Chapter 1
Commutator Calculus

In this chapter, we introduce the commutator calculus. This is one of the most important tools for studying nilpotent groups. In Section 1.1, the center of a group and other notions surrounding the concept of commutativity are defined. Several results and examples involving central subgroups and central elements are given. Section 1.2 contains the fundamental identities related to commutators of group elements. By definition, the commutator of two elements g and h in a group G is the element $[g, h] = g^{-1}h^{-1}gh$. Clearly, $[g, h] = 1$ whenever g and h commute. This leads to a natural connection between central elements and trivial commutators. The commutator identities allow us to develop properties of commutator subgroups. This is the main focus of Section 1.3.

1.1 The Center of a Group

The commutator calculus is an essential tool which is used for working with nilpotent groups. In this section, we collect various results on commutators which will be used throughout the book. This material can be found in various places in the literature (see [1–6]).

1.1.1 Conjugates and Central Elements

We begin by defining the conjugate of a group element.

Definition 1.1 Let g and h be elements of a group G. The *conjugate* of g by h, denoted by g^h, is the element $h^{-1}gh$ of G.

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The conjugate of g^{-1} by h is written as g^{-h}. Notice that

$$g^{-h} = \left(g^{-1}\right)^h = h^{-1}g^{-1}h = \left(h^{-1}gh\right)^{-1} = \left(g^h\right)^{-1}.$$

Furthermore, if $k \in G$, then

$$(gh)^k = k^{-1}ghk = (k^{-1}gk)(k^{-1}hk) = g^k h^k$$

and

$$\left(g^h\right)^k = \left(h^{-1}gh\right)^k = k^{-1}h^{-1}ghk = (hk)^{-1}g(hk) = g^{hk}.$$

We summarize these in the next lemma.

Lemma 1.1 *Suppose that* g, h, *and* k *are elements of any group. Then* $(gh)^k = g^k h^k$, $\left(g^{-1}\right)^h = \left(g^h\right)^{-1}$, *and* $\left(g^h\right)^k = g^{hk}$.

The notion of conjugacy extends to subgroups in a natural way.

Definition 1.2 Two subgroups H and K of a group G are called *conjugate* if $g^{-1}Hg = K$ for some $g \in G$.

In particular, every normal subgroup of G is conjugate to itself.

Definition 1.3 Let G be a group. An element $g \in G$ is called *central* if it commutes with every element of G. The set of all central elements of G is called the *center* of G and is denoted by $Z(G)$. Thus,

$$\begin{aligned} Z(G) &= \{g \in G \mid gh = hg \text{ for all } h \in G\} \\ &= \{g \in G \mid g^h = g \text{ for all } h \in G\}. \end{aligned}$$

It is easy to verify that $Z(G)$ is a normal abelian subgroup of G, and the conjugate of a central element $g \in G$ by any element of G is just g itself.

If G and H are groups, then the (internal and external) direct product of G and H will be written as $G \times H$.

Lemma 1.2 *If* G_1 *and* G_2 *are groups, then* $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$.

Proof Suppose that $(g_1, g_2) \in Z(G_1 \times G_2)$. Then $(g_1, g_2)(x, y) = (x, y)(g_1, g_2)$ for all $(x, y) \in G_1 \times G_2$. This implies that $(g_1x, g_2y) = (xg_1, yg_2)$, and thus $g_1x = xg_1$ and $g_2y = yg_2$. Hence, $g_1 \in Z(G_1)$ and $g_2 \in Z(G_2)$. Therefore, (g_1, g_2) is contained in $Z(G_1) \times Z(G_2)$. And so, $Z(G_1 \times G_2) \subseteq Z(G_1) \times Z(G_2)$. In a similar way, one can show that $Z(G_1) \times Z(G_2) \subseteq Z(G_1 \times G_2)$. □

Lemma 1.3 *If* G_1 *and* G_2 *are any two groups, then*

$$\frac{G_1 \times G_2}{Z(G_1 \times G_2)} \cong \frac{G_1}{Z(G_1)} \times \frac{G_2}{Z(G_2)}.$$

Proof The map from $G_1 \times G_2$ to $(G_1/Z(G_1)) \times (G_2/Z(G_2))$ defined by

$$(g_1,\ g_2) \mapsto (g_1 Z(G),\ g_2 Z(G))$$

is a surjective homomorphism whose kernel is $Z(G_1 \times G_2)$. The result follows from the First Isomorphism Theorem. □

Let G and H be any two groups. The set of homomorphisms from G to H will be denoted by $Hom(G,\ H)$, and the group of automorphisms of G by $Aut(G)$. The kernel and image of $\varphi \in Hom(G,\ H)$ are abbreviated as $ker\ \varphi$ and $im\ \varphi = \varphi(G)$ respectively. If G and H are isomorphic groups, then we write $G \cong H$.

Let G be a group and $h \in G$. Using Lemma 1.1, it is easy to show that the map

$$\varphi_h : G \to G \ \text{ defined by } \ \varphi_h(g) = g^h$$

is contained in $Aut(G)$.

Definition 1.4 The map φ_h is called the *conjugation map* or *inner automorphism* induced by h.

It is easy to see that the set of all inner automorphisms of G forms a group under composition. This group is denoted by $Inn(G)$. There is a natural connection between the center of a group and the inner automorphisms of the group.

Theorem 1.1 *Let G be a group and $h \in G$. The map*

$$\varrho : G \to Aut(G) \ \textit{ defined by } \ \varrho(h) = \varphi_h, \ \ \textit{where} \ \ \varphi_h(g) = g^h,$$

is a homomorphism with $ker\ \varrho = Z(G)$ and $im\ \varrho = Inn(G)$.

Proof The result follows from Lemma 1.1. □

By Theorem 1.1 and the First Isomorphism Theorem, we have:

Corollary 1.1 *If G is any group, then $G/Z(G) \cong Inn(G)$.*

1.1.2 Examples Involving the Center

In the next few examples, we give the center of various groups.

Example 1.1 A group G is abelian if and only if $Z(G) = G$.

Example 1.2 Let S_n be the symmetric group on the set $S = \{1,\ 2,\ \ldots,\ n\}$, and let "$e$" denote the identity element of S_n. Clearly, S_1 has trivial center because $S_1 = \{e\}$. Furthermore, $Z(S_2) = S_2$ since S_2 is abelian.

We show that $Z(S_n) = \{e\}$ for $n > 2$. Suppose, on the contrary, that $Z(S_n)$ is nontrivial. Let $\sigma \in Z(S_n)$ be a nonidentity element. There exist distinct elements $a,\ b \in S$ such that $\sigma(a) = b$. Choose an element $c \in S$ different from a and b, and let

τ be the transposition $(b \quad c)$. A direct calculation shows that $(\sigma \circ \tau)(a) \neq (\tau \circ \sigma)(a)$, contradicting the assumption that σ is in the center of $Z(S_n)$.

Example 1.3 Let A_n be the alternating group on the set $S = \{1, 2, \ldots, n\}$. This is the subgroup of S_n consisting of all even permutations. Note that $A_1 = A_2 = \{e\}$, and A_3 is cyclic since it has order 3. Thus, $Z(A_n) = A_n$ for $n = 1, 2$, and 3 according to Example 1.1.

The center of A_4 is trivial. The proof is similar to the one used in Example 1.2. Assume that $Z(A_n)$ is nontrivial, and let σ be a nonidentity element of $Z(A_n)$. There exist distinct elements $a, b \in S$ such that $\sigma(a) = b$. Choose two elements c and d in S different from a and b, and let $\tau = (b \quad c \quad d)$. It is easy to see that $(\sigma \circ \tau)(a) \neq (\tau \circ \sigma)(a)$, contradicting the assumption that the center is nontrivial.

Using the same argument as above, one can show that A_n has trivial center whenever $n \geq 5$. We provide an alternative proof which uses the fact that A_n is a simple group whenever $n \geq 5$. Since this is the case, either $Z(A_n) = \{e\}$ or $Z(A_n) = A_n$. If it were true that $Z(A_n) = A_n$, then A_n would be abelian by Example 1.1. However, a quick calculation shows that

$$(1 \quad 2 \quad 3)(3 \quad 4 \quad 5) \neq (3 \quad 4 \quad 5)(1 \quad 2 \quad 3).$$

Thus, A_n is non-abelian and $Z(A_n) \neq A_n$. We conclude that $Z(A_n) = \{e\}$ for $n \geq 5$.

Example 1.4 Let D_n be the dihedral group of order $2n$, the group of isometries of the plane which preserve a regular n-gon. If y is a reflection across a line through a vertex and x is the counterclockwise rotation by $2\pi/n$ radians, then the elements of D_n are

$$1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y,$$

and the equalities

$$x^n = 1, \; y^2 = 1, \; \text{and } xy = yx^{-1}$$

hold in D_n.

Both D_1 and D_2 are abelian, so $Z(D_1) = D_1$ and $Z(D_2) = D_2$. We determine $Z(D_n)$ when $n \geq 3$. Since $xy = yx^{-1}$, we have

$$x^r y = yx^{-r} \quad (r \in \mathbb{Z}). \tag{1.1}$$

We claim that no element of the form $x^t y$ for any $t \in \{0, 1, \ldots, n-1\}$ is central. Assume, on the contrary, that $x^t y \in Z(D_n)$ for some such t. Then $x^t y$ commutes with x. Hence, $x^{-1}(x^t y)x = x^t y$, and thus $x^{t-1}yx = x^t y$. Applying (1.1) to both sides of this equality yields $yx^{1-t}x = yx^{-t}$. After canceling the y's, we get $x^{2-t} = x^{-t}$. This means that $x^2 = 1$, a contradiction. Therefore, $x^t y \notin Z(D_n)$ for any

$t \in \{0, 1, \ldots, n-1\}$. Consequently, an element of $Z(D_n)$ must take the form x^t for some $t \in \{0, 1, \ldots, n-1\}$. Clearly, $x^0 = 1 \in Z(D_n)$.

Suppose $x^t \in Z(D_n)$ for some $t \in \{1, \ldots, n-1\}$. By (1.1), we have

$$yx^t = x^t y = yx^{-t}.$$

Hence, $x^t = x^{-t}$; that is, $x^{2t} = 1$. Since x has order n, it must be that n divides $2t$. Hence, there exists $k \in \mathbb{N}$ such that $2t = nk$. If $k \geq 2$, then $2t \geq 2n$. This cannot happen since $1 \leq t \leq n-1$. This means that $k = 1$, and thus $2t = n$. Now, if n is odd, then no such t exists. We conclude that $Z(D_n)$ is trivial when n is odd. If n is even, then $t = \frac{n}{2}$, and consequently, $x^{n/2} \in Z(D_n)$. Therefore, $Z(D_n)$ is the cyclic group of order 2 generated by $x^{n/2}$ when n is even.

Example 1.5 Let $\mathscr{H}$ be the group of 3×3 upper unitriangular matrices over $\mathbb{Z}$ with the group operation being matrix multiplication. Thus,

$$\mathscr{H} = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a_{ij} \in \mathbb{Z} \right\}.$$

This group is called the *Heisenberg group*. The identity element in $\mathscr{H}$ is clearly the 3×3 identity matrix and will be denoted by I_3. It is easy to show that

$$Z(\mathscr{H}) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; c \in \mathbb{Z} \right\}.$$

1.1.3 Central Subgroups and the Centralizer

Definition 1.5 A subgroup H of a group G is called *central* if $H \leq Z(G)$.

Related to the center of a group is the centralizer of a subset of a group.

Definition 1.6 The *centralizer* of a nonempty subset X of a group G is

$$C_G(X) = \{g \in G \mid g^{-1}xg = x \text{ for all } x \in X\}.$$

It is easy to verify that $C_G(X)$ is a subgroup of G. If $X = \{x\}$, then we write $C_G(x)$ for the centralizer of x. Clearly,

$$C_G(G) = \bigcap_{x \in G} C_G(x) = Z(G).$$

Notice that $C_G(x)$ is just the stabilizer of x under the action of G on itself by conjugation. The orbit of x under this action, called the *conjugacy class* of x, is the set $\{g^{-1}xg \mid g \in G\}$. When G is finite, we get the *class equation* of G :

$$|G| = |Z(G)| + \sum_k [G : C_G(x_k)], \tag{1.2}$$

where one x_k is chosen from each conjugacy class containing at least two elements. Here, $|G|$ stands for the order of G and $[G : H]$ is the index of a subgroup H in G. These notations are standard and will be used throughout the book. We will also write $|g|$ for the order of an element $g \in G$.

1.1.4 The Center of a p-Group

Definition 1.7 Let p be any prime. A group G is called a *p-group* if every element of G has order a power of p.

Finite p-groups are the building blocks of finite groups. The next fact regarding their central structure is important in the study of finite groups.

Theorem 1.2 *If G is a nontrivial finite p-group for some prime p, then* $Z(G) \neq 1$.

Proof Suppose that $|G| = n$. Consider the class equation (1.2) of G. If $x_k \in G$ is not central for some $1 \leq k \leq n$, then $C_G(x_k)$ is a proper subgroup of G. Hence, $[G : C_G(x_k)]$ is a positive power of p. Consequently, each summand in the sum

$$\sum_k [G : C_G(x_k)]$$

is divisible by p. Since p divides $|G|$ by hypothesis, p also divides $|Z(G)|$. Therefore, $Z(G)$ contains nontrivial elements. □

Remark 1.1 It is important to emphasize that G must be finite in Theorem 1.2. An infinite p-group does not necessarily have nontrivial center. This notion is discussed in Remark 2.8.

1.2 The Commutator of Group Elements

One can determine whether or not two group elements commute by calculating their commutator.

Definition 1.8 Let g and h be elements of a group G. The *commutator* of g and h, written as $[g,\ h]$, is

$$[g,\ h] = g^{-1}h^{-1}gh = g^{-1}g^h.$$

Clearly, g and h commute if and only if $[g, h] = 1$. Thus, the center of G can also be characterized as

$$Z(G) = \{g \in G \mid [g, h] = 1 \text{ for all } h \in G\}.$$

Definition 1.9 Let $S = \{g_1, g_2, \ldots, g_n\}$ be a set of elements of a group G. A *simple commutator*, or *left-normed commutator*, of *weight* $n \geq 1$ is defined recursively as follows:

1. The simple commutators of weight 1 are the elements of S, written as $g_j = [g_j]$.
2. The simple commutators of weight $n > 1$ are $[g_1, \ldots, g_n] = [[g_1, \ldots, g_{n-1}], g_n]$.

We collect some commutator identities which are of utmost importance.

Lemma 1.4 *Let x, y, and z be elements of a group G.*

(i) $xy = yx[x, y]$.
(ii) $x^y = x[x, y]$.
(iii) $[x, y] = [y, x]^{-1}$.
(iv) $[x, y]^z = [x^z, y^z]$.
(v) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$.
(vi) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$.
(vii) $\left[x, y^{-1}\right] = \left([x, y]^{y^{-1}}\right)^{-1}$.
(viii) $\left[x^{-1}, y\right] = \left([x, y]^{x^{-1}}\right)^{-1}$.

Proof

(i) $xy = yx\left(x^{-1}y^{-1}xy\right) = yx[x, y]$.
(ii) $x^y = y^{-1}xy = x\left(x^{-1}y^{-1}xy\right) = x[x, y]$.
(iii) $[x, y] = x^{-1}y^{-1}xy = \left(y^{-1}x^{-1}yx\right)^{-1} = [y, x]^{-1}$.
(iv) We have

$$\begin{aligned}
[x, y]^z &= z^{-1}\left(x^{-1}y^{-1}xy\right)z \\
&= \left(z^{-1}x^{-1}z\right)\left(z^{-1}y^{-1}z\right)\left(z^{-1}xz\right)\left(z^{-1}yz\right) \\
&= \left(z^{-1}xz\right)^{-1}\left(z^{-1}yz\right)^{-1}\left(z^{-1}xz\right)\left(z^{-1}yz\right) \\
&= [x^z, y^z].
\end{aligned}$$

(v) Observe that

$$\begin{aligned}[xy,\ z] &= (xy)^{-1}z^{-1}xyz \\ &= y^{-1}x^{-1}z^{-1}xyz \\ &= y^{-1}\left(x^{-1}z^{-1}xz\right)y\left(y^{-1}z^{-1}yz\right) \\ &= y^{-1}[x,\ z]y[y,\ z] \\ &= [x,\ z]^{y}[y,\ z] \\ &= [x,\ z][x,\ z,\ y][y,\ z] \text{ by (ii).}\end{aligned}$$

A similar computation gives (vi). By (vi), we have

$$1 = \left[x,\ yy^{-1}\right] = \left[x,\ y^{-1}\right][x,\ y]^{y^{-1}}. \tag{1.3}$$

This establishes (vii), and (viii) follows from (v) in a similar way. □

Lemma 1.5 (The Hall-Witt Identities) *If x, y, and z are elements of a group, then*

$$\left[x,\ y^{-1},\ z\right]^{y}\left[y,\ z^{-1},\ x\right]^{z}\left[z,\ x^{-1},\ y\right]^{x} = 1$$

and

$$\left[x,\ y,\ z^{x}\right]\left[z,\ x,\ y^{z}\right]\left[y,\ z,\ x^{y}\right] = 1.$$

Proof By Lemma 1.4 (iii), we have

$$\begin{aligned}\left[x,\ y^{-1},\ z\right]^{y} &= y^{-1}\left[\left[x,\ y^{-1}\right],\ z\right]y \\ &= y^{-1}\left[x,\ y^{-1}\right]^{-1}z^{-1}\left[x,\ y^{-1}\right]zy \\ &= y^{-1}\left[y^{-1},\ x\right]z^{-1}\left[x,\ y^{-1}\right]zy \\ &= x^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy \\ &= \left(xzx^{-1}yx\right)^{-1}yxy^{-1}zy.\end{aligned}$$

Similarly,

$$\left[y,\ z^{-1},\ x\right]^{z} = \left(yxy^{-1}zy\right)^{-1}zyz^{-1}xz$$

and

$$\left[z,\ x^{-1},\ y\right]^x = \left(zyz^{-1}xz\right)^{-1} xzx^{-1}yx.$$

It follows that $\left[x,\ y^{-1},\ z\right]^y \left[y,\ z^{-1},\ x\right]^z \left[z,\ x^{-1},\ y\right]^x = 1$. One can prove the other identity in a similar way. □

1.3 Commutator Subgroups

The notion of the commutator of elements of a group can be generalized to the commutator of subsets of a group.

Definition 1.10 Let G be a group with subset $S = \{s_1,\ s_2,\ \ldots\}$. The subgroup of G *generated* by S, denoted by

$$gp(S) = gp(s_1,\ s_2,\ \ldots),$$

is the smallest subgroup of G containing S. We call S a set of *generators* for $gp(S)$.

The subgroup $gp(S)$ of G can be obtained by taking the intersection of all subgroups of G that contain S. A typical element of $gp(S)$ is of the form

$$s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} \cdots s_{i_n}^{\varepsilon_n},$$

where $s_{i_j} \in S$ and $\varepsilon_j \in \{-1,\ 1\}$ for $1 \le j \le n$. If $g \in G$, then $gp(g)$ is just the cyclic subgroup of G generated by g. If $S_1,\ \ldots,\ S_n$ are subsets of G, then the subgroup $gp(S_1 \bigcup \cdots \bigcup S_n)$ is written as $gp(S_1,\ \ldots,\ S_n)$.

Definition 1.11 Let X_1 and X_2 be nonempty subsets of a group G. The *commutator subgroup* of X_1 and X_2 is defined as

$$[X_1,\ X_2] = gp\left([x_1,\ x_2] \mid x_1 \in X_1,\ x_2 \in X_2\right).$$

Thus, $[X_1,\ X_2]$ is the subgroup of G generated by *all* commutators $[x_1,\ x_2]$, where x_1 varies over X_1 and x_2 varies over X_2. In particular, $[G,\ G] = G'$ is the *commutator subgroup* or *derived subgroup* of G.

Remark 1.2 The *set* of all commutators

$$S = \{[x_1,\ x_2] \mid x_1 \in X_1,\ x_2 \in X_2\}$$

does not necessarily form a subgroup of G. For instance, $[x_1,\ x_2]^{-1}$ may not be in S for some $[x_1,\ x_2] \in S$.

If $X_1 = X_2 = G$, then the inverse of every element of S is contained in S by Lemma 1.4 (iii). However, it may be that S is not a subgroup of G because the product of two or more commutators in S is not necessarily a commutator in S. Consider, for example, the *special linear group* $SL_2(\mathbb{R})$ whose elements are the 2×2 matrices with real entries and determinant 1 (the group operation is matrix multiplication). Let I_2 denote the 2×2 identity matrix, and set

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A routine check shows that $-I_2 = (ABA)^2$,

$$A = \left[\begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \right], \text{ and } B = \left[\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix} \right].$$

Thus, $-I_2$ is a product of commutators. However, $-I_2$ is not the commutator of two elements of $SL_2(\mathbb{R})$. To see this, assume, on the contrary, that $-I_2 = [C, D]$ for some $C, D \in SL_2(\mathbb{R})$. Rewriting this gives $C^{-1}DC = -D$, and thus D and $-D$ are similar matrices. Since the trace of a square matrix equals the trace of any matrix similar to it, D and $-D$ have equal trace. Consequently, the trace of D equals 0. Since the determinant of D equals 1, the characteristic polynomial of D is $f(\lambda) = \lambda^2 + 1$. And so, D has eigenvalues $\pm i$. This means that D is similar to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Without loss of generality, we may as well assume that $D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $CD = -DC$ by assumption, a computation shows that $d = -a$ and $c = b$. Using the fact that C has determinant 1, it follows that $-a^2 - b^2 = 1$. This contradicts the fact that $a, b \in \mathbb{R}$.

Definition 1.11 can be generalized. If $\{X_1, X_2, \ldots\}$ is a collection of nonempty subsets of G, then

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n],$$

where $n \geq 2$. Note that $[X_1, \ldots, X_n]$ contains all simple commutators of the form $[x_1, \ldots, x_n]$, where $x_1 \in X_1, \ldots, x_n \in X_n$. Thus,

$$[X_1, \ldots, X_n] \geq gp([x_1, \ldots, x_n] \mid x_1 \in X_1, \ldots, x_n \in X_n).$$

However, $[X_1, \ldots, X_n]$ may not equal $gp([x_1, \ldots, x_n] \mid x_1 \in X_1, \ldots, x_n \in X_n)$ if $n \geq 3$. For example (see [6]), consider the cyclic subgroups

$$H_1 = gp(\,(1 \;\; 2)\,),\; H_2 = gp(\,(2 \;\; 3)\,), \text{ and } H_3 = gp(\,(3 \;\; 4)\,)$$

of the symmetric group S_4. A routine check confirms that $[H_1, H_2, H_3]$ equals A_4, while $gp([h_1, h_2, h_3] \mid h_1 \in H_1, h_2 \in H_2, h_3 \in H_3)$ equals $gp(\,(1\ \ 3\ \ 4)\,)$. Thus,

$$[H_1, H_2, H_3] \neq gp([h_1, h_2, h_3] \mid h_1 \in H_1, h_2 \in H_2, h_3 \in H_3).$$

Lemma 1.6 *Let G be any group.*

(i) If $H \leq G$ and $[G, G] \leq H$, then $H \trianglelefteq G$ and G/H is abelian. Thus, $[G, G] \trianglelefteq G$ and $G/[G, G]$ is abelian.
(ii) If $N \trianglelefteq G$ and G/N is abelian, then $[G, G] \trianglelefteq N$.

Thus, the commutator subgroup of a group is the smallest normal subgroup inducing an abelian quotient. The factor group $Ab(G) = G/[G, G]$ is called the *abelianization* of G.

Proof

(i) Let $g \in G$ and $h \in H$. By Lemma 1.4 (ii),

$$h^g = g^{-1}hg = h[h, g] \in H$$

because H contains $[G, G]$. Therefore, $g^{-1}Hg = H$, and thus H is normal in G. If g_1H and g_2H are elements of G/H, then

$$(g_1H)(g_2H) = g_1g_2H = g_2g_1[g_1, g_2]H = g_2g_1H = (g_2H)(g_1H)$$

by Lemma 1.4 (i). Therefore, G/H is abelian.

(ii) If $gN, hN \in G/N$, then $(gN)(hN) = (hN)(gN)$. Hence,

$$(gN)^{-1}(hN)^{-1}(gN)(hN) = N.$$

We thus have $g^{-1}h^{-1}gh = [g, h] \in N$. It follows that $[G, G] \trianglelefteq N$. □

Lemma 1.6 allows one to conveniently calculate the derived subgroup. This is illustrated in the next few examples.

Example 1.6 Any two elements of an abelian group G commute. Thus, $[G, G] = 1$.

Example 1.7 We compute the commutator subgroup of the alternating group A_n on the set $S = \{1, 2, \ldots, n\}$. Clearly, $[A_n, A_n] = \{e\}$ for $n = 1, 2, 3$ by Example 1.6.

We find the commutator subgroup of A_4. It is well known that A_4 contains a unique nontrivial normal subgroup

$$K = \{e, (1\ \ 2)(3\ \ 4), (1\ \ 3)(2\ \ 4), (1\ \ 4)(2\ \ 3)\},$$

which is an isomorphic copy of the Klein 4-group (see [1]). Since $[A_4 : K] = 3$, the quotient A_4/K is abelian. Therefore, $[A_4, A_4] \trianglelefteq K$, and thus $[A_4, A_4] = K$.

Lastly, we consider the case when $n \geq 5$. In this case, A_n is simple. Thus, the only normal subgroups of A_n are $\{e\}$ and A_n. Since A_n is not abelian, $[A_n, A_n] = A_n$.

Example 1.8 We find the commutator subgroup of the symmetric group S_n on the set $S = \{1, 2, \dots, n\}$. By Example 1.6, $[S_n, S_n] = \{e\}$ for $n = 1, 2$.

In order to find $[S_n, S_n]$ for $n \geq 3$, we use the fact that A_n is a normal subgroup of index 2 in S_n, and thus S_n/A_n is an abelian group. First, we find $[S_3, S_3]$. Since S_3/A_3 is abelian, we know that $[S_3, S_3] \trianglelefteq A_3$. Furthermore, each element of A_3 can be written as a commutator of elements in S_3 (this is obvious for the identity permutation):

$$(1\ \ 2\ \ 3) = [(2\ \ 3), (1\ \ 3\ \ 2)] \text{ and } (1\ \ 3\ \ 2) = [(2\ \ 3), (1\ \ 2\ \ 3)]\,.$$

Therefore, A_3 is contained in $[S_3, S_3]$, and consequently, $[S_3, S_3] = A_3$.

Next, we show that $[S_4, S_4] = A_4$. Let $(a\ \ b\ \ c)$ be any 3-cycle for some distinct elements $a, b, c \in S$. This 3-cycle can be written as a commutator of elements in S_4 as

$$(a\ \ b\ \ c) = [(a\ \ b), (a\ \ c\ \ b)]\,.$$

It follows that $A_4 \leq [S_4, S_4]$ because A_4 is generated by 3-cycles. Since S_4/A_4 is abelian, $[S_4, S_4] \trianglelefteq A_4$. We conclude that $[S_4, S_4] = A_4$.

Finally, consider the case when $n \geq 5$. Once again, $[S_n, S_n] \trianglelefteq A_n$ because S_n/A_n is abelian. Since the only nontrivial normal subgroup of S_n is A_n, it must be that $[S_n, S_n] = A_n$.

Example 1.9 We find the derived subgroup of the dihedral group D_n. Recall from Example 1.6 that

$$D_n = \{1, x, x^2, \dots, x^{n-1}, y, xy, x^2y, \dots, x^{n-1}y\},$$

where

$$x^n = 1,\ y^2 = 1, \text{ and } xy = yx^{-1}. \tag{1.4}$$

It follows from the last equality in (1.4) that

$$x^r y = yx^{-r} \text{ and } x^r y = (x^r y)^{-1} \quad (r \in \mathbb{Z}). \tag{1.5}$$

Now, it is clear that $[D_1, D_1] = [D_2, D_2] = 1$ by Example 1.6 because D_1 and D_2 are abelian. We claim that $[D_n, D_n] = gp\left(x^2\right)$ for $n \geq 3$.

From this point on, suppose $n \geq 3$ and let r and s denote integers. Choose $x^{2r} \in gp\left(x^2\right)$, and observe that this element can be written as a commutator as follows:

$$x^{2r} = x^r y^{-1} y x^r = x^r y^{-1} x^{-r} y = [x^{-r}, y],$$

where the second equality is a consequence of (1.5). Thus, $gp\left(x^2\right) \leq [D_n, D_n]$. To prove that $[D_n, D_n] \leq gp\left(x^2\right)$, we use (1.4) and (1.5). Suppose that $[a, b] \in [D_n, D_n]$. There are four possible cases for a and b.

- If $a = x^r$ and $b = x^s$, then $[x^r, x^s] = 1 \in gp\left(x^2\right)$.
- If $a = x^r$ and $b = x^s y$, then

$$\begin{aligned}[x^r, x^s y] &= x^{-r}\left(x^s y\right)^{-1} x^r x^s y = x^{-r} y^{-1} x^{-s} x^r x^s y \\ &= x^{-r} y^{-1} x^r y = x^{-r} y^{-1} y x^{-r} = x^{-2r} \in gp\left(x^2\right).\end{aligned}$$

- Suppose $a = x^r y$ and $b = x^s$. Then

$$[x^r y, x^s] = [x^s, x^r y]^{-1} \in gp\left(x^2\right)$$

  by the previous case and Lemma 1.4 (iii).
- Suppose $a = x^r y$ and $b = x^s y$. Then

$$\begin{aligned}[x^r y, x^s y] &= \left(x^r y\right)^{-1}\left(x^s y\right)^{-1} x^r y x^s y = x^r y x^s y x^r y x^s y \\ &= x^r x^{-s} y y x^r x^{-s} y y = x^{2r-2s} \in gp\left(x^2\right).\end{aligned}$$

It follows that $[D_n, D_n] \leq gp\left(x^2\right)$. And so, $[D_n, D_n] = gp\left(x^2\right)$ for $n \geq 3$ as claimed. In fact, $[D_n, D_n] = gp\left(x^2\right) = gp\,(x)$ whenever $n \geq 3$ is odd.

Example 1.10 We show that the derived subgroup of the Heisenberg group $\mathscr{H}$ equals its center. By Example 1.5, the center of $\mathscr{H}$ is

$$Z(\mathscr{H}) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, c \in \mathbb{Z} \right\} = gp\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right). \tag{1.6}$$

Let

$$a = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix}$$

be elements of $\mathscr{H}$. A simple calculation shows that

$$[a, b] = \begin{pmatrix} 1 & 0 & a_1 b_3 - b_1 a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, each commutator of elements of $\mathscr{H}$ is central, and thus $[\mathscr{H}, \mathscr{H}] \leq Z(\mathscr{H})$. In addition, the generator of $Z(\mathscr{H})$ in (1.6) is a commutator of elements of $\mathscr{H}$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right].$$

It follows that $[\mathscr{H}, \mathscr{H}] = Z(\mathscr{H})$.

1.3.1 Properties of Commutator Subgroups

We collect several properties of commutator subgroups.

Definition 1.12 Let G be any group, and let S be a nonempty subset of G. The *normalizer* of S in G, denoted by $N_G(S)$, is

$$N_G(S) = \{g \in G \mid gS = Sg\}.$$

If H is a subgroup of G, then $N_G(H)$ is the largest subgroup of G in which H is normal. If K is another subgroup of G, then K *normalizes* H if $K \leq N_G(H)$. Clearly, $N_G(H) = G$ if and only if $H \trianglelefteq G$.

Theorem 1.3 *Let G be a group and $H \leq G$. Then $C_G(H) \trianglelefteq N_G(H)$ and the factor group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.*

In particular, we obtain Corollary 1.1 when $H = G$.

Proof By Theorem 1.1, the map

$$\varrho : G \to Aut(H) \text{ defined by } \varrho(h) = \varphi_h, \text{ where } \varphi_h(g) = g^h,$$

is a homomorphism. Thus, $\varrho|_{N_G(H)}$, the restriction of ϱ to $N_G(H)$, is a homomorphism. It is easy to verify that $\varrho|_{N_G(H)}$ has kernel $C_G(H)$. The result follows from the First Isomorphism Theorem. □

Proposition 1.1 *Let G be any group with subgroups H and K.*

(i) $[H, K] = [K, H]$.
(ii) $[H, K] \leq H$ *if and only if K normalizes H. In particular,* $[H, G] < H$ *if and only if $H \trianglelefteq G$.*
(iii) *If $H_1 < G$ and $K_1 < G$ such that $H_1 \leq H$ and $K_1 \leq K$, then* $[H_1, K_1] \leq [H, K]$.

We point out that (i) is valid for any two *subsets* H and K of G.

Proof

(i) By Lemma 1.4 (iii),

$$\begin{aligned}[H, K] &= gp\,([h, k] \mid h \in H, k \in K) \\ &= gp\left([k, h]^{-1} \mid h \in H, k \in K\right) \\ &= [K, H].\end{aligned}$$

(ii) If $[H, K] \leq H$, then $[h, k] \in H$ for any $h \in H$ and $k \in K$. This means that $k^{-1}hk \in H$, and consequently, $k^{-1}Hk < H$. Similarly, we have $kHk^{-1} \leq H$. Therefore, $k^{-1}Hk = H$; that is, $k \in N_G(H)$. Conversely, if $k \in N_G(H)$, then $[h, k] \in H$ for all $h \in H$. A routine check confirms that $[H, K] \leq H$.

(iii) The proof is straightforward. □

Definition 1.13 Let G be a group with $H \leq G$, and let $A \subseteq Aut(G)$.

(i) If $\varphi(h) \in H$ for every $\varphi \in A$ and $h \in H$, then H is called *A-invariant*.
(ii) If H is $Aut(G)$-invariant, then H is called *characteristic* in G.
(iii) If every endomorphism of G restricts to an endomorphism of H, then H is *fully invariant*.

Clearly, every fully invariant subgroup must be characteristic. Furthermore, every characteristic subgroup is normal. We record this as a lemma.

Lemma 1.7 *Let G be any group. If H is a characteristic subgroup of G, then $H \trianglelefteq G$.*

Proof If H is a characteristic subgroup of G, then $\varphi(H) = H$ for every $\varphi \in Aut(G)$. In particular, $\varphi_g(H) = H$, where $g \in G$ and φ_g is the inner automorphism induced by g. Thus, $g^{-1}Hg = H$ for every $g \in G$; that is, $H \trianglelefteq G$. □

The next property of characteristic subgroups will be useful later.

Lemma 1.8 *Let G be a group with subgroups H and K. If H is characteristic in K and $K \triangleleft G$, then $H \triangleleft G$.*

Proof Choose any element $g \in G$. Since $K \triangleleft G$, there is an endomorphism

$$\varphi_g : K \to K \ \text{ defined by } \ \varphi_g(x) = x^g.$$

It is easy to verify that $\varphi_g \in Aut(K)$. Since H is characteristic in K, $\varphi_g(H) = H$. And so, $g^{-1}Hg = H$. This is true for all $g \in G$ since g was arbitrarily chosen. □

Proposition 1.2 *Let G and H be groups, and let G_1 and G_2 be subgroups of G.*

(i) If $\theta \in Hom(G, H)$, then $\theta([G_1, G_2]) = [\theta(G_1), \theta(G_2)]$.
(ii) Let $A \subseteq Aut(G)$. If G_1 and G_2 are A-invariant, then $[G_1, G_2]$ is also A-invariant.

Proof

(i) If $g_{1_j} \in G_1,\ g_{2_j} \in G_2$, and $\varepsilon_j \in \{-1,\ 1\}$ for $1 \le j \le k$, then

$$\begin{aligned}\theta\left(\prod_{j=1}^{k}\left[g_{1_j},\ g_{2_j}\right]^{\varepsilon_j}\right) &= \prod_{j=1}^{k}\theta\left(\left[g_{1_j},\ g_{2_j}\right]\right)^{\varepsilon_j}\\ &= \prod_{j=1}^{k}\theta\left(g_{1_j}^{-1}g_{2_j}^{-1}g_{1_j}g_{2_j}\right)^{\varepsilon_j}\\ &= \prod_{j=1}^{k}\left[\theta\left(g_{1_j}\right)^{-1}\theta\left(g_{2_j}\right)^{-1}\theta\left(g_{1_j}\right)\theta\left(g_{2_j}\right)\right]^{\varepsilon_j}\\ &= \prod_{j=1}^{k}\left[\theta\left(g_{1_j}\right),\ \theta\left(g_{2_j}\right)\right]^{\varepsilon_j}.\end{aligned}$$

(ii) We show that $\varphi\left([G_1,\ G_2]\right) \le [G_1,\ G_2]$ for any $\varphi \in A$. Let

$$\prod_{j=1}^{k}\left[g_{1_j},\ g_{2_j}\right]^{\varepsilon_j} \in [G_1,\ G_2]$$

as above. Since G_1 and G_2 are A-invariant subgroups, a computation similar to (i) gives

$$\varphi\left(\prod_{j=1}^{k}\left[g_{1_j},\ g_{2_j}\right]^{\varepsilon_j}\right) = \prod_{j=1}^{k}\left[\varphi\left(g_{1_j}\right),\ \varphi\left(g_{2_j}\right)\right]^{\varepsilon_j} \in [G_1,\ G_2].$$

This completes the proof. □

It follows from Proposition 1.2 (i) that the derived subgroup of any group is always fully invariant.

Corollary 1.2 *Let G be a group and $N \trianglelefteq G$. If $H \le G$ and $K \le G$, then*

$$[HN/N,\ KN/N] = [H,\ K]N/N.$$

Proof If $\pi : G \to G/N$ is the natural homomorphism, then $\pi(H) = HN/N$ and $\pi(K) = KN/N$. Apply Proposition 1.2 (i). □

Lemma 1.9 *Let G be a group, and suppose that $N \trianglelefteq G$ and $H < G$. Then $[H,\ G] \le N$ if and only if $HN/N \le Z(G/N)$.*

Proof Suppose that $HN/N \leq Z(G/N)$. If $h \in H$, then $(hN)(gN) = (gN)(hN)$ for any $g \in G$. This means that $[hN,\ gN] = N$. Since $[hN,\ gN] = [h,\ g]N$, it follows that $[h,\ g] \in N$. Consequently, every commutator of an element of H and an element of G is contained in N. It follows that $[H,\ G]$ is a subgroup of N.

Conversely, suppose that $[H,\ G] \leq N$ and let $hN \in HN/N$ and $gN \in G/N$. Since $[hN,\ gN] = [h,\ g]N$ and $[h,\ g] \in N$ by hypothesis, we have $[hN,\ gN] = N$. Thus, $hN \in Z(G/N)$. □

Theorem 1.4 *Let G be a group. If H and K are normal subgroups of G, then $[H,\ K] \trianglelefteq G$ and $[H,\ K] \leq H \cap K$. In particular, every element of H commutes with every element of K whenever $H \cap K = 1$.*

Proof Suppose that $g \in G$ and $\prod_{i=1}^{n}[h_i,\ k_i]^{\varepsilon_i} \in [H,\ K]$, where $h_i \in H,\ k_i \in K$, and $\varepsilon_i \in \{-1,\ 1\}$. By Lemmas 1.1 and 1.4 (iv),

$$\left([h_1,\ k_1]^{\varepsilon_1}[h_2,\ k_2]^{\varepsilon_2}\cdots[h_n,\ k_n]^{\varepsilon_n}\right)^g = \left([h_1,\ k_1]^{\varepsilon_1}\right)^g\left([h_2,\ k_2]^{\varepsilon_2}\right)^g\cdots\left([h_n,\ k_n]^{\varepsilon_n}\right)^g$$
$$= \left(\left[h_1^g,\ k_1^g\right]^{\varepsilon_1}\right)\left(\left[h_2^g,\ k_2^g\right]^{\varepsilon_2}\right)\cdots\left(\left[h_n^g,\ k_n^g\right]^{\varepsilon_n}\right)$$

is contained in $[H,\ K]$ since H and K are normal in G. Thus, $[H, K] \trianglelefteq G$. Furthermore,

$$\prod_{i=1}^{n}[h_i,\ k_i]^{\varepsilon_i} = \prod_{i=1}^{n}\left(h_i^{-1}\left(k_i^{-1}h_ik_i\right)\right)^{\varepsilon_i} \in H$$

and

$$\prod_{i=1}^{n}[h_i,\ k_i]^{\varepsilon_i} = \prod_{i=1}^{n}\left(\left(h_i^{-1}k_i^{-1}h_i\right)k_i\right)^{\varepsilon_i} \in K.$$

Thus, $\prod_{i=1}^{n}[h_i,\ k_i]^{\varepsilon_i} \in H \cap K$, and therefore, $[H,\ K] \leq H \cap K$. □

An easy induction argument gives:

Corollary 1.3 *If $G_1,\ \ldots,\ G_n$ are normal subgroups of a group G, then the subgroup $[G_1,\ \ldots,\ G_n]$ is normal in G.*

Lemma 1.10 *If $H,\ K$, and L are normal subgroups of a group G, then*

$$[HK,\ L] = [H,\ L][K,\ L] \text{ and } [H,\ KL] = [H,\ K][H,\ L].$$

Proof The result follows from Lemma 1.4 (v) and (vi), together with Theorem 1.4. □

More generally, we have:

Lemma 1.11 *If $\{G_1, \ldots, G_n, H_1, H_2\}$ is a set of normal subgroups of a group G, then*

(i) $[G_1, \ldots, G_n, H_1H_2] = \prod_{i=1}^{2}[G_1, \ldots, G_n, H_i]$;

(ii) $[H_1H_2, G_1, \ldots, G_n] = \prod_{i=1}^{2}[H_i, G_1, \ldots, G_n]$;

(iii) $[G_1, \ldots, G_{m-1}, H_1H_2, G_{m+1}, \ldots, G_n] = \prod_{i=1}^{2}[G_1, \ldots, G_{m-1}, H_i, G_{m+1}, \ldots, G_n]$ *for* $1 < m < n$.

Proof

(i) Note that $[G_1, \ldots, G_n] \trianglelefteq G$ by Corollary 1.3. The result follows from Lemma 1.10.

(ii) Set $n = 2$. By Lemma 1.10,

$$\begin{aligned}[H_1H_2, G_1, G_2] &= [[H_1H_2, G_1], G_2] \\ &= [[H_1, G_1][H_2, G_1], G_2].\end{aligned}$$

Now, $[H_1, G_1]$ and $[H_2, G_1]$ are normal in G by Theorem 1.4. Another application of Lemma 1.10 gives

$$[[H_1, G_1][H_2, G_1], G_2] = [H_1, G_1, G_2][H_2, G_1, G_2].$$

We iterate this procedure for any n to obtain the desired result.

(iii) Let $C = [G_1, \ldots, G_{m-1}]$. By Corollary 1.3, Lemma 1.10, and (ii) above, we have

$$\begin{aligned}[C, H_1H_2, G_{m+1}, \ldots, G_n] &= [[C, H_1H_2], G_{m+1}, \ldots, G_n] \\ &= [[C, H_1][C, H_2], G_{m+1}, \ldots, G_n] \\ &= [[C, H_1], G_{m+1}, \ldots, G_n][[C, H_2], G_{m+1}, \ldots, G_n] \\ &= [C, H_1, G_{m+1}, \ldots, G_n][C, H_2, G_{m+1}, \ldots, G_n].\end{aligned}$$

This completes the proof. □

The next two lemmas pertain to central commutator subgroups.

Lemma 1.12 (P. Hall) *Let G be a group with subgroups H and K, and suppose that $[H, K] \le Z(G)$. For any $a \in H$ and $b \in K$, the maps*

$$\varphi_a : K \to Z(G) \text{ defined by } \varphi_a(k) = [a, k]$$

and

$$\varphi_b : H \to Z(G) \text{ defined by } \varphi_b(h) = [h, b]$$

are homomorphisms.

Proof Suppose that $k_1, k_2 \in K$. By Lemma 1.4 (vi),

$$[a, k_1k_2] = [a, k_2][a, k_1]^{k_2} = [a, k_2][a, k_1].$$

Therefore, φ_a is a homomorphism. In a similar way, one can show that φ_b is a homomorphism. □

Lemma 1.13 *Let G be any group. If $[g, h] \in Z(G)$ for some $g, h \in G$ and $n \in \mathbb{Z}$, then*

$$[g^n, h] = [g, h]^n = [g, h^n].$$

Proof The result is obvious for $n = 0$ and $n = 1$, and Lemma 1.12 gives the result when $n \ge 2$. Suppose that $n < 0$. Since $[g, h]$ is central, so is $[g, h]^{-n}$. This, together with Lemma 1.4 (viii), implies

$$\begin{aligned}[g^n, h] &= \left[(g^{-n})^{-1}, h\right] = \left([g^{-n}, h]^{g^n}\right)^{-1} \\ &= \left(([g, h]^{-n})^{g^n}\right)^{-1} = ([g, h]^{-n})^{-1} \\ &= [g, h]^n .\end{aligned}$$

In a similar way, one can show that $[g, h^n] = [g, h]^n$. □

1.3.2 The Normal Closure

Let S and T be nonempty subsets of a group G. Denote by S^T, the subgroup of G generated by all conjugates of elements of S by elements of T :

$$S^T = gp\left(t^{-1}st \,\middle|\, s \in S, t \in T\right).$$

It is easy to see that if $H \leq G$, then S^H is the smallest normal subgroup of $gp(S,\ H)$ containing S. We call S^H the *normal closure* of S in $gp(S,\ H)$.

We record some fundamental properties on normal closures and commutator subgroups.

Proposition 1.3 *Let G be a group with $H \leq G$ and $\emptyset \neq S \subseteq G$.*

(i) $S^H = gp\,(S,\ [S,\ H])$.
(ii) $[S,\ H]^H = [S,\ H]$.
(iii) *If $H = gp(T)$ for some $\emptyset \neq T \subseteq G$, then $[S,\ H] = [S,\ T]^H$ and $[H,\ S] = [T,\ S]^H$.*

Proof

(i) Note first that $S \subseteq S^H$ because $H \leq G$. Moreover, any generator of $[S,\ H]$ can be written as $[s,\ h] = s^{-1}s^h$ with $s \in S$ and $h \in H$. Thus, $gp\,(S,\ [S,\ H]) \leq S^H$. It follows from Lemma 1.4 (ii) that $S^H \leq gp\,(S,\ [S,\ H])$.

(ii) Since $H \leq G$, $[S,\ H] \subseteq [S,\ H]^H$. We establish the reverse inclusion. By definition and Lemma 1.1,

$$\begin{aligned}[S,\ H]^H &= gp\left(x^h \,\middle|\, x \in [S,\ H],\ h \in H\right)\\ &= gp\left([s,\ h_1]^{h_2} \,\middle|\, s \in S,\ h_1 \in H,\ h_2 \in H\right).\end{aligned}$$

By Lemma 1.4 (vi),

$$[s,\ h_1]^{h_2} = [s,\ h_2]^{-1}[s,\ h_1h_2].$$

Consequently, $[s,\ h_1]^{h_2} \in [S,\ H]$, and thus $[S,\ H]^H = [S,\ H]$.

(iii) It is enough to prove that $[S,\ H] = [S,\ T]^H$. First, observe that $[S,\ T] \leq [S,\ H]$ because $T \subseteq H$. This implies that $[S,\ T]^H \leq [S,\ H]^H$. By (ii), $[S,\ T]^H \leq [S,\ H]$. It suffices to show that $[s,\ h] \in [S, T]^H$ for any $s \in S$ and $h \in H$. Since $H = gp(T)$, we can write

$$h = t_1^{\varepsilon_1}t_2^{\varepsilon_2}\cdots t_m^{\varepsilon_m}$$

for $t_i \in T$ and $\varepsilon_i \in \{-1,\ 1\}$. The proof is done by induction on m. If $m = 1$ and $\varepsilon_1 = 1$, then $[s,\ t_1] \in [S,\ T]^H$. If $m = 1$ and $\varepsilon_1 = -1$, then $\left[s,\ t_1^{-1}\right] = \left([s,\ t_1]^{t_1^{-1}}\right)^{-1}$ is also contained in $[S,\ T]^H$.

Assume that the result is true for $m - 1$. If $m > 1$, then Lemma 1.4 (vi), together with induction, implies that

$$[s,\ h] = \left[s,\ t_1^{\varepsilon_1}t_2^{\varepsilon_2}\cdots t_{m-1}^{\varepsilon_{m-1}}t_m^{\varepsilon_m}\right] = \left[s,\ t_m^{\varepsilon_m}\right]\left[s,\ t_1^{\varepsilon_1}t_2^{\varepsilon_2}\cdots t_{m-1}^{\varepsilon_{m-1}}\right]^{t_m^{\varepsilon_m}}$$

is contained in $[S,\ T]^H$. □

Corollary 1.4 *If G is a group with $H \leq G$ and $K \leq G$, then $[H,\ K] \trianglelefteq gp(H,\ K)$.*

Proof By Proposition 1.3 (ii), $[H,\ K]^K = [H,\ K]$ and $[K,\ H]^H = [K,\ H]$. Hence,

$$[H,\ K]^K = [H,\ K] = [H,\ K]^H$$

by Proposition 1.1 (i). Consequently, both H and K normalize $[H,\ K]$. □

The next two corollaries follow from Proposition 1.3 (iii).

Corollary 1.5 *Let H and K be subgroups of a group G, and let S and T be nonempty subsets of G. If $H = gp(S)$ and $K = gp(T)$, then $[H,\ K] = \left([S,\ T]^H\right)^K$.*

Corollary 1.6 *If G is a group and $H_1,\ \ldots,\ H_n$ are normal subgroups of G, then*

$$[H_1,\ \ldots,\ H_n] = gp\left([h_1,\ \ldots,\ h_n] \mid h_i \in H_i \ \text{for}\ \ i = 1,\ \ldots,\ n\right).$$

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