A Note on Analytically Irreducible Domains

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Abstract If D is a one-dimensional, Noetherian, local domain, it is well known that D is analytically irreducible if and only if D is unibranched and the integral closure D' of D is finitely generated as D-module. However, the proof of this result is split into pieces and spread over the literature. This paper collects the pieces and assembles them to a complete proof. Next to several results on integral extensions and completions of modules, we use Cohen's structure theorem for complete, Noetherian, local domains to prove the main result. The purpose of this survey is to make this characterization of analytically irreducible domains more accessible.

Keywords Unibranched • Analytically irreducible • Noetherian • Local • One-dimensional

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1 Introduction

Let $(D, \mathbf{m})$ be a Noetherian, one-dimensional, local domain. It is well known that the question of whether $\widehat{D}$ has zero-divisors or nilpotents is strongly connected to certain properties of the integral closure D' of D.

Definition 1.1 Let $(D, \mathbf{m})$ be a Noetherian, local domain with integral closure D' and $\mathbf{m}$-adic completion $\widehat{D}$. We say D is

1. *unibranched*, if D' is local,
2. *analytically unramified*, if $\widehat{D}$ is a reduced ring and
3. *analytically irreducible*, if $\widehat{D}$ is a domain.

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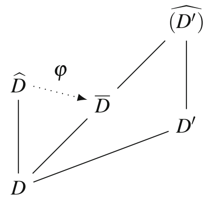

Fig. 1 Completions of D with respect to m and m' in case D' is local with maximal ideal m'

The aim of this paper is to give a complete proof of the following well-known theorem.

Theorem 1 *Let* $(D, \mathbf{m})$ *be a one-dimensional, Noetherian, local domain with integral closure* D'*. Then the following assertions are equivalent:*

1. *D is analytically irreducible.*
2. *D is unibranched and analytically unramified.*
3. *D is unibranched and* D' *is finitely generated as D-module.*
4. *D is unibranched and if* $\mathbf{m}'$ *denotes the maximal ideal of the integral closure, then the* $\mathbf{m}$*-adic topology on D coincides with subspace topology induced by* $\mathbf{m}'$*.*

Assume that D is unibranched and let $\mathbf{m}'$ be the unique maximal ideal of the integral closure D'. It follows from the Krull-Akizuki theorem that D' is a discrete valuation domain (see Corollary 2.5 below). In particular, D' is Noetherian and can be embedded into the $\mathbf{m}'$-adic completion $\widehat{(D')}$ of D'. Moreover, the valuation on D' can be extended to a valuation on $\widehat{(D')}$ which implies that $\widehat{(D')}$ is a domain (see Example 3.1). Since the completion $\overline{D}$ of D considered as a topological subspace of D' is the topological closure of D in $\widehat{(D')}$, it follows that $\overline{D} \subseteq \widehat{(D')}$ is a domain too.

On the other hand, D can be embedded into the $\mathbf{m}$-adic completion $\widehat{D}$ of D. Figure 1 above demonstrates the relationship between D, D' and the completions with respect to the different topologies. As usual, the solid lines represent inclusions. However, the dotted arrow deserves some additional explanation. Since $\mathbf{m}^n \subseteq \mathbf{m}'^n \cap D$ it follows that the $\mathbf{m}$-adic topology is finer than the $\mathbf{m}'$-adic subspace topology on D. This further implies that the inclusion $D \longrightarrow \overline{D}$ is a uniformly continuous homomorphism (where D is equipped with the $\mathbf{m}$-adic topology). Since $\overline{D}$ is complete, the inclusion can be uniquely extended to a uniformly continuous map $\varphi : \widehat{D} \longrightarrow \overline{D}$. Theorem 1 implies that D is analytically irreducible if and only if φ is an isomorphism. However, if $\widehat{D}$ is not a domain φ is not even injective.

Theorem 1 is well known but its proof is split into pieces and has to be assembled from several sources. This survey collects known results from different references in order to present a complete proof. We follow the approach of Nagata's textbook [7, (32.2)] for the implication (2) $\Rightarrow$ (3). For the remaining implications, we pursue the suggestions of [3, Theorem III.5.2]. One can also refer to [8, Theorem 8] for the implication (4) $\Rightarrow$ (1).

In order to make this survey more self-contained, we give a short introduction to integral ring extensions and completions in Sects. 2 and 3, respectively. Then, in Sect. 4 we discuss Cohen's structure theorem which allows us to prove that the integral closure of a complete, Noetherian, local, reduced ring R is a finitely generated R-module in Sect. 5. Finally, we give a proof of Theorem 1 in Sect. 6. It is worth mentioning that Sects. 4 and 5 are only needed for the implication (2) $\Rightarrow$ (3) whereas the remaining implications can be shown using the results in Sects. 2 and 3.

2 Integral Ring Extensions

In this section we recall some facts on integral ring extensions which we use throughout this paper.

Fact 2.1 (cf. [1, Proposition 5.1, Corollaries 5.3, 5.4]) *Let $R \subseteq S$ be a ring extension. We call $s \in S$* integral over *R if the following equivalent assertions are satisfied:*

1. *There exists a monic polynomial $f \in R[X]$ such that $f(s) = 0$.*
2. *$R[s]$ is finitely generated as R-module.*
3. *There exists a ring T containing $R[s]$ which is finitely generated as R-module.*

Let $R'_S = \{s \in S \mid s \text{ integral over } R\}$ denote the set of elements of S which are integral over R. Then $R \subseteq R'_S$ is a ring extension.

We call R'_S the integral closure of *R* in *S and if $R = R'_S$ we say R is* integrally closed in *S. If $S = R'_S$, we say $R \subseteq S$ is an* integral extension.

If $R \subseteq T \subseteq S$ is an intermediate ring such that both $R \subseteq T$ and $T \subseteq S$ are integral extensions, then $R \subseteq S$ is an integral extension. In particular, $R'_S = (R'_T)'_S$.

In case S is the total ring of quotients of R, we simplify and say $R' := R'_S$ is the integral closure of *R and R is* integrally closed *if $R = R'$.*

Fact 2.2 (Cohen-Seidenberg, cf. [1, Corollary 5.9, Theorem 5.11]) *Let $R \subseteq S$ be an integral extension. Then the following assertions hold:*

1. *If $Q_1 \subseteq Q_2$ are prime ideals of S such that $Q_1 \cap R = Q_2 \cap R$, then $Q_1 = Q_2$.*
2. *If $P_1, P_2 \in \mathsf{spec}(R)$ with $P_1 \subseteq P_2$ and $Q_1 \in \mathsf{spec}(S)$ with $Q_1 \cap R = P_1$, then there exists $Q_2 \in \mathsf{spec}(S)$ such that $Q_1 \subseteq Q_2$ and $Q_2 \cap R = P_2$.*
3. *$\mathsf{dim}(R) = \mathsf{dim}(S)$ and $\mathsf{max}(S) = \{P \in \mathsf{spec}(S) \mid P \cap R \in \mathsf{max}(R)\}$.*

As a first result we prove the so-called *Krull-Akizuki* theorem which is central to the remainder of this paper.

Proposition 2.3 (Krull-Akizuki, cf. [6, Theorem 11.7], [4, Theorem 4.9.2]) *Let D be a one-dimensional, Noetherian domain with quotient field K and L a finite field extension of K.*

Then the integral closure D'_L of D in L is a Dedekind domain. Moreover, if I is a nonzero ideal of D', then D'/I is a finitely generated D-module.

Proof We can reduce the proof to the case $L = K$ with the following argument. Let $b_1, \dots, b_n$ form a K-basis of L. Without restriction we can assume that $b_i \in D'_L$.

Then the domain $R = D[b_1, \dots, b_n]$ is finitely generated as D-module and therefore $D \subseteq R$ is an integral extension according to Fact 2.1. Further, R is Noetherian and since $L = K[b_1, \dots, b_n]$ is the quotient field of R it follows that $R' = D'_L$ is the integral closure of R. Moreover, if I is a nonzero ideal of R' and R'/I is finitely generated as R-module, then $R'/I = D'_L/I$ is a finitely generated D-module.

Hence from this point on we assume that $L = K$ and $D = R$. Since $1 = \mathsf{dim}(D) = \mathsf{dim}(D')$ by Fact 2.2 and D' is integrally closed, we only need to prove that D' is Noetherian to conclude that D' is a Dedekind domain.

Let I be a nonzero ideal of D' and $s = \frac{a}{t} \in I$ be a nonzero element. Then $a = ts \in I \cap D$ is a nonzero element which implies that D/aD is a zero-dimensional, Noetherian ring and thus Artinian. Since $I_n = (a^n D') \cap D + aD$ for $n \in \mathbb{N}$ form a descending chain of ideals of D/aD, there exists an $m \in \mathbb{N}$ such that $I_m = I_n$ for all $n \geq m$.

If $a^m D' \subseteq a^{m+1} D' + D$, then

$$D'/aD' \simeq a^m D'/a^{m+1} D' \subseteq (a^{m+1} D' + D)/a^{m+1} D' \simeq D/(D \cap a^{m+1} D')$$

holds. This further implies that D'/aD' is a submodule of the Noetherian module $D/(D \cap a^{m+1} D')$ and hence a finitely generated D-module. Hence D'/aD' is Noetherian and the submodule I/aD' is finitely generated. Consequently I is a finitely generated ideal of D'. In addition, $D'/I \simeq (D'/aD')/(I/aD')$ is a quotient of the finitely generated D-module D'/aD' and therefore finitely generated.

It remains to prove that $a^m D' \subseteq a^{m+1} D' + D$. We can localize at each maximal ideal of D and prove the inclusion locally. So assume that D is local with maximal ideal $\mathbf{m}$.

If $a \notin \mathbf{m}$, then a is a unit in D and therefore $a^m D' = D' = a^{m+1} D' + D$. Now assume $a \in \mathbf{m}$ and let $x = \frac{b}{c} \in D' \setminus D$ where $b \in D$ and $c \in \mathbf{m}$. The radical of the nonzero ideal cD is then $\mathbf{m}$ and therefore there exists $n \geq m$ with $\mathbf{m}^{n+1} \subseteq cD$. It follows that

$$a^{n+1} x \in (a^{n+1} D') \cap D \subseteq I_{n+1} = I_{n+2} = (a^{n+2} D') \cap D + aD$$

and hence $a^n x \in a^{n+1} D' + D$. If $n > m$, then

$$a^n x \in (a^{n+1} D' + D) \cap a^n D' = a^{n+1} D' + \underbrace{D \cap a^n D'}_{\subseteq I_n = I_{n+1}} \subseteq a^{n+1} D' + aD$$

and therefore $a^{n-1} x \in a^n D' + D$. Repeating this argument completes the proof.

Corollary 2.4 *Let D be a one-dimensional, Noetherian, local domain with quotient field K and L a finite field extension of K.*

If $\mathbf{m}$ is a maximal ideal of D and $\mathbf{m}'$ a maximal ideal of the integral closure D'_L of D in L with $\mathbf{m}' \cap D = \mathbf{m}$, then the field extension $D/\mathbf{m} \subseteq D'_L/\mathbf{m}'$ is finite.

Proof It follows from Proposition 2.3, that $D'_L/\mathbf{m}'$ is a finitely generated D-module. Therefore $D'_L/\mathbf{m}'$ is finitely generated as $D/\mathbf{m}$-vector space as well.

Corollary 2.5 *Let D be a one-dimensional, Noetherian, local domain.*
If D is unibranched, then the integral closure D' is a discrete valuation domain.

Remark If D is a local, one-dimensional, Noetherian domain with maximal ideal $\mathbf{m}$, then Fact 2.2 implies that the maximal ideals of D' are the minimal primes of $\mathbf{m}D$ and therefore D' is always a semilocal Dedekind domain.

3 Completions

In this section we recall the necessary facts on topologies on rings and modules which are induced by ideals. Let R be a Noetherian ring, I an ideal of R and M an R-module. Then the submodules $(I^nM)_{n\in\mathbb{N}}$ form a filtration on M which induces a linear topology on M, that is, the sets $m + I^nM$ for $m \in M$ and $n \in \mathbb{N}$ form a basis of this topology. We call this the *I-adic topology* on M.

Addition, subtraction and scalar multiplication are continuous with respect to this topology. If M is a ring extension of R, then multiplication in M is continuous too.

Moreover, $M \setminus m + I^nM = \bigcup_y y + I^nM$ where the union runs over all $y \in M$ with $m - y \notin I^nM$ and hence each $m + I^nM$ is both open and closed.

The completion $\widehat{M}$ of M is the inverse limit of the inverse system M/I^nM together with the canonical projections $M/I^nM \longrightarrow M/I^mM$ for $n \geq m$, that is,

$$\widehat{M} = \varprojlim M/I^nM = \left\{ (a_n + I^nM)_n \in \prod_{n\in\mathbb{N}} M/I^nM \;\middle|\; a_{n+1} \equiv a_n \pmod{I^nM} \right\}$$

A sequence $(x_k)_k$ in M is an I-adic Cauchy sequence, if for each n there exists k_n such that $x_{k_n} - x_{k_n+m} \in I^nM$ for all $m \in \mathbb{N}$. As usual, we say two Cauchy sequences $(x_k)_k$, $(y_k)_k$ are equivalent if $(x_k - y_k)_k$ converges to 0. In particular, (x_k) is equivalent to $(x_{k_n})_n$. Hence each equivalence class of Cauchy sequences in M contains the so-called *coherent* sequence $(a_n)_n$ which satisfies $a_{n+1} \equiv a_n \pmod{I^nM}$ for all n. Thus $\widehat{M}$ is isomorphic to the set of equivalence classes of Cauchy sequences.

If $\overline{\mathbf{0}} = \bigcap_{n\in\mathbb{N}} I^nM = \mathbf{0}$, then M is I-adically separated and we can embed M into $\widehat{M}$ via $m \longmapsto (m)_n$. We say that M is complete if $M \simeq \widehat{M}$.

Example 3.1 Let V be a discrete valuation domain with maximal ideal (t) and valuation v. By $\widehat{V}$ we denote the (t)-adic completion of V.

Moreover, let $(a_n)_n$ be a (t)-adic Cauchy sequence with limit $a \in \widehat{V}$. If $a = 0$, then for each $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $a_n \in (t^k)$ for all $n \geq n_0$ and hence $\lim v(a_n) = \infty$.

If $a \neq 0$, then there exist $k, m_0 \in \mathbb{N}$ such that $a_n \notin (t^k)$ for $n \geq m_0$. However, the sequence $(a_{n+1} - a_n)_n$ converges to 0 and hence there exists $m_1 \in \mathbb{N}$ such that

$a_{n+1} - a_n \in (t^k)$ for all $n \geq m_1$. If $m = \max\{m_0, m_1\}$, then for all $n \geq m$,

$$v(a_{n+1}) = v(a_{n+1} - a_n + a_n) \geq \min\{v(a_{n+1} - a_n), v(a_n)\} = v(a_n) < k$$

holds and the sequence $(v(a_n))_n$ stabilizes at $v(a_m)$.

For $a \in \widehat{V}$, we set $v(a) = \lim v(a_n) = v(a_m)$. This extends v to a discrete valuation on $\widehat{V}$.

Next, we present some basic results on the completion of finitely generated modules over a Noetherian ring R.

Fact 3.2 (Artin-Rees, cf. [6, Theorem 8.5]) *Let R be a Noetherian ring, I an ideal of R and M a finitely generated R-module.*

If N is an R-submodule of M, then there exists an integer r such that for all $k \geq 0$

$$I^{r+k}M \cap N = I^k(I^r M \cap N).$$

Corollary 3.3 (cf. [6, Theorem 8.9, Theorem 8.10]) *Let R be a Noetherian ring, I an ideal of R and M a finitely generated R-module.*

1. *If $N = \bigcap_{n\in\mathbb{N}} I^n M$, then there exists an $a \in R$ with $aN = \mathbf{0}$ and $1 - a \in I$.*
2. *If $I \subseteq \mathsf{Jac}(R)$, then M is I-adically separated and every submodule of M is I-adically closed.*

Proof (1): According to Fact 3.2, there exists $r \in \mathbb{N}$ such that $(I^{r+1}M) \cap N = I(I^r M \cap N) \subseteq IN$ and hence

$$IN \subseteq N = \bigcap_{n\in\mathbb{N}} I^n M \subseteq (I^{r+1}M) \cap N \subseteq IN.$$

Consequently, $N = IN$ and it follows from Nakayama's lemma that there exists an $a \in R$ with $1 - a \in I$ and $aN = \mathbf{0}$. (2): Since $I \subseteq \mathsf{Jac}(R)$, the element a from (1) is a unit of R. Therefore $\bigcap_{n\in\mathbb{N}} I^n M = \mathbf{0}$ and M is separated. Consequently, if P is a submodule of M, it follows that $\bigcap_{n\in\mathbb{N}} I^n(M/P) = \mathbf{0} = PM/P$ and therefore $\bigcap_{n\in\mathbb{N}}(P + I^n M) = P$.

Fact 3.4 (cf. [6, Theorem 8.7]) *Let R be a Noetherian ring, I an ideal and M a finitely generated module. Further, let $\widehat{R}$, $\widehat{M}$ be the I-adic completions of R and M, respectively.*

Then $\widehat{R} \otimes_R M \simeq \widehat{M}$ via $(\lim r_n, m) \mapsto \lim r_n m$. In particular, if R is I-adically complete, then M is I-adically complete.

Proposition 3.5 (cf. [6, Theorem 8.4]) *Let R be a complete ring with respect to an ideal I of R and M an I-adically separated R-module.*

If M/IM is a finitely generated R/I-module, then M is a finitely generated R-module.

Proof Let $m_1, \ldots, m_t \in M$ be elements such that their projections modulo IM generate M/IM as R/I-module. Then $M = \sum_{i=1}^{t} Rm_i + IM$ and for $x \in M$, there exist $r_{0,i} \in R$, $i_1 \in I$ and $x_1 \in M$ such that $x = \sum_{i=1}^{t} r_{0,i}m_i + i_1x_1$. Then again, for $x_1 \in M$, there exist $r_{1,i} \in R$, $i_2 \in I$ and $x_2 \in M$ such that $x_1 = \sum_{i=1}^{t} r_{1,i}m_i + i_2x_2$. For $j > 2$, we successively choose $r_{j-1,i} \in R$, $i_j \in I$ and $x_j \in M$ such that $x_{j-1} = \sum_{i=1}^{t} r_{j-1,i}m_i + i_jx_j$. Then $\left(\sum_{j=0}^{n} \left(\prod_{t=1}^{j} i_t\right) r_{j,i}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in R which has a limit $r_i \in R$. Moreover,

$$x - \sum_{i=1}^{t} r_im_i \in \bigcap_{n\in\mathbb{N}} I^nM = \mathbf{0}$$

and therefore M is generated by $m_1, \ldots, m_t$.

Zero-Divisors in the Completion of R

Let V be a discrete valuation domain with valuation v and $\widehat{V}$ its completion. If $a, b \in \widehat{V}$ are nonzero elements, then $v(a), v(b) < \infty$ according to Example 3.1. Consequently, $v(ab) = v(a) + v(b) < \infty$ which implies that $\widehat{V}$ is a domain.

However, in general the completion of a domain may not be a domain.

Proposition 3.6 (cf. [3, Lemma III.3.4]) *Let R be a Noetherian domain and I an ideal of R.*

If there exist ideals J_1 and J_2 of R such that $I = J_1 \cap J_2$ and $R = J_1 + J_2$, then the I-adic completion $\widehat{R}$ of R is not a domain.

Proof Since J_1 and J_2 are coprime it follows that J_1^k and J_2^k are coprime as well. Hence there exist $b_k, c_k \in R$ such that

$$b_k \equiv 0 \bmod J_1^k, \qquad b_k \equiv 1 \bmod J_2^k$$
$$c_k \equiv 1 \bmod J_1^k, \qquad c_k \equiv 0 \bmod J_2^k$$

for all $k \in \mathbb{N}$. Since $b_{k+1} - b_k \equiv c_{k+1} - c_k \equiv 0 \bmod J_1^k \cap J_2^k = J_1^kJ_2^k = I^k$, the sequences $(b_k)_k$ and $(c_k)_k$ converge I-adically. Let $b = \lim b_k \in \widehat{R}$ and $c = \lim c_k \in \widehat{R}$ their I-adic limits. By construction, $b \neq 0$ and $c \neq 0$. However, since $b_kc_k \equiv 0 \bmod J_1^k \cap J_2^k = I^k$ for all k, it follows that $bc = 0$. Hence b and c are nonzero zero-divisors.

It follows from Proposition 3.6 that the completion of a domain may contain zero-divisors (see also Proposition 3.8). However, constant sequences behave well as the next lemma states.

Lemma 3.7 *Let R be a Noetherian ring, I a proper ideal of R and $\widehat{R}$ the I-adic completion of R. If $d \in R$ is not a zero-divisor in R, then d is not a zero-divisor in $\widehat{R}$.*

Proof By Fact 3.2, there exists $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$I^{n+r} \cap dR = I^n(I^r \cap dD) \subseteq dI^n \tag{1}$$

Let $x = \lim_n x_n \in \widehat{R}$ such that $dx = 0$. Then $(dx_n)_n$ is an I-adic Cauchy sequence with limit 0. Hence for each $n \in \mathbb{N}$ there exists $t_0 \in \mathbb{N}$ such that $dx_t \in I^{n+r}$ for all $t \geq t_0$. Then $dx_t \in dI^n$ by Equation (1) and therefore $x_t \in I^n$ which implies $0 = \lim_n x_n = x$.

Proposition 3.8 *Let $(D, \mathbf{m})$ be a Noetherian, local domain with quotient field K and $D \subset R \subset K$ be an intermediate ring such that R is finitely generated as D-module.*

Then the following assertions hold:

1. *R is semilocal,*
2. *the $\mathbf{m}$-adic topology on D coincides with subspace topology induced by the $\mathsf{Jac}(R)$-adic topology on R.*
3. *If R is not local, then the $\mathbf{m}$-adic completion $\widehat{D}$ of D is not a domain.*

Proof R is finitely generated as module over the Noetherian domain D and hence a Noetherian domain. Moreover, the ring extension $D \subseteq R$ is integral by Fact 2.1. Therefore, according to Fact 2.2, all prime ideals of R which lie over $\mathbf{m}$ are maximal and thus minimal prime ideals of $\mathbf{m}R$. Hence there are only finitely many maximal ideals $N_1, \ldots, N_n$ in R which proves (1).

(2): Since $\sqrt{\mathbf{m}R} = \bigcap_{i=1}^n N_i = \mathsf{Jac}(R)$, there exists $\ell \in \mathbb{N}$ such that $\mathsf{Jac}(R)^\ell \subseteq \mathbf{m}R$. Then

$$\mathsf{Jac}(R)^{\ell k} \subseteq (\mathbf{m}R)^k \subseteq \mathsf{Jac}(R)^k$$

and hence the $\mathsf{Jac}(R)$-adic and the $\mathbf{m}R$-adic topology coincide on R. Thus it suffices to prove that the $\mathbf{m}$-adic topology on D coincides with subspace topology induced by the $\mathbf{m}R$-adic topology on R. Clearly, $\mathbf{m}^k \subseteq \mathbf{m}^k R \cap D$ holds for all k. On the other hand, Fact 3.2 implies that there exists an integer r such that

$$\mathbf{m}^{k+r} R \cap D = \mathbf{m}^k(\mathbf{m}^r R \cap D) \subseteq \mathbf{m}^k$$

for all $k \geq 0$ and hence the topologies coincide.

(3): Let $M_1, \ldots, M_n$ be the maximal ideals of R with $n > 1$. We set $J_1 = M_1 \cdots M_{n-1} = M_1 \cap \cdots \cap M_{n-1}$ and $J_2 = M_n$. Then $J_1 \cap J_2 = \mathsf{Jac}(R)$ and $J_1 + J_2 = R$. By Proposition 3.6, the $\mathsf{Jac}(R)$-adic completion $\widehat{R}$ of R is not a domain.

By (2), $\widehat{D}$ is a topological subspace of $\widehat{R}$. Since $\widehat{R}$ is not a domain, there exist nonzero $b, c \in \widehat{R}$ with $bc = 0$. However, R is a finitely generated D-module and therefore there exists a nonzero element $d \in D$ with $d\widehat{R} \subseteq \widehat{D}$. Hence $(db)(dc) = 0$ with $db, dc \in \widehat{D}$. By Lemma 3.7, d is not a zero-divisor in $\widehat{D}$ and therefore $db \neq 0$ is a zero-divisor in $\widehat{D}$.

4 Structure Theorem for One-Dimensional Complete Local Domains

In this section we discuss the structure theorem for complete, Noetherian, local domains. We restrict our study to the one-dimensional case, since this is what we need later on. Nevertheless, it is worth mentioning that the results can be extended to higher dimensions but the proofs become more technical.

As Proposition 4.3 states, a complete, Noetherian, one-dimensional local domain $(D, \mathbf{m})$ contains a subring S such that D is finitely generated as S-module. Moreover, S is a certain complete discrete valuation domain whose residue field is isomorphic to $D/\mathbf{m}$. This result allows us in the next section to reduce the investigation to domains of this form.

Definition 4.1 Let $(D, \mathbf{m})$ be a complete, Noetherian, local domain. We say

1. D is of *equal characteristic*, if $\mathrm{char}(D) = \mathrm{char}(D/\mathbf{m})$ and
2. D is of *unequal characteristic*, if $\mathrm{char}(D) \neq \mathrm{char}(D/\mathbf{m})$.

If $\mathrm{char}(D/\mathbf{m}) = 0$, it follows that $\mathrm{char}(D) = 0$ and therefore $\mathbb{Z} \subseteq D$. However, $\mathbb{Z} \cap \mathbf{m} = \mathbf{0}$ which implies that every integer is invertible in D and thus $\mathbb{Q} \subseteq D$. Similarly, if $\mathrm{char}(D) = p > 0$, then $\mathrm{char}(D/\mathbf{m}) = p$ and $\mathbb{Z}/p\mathbb{Z}$ is contained in D. Hence, a domain of equal characteristic contains a field. On the other hand, if D contains a field k, then $\mathrm{char}(k) = \mathrm{char}(D)$. Let $\pi : D \longrightarrow D/\mathbf{m}$ denote the canonical projection. Then $\pi(k)$ is a subfield of $D/\mathbf{m}$ and since $\mathrm{char}(\pi(k)) = \mathrm{char}(k)$ it follows that D is of equal characteristic. Indeed, it is possible to show that a domain D of equal characteristic contains a field k with $\pi(k) = D/\mathbf{m}$, cf. Fact 4.2.

If D is a domain of unequal characteristic, then $\mathrm{char}(D) = 0$ and $\mathrm{char}(D/\mathbf{m}) = p > 0$. In this case it is possible to show that D contains a complete discrete valuation domain (R, pR) such that the residue fields of R and D are isomorphic. We summarize these results in Fact 4.2. However, the proof goes beyond the scope of this paper. We refer to Matsumura's textbook [6] for details.

Fact 4.2 (cf. [6, Theorem 28.3, Theorem 29.3]) *Let $(D, \mathbf{m})$ be a complete, Noetherian, local domain.*

1. *If D is of equal characteristic, then D contains a field k which is isomorphic to $D/\mathbf{m}$ via $d \longmapsto d + \mathbf{m}$. We say k is a* coefficient field of D.
2. *If D is of unequal characteristic and $\mathrm{char}(D/\mathbf{m}) = p$, then D contains a complete discrete valuation domain (R, pR) such that R/pR is isomorphic to $D/\mathbf{m}$ via $r + pR \longmapsto r + \mathbf{m}$. We say R is a* coefficient ring of D.

The existence of a coefficient field or coefficient ring, respectively, is crucial for the proof of the structure theorem which we state in the next proposition.

Proposition 4.3 (cf. [6, Theorem 29.4.(iii)]) *Let $(D, \mathbf{m})$ be a complete, Noetherian, one-dimensional, local domain.*

Then D contains a complete discrete valuation domain S such that D is finitely generated over S and

1. *in equal characteristic* $S \simeq k[\![X]\!]$ *where k is a coefficient field of D.*
2. *in unequal characteristic S is a coefficient ring of D.*

Proof Let $\mathbf{m}$ be the maximal ideal of D. First, we consider the case where D is of unequal characteristic and let $p = \mathsf{char}(D/\mathbf{m}) > 0$. According to Fact 4.2, D contains a coefficient ring S which is a complete discrete valuation domain with maximal ideal pS such that $S/pS \simeq D/\mathbf{m}$ via π.

Further, D/pD is a zero-dimensional, Noetherian ring and hence Artinian. Therefore D/pD has finite length as (D/pD)-module and hence as D-module. However, this is equivalent to the existence of a composition series $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = D/pD$ of the D-module D/pD. Since N_{i+1}/N_i is simple for all $0 \leq i \leq r-1$ it follows that $N_{i+1}/N_i \simeq D/\mathbf{m} \simeq S/pS$ which implies that $(N_i)_{i=0}^{r}$ is a composition series of D/pD as S-module. Thus D/pD has finite length as S-module and is therefore a finite dimensional (S/pS)-vector space. Further, D is p-adically separated since D is Noetherian and we can conclude that D is a finitely generated S-module by Proposition 3.5.

If D is of equal characteristic, then D contains a coefficient field k which is a subfield of D such that $k \simeq D/\mathbf{m}$ via π according to Fact 4.2.

Let $T = k[\![X]\!]$ be the power series ring in the variable X and let $y \in D$ be a nonzero non-unit. We define the k-homomorphism $\varphi : T \longrightarrow D$ by $\varphi(X) = y$ and set $S = k[\![y]\!]$ to be the image of T under φ.

With the same argument as above we can conclude that D/yD is a finitely generated (S/yS)-module. Moreover, S is complete and D is separated with respect to the ideal yS. Hence D is finitely generated as S-module by Proposition 3.5. Furthermore, this implies $\dim(S) = \dim(D) = 1$ by Fact 2.2. However, since $S \simeq k[\![X]\!]/\ker(\varphi)$ and $\dim(k[\![X]\!]) = 1$ it follows that $\ker(\varphi) = \mathbf{0}$ and $S \simeq k[\![X]\!]$.

Remark The domain S is the so-called *regular local ring*, that is, a local, Noetherian domain S such that its maximal ideal is generated by $\mathsf{dim}(S)$ elements. Proposition 4.3 is a special case of Cohen's structure theorem which states that every complete Noetherian local domain D contains a regular local subring S such that D is finitely generated as S-module. Moreover, $S = R[\![X_1, \ldots, X_n]\!]$ is a power series ring where in equal characteristic R is a coefficient field and $n = \mathsf{dim}(D)$ and in unequal characteristic R is a coefficient ring and $n = \mathsf{dim}(D) - 1$. For details, we refer Matsumura's textbook [6, §28, §29].

5 Finiteness of the Integral Closure

Let D be a complete, one-dimensional, Noetherian, local domain with quotient field K and let $K \subseteq L$ be a finite field extension. The goal of this section is to prove that the integral closure D'_L of D in L is finitely generated as D-module (see Proposition 5.3). This allows us to conclude in Corollary 5.4 that the integral closure R' of a complete, one-dimensional, Noetherian, local, reduced ring R is finitely generated as R-module. The latter result is essential in the proof of Theorem 1 in the next section.

Following Nagata's textbook [7], we exploit the structure of complete, Noetherian, local domains. According to Proposition 4.3, D contains a subring S such that D is finitely generated as S-module (see Figure 2). If F is the quotient field of S, then the extension $F \subseteq L$ is finite. Moreover, by Fact 2.1, the extension $S \subseteq D$ is integral and hence $D'_L = S'_L$ is the integral closure of S in L.

If we show that S'_L is finitely generated as S-module, then it follows that D'_L is a finitely generated D-module. Therefore, Proposition 4.3 allows us to reduce the investigation to the case where S is a certain complete discrete valuation domain.

To prove that the integral closure of S in L is finitely generated, we distinguish between two cases, either the field extension $F \subset L$ is separable or it is inseparable.

Proposition 5.1 (cf. [2, Ch. V, 1.6, Corollary 1 of Proposition 18]) *Let S be an integrally closed, Noetherian domain with quotient field F and $F \subseteq L$ a finite field extension.*

If $F \subseteq L$ is separable, then the integral closure S'_L of S in L is finitely generated as S-module.

Proof Let $w_1, \dots, w_n \in L$ be a K-basis of L. Without restriction we can assume that $w_j \in S'_L$ for $1 \leq j \leq n$. Further, let $L^\star = \mathsf{Hom}_K(L, K)$ be the dual space of L and $w'_i \in L^\star$ be the K-basis of $L^\star$ which is defined by $w'_i(w_j) = \delta_{ij}$ (Kronecker-delta) for $1 \leq i, j \leq n$.

Since L is a finite separable extension of K, L is isomorphic to its dual space $L^\star$ via the K-linear map

$$\begin{aligned} T : L &\longrightarrow L^\star \\ x &\longmapsto (y \mapsto \mathsf{tr}_{L/K}(xy)) \end{aligned}$$

where $\mathsf{tr}_{L/K} : L \longrightarrow K$ is the field trace with respect to the extension $K \subseteq L$ (cf. [5, Theorem 5.2]).

For $1 \leq i \leq n$, set $w_i^\star = T^{-1}(w'_i)$. Then $w_1^\star, \dots, w_n^\star$ form a K-basis of L and $\mathsf{tr}_{L/K}(w_i^\star w_j) = \delta_{ij}$ holds for all $1 \leq i, j \leq n$. Hence, for $a \in S'_L$ there exist $a_i \in K$ such that $a = \sum_{i=1}^n a_i w_i^\star$. Moreover,

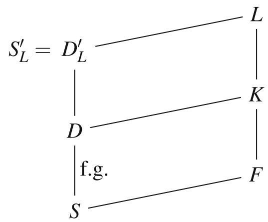

Fig. 2 D contains a subring S such that D is finitely generated as S-module which is either a complete discrete valuation ring or isomorphic to $k[\![X]\!]$ where k is a coefficient field of D

$$a_j = \sum_{i=1}^{n} a_i \,\mathsf{tr}_{L/K}(w_i^\star w_j) = \mathsf{tr}_{L/K}(aw_j)$$

holds for $1 \leq j \leq n$.

If $g_j \in K[X]$ is the minimal polynomial of aw_j and $z_1, \dots, z_m$ are all roots of g_j in some field extension $\tilde{L}$ of K, then $\mathsf{tr}_{L/K}(aw_j) = \sum_{i=1}^m z_i \in K$ holds (cf. [5, p. 284]). Moreover, since aw_j is integral over S it follows that $g_j \in S[X]$ and $z_1, \dots, z_m \in S'_{\tilde{L}}$ are integral over S as well. Therefore

$$a_j = \mathsf{tr}_{L/K}(aw_j) \in K \cap S'_{\tilde{L}} = S$$

where the last equality holds since S is integrally closed. It follows that S'_L is an S-submodule of the Noetherian module $\sum_{i=1}^n Sw_i^\star$ and therefore finitely generated.

Proposition 5.2 (cf. [6, p. 263]) *Let $S = k[\![X]\!]$ be a power series ring over a field k with quotient field F and L a finite purely inseparable field extension of F.*

Then the integral closure S'_L of S in L is finitely generated as S-module.

Proof Let $p > 0$ be the characteristic of the field F and $q = p^e = [L : F] < \infty$ be the degree of the field extension $F \subseteq L$. Since the extension is purely inseparable, every element $a \in L$ is a q-th root of an element in F.

Let $\overline{F}$ be an algebraically closed extension of F that contains L. Then $\overline{F}$ contains an element Y such that $X = Y^q$ and $\tilde{L} = L(Y)$ is a finite, purely inseparable field extension of K. Moreover, S'_L is an S-submodule of $S'_{\tilde{L}}$ and it therefore suffices to show that $S'_{\tilde{L}}$ is a finitely generated S-module. This allows us to assume that $L = \tilde{L}$ and $Y \in L$ from this point on.

If $a \in S'_L$ is an integral element, then $a^q \in S'_L \cap F$. However, S is a discrete valuation domain, so it is integrally closed and therefore $S'_L = \{a \in L \mid a^q \in S\}$.

If M is a maximal ideal of S'_L, then $M \cap S = XS$ by Fact 2.2. Hence $M = \{a \in L \mid a^q \in XS\}$ which implies that $M = YS'_L$ is the unique maximal ideal of S'_L. In addition, it follows from Corollary 2.4 that the field extension $k \simeq S/XS \subseteq S'_L/YS'_L$ is finite. Further, Corollary 3.3 implies that S'_L is X-adically separated. Finally, S is X-adically complete and we can conclude that S'_L is a finitely generated S-module by Proposition 3.5.

Proposition 5.3 *Let D be a complete, local, one-dimensional, Noetherian domain with quotient field K and $K \subseteq L$ a finite field extension.*

Then the integral closure D'_L of D in L is finitely generated as D-module.

Proof According to Proposition 4.3, D contains a complete discrete valuation domain S such that D is a finitely generated S-module. Let F denote the quotient field of S. Hence $F \subseteq L$ is a finite field extension, $S \subseteq D$ is integral and $S'_L = D'_L$ (see Fact 2.1), cf. Figure 2. Therefore, it suffices to show that S'_L is a finitely generated S-module. Moreover, S is a complete, discrete valuation domain and in the equicharacteristic case $S \simeq k[\![X]\!]$ where k is a field by Proposition 4.3 and $\dim(S) = 1$.

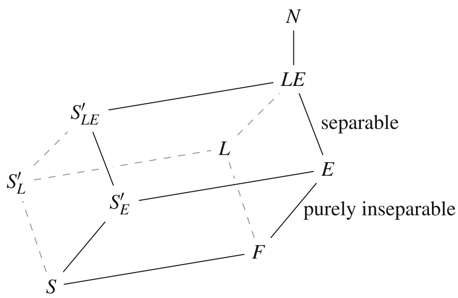

Fig. 3 Integral closures of S in the field extensions L, E and LE where E is a finite inseparable field extension of F such that LE is a finite separable extension of E

In particular, S is integrally closed. Consequently, if the field extension $F \subseteq L$ is separable, then the assertion follows from Proposition 5.1.

Let us assume that $F \subseteq L$ is inseparable. Then $\mathsf{char}(F) = p > 0$ and hence $\mathsf{char}(S) = \mathsf{char}(D) = p$ which implies that D is of equal characteristic. Therefore $S \simeq k[\![X]\!]$ where k is a field and $k \simeq D/\mathbf{m}$.

Let N be the normal hull of L and E be the fixed field of the automorphism group $\mathsf{Aut}_F(N)$ of $F \subseteq N$. Then $F \subseteq E$ is a purely inseparable extension and $E \subseteq N$ is a separable extension, cf. [5, Proposition V.6.11], see Figure 3.

Moreover, since $F \subseteq L$ is a finite extension, it follows that $F \subseteq N$ is finite which in turn implies that $F \subseteq E$ is a finite extension too.

Hence $E \subseteq LE$ is a finite separable extension and it follows from Proposition 5.1 that $S'_{LE} = (S'_E)'_{LE}$ is finitely generated as S'_E-module. In addition, S'_E is finitely generated as S-module according to Proposition 5.2.

Consequently, S'_{LE} is finitely generated as S-module and therefore a Noetherian S-module. However, S'_L is an S-submodule of S'_{LE} and thus finitely generated.

We conclude this section with the analogous assertion for complete, Noetherian, local, reduced rings.

Corollary 5.4 *Let R be a complete, Noetherian, one-dimensional, local ring. If R is reduced, then the integral closure R' of R is finitely generated as R-module.*

Proof Let $P_1, \ldots, P_n$ be the minimal prime ideals of R. For $1 \leq i \leq n$, let Q_i be the quotient field of the Noetherian, one-dimensional, local domain R/P_i. Then $Q = Q_1 \times \cdots \times Q_n$ is the total ring of quotients of $R/P_1 \times \cdots \times R/P_n$. Moreover,

$$(R/P_1 \times \cdots \times R/P_n)'_Q = (R/P_1)'_{Q_1} \times \cdots \times (R/P_n)'_{Q_n}. \quad (2)$$

The Noetherian, local domain R/P_i is a finitely generated R-module and hence $\mathbf{m}$-adically complete by Fact 3.4. As the $\mathbf{m}$-adic topology coincides with the $\mathbf{m}/P_i$-adic topology on R/P_i, it follows that $(R/P_i)'_{Q_i}$ is a finitely generated (R/P_i)-module according to Proposition 5.3. Together with Equation (2), it now follows that $(\prod_{i=1}^n R/P_i)'_Q$ is a finitely generated $\left(\prod_{i=1}^n R/P_i\right)$-module.

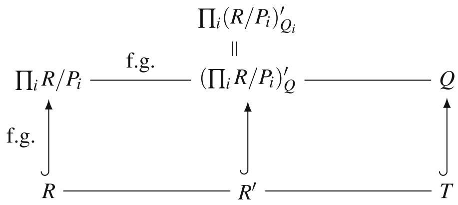

Fig. 4 Embeddings via ε

Since $\prod_{i=1}^{n} R/P_i$ is a finitely generated R-module, it follows that $(\prod_{i=1}^{n} R/P_i)'_Q$ is finitely generated as R-module too and therefore Noetherian. Let

$$\begin{aligned} \varepsilon : R &\longrightarrow R/P_1 \times \cdots \times R/P_n \\ r &\longmapsto (r + P_1, \ldots, r + P_n). \end{aligned}$$

Then $\ker(\varepsilon) = \bigcap_{i=1}^{n} P_i = \mathsf{nil}(R) = \mathbf{0}$ since R is reduced by hypothesis. Hence we can embed R into $\prod_{i=1}^{n} R/P_i$ via ε. Similarly, we can embed R' into $(\prod_{i=1}^{n} R/P_i)'_Q$ since ε can be canonically extended to the total ring of quotients T of R, see Figure 4.

Thus R' is isomorphic to a submodule of the Noetherian R-module $(\prod_{i=1}^{n} R/P_i)'_Q$ and hence finitely generated.

6 Proof of the Theorem

Finally, we are ready to give a proof of Theorem 1. For the reader's convenience we restate it here. Recall that a Noetherian, local domain $(D, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{D}$ and integral closure D' is called

- *unibranched*, if D' is local,
- *analytically unramified*, if $\widehat{D}$ is a reduced ring and
- *analytically irreducible*, if $\widehat{D}$ is a domain

(cf. Definition 1.1).

Theorem 1 *Let $(D, \mathbf{m})$ be a one-dimensional, Noetherian, local domain with integral closure D'. Then the following assertions are equivalent:*

1. *D is analytically irreducible.*
2. *D is unibranched and analytically unramified.*
3. *D is unibranched and D' is finitely generated as D-module.*
4. *D is unibranched and if $\mathbf{m}'$ denotes the maximal ideal of the integral closure, then the $\mathbf{m}$-adic topology on D coincides with subspace topology induced by $\mathbf{m}'$.*

Proof (1) $\Rightarrow$ (2): By assumption, $\widehat{D}$ is a domain and therefore is a reduced ring.

Assume that D is not unibranched and let $M_1 \neq M_2$ be two different maximal ideals of the integral closure D' of D. Let $a_1 \in M_1 \setminus M_2$ and $a_2 \in M_2 \setminus M_1$. Then

$R = D[a_1, a_2] \subseteq D'$ is an integral extension of D and therefore finitely generated as D-module by Fact 2.1. Hence, according to Proposition 3.8, R is a semilocal domain. However, the extension $R \subseteq D'$ is integral and therefore $N_i = M_i \cap R$ are maximal ideals of R for $i = 1, 2$ according to Fact 2.2. Due to the choice of a_1 and a_2, $N_1 \neq N_2$ and R is semilocal but not local. It follows from Proposition 3.8 that $\widehat{D}$ is not a domain.

(2) $\Rightarrow$ (3): If D' is not finitely generated as D-module, then there exists an infinite strictly ascending chain of intermediate rings $D \subseteq D_i \subseteq D'$ which are finitely generated as D-modules.

Let K be the quotient field of D and D_i for all i and $\widehat{D_i}$ denote the $\mathbf{m}$-adic completion of D_i. If $\frac{a}{b} \in \widehat{D_i} \cap K$, then $a \in b\widehat{D_i} \cap K$. However, by Corollary 3.3, $b\widehat{D_i} \cap K = bD_i$ and hence $\frac{a}{b} \in D_i$. Hence $\widehat{D_i} \cap K = D_i \subsetneq D_{i+1} = \widehat{D_{i+1}} \cap K$ which implies that $\widehat{D_i} \subsetneq \widehat{D_{i+1}}$ for all i.

Moreover, according to Fact 3.4, $\widehat{D_i} \simeq D_i \otimes_D \widehat{D}$ is a finitely generated $\widehat{D}$-module. Hence $\widehat{D_i}$ is contained in the integral closure $(\widehat{D})'$ of $\widehat{D}$ in its total ring of quotients. Consequently, the extension $\widehat{D} \subseteq (\widehat{D})'$ contains the infinite strictly ascending chain of intermediate rings $\widehat{D_i}$. Thus $(\widehat{D})'$ is not finitely generated as $\widehat{D}$-module which implies that $\widehat{D}$ is not reduced by Corollary 5.4.

(3) $\Rightarrow$ (4): The assertion immediately follows from Proposition 3.8.

(4) $\Rightarrow$ (1): Let $\widehat{(D')}$ be the $\mathbf{m}'$-adic completion of D'. Since the $\mathbf{m}'$-adic topology induces the $\mathbf{m}$-adic topology on D, it follows that D is a topological subspace of $\widehat{(D')}$ and $\widehat{D}$ is the topological closure of D in $\widehat{(D')}$.

By assumption D' is local, so D' is a discrete valuation domain according to Corollary 2.5. As shown in Example 3.1, the $\mathbf{m}'$-adic completion $\widehat{(D')}$ of D' is also a discrete valuation domain.

Hence $\widehat{D}$ is a subring of the domain $\widehat{(D')}$ and thus it is a domain itself.

Remark There are examples of one-dimensional, Noetherian, local domains which are unibranched but not analytically irreducible, cf. [9].

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