal meshes.

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