Quasi-Prüfer Extensions of Rings

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Abstract We introduce quasi-Prüfer ring extensions, in order to relativize quasi-Prüfer domains and to take also into account some contexts in recent papers. An extension is quasi-Prüfer if and only if it is an INC pair. The class of these extensions has nice stability properties. We also define almost-Prüfer extensions that are quasi-Prüfer, the converse being not true. Quasi-Prüfer extensions are closely linked to finiteness properties of fibers. Applications are given for FMC extensions, because they are quasi-Prüfer.

Keywords Flat epimorphism • FIP • FCP Extension • Minimal extension • Integral extension • Morita • Prüfer hull • Support of a module • Fiber

2010 *Mathematics Subject Classification* Primary:13B02; Secondary:13B22, 13A18, 13F05, 14A05

1 Introduction and Notation

We consider the category of commutative and unital rings. An epimorphism is an epimorphism of this category. Let $R \subseteq S$ be a (ring) extension. The set of all R-subalgebras of S is denoted by $[R, S]$. A *chain* of R-subalgebras of S is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that the extension $R \subseteq S$ has FCP (for the "finite chain property") if each chain in $[R, S]$ is finite. Dobbs and the authors characterized FCP extensions [13]. An extension $R \subseteq S$ is called FMC if there is a finite maximal chain of extensions from R to S.

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M. Fontana et al. (eds.), *Rings, Polynomials, and Modules*,
https://doi.org/10.1007/978-3-319-65874-2_16

We begin by explaining our motivations and aims. The reader who is not familiar with the notions used will find some Scholia in the sequel, as well as necessary definitions that exist in the literature. Knebusch and Zhang introduced Prüfer extensions in their book [25]. Actually, these extensions are nothing but normal pairs, that are intensively studied in the literature. We do not intend to give an extensive list of recent papers, written by Ayache, Ben Nasr, Dobbs, Jaballah, Jarboui, and some others. We are indebted to these authors because their papers are a rich source of suggestions. We observed that some of them are dealing with FCP (FMC) extensions, followed by a Prüfer extension, perhaps under a hidden form. These extensions reminded us quasi-Prüfer domains (see [18] for a comprehensive study). Therefore, we introduced in [38] *quasi-Prüfer* extensions $R \subseteq S$ as extensions that can be factored $R \subseteq R' \subseteq S$, where the first extension is integral and the second is Prüfer. Note that FMC extensions are quasi-Prüfer.

We give a systematic study of quasi-Prüfer extensions in Sects. 2 and 3. The class of quasi-Prüfer extensions has a nice behavior with respect to the classical operations of commutative algebra. An important result is that quasi-Prüfer extensions coincide with INC-pairs. Another one is that this class is stable under forming subextensions and composition. A striking result is the stability of the class of quasi-Prüfer extensions by absolutely flat base change, like localizations and Henselizations. An arbitrary ring extension $R \subseteq S$ admits a quasi-Prüfer closure, contained in S. Examples are provided by Laskerian pairs, open pairs, and the pseudo-Prüfer pairs of Dobbs-Shapiro [12].

Section 4 deals with *almost-Prüfer* extensions, a special kind of quasi-Prüfer extensions. They are of the form $R \subseteq T \subseteq S$, where the first extension is Prüfer and the second is integral. An arbitrary ring extension $R \subseteq S$ admits an almost-Prüfer closure, contained in S. The class of almost-Prüfer extensions seems to have less properties than the class of quasi-Prüfer extensions but has the advantage that almost-Prüfer closures commute with localizations at prime ideals. We examine the transfer of the quasi (almost)-Prüfer properties to subextensions. It is noteworthy that the class of FCP almost-Prüfer extensions is stable under the formation of subextensions, although this does not hold for arbitrary almost-Prüfer extensions.

In Sect. 5, we complete and generalize the results of Ayache-Dobbs in [5], with respect to the finiteness of fibers. These authors have evidently considered particular cases of quasi-Prüfer extensions. A main result is that if $R \subseteq S$ is quasi-Prüfer with finite fibers, then so is $R \subseteq T$ for $T \in [R, S]$. In particular, we recover a result of [5] about FMC extensions.

1.1 Recalls About Some Results and Definitions

The reader is warned that we will mostly use the definition of Prüfer extensions by flat epimorphic subextensions investigated in [25]. The results needed may be found in Scholium A for flat epimorphic extensions and some results of [25] are summarized in Scholium B. Their powers give quick proofs of results that are generalizations of results of the literature.

As long as FCP or FMC extensions are concerned, we use minimal (ring) extensions, a concept introduced by Ferrand-Olivier [17]. An extension $R \subset S$ is called *minimal* if $[R, S] = \{R, S\}$. It is known that a minimal extension is either module-finite or a flat epimorphism [17] and these conditions are mutually exclusive. There are three types of integral minimal (module-finite) extensions: ramified, decomposed, or inert [36, Theorem 3.3]. A minimal extension $R \subset S$ admits a crucial ideal $\mathcal{C}(R, S) =: M$ which is maximal in R and such that $R_P = S_P$ for each $P \neq M, P \in \mathrm{Spec}(R)$. Moreover, $\mathcal{C}(R, S) = (R : S)$ when $R \subset S$ is an integral minimal extension. The key connection between the above ideas is that if $R \subseteq S$ has FCP or FMC, then any maximal (necessarily finite) chain of R-subalgebras of S, $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, with *length* $n < \infty$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1},\ 0 \leq i \leq n-1$.

We define the *length* $\ell[R, S]$ of $[R, S]$ as the supremum of the lengths of chains in $[R, S]$. In particular, if $\ell[R, S] = r$, for some integer r, there exists a maximal chain in $[R, S]$ with length r.

As usual, $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$, $\mathrm{Min}(R)$, $\mathrm{U}(R)$, $\mathrm{Tot}(R)$ are, respectively, the set of prime ideals, maximal ideals, minimal prime ideals, units, total ring of fractions of a ring R and $\kappa(P) = R_P/PR_P$ is the residual field of R at $P \in \mathrm{Spec}(R)$.

If $R \subseteq S$ is an extension, then $(R : S)$ is its conductor and if $P \in \mathrm{Spec}(R)$, then S_P is the localization $S_{R \setminus P}$. We denote the integral closure of R in S by $\overline{R}^S$ (or $\overline{R}$).

A local ring is here what is called elsewhere a quasi-local ring. The *support* of an R-module E is $\mathrm{Supp}_R(E) := \{P \in \mathrm{Spec}(R) \mid E_P \neq 0\}$ and $\mathrm{MSupp}_R(E) := \mathrm{Supp}_R(E) \cap \mathrm{Max}(R)$. Finally, $\subset$ denotes proper inclusion and $|X|$ the cardinality of a set X.

Scholium A We give some recalls about flat epimorphisms (see [26, Chapitre IV], except (2) which is [30, Proposition 2]).

(1) $R \to S$ is a flat epimorphism $\Leftrightarrow$ for all $P \in \mathrm{Spec}(R)$, either $R_P \to S_P$ is an isomorphism or $S = PS \Leftrightarrow R_P \subseteq S_P$ is a flat epimorphism for all $P \in \mathrm{Spec}(R) \Leftrightarrow R_{(Q \cap R)} \to S_Q$ is an isomorphism for all $Q \in \mathrm{Spec}(S)$ and $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective.
(2) (S) A flat epimorphism, with a zero-dimensional domain, is surjective.
(3) If $f : A \to B$ and $g : B \to C$ are ring morphisms such that $g \circ f$ is injective and f is a flat epimorphism, then g is injective.
(4) Let $R \subseteq T \subseteq S$ be a tower of extensions, such that $R \subseteq S$ is a flat epimorphism. Then $T \subseteq S$ is a flat epimorphism but $R \subseteq T$ does not need. A Prüfer extension remedies this defect.
(5) (L) A faithfully flat epimorphism is an isomorphism. Hence, $R = S$ if $R \subseteq S$ is an integral flat epimorphism.
(6) If $f : R \to S$ is a flat epimorphism and J an ideal of S, then $J = f^{-1}(J)S$.
(7) If $f : R \to S$ is an epimorphism, then f is spectrally injective (i.e., ${}^af : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an injection) and its residual extensions are isomorphisms.

(8) Flat epimorphisms remain flat epimorphisms under base change (in particular, after a localization with respect to a multiplicatively closed subset).
(9) Flat epimorphisms are descended by faithfully flat morphisms.

1.2 Recalls and Results on Prüfer Extensions

There are a lot of characterizations of Prüfer extensions. We keep only those that are useful in this paper. Let $R \subseteq S$ be an extension.

Scholium B

(1) [25] $R \subseteq S$ is called Prüfer if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$.
(2) $R \subseteq S$ is called a *normal* pair if $T \subseteq S$ is integrally closed for each $T \in [R, S]$.
(3) $R \subseteq S$ is Prüfer if and only if it is a normal pair [25, Theorem 5.2(4)].
(4) R is called Prüfer if its finitely generated regular ideals are invertible, or equivalently, $R \subseteq \mathrm{Tot}(R)$ is Prüfer [21, Theorem 13((5)(9))].

Hence Prüfer extensions are a relativization of Prüfer rings. Clearly, a minimal extension is a flat epimorphism if and only if it is Prüfer. We will then use for such extensions the terminology: *Prüfer minimal* extensions. The reader may find some properties of Prüfer minimal extensions in [36, Proposition 3.2, Lemma 3.4 and Proposition 3.5], where in addition R must be supposed local. The reason why is that this word has disappeared during the printing process of [36].

We will need the two next results. Some of them do not explicitly appear in [25] but deserve to be emphasized. We refer to [25, Definition 1, p.22] for a definition of Manis extensions and remark that Proposition 1.1(1) was also noted in [12].

Proposition 1.1 *Let $R \subseteq S$ be a ring extension.*

(1) $R \subseteq S$ is Prüfer if and only if $R_P \subseteq S_P$ is Prüfer for each $P \in \mathrm{Spec}(R)$ (respectively, $P \in \mathrm{Supp}(S/R)$).
(2) $R \subseteq S$ is Prüfer if and only if $R_M \subseteq S_M$ is Manis for each $M \in \mathrm{Max}(R)$.

Proof (1) The class of Prüfer extensions is stable under localization [25, Proposition 5.1(ii), p.46-47]. To get the converse, use Scholium A(1). (2) follows from [25, Proposition 2.10, p.28, Definition 1, p.46]. □

Proposition 1.2 *Let $R \subseteq S$ be a ring extension, where R is local.*

(1) $R \subseteq S$ is Manis if and only if $S \setminus R \subseteq \mathrm{U}(S)$ and $x \in S \setminus R \Rightarrow x^{-1} \in R$. In that case, $R \subseteq S$ is integrally closed.
(2) $R \subseteq S$ is Manis if and only if $R \subseteq S$ is Prüfer.
(3) $R \subseteq S$ is Prüfer if and only if there exists $P \in \mathrm{Spec}(R)$ such that $S = R_P$, $P = SP$ and R/P is a valuation domain. Under these conditions, S/P is the quotient field of R/P.

Proof (1) is [25, Theorem 2.5, p.24]. (2) is [25, Scholium 10.4, p. 147]. Then (3) is [13, Theorem 6.8]. □

Next result shows that Prüfer FCP extensions can be described in a special manner.

Proposition 1.3 *Let $R \subset S$ be a ring extension.*

(1) If $R \subset S$ has FCP, then $R \subset S$ is integrally closed $\Leftrightarrow$ $R \subset S$ is Prüfer $\Leftrightarrow$ $R \subset S$ is a composite of Prüfer minimal extensions.
(2) If $R \subset S$ is integrally closed, then $R \subset S$ has FCP $\Leftrightarrow$ $R \subset S$ is Prüfer and $\mathrm{Supp}(S/R)$ *is finite.*

Proof (1) Assume that $R \subset S$ has FCP. If $R \subset S$ is integrally closed, then, $R \subset S$ is composed of Prüfer minimal extensions by [13, Lemma 3.10]. We know that a composite of Prüfer extensions is a Prüfer extension [25, Theorem 5.6]. Thus, by [25], $R \subset S$ is a normal pair. Conversely, if $R \subset S$ is composed of Prüfer minimal extensions, $R \subset S$ is integrally closed, since so is each Prüfer minimal extension. A Prüfer extension is obviously integrally closed, and an FCP integrally closed extension is Prüfer by [13, Theorem 6.3].

(2) The logical equivalence is [13, Theorem 6.3]. □

Definition 1.4 [25] A ring extension $R \subseteq S$ has:

(1) a greatest flat epimorphic subextension $R \subseteq \widehat{R}^S$, called the **Morita hull** of R in S.
(2) a greatest Prüfer subextension $R \subseteq \widetilde{R}^S$, called the **Prüfer hull** of R in S.

We set $\widehat{R} := \widehat{R}^S$ and $\widetilde{R} := \widetilde{R}^S$, if no confusion can occur. $R \subseteq S$ is called Prüfer-closed if $R = \widetilde{R}$.

Note that $\widetilde{R}^S$ is denoted by $\mathrm{P}(R, S)$ in [25] and $\widehat{R}^S$ is the weakly surjective hull $\mathrm{M}(R, S)$ of [25]. Our terminology is justified because Morita's work is earlier [29, Corollary 3.4]. The Morita hull can be computed by using a (transfinite) induction [29]. Let S' be the set of all $s \in S$ such that there is some ideal I of R, such that $IS = S$ and $Is \subseteq R$. Then $R \subseteq S'$ is a subextension of $R \subseteq S$. We set $S_1 := S'$ and $S_{i+1} := (S_i)' \subseteq S_i$. By [29, p. 36], if $R \subset S$ is an FCP extension, then $\widehat{R} = S_n$ for some integer n.

At this stage it is interesting to point out a result showing again that integral closedness and Prüfer extensions are closely related.

Proposition 1.5 *Olivier [32, Corollary, p. 56] An extension $R \subseteq S$ is integrally closed if and only if there is a pullback square:*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ V & \longrightarrow & K \end{array}$$

where V is a semi-hereditary ring and K its total quotient ring.

In that case $V \subseteq K$ is a Prüfer extension, since V is a Prüfer ring, whose localizations at prime ideals are valuation domains and K is an absolutely flat ring. As there exist integrally closed extensions that are not Prüfer, we see in passing that the pullback construction may not descend Prüfer extensions. The above result has a companion for minimal extensions that are Prüfer [20, Proposition 3.2].

Proposition 1.6 *Let $R \subseteq S$ be an extension and $T \in [R, S]$, then $\widetilde{R}^T = \widetilde{R} \cap T$. Therefore, for $T, U \in [R, S]$ with $T \subseteq U$, then $\widetilde{R}^T \subseteq \widetilde{R}^U$.*

Proof Obvious, since the Prüfer hull $\widetilde{R}^T$ is the greatest Prüfer extension $R \subseteq V$ contained in T. □

We will show later that in some cases $\widetilde{T} \subseteq \widetilde{U}$ if $R \subseteq S$ has FCP.

2 Quasi-Prüfer Extensions

We introduced the following definition in [38, p. 10].

Definition 2.1 An extension of rings $R \subseteq S$ is called quasi-Prüfer if one of the following equivalent statements holds:

(1) $\overline{R} \subseteq S$ is a Prüfer extension;
(2) $R \subseteq S$ can be factored $R \subseteq T \subseteq S$, where $R \subseteq T$ is integral and $T \subseteq S$ is Prüfer. In that case $\overline{R} = T$.

To see that (2) $\Rightarrow$ (1) observe that if (2) holds, then $T \subseteq \overline{R}$ is integral and a flat injective epimorphism, so that $\overline{R} = T$ by (L) (Scholium A(5)).

We observe that quasi-Prüfer extensions are akin to quasi-finite extensions if we refer to Zariski Main Theorem. This will be explored in Sect. 5, see, for example, Theorem 5.2.

Hence integral or Prüfer extensions are quasi-Prüfer. An extension is clearly Prüfer if and only if it is quasi-Prüfer and integrally closed. Quasi-Prüfer extensions allow us to avoid FCP hypotheses.

We give some other definitions involved in ring extensions $R \subseteq S$. The *fiber* at $P \in \mathrm{Spec}(R)$ of $R \subseteq S$ is $\mathrm{Fib}_{R,S}(P) := \{Q \in \mathrm{Spec}(S) \mid Q \cap R = P\}$. The subspace $\mathrm{Fib}_{R,S}(P)$ of $\mathrm{Spec}(S)$ is homeomorphic to the spectrum of the fiber ring at P, $\mathrm{F}_{R,S}(P) := \kappa(P) \otimes_R S$. The homeomorphism is given by the spectral map of $S \to \kappa(P) \otimes_R S$ and $\kappa(P) \to \kappa(P) \otimes_R S$ is the *fiber morphism* at P.

Definition 2.2 A ring extension $R \subseteq S$ is called:

(1) *incomparable* if for each pair $Q \subseteq Q'$ of prime ideals of S, then $Q \cap R = Q' \cap R \Rightarrow Q = Q'$, or equivalently, $\kappa(P) \otimes_R T$ is a zero-dimensional ring for each $T \in [R, S]$ and $P \in \mathrm{Spec}(R)$, such that $\kappa(P) \otimes_R T \neq 0$.
(2) an *INC-pair* if $R \subseteq T$ is incomparable for each $T \in [R, S]$ $\Leftrightarrow$ $T \subseteq U$ is incomparable for all $T \subseteq U$ in $[R, S]$.
(3) *residually algebraic* if $R/(Q \cap R) \subseteq S/Q$ is algebraic for each $Q \in \mathrm{Spec}(S)$.

(4) a *residually algebraic pair* if the extension $R \subseteq T$ is residually algebraic for each $T \in [R, S]$.

An extension $R \subseteq S$ is an INC-pair if and only if $R \subseteq S$ is a residually algebraic pair. This fact is an easy consequence of [10, Theorem] (via a short proof that was explicitly given in [9]). This fact was given for the particular case where S is an integral domain in [4].

The following characterization was announced in [38]. We were unaware that this result is also proved in [6, Corollary 1], when we presented it in ArXiv. However, our proof is largely shorter because we use the powerful results of [25].

Theorem 2.3 *An extension $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq S$ is an INC-pair and, if and only if, $R \subseteq S$ is a residually algebraic pair.*

Proof Suppose that $R \subseteq S$ is quasi-Prüfer and let $T \in [R, S]$. We set $U := \overline{R}T$. Then $\overline{R} \subseteq U$ is a flat epimorphism by definition of a Prüfer extension and hence is incomparable as is $R \subseteq \overline{R}$. It follows that $R \subseteq U$ is incomparable. Since $T \subseteq U$ is integral, it has going-up. It follows that $R \subseteq T$ is incomparable. Conversely, if $R \subseteq S$ is an INC-pair, then so is $\overline{R} \subseteq S$. Since $\overline{R} \subseteq S$ is integrally closed, $\overline{R} \subseteq S$ is Prüfer [25, Theorem 5.2,(9'), p. 48]. The second equivalence is given by the above comments about [10] and [9]. □

Corollary 2.4 *An extension $R \subseteq S$ is quasi-Prüfer if and only if $\overline{R} \subseteq \overline{T}$ is Prüfer for each $T \in [R, S]$. In this case, $\overline{R}$ is the least $T \in [R, S]$ such that $T \subseteq S$ is Prüfer.*

It follows that most of the properties described in [4] for integrally closed INC-pairs of domains are valid for arbitrary ring extensions. Moreover, a result of Dobbs is easily gotten as a consequence of Corollary 2.4: an INC-pair $R \subseteq S$ is an integral extension if and only if $\overline{R} \subseteq S$ is spectrally surjective [11, Theorem 2.2]. This follows from Corollary 2.4 and Scholium A, Property (L).

Example 2.5 Quasi-Prüfer domains R with quotient fields K can be characterized by $R \subseteq K$ is quasi-Prüfer. The reader may consult [7, Theorem 1.1] or [18].

We give here another example of quasi-Prüfer extension. An extension $R \subset S$ is called a *going-down pair* if each of its subextensions has the going-down property. For such a pair, $R \subseteq T$ has incomparability for each $T \in [R, S]$, at each non-maximal prime ideal of R [2, Lemma 5.8](ii). Now let M be a maximal ideal of R, whose fiber is not void in T. Then $R \subseteq T$ is a going-down pair, and so is $R/M \subseteq T/MT$ because $MT \cap R = M$. By [2, Corollary 5.6], the dimension of T/MT is ≤ 1. Therefore, if $R \subset S$ is a going-down pair, then $R \subset S$ is quasi-Prüfer if and only if $\dim(T/MT) \neq 1$ for each $T \in [R, S]$ and $M \in \mathrm{Max}(R)$.

Also *open-ring pairs* $R \subset S$ are quasi-Prüfer by [8, Proposition 2.13].

An *i-pair* is an extension $R \subseteq S$ such that $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is injective for each $T \in [R, S]$, or equivalently if and only if $R \subseteq S$ is quasi-Prüfer and $R \subseteq \overline{R}$ is spectrally injective [38, Proposition 5.8]. These extensions appear frequently in the integral domains context. Another examples are given by some extensions $R \subseteq S$, such that $\mathrm{Spec}(S) = \mathrm{Spec}(R)$ as sets, as we will see later.

We proved that Δ-extensions $R \subseteq S$ (such that $U, V \in [R, S] \Rightarrow U + V \in [R, S]$) are quasi-Prüfer [38, Proposition 5.15].

3 Properties of quasi-Prüfer Extensions

We now develop the machinery of quasi-Prüfer extensions.

Proposition 3.1 *An extension $R \subset S$ is (quasi-)Prüfer if and only if $R_P \subseteq S_P$ is (quasi-)Prüfer for any $P \in \mathrm{Spec}(R)$ ($P \in \mathrm{MSupp}(S/R)$).*

Proof The proof is easy if we use the INC-pair property definition of quasi-Prüfer extension (see also [4, Proposition 2.4]). □

Proposition 3.2 *Let $R \subseteq S$ be a quasi-Prüfer extension and $\varphi : S \to S'$ an integral ring morphism. Then $\varphi(R) \subseteq S'$ is quasi-Prüfer and $S' = \varphi(S)\overline{\varphi(\overline{R})}$, where $\overline{\varphi(\overline{R})}$ is the integral closure of $\varphi(\overline{R})$ in S'.*

Proof It is enough to apply [25, Theorem 5.9] to the Prüfer extension $\overline{R} \subseteq S$ and to use Definition 2.1. □

This result applies with $S' := S \otimes_R R'$, where $R \to R'$ is an integral morphism. Therefore integrality ascends the quasi-Prüfer property.

Recall that a composite of Prüfer extensions is Prüfer [25, Theorem 5.6, p. 51]. We next give a result that will be used frequently. The following Corollary 3.3 contains [6, Theorem 3].

Corollary 3.3 *Let $R \subseteq T \subseteq S$ be a tower of extensions. Then $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq T$ and $T \subseteq S$ are quasi-Prüfer. Hence, $R \subseteq T$ is quasi-Prüfer if and only if $R \subseteq \overline{R}T$ is quasi-Prüfer.*

Proof Consider a tower $(\mathcal{T})$ of extensions $R \subseteq \overline{R} \subseteq S := R' \subseteq \overline{R'} \subseteq S'$ (a composite of two quasi-Prüfer extensions). By using Proposition 3.2 we see that $\overline{R} \subseteq S = R' \subseteq \overline{R'}$ is quasi-Prüfer. Then $(\mathcal{T})$ is obtained by writing on the left an integral extension and on the right a Prüfer extension. Therefore, $(\mathcal{T})$ is quasi-Prüfer. We prove the converse.

If $R \subseteq T \subseteq S$ is a tower of extensions, then $R \subseteq T$ and $T \subseteq S$ are INC-pairs whenever $R \subseteq S$ is an INC-pair. The converse is then a consequence of Theorem 2.3.

The last statement is [6, Corollary 4]. □

Using the above corollary, we can exhibit new examples of quasi-Prüfer extensions. We recall that a ring R is called *Laskerian* if each of its ideals is a finite intersection of primary ideals and a ring extension $R \subset S$ a *Laskerian pair* if each $T \in [R, S]$ is a Laskerian ring. Then [41, Proposition 2.1] shows that if R is an integral domain with quotient field $F \neq R$ and $F \subset K$ is a field extension, then $R \subset K$ is a Laskerian pair if and only if K is algebraic over R and $\overline{R}$ (in K) is a Laskerian Prüfer domain. It follows easily that $R \subset K$ is quasi-Prüfer under these conditions.

Next result generalizes [24, Proposition 1].

Corollary 3.4 *An FMC extension $R \subset S$ is quasi-Prüfer.*

Proof Because $R \subset S$ is a composite of finitely many minimal extensions, by Corollary 3.3, it is enough to observe that a minimal extension is either Prüfer or integral. □

Corollary 3.5 *Let $R \subseteq S$ be a quasi-Prüfer extension and a tower $R \subseteq T \subseteq S$, where $R \subseteq T$ is integrally closed. Then $R \subseteq T$ is Prüfer.*

Proof Observe that $R \subseteq T$ is quasi-Prüfer and then that $R = \overline{R}^T$. □

Next result deals with the Dobbs-Shapiro *pseudo-Prüfer* extensions of integral domains [12], that they called pseudo-normal pairs. Suppose that R is local, we call here pseudo-Prüfer an extension $R \subseteq S$ such that there exists $T \in [R, S]$ with $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ and $T \subseteq S$ is Prüfer [12, Corollary 2.5]. If R is arbitrary, the extension $R \subseteq S$ is called pseudo-Prüfer if $R_M \subseteq S_M$ is pseudo-Prüfer for each $M \in \mathrm{Max}(R)$. In view of the Corollary 3.3, it is enough, if one wishes to characterize quasi-Prüfer extensions, to characterize quasi-Prüfer extensions of the type $R \subseteq T$ with $\mathrm{Spec}(R) = \mathrm{Spec}(T)$.

Corollary 3.6 *Let $R \subseteq T$ be an extension with* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ *and* (R, M) *local. Then $R \subseteq T$ is quasi-Prüfer if and only if* $\mathrm{Spec}(R) = \mathrm{Spec}(U)$ *for all* $U \in [R, T]$ *and, if and only if $R/M \subseteq T/M$ is an algebraic field extension. In such a case, $R \subseteq T$ is integral, hence Prüfer-closed.*

Proof It follows from [1] that $M \in \mathrm{Max}(T)$. Part of the proof is gotten by observing that $R/M \subseteq T/M$ is an algebraic field extension $\Rightarrow$ $\mathrm{Spec}(R) = \mathrm{Spec}(U)$ for all $U \in [R, T] \Rightarrow R \subseteq T$ is quasi-Prüfer $\Rightarrow$ ($R \subseteq T$ is integral and) $R/M \subseteq T/M$ is an algebraic field extension. Now $R \subseteq \widetilde{R}$ is a spectrally surjective flat epimorphism and then, by Scholium A, $R = \widetilde{R}$. □

Let $R \subseteq S$ be an extension and I an ideal shared by R and S. It is easy to show that $R \subseteq S$ is quasi-Prüfer if and only if $R/I \subseteq S/I$ is quasi-Prüfer by using [25, Proposition 5.8] in the Prüfer case. We are able to give a more general statement.

Lemma 3.7 *Let $R \subseteq S$ be a (quasi-)Prüfer extension and J an ideal of S with $I = J \cap R$. Then $R/I \subseteq S/J$ is a (quasi-)Prüfer extension. If $R \subseteq S$ is Prüfer and N is a maximal ideal of S, then $R/(N \cap R)$ is a valuation domain with quotient field S/N.*

Proof It follows from [25, Proposition 5.8] that if $R \subseteq S$ is Prüfer, then $R/I \cong (R + J)/J \subseteq S/J$ is Prüfer. Then the quasi-Prüfer case is an easy consequence. □

With this lemma we generalize and complete [23, Proposition 1.1].

Proposition 3.8 *Let $R \subseteq S$ be an extension of rings. The following statements are equivalent:*

(1) $R \subseteq S$ is quasi-Prüfer;
(2) $R/(Q \cap R) \subseteq S/Q$ is quasi-Prüfer for each $Q \in \mathrm{Spec}(S)$*;*

(3) $(X - s)S[X] \cap R[X] \not\subseteq M[X]$ *for each* $s \in S$ *and* $M \in \mathrm{Max}(R)$*;*
(4) For each $T \in [R, S]$*, the fiber morphisms of* $R \subseteq T$ *are integral.*

Proof (1) ⇒ (2) is entailed by Lemma 3.7. Assume that (2) holds and let $M \in \mathrm{Max}(R)$ that contains a minimal prime ideal P lain over by a minimal prime ideal Q of S. Then (2) ⇒ (3) follows from [23, Proposition 1.1(1)], applied to $R/(Q \cap R) \subseteq S/Q$. If (3) holds, argue as in the paragraph before [23, Proposition 1.1] to get that $R \subseteq S$ is a $\mathcal{P}$-extension, whence an INC-pair, cf. [11]. Then $R \subseteq S$ is quasi-Prüfer by Theorem 2.3, giving (3) ⇒ (1). Because integral extensions have incomparability, we see that (4) ⇒ (1). Corollary 3.3 shows that the reverse implication holds, if any quasi-Prüfer extension $R \subseteq S$ has integral fiber morphisms. For $P \in \mathrm{Spec}(R)$, the extension $R_P/PR_P \subseteq S_P/PS_P$ is quasi-Prüfer by Lemma 3.7. The ring $\overline{R}_P/P\overline{R}_P$ is zero-dimensional and $\overline{R}_P/P\overline{R}_P \to S_P/PS_P$, being a flat epimorphism, is therefore surjective by Scholium A (S). It follows that the fiber morphism at P is integral. □

Remark 3.9 The logical equivalence (1) ⇔ (2) is still valid if we replace quasi-Prüfer with integral in the above proposition. It is enough to show that an extension $R \subseteq S$ is integral when $R/P \subseteq S/Q$ is integral for each $Q \in \mathrm{Spec}(S)$ and $P := Q \cap R$. We can suppose that $S = R[s] \cong R[X]/I$, where X is an indeterminate, I an ideal of $R[X]$, and Q varies in $\mathrm{Min}(S)$, because for an extension $A \subseteq B$, any element of $\mathrm{Min}(A)$ is lain over by some element of $\mathrm{Min}(B)$. If Σ is the set of unitary polynomials of $R[X]$, the assumptions show that any element of $\mathrm{Spec}(R[X])$, containing I, meets Σ. As Σ is a multiplicatively closed subset, $I \cap \Sigma \neq \emptyset$, whence s is integral over R.

But a similar result does not hold if we replace quasi-Prüfer with Prüfer, except if we suppose that $R \subseteq S$ is integrally closed. To see this, apply the above proposition to get a quasi-Prüfer extension $R \subseteq S$ if each $R/P \subseteq S/Q$ is Prüfer. Actually, this situation already occurs for Prüfer rings and their factor domains, as Lucas's paper [28] shows. More precisely, [28, Proposition 2.7] and the third paragraph of [28, p. 336] shows that if R is a ring with $\mathrm{Tot}(R)$ absolutely flat, then R is a quasi-Prüfer ring if R/P is a Prüfer domain for each $P \in \mathrm{Spec}(R)$. Now example [28, Example 2.4] shows that R is not necessarily Prüfer.

We observe that if $R \subseteq S$ is quasi-Prüfer, then R/M is a quasi-Prüfer domain for each $N \in \mathrm{Max}(S)$ and $M := N \cap R$ (in case $R \subseteq S$ is integral, R/M is a field). To prove this, observe that $R/M \subseteq S/N$ can be factored $R/M \subseteq \kappa(M) \subseteq S/N$. As we will see, $R/M \subseteq \kappa(M)$ is quasi-Prüfer because $R/M \subseteq S/N$ is quasi-Prüfer.

The class of Prüfer extensions is not stable by (flat) base change. For example, let V be a valuation domain with quotient field K. Then $V[X] \subseteq K[X]$ is not Prüfer [25, Example 5.12, p. 53].

Proposition 3.10 *Let* $R \subseteq S$ *be a (quasi)-Prüfer extension and* $R \to T$ *a flat epimorphism, then* $T \subseteq S \otimes_R T$ *is (quasi)-Prüfer. If in addition* S *and* T *are both subrings of some ring and* $R \subseteq T$ *is an extension, then* $T \subseteq TS$ *is (quasi)-Prüfer.*

Proof For the first part, it is enough to consider the Prüfer case. It is well known that the following diagram is a pushout if $Q \in \mathrm{Spec}(T)$ is lying over P in R:

$$\begin{array}{ccc} R_P & \longrightarrow & S_P \\ \downarrow & & \downarrow \\ T_Q & \longrightarrow & (T \otimes_R S)_Q \end{array}$$

As $R_P \to T_Q$ is an isomorphism since $R \to T$ is a flat epimorphism by Scholium A (1), it follows that $R_P \subseteq S_P$ identifies to $T_Q \to (T \otimes_R S)_Q$. The first assertion follows because Prüfer extensions localize and globalize.

The final assertion is then a special case because, under its hypotheses, $TS \cong T \otimes_R S$ canonically. □

The reader may find in [25, Corollary 5.11, p. 53] that if $R \subseteq A \subseteq S$ and $R \subseteq B \subseteq S$ are extensions and $R \subseteq A$ and $R \subseteq B$ are both Prüfer, then $R \subseteq AB$ is Prüfer.

Proposition 3.11 *Let $R \subseteq A$ and $R \subseteq B$ be two extensions, where A and B are subrings of a ring S. If they are both quasi-Prüfer, then $R \subseteq AB$ is quasi-Prüfer.*

Proof Let U and V be the integral closures of R in A and B. Then $R \subseteq A \subseteq AV$ is quasi-Prüfer because $A \subseteq AV$ is integral and Corollary 3.3 applies. Using again Corollary 3.3 with $R \subseteq V \subseteq AV$, we find that $V \subseteq AV$ is quasi-Prüfer. Now Proposition 3.10 entails that $B \subseteq AB$ is quasi-Prüfer because $V \subseteq B$ is a flat epimorphism. Finally $R \subseteq AB$ is quasi-Prüfer, since a composite of quasi-Prüfer extensions. □

It is known that an arbitrary direct product of extensions is Prüfer if and only if each of its components is Prüfer [25, Proposition 5.20, p. 56]. The following result is an easy consequence.

Proposition 3.12 *Let $\{R_i \subseteq S_i | i = 1, \dots, n\}$ be a finite family of quasi-Prüfer extensions, then $R_1 \times \cdots \times R_n \subseteq S_1 \times \cdots \times S_n$ is quasi-Prüfer. In particular, by Corollary 3.3, if $\{R \subseteq S_i | i = 1, \dots, n\}$ is a finite family of quasi-Prüfer extensions, then $R \subseteq S_1 \times \cdots \times S_n$ is quasi-Prüfer.*

In the same way we have the following result deduced from [25, Remark 5.14, p. 54].

Proposition 3.13 *Let $R \subseteq S$ be an extension of rings and an upward directed family $\{R_\alpha | \alpha \in I\}$ of elements of $[R, S]$ such that $R \subseteq R_\alpha$ is quasi-Prüfer for each $\alpha \in I$. Then $R \subseteq \cup[R_\alpha | \alpha \in I]$ is quasi-Prüfer.*

Proof It is enough to use [25, Proposition 5.13, p. 54] where A_α is the integral closure of R in R_α. □

Here are some descent results used later on.

Proposition 3.14 *Let $R \subseteq S$ be a ring extension and $R \to R'$ a spectrally surjective ring morphism (for example, either faithfully flat or injective and integral). Then $R \subseteq S$ is quasi-Prüfer if $R' \to R' \otimes_R S$ is injective (for example, if $R \to R'$ is faithfully flat) and quasi-Prüfer.*

Proof Let $T \in [R, S]$ and $P \in \mathrm{Spec}(R)$ and set $T' := T \otimes_R R'$. There is some $P' \in \mathrm{Spec}(R')$ lying over P, because $R \to R'$ is spectrally surjective. By [22, Corollaire 3.4.9], there is a faithfully flat morphism $\mathrm{F}_{R,T}(P) \to \mathrm{F}_{R',T'}(P') \cong \mathrm{F}_{R,T}(P) \otimes_{\mathbf{k}(P)} \kappa(P')$, inducing a surjective map $\mathrm{Fib}_{R',T'}(P') \to \mathrm{Fib}_{R,T}(P)$ since it satisfies lying over. By Theorem 2.3, the result follows from the faithful flatness of $\mathrm{F}_{R,T}(P) \to \mathrm{F}_{R',T\otimes_R R'}(P')$. □

Corollary 3.15 *Let $R \subseteq S$ be an extension of rings, $R \to R'$ a faithfully flat ring morphism and set $S' := R' \otimes_R S$. If $R' \subseteq S'$ is (quasi-) Prüfer (respectively, FCP), then so is $R \subseteq S$.*

Proof The Prüfer case is clear, because faithfully flat morphisms descend flat epimorphisms (Scholium A (9)). For the quasi-Prüfer case, we use Proposition 3.14. The FCP case is proved in [15, Theorem 2.2]. □

The integral closure of a ring morphism $f : R \to T$ is the integral closure of the extension $f(R) \subseteq T$. By definition, a ring morphism $R \to T$ preserves the integral closure of ring morphisms $R \to S$ if $\overline{T}^{T\otimes_R S} \cong T \otimes_R \overline{R}$ for every ring morphism $R \to S$. An absolutely flat morphism $R \to T$ ($R \to T$ and $T \otimes_R T \to T$ are both flat) preserves integral closure [32, Theorem 5.1]. Flat epimorphisms, Henselizations, and étale morphisms are absolutely flat. Another examples are morphisms $R \to T$ that are essentially of finite type and (absolutely) reduced [34, Proposition 5.19](2). Such morphisms are flat if R is reduced [27, Proposition 3.2].

We will prove an ascent result for absolutely flat ring morphisms. This will be proved by using base changes. For this we need to introduce some concepts. A ring A is called an AIC ring if each monic polynomial of $A[X]$ has a zero in A. The first author recalled in [35, p. 4662] that any ring A has a faithfully flat integral extension $A \to A^*$, where A^* is an AIC ring. Moreover, if A is an AIC ring, each localization A_P at a prime ideal P of A is a strict Henselian ring [35, Lemma II.2].

Theorem 3.16 *Let $R \subseteq S$ be a (quasi-) Prüfer extension and $R \to T$ an absolutely flat ring morphism. Then $T \to T \otimes_R S$ is a (quasi-) Prüfer extension.*

Proof We can suppose that R is an AIC ring. To see this, it is enough to use the base change $R \to R^*$. We set $T^* := T \otimes_R R^*$, $S^* := S \otimes_R R^*$. We first observe that $R^* \subseteq S^*$ is quasi-Prüfer for the following reason: the composite extension $R \subseteq S \subseteq S^*$ is quasi-Prüfer by Corollary 3.3 because the last extension is integral. Moreover, $R^* \to T^*$ is absolutely flat. In case $T^* \subseteq T^* \otimes_{R^*} S^*$ is quasi-Prüfer, so is $T \subseteq T \otimes_R S$, because $T \to T^* = T \otimes_R R^*$ is faithfully flat and $T^* \subseteq T^* \otimes_{R^*} S^*$ is deduced from $T \subseteq T \otimes_R S$ by the faithfully flat base change $T \to T \otimes_R R^*$. It is then enough to apply Proposition 3.14.

We thus assume from now on that R is an AIC ring.

Let $N \in \operatorname{Spec}(T)$ be lying over M in R. Then $R_M \to T_N$ is absolutely flat [31, Proposition f] and $R_M \subseteq S_M$ is quasi-Prüfer. Now observe that $(T \otimes_R S)_N \cong T_N \otimes_{R_M} S_M$. Therefore, we can suppose that R and T are local and $R \to T$ is local and injective. We deduce from [32, Theorem 5.2] that $R_M \to T_N$ is an isomorphism because R_M is a strict Henselian ring. Therefore the proof is complete in the quasi-Prüfer case. For the Prüfer case, we need only to observe that absolutely flat morphisms preserve integral closure and a quasi-Prüfer extension is Prüfer if it is integrally closed. □

Lemma 3.17 *Let $R \subseteq S$ be an extension of rings and $R \to T$ a base change which preserves integral closure. If $T \subseteq T \otimes_R S$ has FCP and $R \subseteq S$ is Prüfer, then $T \subseteq T \otimes_R S$ is Prüfer.*

Proof The result holds because an FCP extension is Prüfer if and only if it is integrally closed. □

We observe that $T \otimes_R \widetilde{R} \subseteq \widetilde{T}$ need not to be an isomorphism, since this property may fail even for a localization $R \to R_P$, where P is a prime ideal of R.

Theorem 3.18 *Let $R \subseteq S$ be a ring extension.*

(1) $R \subseteq S$ has a greatest quasi-Prüfer subextension $R \subseteq \overset{\Longrightarrow}{R} = \widetilde{\widetilde{R}}$.

(2) $R \subseteq \overline{R}\widetilde{R} =: \vec{R}$ is quasi-Prüfer and then $\vec{R} \subseteq \overset{\Longrightarrow}{R}$.

(3) $\overline{R}^{\overset{\Longrightarrow}{R}} = \overline{R}$ and $\widetilde{R}^{\overset{\Longrightarrow}{R}} = \widetilde{R}$.

Proof To see (1), use Proposition 3.11 which tells us that the set of all quasi-Prüfer subextensions is upward directed and then use Proposition 3.13 to prove the existence of $\overset{\Longrightarrow}{R}$. Then let $R \subseteq T \subseteq \overset{\Longrightarrow}{R}$ be a tower with $R \subseteq T$ integral and $T \subseteq \overset{\Longrightarrow}{R}$ Prüfer. From $T \subseteq \overline{R} \subseteq \widetilde{\widetilde{R}} \subseteq \overset{\Longrightarrow}{R}$, we deduce that $T = \overline{R}$ and then $\overset{\Longrightarrow}{R} = \widetilde{\widetilde{R}}$.

(2) Now $R \subseteq \overline{R}\widetilde{R}$ can be factored $R \subseteq \widetilde{R} \subseteq \overline{R}\widetilde{R}$ and is a tower of quasi-Prüfer extensions, because $\widetilde{R} \to \overline{R}\widetilde{R}$ is integral.

(3) Clearly, the integral closure and the Prüfer closure of R in $\overset{\Longrightarrow}{R}$ are the respective intersections of $\overline{R}$ and $\widetilde{R}$ with $\overset{\Longrightarrow}{R}$, and $\overline{R}, \widetilde{R} \subseteq \overset{\Longrightarrow}{R}$. □

This last result means that, as far as properties of integral closures and Prüfer closures of subsets of $\overset{\Longrightarrow}{R}$ are concerned, we can suppose that $R \subseteq S$ is quasi-Prüfer.

4 Almost-Prüfer Extensions

We next give a definition "dual" of the definition of a quasi-Prüfer extension.

4.1 Arbitrary Extensions

Definition 4.1 A ring extension $R \subseteq S$ is called an *almost-Prüfer* extension if it can be factored $R \subseteq T \subseteq S$, where $R \subseteq T$ is Prüfer and $T \subseteq S$ is integral.

Proposition 4.2 *An extension $R \subseteq S$ is almost-Prüfer if and only if $\widetilde{R} \subseteq S$ is integral. It follows that the subring T of the above definition is $\widetilde{R} = \widehat{R}$ when $R \subseteq S$ is almost-Prüfer.*

Proof If $R \subseteq S$ is almost-Prüfer, there is a factorization $R \subseteq T \subseteq \widetilde{R} \subseteq \widehat{R} \subseteq S$, where $T \subseteq \widehat{R}$ is both integral and a flat epimorphism by Scholium A (4). Therefore, $T = \widetilde{R} = \widehat{R}$ by Scholium A (5) (L). □

Corollary 4.3 *Let $R \subseteq S$ be a quasi-Prüfer extension, and let $T \in [R, S]$. Then, $T \cap \overline{R} \subseteq T\overline{R}$ is almost-Prüfer and $T = \widetilde{\overline{R} \cap T}^{T\overline{R}}$. Moreover, if $T \cap \overline{R} = R$, then, $T = T\overline{R} \cap \widetilde{R}$.*

Proof $T \cap \overline{R} \subseteq T$ is quasi-Prüfer by Corollary 3.3. Being integrally closed, it is Prüfer by Corollary 3.5. Moreover, $T \subseteq T\overline{R}$ is an integral extension. Then, $T \cap \overline{R} \subseteq T\overline{R}$ is almost-Prüfer and $T = \widetilde{\overline{R} \cap T}^{T\overline{R}}$. If $T \cap \overline{R} = R$, then $T \subseteq T\overline{R} \cap \widetilde{R}$ is both Prüfer and integral, so that $T = T\overline{R} \cap \widetilde{R}$. □

We note that integral extensions and Prüfer extensions are almost-Prüfer and hence minimal extensions are almost-Prüfer. There are quasi-Prüfer extensions that are not almost-Prüfer. It is enough to consider [37, Example 3.5(1)]. Let $R \subseteq T \subseteq S$ be two minimal extensions, where R is local, $R \subseteq T$ integral and $T \subseteq S$ is Prüfer. Then $R \subseteq S$ is quasi-Prüfer but not almost-Prüfer, because $S = \widehat{R}$ and $R = \widetilde{R}$. The same example shows that a composite of almost-Prüfer extensions may not be almost-Prüfer.

But the reverse implication holds.

Theorem 4.4 *Let $R \subseteq S$ be an almost-Prüfer extension. Then $R \subseteq S$ is quasi-Prüfer. Moreover, $\widetilde{R} = \widehat{R}$, $(\widetilde{R})_P = \widetilde{R_P}$ for each $P \in \mathrm{Spec}(R)$. In this case, any flat epimorphic subextension $R \subseteq T$ is Prüfer.*

Proof Let $R \subseteq \widetilde{R} \subseteq S$ be an almost-Prüfer extension, that is $\widetilde{R} \subseteq S$ is integral. The first assertion follows from Corollary 3.3 because $R \subseteq \widetilde{R}$ is Prüfer. Now the Morita hull and the Prüfer hull coincide by Proposition 4.2. In the same way, $(\widetilde{R})_P \to \widetilde{R_P}$ is a flat epimorphism and $(\widetilde{R})_P \to S_P$ is integral. □

We could define almost-Prüfer rings as the rings R such that $R \subseteq \mathrm{Tot}(R)$ is almost-Prüfer. But in that case $\widetilde{R} = \mathrm{Tot}(R)$ (by Theorem 4.4), so that R is a Prüfer ring. The converse evidently holds. Therefore, this concept does not define something new.

It was observed in [13, Remark 2.9(c)] that there is an almost-Prüfer FMC extension $R \subseteq S \subseteq T$, where $R \subseteq S$ is a Prüfer minimal extension and $S \subseteq T$ is minimal and integral, but $R \subseteq T$ is not an FCP extension.

Proposition 4.5 *Let $R \subseteq S$ be an extension verifying the hypotheses:*

(i) $R \subseteq S$ is quasi-Prüfer.
(ii) $R \subseteq S$ can be factored $R \subseteq T \subseteq S$, where $R \subseteq T$ is a flat epimorphism.

(1) Then the following commutative diagram (D) is a pushout,

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \\ T & \longrightarrow & T\overline{R} \end{array}$$

$T\overline{R} \subseteq S$ is Prüfer and $R \subseteq T\overline{R}$ is quasi-Prüfer. Moreover, $\mathrm{F}_{R,\overline{R}}(P) \cong \mathrm{F}_{T,T\overline{R}}(Q)$ for each $Q \in \mathrm{Spec}(T)$ and $P := Q \cap R$.
(2) If in addition $R \subseteq T$ is integrally closed, (D) is a pullback, $T \cap \overline{R} = R$, $(R : \overline{R}) = (T : T\overline{R}) \cap R$ and $(T : T\overline{R}) = (R : \overline{R})T$.

Proof (1) Consider the injective composite map $\overline{R} \to \overline{R} \otimes_R T \to T\overline{R}$. As $\overline{R} \to \overline{R} \otimes_R T$ is a flat epimorphism, because deduced by a base change of $R \to T$, we get that the surjective map $\overline{R} \otimes_R T \to T\overline{R}$ is an isomorphism by Scholium A (3). By fibers transitivity, we have $\mathrm{F}_{T,\overline{R}T}(Q) \cong \kappa(Q) \otimes_{\mathbf{k}(P)} \mathrm{F}_{R,\overline{R}}(P)$ [22, Corollaire 3.4.9]. As $\kappa(P) \to \kappa(Q)$ is an isomorphism by Scholium A, we get that $\mathrm{F}_{R,\overline{R}}(P) \cong \mathrm{F}_{T,\overline{R}T}(Q)$.

(2) As in [5, Lemma 3.5], $R = T \cap \overline{R}$. The first statement on the conductors has the same proof as in [5, Lemma 3.5]. The second holds because $R \subseteq T$ is a flat epimorphism (see Scholium A (6)). □

Theorem 4.6 *Let $R \subset S$ be a quasi-Prüfer extension and the diagram (D'):*

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \\ \widetilde{R} & \longrightarrow & \widetilde{R}\overline{R} \end{array}$$

(1) (D') is a pushout and a pullback, such that $\overline{R} \cap \widetilde{R} = R$ and $(R : \overline{R}) = (\widetilde{R} : \widetilde{R}\overline{R}) \cap R$ so that $(\widetilde{R} : \widetilde{R}\overline{R}) = (R : \overline{R})\widetilde{R}$.

(2) $R \subset S$ can be factored $R \subseteq \widetilde{R}\overline{R} = \overline{\widetilde{R}} = \vec{R} \subseteq \overset{\Longrightarrow}{R} = \widetilde{\overline{R}} = S$, where the first extension is almost-Prüfer and the second is Prüfer.
(3) $R \subset S$ is almost-Prüfer $\Leftrightarrow S = \overline{R}\widetilde{R} \Leftrightarrow \widetilde{\overline{R}} = \overline{\widetilde{R}}$.
(4) $R \subseteq \widetilde{R}\overline{R} = \overline{\widetilde{R}} = \vec{R}$ is the greatest almost-Prüfer subextension of $R \subseteq S$ and $\widetilde{R} = \widetilde{R}^{\vec{R}}$.
(5) $\mathrm{Spec}(\vec{R})$ is homeomorphic to $\mathrm{Spec}(\overline{R}) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\widetilde{R})$.
(6) $\mathrm{Supp}(S/R) = \mathrm{Supp}(\widetilde{R}/R) \cup \mathrm{Supp}(\overline{R}/R)$ if $R \subseteq S$ is almost-Prüfer. (Supp can be replaced with MSupp).

Proof To show (1), (2), in view of Theorem 3.18, it is enough to apply Proposition 4.5 with $T = \widetilde{R}$ and $S = \overset{\Longrightarrow}{R}$, because $R \subseteq \widetilde{R}\overline{R}$ is almost-Prüfer whence quasi-Prüfer, keeping in mind that a Prüfer extension is integrally closed, whereas an integral Prüfer extension is trivial. Moreover, $\widetilde{\overline{R}} = \overline{R}\widetilde{R}$ because $\overline{R}\widetilde{R} \subseteq \widetilde{\overline{R}}$ is both integral and integrally closed.

(3) is obvious.

(4) Now consider an almost-Prüfer subextension $R \subseteq T \subseteq U$, where $R \subseteq T$ is Prüfer and $T \subseteq U$ is integral. Applying (3), we see that $U = \overline{R}^U \widetilde{R}^U \subseteq \overline{R}\widetilde{R}$ in view of Proposition 1.6.

(5) Recall from [33] that a ring morphism $A \to A'$ is called a subtrusion if for each pair of prime ideals $P \subseteq Q$ of A, there is a pair of prime ideals $P' \subseteq Q'$ above $P \subseteq Q$. A subtrusion defines a submersion $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. We refer to [33, First paragraph of p. 570] for the definition of the property $P(\Delta)$ of a pushout diagram (Δ). Then [33, Lemme 2,(b), p. 570] shows that $P(D')$ holds, because $R \to \widetilde{R}$ is a flat epimorphism. Now [33, Proposition 2, p. 576] yields that $\mathrm{Spec}(\vec{R}) \to \mathrm{Spec}(\overline{R}) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\widetilde{R})$ is subtrusive. This map is also injective because $R \to \widetilde{R}$ is spectrally injective. Observing that an injective submersion is an homeomorphism, the proof is complete.

(6) Obviously, $\mathrm{Supp}(\widetilde{R}/R) \cup \mathrm{Supp}(\overline{R}/R) \subseteq \mathrm{Supp}(S/R)$. Conversely, let $M \in \mathrm{Spec}(R)$ be such that $R_M \neq S_M$, and $R_M = (\widetilde{R})_M = \overline{R}_M$. Then (3) entails that $S_M = (\overline{R}\widetilde{R})_M = (\overline{R})_M(\widetilde{R})_M = R_M$, which is absurd. □

Corollary 4.7 *Let $R \subseteq S$ be an almost-Prüfer extension. The following conditions are equivalent:*

(1) $\mathrm{Supp}(S/\overline{R}) \cap \mathrm{Supp}(\overline{R}/R) = \emptyset$.
(2) $\mathrm{Supp}(S/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R) = \emptyset$.
(3) $\mathrm{Supp}(\widetilde{R}/R) \cap \mathrm{Supp}(\overline{R}/R) = \emptyset$.

Proof Since $R \subseteq S$ is almost-Prüfer, we get $(\widetilde{R})_P = \widetilde{R_P}$ for each $P \in \mathrm{Spec}(R)$. Moreover, $\mathrm{Supp}(S/R) = \mathrm{Supp}(\widetilde{R}/R) \cup \mathrm{Supp}(\overline{R}/R) = \mathrm{Supp}(S/\overline{R}) \cup \mathrm{Supp}(\overline{R}/R) = \mathrm{Supp}(S/\widetilde{R}) \cup \mathrm{Supp}(\widetilde{R}/R)$.

(1) $\Rightarrow$ (2): Assume that there exists $P \in \mathrm{Supp}(S/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R)$. Then, $(\widetilde{R})_P \neq S_P, R_P$, so that $R_P \subset S_P$ is neither Prüfer nor integral. But, $P \in \mathrm{Supp}(S/R) = \mathrm{Supp}(S/\overline{R}) \cup \mathrm{Supp}(\overline{R}/R)$. If $P \in \mathrm{Supp}(S/\overline{R})$, then $P \notin \mathrm{Supp}(\overline{R}/R)$, so that $(\overline{R})_P = R_P$ and $R_P \subset S_P$ is Prüfer, a contradiction. If $P \in \mathrm{Supp}(\overline{R}/R)$, then $P \notin \mathrm{Supp}(S/\overline{R})$, so that $(\overline{R})_P = S_P$ and $R_P \subset S_P$ is integral, a contradiction.

(2) $\Rightarrow$ (3): Assume that there exists $P \in \mathrm{Supp}(\widetilde{R}/R) \cap \mathrm{Supp}(\overline{R}/R)$. Then, $R_P \neq (\widetilde{R})_P, (\overline{R})_P$, so that $R_P \subset S_P$ is neither Prüfer nor integral. But, $P \in \mathrm{Supp}(S/R) = \mathrm{Supp}(S/\widetilde{R}) \cup \mathrm{Supp}(\widetilde{R}/R)$. If $P \in \mathrm{Supp}(S/\widetilde{R})$, then $P \notin \mathrm{Supp}(\widetilde{R}/R)$, so that $(\widetilde{R})_P = R_P$ and $R_P \subset S_P$ is integral, a contradiction. If $P \in \mathrm{Supp}(\widetilde{R}/R)$, then $P \notin \mathrm{Supp}(S/\widetilde{R})$, so that $(\widetilde{R})_P = S_P$ and $R_P \subset S_P$ is Prüfer, a contradiction.

(3) $\Rightarrow$ (1): Assume that there exists $P \in \mathrm{Supp}(S/\overline{R}) \cap \mathrm{Supp}(\overline{R}/R)$. Then, $(\overline{R})_P \neq R_P, S_P$, so that $R_P \subset S_P$ is neither Prüfer nor integral. But, $P \in \mathrm{Supp}(S/R) = \mathrm{Supp}(\overline{R}/R) \cup \mathrm{Supp}(\widetilde{R}/R)$. If $P \in \mathrm{Supp}(\widetilde{R}/R)$, then $P \notin \mathrm{Supp}(\overline{R}/R)$, so that $(\overline{R})_P =$

R_P and $R_P \subset S_P$ is Prüfer, a contradiction. If $P \in \mathrm{Supp}(\overline{R}/R)$, then $P \notin \mathrm{Supp}(\widetilde{R}/R)$, so that $(\widetilde{R})_P = R_P$ and $R_P \subset S_P$ is integral, a contradiction. □

Proposition 4.5 has the following similar statement proved by Ayache and Dobbs. It reduces to Theorem 4.6 in case $R \subseteq S$ has FCP because of Proposition 1.3.

Proposition 4.8 *Let $R \subseteq T \subseteq S$ be a quasi-Prüfer extension, where $T \subseteq S$ is an integral minimal extension and $R \subseteq T$ is integrally closed. Then the diagram (D) is a pullback, $S = T\overline{R}$ and $(T : S) = (R : \overline{R})T$.*

Proof [5, Lemma 3.5]. □

Proposition 4.9 *Let $R \subseteq U \subseteq S$ and $R \subseteq V \subseteq S$ be two towers of extensions, such that $R \subseteq U$ and $R \subseteq V$ are almost-Prüfer. Then $R \subseteq UV$ is almost-Prüfer and $\widetilde{UV} = \widetilde{U}\widetilde{V}$.*

Proof Denote by U', V', and W' the Prüfer hulls of R in U, V, and $W = UV$. We deduce from [25, Corollary 5.11, p. 53], that $R \subseteq U'V'$ is Prüfer. Moreover, $U'V' \subseteq UV$ is clearly integral and $U'V' \subseteq W'$ because the Prüfer hull is the greatest Prüfer subextension. We deduce that $R \subseteq UV$ is almost-Prüfer and that $\widetilde{UV} = \widetilde{U}\widetilde{V}$. □

Proposition 4.10 *Let $R \subseteq U \subseteq S$ and $R \subseteq V \subseteq S$ be two towers of extensions, such that $R \subseteq U$ is almost-Prüfer and $R \subseteq V$ is a flat epimorphism. Then $U \subseteq UV$ is almost-Prüfer.*

Proof Mimic the proof of Proposition 4.9, using [25, Theorem 5.10, p. 53]. □

Proposition 4.11 *Let $R \subseteq S$ be an almost-Prüfer extension and $R \to T$ a flat epimorphism. Then $T \subseteq T \otimes_R S$ is almost-Prüfer.*

Proof It is enough to use Proposition 3.10 and Definition 4.1. □

Proposition 4.12 *An extension $R \subseteq S$ is almost-Prüfer if and only if $R_P \subseteq S_P$ is almost-Prüfer and $\widetilde{R_P} = (\widetilde{R})_P$ for each $P \in \mathrm{Spec}(R)$.*

Proof For an arbitrary extension $R \subseteq S$ we have $(\widetilde{R})_P \subseteq \widetilde{R_P}$. Suppose that $R \subseteq S$ is almost-Prüfer, then so is $R_P \subseteq S_P$ and $(\widetilde{R})_P = \widetilde{R_P}$ by Theorem 4.4. Conversely, if $R \subseteq S$ is locally almost-Prüfer, whence locally quasi-Prüfer, then $R \subseteq S$ is quasi-Prüfer. If $\widetilde{R_P} = (\widetilde{R})_P$ holds for each $P \in \mathrm{Spec}(R)$, we have $S_P = (\overline{R}\widetilde{R})_P$ so that $S = \overline{R}\widetilde{R}$ and $R \subseteq S$ is almost-Prüfer by Theorem 4.6. □

Corollary 4.13 *An FCP extension $R \subseteq S$ is almost-Prüfer if and only if $R_P \subseteq S_P$ is almost-Prüfer for each $P \in \mathrm{Spec}(R)$.*

Proof It is enough to show that $R \subseteq S$ is almost-Prüfer if $R_P \subseteq S_P$ is almost-Prüfer for each $P \in \mathrm{Spec}(R)$ using Proposition 4.12. Any minimal extension $\widetilde{R} \subset R_1$ is integral by definition of $\widetilde{R}$. Assume that $(\widetilde{R})_P \subset \widetilde{(R_P)}$, so that there exists $R'_2 \in [\widetilde{R}, S]$ such that $(\widetilde{R})_P \subset (R'_2)_P$ is a Prüfer minimal extension with crucial maximal ideal $Q(\widetilde{R})_P$, for some $Q \in \mathrm{Max}(\widetilde{R})$ with $Q \cap R \subseteq P$. In particular, $\widetilde{R} \subset R'_2$ is not integral. We may assume that there exists $R'_1 \in [\widetilde{R}, R'_2]$ such that $R'_1 \subset R'_2$ is a Prüfer minimal extension with $P \notin \mathrm{Supp}(R'_1/\widetilde{R})$. Using [37, Lemma 1.10], there exists $R_2 \in [\widetilde{R}, R'_2]$ such that $\widetilde{R} \subset R_2$ is a Prüfer minimal extension with crucial maximal ideal Q, a contradiction. Then, $(\widetilde{R})_P \subset S_P$ is integral for each P, whence $(\widetilde{R})_P = \widetilde{(R_P)}$. □

We now intend to demonstrate that our methods allow us to prove easily some results. For instance, next statement generalizes [5, Corollary 4.5] and can be fruitful in algebraic number theory.

Proposition 4.14 *Let (R, M) be a one-dimensional local ring and $R \subseteq S$ a quasi-Prüfer extension. Suppose that there is a tower $R \subset T \subseteq S$, where $R \subset T$ is integrally closed. Then $R \subseteq S$ is almost-Prüfer, $T = \widetilde{R}$ and S is zero-dimensional.*

Proof Because $R \subset T$ is quasi-Prüfer and integrally closed, it is Prüfer. If some prime ideal of T is lying over M, $R \subset T$ is a faithfully flat epimorphism, whence an isomorphism by Scholium A, which is absurd. Now let N be a prime ideal of T and $P := N \cap R$. Then R_P is zero-dimensional and isomorphic to T_N. Therefore, T is zero-dimensional. It follows that $T\overline{R}$ is zero-dimensional. Since $R\overline{T} \subseteq S$ is Prüfer, we deduce from Scholium A, that $\overline{R}T = S$. The proof is now complete. □

We also generalize [5, Proposition 5.2] as follows.

Proposition 4.15 *Let $R \subset S$ be a quasi-Prüfer extension, such that $\overline{R}$ is local with maximal ideal $N := \sqrt{(R : \overline{R})}$. Then R is local and $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$. If in addition R is one-dimensional, then either $R \subset S$ is integral or there is some minimal prime ideal P of $\overline{R}$, such that $S = (\overline{R})_P$, $P = SP$ and $\overline{R}/P$ is a one-dimensional valuation domain with quotient field S/P.*

Proof R is obviously local. Let $T \in [R, S] \setminus [R, \overline{R}]$ and $s \in T \setminus \overline{R}$. Then $s \in \mathrm{U}(S)$ and $s^{-1} \in \overline{R}$ by Proposition 1.2 (1). But $s^{-1} \notin \mathrm{U}(\overline{R})$, so that $s^{-1} \in N$. It follows that there exists some integer n such that $s^{-n} \in (R : \overline{R})$, giving $s^{-n}\overline{R} \subseteq R$, or, equivalently, $\overline{R} \subseteq Rs^n \subseteq T$. Then, $T \in [\overline{R}, S]$ and we obtain $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$.

Assume that R is one-dimensional. If $R \subset S$ is not integral, then $\overline{R} \subset S$ is Prüfer and $\overline{R}$ is one-dimensional. To complete the proof, use Proposition 1.2 (3). □

4.2 FCP Extensions

In case we consider only FCP extensions, we obtain more results.

Proposition 4.16 *Let $R \subseteq S$ be an FCP extension. The following statements are equivalent:*

(1) $R \subseteq S$ *is almost-Prüfer.*
(2) $R_P \subseteq S_P$ *is either integral or Prüfer for each* $P \in \mathrm{Spec}(R)$.
(3) $R_P \subseteq S_P$ *is almost-Prüfer for each* $P \in \mathrm{Spec}(R)$ *and* $\mathrm{Supp}(S/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R) = \emptyset$.
(4) $\mathrm{Supp}(\overline{R}/R) \cap \mathrm{Supp}(S/\overline{R}) = \emptyset$.

Proof The equivalence of Proposition 4.12 shows that (2) $\Leftrightarrow$ (1) holds because $\widehat{T} = \widetilde{T}$ and over a local ring T, an almost-Prüfer FCP extension $T \subseteq U$ is either integral or Prüfer [37, Proposition 2.4] . Moreover when $R_P \subseteq S_P$ is either integral or Prüfer, it is easy to show that $(\widetilde{R})_P = \widetilde{R_P}$

Next we show that (3) is equivalent to (2) of Proposition 4.12.

Let $P \in \mathrm{Supp}(S/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R)$ be such that $R_P \subseteq S_P$ is almost-Prüfer. Then, $(\widetilde{R})_P \neq R_P, S_P$, so that $R_P \subset (\widetilde{R})_P \subset S_P$. Since $R \subset \widetilde{R}$ is Prüfer, so is $R_P \subset (\widetilde{R})_P$, giving $(\widetilde{R})_P \subseteq \widetilde{R_P}$ and $R_P \neq \widetilde{R_P}$. It follows that $\widetilde{R_P} = S_P$ in view of the dichotomy principle [37, Proposition 3.3] since R_P is a local ring, and then $\widetilde{R_P} \neq (\widetilde{R})_P$.

Conversely, assume that $\widetilde{R_P} \neq (\widetilde{R})_P$, *i.e.* $P \in \mathrm{Supp}(S/R)$. Then, $R_P \neq \widetilde{R_P}$, so that $\widetilde{R_P} = S_P$, as we have just seen. Hence $R_P \subset S_P$ is integrally closed. It follows that $\overline{R_P} = \overline{R}_P = R_P$, so that $P \notin \mathrm{Supp}(\overline{R}/R)$ and $P \in \mathrm{Supp}(\widetilde{R}/R)$ by Theorem 4.6(5). Moreover, $\widetilde{R}_P \neq S_P$ implies that $P \in \mathrm{Supp}(S/\widetilde{R})$. To conclude, $P \in \mathrm{Supp}(S/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R)$.

(1) $\Leftrightarrow$ (4) An FCP extension is quasi-Prüfer by Corollary 3.4. Suppose that $R \subseteq S$ is almost-Prüfer. By Theorem 4.6, letting $U := \widetilde{R}$, we get that $U \cap \overline{R} = R$ and $S = \overline{R}U$. We deduce from [37, Proposition 3.6] that $\mathrm{Supp}(\overline{R}/R) \cap \mathrm{Supp}(S/\overline{R}) = \emptyset$. Suppose that this last condition holds. Then by [37, Proposition 3.6] $R \subseteq S$ can be factored $R \subseteq U \subseteq S$, where $R \subseteq U$ is integrally closed, whence Prüfer by Proposition 1.3, and $U \subseteq S$ is integral. Therefore, $R \subseteq S$ is almost-Prüfer. ◻

Proposition 4.17 *Let $R \subset S$ be an FCP almost-Prüfer extension. Then, $\widetilde{R} = \widehat{R}$ and $\widetilde{R}$ is the least $T \in [R, S]$ such that $T \subseteq S$ is integral.*

Proof Let $T \in [R, S]$ be such that $T \subseteq S$ is integral. So is $T_M \subseteq S_M$ for each $M \in \mathrm{Max}(R)$. But $R_M \subseteq S_M$ is either integral (1) or Prüfer (2). In case (1), we get $R_M = \widetilde{R}_M \subseteq T_M$ and in case (2), we get $\widetilde{R}_M = S_M = T_M$, so that $\widetilde{R}_M \subseteq T_M$. By globalization, $\widetilde{R} \subseteq T$. ◻

We will need a relative version of the support. Let $f : R \to T$ be a ring morphism and E a T-module. The relative support of E over R is $\mathscr{S}_R(E) := {}^a f(\mathrm{Supp}_T(E))$ and $\mathrm{M}\mathscr{S}_R(E) := \mathscr{S}_R(E) \cap \mathrm{Max}(R)$. In particular, for a ring extension $R \subset S$, we have $\mathscr{S}_R(S/R) := \mathrm{Supp}_R(S/R))$.

Proposition 4.18 *Let $R \subseteq S$ be an FCP extension. The following statements hold:*

(1) $\mathrm{Supp}(\overline{\widetilde{R}}/\overline{R}) \cap \mathrm{Supp}(\overline{R}/R) = \emptyset$.
(2) $\mathrm{Supp}(\widetilde{R}/R) \cap \mathrm{Supp}(\overline{R}/R) = \mathrm{Supp}(\overline{\widetilde{R}}/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R) = \emptyset$.
(3) $\mathrm{MSupp}(S/R) = \mathrm{MSupp}(\widetilde{R}/R) \cup \mathrm{MSupp}(\overline{R}/R)$.

Proof (1) is a consequence of Proposition 4.16(4) because $R \subseteq \overline{\widetilde{R}}$ is almost-Prüfer.

We prove the first part of (2). If some $M \in \mathrm{Supp}(\widetilde{R}/R) \cap \mathrm{Supp}(\overline{R}/R)$, it can be supposed in $\mathrm{Max}(R)$ because supports are stable under specialization. Set $R' := R_M, U := (\widetilde{R})_M, T := (\overline{R})_M$ and $M' := MR_M$. Then, $R' \neq U, T$, with $R' \subset U$ FCP Prüfer and $R' \subset T$ FCP integral, an absurdity [37, Proposition 3.3].

To show the second part, assume that some $P \in \mathrm{Supp}(\overline{\widetilde{R}}/\widetilde{R}) \cap \mathrm{Supp}(\widetilde{R}/R)$. Then, $P \notin \mathrm{Supp}(\overline{R}/R)$ by the first part of (2), so that $\overline{R}_P = R_P$, giving $(\overline{\widetilde{R}})_P = \overline{R}_P\widetilde{R}_P = \widetilde{R}_P$, a contradiction.

(3) Obviously, $\mathrm{MSupp}(S/R) = \mathrm{M}\mathscr{S}(S/R) = \mathrm{M}\mathscr{S}(S/\overline{T}^S) \cup \mathrm{M}\mathscr{S}(\overline{T}^S/T) \cup \mathrm{M}\mathscr{S}(T/\overline{U}^T) \cup \mathrm{M}\mathscr{S}(\overline{U}^T/U) \cup \mathrm{M}\mathscr{S}(U/R)$. By [37, Propositions 2.3 and 3.2], we have $\mathrm{M}\mathscr{S}(S/\overline{T}^S) \subseteq \mathscr{S}(\overline{T}^S/T) = \mathscr{S}(\overline{R}/\overline{R}^T) = \mathrm{M}\mathscr{S}(\overline{R}/R) = \mathrm{MSupp}(\overline{R}/R)$, $\mathrm{M}\mathscr{S}(T/\overline{U}^T) = \mathscr{S}(\overline{R}^T/R) \subseteq \mathscr{S}(\overline{R}/R) = \mathrm{Supp}(\overline{R}/R)$ and $\mathrm{M}\mathscr{S}(\overline{U}^T/U) = \mathscr{S}(\overline{R}^T/R) = \mathrm{Supp}(\overline{R}/R)$. To conclude, $\mathrm{MSupp}(S/R) = \mathrm{MSupp}(\widetilde{R}/R) \cup \mathrm{MSupp}(\overline{R}/R)$. □

Proposition 4.19 *Let $R \subset S$ be an FCP extension and $M \in \mathrm{MSupp}(S/R)$, then $\widetilde{R_M} = (\widetilde{R})_M$ if and only if $M \notin \mathrm{MSupp}(S/\widetilde{R}) \cap \mathrm{MSupp}(\widetilde{R}/R)$.*

Proof In fact, we are going to show that $\widetilde{R_M} \neq (\widetilde{R})_M$ if and only if $M \in \mathrm{MSupp}(S/\widetilde{R}) \cap \mathrm{MSupp}(\widetilde{R}/R)$.

Let $M \in \mathrm{MSupp}(S/\widetilde{R}) \cap \mathrm{MSupp}(\widetilde{R}/R)$. Then, $\widetilde{R_M} \neq R_M, S_M$ and $R_M \subset \widetilde{R_M} \subset S_M$. Since $R \subset \widetilde{R}$ is Prüfer, so is $R_M \subset \widetilde{R}_M$ by Proposition 1.2, giving $(\widetilde{R})_M \subseteq \widetilde{R_M}$ and $R_M \neq \widetilde{R_M}$. Therefore, $\widetilde{R_M} = S_M$ [37, Proposition 3.3] since R_M is local, and then $\widetilde{R_M} \neq (\widetilde{R})_M$.

Conversely, if $\widetilde{R_M} \neq (\widetilde{R})_M$, then, $R_M \neq \widetilde{R_M}$, so that $\widetilde{R_M} = S_M$, as we have just seen and then $R_M \subset S_M$ is integrally closed. It follows that $\overline{R_M} = \overline{R}_M = R_M$, so that $M \notin \mathrm{MSupp}(\overline{R}/R)$. Hence, $M \in \mathrm{MSupp}(\widetilde{R}/R)$ by Proposition 4.18(3). Moreover, $\widetilde{R}_M \neq S_M \Rightarrow M \in \mathrm{MSupp}(S/\widetilde{R})$. To conclude, $M \in \mathrm{MSupp}(S/\widetilde{R}) \cap \mathrm{MSupp}(\widetilde{R}/R)$. □

If $R \subseteq S$ is an extension, with $\dim(R) = 0$, $\widetilde{R_M} = (\widetilde{R})_M$ for any $M \in \mathrm{Max}(R)$. Indeed by Scholium A (2), the flat epimorphisms $R \to \widetilde{R}$ and $R_M \to (\widetilde{R})_M$ are bijective. This conclusion holds in another context.

Corollary 4.20 *Let $R \subset S$ be an FCP extension. Assume that one of the following conditions is satisfied:*

(1) $\mathrm{MSupp}(S/\widetilde{R}) \cap \mathrm{MSupp}(\widetilde{R}/R) = \emptyset$.
(2) $S = \overline{R}\widetilde{R}$, *or equivalently, $R \subseteq S$ is almost-Prüfer.*

Then, $\widetilde{R_M} = (\widetilde{R})_M$ for any $M \in \mathrm{Max}(R)$.

Proof (1) is Proposition 4.19. (2) is Proposition 4.12. □

Proposition 4.21 *Let $R \subset S$ be an almost-Prüfer FCP extension. Then, any $T \in [R, S]$ is the integral closure of $T \cap \widetilde{R}$ in $T\widetilde{R}$. Moreover, if $T \cap \widetilde{R} = R$, then $T = T\widetilde{R} \cap \overline{R}$; if $T\overline{R} = S$, then $T = (T \cap \overline{R})\widetilde{R}$; if $T\widetilde{R} = S$, then $T = (T \cap \widetilde{R})\overline{R}$.*

Proof Set $U := T \cap \widetilde{R}$ and $V := T\widetilde{R}$. Since $R \subset S$ is almost-Prüfer, $U \subseteq \widetilde{R}$ is Prüfer and $\widetilde{R} \subseteq V$ is integral and $\widetilde{R}$ is also the Prüfer hull of $U \subseteq V$. Because $R \subset S$ is almost-Prüfer, for each $M \in \mathrm{MSupp}_R(S/R)$, $R_M \subseteq S_M$ is either integral or Prüfer by Proposition 4.16, and so is $U_M \subseteq V_M$. But $\widetilde{R_M} = (\widetilde{R})_M$ by Corollary 4.20 is also the Prüfer hull of $U_M \subseteq V_M$. Let T' be the integral closure of U in V. Then, T'_M is the integral closure of U_M in V_M.

Assume that $U_M \subseteq V_M$ is integral. Then $V_M = T'_M$ and $U_M = (\widetilde{R})_M$, so that $V_M = T_M(\widetilde{R})_M = T_M$, giving $T_M = T'_M$.

Assume that $U_M \subseteq V_M$ is Prüfer. Then $U_M = T'_M$ and $V_M = (\widetilde{R})_M$, so that $U_M = T_M \cap (\widetilde{R})_M = T_M$, giving $T_M = T'_M$.

To conclude, $T_M = T'_M$ follows for each $M \in \mathrm{MSupp}_R(S/R)$. Since $R_M = S_M$, with $T_M = T'_M$ for each $M \in \mathrm{Max}(R) \setminus \mathrm{MSupp}_R(S/R)$, we get $T = T'$, whence T is the integral closure of $U \subseteq V$.

The last results are then obvious. □

We build an example of an FCP extension $R \subset S$ where $\widetilde{R_M} \neq (\widetilde{R})_M$ for some $M \in \mathrm{Max}(R)$. In particular, $R \subset S$ is not almost-Prüfer.

Example 4.22 Let R be an integral domain with quotient field S and $\mathrm{Spec}(R) := \{M_1, M_2, P, 0\}$, where $M_1 \neq M_2$ are two maximal ideals and P a prime ideal satisfying $P \subset M_1 \cap M_2$. Assume that there are R_1, R_2, and R_3 such that $R \subset R_1$ is Prüfer minimal, with $\mathscr{C}(R, R_1) = M_1$, $R \subset R_2$ is integral minimal, with $\mathscr{C}(R, R_2) = M_2$, and $R_2 \subset R_3$ is Prüfer minimal, with $\mathscr{C}(R_2, R_3) = M_3 \in \mathrm{Max}(R_2)$ such that $M_3 \cap R = M_2$ and $M_2 R_3 = R_3$. This last condition is satisfied when $R \subset R_2$ is either ramified or inert. Indeed, in both cases, $M_3 R_3 = R_3$; moreover, in the ramified case, we have $M_3^2 \subseteq M_2$ and in the inert case, $M_3 = M_2$ [36, Theorem 3.3]. We apply [14, Proposition 7.10] and [13, Lemma 2.4] several times. Set $R'_2 := R_1 R_2$. Then, $R_1 \subset R'_2$ is integral minimal, with $\mathscr{C}(R_1, R'_2) =: M'_2 = M_2 R_1$ and $R_2 \subset R'_2$ is Prüfer minimal, with $\mathscr{C}(R_2, R'_2) =: M'_1 = M_1 R_2 \in \mathrm{Max}(R_2)$. Moreover, $M'_1 \neq M_3$, $\mathrm{Spec}(R_1) = \{M'_2, P_1, 0\}$, where P_1 is the only prime ideal of R_1 lying over P. But, $P = (R : R_1)$ by [17, Proposition 3.3], so that $P = P_1$. Set $R'_3 := R_3 R'_2$. Then, $R'_2 \subset R'_3$ is Prüfer minimal, with $\mathscr{C}(R'_2, R'_3) =: M'_3 = M_3 R'_2 \in \mathrm{Max}(R'_2)$ and $R_3 \subset R'_3$ is Prüfer minimal, with $\mathscr{C}(R_3, R'_3) = M''_1 = M_1 R_3 \in \mathrm{Max}(R_3)$. We have therefore $\mathrm{Spec}(R'_3) = \{P', 0\}$ where P' is the only prime ideal of R'_3 lying over P. To end, assume that $R'_3 \subset S$ is Prüfer minimal, with $\mathscr{C}(R'_3, S) = P'$. Hence, R_2 is the integral closure of R in S. In particular, $R \subset S$ has FCP [13, Theorems 6.3 and 3.13] and is quasi-Prüfer. Since $R \subset R_1$ is integrally closed, we have $R_1 \subseteq \widetilde{R}$. Assume that $R_1 \neq \widetilde{R}$. Then, there exists $T \in [R_1, S]$ such that $R_1 \subset T$ is Prüfer minimal and $\mathscr{C}(R_1, T) = M'_2$, a contradiction by Proposition 4.16 since $M'_2 = \mathscr{C}(R_1, R'_2)$, with $R_1 \subset R'_2$ integral minimal. Then, $R_1 = \widetilde{R}$. It follows that $M_1 \in \mathrm{MSupp}(\widetilde{R}/R)$. But, $P = \mathscr{C}(R'_3, S) \cap R \in \mathrm{Supp}(S/\widetilde{R})$ and $P \subset M_1$ give $M_1 \in \mathrm{MSupp}(S/\widetilde{R})$, so that $\widetilde{R_{M_1}} \neq (\widetilde{R})_{M_1}$ by Proposition 4.19 giving that $R \subset S$ is not almost-Prüfer.

We now intend to refine Theorem 4.6, following the scheme used in [3, Proposition 4] for extensions of integral domains.

Proposition 4.23 *Let $R \subseteq S$ and $U, T \in [R, S]$ be such that $R \subseteq U$ is integral and $R \subseteq T$ is Prüfer. Then $U \subseteq UT$ is Prüfer in the following cases and $R \subseteq UT$ is almost-Prüfer.*

(1) $\mathrm{Supp}(\overline{R}/R) \cap \mathrm{Supp}(\widetilde{R}/R) = \emptyset$ *(for example, if $R \subseteq S$ has FCP).*
(2) $R \subseteq U$ preserves integral closure.

Proof (1) We have $\emptyset = \mathrm{MSupp}(U/R) \cap \mathrm{MSupp}(T/R)$, since $U \subseteq \overline{R}$ and $T \subseteq \widetilde{R}$. Let $M \in \mathrm{MSupp}((UT)/R)$. For $M \in \mathrm{MSupp}(U/R)$, we have $R_M = T_M$ and $(UT)_M = U_M$. If $M \notin \mathrm{MSupp}(U/R)$, then $U_M = R_M$ and $(UT)_M = T_M$, so that $U_M \subseteq (UT)_M$ identifies to $R_M \subseteq T_M$.

Let $N \in \mathrm{Max}(U)$ and set $M := N \cap R \in \mathrm{Max}(R)$ since $R \subseteq U$ is integral. If $M \notin \mathrm{Supp}(\overline{R}/R)$, then $R_M = \overline{R}_M = U_M$ and N is the only maximal ideal of U lying over M. It follows that $U_M = U_N$ and $(UT)_M = (UT)_N$ by [13, Lemma 2.4]. Then, $U_N \subseteq (UT)_N$ identifies to $R_M \subseteq T_M$ which is Prüfer. If $M \notin \mathrm{Supp}(\widetilde{R}/R)$, then $R_M = T_M$ gives $U_M = (UT)_M$, so that $U_N = (UT)_N$ by localizing the precedent equality and $U_N \subseteq (UT)_N$ is still Prüfer. Therefore, $U \subseteq UT$ is locally Prüfer, whence Prüfer by Proposition 1.1.

(2) The usual reasoning gives $U \otimes_R T \cong UT$, whence $U \subseteq UT$ is integrally closed. From $U \subseteq \overline{R}^{UT}$, we deduce $U = \overline{R}^{UT}$. Because $R \subseteq UT$ is almost-Prüfer, whence quasi-Prüfer, $U \subseteq UT$ is Prüfer. □

Next propositions generalize Ayache's results of [3, Proposition 11].

Proposition 4.24 *Let $R \subseteq S$ be a quasi-Prüfer extension, $T, T' \in [R, S]$ and $U := T \cap T'$. The following statements hold:*

(1) $\widetilde{T} = \widetilde{(T \cap \overline{R})}$ for each $T \in [R, S]$.
(2) $\widetilde{T} \cap \widetilde{T'} \subseteq \widetilde{T \cap T'}$.
(3) If $\mathrm{Supp}(\overline{T}/T) \cap \mathrm{Supp}(\widetilde{T}/T) = \emptyset$ (this assumption holds if $R \subseteq S$ has FCP), then, $T \subseteq T' \Rightarrow \widetilde{T} \subseteq \widetilde{T'}$.
(4) If $\mathrm{Supp}(\overline{U}/U) \cap \mathrm{Supp}(\widetilde{U}/U) = \emptyset$, then $\widetilde{T} \cap \widetilde{T'} = \widetilde{T \cap T'}$.

Proof (1) We observe that $R \subseteq T$ is quasi-Prüfer by Corollary 3.3. Since $T \cap \overline{R}$ is the integral closure of R in T, we get that $T \cap \overline{R} \subseteq T$ is Prüfer. It follows that $T \cap \overline{R} \subseteq \widetilde{T}$ is Prüfer. We thus have $\widetilde{T} \subseteq \widetilde{T \cap \overline{R}}$. To prove the reverse inclusion, we set $V := T \cap \overline{R}$ and $W := \widetilde{V} \cap \overline{T}$. We have $W \cap \overline{R} = \widetilde{V} \cap \overline{R} = V$, because $V \subseteq \widetilde{V} \cap \overline{R}$ is integral and Prüfer since we have a tower $V \subseteq \widetilde{V} \cap \overline{R} \subseteq \widetilde{V}$. Therefore, $V \subseteq W$ is Prüfer because $W \in [V, \widetilde{V}]$. Moreover, $T \subseteq \widetilde{T} \subseteq \widetilde{V}$, since $V \subseteq \widetilde{T}$ is Prüfer. Then, $T \subseteq W$ is integral because $W \in [T, \overline{T}]$, and we have $V \subseteq T \subseteq W$. This entails that $T = W = \widetilde{V} \cap \overline{T}$, so that $T \subseteq \widetilde{V}$ is Prüfer. It follows that $\widetilde{V} \subseteq \widetilde{T}$ since $T \in [V, \widetilde{V}]$.

(2) A quasi-Prüfer extension is Prüfer if and only if it is integrally closed. We observe that $T \cap T' \subseteq \widetilde{T} \cap \widetilde{T'}$ is integrally closed, whence Prüfer. It follows that $\widetilde{T} \cap \widetilde{T'} \subseteq \widetilde{T \cap T'}$.

(3) Set $U = T \cap \overline{R}$ and $U' = T' \cap \overline{R}$, so that $U, U' \in [R, \overline{R}]$ with $U \subseteq U'$. In view of (1), we thus can suppose that $T, T' \in [R, \overline{R}]$. It follows that $T \subseteq T'$ is integral and $T \subseteq \widetilde{T}$ is Prüfer. We deduce from Proposition 4.23(1) that $T' \subseteq T'\widetilde{T}$ is Prüfer, so that $\widetilde{T}T' \subseteq \widetilde{T'}$, because $\mathrm{Supp}(\overline{T}/T) \cap \mathrm{Supp}(\widetilde{T}/T) = \emptyset$ and $\overline{T} = \overline{R}$. Therefore, we have $\widetilde{T} \subseteq \widetilde{T'}$.

(4) Assume that $\mathrm{Supp}(\overline{U}/U) \cap \mathrm{Supp}(\widetilde{U}/U) = \emptyset$. Then, $T \cap T' \subset T, T'$ gives $\widetilde{T \cap T'} \subseteq \widetilde{T} \cap \widetilde{T'}$ in view of (3), so that $\widetilde{T \cap T'} = \widetilde{T} \cap \widetilde{T'}$ by (2). □

Proposition 4.25 *Let $R \subseteq S$ be a quasi-Prüfer extension and $T \subseteq T'$ a subextension of $R \subseteq S$. Set $U := T \cap \overline{R}$, $U' := T' \cap \overline{R}$, $V := T\overline{R}$ and $V' := T'\overline{R}$. The following statements hold:*

(1) $T \subseteq T'$ is integral if and only if $V = V'$.
(2) $T \subseteq T'$ is Prüfer if and only if $U = U'$.
(3) Assume that $U \subset U'$ is integral minimal and $V = V'$. Then, $T \subset T'$ is integral minimal, of the same type as $U \subset U'$.
(4) Assume that $V \subset V'$ is Prüfer minimal and $U = U'$. Then, $T \subset T'$ is Prüfer minimal.
(5) Assume that $T \subset T'$ is minimal and set $P := \mathcal{C}(T, T')$.

(a) If $T \subset T'$ is integral, then $U \subset U'$ is integral minimal if and only if $P \cap U \in$ $\mathrm{Max}(U)$.
(b) If $T \subset T'$ is Prüfer, then $V \subset V'$ is Prüfer minimal if and only if there is exactly one prime ideal in V lying over P.

Proof In $[R, S]$, the extensions $U \subseteq U'$, $T \subseteq V$, $T' \subseteq V'$ are integral and $V \subseteq V'$, $U \subseteq T$, $U' \subseteq T'$ are Prüfer. Moreover, $\overline{R}$ is also the integral closure of $U \subseteq V'$.

(1) is gotten by considering the extension $T \subseteq V'$, which is both $T \subseteq V \subseteq V'$ and $T \subseteq T' \subseteq V'$.

(2) is gotten by considering the extension $U \subseteq T'$, which is both $U \subseteq T \subseteq T'$ and $U \subseteq U' \subseteq T'$.

(3) Assume that $U \subset U'$ is integral minimal and $V = V'$. Then, $T \subset T'$ is integral by (1) and $T \neq T'$ because of (2). Set $M := (U : U') \in \mathrm{Supp}_U(U'/U)$. For any $M' \in \mathrm{Max}(U)$ such that $M' \neq M$, we have $U_{M'} = U'_{M'}$, so that $T_{M'} = T'_{M'}$ because $U_{M'} \subseteq T'_{M'}$ is Prüfer. But, $U \subseteq T'$ is almost-Prüfer, giving $T' = TU'$. By Theorem 4.6, $(T : T') = (U : U')T = MT \neq T$ because $T \neq T'$. We get that $U \subseteq T$ Prüfer implies that $M \notin \mathrm{Supp}_U(T/U)$ and $U_M = T_M$. It follows that $T'_M = T_M U'_M = U'_M$. Therefore, $T_M \subseteq T'_M$ identifies to $U_M \subseteq U'_M$, which is minimal of the same type as $U \subset U'$ by [14, Proposition 4.6]. Then, $T \subset T'$ is integral minimal of the same type as $U \subset U'$.

(4) Assume that $V \subset V'$ is Prüfer minimal and $U = U'$. Then, $T \subset T'$ is Prüfer by (2) and $T \neq T'$ because of (1). Set $Q := \mathcal{C}(V, V')$ and $P := Q \cap T \in \mathrm{Max}(T)$ since $Q \in \mathrm{Max}(V)$. For any $P' \in \mathrm{Max}(T)$ such that $P' \neq P$, and $Q' \in \mathrm{Max}(V)$ lying over P', we have $V_{Q'} = V'_{Q'}$, so that $V_{P'} = V'_{P'}$. Therefore, $T'_{P'} \subseteq V'_{P'}$ is integral, so that $T_{P'} = T'_{P'}$ and $P' \notin \mathrm{Supp}_T(T'/T)$. Hence $T \subset T'$ is Prüfer minimal [13, Proposition 6.12].

(5) Assume that $T \subset T'$ is a minimal extension and set $P := \mathcal{C}(T, T')$.

(a) Assume that $T \subset T'$ is integral. Then, $V = V'$ and $U \neq U'$ by (1) and (2). We can use Proposition 4.5 getting that $P = (U : U')T \in \mathrm{Max}(T)$ and $Q := (U : U') = P \cap U \in \mathrm{Spec}(U)$. It follows that $Q \notin \mathrm{Supp}_U(T/U)$, so that $U_Q = T_Q$ and $U'_Q = T'_Q$. Then, $U_Q \subset U'_Q$ is integral minimal, with $Q \in \mathrm{Supp}_U(U'/U)$.

If $Q \notin \mathrm{Max}(U)$, then $U \subset U'$ is not minimal by the properties of the crucial maximal ideal.

Assume that $Q \in \text{Max}(U)$ and let $M \in \text{Max}(U)$, with $M \neq Q$. Then, $U_M = U'_M$ because $M + Q = U$, so that $U \subset U'$ is a minimal extension and (a) is gotten.

(b) Assume that $T \subset T'$ is Prüfer. Then, $V \neq V'$ and $U = U'$ by (1) and (2). Moreover, $PT' = T'$ gives $PV' = V'$. Let $Q \in \text{Max}(V)$ lying over P. Then, $QV' = V'$ gives that $Q \in \text{Supp}_V(V'/V)$. Moreover, we have $V' = VT'$. Let $P' \in \text{Max}(T)$, $P' \neq P$. Then, $T_{P'} = T'_{P'}$ gives $V_{P'} = V'_{P'}$. It follows that $\text{Supp}_T(V'/V) = \{P\}$ and $\text{Supp}_V(V'/V) = \{Q \in \text{Max}(V) \mid Q \cap T = P\}$. But, by [13, Proposition 6.12], $V \subset V'$ is Prüfer minimal if and only if $|\text{Supp}_V(V'/V)| = 1$, and then if and only if there is exactly one prime ideal in V lying over P. □

This proposition has a simpler dual form in the FCP almost-Prüfer case.

Proposition 4.26 *Let $R \subseteq S$ be an FCP almost-Prüfer extension and $T \subseteq T'$ a subextension of $R \subseteq S$. Set $U := T \cap \widetilde{R}$, $U' := T' \cap \widetilde{R}$, $V := T\widetilde{R}$, and $V' := T'\widetilde{R}$. The following statements hold:*

(1) $T \subseteq T'$ is integral (and minimal) if and only if $U = U'$ (and $V \subset V'$ is minimal).
(2) $T \subseteq T'$ is Prüfer (and minimal) if and only if $V = V'$ (and $U \subset U'$ is minimal).

Proof In view of Proposition 4.21, T (resp. T') is the integral closure of U (resp. U') in V (resp. V'). The result is gotten by localizing at the elements of $\text{MSupp}_U(V'/U)$ and using Proposition 4.16. □

Lemma 4.27 *Let $R \subseteq S$ be an FCP almost-Prüfer extension and $U \in [R, \overline{R}]$, $V \in [\overline{R}, S]$. Then $U \subseteq V$ has FCP and is almost-Prüfer. The same result holds when $U \in [R, \widetilde{R}]$ and $V \in [\widetilde{R}, S]$.*

Proof Assume first that $U \in [R, \overline{R}]$ and $V \in [\overline{R}, S]$. Obviously, $U \subseteq V$ has FCP and $\overline{R}$ is the integral closure of U in V. Proposition 4.16 entails that $\text{Supp}_R(\overline{R}/R) \cap \text{Supp}_R(S/\overline{R}) = \emptyset$. We claim that $\text{Supp}_U(\overline{R}/U) \cap \text{Supp}_U(V/\overline{R}) = \emptyset$. Deny and let $Q \in \text{Supp}_U(\overline{R}/U) \cap \text{Supp}_U(V/\overline{R})$. Then, $\overline{R}_Q \neq U_Q, V_Q$. If $P := Q \cap R$, we get that $\overline{R}_P \neq U_P, V_P$, giving $\overline{R}_P \neq R_P, S_P$, a contradiction. Another use of Proposition 4.16 shows that $U \subseteq V$ is almost-Prüfer. The second result is obvious. □

Theorem 4.28 *Let $R \subseteq S$ be an FCP almost-Prüfer extension and $T \subseteq T'$ a subextension of $R \subseteq S$. Set $U := T \cap \overline{R}$ and $V' := T'\overline{R}$. Let W be the Prüfer hull of $U \subseteq V'$. Then, W is also the Prüfer hull of $T \subseteq T'$ and $T \subseteq T'$ is an FCP almost-Prüfer extension.*

Proof By Lemma 4.27, we get that $U \subseteq V'$ is an FCP almost-Prüfer extension. Let $\widetilde{T}$ be the Prüfer hull of $T \subseteq T'$. Since $U \subseteq T$ and $T \subseteq \widetilde{T}$ are Prüfer, so is $U \subseteq \widetilde{T}$ and $\widetilde{T} \subseteq V'$ gives that $\widetilde{T} \subseteq W$. Then, $T \subseteq W$ is Prüfer as a subextension of $U \subseteq W$.

Moreover, in view of Proposition 4.17, W is the least U-subalgebra of V' over which V' is integral. Since $T' \subseteq V'$ is integral, we get that $W \subseteq T'$, so that $W \in [T, T']$, with $W \subseteq T'$ integral as a subextension of $W \subseteq V'$. It follows that W is also the Prüfer hull of $T \subseteq T'$ and $T \subseteq T'$ is an FCP almost-Prüfer extension. □

Remark 4.29 The result of this theorem may not hold if the FCP hypothesis is lacking. Take the example of [13, Remark 2.9(c)], where $R \subseteq S \subseteq T$ is almost-Prüfer, $R \subseteq S$ Prüfer, $S \subseteq T$ integral and $R \subseteq T$ has not FCP. Here, (R, M) is a one-dimensional valuation domain with quotient field S and $T = S[X]/(X^2) = S[x]$. Set $R' := R[x]$. Then, R' is local, with $\mathrm{Spec}(R') = \{P' := Rx,\ M' := M \widetilde{+} Rx\}$. By the characterization of a Prüfer extension in Proposition 1.2 (3), $R' = \widetilde{R'}$, but $R' \subset T$ is not integral, so that $R' \subset T$ is not almost-Prüfer.

5 Fibers of Quasi-Prüfer Extensions

We intend to complete some results of Ayache-Dobbs [5]. We begin by recalling some features about quasi-finite ring morphisms. A ring morphism $R \to S$ is called quasi-finite by [39] if it is of finite type and $\kappa(P) \to \kappa(P) \otimes_R S$ is finite (as a $\kappa(P)$-vector space), for each $P \in \mathrm{Spec}(R)$ [39, Proposition 3, p. 40].

Proposition 5.1 *A ring morphism of finite type is incomparable if and only if it is quasi-finite and, if and only if its fibers are finite.*

Proof Use [40, Corollary 1.8] and the above definition. □

Theorem 5.2 *An extension $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq T$ is quasi-finite (equivalently, has finite fibers) for each $T \in [R, S]$ such that T is of finite type over R, if and only if $R \subseteq T$ has integral fiber morphisms for each $T \in [R, S]$.*

Proof Clearly, $R \subseteq S$ is an INC-pair implies the condition by Proposition 5.1. To prove the converse, write $T \in [R, S]$ as the union of its finite type R-subalgebras T_α. Now let $Q \subseteq Q'$ be prime ideals of T, lying over a prime ideal P of R and set $Q_\alpha := Q \cap T_\alpha$ and $Q'_\alpha := Q' \cap T_\alpha$. If $R \subseteq T_\alpha$ is quasi-finite, then $Q_\alpha = Q'_\alpha$, so that $Q = Q'$ and then $R \subseteq T$ is incomparable. The last statement is Proposition 3.8. □

Corollary 5.3 *An integrally closed extension is Prüfer if and only if each of its subextensions $R \subseteq T$ of finite type has finite fibers.*

Proof It is enough to observe that the fibers of a (flat) epimorphism have a cardinal ≤ 1, because an epimorphism is spectrally injective. □

An extension $R \subseteq S$ is called *strongly affine* if each of its subextensions $R \subseteq T$ is of finite type. The above considerations show that in this case $R \subseteq S$ is quasi-Prüfer if and only if each of its subextensions has finite fibers. For example, an FCP extension is strongly affine and quasi-Prüfer. We are also interested in extensions $R \subseteq S$ that are not necessarily strongly affine, whose subextensions have finite fibers.

Next lemma will be useful, its proof is obvious.

Lemma 5.4 *Let $R \subseteq S$ be an extension and $T \in [R, S]$.*

(1) If $T \subseteq S$ is spectrally injective and $R \subseteq T$ has finite fibers, then $R \subseteq S$ has finite fibers.
(2) If $R \subseteq T$ is spectrally injective, then $T \subseteq S$ has finite fibers if and only if $R \subseteq S$ has finite fibers.

Remark 5.5 Let $R \subseteq S$ be an almost-Prüfer extension, such that the integral extension $T := \widetilde{R} \subseteq S$ has finite fibers and let $P \in \mathrm{Spec}(R)$. The study of the finiteness of $\mathrm{Fib}_{R,S}(P)$ can be reduced as follows. As $\overline{R} \subseteq S$ is an epimorphism, because it is Prüfer, it is spectrally injective (see Scholium A). The hypotheses of Proposition 4.5 hold. We examine three cases. In case $(R : \overline{R}) \not\subseteq P$, it is well known that $R_P = (\overline{R})_P$ so that $|\mathrm{Fib}_{R,S}(P)| = 1$, because $\overline{R} \to S$ is spectrally injective. Suppose now that $(R : \overline{R}) = P$. From $(R : \overline{R}) = (T : S) \cap R$, we deduce that P is lain over by some $Q \in \mathrm{Spec}(T)$ and then $\mathrm{Fib}_{R,\overline{R}}(P) \cong \mathrm{Fib}_{T,S}(Q)$. The conclusion follows as above. Thus the remaining case is $(R : \overline{R}) \subset P$ and we can assume that $PT = T$ for if not $\mathrm{Fib}_{R,\overline{R}}(P) \cong \mathrm{Fib}_{T,S}(Q)$ for some $Q \in \mathrm{Spec}(T)$ by Scholium A (1).

Proposition 5.6 *Let $R \subseteq S$ be an almost-Prüfer extension. If $\widetilde{R} \subseteq S$ has finite fiber morphisms and $(\widetilde{R}_P : S_P)$ is a maximal ideal of $\widetilde{R}_P$ for each $P \in \mathrm{Supp}_R(S/\widetilde{R})$, then $R \subseteq \overline{R}$ and $R \subseteq S$ have finite fibers.*

Proof The Prüfer closure commutes with the localization at prime ideals by Proposition 4.12. We set $T := \widetilde{R}$. Let P be a prime ideal of R and $\varphi : R \to R_P$ the canonical morphism. We clearly have $\mathrm{Fib}_{R,.}(P) = {}^a\varphi(\mathrm{Fib}_{R_P,.P}(PR_P))$. Therefore, we can localize the data at P and we can assume that R is local.

In case $(T : S) = T$, we get a factorization $R \to \overline{R} \to T$. Since $R \to T$ is Prüfer so is $R \to \overline{R}$ and it follows that $R = \overline{R}$ because a Prüfer extension is integrally closed.

From Proposition 1.2 applied to $R \subseteq T$, we get that there is some $\mathfrak{P} \in \mathrm{Spec}(R)$ such that $T = R_{\mathfrak{P}}$, $R/\mathfrak{P}$ is a valuation ring with quotient field $T/\mathfrak{P}$ and $\mathfrak{P} = \mathfrak{P}T$. It follows that $(T : S) = \mathfrak{P}T = \mathfrak{P} \subseteq R$, and hence $(T : S) = (T : S) \cap R = (R : \overline{R})$. We have therefore a pushout diagram by Theorem 4.6:

$$\begin{array}{ccc} R' := R/\mathfrak{P} & \longrightarrow & \overline{R}/\mathfrak{P} := \overline{R'} \\ \downarrow & & \downarrow \\ T' := T/\mathfrak{P} & \longrightarrow & S/\mathfrak{P} := S' \end{array}$$

where $R/\mathfrak{P}$ is a valuation domain, $T/\mathfrak{P}$ is its quotient field, and $\overline{R}/\mathfrak{P} \to S/\mathfrak{P}$ is Prüfer by [25, Proposition 5.8, p. 52].

Because $\overline{R'} \to S'$ is injective and a flat epimorphism, there is a bijective map $\mathrm{Min}(S') \to \mathrm{Min}(\overline{R'})$. But $T' \to S'$ is the fiber at $\mathfrak{P}$ of $T \to S$ and is therefore finite. Therefore, $\mathrm{Min}(S')$ is a finite set $\{N_1, \dots, N_n\}$ of maximal ideals lying over the minimal prime ideals $\{M_1, \dots, M_n\}$ of $\overline{R'}$ lying over 0 in R'. We infer from Lemma 3.7 that $\overline{R'}/M_i \to S'/N_i$ is Prüfer, whence integrally closed. Therefore, $\overline{R'}/M_i$ is an integral domain and the integral closure of R' in S'/N_i. Any maximal ideal M of $\overline{R'}$ contains some M_i. To conclude it is enough to use a result of Gilmer [19, Corollary 20.3] because the number of maximal ideals in $\overline{R'}/M_i$ is less than the separable degree of the extension of fields $T' \subseteq S'/N_i$. ◻

Remark 5.7

(1) Suppose that $(\widetilde{R} : S)$ is a maximal ideal of $\widetilde{R}$. We clearly have $(\widetilde{R} : S)_P \subseteq (\widetilde{R}_P : S_P)$ and the hypotheses on $(\widetilde{R} : S)$ of the above proposition hold.
(2) In case $\widetilde{R} \subseteq S$ is a tower of finitely many integral minimal extensions $R_{i-1} \subseteq R_i$ with $M_i = (R_{i-1} : R_i)$, then $\mathrm{Supp}_{\widetilde{R}}(S/\widetilde{R}) = \{N_1, \dots, N_n\} \subseteq \mathrm{Max}(\widetilde{R})$ where $N_i = M_i \cap \widetilde{R}$. If the ideals N_i are different, each localization at N_i of $\widetilde{R} \subseteq S$ is integral minimal and the above result may apply. This generalizes the Ayache-Dobbs result [5, Lemma 3.6], where $\widetilde{R} \subseteq S$ is supposed to be integral minimal.

Theorem 5.8 *Let $R \subseteq S$ be a quasi-Prüfer ring extension. The following three conditions are equivalent:*

(1) $R \subseteq S$ has finite fibers.
(2) $R \subseteq \overline{R}$ has finite fibers.
(3) Each extension $R \subseteq T$, where $T \in [R, S]$ has finite fibers.

Proof (1) ⇔ (2) Let $P \in \mathrm{Spec}(R)$ and the morphisms $\kappa(P) \to \kappa(P) \otimes_R \overline{R} \to \kappa(P) \otimes_R S$. The first (second) morphism is integral (a flat epimorphism) because deduced by base change from the integral morphism $R \to \overline{R}$ (the flat epimorphism $\overline{R} \to S$). Therefore, the ring $\kappa(P) \otimes_R \overline{R}$ is zero-dimensional, so that the second morphism is surjective by Scholium A (2). Set $A := \kappa(P) \otimes_R \overline{R}$ and $B := \kappa(P) \otimes_R S$, we thus have a module finite flat ring morphism $A \to B$. Hence, $A_Q \to B_Q$ is free for each $Q \in \mathrm{Spec}(A)$ [16, Proposition 9] and $B_Q \neq 0$ because it contains $\kappa(P) \neq 0$. Therefore, $A_Q \to B_Q$ is injective and it follows that $A \cong B$ giving (1) ⇔ (2).

(2) ⇒ (3) Suppose that $R \subseteq \overline{R}$ has finite fibers and let $T \in [R, S]$, then $\overline{R} \subseteq \overline{R}T$ is a flat epimorphism by Proposition 4.5(1) and so is $\kappa(P) \otimes_R \overline{R} \to \kappa(P) \otimes_R \overline{R}T$. Since $\mathrm{Spec}(\kappa(P) \otimes_R \overline{R}T) \to \mathrm{Spec}(\kappa(P) \otimes_R \overline{R})$ is injective, $R \subseteq \overline{R}T$ has finite fibers. Now $R \subseteq T$ has finite fibers because $T \subseteq \overline{R}T$ is integral and is therefore spectrally surjective.

(3) ⇒ (1) is obvious. □

Remark 5.9 Actually, the statement (1) ⇔ (2) is valid if we only suppose that $\overline{R} \subseteq S$ is a flat epimorphism. But this equivalence fails in case $\overline{R} \subseteq S$ is not a flat epimorphism as we can see in the following example. Let R be an integral domain with quotient field K and integral closure $\overline{R}$ such that $R \subset \overline{R}$ is a minimal extension. Then $R \subset \overline{R}$ has finite fibers. Consider the polynomial ring $S := K[X]$. It follows that $\overline{R}$ is also the integral closure of R in S. Moreover, $K \subset S$ and then $R \subset S$ have not finite fibers. Actually, $K \subset S$ and $\overline{R} \subset S$ are not flat epimorphisms.

Next result contains [5, Lemma 3.6], gotten after a long proof.

Corollary 5.10 *Let $R \subseteq S$ be an almost-Prüfer extension. Then $R \subseteq S$ has finite fibers if and only if $R \subseteq \overline{R}$ has finite fibers, and if and only if $\widetilde{R} \subseteq S$ has finite fibers.*

Proof By Theorem 5.8 the first equivalence is clear. The second is a consequence of Lemma 5.4(2). □

The following result is then clear and obviates any need to assume FCP or FMC.

Theorem 5.11 *Let $R \subseteq S$ be a quasi-Prüfer extension with finite fibers, then $R \subseteq T$ has finite fibers for each $T \in [R, S]$.*

Corollary 5.12 *If $R \subseteq S$ is quasi-finite and quasi-Prüfer, then $R \subseteq T$ has finite fibers for each $T \in [R, S]$ and $\widetilde{R} \subseteq S$ is module finite.*

Proof By the Zariski Main Theorem, there is a factorization $R \subseteq F \subseteq S$ where $R \subseteq F$ is module finite and $F \subseteq S$ is a flat epimorphism [39, Corollaire 2, p. 42]. To conclude, we use Scholium A in the rest of the proof. The map $\widetilde{R} \otimes_R F \to S$ is injective because $F \to \widetilde{R} \otimes_R F$ is a flat epimorphism and is surjective, since it is integral and a flat epimorphism because $\widetilde{R} \otimes_R F \to S$ is a flat epimorphism. □

Corollary 5.13 *An FMC extension $R \subseteq S$ is such that $R \subseteq T$ has finite fibers for each $T \in [R, S]$.*

Proof Such an extension is quasi-finite and quasi-Prüfer. Then use Corollary 5.12. □

[5, Example 4.7] exhibits some FMC extension $R \subseteq S$, such that $R \subseteq \overline{R}$ has not FCP. Actually, $[R, \overline{R}]$ is an infinite (maximal) chain.

Proposition 5.14 *Let $R \subseteq S$ be a quasi-Prüfer extension such that $R \subseteq \overline{R}$ has finite fibers and R is semi-local. Then T is semi-local for each $T \in [R, S]$.*

Proof Obviously $\overline{R}$ is semi-local. From the tower $\overline{R} \subseteq T\overline{R} \subseteq S$ we deduce that $\overline{R} \subseteq T\overline{R}$ is Prüfer. It follows that $T\overline{R}$ is semi-local [5, Lemma 2.5 (f)]. As $T \subseteq T\overline{R}$ is integral, we get that T is semi-local. □

The next proposition gives a kind of converse, but, before, we rewrite [4, Theorem 3.10] proved in the integral domains context, which holds in a more general context.

Theorem 5.15 *Let $R \subseteq S$ be an integrally closed extension with R semi-local. The following three conditions are equivalent:*

(1) $R \subseteq S$ is a Prüfer extension.
(2) $|\mathrm{Max}(T)| \leq |\mathrm{Max}(R)|$ for each $T \in [R, S]$.
(3) Each $T \in [R, S]$ is a semi-local ring.

Proof It is enough to mimic the proof of [4, Theorem 3.10] which is still valid for an arbitrary integrally closed extension of rings $R \subseteq S$. Indeed, $R \subseteq S$ is a Prüfer extension if and only if (R, S) is a residually algebraic pair such that $R \subseteq S$ is an integrally closed extension by Theorem 2.3 and Definition 2.1 . □

Proposition 5.16 *Let $R \subseteq S$ be an extension with $\overline{R}$ semi-local. Then $R \subseteq S$ is quasi-Prüfer if and only if T is semi-local for each $T \in [R, S]$.*

Proof If $R \subseteq S$ is quasi-Prüfer, $\overline{R} \subseteq S$ is Prüfer. Let $T \in [R, S]$ and set $T' := T\overline{R}$, so that $T \subseteq T'$ is integral, and $\overline{R} \subseteq T'$ is Prüfer (and then a normal pair). It follows from [5, Lemma 2.5 (f)] that T' is semi-local, and so is T.

If T is semi-local for each $T \in [R, S]$, so is any $T \in [\overline{R}, S]$. Then, $\overline{R} \subseteq S$ is Prüfer by Theorem 5.15 and $R \subseteq S$ is quasi-Prüfer. □

Acknowledgements The authors wish to thank the referee whose many suggestions improved the presentation of the paper.

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