# **Quasi-Prüfer Extensions of Rings**

**Gabriel Picavet and Martine Picavet-L'Hermitte**

**Abstract** We introduce quasi-Prüfer ring extensions, in order to relativize quasi-Prüfer domains and to take also into account some contexts in recent papers. An extension is quasi-Prüfer if and only if it is an INC pair. The class of these extensions has nice stability properties. We also define almost-Prüfer extensions that are quasi-Prüfer, the converse being not true. Quasi-Prüfer extensions are closely linked to finiteness properties of fibers. Applications are given for FMC extensions, because they are quasi-Prüfer.

**Keywords** Flat epimorphism • FIP • FCP Extension • Minimal extension • Integral extension • Morita • Prüfer hull • Support of a module • Fiber

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# **1 Introduction and Notation**

We consider the category of commutative and unital rings. An epimorphism is an epimorphism of this category. Let  $R \subseteq S$  be a (ring) extension. The set of all  $R$ -subalgebras of  $S$  is a set of *R*-subalgebras of *S* is denoted by [*R*, *S*]. A *chain* of *R*-subalgebras of *S* is a set of elements of [R, S] that are pairwise comparable with respect to inclusion. We say that the extension  $R \subseteq S$  has FCP (for the "finite chain property") if each chain<br>in [*R* S] is finite. Dobbs and the authors characterized ECP extensions [13]. An in  $[R, S]$  is finite. Dobbs and the authors characterized FCP extensions [\[13\]](#page-28-0). An extension  $R \subseteq S$  is called FMC if there is a finite maximal chain of extensions from  $R$  to  $S$ *R* to *S*.

G. Picavet ( $\boxtimes$ ) • M. Picavet-L'Hermitte

Laboratoire de Mathématiques, Université Blaise Pascal, 24, avenue des Landais, UMR6620 CNRS, BP 80026, 63177 Aubière Cedex, France e-mail: [Gabriel.Picavet@math.univ-bpclermont.fr](mailto:Gabriel.Picavet@math.univ-bpclermont.fr)

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We begin by explaining our motivations and aims. The reader who is not familiar with the notions used will find some Scholia in the sequel, as well as necessary definitions that exist in the literature. Knebusch and Zhang introduced Prüfer extensions in their book [\[25\]](#page-28-1). Actually, these extensions are nothing but normal pairs, that are intensively studied in the literature. We do not intend to give an extensive list of recent papers, written by Ayache, Ben Nasr, Dobbs, Jaballah, Jarboui, and some others. We are indebted to these authors because their papers are a rich source of suggestions. We observed that some of them are dealing with FCP (FMC) extensions, followed by a Prüfer extension, perhaps under a hidden form. These extensions reminded us quasi-Prüfer domains (see [\[18\]](#page-28-2) for a comprehensive study). Therefore, we introduced in [\[38\]](#page-29-0) *quasi-Prüfer* extensions  $R \subseteq S$  as extensions that can be factored  $R \subseteq R' \subseteq S$ , where the first extension is integral and the second is Prijfer. Note that FMC extensions are quasi-Prijfer integral and the second is Prüfer. Note that FMC extensions are quasi-Prüfer.

We give a systematic study of quasi-Prüfer extensions in Sects. [2](#page-5-0) and [3.](#page-7-0) The class of quasi-Prüfer extensions has a nice behavior with respect to the classical operations of commutative algebra. An important result is that quasi-Prüfer extensions coincide with INC-pairs. Another one is that this class is stable under forming subextensions and composition. A striking result is the stability of the class of quasi-Prüfer extensions by absolutely flat base change, like localizations and Henselizations. An arbitrary ring extension  $R \subseteq S$  admits a quasi-Prüfer closure, contained in S. Examples are provided by Laskerian pairs, open pairs, and the contained in *S*. Examples are provided by Laskerian pairs, open pairs, and the pseudo-Prüfer pairs of Dobbs-Shapiro [\[12\]](#page-28-3).

Section [4](#page-12-0) deals with *almost-Prüfer* extensions, a special kind of quasi-Prüfer extensions. They are of the form  $R \subseteq T \subseteq S$ , where the first extension is Prüfer<br>and the second is integral. An arbitrary ring extension  $R \subseteq S$  admits an almostand the second is integral. An arbitrary ring extension  $R \subseteq S$  admits an almost-<br>Prijfer closure, contained in *S*. The class of almost-Prijfer extensions seems to have Prüfer closure, contained in *S*. The class of almost-Prüfer extensions seems to have less properties than the class of quasi-Prüfer extensions but has the advantage that almost-Prüfer closures commute with localizations at prime ideals. We examine the transfer of the quasi (almost)-Prüfer properties to subextensions. It is noteworthy that the class of FCP almost-Prüfer extensions is stable under the formation of subextensions, although this does not hold for arbitrary almost-Prüfer extensions.

In Sect. [5,](#page-24-0) we complete and generalize the results of Ayache-Dobbs in [\[5\]](#page-28-4), with respect to the finiteness of fibers. These authors have evidently considered particular cases of quasi-Prüfer extensions. A main result is that if  $R \subseteq S$  is quasi-Prüfer with finite fibers, then so is  $R \subset T$  for  $T \in [R \times S]$ . In particular, we recover a result of [5] finite fibers, then so is  $R \subseteq T$  for  $T \in [R, S]$ . In particular, we recover a result of [\[5\]](#page-28-4) about **FMC** extensions about FMC extensions.

#### *1.1 Recalls About Some Results and Definitions*

The reader is warned that we will mostly use the definition of Prüfer extensions by flat epimorphic subextensions investigated in [\[25\]](#page-28-1). The results needed may be found in Scholium A for flat epimorphic extensions and some results of [\[25\]](#page-28-1) are summarized in Scholium B. Their powers give quick proofs of results that are generalizations of results of the literature.

As long as FCP or FMC extensions are concerned, we use minimal (ring) extensions, a concept introduced by Ferrand-Olivier [\[17\]](#page-28-5). An extension  $R \subset S$ is called *minimal* if  $[R, S] = \{R, S\}$ . It is known that a minimal extension is<br>either module-finite or a flat enimorphism [17] and these conditions are mutually either module-finite or a flat epimorphism [\[17\]](#page-28-5) and these conditions are mutually exclusive. There are three types of integral minimal (module-finite) extensions: ramified, decomposed, or inert [\[36,](#page-29-1) Theorem 3.3]. A minimal extension  $R \subset S$ admits a crucial ideal  $C(R, S) =: M$  which is maximal in *R* and such that  $R_P = S_P$  for each  $P \neq M, P \in \text{Spec}(R)$ . Moreover,  $C(R, S) = (R : S)$  when  $R \subset S$  is an integral minimal extension. The key connection between the above ideas is that if  $R \subseteq S$ <br>has ECP or EMC, then any maximal (necessarily finite) chain of *R*-subalgebras of has FCP or FMC, then any maximal (necessarily finite) chain of *R*-subalgebras of  $S, R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$ , with *length n* <  $\infty$ , results from instanceing *n* minimal extensions *R*.  $\subset R_{n+1} \cup 0 \le i \le n-1$ juxtaposing *n* minimal extensions  $R_i \subset R_{i+1}$ ,  $0 \le i \le n-1$ .

We define the *length*  $\ell[R, S]$  of  $[R, S]$  as the supremum of the lengths of chains in  $[R, S]$ . In particular, if  $\ell[R, S] = r$ , for some integer *r*, there exists a maximal chain in  $[R, S]$  with length *r* in  $[R, S]$  with length  $r$ .

As usual,  $Spec(R)$ ,  $Max(R)$ ,  $Min(R)$ ,  $U(R)$ ,  $Tot(R)$  are, respectively, the set of prime ideals, maximal ideals, minimal prime ideals, units, total ring of fractions of a ring *R* and  $\kappa(P) = R_P/PR_P$  is the residual field of *R* at  $P \in \text{Spec}(R)$ .

If  $R \subseteq S$  is an extension, then  $(R : S)$  is its conductor and if  $P \in \text{Spec}(R)$ , then<br>is the localization  $S$ . We denote the integral algebra of *R* in  $S$  by  $\overline{B}^S$  (or  $\overline{B}$ ). *S<sub>P</sub>* is the localization *S<sub>R\P</sub>*. We denote the integral closure of *R* in *S* by  $\overline{R}^S$  (or  $\overline{R}$ ).

A local ring is here what is called elsewhere a quasi-local ring. The *support* of an *R*-module *E* is  $\text{Supp}_R(E) := \{ P \in \text{Spec}(R) \mid E_P \neq 0 \}$  and  $\text{MSupp}_R(E) :=$  $\text{Supp}_R(E) \cap \text{Max}(R)$ . Finally,  $\subset$  denotes proper inclusion and |*X*| the cardinality of a set *X*.

**Scholium A** We give some recalls about flat epimorphisms (see [\[26,](#page-28-6) Chapitre IV], except (2) which is [\[30,](#page-29-2) Proposition 2]).

- (1)  $R \rightarrow S$  is a flat epimorphism  $\Leftrightarrow$  for all  $P \in Spec(R)$ , either  $R_P \rightarrow Sp$ is an isomorphism or  $S = PS \Leftrightarrow R_P \subseteq S_P$  is a flat epimorphism for all  $P \in \text{Spec}(R) \Leftrightarrow R_{QQCD} \rightarrow S_Q$  is an isomorphism for all  $Q \in \text{Spec}(S)$  and  $P \in \text{Spec}(R) \Leftrightarrow R_{(Q \cap R)} \rightarrow S_Q$  is an isomorphism for all  $Q \in \text{Spec}(S)$  and  $Spec(S) \rightarrow Spec(R)$  is injective.
- (2) (S) A flat epimorphism, with a zero-dimensional domain, is surjective.
- (3) If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are ring morphisms such that  $g \circ f$  is injective and *f* is a flat epimorphism, then *g* is injective.
- (4) Let  $R \subseteq T \subseteq S$  be a tower of extensions, such that  $R \subseteq S$  is a flat epimorphism.<br>Then  $T \subseteq S$  is a flat epimorphism but  $R \subseteq T$  does not need. A Prijfer extension Then  $T \subseteq S$  is a flat epimorphism but  $R \subseteq T$  does not need. A Prüfer extension remedies this defect remedies this defect.
- (5) (L) A faithfully flat epimorphism is an isomorphism. Hence,  $R = S$  if  $R \subseteq S$  is an integral flat epimorphism an integral flat epimorphism.
- (6) If  $f : R \to S$  is a flat epimorphism and *J* an ideal of *S*, then  $J = f^{-1}(J)S$ .<br>(7) If  $f : R \to S$  is an enimorphism then *f* is spectrally injective
- (7) If  $f : R \rightarrow S$  is an epimorphism, then  $f$  is spectrally injective (i.e.,  $a_f$  : Spec $(S) \rightarrow Spec(R)$  is an injection) and its residual extensions are isomorphisms.
- (8) Flat epimorphisms remain flat epimorphisms under base change (in particular, after a localization with respect to a multiplicatively closed subset).
- (9) Flat epimorphisms are descended by faithfully flat morphisms.

## *1.2 Recalls and Results on Prüfer Extensions*

There are a lot of characterizations of Prüfer extensions. We keep only those that are useful in this paper. Let  $R \subseteq S$  be an extension.<br>**Scholium B** 

#### **Scholium B**

- (1)  $[25]$  *R*  $\subseteq$  *S* is called Prüfer if *R*  $\subseteq$  *T* is a flat epimorphism for each *T*  $\in$  *[R, S]*.<br>(2) *R*  $\subset$  *S* is called a normal pair if *T*  $\subset$  *S* is integrally closed for each *T*  $\in$  *[R S]*
- (2)  $R \subseteq S$  is called a *normal* pair if  $T \subseteq S$  is integrally closed for each  $T \in [R, S]$ .<br>(3)  $R \subset S$  is Prijfer if and only if it is a normal pair [25] Theorem 5.2(4)]
- (3)  $R \subseteq S$  is Prüfer if and only if it is a normal pair [\[25,](#page-28-1) Theorem 5.2(4)].<br>(4) R is called Prijfer if its finitely generated regular ideals are inve
- (4) *R* is called Prüfer if its finitely generated regular ideals are invertible, or equivalently,  $R \subseteq \text{Tot}(R)$  is Prüfer [\[21,](#page-28-7) Theorem 13((5)(9))].

Hence Prüfer extensions are a relativization of Prüfer rings. Clearly, a minimal extension is a flat epimorphism if and only if it is Prüfer. We will then use for such extensions the terminology: *Prüfer minimal* extensions. The reader may find some properties of Prüfer minimal extensions in [\[36,](#page-29-1) Proposition 3.2, Lemma 3.4 and Proposition 3.5], where in addition *R* must be supposed local. The reason why is that this word has disappeared during the printing process of [\[36\]](#page-29-1).

We will need the two next results. Some of them do not explicitly appear in [\[25\]](#page-28-1) but deserve to be emphasized. We refer to [\[25,](#page-28-1) Definition 1, p.22] for a definition of Manis extensions and remark that Proposition [1.1\(](#page-3-0)1) was also noted in [\[12\]](#page-28-3).

<span id="page-3-0"></span>**Proposition 1.1** *Let*  $R \subseteq S$  *be a ring extension.* 

- *(1)*  $R \subseteq S$  *is Prüfer if and only if*  $R_P \subseteq S_P$  *is Prüfer for each*  $P \in \text{Spec}(R)$ <br>*(respectively*  $P \in \text{Sum}(S/R)$ ) *(respectively, P*  $\in$  Supp $(S/R)$ ).
- (2)  $R \subseteq S$  *is Prüfer if and only if*  $R_M \subseteq S_M$  *is Manis for each*  $M \in \text{Max}(R)$ *.*

*Proof* (1) The class of Prüfer extensions is stable under localization [\[25,](#page-28-1) Proposition] 5.1(ii), p.46-47]. To get the converse, use Scholium  $A(1)$ . (2) follows from [\[25,](#page-28-1) Proposition 2.10, p.28, Definition 1, p.46].  $\Box$ 

<span id="page-3-1"></span>**Proposition 1.2** *Let*  $R \subseteq S$  *be a ring extension, where*  $R$  *is local.* 

- *(1)*  $R \subseteq S$  *is Manis if and only if*  $S \setminus R \subseteq U(S)$  *and*  $x \in S \setminus R \Rightarrow x^{-1} \in R$ . In that case  $R \subseteq S$  is integrally closed *case,*  $R \subseteq S$  *is integrally closed.*<br> $R \subseteq S$  *is Manis if and only if*  $R$ .
- (2)  $R \subseteq S$  is Manis if and only if  $R \subseteq S$  is Prüfer.<br>(3)  $R \subseteq S$  is Prüfer if and only if there exists
- (3)  $R \subseteq S$  is Prüfer if and only if there exists  $P \in \text{Spec}(R)$  such that  $S = R_P$ ,<br> $P \subseteq SP$  and  $R/P$  is a valuation domain. Under these conditions,  $S/P$  is the  $P = SP$  and  $R/P$  is a valuation domain. Under these conditions,  $S/P$  is the *quotient field of R/P.*

*Proof* (1) is [\[25,](#page-28-1) Theorem 2.5, p.24]. (2) is [25, Scholium 10.4, p. 147]. Then (3) is  $[13,$  Theorem 6.8].

Next result shows that Prüfer FCP extensions can be described in a special manner.

#### <span id="page-4-0"></span>**Proposition 1.3** *Let*  $R \subset S$  *be a ring extension.*

- *(1)* If  $R \subset S$  has FCP, then  $R \subset S$  is integrally closed  $\Leftrightarrow R \subset S$  is Prüfer  $\Leftrightarrow R \subset S$ *is a composite of Prüfer minimal extensions.*
- *(2)* If  $R \subset S$  is integrally closed, then  $R \subset S$  has  $FCP \Leftrightarrow R \subset S$  is Prüfer and  $\text{Supp}(S/R)$  *is finite.*

*Proof* (1) Assume that  $R \subset S$  has FCP. If  $R \subset S$  is integrally closed, then,  $R \subset S$ is composed of Prüfer minimal extensions by [\[13,](#page-28-0) Lemma 3.10]. We know that a composite of Prüfer extensions is a Prüfer extension [\[25,](#page-28-1) Theorem 5.6]. Thus, by [\[25\]](#page-28-1),  $R \subset S$  is a normal pair. Conversely, if  $R \subset S$  is composed of Prüfer minimal extensions,  $R \subset S$  is integrally closed, since so is each Prüfer minimal extension. A Prüfer extension is obviously integrally closed, and an FCP integrally closed extension is Prüfer by [\[13,](#page-28-0) Theorem 6.3].

(2) The logical equivalence is  $[13,$  Theorem 6.3].

# **Definition 1.4** [\[25\]](#page-28-1) A ring extension  $R \subseteq S$  has:

- (2) The logical equivalence is [13, Theorem 6.3].<br> **Definition 1.4** [25] A ring extension  $R \subseteq S$  has:<br>
(1) a greatest flat epimorphic subextension  $R \subseteq \hat{R}^S$ , called the **Morita hull** of *R* in *S*. ension  $R \subseteq \widehat{R}^S$ , called the **Morita hull**<br> $\subseteq \widetilde{R}^S$ , called the **Prüfer hull** of *R* in *S*.
- (2) a greatest Prüfer subextension  $R \subseteq \widehat{R}$

S.<br>
(2) a greatest Prüfer subextension  $R \subseteq \widetilde{R}^S$ , called the **Prüfer hi**<br>
We set  $\widehat{R} := \widehat{R}^S$  and  $\widetilde{R} := \widetilde{R}^S$ , if no confusion can occur.  $R \subseteq$ <br>
closed if  $R = \widetilde{R}$  $\subseteq$  *S* is called Prüfer-(2) a greatest Pr<br>We set  $\hat{R} := \hat{R}^S$ <br>closed if  $R = \tilde{R}$ .<br>Note that  $\tilde{R}^S$  $\hat{R} := \hat{R}^S$  and  $\tilde{R} := \tilde{R}^S$ , if no confusion can occur.  $R \subseteq S$  is called Prüfersed if  $R = \tilde{R}$ .<br>Note that  $\tilde{R}^S$  is denoted by P(*R*, *S*) in [\[25\]](#page-28-1) and  $\hat{R}^S$  is the weakly surjective hull

 $M(R, S)$  of [\[25\]](#page-28-1). Our terminology is justified because Morita's work is earlier [\[29,](#page-29-3) Corollary 3.4]. The Morita hull can be computed by using a (transfinite) induction [\[29\]](#page-29-3). Let S' be the set of all  $s \in S$  such that there is some ideal *I* of *R*, such that  $IS = S$  and  $Is \subseteq R$ . Then  $R \subseteq S'$  is a subextension of  $R \subseteq S$ . We set  $S_1 := S'$  and  $S_{\text{max}} := (S_1)' \subset S$ . By [29 n 36] if  $R \subset S$  is an ECP extension then  $\widehat{R} = S$  for *Si* Corollary 3.4]. The Morita hull can be computed by using a (transfinite) induction [29]. Let S' be the set of all  $s \in S$  such that there is some ideal I of *R*, such that  $IS = S$  and  $Is \subseteq R$ . Then  $R \subseteq S'$  is a subextens some integer *n*.

At this stage it is interesting to point out a result showing again that integral closedness and Prüfer extensions are closely related.

**Proposition 1.5** *Olivier* [\[32,](#page-29-4) *Corollary, p. 56]* An extension  $R \subseteq S$  is integrally closed if and only if there is a pullback square: *closed if and only if there is a pullback square:*



*where V is a semi-hereditary ring and K its total quotient ring.*

In that case  $V \subseteq K$  is a Prüfer extension, since V is a Prüfer ring, whose localizations<br>at prime ideals are valuation domains and K is an absolutely flat ring. As there exist at prime ideals are valuation domains and  $K$  is an absolutely flat ring. As there exist integrally closed extensions that are not Prüfer, we see in passing that the pullback construction may not descend Prüfer extensions. The above result has a companion for minimal extensions that are Prüfer [\[20,](#page-28-8) Proposition 3.2]. **Proposition 1.6** *Let*  $R \subseteq S$  *be an extensions.* The above result has a companion for minimal extensions that are Prüfer [20, Proposition 3.2].<br>**Proposition 1.6** *Let*  $R \subseteq S$  *be an extension and*  $T \in [R, S]$ , *then*  $\wid$ 

<span id="page-5-2"></span>*Therefore, for*  $T, U \in [R, S]$  with  $T \subseteq$ *U. Extension and*  $T \in \subseteq U$ , then  $\widetilde{R}^T \subseteq \widetilde{R}^U$ . **Proposition 1.6** Let  $R \subseteq S$  be an extension and  $T \in [R, S]$ , then  $\widetilde{R}^T = \widetilde{R} \cap$ <br>*Therefore, for*  $T, U \in [R, S]$  with  $T \subseteq U$ , then  $\widetilde{R}^T \subseteq \widetilde{R}^U$ .<br>*Proof* Obvious, since the Prüfer hull  $\widetilde{R}^T$  is the gre

 $\subseteq$  *V* contained in *T*. We will show later that in some cases  $\widetilde{T} \subseteq \widetilde{U}$  if  $R \subseteq S$  has FCP. *oof* Obvious, since the Prüfer hull  $\widetilde{R}^T$  is the trained in  $T$ .<br>We will show later that in some cases  $\widetilde{T} \subseteq \widetilde{U}$ 

## <span id="page-5-0"></span>**2 Quasi-Prüfer Extensions**

We introduced the following definition in [\[38,](#page-29-0) p. 10].

<span id="page-5-1"></span>**Definition 2.1** An extension of rings  $R \subseteq S$  is called quasi-Prüfer if one of the following equivalent statements holds: following equivalent statements holds:

- (1)  $R \subseteq S$  is a Prüfer extension;<br>(2)  $R \subseteq S$  can be factored  $R \subseteq S$
- (2)  $R \subseteq S$  can be factored  $R \subseteq T \subseteq S$ , where  $R \subseteq T$  is integral and  $T \subseteq S$  is Prüfer.<br>In that case  $\overline{R} T$ In that case  $\overline{R} = T$ .

To see that  $(2) \Rightarrow (1)$  observe that if  $(2)$  holds, then  $T \subseteq R$  is integral and a flat injective enjoymhism so that  $\overline{R} - T$  by  $(1)$  (Scholium A(5)) injective epimorphism, so that  $\overline{R} = T$  by (L) (Scholium A(5)).

We observe that quasi-Prüfer extensions are akin to quasi-finite extensions if we refer to Zariski Main Theorem. This will be explored in Sect. [5,](#page-24-0) see, for example, Theorem [5.2.](#page-24-1)

Hence integral or Prüfer extensions are quasi-Prüfer. An extension is clearly Prüfer if and only if it is quasi-Prüfer and integrally closed. Quasi-Prüfer extensions allow us to avoid FCP hypotheses.

We give some other definitions involved in ring extensions  $R \subseteq S$ . The *fiber*<br> $P \in \text{Spec}(R)$  of  $R \subseteq S$  is  $\text{Fib}_{\text{B}}(P) := \{Q \in \text{Spec}(S) \mid Q \cap R = P\}$ . The at  $P \in \text{Spec}(R)$  of  $R \subseteq S$  is  $\text{Fib}_{R,S}(P) := \{Q \in \text{Spec}(S) \mid Q \cap R = P\}$ . The subspace  $\text{Fib}_{R,S}(P)$  of  $\text{Spec}(S)$  is homeomorphic to the spectrum of the fiber ring subspace  $\text{Fib}_{R,S}(P)$  of  $\text{Spec}(S)$  is homeomorphic to the spectrum of the fiber ring at *P*,  $F_{R,S}(P) := \kappa(P) \otimes_R S$ . The homeomorphism is given by the spectral map of  $S \to \kappa(P) \otimes_R S$  and  $\kappa(P) \to \kappa(P) \otimes_R S$  is the *fiber morphism* at *P*.

**Definition 2.2** A ring extension  $R \subseteq S$  is called:

- (1) *incomparable* if for each pair  $Q \subseteq Q'$  of prime ideals of *S*, then  $Q \cap R =$ <br>  $Q' \cap R \rightarrow Q Q'$  or equivalently  $\kappa(P) \otimes_R T$  is a zero-dimensional ring for  $Q' \cap R \Rightarrow Q = Q'$ , or equivalently,  $\kappa(P) \otimes_R T$  is a zero-dimensional ring for each  $T \in [R \times S]$  and  $P \in \text{Spec}(R)$ , such that  $\kappa(P) \otimes_R T \neq 0$ each  $T \in [R, S]$  and  $P \in \text{Spec}(R)$ , such that  $\kappa(P) \otimes_R T \neq 0$ .<br>
an *INC-pair* if  $R \subset T$  is incomparable for each  $T \in [R, S]$
- (2) an *INC-pair* if  $R \subseteq T$  is incomparable for each  $T \in [R, S] \Leftrightarrow T \subseteq U$  is incomparable for all  $T \subset U$  in  $[R, S]$ incomparable for all  $T \subseteq U$  in  $[R, S]$ .<br>residually algebraic if  $R/(Q \cap R) \subseteq$
- (3) *residually algebraic* if  $R/(Q \cap R) \subseteq S/Q$  is algebraic for each  $Q \in \text{Spec}(S)$ .

(4) a *residually algebraic pair* if the extension  $R \subseteq T$  is residually algebraic for each  $T \in [R, S]$ each  $T \in [R, S]$ .

An extension  $R \subseteq S$  is an INC-pair if and only if  $R \subseteq S$  is a residually algebraic<br>r. This fact is an easy consequence of [10] Theoreml (via a short proof that was pair. This fact is an easy consequence of  $[10,$  Theorem] (via a short proof that was explicitly given in [\[9\]](#page-28-10)). This fact was given for the particular case where *S* is an integral domain in [\[4\]](#page-28-11).

The following characterization was announced in [\[38\]](#page-29-0). We were unaware that this result is also proved in [\[6,](#page-28-12) Corollary 1], when we presented it in ArXiv. However, our proof is largely shorter because we use the powerful results of [\[25\]](#page-28-1).

<span id="page-6-1"></span>**Theorem 2.3** An extension  $R \subseteq S$  is quasi-Prüfer if and only if  $R \subseteq S$  is an INC-<br>pair and if and only if  $R \subseteq S$  is a residually algebraic pair  $p$ air and, if and only if,  $R \subseteq S$  is a residually algebraic pair.

*Proof* Suppose that  $R \subseteq S$  is quasi-Prüfer and let  $T \in [R, S]$ . We set  $U := RT$ .<br>Then  $\overline{R} \subseteq U$  is a flat enimorphism by definition of a Prüfer extension and hence is Then  $R \subseteq U$  is a flat epimorphism by definition of a Prüfer extension and hence is<br>incomparable as is  $R \subseteq \overline{R}$ . It follows that  $R \subseteq U$  is incomparable. Since  $T \subseteq U$ incomparable as is  $R \subseteq R$ . It follows that  $R \subseteq U$  is incomparable. Since  $T \subseteq U$ <br>is integral, it has going-un. It follows that  $R \subseteq T$  is incomparable. Conversely, it is integral, it has going-up. It follows that  $R \subseteq T$  is incomparable. Conversely, if  $R \subset S$  is an INC-pair then so is  $\overline{R} \subset S$  since  $\overline{R} \subset S$  is integrally closed  $\overline{R} \subset S$  is  $R \subseteq S$  is an INC-pair, then so is  $R \subseteq S$ . Since  $R \subseteq S$  is integrally closed,  $R \subseteq S$  is Prijfer [25]. Theorem 5.2.(9'), p. 481. The second equivalence is given by the above Prüfer [\[25,](#page-28-1) Theorem 5.2,(9'), p. 48]. The second equivalence is given by the above comments about [\[10\]](#page-28-9) and [\[9\]](#page-28-10).  $\Box$ 

<span id="page-6-0"></span>**Corollary 2.4** An extension  $R \subseteq S$  is quasi-Prüfer if and only if  $R \subseteq T$  is Prüfer<br>for each  $T \in [R \ S]$  In this case  $\overline{R}$  is the least  $T \subseteq [R \ S]$  such that  $T \subseteq S$  is Prüfer *for each T*  $\in$  [*R*, *S*]. *In this case, R is the least*  $T \in$  [*R*, *S*] *such that*  $T \subseteq S$  *is Prüfer.*<br>It follows that most of the properties described in [4] for integrally closed INC

It follows that most of the properties described in [\[4\]](#page-28-11) for integrally closed INCpairs of domains are valid for arbitrary ring extensions. Moreover, a result of Dobbs is easily gotten as a consequence of Corollary 2.4: an INC-pair  $R \subseteq S$  is an integral<br>extension if and only if  $\overline{R} \subseteq S$  is spectrally surjective 111. Theorem 2.21. This extension if and only if  $R \subseteq S$  is spectrally surjective [\[11,](#page-28-13) Theorem 2.2]. This follows from Corollary 2.4 and Scholium A. Property (1) follows from Corollary [2.4](#page-6-0) and Scholium A, Property (L).

*Example 2.5* Quasi-Prüfer domains *R* with quotient fields *K* can be characterized by  $R \subseteq K$  is quasi-Prüfer. The reader may consult [\[7,](#page-28-14) Theorem 1.1] or [\[18\]](#page-28-2).<br>We give here another example of quasi-Prüfer extension. An extension is

We give here another example of quasi-Prüfer extension. An extension  $R \subset S$  is called a *going-down pair* if each of its subextensions has the going-down property. For such a pair,  $R \subseteq T$  has incomparability for each  $T \in [R, S]$ , at each non-maximal<br>prime ideal of  $R$  [2] Lemma 5.81(ii) Now let M be a maximal ideal of  $R$ , whose prime ideal of  $R$  [\[2,](#page-28-15) Lemma 5.8](ii). Now let  $M$  be a maximal ideal of  $R$ , whose fiber is not void in *T*. Then  $R \subseteq T$  is a going-down pair, and so is  $R/M \subseteq T/MT$ <br>because  $MT \cap R = M$  By  $[2 \text{ Corollary 5-6}]$  the dimension of  $T/MT$  is  $\lt 1$ because  $MT \cap R = M$ . By [\[2,](#page-28-15) Corollary 5.6], the dimension of  $T/MT$  is  $\leq 1$ . Therefore, if  $R \subset S$  is a going-down pair, then  $R \subset S$  is quasi-Prüfer if and only if  $\dim(T/MT) \neq 1$  for each  $T \in [R, S]$  and  $M \in \text{Max}(R)$ .<br>Also open-ring pairs  $R \subseteq S$  are quasi-Prijfer by [8]

Also *open-ring pairs*  $R \subset S$  are quasi-Prüfer by [\[8,](#page-28-16) Proposition 2.13].

An *i*-*pair* is an extension  $R \subseteq S$  such that  $Spec(T) \to Spec(R)$  is injective for  $T \subseteq [R \times S]$  or equivalently if and only if  $R \subseteq S$  is quasi-Prijfer and  $R \subseteq \overline{R}$  is each  $T \in [R, S]$ , or equivalently if and only if  $R \subseteq S$  is quasi-Prüfer and  $R \subseteq R$  is spectrally injective [38. Proposition 5.8]. These extensions appear frequently in the spectrally injective [\[38,](#page-29-0) Proposition 5.8]. These extensions appear frequently in the integral domains context. Another examples are given by some extensions  $R \subseteq S$ , such that  $\text{Spec}(S) = \text{Spec}(R)$  as sets, as we will see later such that  $Spec(S) = Spec(R)$  as sets, as we will see later.

We proved that  $\Delta$ -extensions  $R \subseteq S$  (such that  $U, V \in [R, S] \Rightarrow U + V \in [R, S]$ )<br>
quasi-Prifer [38] Proposition 5.15] are quasi-Prüfer [\[38,](#page-29-0) Proposition 5.15].

## <span id="page-7-0"></span>**3 Properties of quasi-Prüfer Extensions**

We now develop the machinery of quasi-Prüfer extensions.

**Proposition 3.1** An extension  $R \subset S$  is (quasi-)Prüfer if and only if  $R_P \subseteq S_P$  is (quasi-)Prüfer for any  $P \in$  Spec $(R)$  ( $P \in$  MSupp $(S/R)$ ) *(quasi-)Prüfer for any P*  $\in$  Spec $(R)$  *(P*  $\in$  MSupp $(S/R)$ ).

*Proof* The proof is easy if we use the INC-pair property definition of quasi-Prüfer extension (see also  $[4,$  Proposition 2.4]).  $\square$ 

<span id="page-7-2"></span>**Proposition 3.2** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension and*  $\varphi : S \to S'$  *an integral*<br>vive morphism  $T_1$  or  $\varphi(D) \subseteq S'$  is most Prifer and  $S'$  and  $\overline{S}$  and  $\overline{S}$  is an  $\overline{\varphi(D)}$  is *ring morphism. Then*  $\varphi(R) \subseteq S'$  *is quasi-Prüfer and*  $S' = \varphi(S)\varphi(R)$ *, where*  $\varphi(R)$  *is* the integral closure of  $\varphi(\overline{R})$  in  $S'$ *the integral closure of*  $\varphi$  *(R) in S'*.

*Proof* It is enough to apply [\[25,](#page-28-1) Theorem 5.9] to the Prüfer extension  $R \subseteq S$  and to use Definition 2.1 use Definition [2.1.](#page-5-1)  $\square$ <br>This result annlies with  $S' := S \otimes_R R'$  where  $R \to R'$  is an integral morphism

This result applies with  $S' := S \otimes_R R'$ , where  $R \to R'$  is an integral morphism.<br>erefore integrality ascends the quasi-Prijfer property Therefore integrality ascends the quasi-Prüfer property.

Recall that a composite of Prüfer extensions is Prüfer [\[25,](#page-28-1) Theorem 5.6, p. 51]. We next give a result that will be used frequently. The following Corollary [3.3](#page-7-1) contains [\[6,](#page-28-12) Theorem 3].

<span id="page-7-1"></span>**Corollary 3.3** *Let*  $R \subseteq T \subseteq S$  *be a tower of extensions. Then*  $R \subseteq S$  *is quasi-Prüfer if and only if*  $R \subseteq T$  *and*  $T \subseteq S$  *are quasi-Prüfer Hence*  $R \subseteq T$  *is quasi-Prüfer if* if and only if  $R \subseteq T$  and  $T \subseteq S$  are quasi-Prüfer. Hence,  $R \subseteq T$  is quasi-Prüfer if and only if  $R \subset \overline{R}T$  is quasi-Prüfer and only if  $R \subseteq RT$  is quasi-Prüfer.

*Proof* Consider a tower (*T*) of extensions  $R \subseteq R \subseteq S := R' \subseteq R' \subseteq S'$  (a composite of two quasi-Prifer extensions). By using Proposition 3.2 we see that  $\overline{R} \subset S - R' \subset$ of two quasi-Prüfer extensions). By using Proposition [3.2](#page-7-2) we see that  $R \subseteq S = R' \subseteq \overline{R'}$  is quasi-Prüfer. Then (*T*) is obtained by writing on the left an integral extension  $\overline{R}$ <sup>0</sup> is quasi-Prüfer. Then (*T*) is obtained by writing on the left an integral extension and on the right a Prüfer extension. Therefore,  $(\mathcal{T})$  is quasi-Prüfer. We prove the converse.

If  $R \subseteq T \subseteq S$  is a tower of extensions, then  $R \subseteq T$  and  $T \subseteq S$  are INC-pairs<br>enever  $R \subseteq S$  is an INC-pair. The converse is then a consequence of Theorem 2.3. whenever  $R \subseteq S$  is an INC-pair. The converse is then a consequence of Theorem [2.3.](#page-6-1)<br>The last statement is 16. Corollary 41

The last statement is [\[6,](#page-28-12) Corollary 4].  $\square$ <br>Using the above corollary, we can exhibit new examples of quasi-Prüfer extensions. We recall that a ring *R* is called *Laskerian* if each of its ideals is a finite intersection of primary ideals and a ring extension  $R \subset S$  a *Laskerian pair* if each  $T \in [R, S]$  is a Laskerian ring. Then [\[41,](#page-29-5) Proposition 2.1] shows that if *R* is a integral domain with quotient field  $F \neq R$  and  $F \subseteq K$  is a field extension, then integral domain with quotient field  $F \neq R$  and  $F \subset K$  is a field extension, then  $R \subset K$  is a Laskerian pair if and only if *K* is algebraic over *R* and  $\overline{R}$  (in *K*) is a Laskerian Prüfer domain. It follows easily that  $R \subset K$  is quasi-Prüfer under these conditions.

<span id="page-8-2"></span>Next result generalizes [\[24,](#page-28-17) Proposition 1].

**Corollary 3.4** An FMC extension  $R \subset S$  is quasi-Prüfer.

*Proof* Because  $R \subset S$  is a composite of finitely many minimal extensions, by Corollary [3.3,](#page-7-1) it is enough to observe that a minimal extension is either Prüfer or  $\Box$ integral.  $\Box$ 

<span id="page-8-1"></span>**Corollary 3.5** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension and a tower*  $R \subseteq T \subseteq S$ , where  $R \subseteq T$  is integrally closed. Then  $R \subseteq T$  is Prüfer where  $R \subseteq T$  is integrally closed. Then  $R \subseteq T$  is Prüfer.

*Proof* Observe that  $R \subseteq T$  is quasi-Prüfer and then that  $R = \overline{R}^T$ <br>Next result deals with the Dobbs-Shapiro *pseudo-Prüfer* ex

 $\log f$  Observe that  $R \subseteq T$  is quasi-Prüfer and then that  $R = R$ .  $\square$ <br>Next result deals with the Dobbs-Shapiro *pseudo-Prüfer* extensions of integral domains  $[12]$ , that they called pseudo-normal pairs. Suppose that *R* is local, we call here pseudo-Prüfer an extension  $R \subseteq S$  such that there exists  $T \in [R, S]$  with  $S_{\text{pec}}(R) = S_{\text{pec}}(T)$  and  $T \subseteq S$  is Prüfer [12] Corollary 2.51. If R is arbitrary  $Spec(R) = Spec(T)$  and  $T \subseteq S$  is Prüfer [\[12,](#page-28-3) Corollary 2.5]. If *R* is arbitrary, the extension  $R \subseteq S$  is called pseudo-Prüfer if  $R_{11} \subseteq S_{12}$  is pseudo-Prüfer for each the extension  $R \subseteq S$  is called pseudo-Prüfer if  $R_M \subseteq S_M$  is pseudo-Prüfer for each  $M \in \text{Max}(R)$ . In view of the Corollary 3.3, it is enough if one wishes to characterize  $M \in \text{Max}(R)$ . In view of the Corollary [3.3,](#page-7-1) it is enough, if one wishes to characterize quasi-Prüfer extensions, to characterize quasi-Prüfer extensions of the type  $R \subseteq T$ <br>with  $\text{Spec}(R) - \text{Spec}(T)$ with  $Spec(R) = Spec(T)$ .

**Corollary 3.6** *Let*  $R \subseteq T$  *be an extension with*  $Spec(R) = Spec(T)$  *and*  $(R, M)$  *local Then*  $R \subseteq T$  *is quasi-Priifer if and only if*  $Spec(R) = Spec(I)$  *for all U local. Then R*  $\subseteq$  *T* is quasi-Prüfer if and only if Spec(*R*)  $=$  Spec(*U*) for all *U*  $\in$  *R T*) and if and only if *R*/*M*  $\subset$  *T*/*M* is an algebraic field extension. In such a case  $[R, T]$  and, if and only if  $R/M \subseteq T/M$  is an algebraic field extension. In such a case,<br> $R \subset T$  is integral, hence Priifer-closed  $R \subseteq T$  is integral, hence Prüfer-closed.

*Proof* It follows from [\[1\]](#page-28-18) that  $M \in Max(T)$ . Part of the proof is gotten by observing that  $R/M \subseteq T/M$  is an algebraic field extension  $\Rightarrow$  Spec $(R) = \text{Spec}(U)$  for all  $U \in [R, T] \rightarrow R \subset T$  is quasi-Prijfer  $\rightarrow (R \subset T$  is integral and  $R/M \subset T/M$  is  $U \in [R, T] \Rightarrow R \subseteq T$  is quasi-Prüfer  $\Rightarrow (R \subseteq T$  is integral and)  $R/M \subseteq T/M$  is<br>an algebraic field extension. Now  $R \subseteq \widetilde{R}$  is a spectrally surjective flat enimorphism an algebraic field extension. Now  $R \subseteq \overline{R}$  is a spectrally surjective flat epimorphism and then, by Scholium A,  $R = \overline{R}$ . flax(*I*). Part of the proof is gotten by observing<br>eld extension  $\Rightarrow$  Spec(*R*) = Spec(*U*) for all<br> $er \Rightarrow (R \subseteq T$  is integral and)  $R/M \subseteq T/M$  is<br> $\subseteq \widetilde{R}$  is a spectrally surjective flat epimorphism that  $R/M \subseteq T/M$  is an algebraic field extension  $\Rightarrow$  Spec( $R$ ) = Spec( $U$ ) for all  $U \in [R, T] \Rightarrow R \subseteq T$  is quasi-Prüfer  $\Rightarrow (R \subseteq T$  is integral and)  $R/M \subseteq T/M$  is an algebraic field extension. Now  $R \subseteq \widetilde{R}$  is a spectrally surje

Let  $R \subseteq S$  be an extension and *I* an ideal shared by *R* and *S*. It is easy to show  $R \subseteq S$  is quasi-Prifer if and only if  $R/I \subseteq S/I$  is quasi-Prifer by using [25] that  $R \subseteq S$  is quasi-Prüfer if and only if  $R/I \subseteq S/I$  is quasi-Prüfer by using [\[25,](#page-28-1) Proposition 5.81 in the Prüfer case. We are able to give a more general statement Proposition 5.8] in the Prüfer case. We are able to give a more general statement.

<span id="page-8-0"></span>**Lemma 3.7** *Let*  $R \subseteq S$  *be a (quasi-)Prüfer extension and J an ideal of S with*  $I - I \cap R$  *Then*  $R/I \subseteq S/I$  *is a (quasi-)Prüfer extension If*  $R \subseteq S$  *is Prüfer and*  $I = J \cap R$ . Then  $R/I \subseteq S/J$  is a (quasi-)Prüfer extension. If  $R \subseteq S$  is Prüfer and N is a maximal ideal of S, then  $R/(N \cap R)$  is a valuation domain with augtient field *N* is a maximal ideal of *S*, then  $R/(N \cap R)$  is a valuation domain with quotient field *S*=*N.*

*Proof* It follows from [\[25,](#page-28-1) Proposition 5.8] that if  $R \subseteq S$  is Prüfer, then  $R/I \cong (R + I)/I \subseteq S/I$  is Prüfer. Then the quasi-Prüfer case is an easy consequence  $(R + J)/J \subseteq S/J$  is Prüfer. Then the quasi-Prüfer case is an easy consequence.  $\Box$ <br>With this lemma we generalize and complete [23] Proposition 1.11 With this lemma we generalize and complete [\[23,](#page-28-19) Proposition 1.1].

<span id="page-8-3"></span>**Proposition 3.8** *Let*  $R \subseteq S$  *be an extension of rings. The following statements are equivalent: equivalent:*

(1)  $R \subseteq S$  is quasi-Prüfer;<br>(2)  $R/(Q \cap R) \subseteq S/Q$  is a

*(2)*  $R/(Q \cap R) \subseteq S/Q$  is quasi-Prüfer for each  $Q \in \text{Spec}(S)$ ;

- (3)  $(X s)S[X] \cap R[X] \nsubseteq M[X]$  for each  $s \in S$  and  $M \in \text{Max}(R)$ ;<br>(4) For each  $T \in \mathbb{R}$ . S the fiber morphisms of  $R \subseteq T$  are integral
- *(4) For each*  $T \in [R, S]$ , the fiber morphisms of  $R \subseteq T$  are integral.

*Proof* (1)  $\Rightarrow$  (2) is entailed by Lemma [3.7.](#page-8-0) Assume that (2) holds and let  $M \in$  $Max(R)$  that contains a minimal prime ideal *P* lain over by a minimal prime ideal *Q* of *S*. Then (2)  $\Rightarrow$  (3) follows from [\[23,](#page-28-19) Proposition 1.1(1)], applied to  $R/(Q \cap R) \subseteq$ <br> $S/O$ . If (3) holds, argue as in the paragraph before [23. Proposition 1.11 to get that  $S/O$ . If (3) holds, argue as in the paragraph before [\[23,](#page-28-19) Proposition 1.1] to get that  $R \subseteq S$  is a *P*-extension, whence an INC-pair, cf. [\[11\]](#page-28-13). Then  $R \subseteq S$  is quasi-Prüfer by<br>Theorem 2.3, giving (3)  $\rightarrow$  (1). Because integral extensions have incomparability Theorem [2.3,](#page-6-1) giving (3)  $\Rightarrow$  (1). Because integral extensions have incomparability, we see that (4)  $\Rightarrow$  (1). Corollary [3.3](#page-7-1) shows that the reverse implication holds, if any quasi-Prüfer extension  $R \subseteq S$  has integral fiber morphisms. For  $P \in \text{Spec}(R)$ , the extension  $R_R/PR_R \subseteq S_R/PS_R$  is quasi-Prüfer by Lemma 3.7. The ring  $R_R/PR_R$  is extension  $R_P/PR_P \subseteq S_P/PS_P$  is quasi-Prüfer by Lemma [3.7.](#page-8-0) The ring  $R_P/PR_P$  is zero-dimensional and  $\overline{R}_P/PR_P \to S_P/PS_P$  being a flat enimorphism is therefore zero-dimensional and  $\bar{R}_P/P\bar{R}_P \rightarrow S_P/PS_P$ , being a flat epimorphism, is therefore surjective by Scholium A (S). It follows that the fiber morphism at  $P$  is integral.

*Remark 3.9* The logical equivalence (1)  $\Leftrightarrow$  (2) is still valid if we replace quasi-Prüfer with integral in the above proposition. It is enough to show that an extension  $R \subseteq S$  is integral when  $R/P \subseteq S/Q$  is integral for each  $Q \in \text{Spec}(S)$  and  $P \subset Q \cap R$ . We can suppose that  $S - R[s] \simeq R[X]/I$  where *X* is an indeterminate  $P := Q \cap R$ . We can suppose that  $S = R[s] \cong R[X]/I$ , where *X* is an indeterminate, *I* an ideal of  $R[X]$  and *O* varies in Min(*S*) because for an extension  $A \subseteq R$  any *I* an ideal of *R*[*X*], and *Q* varies in Min(*S*), because for an extension  $A \subseteq B$ , any element of Min(*A*) is lain over by some element of Min(*R*). If  $\Sigma$  is the set of element of Min(*A*) is lain over by some element of Min(*B*). If  $\Sigma$  is the set of unitary polynomials of  $R[X]$ , the assumptions show that any element of  $Spec(R[X])$ , containing *I*, meets  $\Sigma$ . As  $\Sigma$  is a multiplicatively closed subset,  $I \cap \Sigma \neq \emptyset$ , whence *s* is integral over *R*.

But a similar result does not hold if we replace quasi-Prüfer with Prüfer, except if we suppose that  $R \subseteq S$  is integrally closed. To see this, apply the above proposition to get a quasi-Prijfer extension  $R \subseteq S$  if each  $R/P \subseteq S/O$  is Prijfer. Actually, this to get a quasi-Prüfer extension  $R \subseteq S$  if each  $R/P \subseteq S/Q$  is Prüfer. Actually, this situation already occurs for Prüfer rings and their factor domains as I ucas's paper situation already occurs for Prüfer rings and their factor domains, as Lucas's paper [\[28\]](#page-29-6) shows. More precisely, [\[28,](#page-29-6) Proposition 2.7] and the third paragraph of [\[28,](#page-29-6) p. 336] shows that if *R* is a ring with  $Tot(R)$  absolutely flat, then *R* is a quasi-Prüfer ring if *R*/*P* is a Prüfer domain for each  $P \in \text{Spec}(R)$ . Now example [\[28,](#page-29-6) Example 2.4] shows that *R* is not necessarily Prüfer.

We observe that if  $R \subseteq S$  is quasi-Prüfer, then  $R/M$  is a quasi-Prüfer domain for  $h N \in \text{Max}(S)$  and  $M := N \cap R$  (in case  $R \subset S$  is integral  $R/M$  is a field). To each  $N \in \text{Max}(S)$  and  $M := N \cap R$  (in case  $R \subseteq S$  is integral,  $R/M$  is a field). To<br>prove this observe that  $R/M \subseteq S/N$  can be factored  $R/M \subseteq \kappa(M) \subseteq S/N$ . As we prove this, observe that  $R/M \subseteq S/N$  can be factored  $R/M \subseteq \kappa(M) \subseteq S/N$ . As we will see  $R/M \subset \kappa(M)$  is quasi-Prijfer because  $R/M \subset S/N$  is quasi-Prijfer will see,  $R/M \subseteq \kappa(M)$  is quasi-Prüfer because  $R/M \subseteq S/N$  is quasi-Prüfer.<br>The class of Prüfer extensions is not stable by (flat) base change. For e

The class of Prüfer extensions is not stable by (flat) base change. For example, let *V* be a valuation domain with quotient field *K*. Then  $V[X] \subseteq K[X]$  is not Prüfer  $[25 \text{ Fyample } 5, 12, \text{p, } 531$ [\[25,](#page-28-1) Example 5.12, p. 53].

<span id="page-9-0"></span>**Proposition 3.10** *Let*  $R \subseteq S$  *be a (quasi)-Prüfer extension and*  $R \rightarrow T$  *a flat enimorphism then*  $T \subseteq S \otimes_S T$  *is (quasi)-Prüfer If in addition S and*  $T$  *are both epimorphism, then*  $T \subseteq S \otimes_R T$  *is (quasi)-Prüfer. If in addition S and T are both*<br>subrings of some ring and  $R \subseteq T$  is an extension, then  $T \subseteq TS$  is (quasi)-Prüfer subrings of some ring and  $R \subseteq T$  is an extension, then  $T \subseteq TS$  is (quasi)-Prüfer.

*Proof* For the first part, it is enough to consider the Prüfer case. It is well known that the following diagram is a pushout if  $Q \in Spec(T)$  is lying over *P* in *R*:



As  $R_P \rightarrow T_Q$  is an isomorphism since  $R \rightarrow T$  is a flat epimorphism by Scholium A (1), it follows that  $R_P \subseteq S_P$  identifies to  $T_Q \to (T \otimes_R S)_Q$ . The first assertion follows because Prifer extensions localize and globalize follows because Prüfer extensions localize and globalize.

The final assertion is then a special case because, under its hypotheses, *TS*  $\cong$  *T*  $\otimes_R S$  canonically. *T*  $\otimes_R S$  canonically.  $\Box$ <br>The reader may find in [25] Corollary 5.11 n. 531 that if  $R \subset A \subset S$  and  $R \subset$ 

The reader may find in [\[25,](#page-28-1) Corollary 5.11, p. 53] that if  $R \subseteq A \subseteq S$  and  $R \subseteq$ <br>  $\subseteq S$  are extensions and  $R \subseteq A$  and  $R \subseteq R$  are both Prijfer, then  $R \subseteq AR$  is Prijfer *B*  $\subseteq$  *S* are extensions and *R*  $\subseteq$  *A* and *R*  $\subseteq$  *B* are both Prüfer, then *R*  $\subseteq$  *AB* is Prüfer.

<span id="page-10-0"></span>**Proposition 3.11** *Let*  $R \subseteq A$  and  $R \subseteq B$  be two extensions, where A and B are subrings of a ring S. If they are both quasi-Priifer then  $R \subseteq AB$  is quasi-Priifer  $subrings$  of a ring S. If they are both quasi-Prüfer, then  $R \subseteq AB$  is quasi-Prüfer.

*Proof* Let *U* and *V* be the integral closures of *R* in *A* and *B*. Then  $R \subseteq A \subseteq AV$  is quasi-Prifer because  $A \subseteq AV$  is integral and Corollary 3.3 applies. Using again quasi-Prüfer because  $A \subseteq AV$  is integral and Corollary [3.3](#page-7-1) applies. Using again<br>Corollary 3.3 with  $R \subseteq V \subseteq AV$  we find that  $V \subseteq AV$  is quasi-Prüfer Now Corollary [3.3](#page-7-1) with  $R \subseteq V \subseteq AV$ , we find that  $V \subseteq AV$  is quasi-Prüfer. Now<br>Proposition 3.10 entails that  $R \subseteq AR$  is quasi-Prüfer because  $V \subseteq R$  is a flat Proposition [3.10](#page-9-0) entails that  $B \subseteq AB$  is quasi-Prüfer because  $V \subseteq B$  is a flat primorphism Finally  $R \subseteq AB$  is quasi-Prüfer since a composite of quasi-Prüfer epimorphism. Finally  $R \subseteq AB$  is quasi-Prüfer, since a composite of quasi-Prüfer<br>extensions

extensions. □<br>It is known that an arbitrary direct product of extensions is Prüfer if and only if each of its components is Prüfer [\[25,](#page-28-1) Proposition 5.20, p. 56]. The following result is an easy consequence.

**Proposition 3.12** *Let*  $\{R_i \subseteq S_i | i = 1, ..., n\}$  *be a finite family of quasi-Prüfer*<br>extensions, then  $R_1 \times \cdots \times R_n \subseteq S_1 \times \cdots \times S_n$  is quasi-Prüfer. In particular, by *extensions, then*  $R_1 \times \cdots \times R_n \subseteq S_1 \times \cdots \times S_n$  *is quasi-Prüfer. In particular, by*<br>Corollary 3.3 if  $\{R \subset S_i | i = 1, \ldots, n\}$  is a finite family of quasi-Prüfer extensions *Corollary [3.3,](#page-7-1) if*  $\{R \subseteq S_i | i = 1, ..., n\}$  *is a finite family of quasi-Prüfer extensions,*<br>then  $R \subseteq S_i \times \ldots \times S_j$  is quasi-Prüfer *then*  $R \subseteq S_1 \times \cdots \times S_n$  *is quasi-Prüfer.*<br>In the same way we have the follow.

In the same way we have the following result deduced from [\[25,](#page-28-1) Remark 5.14, p. 54].

<span id="page-10-1"></span>**Proposition 3.13** *Let*  $R \subseteq S$  *be an extension of rings and an upward directed family*<br>*IR*  $\cup \alpha \in R$  of elements of  $[R, S]$  such that  $R \subseteq R$  is quasi-Prifer for each  $\alpha \in I$  ${R_\alpha|\alpha \in I}$  *of elements of*  $[R, S]$  *such that*  $R \subseteq R_\alpha$  *is quasi-Prüfer for each*  $\alpha \in I$ .<br>Then  $R \subseteq I$   $|\alpha \in I|$  *is quasi-Prüfer Then*  $R \subseteq \bigcup [R_\alpha | \alpha \in I]$  *is quasi-Prüfer.* 

*Proof* It is enough to use  $[25,$  Proposition 5.13, p. 54] where  $A_{\alpha}$  is the integral closure of *R* in  $R_\alpha$ .  $\square$ <br>Here are some descent results used later on.

<span id="page-11-0"></span>**Proposition 3.14** *Let*  $R \subseteq S$  *be a ring extension and*  $R \rightarrow R'$  *a spectrally surjective ring morphism (for example, either faithfully flat or injective and integral). Then ring morphism (for example, either faithfully flat or injective and integral). Then*  $R \subseteq S$  is quasi-Prüfer if  $R' \to R' \otimes_R S$  is injective (for example, if  $R \to R'$  is faithfully flat) and quasi-Prüfer *faithfully flat) and quasi-Prüfer.*

*Proof* Let  $T \in [R, S]$  and  $P \in \text{Spec}(R)$  and set  $T' := T \otimes_R R'$ . There is some  $P' \in \text{Spec}(R')$  lying over P because  $R \to R'$  is spectrally surjective. By [22] Corollaire Spec(*R'*) lying over *P*, because  $R \to R'$  is spectrally surjective. By [\[22,](#page-28-20) Corollaire 3.4.9] there is a faithfully flat morphism  $F_R \nI(P) \to F_R \nI(P') \simeq F_R \nI(P) \otimes_{\mathbb{R}(R)} \mathbb{R}$ 3.4.9], there is a faithfully flat morphism  $F_{R,T}(P) \to F_{R',T'}(P') \cong F_{R,T}(P) \otimes_{\mathbf{k}(P)} \mathbb{R}(P)$ .<br> $\kappa(P')$  inducing a surjective man  $\text{Fib}_{R,T}(P') \to \text{Fib}_{R,T}(P)$  since it satisfies lying  $\kappa(P')$ , inducing a surjective map  $\text{Fib}_{R',T'}(P') \to \text{Fib}_{R,T}(P)$  since it satisfies lying<br>over By Theorem 2.3, the result follows from the faithful flatness of  $\text{F}_{R,T}(P) \to$ over. By Theorem [2.3,](#page-6-1) the result follows from the faithful flatness of  $F_{R,T}(P) \rightarrow$ <br> $F_{P', T \otimes_R P'}(P')$ .  $F_{R',T\otimes_R R'}(P').$  $\Box$ 

**Corollary 3.15** *Let*  $R \subseteq S$  *be an extension of rings,*  $R \rightarrow R'$  *a faithfully flat ring* morphism and set  $S' := R' \otimes_R S$  *If*  $R' \subseteq S'$  is (quasi-) Priifer (respectively FCP) *morphism and set*  $S' := R' \otimes_R S$ . If  $R' \subseteq S'$  *is (quasi-) Prüfer (respectively, FCP),*<br>then so is  $R \subset S$ *then so is*  $R \subseteq S$ .

*Proof* The Prüfer case is clear, because faithfully flat morphisms descend flat epimorphisms (Scholium A  $(9)$ ). For the quasi-Prüfer case, we use Proposition [3.14.](#page-11-0) The FCP case is proved in [\[15,](#page-28-21) Theorem 2.2].<br>The integral closure of a ring morphism  $f : R \to T$  is the integral closure of the

The integral closure of a ring morphism  $f : R \to T$  is the integral closure of the ension  $f(R) \subset T$ . By definition a ring morphism  $R \to T$  preserves the integral extension  $f(R) \subseteq T$ . By definition, a ring morphism  $R \to T$  preserves the integral<br>classes of size morphisms  $R \to S$  is  $\overline{T}^T \otimes_R S$  as  $T \otimes_R \overline{R}$  for some size morphism closure of ring morphisms  $R \to S$  if  $\overline{T}^{T \otimes_R S} \cong T \otimes_R \overline{R}$  for every ring morphism  $R \to S$ . An absolutely flat morphism  $R \to T (R \to T$  and  $T \otimes_R T \to T$  are both flat)  $R \rightarrow S$ . An absolutely flat morphism  $R \rightarrow T (R \rightarrow T$  and  $T \otimes_R T \rightarrow T$  are both flat) preserves integral closure [\[32,](#page-29-4) Theorem 5.1]. Flat epimorphisms, Henselizations, and étale morphisms are absolutely flat. Another examples are morphisms  $R \to T$ that are essentially of finite type and (absolutely) reduced [\[34,](#page-29-7) Proposition 5.19](2). Such morphisms are flat if *R* is reduced [\[27,](#page-28-22) Proposition 3.2].

We will prove an ascent result for absolutely flat ring morphisms. This will be proved by using base changes. For this we need to introduce some concepts. A ring *A* is called an AIC ring if each monic polynomial of *A*[*X*] has a zero in *A*. The first author recalled in [\[35,](#page-29-8) p. 4662] that any ring *A* has a faithfully flat integral extension  $A \rightarrow A^*$ , where  $A^*$  is an AIC ring. Moreover, if *A* is an AIC ring, each localization *AP* at a prime ideal *P* of *A* is a strict Henselian ring [\[35,](#page-29-8) Lemma II.2].

**Theorem 3.16** *Let*  $R \subseteq S$  *be a (quasi-) Prüfer extension and*  $R \rightarrow T$  *an absolutely*<br>flat ring mornhism Then  $T \rightarrow T \otimes_S S$  is a (quasi-) Prüfer extension *flat ring morphism. Then*  $T \to T \otimes_R S$  *is a (quasi-) Prüfer extension.* 

*Proof* We can suppose that *R* is an AIC ring. To see this, it is enough to use the base change  $R \to R^*$ . We set  $T^* := T \otimes_R R^*$ ,  $S^* := S \otimes_R R^*$ . We first observe that  $R^* \subseteq S^*$  is quasi-Prüfer for the following reason: the composite extension  $R \subset S \subset S^*$  is quasi-Prüfer by Corollary 3.3 because the last extension is integral.  $R \subseteq S \subseteq S^*$  is quasi-Prüfer by Corollary [3.3](#page-7-1) because the last extension is integral.<br>Moreover  $R^* \to T^*$  is absolutely flat. In case  $T^* \subseteq T^* \otimes_{\mathbb{Z}^*} S^*$  is quasi-Prüfer so Moreover,  $R^* \to T^*$  is absolutely flat. In case  $T^* \subseteq T^* \otimes_{R^*} S^*$  is quasi-Prüfer, so is  $T \subset T \otimes_R S$  because  $T \to T^* \subset T \otimes_R R^*$  is faithfully flat and  $T^* \subset T^* \otimes_{R^*} S^*$ is  $T \subseteq T \otimes_R S$ , because  $T \to T^* = T \otimes_R R^*$  is faithfully flat and  $T^* \subseteq T^* \otimes_{R^*} S^*$ <br>is deduced from  $T \subset T \otimes_R S$  by the faithfully flat base change  $T \to T \otimes_R R^*$ . It is is deduced from  $T \subseteq T \otimes_R S$  by the faithfully flat base change  $T \to T \otimes_R R^*$ . It is then enough to apply Proposition 3.14 then enough to apply Proposition [3.14.](#page-11-0)

We thus assume from now on that *R* is an AIC ring.

Let  $N \in \text{Spec}(T)$  be lying over *M* in *R*. Then  $R_M \to T_N$  is absolutely flat [\[31,](#page-29-9) Proposition f] and  $R_M \subseteq S_M$  is quasi-Prüfer. Now observe that  $(T \otimes_R S)_N \cong T_N \otimes_R S_M$ . Therefore, we can suppose that R and T are local and  $R \to T$  is  $T_N \otimes_{R_M} S_M$ . Therefore, we can suppose that *R* and *T* are local and  $R \rightarrow T$  is local and injective. We deduce from [\[32,](#page-29-4) Theorem 5.2] that  $R_M \rightarrow T_N$  is an isomorphism because  $R_M$  is a strict Henselian ring. Therefore the proof is complete in the quasi-Prüfer case. For the Prüfer case, we need only to observe that absolutely flat morphisms preserve integral closure and a quasi-Prüfer extension is Prüfer if it is integrally closed.

**Lemma 3.17** *Let*  $R \subseteq S$  *be an extension of rings and*  $R \rightarrow T$  *a base change which preserves integral closure* If  $T \subseteq T \otimes_R S$  *has FCP and*  $R \subseteq S$  *is Prifer then preserves integral closure. If*  $T \subseteq T \otimes_R S$  *has FCP and*  $R \subseteq S$  *is Prüfer, then*  $T \subset T \otimes_R S$  *is Prüfer*  $T \subseteq T \otimes_R S$  is Prüfer.

*Proof* The result holds because an FCP extension is Prüfer if and only if it is integrally closed.<br>We observe that  $T \otimes_{\mathbb{R}} \widetilde{R} \subset \widetilde{T}$  need not to be an isomorphism since this property *oof* The result holds because an FCP extension is Prüfer if and only if it is egrally closed.<br>We observe that  $T \otimes_R \widetilde{R} \subseteq \widetilde{T}$  need not to be an isomorphism, since this property v fail even for a localization  $R \to R$ 

may fail even for a localization  $R \to R_P$ , where *P* is a prime ideal of *R*.

<span id="page-12-1"></span>**Theorem 3.18** *Let*  $R \subseteq S$  *be a ring extension.* 

(1)  $R \subseteq S$  has a greatest quasi-Prüfer subextension  $R \subseteq \overline{R}$  $\overrightarrow{R} = \overrightarrow{\overline{R}}.$ (2)  $R \subseteq \overline{R} \widetilde{R} =: \overrightarrow{R}$  is quasi-Priifer and then  $\overrightarrow{R} \subseteq \overrightarrow{R}$ .  $\leq$  *S* has a greatest quasi-Prüfer subextens<br>  $\leq$   $\overline{R}$   $\widetilde{R}$  =:  $\overline{R}$  is quasi-Prüfer and then  $\overline{R}$   $\leq$ *(3) R*  $\Rightarrow R$  $R\overline{R}R =: \overline{R}$  is<br> $R\overline{R} = R$  and  $\overline{R}$  $\Rightarrow R$  $\begin{aligned} \n\text{as } i-P_i \\
&= \widetilde{R}. \n\end{aligned}$ 

*Proof* To see (1), use Proposition [3.11](#page-10-0) which tells us that the set of all quasi-Prüfer subextensions is upward directed and then use Proposition [3.13](#page-10-1) to prove the existence of  $\overrightarrow{R}$ . Then let  $R \subseteq T \subseteq \overrightarrow{R}$  be a tower with  $R \subseteq T$  integral and  $T \subseteq \overrightarrow{R}$ <br>Prüfer. From  $T \subseteq \overline{R} \subseteq \overrightarrow{R} \subseteq \overrightarrow{R}$ , we deduce that  $T = \overline{R}$  and then  $\overrightarrow{R} = \overrightarrow{R}$ .<br>(2) Now  $R \subseteq \overrightarrow{R} \overrightarrow{R}$  can be Prüfer subextensions is upv<br>existence of  $\overrightarrow{R}$ . Then let R<br>Prüfer. From  $T \subseteq \overline{R} \subseteq \widetilde{\overline{R}} \subseteq$ <br>(2) Now  $R \subseteq \overline{R} \overline{R}$  can b  $\subseteq \overrightarrow{R}$ , we deduce that  $T = \overline{R}$  and then  $\overrightarrow{R}$ <br>be factored  $R \subseteq \widetilde{R} \subseteq \overline{R} \widetilde{R}$  and is a tower  $\frac{3.13 \text{ to}}{R}$ <br>  $\Rightarrow R = \widetilde{R}.$ <br>
From t and Then let  $R \subseteq T \subseteq \overline{R}$  be a tow<br>  $\subseteq \overline{R} \subseteq \overline{\widetilde{R}} \subseteq \overline{R}$ , we deduce that<br>  $\subseteq \overline{R}\widetilde{R}$  can be factored  $R \subseteq \widetilde{R} \subseteq$ <br>
rause  $\widetilde{R} \to \widetilde{R}\overline{R}$  is integral Prüfer. From  $T \subseteq \overline{R} \subseteq \overline{\widetilde{R}} \subseteq \overline{R}$ , we ded<br>
(2) Now  $R \subseteq \overline{R}\widetilde{R}$  can be factored R<br>
extensions, because  $\widetilde{R} \to \overline{R}\overline{R}$  is integral.

(2) Now  $R \subseteq \overline{RR}$  can be factored  $R \subseteq \overline{R} \subseteq \overline{RR}$  and is a tower of quasi-Prüfer extensions, because  $\overline{R} \to \overline{RR}$  is integral.

(3) Clearly, the integral closure and the Prüfer closure of *R* in  $\overrightarrow{R}$  are the respective intersections of *R* and *R* with  $\overrightarrow{R}$ , and *R*;  $\overrightarrow{R}$   $\leq$ <br>respective intersections of  $\overrightarrow{R}$  and  $\overrightarrow{R}$  with  $\overrightarrow{R}$ , and  $\overrightarrow{R}$ ,  $\overrightarrow{R}$   $\leq$ <br>*This last result means that as far as propertie*  $\overrightarrow{R}$ , and  $\overline{R}$ ,  $\widetilde{R} \subseteq \overrightarrow{R}$ respective intersections of R and R with R, and R,  $R \subseteq R$ .  $\square$ <br>This last result means that, as far as properties of integral closures and Prüfer closures of subsets of  $\overrightarrow{R}$  are concerned, we can suppose that  $R \subseteq S$  is quasi-Prüfer.

# <span id="page-12-0"></span>**4 Almost-Prüfer Extensions**

We next give a definition "dual" of the definition of a quasi-Prüfer extension.

## *4.1 Arbitrary Extensions*

<span id="page-13-2"></span>**Definition 4.1** A ring extension  $R \subseteq S$  is called an *almost-Prüfer* extension if it can<br>be factored  $R \subset T \subset S$  where  $R \subset T$  is Prüfer and  $T \subset S$  is integral be factored  $R \subseteq T \subseteq S$ , where  $R \subseteq T$  is Prüfer and  $T \subseteq S$  is integral. **Definition 4.1** A ring extension  $R \subseteq S$  is called an *almost-Prüfer* extension if it can be factored  $R \subseteq T \subseteq S$ , where  $R \subseteq T$  is Prüfer and  $T \subseteq S$  is integral.<br>**Proposition 4.2** *An extension*  $R \subseteq S$  *is almost-Prüfer if* 

<span id="page-13-0"></span>*integral. integral. <i>integral. <i>i***ntegral.** *<i>i***ntegral.** *i***ntegral.** *It**antextension* $R \subseteq S$ *is almost-Prüfer if and only if* $\widetilde{R} \subseteq S$ *integral. It follows that the subring**T**of the above definit* integral. It follows that the subring T of the above definition is  $\widetilde{R} = \widehat{R}$  when  $R \subseteq S$ *is almost-Prüfer. integral. It follows that the subring T* of the above definition is  $\widetilde{R} = \widehat{R}$  when  $R \subseteq S$ <br>*is almost-Priifer.*<br>*Proof* If  $R \subseteq S$  is almost-Priifer, there is a factorization  $R \subseteq T \subseteq \widetilde{R} \subseteq \widehat{R} \subseteq S$ ,<br>where  $T \subseteq$ 

where  $T \subseteq \widehat{R}$  is both integral and a flat epimorphism by Scholium A (4). Therefore,<br>  $T = \widetilde{R} = \widehat{R}$  by Scholium A (5) (L). Priifer.<br>  $R \subseteq S$  is almost-Prüfer, there is a factorization  $R \subseteq T \subseteq \widetilde{R} \subseteq \widehat{R} \subseteq S$ ,<br>  $\subseteq \widehat{R}$  is both integral and a flat epimorphism by Scholium A (4). Therefore, *Proof* If  $R \subseteq S$  is almost-Prüfer, there is a factorization  $R \subseteq T \subseteq \widetilde{R} \subseteq S$ , where  $T \subseteq \widehat{R}$  is both integral and a flat epimorphism by Scholium A (4). Therefore,  $T = \widetilde{R} = \widehat{R}$  by Scholium A (5) (L).

**Corollary 4.3** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension, and let*  $T \in [R, S]$ <br> $\overline{IR}$ *i* quasi-Prüfer extension, and let  $T \in [R, S]$ . Then,  $T \cap \overline{R} \subseteq T\overline{R}$  is almost-Prüfer and  $T = \widetilde{R \cap T}^T$ . Moreover, if  $T \cap \overline{R} = R$ , then,<br> $T - T\overline{R} \cap \widetilde{R}$ **Corollary 4..**<br>  $T \cap \overline{R} \subseteq T\overline{R}$ <br>  $T = T\overline{R} \cap \overline{\widetilde{R}}$ .

*Proof T*  $\cap$  *R*  $\subseteq$  *T* is quasi-Prüfer by Corollary [3.3.](#page-7-1) Being integrally closed, it is prijfer by Corollary 3.5. Moreover  $T \subset T\overline{R}$  is an integral extension. Then  $T \cap \overline{R} \subset T$ Prüfer by Corollary [3.5.](#page-8-1) Moreover,  $T \subseteq TR$  is an integral extension. Then,  $T \cap R \subseteq T\overline{R}$ *Proof*  $T \cap \overline{R} \subseteq T$  is quasi-Prüfer by Corollary 3.3. Being integrally closed, it is Prüfer by Corollary 3.5. Moreover,  $T \subseteq T\overline{R}$  is an integral extension. Then,  $T \cap \overline{R} \subseteq T\overline{R}$  is almost-Prüfer and  $T = \overline{R} \cap$ 

Prüfer by Corollary 3.5. Moreover,  $T \subseteq T\overline{R}$  is an integral extension. Then,  $T \cap \overline{R} \subseteq T\overline{R}$  is almost-Prüfer and  $T = \overline{R} \cap T$ . If  $T \cap \overline{R} = R$ , then  $T \subseteq T\overline{R} \cap \overline{R}$  is both Prüfer and integral, so that  $T =$ 

We note that integral extensions and Prüfer extensions are almost-Prüfer and hence minimal extensions are almost-Prüfer. There are quasi-Prüfer extensions that are not almost-Prüfer. It is enough to consider [\[37,](#page-29-10) Example 3.5(1)]. Let  $R \subseteq T \subseteq S$ <br>be two minimal extensions, where R is local,  $R \subseteq T$  integral and  $T \subseteq S$  is Prifer be two minimal extensions, where *R* is local,  $R \subseteq T$  integral and  $T \subseteq S$  is Prüfer.<br>Then  $R \subseteq S$  is quasi-Prüfer but not almost-Prüfer, because  $S = \hat{R}$  and  $R = \tilde{R}$ . Then  $R \subseteq S$  is quasi-Prüfer but not almost-Prüfer, because  $S = R$  and  $R = R$ . mmal extensions are almost-Prufer. There are quasi-Prufer extensions that<br> *S* immal extensions, where *R* is local,  $R \subseteq T$  integral and  $T \subseteq S$  is Prüfer.<br>  $\subseteq S$  is quasi-Prüfer but not almost-Prüfer, because  $S = \hat{R}$  an The same example shows that a composite of almost-Prüfer extensions may not be almost-Prüfer.

<span id="page-13-1"></span>But the reverse implication holds. f

**Theorem 4.4** Let  $R \subseteq S$  be an almost-Prüfer extension. Then  $R \subseteq S$  is quasi-<br>Prüfer Moreover  $\widetilde{R} = \widehat{R}$  ( $\widetilde{R}$ )<sub>n</sub>  $\cong \widetilde{R}$  for each  $P \in$  Spec(R). In this case, any flat **Property By** *Prüfer. Moreover,*  $\widetilde{R} = \widehat{R}$ ,  $(\widetilde{R})_P = \widehat{R}$ <br>*Prüfer. Moreover,*  $\widetilde{R} = \widehat{R}$ ,  $(\widetilde{R})_P = \widehat{R}$ <br>*Prüfer. Moreover,*  $\widetilde{R} = \widehat{R}$ ,  $(\widetilde{R})_P = \widehat{R}$ *Prüfer. Moreover,*  $\widetilde{R} = \widehat{R}$ ,  $(\widetilde{R})_P = \widetilde{R_P}$  for each  $P \in \text{Spec}(R)$ . In this case, any flat  $e$ *pimorphic subextension R*  $\subseteq$  *T is Prüfer. Priifer. Moreover,*  $\widetilde{R} = \widehat{R}$ ,  $(\widetilde{R})_P = \widetilde{R}_P$  *for each*  $P \in \text{Spec}(R)$ . In this case, any flat epimorphic subextension  $R \subseteq T$  *is Priifer.*<br>*Proof* Let  $R \subseteq \widetilde{R} \subseteq S$  be an almost-Prüfer extension, that is

first assertion follows from Corollary [3.3](#page-7-1) because  $R \subseteq \overline{R}$  is Prüfer. Now the Morita efficient is  $\widetilde{R} \subseteq S$  is integral. The  $\subseteq \widetilde{R}$  is Prüfer. Now the Morita **Proof** Let  $R \subseteq \widetilde{R} \subseteq S$  be an almost-Prüfer extension, that is  $\widetilde{R} \subseteq S$  is integral. The first assertion follows from Corollary 3.3 because  $R \subseteq \widetilde{R}$  is Prüfer. Now the More hull and the Prüfer hull coincide by *<sup>P</sup>* is **Proof** Let  $R \subseteq R \subseteq S$  be an almost-Pruter extension, that is  $R \subseteq S$  is integral. The first assertion follows from Corollary 3.3 because  $R \subseteq \tilde{R}$  is Prüfer. Now the Morita hull and the Prüfer hull coincide by Propositio

We could define almost-Prüfer rings as the rings *R* such that  $R \subseteq \text{Tot}(R)$  is<br>nost-Prüfer. But in that case  $\widetilde{R}$  –  $\text{Tot}(R)$  (by Theorem 4.4), so that *R* is a a flat epimorphism and  $(R)_P \rightarrow S_P$  is integral.<br>We could define almost-Prüfer rings as the rings *R* such that  $R \subseteq \text{Tot}(R)$  is almost-Prüfer. But in that case  $\widetilde{R} = \text{Tot}(R)$  (by Theorem [4.4\)](#page-13-1), so that *R* is a Prüfer ring. The converse evidently holds. Therefore, this concept does not define something new.

It was observed in  $[13,$  Remark 2.9(c)] that there is an almost-Prüfer FMC extension  $R \subseteq S \subseteq T$ , where  $R \subseteq S$  is a Prüfer minimal extension and  $S \subseteq T$ <br>is minimal and integral but  $R \subseteq T$  is not an ECP extension is minimal and integral, but  $R \subseteq T$  is not an FCP extension.

<span id="page-14-0"></span>**Proposition 4.5** *Let*  $R \subseteq S$  *be an extension verifying the hypotheses:* 

- $(i)$   $R \subseteq$ <br>*ii*)  $R \subset$ *(i)*  $R \subseteq S$  *is quasi-Prüfer.*
- (*ii*)  $R \subseteq S$  can be factored  $R \subseteq T \subseteq S$ , where  $R \subseteq T$  is a flat epimorphism.
- *(1) Then the following commutative diagram (D) is a pushout,*



 $TR \subseteq S$  is Prüfer and  $R \subseteq TR$  is quasi-Prüfer. Moreover,  $F_{R,\overline{R}}(P) \cong F_{T,T\overline{R}}(Q)$ <br>for each  $Q \in \text{Spec}(T)$  and  $P := Q \cap R$ *for each*  $Q \in Spec(T)$  *and*  $P := Q \cap R$ .

(2) If in addition  $R \subseteq T$  is integrally closed, (D) is a pullback,  $T \cap R = R$ ,  $(R : \overline{R}) - (T \cdot T\overline{R}) \cap R$  and  $(T \cdot T\overline{R}) - (R \cdot \overline{R})T$  $\overline{R}$  =  $(T : T\overline{R}) \cap R$  and  $(T : T\overline{R}) = (R : \overline{R})T$ .

*Proof* (1) Consider the injective composite map  $\overline{R} \rightarrow \overline{R} \otimes_R T \rightarrow T\overline{R}$ . As  $\overline{R} \rightarrow$  $\overline{R} \otimes_R T$  is a flat epimorphism, because deduced by a base change of  $R \to T$ , we get that the surjective map  $\overline{R} \otimes_R T \to T\overline{R}$  is an isomorphism by Scholium A (3). By fibers transitivity, we have  $F_{T,\overline{RT}}(Q) \cong \kappa(Q) \otimes_{k(P)} F_{R,\overline{R}}(P)$  [\[22,](#page-28-20) Corollaire 3.4.9]. As  $\kappa(P) \to \kappa(Q)$  is an isomorphism by Scholium A, we get that  $F_{R\bar{R}}(P) \cong F_{T\bar{R}T}(Q)$ .

(2) As in [\[5,](#page-28-4) Lemma 3.5],  $R = T \cap \overline{R}$ . The first statement on the conductors has the same proof as in [\[5,](#page-28-4) Lemma 3.5]. The second holds because  $R \subseteq T$  is a flat<br>enimorphism (see Scholium A (6)) epimorphism (see Scholium A (6)).  $\Box$ 

<span id="page-14-1"></span>**Theorem 4.6** *Let*  $R \subset S$  *be a quasi-Prüfer extension and the diagram*  $(D')$ *:* 



- $\widetilde{R} \longrightarrow \widetilde{R}\overline{R}$ <br>(1) (D') is a pushout and a pullback, such that  $\overline{R}\cap \widetilde{R} = R$  and  $(R: \overline{R}) = (\widetilde{R}: \widetilde{R}\overline{R})\cap R$ <br>so that  $(R: \widetilde{R}R) = (R: \overline{R})\widetilde{R}$ *(D') is a pushout and a pullb.*<br>*so that*  $(R : \overline{RR}) = (R : \overline{R})\overline{R}$ . *(1) (D')* is a pushout and a pullback, such that  $\overline{R} \cap \widetilde{R}$ <br>so that  $(\widetilde{R} : \widetilde{R}\overline{R}) = (R : \overline{R})\widetilde{R}$ .<br>(2)  $R \subset S$  can be factored  $R \subseteq \widetilde{R}\overline{R} = \overline{\widetilde{R}} = \overline{R} \subseteq R$ <br>extension is almost-Pritier and the *R* and  $(R : \overline{R}) = (\overline{R} : \overline{R}\overline{R}) \cap R$ <br>  $\overrightarrow{R} = \overline{\overline{R}} = S$ , where the first
- $\subseteq \overrightarrow{R}$ *extension is almost-Prüfer and the second is Prüfer.* (2)  $R \subset S$  *can be factored*  $R \subseteq \overline{RR} = \overline{R} = \overline{R}$  *s*<br>*extension is almost-Prüfer and the second is P*<br>(3)  $R \subset S$  *is almost-Prüfer*  $\Leftrightarrow S = \overline{RR} \Leftrightarrow \overline{R} = \overline{R}$ .<br>(4)  $R \subset \widetilde{PD} = \overline{R} = \overline{R}$ , is the symptotic
- (3)  $R \subset S$  is almost-Prüfer  $\Leftrightarrow S = \overline{R}\overline{R} \Leftrightarrow \overline{R} = \overline{\overline{R}}$ .
- *(2)*  $R \subseteq B$  can be gatibled  $R \subseteq RR \implies R = R = R$ .  $\Rightarrow R = B$ , where the grist extension is almost-Prüfer and the second is Prüfer.<br>
(3)  $R \subseteq S$  is almost-Prüfer  $\Leftrightarrow S = \overline{R}R \Leftrightarrow \overline{R} = \overline{R}$ .<br>
(4)  $R \subseteq \overline{R}R = \overline{R} = \overline{R}$  i  $R \subseteq S$  i<br>  $R \subseteq \widetilde{R}$ <br>  $\widetilde{R} = \widetilde{R}^{\widetilde{R}}$ <br>  $S$  pec ( $\widetilde{R}$ )  $\widetilde{R} = \widetilde{R}^R$ . (4)  $R \subseteq \overline{RR} = \overline{R} = \overline{R}$  is the greatest almost-Prüfer sube<br>  $\widetilde{R} = \widetilde{R}^{\overline{R}}$ .<br>
(5)  $\text{Spec}(\overline{R})$  is homeomorphic to  $\text{Spec}(\overline{R}) \times \text{Spec}(\overline{R})$ .<br>
(6)  $\text{Sunn}(S/R) = \text{Sunn}(\overline{R}/R) + \text{Sunn}(\overline{R}/R)$  if  $R \subseteq S$  is all
- *(5)* Spec $(\overline{R})$  is homeomorphic to Spec $(\overline{R}) \times_{Spec(R)} Spec(\overline{R})$ .
- *(6)*  $\text{Supp}(S/R) = \text{Supp}(\overline{R}/R) \cup \text{Supp}(\overline{R}/R)$  *if*  $R \subseteq S$  *is almost-Prüfer.* (Supp can *be replaced with* MSupp*).*

*Proof* To show (1), (2), in view of Theorem [3.18,](#page-12-1) it is enough to apply Proposi-*Proof* To show (1), (2), in view of Theorem 3.18, it is enough to apply Proposition [4.5](#page-14-0) with  $T = \widetilde{R}$  and  $S = \overline{R}$ , because  $R \subseteq \widetilde{R}R$  is almost-Prüfer whence quasi-Prüfer keeping in mind that a Prüfer extension quasi-Prüfer, keeping in mind that a Prüfer extension is integrally closed, whereas tion 4.5 with  $T = \overline{R}$  and  $S = \overline{R}$ , because  $R \subseteq \overline{R}R$  is almost-Prüfer whence<br>quasi-Prüfer, keeping in mind that a Prüfer extension is integrally closed, whereas<br>an integral Prüfer extension is trivial. Moreover, integral and integrally closed.

(3) is obvious.

(4) Now consider an almost-Prüfer subextension  $R \subseteq T \subseteq U$ , where  $R \subseteq$ <br>ifor and  $T \subseteq U$  is integral. Applying (3), we see that  $U = \overline{P}^U \widetilde{P}^U \subseteq \widetilde{P} \widetilde{P}$  in (4) Now consider an almost-Prüfer subextension  $R \subseteq T \subseteq U$ , where  $R \subseteq T$  is Figure and *I* integrally closed.<br>
(3) is obvious.<br>
(4) Now consider an almost-Prüfer subextension  $R \subseteq T \subseteq U$ , where  $R \subseteq T$  is<br>
Prüfer and  $T \subseteq U$  is integral. Applying (3), we see that  $U = \overline{R}^U \overline{R}^U \subseteq \overline{R} \overline{R}$  in of Proposition [1.6.](#page-5-2)

(5) Recall from [\[33\]](#page-29-11) that a ring morphism  $A \rightarrow A'$  is called a subtrusion if for each pair of prime ideals  $P \subseteq Q$  of *A*, there is a pair of prime ideals  $P' \subseteq Q'$ <br>above  $P \subseteq Q$  A subtrusion defines a submersion  $Spec(A') \rightarrow Spec(A)$  We refer to above  $P \subseteq Q$ . A subtrusion defines a submersion  $Spec(A') \rightarrow Spec(A)$ . We refer to 133. First paragraph of p. 5701 for the definition of the property  $P(A)$  of a pushout [\[33,](#page-29-11) First paragraph of p. 570] for the definition of the property  $P(\Delta)$  of a pushout diagram  $(\Delta)$ . Then [\[33,](#page-29-11) Lemme 2,(b), p. 570] shows that  $P(D')$  holds, because  $R \rightarrow \widetilde{R}$  is a flat enimorphism. Now [33, Proposition 2, p. 576] vialds that Spec( $\widetilde{R}$ )  $\rightarrow$  $\widetilde{R}$  is a flat epimorphism. Now [\[33,](#page-29-11) Proposition 2, p. 576] yields that  $Spec(\widetilde{R}) \rightarrow$  $Spec(\overline{R}) \times_{Spec(R)} Spec(\overline{R})$  is subtrusive. This map is also injective because  $R \to \overline{R}$  is spectrally injective. Observing that an injective submersion is an homeomorphism, the proof is complete.  $\text{gcd}(R) \times \text{Spec}(R)$  Spec(*R*) is subtrusive. This map is also injective because  $R \to R$  is<br>certaily injective. Observing that an injective submersion is an homeomorphism,<br>proof is complete.<br>(6) Obviously, Supp $(\overline{R}/R) \cup \text{$ 

spectrally injective. Observing that an injective submersion is an homeomorphism,<br>the proof is complete.<br>(6) Obviously, Supp $(\overline{R}/R) \cup \text{Supp}(\overline{R}/R) \subseteq \text{Supp}(S/R)$ . Conversely, let  $M \in$ <br>Spec $(R)$  be such that  $R_M \neq S_M$ , and the proof is complete.<br>
(6) Obviously, Supp $(\overline{R}/R) \cup \text{Supp}(\overline{R}/R) \subseteq \text{Supp}(S/R)$ . Conversely, let  $M \in$ <br>
Spec $(R)$  be such that  $R_M \neq S_M$ , and  $R_M = (\overline{R})_M = \overline{R}_M$ . Then (3) entails that<br>  $S_M = (\overline{R}\overline{R})_M = (\overline{R})_M(\overline{R})_$ 

**Corollary 4.7** *Let*  $R \subseteq S$  *be an almost-Prüfer extension. The following conditions are equivalent are equivalent:*

*(1)* Supp $(S/\overline{R}) \cap$  Supp $(\overline{R}/R) = \emptyset$ .

*(2)* Supp $(S/\overline{R}) \cap$  Supp $(\overline{R}/R) = \emptyset$ .

*(3)* Supp $(\overline{R}/R) \cap$  Supp $(\overline{R}/R) = \emptyset$ .

*(2)*  $\text{Supp}(S/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R) = \emptyset.$ <br> *(3)*  $\text{Supp}(\widetilde{R}/R) \cap \text{Supp}(\overline{R}/R) = \emptyset.$ <br> *Proof* Since  $R \subseteq S$  is almost-Prüfer, we get  $(\widetilde{R})_P = \widetilde{R}_P$  for each  $P \in \text{Spec}(R)$ .<br> *Proof* Since  $R \subseteq S$  is almost-Prüfer, (3)  $\text{Supp}(\overline{R}/R) \cap \text{Supp}(\overline{R}/R) = \emptyset.$ <br>*Proof* Since  $R \subseteq S$  is almost-Prüfer, we get  $(\overline{R})_P = \overline{R_P}$  for each  $P \in \text{Spec}(R)$ .<br>Moreover,  $\text{Supp}(S/R) = \text{Supp}(\overline{R}/R) \cup \text{Supp}(\overline{R}/R) = \text{Supp}(S/\overline{R}) \cup \text{Supp}(\overline{R}/R) = \text{Supp}(S/\over$ Proof Since  $R \subseteq S$  is all<br>Moreover, Supp $(S/R) =$ <br>Supp $(S/\overline{R}) \cup$  Supp $(\overline{R}/R)$ .<br>(1)  $\rightarrow$  (2): Assume that for Since  $R \subseteq S$  is almost-Pruter, we get  $(R)_P = R_P$  for each  $P \in \text{Spec}(R)$ .<br>
foreover,  $\text{Supp}(S/R) = \text{Supp}(\widetilde{R}/R) \cup \text{Supp}(\overline{R}/R) = \text{Supp}(S/\overline{R}) \cup \text{Supp}(\overline{R}/R) = \text{supp}(S/\overline{R}) \cup \text{Supp}(\widetilde{R}/R)$ .<br>
(1)  $\Rightarrow$  (2): Assume that the

 $S_P$ ,  $R_P$ , so that  $R_P \subset S_P$  is neither Prüfer nor integral. But,  $P \in \text{Supp}(S/R) =$  $\text{Supp}(S/\overline{R}) \cup \text{Supp}(\overline{R}/R)$ . If  $P \in \text{Supp}(S/\overline{R})$ , then  $P \notin \text{Supp}(\overline{R}/R)$ , so that  $(\overline{R})_P =$  $R_P$  and  $R_P \subset S_P$  is Prüfer, a contradiction. If  $P \in \text{Supp}(R/R)$ , then  $P \notin \text{Supp}(S/R)$ , so that  $(\overline{R})_P = S_P$  and  $R_P \subset S_P$  is integral, a contradiction so that  $(R)_P = S_P$  and  $R_P \subset S_P$  is integral, a contradiction.<br>  $(2) \Rightarrow (3)$ : Assume that there exists  $P \in \text{Supp}(\overline{R}/R) \cap \text{Supp}(\overline{R}/R)$ . Then,  $R_P \neq$ pp(*S*/*R*)  $\cup$  Supp(*R*/*R*). If  $P \in \text{Supp}(S/R)$ , then  $P \notin \text{Supp}(R/R)$ , so that  $(R)_P =$ <br>and  $R_P \subset S_P$  is Prüfer, a contradiction. If  $P \in \text{Supp}(\overline{R}/R)$ , then  $P \notin \text{Supp}(S/\overline{R})$ ,<br>that  $(\overline{R})_P = S_P$  and  $R_P \subset S_P$  is integral,

 $R_P$  and  $R_P \subset S_P$  is Pruter, a contradiction. If  $P \in \text{Supp}(R/R)$ , then  $P \notin \text{Supp}(S/R)$ ,<br>so that  $(\overline{R})_P = S_P$  and  $R_P \subset S_P$  is integral, a contradiction.<br>(2)  $\Rightarrow$  (3): Assume that there exists  $P \in \text{Supp}(\overline{R}/R) \cap \text{Supp}(\overline{R$ so that  $(R)_P = S_P$  and  $R_P \subset S_P$  is integral, a contradiction.<br>  $(2) \Rightarrow (3)$ : Assume that there exists  $P \in \text{Supp}(\overline{R}/R) \cap \text{Supp}(\overline{R}/R)$ . Then,  $R_P \neq (\overline{R})_P$ , so that  $R_P \subset S_P$  is neither Prüfer nor integral. But,  $P \in \text{Supp$ *R*)<sub>*P*</sub>, (*R*)<sub>*P*</sub>, so that *R<sub>P</sub>* ⊂ *S<sub>P</sub>* is neither Prüfer nor integral. But, *P* ∈ Supp $(S/R)$  = Supp $(S/R)$  ∪ Supp $(\widetilde{R}/R)$ . If *P* ∈ Supp $(S/\widetilde{R})$ , then *P* ∉ Supp $(\widetilde{R}/R)$ , so that  $(\widetilde{R})_P$  = *R<sub>P</sub>* and *R<sub>*</sub> Supp( $S/R$ )  $\cup$  Supp( $R/R$ ). If  $P \in \text{Supp}(S/R)$ , then  $P \notin \text{Supp}(R/R)$ , so that  $(R)_P = R_P$  and  $R_P \subset S_P$  is integral, a contradiction. If  $P \in \text{Supp}(\widetilde{R}/R)$ , then  $P \notin \text{Supp}(S/\widetilde{R})$ , so that  $(\widetilde{R})_P = S_P$  and  $R_P \subset S_P$  is Prü

 $(3) \Rightarrow (1)$ : Assume that there exists  $P \in \text{Supp}(S/R) \cap \text{Supp}(R/R)$ . Then,  $(R)_P \neq R$  $R_P$ ,  $S_P$ , so that  $R_P \subset S_P$  is neither Prüfer nor integral. But,  $P \in \text{Supp}(S/R) =$  $\text{Supp}(\overline{R}/R) \cup \text{Supp}(\overline{R}/R)$ . If  $P \in \text{Supp}(\overline{R}/R)$ , then  $P \notin \text{Supp}(\overline{R}/R)$ , so that  $(\overline{R})_P =$  Quasi-Prüfer Extensions of Rings<br>  $R_P$  and  $R_P \subset S_P$  is Prüfer, a contradiction. If  $P \in \text{Supp}(\overline{R}/R)$ , then  $P \notin \text{Supp}(\overline{R}/R)$ , so that  $(\overline{R})_P = R_P$  and  $R_P \subset S_P$  is integral a contradiction  $R_P$  and  $R_P \subset S_P$  is Prüfer, a contradiction. If  $P \in \text{Supp}(\overline{R}/R)$ , then  $P \notin \text{Supp}(\overline{R}/R)$ , so that  $(\overline{R})_P = R_P$  and  $R_P \subset S_P$  is integral, a contradiction.  $\Box$ <br>Proposition 4.5 has the following similar statement pro

Proposition [4.5](#page-14-0) has the following similar statement proved by Ayache and Dobbs. It reduces to Theorem [4.6](#page-14-1) in case  $R \subseteq S$  has FCP because of Proposition [1.3.](#page-4-0)

**Proposition 4.8** *Let*  $R \subseteq T \subseteq S$  *be a quasi-Prüfer extension, where*  $T \subseteq S$  *is an integral minimal extension and*  $R \subseteq T$  *is integrally closed. Then the diagram (D) is a pullback,*  $S = T\overline{R}$  *and*  $(T : S) = (R : \overline{R})T$ .

*Proof* [\[5,](#page-28-4) Lemma 3.5].

*integral minimal extension and*  $R \subseteq T$  *is integrally closed. Then the diagram (D) is a pullback,*  $S = T\overline{R}$  *and*  $(T : S) = (R : \overline{R})T$ .<br>*Proof* [5, Lemma 3.5]. <br>*Proposition 4.9 Let*  $R \subseteq U \subseteq S$  *and*  $R \subseteq V \subseteq S$  *be two* **Proposition 4.9** *Let*  $R \subseteq U \subseteq S$  and  $R \subseteq V \subseteq S$  be two towers of extensions, such that  $R \subseteq U$  and  $R \subseteq V$  are almost-Prifer Then  $R \subseteq UV$  is almost-Prifer and  $\textit{such that } R \subseteq U \textit{ and } R \subseteq V \textit{ are almost-Prüfer. Then } R \subseteq UV \textit{ is almost-Prüfer and } \widetilde{UV} - \widetilde{IVV}$  $\widetilde{UV} = \widetilde{UV}$ .

<span id="page-16-0"></span>*UV*  $\subseteq$  *V*  $\subseteq$  *S be two towers of extensic-Prüfer. Then*  $R \subseteq UV$  *is almost-Prüfer c*<br>Prüfer hulls of *R* in *U*, *V*, and *W* = *U*<br>0. 53], that  $R \subseteq U'V'$  is Prüfer. Moreov<br> $\subseteq W'$  because the Prüfer hull is the gre *Proof* Denote by *U'*, *V'*, and *W'* the Prüfer hulls of *R* in *U*, *V*, and *W* = *UV*. We deduce from [25] Corollary 5.11, p. 531, that  $R \text{ } \subset U'V'$  is Prifer Moreover We deduce from [\[25,](#page-28-1) Corollary 5.11, p. 53], that  $R \subseteq U'V'$  is Prüfer. Moreover,  $U'V' \subset UV$  is clearly integral and  $U'V' \subset W'$  because the Prüfer bull is the greatest  $U'V' \subseteq UV$  is clearly integral and  $U'V' \subseteq W'$  because the Prüfer hull is the greatest<br>Prijfer subextension. We deduce that  $R \subseteq UV$  is almost-Prüfer and that  $\widetilde{UV} = \widetilde{UV}$ Prüfer subextension. We deduce that  $R \subseteq UV$  is almost-Prüfer and that  $UV = UV$ .  $\Box$ 

**Proposition 4.10** *Let R*  $\subseteq$ <br>such that *R*  $\subseteq$  *U* is almost-*P*  $\subseteq U \subseteq S$  and  $R \subseteq V \subseteq S$  be two towers of extensions,<br>Prijfer and  $R \subseteq V$  is a flat enimorphism. Then  $U \subseteq UV$  $such that R \subseteq U$  is almost-Prüfer and  $R \subseteq V$  is a flat epimorphism. Then  $U \subseteq UV$ <br>is almost-Priifer *is almost-Prüfer.*

*Proof* Mimic the proof of Proposition [4.9,](#page-16-0) using  $[25$ , Theorem 5.10, p. 53].  $\Box$ 

**Proposition 4.11** *Let*  $R \subseteq S$  *be an almost-Prüfer extension and*  $R \rightarrow T$  *a flat enimorphism Then*  $T \subseteq T \otimes_R S$  *is almost-Prüfer epimorphism. Then*  $T \subseteq T \otimes_R S$  *is almost-Prüfer.* 

*Proof* It is enough to use Proposition [3.10](#page-9-0) and Definition [4.1.](#page-13-2)

**Proposition 4.12** *An extension*  $R \subseteq S$  *is almost-Prüfer if and only if*  $R_P \subseteq S_P$  *is almost-Prüfer and*  $\widetilde{R}_P - (\widetilde{R})_P$  *for each*  $P \in \text{Spec}(R)$ *almost-Prüfer and*  $\widetilde{R_P} = (\widetilde{R})_P$  for each  $P \in \text{Spec}(R)$ . *Proposition 3.10 and Defin*<br>*P <i>n extension R*  $\subseteq$  *S is almost-Pr*<br>*P* =  $(\widetilde{R})$ *P for each P*  $\in$  Spec(*R*). **Proposition 4.12** An extension  $R \subseteq S$  is almost-Prüfer if and only if  $R_P \subseteq S_P$  is almost-Prüfer and  $\overline{R_P} = (\overline{R})_P$  for each  $P \in \text{Spec}(R)$ .<br>*Proof* For an arbitrary extension  $R \subseteq S$  we have  $(\overline{R})_P \subseteq \overline{R_P}$ . Suppose  $\epsilon_{di}$ 

<span id="page-16-1"></span>is almost-Prüfer, then so is  $R_P \subseteq S_P$  and  $(R)_P = R_P$  by Theorem [4.4.](#page-13-1) Conversely,<br>if  $R \subset S$  is locally almost-Prüfer, whence locally quasi-Prüfer, then  $R \subset S$  is quasi-*SPECAP*<br>  $R \subseteq S$  we have  $\hat{\theta}$ <br>  $\subseteq S_P$  and  $\hat{R}$  $\hat{R}$  $P = \hat{R}$ <br>  $\therefore$  whence locally guess. if  $R \subseteq S$  is locally almost-Prüfer, whence locally quasi-Prüfer, then  $R \subseteq S$  is quasi-<br>Prüfer, If  $\widetilde{R_{\rho}} = (\widetilde{R})_P$  holds for each  $P \in \text{Spec}(R)$ , we have  $S_p = (\widetilde{R}\widetilde{R})_P$  so that *Proof* For an arbitrary extension  $R \subseteq S$  we have  $(R)_P \subseteq R_P$ . Suppose that  $R \subseteq S$  is almost-Prüfer, then so is  $R_P \subseteq S_P$  and  $(\widetilde{R})_P = \widetilde{R}_P$  by Theorem 4.4. Conversely, if  $R \subseteq S$  is locally almost-Prüfer, whence locall <sup>ar</sup> *S* almost-Pruter,<br>if  $R \subseteq S$  is locall<br>Prüfer. If  $\widetilde{R_P} = S = \overline{R}R$  and  $R \subseteq S$  $S = S$  is almost-Prüfer by Theorem [4.6.](#page-14-1)

**Corollary 4.13** *An FCP extension*  $R \subseteq S$  *is almost-Prüfer if and only if*  $R_P \subseteq S_P$  *is almost-Prüfer for each P*  $\in$  Spec $(R)$ *almost-Prüfer for each P*  $\in$  Spec $(R)$ *.* 

*Proof* It is enough to show that  $R \subseteq S$  is almost-Prüfer if  $R_P \subseteq S_P$  is almost-Prüfer for each  $P \in$  Spec $(R)$  using Proposition 4.12. Any minimal extension  $\widetilde{R} \subset R$ , is for each  $P \in \text{Spec}(R)$  using Proposition [4.12.](#page-16-1) Any minimal extension  $\overline{R} \subset R_1$  is *Proof* It is enough to show that  $R \subseteq S$  is almost-Prüfer if  $R_P \subseteq S_P$  is almost-Prüfer for each  $P \in \text{Spec}(R)$  using Proposition 4.12. Any minimal extension  $\widetilde{R} \subset R_1$  is integral by definition of  $\widetilde{R}$ . Assume that **EXPROOF IT IS ENDINE TO SHOW THAT APPLATE THAT APPLATE TO APPLAT ANY MINIMAL EXTEND FOR A LABY MINIMAL EXTENDS INTEGRAL SINCE**  $R_1$  **is integral by definition of**  $\widetilde{R}$ **. Assume that**  $(\widetilde{R})_P \subset (\widetilde{R}_P)$ **, so that ther** *Q*. For some *Q* 2 Max. The *Q R*, *A* assume that  $\overline{R}$ ,  $\overline{R}$   $\subset$   $\overline{R}$ , so that there exists that  $\overline{R}$ ,  $\overline{R}$   $\subset$   $\overline{R}'_2$ ,  $\overline{R}$  is a Prüfer minimal extension with crucial  $Q(\overline{R})_P$ , for so  $Q(\widetilde{R})_P$ , for some  $Q \in \text{Max}(\widetilde{R})$  with  $Q \cap R \subseteq P$ . In particular,  $\widetilde{R} \subset R'_2$  is not integral. We may assume that there exists  $R'_1 \in [R, R'_2]$  such that  $R'_1 \subset R'_2$  is a Prüfer minimal Exercise<br>  $\Omega \cap R \subseteq$ <br>  $\Omega \cap R \subseteq$ <br>  $\{R, R\}$ 

$$
\Box
$$

324<br>
extension with *P*  $\notin$  Supp $(R'_1/\widetilde{R})$ . Using [\[37,](#page-29-10) Lemma 1.10], there exists  $R_2 \in [\widetilde{R}, R'_2]$ <br>
such that  $\widetilde{R} \subset R_2$  is a Prijfer minimal extension with crucial maximal ideal *Q* a such that  $\widetilde{R} \subset R_2$  is a Prüfer minimal extension with crucial maximal ideal *Q*, a<br>contradiction Then  $(\widetilde{R})_0 \subset S_2$  is integral for each *P* whence  $(\widetilde{R})_0 = (\widetilde{R}_1)$ . extension with  $P \notin \text{Supp}(R'_1/\widetilde{R})$ . Using [37, Lemma 1.10], there exists such that  $\widetilde{R} \subset R_2$  is a Prüfer minimal extension with crucial maxima contradiction. Then,  $(\widetilde{R})_P \subset S_P$  is integral for each *P*, whence

*Report Then,*  $(R)_P \subset S_P$  is integral for each *P*, whence  $(R)_P = (R_P)$ .  $\square$ <br>We now intend to demonstrate that our methods allow us to prove easily some results. For instance, next statement generalizes [\[5,](#page-28-4) Corollary 4.5] and can be fruitful in algebraic number theory.

**Proposition 4.14** *Let*  $(R, M)$  *be a one-dimensional local ring and*  $R \subseteq S$  *a quasi-*<br>*Prifer extension. Suppose that there is a tower*  $R \subset T \subseteq S$  where  $R \subset T$  is *Prüfer extension. Suppose that there is a tower*  $R \subset T \subseteq S$ *, where*  $R \subset T$  *is integrally closed. Then*  $R \subset S$  *is almost-Prüfer*  $T = \widetilde{R}$  *and S is zero-dimensional integrally closed. Then*  $R \subseteq S$  *is almost-Prüfer,*  $T = \overline{R}$  *and* S *is zero-dimensional. S S S A S A C B a one-dimensional local ring and*  $R \subseteq S$  *<i>a quase R E T E S is almost-Prüfer, T* =  $\widetilde{R}$  *and S is zero-dimensional.* 

*Proof* Because  $R \subset T$  is quasi-Prüfer and integrally closed, it is Prüfer. If some prime ideal of *T* is lying over *M*,  $R \subset T$  is a faithfully flat epimorphism, whence an isomorphism by Scholium A, which is absurd. Now let *N* be a prime ideal of *T* and  $P := N \cap R$ . Then  $R_P$  is zero-dimensional and isomorphic to  $T_N$ . Therefore, *T* is zero-dimensional. It follows that *TR* is zero-dimensional. Since  $RT \subseteq S$  is Prüfer,<br>we deduce from Scholium A, that  $\overline{RT} = S$ . The proof is now complete we deduce from Scholium A, that  $\overline{RT} = S$ . The proof is now complete.  $\Box$ We also generalize [\[5,](#page-28-4) Proposition 5.2] as follows.

**Proposition 4.15** *Let*  $R \subset S$  *be a quasi-Prüfer extension, such that*  $\overline{R}$  *is local with maximal ideal N* :=  $\sqrt{(R : \overline{R})}$ . Then *R* is local and  $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$ . If in addition *R* is one-dimensional then either *R*  $\subset$  *S* is integral or there is some minimal ideal *N* :=  $\sqrt{(R : \overline{R})}$ . Then *addition R is one-dimensional, then either R*  $\subset$  *S is integral or there is some minimal prime ideal P of*  $\overline{R}$ *, such that*  $S = (\overline{R})_P$ *, P = SP and*  $\overline{R}/P$  *is a one-dimensional valuation domain with quotient field S/P.* 

*Proof R* is obviously local. Let  $T \in [R, S] \setminus [R, R]$  and  $s \in T \setminus R$ . Then  $s \in U(S)$  and  $s^{-1} \in \overline{R}$  by Proposition 1.2.(1). But  $s^{-1} \notin U(\overline{R})$  so that  $s^{-1} \in N$ . It follows that there  $s^{-1} \in \overline{R}$  by Proposition [1.2](#page-3-1) (1). But  $s^{-1} \notin U(\overline{R})$ , so that  $s^{-1} \in N$ . It follows that there exists some integer *n* such that  $s^{-n} \in (R \cdot \overline{R})$  giving  $s^{-n} \overline{R} \subset R$  or equivalently exists some integer *n* such that  $s^{-n} \in (R : \overline{R})$ , giving  $s^{-n}\overline{R} \subseteq R$ , or, equivalently,  $\overline{R} \subset R s^n \subset T$  Then  $T \in [\overline{R} \times I]$  and we obtain  $[R \times I] - [R \times R] + [R \times I]$  $\overline{R} \subseteq Rs^n \subseteq T$ . Then,  $T \in [\overline{R}, S]$  and we obtain  $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$ .<br>Assume that *R* is one-dimensional If  $R \subset S$  is not integral then  $\overline{R}$ 

Assume that *R* is one-dimensional. If  $R \subset S$  is not integral, then  $\overline{R} \subset S$  is Prüfer  $\overline{R}$  is one-dimensional. To complete the proof, use Proposition 1.2 (3). and  $\overline{R}$  is one-dimensional. To complete the proof, use Proposition [1.2](#page-3-1) (3).

# *4.2 FCP Extensions*

In case we consider only FCP extensions, we obtain more results.

<span id="page-17-0"></span>**Proposition 4.16** *Let*  $R \subseteq S$  *be an FCP extension. The following statements are equivalent: equivalent:*

- *(1)*  $R \subseteq S$  is almost-Prüfer.<br>*(2)*  $R_R \subseteq S_R$  is either integr
- (2)  $R_P \subseteq S_P$  *is either integral or Prüfer for each P*  $\in$  Spec $(R)$ *.*<br>(3)  $R_P \subset S_P$  *is almost-Prüfer for each P*  $\in$  Spec $(R)$  *and*
- (3)  $R_P \subseteq S_P$  *is almost-Prüfer for each P*  $\in$  Spec(*R*) *and*<br>Supp( $S/\widetilde{R}$ )  $\cap$  Supp( $\widetilde{R}/R$ )  $\to$   $\emptyset$
- $R \subseteq S$  is almost-Pruter.<br>  $R_P \subseteq S_P$  is either integral or P<br>  $R_P \subseteq S_P$  is almost-Prüfer for e<br>  $\text{Supp}(S/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R) = \emptyset$ .<br>  $\text{Supp}(R/R) \cap \text{Supp}(S/\overline{R}) = \emptyset$ .
- *(4)* Supp $(\overline{R}/R) \cap$  Supp $(S/\overline{R}) = \emptyset$ .

*Proof* The equivalence of Proposition [4.12](#page-16-1) shows that  $(2) \Leftrightarrow (1)$  holds because  $\hat{T} = \tilde{T}$  and over a local ring *T*, an almost-Prüfer FCP extension  $T \subseteq U$  is either integral or Prüfer [37] Proposition 2.41. Moreover integral or Prüfer [\[37,](#page-29-10) Proposition 2.4] . Moreover when  $R_P \subseteq S_P$  is either integral or Prüfer it is easy to show that  $(\widetilde{R})_P = \widetilde{R}_P$ *Proof* The equivalence of Proposition 4.<br>  $\hat{T} = \tilde{T}$  and over a local ring *T*, an almos<br>
integral or Prüfer [37, Proposition 2.4] . No<br>
or Prüfer, it is easy to show that  $(\tilde{R})_P = \tilde{R}$ <br>
Next we show that (3) is eq *Proof* The equivalence of Proposition 4.12 shows that (2)  $\Leftrightarrow$  (1) holds because or Prüfer, it is easy to show that  $(R)_P = R_P$ 

Next we show that  $(3)$  is equivalent to  $(2)$  of Proposition [4.12.](#page-16-1)

Equal or Pruter [37, Proposition 2.4]. Moreover when  $R_P \subseteq S_P$  is either integral<br>Prüfer, it is easy to show that  $(\widetilde{R})_P = \widetilde{R_P}$ <br>Next we show that (3) is equivalent to (2) of Proposition 4.12.<br>Let  $P \in \text{Supp}(S/\widetilde{R}) \$ or Pruter, it is easy to show that  $(R)_P = R_P$ <br>
Next we show that (3) is equivalent to (2) of Proposition 4.12.<br>
Let  $P \in \text{Supp}(S/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R)$  be such that  $R_P \subseteq S_P$  is almost-Prüfer. Then,<br>  $(\widetilde{R})_P \neq R_P, S_P$ , so Next we sh<br>
Let  $P \in$  Su<br>  $(\widetilde{R})_P \neq R_P, S_P$ <br>
giving  $(\widetilde{R})_P \subseteq$ <br>
principle [37] giving  $(R)_P \subseteq R_P$  and  $R_P \neq R_P$ . It follows that  $R_P = S_P$  in view of the dichotomy<br>principle [37] Proposition 3.31 since  $R_P$  is a local ring, and then  $\widetilde{R_P} \neq (\widetilde{R})_P$ Let  $P \in \text{Supp}(S/R) \cap \text{Supp}(K/R)$  be such that  $R_P \subseteq S_P$  is almost-Prute  $(\overline{R})_P \neq R_P$ ,  $S_P$ , so that  $R_P \subset (\overline{R})_P \subset S_P$ . Since  $R \subset \overline{R}$  is Prüfer, so is  $R_P$  giving  $(\overline{R})_P \subseteq \overline{R}_P$  and  $R_P \neq \overline{R}_P$ . It follows that  $\subseteq$   $(R)_P \subseteq$   $S_P$ . Since  $R \subseteq R$  is Prufer, so is  $R_P$ *R<sub>P</sub>*. It follows that  $\overline{R}_P = S_P$  in view of the diagram  $\overline{R}_P$ . It follows that  $\overline{R}_P = S_P$  in view of the diagram is  $\overline{R}_P \neq (\overline{R})_P$ , *i.e.*  $P \in \text{Supp}(S/R)$ . Then,  $R_P \neq \overline{R}$  is the property of the propert

Conversely, assume that  $R_P \neq (R)_P$ , *i.e.*  $P \in \text{Supp}(S/R)$ . Then,  $R_P \neq R_P$ , so that  $R_P = S_P$ , as we have just seen. Hence  $R_P \subset S_P$  is integrally closed. It follows that  $\overline{R}_P = \overline{R}_P = R_P$  so that  $P \notin \text{Sum}(\overline{R}/R)$  and  $P \in \text{Sum}(\overline{R}/R)$  by Theorem 4.6(5) principle [37, Proposition 3.3] since  $R_P$  is a local ring, and then  $R_P \neq (R)_P$ .<br>
Conversely, assume that  $\overline{R_P} \neq (\overline{R})_P$ , *i.e.*  $P \in \text{Supp}(S/R)$ . Then,  $R_P \neq \overline{R_P}$ , so that  $\overline{R_P} = S_P$ , as we have just seen. Hen Conversely, assume that  $R_P \neq (R)_P$ , *t.e.*  $P \in \text{Supp}(S/R)$ . Then,  $R_P \neq R_P$ , so that  $\overline{R_P} = S_P$ , as we have just seen. Hence  $R_P \subset S_P$  is integrally closed. It follows that  $\overline{R_P} = \overline{R_P} = R_P$ , so that  $P \notin \text{Supp}(\overline{R}/R$  $\overline{R_P} = \overline{R}_P = R_P$ , so that  $P \notin \text{Supp}(\overline{R}/R)$  and  $P \in \text{Supp}(\overline{R}/R)$  by Theorem 4.6(5).<br>Moreover,  $\widetilde{R}_P \neq S_P$  implies that  $P \in \text{Supp}(S/\widetilde{R})$ . To conclude,  $P \in \text{Supp}(S/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R)$ .

(1)  $\Leftrightarrow$  (4) An FCP extension is quasi-Prüfer by Corollary [3.4.](#page-8-2) Suppose that  $R \subseteq S$ <br>almost-Prüfer, By Theorem 4.6, letting  $U \subseteq \widetilde{R}$ , we get that  $U \cap \overline{R} = R$  and is almost-Prüfer. By Theorem [4.6,](#page-14-1) letting  $U := R$ , we get that  $U \cap \overline{R} = R$  and  $S = \overline{R}U$ . We deduce from [\[37,](#page-29-10) Proposition 3.6] that  $\text{Supp}(\overline{R}/R) \cap \text{Supp}(S/\overline{R}) = \emptyset$ . Suppose that this last condition holds. Then by [\[37,](#page-29-10) Proposition 3.6]  $R \subseteq S$  can<br>be factored  $R \subseteq H \subseteq S$  where  $R \subseteq H$  is integrally closed, whence Prijfer by be factored  $R \subseteq U \subseteq S$ , where  $R \subseteq U$  is integrally closed, whence Prüfer by<br>Proposition 1.3, and  $U \subseteq S$  is integral. Therefore,  $R \subseteq S$  is almost-Prüfer.  $\square$ <br>**Proposition 4.17** Let  $R \subset S$  be an FCP almost-Prüfer extension. Proposition [1.3,](#page-4-0) and  $U \subseteq S$  is integral. Therefore,  $R \subseteq S$  is almost-Prüfer.

<span id="page-18-1"></span> $\left| \right.$  such that  $T \subseteq S$  is integral.

*Proof* Let  $T \in [R, S]$  be such that  $T \subseteq S$  is integral. So is  $T_M \subseteq S_M$  for each  $M \in \text{Max}(R)$ . But  $R_M \subseteq S_M$  is either integral (1) or Prijfer (2). In case (1) we get  $M \in \text{Max}(R)$ . But  $R_M \subseteq S_M$  is either integral (1) or Prüfer (2). In case (1), we get  $R_M = \widetilde{R}_M \subset T_M$  and in case (2), we get  $\widetilde{R}_M = S_M = T_M$  so that  $\widetilde{R}_M \subset T_M$ . By *Proof* Let  $T \in [R, S]$  be such that  $T \subseteq S$  is integral. So is  $T_M \subseteq S_M$ <br>  $M \in \text{Max}(R)$ . But  $R_M \subseteq S_M$  is either integral (1) or Prüfer (2). In case (1)<br>  $R_M = \widetilde{R}_M \subseteq T_M$  and in case (2), we get  $\widetilde{R}_M = S_M = T_M$ , so that  $\wid$  $R_M = R_M \subseteq T_M$  and in case (2), we get  $R_M = S_M = T_M$ , so that  $R_M \subseteq T_M$ . By Proof Let  $T \in [R]$ <br>  $M \in \text{Max}(R)$ . But<br>  $R_M = \widetilde{R}_M \subseteq T_M$ <br>
globalization,  $\widetilde{R} \subseteq$ <br>
We will need a  $\subseteq$  *T*.  $\square$ <br>a relative version of the support Let  $f: R \to T$  be a ring morphism

We will need a relative version of the support. Let  $f : R \to T$  be a ring morphism  $R \in T$  *R* and *T* and *R* is  $\mathscr{L}_r(F) := \frac{a_f(\text{Supp}_r(F))}{a}$  and and *E* a *T*-module. The relative support of *E* over *R* is  $\mathscr{S}_R(E) := {}^a f(\text{Supp}_T(E))$  and  $M\mathscr{S}_R(E) := \mathscr{S}_R(E) \cap \text{Max}(R)$ . In particular, for a ring extension  $R \subset S$ , we have  $\mathscr{S}_R(S/R) := \text{Supp}_R(S/R)$ .

**Proposition 4.18** *Let*  $R \subseteq S$  *be an FCP extension. The following statements hold:*<br>-

- *(1)* Supp $(\overline{\overline{R}}/\overline{R})$   $\cap$  Supp $(\overline{R}/R) = \emptyset$ .
- <span id="page-18-0"></span>**Proposition 4.18** Let  $R \subseteq S$  be an FCP extension. The followin<br>
(1) Supp $(\overline{R}/R) \cap \text{Supp}(\overline{R}/R) = \emptyset$ .<br>
(2) Supp $(\widetilde{R}/R) \cap \text{Supp}(\overline{R}/R) = \text{Supp}(\overline{\widetilde{R}}/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R) = \emptyset$ .<br>
(3) MSupp $(S/R) = \text{MSupp}(\widetilde{R}/R) \cup$ *(1)* Supp $(\overline{\widetilde{R}}/\overline{R}) \cap \text{Supp}(\overline{R}/R) = \emptyset$ .<br> *(2)* Supp $(\widetilde{R}/R) \cap \text{Supp}(\overline{R}/R) = \text{Supp}(\overline{\widetilde{R}}/\widetilde{R}) \cap \text{Supp}(3)$ .<br> *(3)* MSupp(*S*/*R*) = MSupp( $\widetilde{R}/R$ )  $\cup$  MSupp( $\overline{R}/R$ ).
- 

(2)  $\text{Supp}(\overline{R}/R) \cap \text{Supp}(\overline{R}/R) = \text{Supp}(\overline{R}/\overline{R}) \cap \text{Supp}(\overline{R}/R) = \emptyset$ .<br>
(3)  $\text{MSupp}(S/R) = \text{MSupp}(\overline{R}/R) \cup \text{MSupp}(\overline{R}/R)$ .<br> *Proof* (1) is a consequence of Proposition [4.16\(](#page-17-0)4) because  $R \subseteq \overline{\overline{R}}$  is almost-Prüfer.<br>

 $MSupp(S/R) = MSupp(\overline{R}/R) \cup MSupp(\overline{R}/R)$ .<br> *oof* (1) is a consequence of Proposition 4.16(4) because  $R \subseteq \overline{\widetilde{R}}$  is almost-Prüfer.<br>
We prove the first part of (2). If some  $M \in Supp(\overline{R}/R) \cap Supp(\overline{R}/R)$ , it can<br>
supposed in Max(*R* be supposed in  $Max(R)$  because supports are stable under specialization. Set  $R' :=$ We prove the first part of (2). If some  $M \in \text{Supp}(\overline{R}/R) \cap \text{Supp}(\overline{R}/R)$ , it can be supposed in  $\text{Max}(R)$  because supports are stable under specialization. Set  $R' := R_M, U := (\overline{R})_M, T := (\overline{R})_M$  and  $M' := MR_M$ . Then,  $R' \neq U, T$ Prüfer and  $R' \subset T$  FCP integral, an absurdity [\[37,](#page-29-10) Proposition 3.3].

To show the second part, assume that some  $P \in \text{Supp}(\overline{\widetilde{R}}/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R)$ . Then,<br>t Supp. $(\overline{R}/R)$  by the first part of (2), so that  $\overline{R} = R$ , giving.  $(\overline{\widetilde{R}}) = \overline{R} \times \widetilde{R} = \widetilde{R}$ . *P*  $\in$  Supp $(\overline{\overline{R}}/\overline{R}) \cap$  Supp $(\overline{\overline{R}}/\overline{R})$ . Then,  $P \notin$  Supp $(\overline{\overline{R}}/R)$  by the first part of (2), so that  $\overline{R}_P = R_P$ , giving  $(\overline{\overline{R}})_P = \overline{R}_P \overline{R}_P = \overline{R}_P$ , a contradiction a contradiction.

(3) Obviously,  $MSupp(S/R) = M\mathscr{S}(S/R) = M\mathscr{S}(S/\overline{T}^S) \cup M\mathscr{S}(\overline{T}^S/T)$ <br>*A*  $\mathscr{S}(\mathcal{I} \mathcal{T}^T) \cup M\mathscr{S}(\overline{U}^T/\mathcal{I}) \cup M\mathscr{S}(\mathcal{I} \mathcal{I}/P)$ . By 137. Propositions 2  $\cup$ M $\mathscr{S}(T/\overline{U}^T)$   $\cup$  M $\mathscr{S}(\overline{U}^T)$ <br>
2.21 *we have M* $\mathscr{S}(S/\overline{T}^S)$  $\bigcup M\mathscr{S}(T/U^*) \cup M\mathscr{S}(U^*/U) \cup M\mathscr{S}(U/R)$ . By [\[37,](#page-29-10) Propositions 2.3 and<br>3.2], we have  $M\mathscr{S}(S/\overline{T}^S) \subseteq \mathscr{S}(\overline{T}^S/T) = \mathscr{S}(\overline{R}/R^T) = M\mathscr{S}(\overline{R}/R) = M\mathscr{S}(\overline{R}/R)$ ,  $M\mathscr{S}(T/\overline{U}^T) = \mathscr{S}(\overline{R}^T/R) \subseteq \mathscr{S}(\overline{R$  $M\mathscr{S}(\overline{U}^T/U) = \mathscr{S}(\overline{R}^T/R) = \text{Supp}(\overline{R}/R)$ . To conclude,  $MSupp(S/R) = \text{MSupp}(\overline{R}/R) \perp \text{MSupp}(\overline{R}/R) = \square$  $MSupp(\overline{R}/R) \cup MSupp(\overline{R}/R)$ .

**Proposition 4.19** *Let*  $R \subset S$  *be an FCP extension and*  $M \in \text{MSupp}(S/R)$ *, then*  $\widetilde{R}_{\mathcal{H}} = (\widetilde{R})_{\mathcal{H}}$  *if and only if*  $M \notin \text{MSupp}(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R)$ *MSupp* $(\widetilde{R}/R) \cup M$ Supp $(\overline{R}/R)$ .<br> **Proposition 4.19** Let  $R \subset S$  be an *FCP* extension and *M*<br>  $\widetilde{R}_M = (\widetilde{R})_M$  if and only if  $M \notin M$ Supp $(S/\widetilde{R}) \cap M$ Supp $(\widetilde{R}/R)$ . **Proposition 4.19** Let  $R \subset S$  be an FCP extension and  $M \in MSupp(S/R)$ , then  $\widetilde{R}_M = (\widetilde{R})_M$  if and only if  $M \notin MSupp(S/\widetilde{R}) \cap MSupp(\widetilde{R}/R)$ .<br>*Proof* In fact, we are going to show that  $\widetilde{R}_M \neq (\widetilde{R})_M$  if and only if  $M \$  $\frac{f(x)}{x}$ 

<span id="page-19-0"></span> $\widetilde{R_M} = (\widetilde{R})_M$  if and only if  $M \notin$ <br>*Proof* In fact, we are going<br>MSupp $(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R)$ .<br>Let  $M \in \text{MSupp}(S/\widetilde{R}) \cap \text{MSup}$ *Coof* In fact, we are going to show that  $\widetilde{R_M} \neq (\widetilde{R})_M$  if and only if  $M \in \text{Supp}(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R)$ .<br>
Let  $M \in \text{MSupp}(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R)$ . Then,  $\widetilde{R_M} \neq R_M$ ,  $S_M$  and  $R_M \subset \widetilde{R_M} \subseteq S_M$ .<br>
Let  $R$ how that  $R_M \neq (R)_M$  if and only if  $\Lambda$  $(S/R) \cap MSupp(K)$ 

*Proof* In fact, we are going to show that  $R_M \neq (R)_M$  if and only  $MSupp(S/\overline{R}) \cap MSupp(\overline{R}/R)$ .<br>
Let  $M \in MSupp(S/\overline{R}) \cap MSupp(\overline{R}/R)$ . Then,  $\overline{R_M} \neq R_M$ ,  $S_M$  and  $R_M \subset \overline{R}$ <br>
Since  $R \subset \overline{R}$  is Prüfer, so is  $R_M \subset \overline{R_M}$  by  $\subseteq R_M$  and  $R_M \neq R_M$ . Therefore,  $R_M = S_M$  [\[37,](#page-29-10) Proposition 3.3] since  $R_M$  is local, and then  $\widetilde{R}_M \neq \widetilde{R}_{M_M}$ .  $R_M \neq (R)_M$ . Let  $M \in \mathbb{R}$ <br>Since  $R \subset \hat{I}$ <br> $R_M \neq \hat{R}_M$ .<br> $\overline{R}_M \neq (\overline{R})_M$ .<br>Converse conversely, if  $\overline{R}_M \subset R_M$  is integrally closed It follows that  $\overline{R}_M = \overline{R}_M$ , as we have just<br>  $\neq (\overline{R})_M$ .<br>
Conversely, if  $\overline{R}_M \neq (\overline{R})_M$ , then,  $R_M \neq \overline{R}_M$ , so that  $\overline{R}_M = S_M$ , as we have just<br>  $\overline{$ rufer, so is  $R_M \subset R_M$  by Proposition 1.2,

seen and then  $R_M \subset S_M$  is integrally closed. It follows that  $\overline{R_M} = \overline{R}_M = R_M$ , so that  $R_M \neq (R)_M$ .<br>Conversely, if  $\widetilde{R_M} \neq (\widetilde{R})_M$ , then,  $R_M \neq \widetilde{R_M}$ , so that  $\widetilde{R_M} = S_M$ , as we have just<br>seen and then  $R_M \subset S_M$  is integrally closed. It follows that  $\overline{R_M} = \overline{R_M} = R_M$ , so that<br> $M \notin \text{MSupp}(\overline{R$ *∉* MSupp(*R/R*). Hence, *M* ∈ MSupp(*R/R*) by Proposition 4.18(3). Moreover,<br>  $\neq S_M \Rightarrow M \in \text{MSupp}(S/\widetilde{R})$ . To conclude, *M* ∈ MSupp( $S/\widetilde{R}$ ) ∩ MSupp( $\widetilde{R}/R$ ).<br>
If *R* ⊆ *S* is an extension, with dim(*R*) = 0,  $\widetilde$  $M \notin \text{MSupp}(\overline{R}/R)$ . Hence,  $M \in \text{MSupp}(R/R)$  by Proposition 4.18(3). Moreover,  $\widetilde{R}_M \neq S_M \Rightarrow M \in \text{MSupp}(S/\widetilde{R})$ . To conclude,  $M \in \text{MSupp}(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R)$ .

 $R_M \neq S_M \Rightarrow M \in \text{MSupp}(S/R)$ . To conclude,  $M \in \text{MSupp}(S/R) \cap \text{MSupp}(R/R)$ .<br>
If  $R \subseteq S$  is an extension, with dim( $R$ ) = 0,  $\widetilde{R_M} = (\widetilde{R})_M$  for any  $M \in \text{Max}(R)$ .<br>
Indeed by Scholium A (2), the flat epimorphisms  $R \to \widetilde{R}$  and bijective. This conclusion holds in another context.

<span id="page-19-1"></span>**Corollary 4.20** *Let*  $R \subset S$  *be an FCP extension. Assume that one of the following*<br> *(1)* MSupp $(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R) = \emptyset$ .<br> *(2)*  $S = \overline{RR}$  *or equivalently*  $R \subset S$  *is almost-Priifer conditions is satisfied:*

*(1)*  $MSupp(S/\widetilde{R}) \cap MSupp(\widetilde{R}/R) = \emptyset.$ <br>
(2)  $S = \overline{RR}$ , or equivalently,  $R \subseteq S$  is almost-Prüfer. (1) MSupp $(S/\overline{R}) \cap \text{MSupp}(\overline{R}/R) = \emptyset$ .  $p(S/\widetilde{R}) \cap \text{MSupp}(\widetilde{R}/R) = \emptyset.$ <br>*R*, *or equivalently*,  $R \subseteq S$  *is almotary*  $M \in \text{Max}(R)$ .

<span id="page-19-2"></span>*Then,*  $\widetilde{R_M} = (\widetilde{R})_M$  for any  $M \in \text{Max}(R)$ .

*Proof* (1) is Proposition [4.19.](#page-19-0) (2) is Proposition [4.12.](#page-16-1)

**Proposition 4.21** *Let*  $R \subset S$  *be an almost-Prüfer FCP extension. Then, any*  $T \in$ *Proof* (1) is Proposition 4.19. (2) is Proposition 4.12.<br> **Proposition 4.21** Let  $R \subset S$  be an almost-Prüfer FCP extension. Then, any  $T \in [R, S]$  is the integral closure of  $T \cap \widetilde{R}$  in  $T\widetilde{R}$ . Moreover, if  $T \cap \widet$ **Proposition 4.21** Let  $R \subset S$  be an almost-Prüfer FCP extens  $[R, S]$  is the integral closure of  $T \cap \widetilde{R}$  in  $T\widetilde{R}$ . Moreover, if  $T \cap \widetilde{R} =$ <br>if  $T\overline{R} = S$ , then  $T = (T \cap \overline{R})\widetilde{R}$ ; if  $T\widetilde{R} = S$ , then  $T = (T$ *Proof Proof Proof Reference Archaeosis, <i>Proof Proof Reference R FR* and *P* =  $T \cap \widetilde{R}$  is also the Prifer bull of  $U \subset V$  *Recause*  $R \subset S$  is Prüfer and  $\widetilde{R} \subset V$  is integral and  $\widetilde{R}$  is also th

if  $T\overline{R} = S$ , then  $T = (T \cap \overline{R})\overline{R}$ ; if  $T\overline{R} = S$ , then  $T = (T \cap \overline{R})\overline{R}$ .<br> *Proof* Set  $U := T \cap \overline{R}$  and  $V := T\overline{R}$ . Since  $R \subset S$  is almost-Prüfer,  $U \subseteq \overline{R}$  is Prüfer<br>
and  $\overline{R} \subseteq V$  is integral and  $\overline{$ almost-Prüfer, for each  $M \in \text{MSupp}_R(S/R), R_M \subseteq S_M$  is either integral or Prüfer by<br>Proposition 4.16, and so is  $U \subseteq V$ . But  $\widetilde{R}_{\cdot\cdot} = (\widetilde{R}) \cup$  by Corollary 4.20 is also the Proposition [4.16,](#page-17-0) and so is  $U_M \subseteq V_M$ . But *R*<br>Prijfer bull of  $U_M \subseteq V_M$ . Let *T'* be the inter-*R*  $\subset$  *S* is almost-Prüter,  $U \subseteq R$  is Prüter<br>ifer hull of  $U \subseteq V$ . Because  $R \subset S$  is<br> $R_M \subseteq S_M$  is either integral or Prüfer by<br> $M = (\widetilde{R})_M$  by Corollary [4.20](#page-19-1) is also the<br>paral closure of *U* in *V*. Then *T'* is the Prüfer hull of  $U_M \subseteq V_M$ . Let *T'* be the integral closure of *U* in *V*. Then,  $T'_M$  is the integral closure of  $U_M$  in  $V_M$ . integral closure of *UM* in *VM*.

asi-Prüfer Extensions of Rings<br>Assume that  $U_M \subseteq V_M$  is integral. Then  $V_M = T'_M$  and  $U_M = (\widetilde{R})_M$ , so that  $U_M = T_M(\widetilde{R})_M = T_M$  giving  $T_M = T'_M$ . Assume that  $U_M \subseteq V_M$  is integral<br>  $V_M = T_M(\widetilde{R})_M = T_M$ , giving  $T_M = T'_M$ <br>
Assume that  $U_M \subseteq V_M$  is Prijfer  $V_M = T_M(\overline{R})_M = T_M$ , giving  $T_M = T'_M$ . Assume that  $U_M \subseteq V_M$  is integral. Then  $V_M = T'_M$  and  $U_M = (\widetilde{R})_M$ , so that  $= T_M(\widetilde{R})_M = T_M$ , giving  $T_M = T'_M$ .<br>Assume that  $U_M \subseteq V_M$  is Prüfer. Then  $U_M = T'_M$  and  $V_M = (\widetilde{R})_M$ , so that  $U = T_M \cap (\widetilde{R})_M = T_M$  so that

Assume that  $U_M \subseteq V_M$  is integral. I<br>  $V_M = T_M(\widetilde{R})_M = T_M$ , giving  $T_M = T'_M$ .<br>
Assume that  $U_M \subseteq V_M$  is Prüfer. Then<br>  $U_M = T_M \cap (\widetilde{R})_M = T_M$ , giving  $T_M = T'_M$ .<br>
To conclude  $T_M = T'_M$  follows for ex- $U_M = T_M \cap (R)_M = T_M$ , giving  $T_M = T'_M$ .

To conclude,  $T_M = T'_M$  follows for each  $M \in \text{MSupp}_R(S/R)$ . Since  $R_M = S_M$ ,<br>b  $T_M = T'_M$  for each  $M \in \text{Max}(R) \setminus \text{MSupp}_R(S/R)$ , we get  $T = T'_M$  whence  $T$  is with  $T_M = T'_M$  for each  $M \in \text{Max}(R) \setminus \text{MSupp}_R(S/R)$ , we get  $T = T'$ , whence *T* is the integral closure of  $U \subseteq V$ the integral closure of  $U \subseteq V$ .<br>The last results are then ob

The last results are then obvious.<br>We build an example of an ECP extension  $R \subset S$  where  $\widetilde{R_{\mathcal{U}}} \neq (\widetilde{R})_{\mathcal{U}}$  for some If  $I_M = I_M^{\prime}$  for each  $M \in \text{Max}(R) \setminus \text{MSupp}_R(S/R)$ , we get  $I = I'$ , whence *I* is<br>integral closure of  $U \subseteq V$ .<br>The last results are then obvious.<br>We build an example of an FCP extension  $R \subset S$  where  $\widetilde{R}_M \neq (\widetilde{R})_M$  fo  $M \in \text{Max}(R)$ . In particular,  $R \subset S$  is not almost-Prüfer.

*Example 4.22* Let *R* be an integral domain with quotient field *S* and  $Spec(R) :=$  ${M_1, M_2, P, 0}$ , where  $M_1 \neq M_2$  are two maximal ideals and *P* a prime ideal satisfying  $P \subset M_1 \cap M_2$ . Assume that there are  $R_1$ ,  $R_2$ , and  $R_3$  such that  $R \subset R_1$ is Prüfer minimal, with  $\mathcal{C}(R, R_1) = M_1, R \subset R_2$  is integral minimal, with  $\mathscr{C}(R, R_2) = M_2$ , and  $R_2 \subset R_3$  is Prüfer minimal, with  $\mathscr{C}(R_2, R_3) = M_3 \in \text{Max}(R_2)$ such that  $M_3 \cap R = M_2$  and  $M_2R_3 = R_3$ . This last condition is satisfied when  $R \subset R_2$  is either ramified or inert. Indeed, in both cases,  $M_3R_3 = R_3$ ; moreover, in the ramified case, we have  $M_3^2 \subseteq M_2$  and in the inert case,  $M_3 = M_2$  [\[36,](#page-29-1) Theorem 3.31 We apply 114 Proposition 7.101 and 113 Lemma 2.41 several times. Set 3.3]. We apply [\[14,](#page-28-23) Proposition 7.10] and [\[13,](#page-28-0) Lemma 2.4] several times. Set  $R'_2 := R_1 R_2$ . Then,  $R_1 \subset R'_2$  is integral minimal, with  $\mathcal{C}(R_1, R'_2) =: M'_2 = M_2 R_1$  and  $R_2 \subset R'_1$  is Prijfer minimal, with  $\mathcal{C}(R_2, R'_1) =: M'_1 = M_1 R_2 \in \text{Max}(R_2)$ . Moreover  $R_2 \subset R'_2$  is Prüfer minimal, with  $\mathcal{C}(R_2, R'_2) =: M'_1 = M_1 R_2 \in \text{Max}(R_2)$ . Moreover,<br>  $M' \neq M_2$ . Spec $(R_1) = M'_1, R_2, 0$ , where  $P_1$  is the only prime ideal of  $R_1$  lying  $M'_1 \neq M_3$ , Spec $(R_1) = \{M'_2, P_1, 0\}$ , where  $P_1$  is the only prime ideal of  $R_1$  lying<br>over *P* But  $P = (R \cdot R_1)$  by [17 Proposition 3.3] so that  $P = P_1$  Set  $R'_1 := R_2 R'_2$ over *P*. But,  $P = (R : R_1)$  by [\[17,](#page-28-5) Proposition 3.3], so that  $P = P_1$ . Set  $R'_3 := R_3 R'_2$ .<br>Then  $R'_1 \subset R'_2$  is Prijfer minimal with  $\mathcal{C}(R'_1, R'_2) = M'_2 = M_3 R'_3 \in \text{Max}(R'_3)$  and Then,  $R'_2 \subset R'_3$  is Prüfer minimal, with  $\mathcal{C}(R'_2, R'_3) =: M'_3 = M_3R'_2 \in \text{Max}(R'_2)$  and  $R_2 \subset R'_1$  is Prüfer minimal, with  $\mathcal{C}(R_2, R'_1) = M'' = M_1R_2 \in \text{Max}(R_2)$ . We have  $R_3 \subset R'_3$  is Prüfer minimal, with  $\mathcal{C}(R_3, R'_3) = M''_1 = M_1R_3 \in \text{Max}(R_3)$ . We have therefore  $\text{Spec}(R') = \{P' \text{ O}\}\$  where P' is the only prime ideal of  $R'$  lying over P therefore  $Spec(R'_3) = {P', 0}$  where *P'* is the only prime ideal of  $R'_3$  lying over *P*.<br>To end, assume that  $R' \subset S$  is Prijfer minimal, with  $\mathcal{C}(R' \cap S) = P'$ . Hence  $R_3$  is the To end, assume that  $R'_3 \subset S$  is Prüfer minimal, with  $\mathcal{C}(R'_3, S) = P'$ . Hence,  $R_2$  is the integral closure of R in S. In particular,  $R \subset S$  has ECP [13] Theorems 6.3 and 3.131 integral closure of *R* in *S*. In particular,  $R \subset S$  has FCP [\[13,](#page-28-0) Theorems 6.3 and 3.13]<br>and is quasi-Priifer. Since  $R \subset R$ , is integrally closed, we have  $R_i \subset \widetilde{R}$ . Assume and is quasi-Prüfer. Since  $R \subset R_1$  is integrally closed, we have  $R_1 \subseteq$ <br>that  $R_1 \neq \widetilde{R}$ . Then, there exists  $T \in [R_1, S]$  such that  $R_2 \subset T$  is Prüfer at Fying over *P*.<br>ence,  $R_2$  is the<br> $(6.3 \text{ and } 3.13]$ <br> $\subseteq \widetilde{R}$ . Assume to end, assume that  $R_3 \text{ }\subset S$  is Pruter minimidegral closure of *R* in *S*. In particular, *R*  $\subset$  and is quasi-Prüfer. Since  $R \subset R_1$  is integrant that  $R_1 \neq \widetilde{R}$ . Then, there exists  $T \in [R_1, S]$ <br> $\mathcal{L}(R_1, T) = M'_$ that  $R_1 \neq R$ . Then, there exists  $T \in [R_1, S]$  such that  $R_1 \subset T$  is Prüfer minimal and  $\mathcal{C}(R_1, T) = M'_2$ , a contradiction by Proposition [4.16](#page-17-0) since  $M'_2 = \mathcal{C}(R_1, R'_2)$ , with  $R_1 \subset R'$  integral minimal. Then  $R_2 = \widetilde{R}$  It follows that  $M_1 \subset M\text{Supp}(\widetilde{R}/R)$ . But and is quasi-Pruter. Since  $R \subset R_1$  is integrally closed, we have  $R_1 \subseteq R$ . Assume<br>that  $R_1 \neq \tilde{R}$ . Then, there exists  $T \in [R_1, S]$  such that  $R_1 \subset T$  is Prüfer minimal and<br> $\mathcal{C}(R_1, T) = M'_2$ , a contradiction by Propo that  $R_1 \neq R$ . Then, there exists  $T \in [R_1, S]$  such that  $R_1 \subset T$  is Pruter minimal and  $\mathcal{C}(R_1, T) = M'_2$ , a contradiction by Proposition 4.16 since  $M'_2 = \mathcal{C}(R_1, R'_2)$ , with  $R_1 \subset R'_2$  integral minimal. Then,  $R_1 = \overline$  $\widetilde{R_{M_1}} \neq (\widetilde{R})_{M_1}$  by Proposition 4.19 giving that  $R \subset S$  is not almost-Prüfer.  $K_1$   $\in$   $R'_2$  integral minimal. Then,  $R_1 = \overline{R}$ . It follows that  $M_1 \in \text{MSupp}(S)$ <br>  $\overline{R}_2 = \mathcal{C}(R'_3, S) \cap R \in \text{Supp}(S/\overline{R})$  and  $P \subset M_1$  give  $M_1 \in \text{MSupp}(S)$ <br>  $\overline{M}_1 \neq (\overline{R})M_1$  by Proposition [4.19](#page-19-0) giving that  $R \$ 

We now intend to refine Theorem [4.6,](#page-14-1) following the scheme used in  $[3, 1]$  $[3, 1]$ Proposition 4] for extensions of integral domains.

<span id="page-20-0"></span>**Proposition 4.23** *Let*  $R \subseteq S$  *and*  $U, T \in [R, S]$  *be such that*  $R \subseteq U$  *is integral and*  $R \subseteq T$  *is Priifer Then*  $U \subseteq UT$  *is Priifer in the following cases and*  $R \subseteq UT$  *is*  $R \subseteq T$  is Prüfer. Then  $U \subseteq UT$  is Prüfer in the following cases and  $R \subseteq UT$  is almost-Prüfer *almost-Prüfer. R*  $\subseteq$  *T* is Prüfer. Then  $\overline{U} \subseteq UT$  is Prüfer in the following cases a<br>almost-Prüfer.<br>(1) Supp $(\overline{R}/R) \cap \text{Supp}(\overline{R}/R) = \emptyset$  (for example, if  $R \subseteq S$  has *FCP*).<br>(2)  $R \subseteq U$  preserves integral closure.

 $(2)$   $R \subseteq U$  preserves integral closure.

*Proof* (1) We have  $\emptyset = \text{MSupp}(U/R) \cap \text{MSupp}(T/R)$ , since  $U \subseteq \overline{R}$  and  $T \subseteq \widetilde{R}$ . Let  $M \in \text{MSupp}((UT)/R)$  For  $M \in \text{MSupp}(U/R)$  we have  $R_U = T_U$  and  $(UT)_U =$  $M \in \text{MSupp}((UT)/R)$ . For  $M \in \text{MSupp}(U/R)$ , we have  $R_M = T_M$  and  $(UT)_M =$ <br>*IIM*  $M \notin \text{MSupp}(U/R)$  then  $I|_M = R_M$  and  $(UT)_M = T_M$  so that  $I|_M \subseteq (UT)_M$  $U_M$ . If  $M \notin \text{MSupp}(U/R)$ , then  $U_M = R_M$  and  $(UT)_M = T_M$ , so that  $U_M \subseteq (UT)_M$ <br>identifies to  $R_M \subseteq T_M$ . identifies to  $R_M \subseteq T_M$ .<br>
Let  $N \in \text{Max}(U)$  a

Let  $N \in \text{Max}(U)$  and set  $M := N \cap R \in \text{Max}(R)$  since  $R \subseteq U$  is integral. If  $\neq$  Suppose  $\overline{R}/R$  then  $R_M = \overline{R}_M = U_M$  and N is the only maximal ideal of U  $M \notin \text{Supp}(R/R)$ , then  $R_M = R_M = U_M$  and *N* is the only maximal ideal of *U* ving over *M* It follows that  $U_M = U_M$  and  $(UT)_M = (UT)_M$  by [13] Lemma 2.41 lying over *M*. It follows that  $U_M = U_N$  and  $(UT)_M = (UT)_N$  by [\[13,](#page-28-0) Lemma 2.4]. Then,  $U_N \subseteq (UT)_N$  identifies to  $R_M \subseteq R_M = T_M$  gives  $U_M = (UT)_M$  so the *N*  $\cap$  *K*  $\in$  *Nax(R)* since  $R \subseteq U$  is integral. If  $U_N$  and *N* is the only maximal ideal of *U*,  $U_N$  and  $(UT)_M = (UT)_N$  by [13, Lemma 2.4].<br> $\subseteq T_M$  which is Prüfer. If  $M \notin \text{Supp}(\overline{R}/R)$ , then that  $U_N = (UT)_N$  by locali  $R_M = T_M$  gives  $U_M = (UT)_M$ , so that  $U_N = (UT)_N$  by localizing the precedent equality and  $U_N \subseteq (UT)_N$  is still Prüfer. Therefore,  $U \subseteq UT$  is locally Prüfer, whence Prüfer by Proposition 1.1 whence Prüfer by Proposition [1.1.](#page-3-0)

(2) The usual reasoning gives  $U \otimes_R T \cong UT$ , whence  $U \subseteq UT$  is integrally<br>and Fram  $U \subseteq \overline{P}^{UT}$  we deduce  $U = \overline{P}^{UT}$ . Because  $R \subseteq UT$  is almost Drifter closed. From  $U \subseteq$ <br>whence quasi-Prijf  $U \subseteq \overline{R}^{UT}$ , we deduce  $U = \overline{R}^{UT}$ . Because  $R \subseteq UT$  is almost-Prüfer,<br>nasi-Prüfer,  $U \subseteq UT$  is Prüfer whence quasi-Prüfer,  $U \subseteq$ <br>Next propositions gene **Exerce quasi-Prüfer,**  $U \subseteq UT$  is Prüfer.<br>Next propositions generalize Ayache's results of [\[3,](#page-28-24) Proposition 11].

**Proposition 4.24** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension,*  $T, T' \in [R, S]$  *and*  $U := T \cap T'$  *The following statements hold:*  $T \cap T'$ . The following statements hold: **Proposition 4.24** Let  $R \subseteq S$  be a quantum  $T \cap T'$ . The following statements hold<br>
(1)  $\widetilde{T} = (\overline{T \cap R})$  for each  $T \in [R, S]$ .<br>
(2)  $\widetilde{T} \cap \widetilde{T'} \subseteq T \cap T'$ whence quasi-Prüfer,<br> *Next* propositions<br> **Proposition 4.24** *Le*<br>  $T \cap T'$ . *The following*<br>
(1)  $\widetilde{T} = (\widetilde{T \cap R})$  *for*<br>
(2)  $\widetilde{T} \cap \widetilde{T'} \subseteq T \cap \widetilde{T'}$ <br>
(3) *If* Supp( $\overline{T}/T$ )  $\cap$ 

- $\overline{1}$
- $\nu \subseteq T \cap T'.$
- $\bar{c}$ *(1)*  $\widetilde{T} = \widetilde{(T \cap R)}$  for each  $T \in [R, S]$ .<br> *(2)*  $\widetilde{T} \cap \widetilde{T}' \subseteq T \cap T'$ .<br> *(3)* If Supp $(\overline{T}/T) \cap \text{Supp}(\widetilde{T}/T) = \emptyset$  *(this assumption holds if*  $R \subseteq S$  *has FCP),*<br> *then*  $T \subseteq T' \implies \widetilde{T} \subseteq T'$ *then, T*  $\subseteq$ *<br>If Sunn(L*  $T$  *T*  $\cap$  *T'*.<br>  $T$   $T$   $\cap$  *T'*.<br>  $T$   $T$   $\cap$   $T$   $\subseteq$ <br>  $T'$   $\Rightarrow$   $T$   $\subseteq$ <br>  $\overline{T}$   $\cap$   $T$   $\cap$   $S$   $\cup$  $\subseteq T'.$ *(2)*  $I \cap I' \subseteq I \cap I'$ .<br> *(3)* If Supp $(\overline{T}/T) \cap \text{Supp}(\overline{T}/T) = \emptyset$  (this assum<br>
then,  $T \subseteq T' \Rightarrow \widetilde{T} \subseteq \widetilde{T}'$ .<br> *(4)* If Supp $(\overline{U}/U) \cap \text{Supp}(\widetilde{U}/U) = \emptyset$ , then  $\widetilde{T} \cap \widetilde{T}$  $\widetilde{T}/T$  = Ø (this assum) fision, 1,<br>
tion holds<br>  $= \widetilde{T} \cap \widetilde{T}$ .
- $\overline{a}$

*Proof* (1) We observe that  $R \subseteq T$  is quasi-Prüfer by Corollary [3.3.](#page-7-1) Since  $T \cap \overline{R}$  is the integral closure of *R* in *T*, we get that  $T \cap \overline{R} \subseteq T$  is Prüfer. It follows that  $T \cap \overline{R} \subseteq \widetilde{T}$  is Prüfer. We thus ha is the integral closure of *R* in *T*, we get that  $T \cap R \subseteq T$  is Prüfer. It follows that  $T \cap \overline{R} = \overline{R} \cap \overline{R}$  $T \cap R \subseteq T$  is Prüfer. We thus have  $T \subseteq T \cap R$ . To prove the reverse inclusion, we set  $V := T \cap \overline{R}$  and  $W := \widetilde{V} \cap \overline{T}$ . We have  $W \cap \overline{R} = \widetilde{V} \cap \overline{R} = V$  because  $V \subset \widetilde{V} \cap \overline{R}$  $\subseteq T$ is the integral closure of *R* in *T*, we get that  $T \cap \overline{R} \subseteq T$  is Prüfer. It follows that  $T \cap \overline{R} \subseteq \overline{T}$  is Prüfer. We thus have  $\widetilde{T} \subseteq T \cap \overline{R}$ . To prove the reverse inclusion, we set  $V := T \cap \overline{R}$  and  $W := \widetilde$ is integral and Prüfer since we have a tower  $V \subseteq$ <br>Prüfer because  $W \in [V \ \widetilde{V}]$  Moreover  $T \subset \widetilde{T} \subset V$ Fig. To prove the reverse inclusion, we<br>  $\overline{R} = \overline{V} \cap \overline{R} = V$ , because  $V \subseteq \overline{V} \cap \overline{R}$ <br>  $\subseteq \overline{V} \cap \overline{R} \subseteq \overline{V}$ . Therefore,  $V \subseteq W$  is<br>  $\subseteq \overline{V}$  since  $V \subseteq \overline{T}$  is Prijfer. Then  $P(X \subseteq T \text{ is } Y) := \overline{V} \cap \overline{R}$  and  $W := \overline{V} \cap \overline{T}$ . We have  $W \cap \overline{R} = \overline{V} \cap \overline{R} = V$ , because  $V \subseteq \overline{V} \cap \overline{R}$ <br>is integral and Prüfer since we have a tower  $V \subseteq \overline{V} \cap \overline{R} \subseteq \overline{V}$ . Therefore,  $V \subseteq W$  is<br>Prüfer . Moreover,  $T \subseteq$ <br> $W \in [T, \overline{T}]$  and  $T \subseteq W$  is integral because  $W \in [T, T]$ , and we have  $V \subseteq T \subseteq W$ . This entails that  $T - W - \widetilde{V} \cap \overline{T}$  so that  $T \subseteq \widetilde{V}$  is Prijfer. It follows that  $\widetilde{V} \subseteq \widetilde{T}$  since  $T \in [V, \widetilde{V}]$ Its integral and Pruter since we have a tower  $V \subseteq V \cap R \subseteq V$ . Therefore,  $V \subseteq W$ <br>
Prüfer because  $W \in [V, \tilde{V}]$ . Moreover,  $T \subseteq \tilde{T} \subseteq \tilde{V}$ , since  $V \subseteq \tilde{T}$  is Prüfer. Therefore,  $T \subseteq W$  is integral because  $W \in [T, \overline{T}]$ , .  $\mu = \mu$  $T$ <br> *T*A and Price and Price and Price and Price  $\overline{V} \cap \overline{T}$ ,<br>  $\overline{T} \cap \overline{T}$ <br>  $T \cap T'$ <br>  $T \cap T'$ 

(2) A quasi-Prüfer extension is Prüfer if and only if it is integrally closed. We observe that  $T \cap T' \subseteq T \cap T'$  is integrally closed, whence Prüfer. It follows that  $\widetilde{T} \cap \widetilde{T'} \subseteq T \cap T'$ cause *W*<br>that  $T \subseteq$ <br> $\colon$  extension:<br> $\subseteq \widetilde{T} \cap \widetilde{T}$  $T \cap T'$ <br>(3)

(3) Set  $U = T \cap \overline{R}$  and  $U' = T' \cap \overline{R}$ , so that  $U, U' \in [R, \overline{R}]$  with  $U \subseteq U'$ . In view of (1), we thus can suppose that  $T, T' \in [R, R]$ . It follows that  $T \subseteq T'$  is integral and  $T \subset \widetilde{T}$  is Prijfer so *T*  $\cap$   $T \subseteq T \cap T$ .<br>
(3) Set  $U = T \cap \overline{R}$  and  $U' = T' \cap \overline{R}$ , so that  $U, U' \in [R, \overline{R}]$  with  $U \subseteq U'$ . In view<br>
of (1), we thus can suppose that *T*,  $T' \in [R, \overline{R}]$ . It follows that  $T \subseteq T'$  is integral and<br>  $T \subseteq \widetilde{T}$  (3) Set  $U = T \cap R$  and  $U' = T' \cap R$ , so that  $U, U' \in [R, R]$  with  $U \subseteq U'$ . In view<br>of (1), we thus can suppose that  $T, T' \in [R, \overline{R}]$ . It follows that  $T \subseteq T'$  is integral and<br> $T \subseteq \widetilde{T}$  is Prüfer. We deduce from Proposition 4.  $\subseteq T'$ , because  $\text{Supp}(T/T) \cap \text{Supp}(T/T) = \emptyset$ of (1), we<br>  $T \subseteq \widetilde{T}$  is<br>
that  $\widetilde{T}T'$ <br>
have  $\widetilde{T} \subseteq$ <br>
(4) As  $\subseteq T'.$ S<sub>1</sub> (3) Set  $U = T \cap \overline{R}$  and  $U' = T' \cap \overline{R}$ , so that  $U, U' \in$ <br>of (1), we thus can suppose that  $T, T' \in [R, \overline{R}]$ . It follow<br> $T \subseteq \widetilde{T}$  is Prüfer. We deduce from Proposition 4.23(1)<br>that  $\widetilde{T}T' \subseteq \widetilde{T}'$ , because Supp $(\overline$ 

(4) Assume that  $\text{Supp}(U/U) \cap \text{Supp}(U/U) = \emptyset$ . Then,  $T \cap T' \subset T$ ,  $T'$  gives  $\subseteq \widetilde{T}'$ .<br>
Assume 1<br>  $\subseteq \widetilde{T} \cap \widetilde{T}$  $\widetilde{T \cap T'} \subset \widetilde{T} \cap \widetilde{T'}$  in view of (3), so that  $\widetilde{T \cap T'} = \widetilde{T} \cap \widetilde{T'}$  by (2).

**Proposition 4.25** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension and*  $T \subseteq T'$  *a subextension* of  $R \subseteq S$  *Set*  $U := T \cap \overline{R}$  *II'*  $:= T' \cap \overline{R}$  *V*  $:= T\overline{R}$  and  $V' := T'\overline{R}$  *The following of R*  $\subseteq$  *S*. *Set U* := *T*  $\cap$  *R*, *U'* := *T'*  $\cap$  *R*, *V* := *TR and V'* := *T'R. The following statements hold statements hold:*

- *(1)*  $T \subseteq T'$  is integral if and only if  $V = V'$ .<br>*(2)*  $T \subset T'$  is Priifer if and only if  $U = U'$
- *(2)*  $T \subseteq T'$  is Prüfer if and only if  $U = U'.$ <br>*(3)* Assume that  $U \subset U'$  is integral minime
- *(3) Assume that*  $U \subset U'$  *is integral minimal and*  $V = V'$ *. Then,*  $T \subset T'$  *is integral minimal of the same type as*  $U \subset U'$ *minimal, of the same type as*  $U \subset U'$ *.*<br>Assume that  $V \subset V'$  is Prüfer minin
- *(4) Assume that*  $V \subset V'$  *is Prüfer minimal and*  $U = U'$ *. Then,*  $T \subset T'$  *is Prüfer* minimal *minimal.*
- *(5)* Assume that  $T \subset T'$  is minimal and set  $P := C(T, T')$ .
	- *(a)* If  $T \subset T'$  *is integral, then*  $U \subset U'$  *is integral minimal if and only if*  $P \cap U \in$  $Max(U)$ .
	- *(b)* If  $T \subset T'$  is Prüfer, then  $V \subset V'$  is Prüfer minimal if and only if there is *exactly one prime ideal in V lying over P.*

*Proof* In [*R*, *S*], the extensions  $U \subseteq U'$ ,  $T \subseteq V$ ,  $T' \subseteq V'$  are integral and  $V \subseteq V'$  *U*  $\subset T$  *U'*  $\subset T'$  are Prifer Moreover  $\overline{R}$  is also the integral closure of  $U \subset V'$ *V'*,  $U \subseteq T$ ,  $U' \subseteq T'$  are Prüfer. Moreover, *R* is also the integral closure of  $U \subseteq V'$ .<br>(1) is gotten by considering the extension  $T \subseteq V'$  which is both  $T \subseteq V \subseteq V'$ .

(1) is gotten by considering the extension  $T \subseteq V'$ , which is both  $T \subseteq V \subseteq V'$ <br>  $T \subset T' \subset V'$ and  $T \subseteq T' \subseteq V'$ .<br>(2) is gotten by

(2) is gotten by considering the extension  $U \subseteq T'$ , which is both  $U \subseteq T \subseteq T'$ <br> $H \cup \subset H' \subset T'$ and  $U \subseteq U' \subseteq T'$ .<br>(3) Assume the

(3) Assume that  $U \subset U'$  is integral minimal and  $V = V'$ . Then,  $T \subset T'$  is equal by (1) and  $T \neq T'$  because of (2) Set  $M := (U \cdot U') \in \text{Supp}_x(U'/U)$ . For integral by (1) and  $T \neq T'$  because of (2). Set  $M := (U : U') \in \text{Supp}_U(U'/U)$ . For  $U \subseteq M \times (U)$  such that  $M' \neq M$  we have  $U \cup U' = U' \cup S$  to that  $T \cup U = T' \cup U$ any  $M' \in \text{Max}(U)$  such that  $M' \neq M$ , we have  $U_{M'} = U'_{M'}$ , so that  $T_{M'} = T'_{M'}$ <br>because  $U_{M'} \subset T'$  is Prijfer. But  $U \subset T'$  is almost-Prijfer, giving  $T' = T U'$ because  $U_{M'} \subseteq T'_{M'}$  is Prüfer. But,  $U \subseteq T'$  is almost-Prüfer, giving  $T' = TU'$ .<br>By Theorem 4.6  $(T \cdot T') - (U \cdot U')T - MT \neq T$  because  $T \neq T'$  We get By Theorem [4.6,](#page-14-1)  $(T : T') = (U : U')T = MT \neq T$  because  $T \neq T'$ . We get that  $U \subset T$  prijfer implies that  $M \notin$  Supp  $(T/U)$  and  $U_{U} = T_{U}$ . It follows that that  $U \subseteq T$  Prüfer implies that  $M \notin \text{Supp}_U(T/U)$  and  $U_M = T_M$ . It follows that  $T' \cup T' \cup T'' \cup T'' \cup T''$  Therefore  $T \cup T' \cup T''$  identifies to  $U \cup T'' \cup T''$  which is  $T'_M = T_M U'_M = U'_M$ . Therefore,  $T_M \subseteq T'_M$  identifies to  $U_M \subseteq U'_M$ , which is minimal of the same type as  $U \subset U'$  by [14] Proposition 4.61. Then  $T \subset T'$  is minimal of the same type as  $U \subset U'$  by [\[14,](#page-28-23) Proposition 4.6]. Then,  $T \subset T'$  is integral minimal of the same type as  $U \subset U'$ .<br>(4) Assume that  $V \subset V'$  is Prijfer minimal

(4) Assume that  $V \subset V'$  is Prüfer minimal and  $U = U'$ . Then,  $T \subset T'$  is Prüfer (2) and  $T \neq T'$  because of (1) Set  $\Omega := C(V, V')$  and  $P := \Omega \cap T \in \text{Max}(T)$ by (2) and  $T \neq T'$  because of (1). Set  $Q := C(V, V')$  and  $P := Q \cap T \in \text{Max}(T)$ <br>since  $Q \in \text{Max}(V)$ . For any  $P' \in \text{Max}(T)$  such that  $P' \neq P$  and  $Q' \in \text{Max}(V)$  lying since  $Q \in \text{Max}(V)$ . For any  $P' \in \text{Max}(T)$  such that  $P' \neq P$ , and  $Q' \in \text{Max}(V)$  lying. over *P'*, we have  $V_{Q'} = V'_{Q'}$ , so that  $V_{P'} = V'_{P'}$ . Therefore,  $T'_{P'} \subseteq V'_{P'}$  is integral, so that  $T_{P'} = T'_{P'}$  and  $P' \notin$  Sunn- $(T'/T)$ . Hence  $T \subset T'$  is Prijfer minimal 113 so that  $T_{P'} = T'_{P'}$  and  $P' \notin \text{Supp}_T(T'/T)$ . Hence  $T \subset T'$  is Prüfer minimal [\[13,](#page-28-0) Proposition 6.12] Proposition 6.12].

(5) Assume that  $T \subset T'$  is a minimal extension and set  $P := \mathcal{C}(T, T')$ .<br>(a) Assume that  $T \subset T'$  is integral. Then  $V = V'$  and  $U \neq U'$  by (

(a) Assume that  $T \subset T'$  is integral. Then,  $V = V'$  and  $U \neq U'$  by (1) and (2). We can use Proposition [4.5](#page-14-0) getting that  $P = (U : U')T \in \text{Max}(T)$  and  $Q := (U : U') - P \cap U \in \text{Spec}(U)$ . It follows that  $O \notin \text{Sum}_U(T/|U|)$  so that  $U_Q = T_Q$  and  $U'$  =  $P \cap U \in \text{Spec}(U)$ . It follows that  $Q \notin \text{Supp}_U(T/U)$ , so that  $U_Q = T_Q$  and  $U' = T'$ . Then  $U_Q \subset U'$  is integral minimal, with  $Q \in \text{Supp}_U(U'/U)$ .  $U_Q' = T_Q'$ . Then,  $U_Q \subset U_Q'$  is integral minimal, with  $Q \in \text{Supp}_U(U'/U)$ .<br>
If  $Q \notin \text{Max}(U)$ , then  $U \subset U'$  is not minimal by the properties of

If  $Q \notin \text{Max}(U)$ , then  $U \subset U'$  is not minimal by the properties of the crucial maximal ideal.

Assume that  $Q \in \text{Max}(U)$  and let  $M \in \text{Max}(U)$ , with  $M \neq Q$ . Then,  $U_M = U'_M$ <br>cause  $M + Q = U$  so that  $U \subset U'$  is a minimal extension and (a) is gotten because  $M + Q = U$ , so that  $U \subset U'$  is a minimal extension and (a) is gotten.

(b) Assume that  $T \subset T'$  is Prüfer. Then,  $V \neq V'$  and  $U = U'$  by (1) and (2). Moreover,  $PT' = T'$  gives  $PV' = V'$ . Let  $O \in \text{Max}(V)$  lying over P. Then, (2). Moreover,  $PT' = T'$  gives  $PV' = V'$ . Let  $Q \in \text{Max}(V)$  lying over *P*. Then,  $OV' = V'$  gives that  $Q \in \text{Sum}_V(V' / V)$ . Moreover, we have  $V' = VT'$ . Let  $P' \in V'$  $QV' = V'$  gives that  $Q \in \text{Supp}_V(V'/V)$ . Moreover, we have  $V' = VT'$ . Let  $P' \in \text{Max}(T)$ ,  $P' \neq P$ . Then  $T_{P'} = T'$ , gives  $V_{P'} = V'$ . It follows that  $\text{Sum}_V(V'/V) =$  $\text{Max}(T), P' \neq P.$  Then,  $T_{P'} = T'_{p'}$  gives  $V_{P'} = V'_{p'}$ . It follows that  $\text{Supp}_T(V'/V) = P$ <br>*P*<sub>2</sub> and Supp<sub>*L*</sub>(*V'*/*V*) = *{Q*  $\in$  Max(*V*) | *Q*  $\cap$  *T* = *P*}. But by [13] Proposition  $\{P\}$  and Supp<sub>*V*</sub> $(V'/V) = \{Q \in \text{Max}(V) \mid Q \cap T = P\}$ . But, by [\[13,](#page-28-0) Proposition 6.121  $V \subset V'$  is Prifer minimal if and only if  $|\text{Sum}_V(V'/V)| = 1$  and then if and 6.12],  $V \subset V'$  is Prüfer minimal if and only if  $|\text{Supp}_V(V'/V)| = 1$ , and then if and only if there is exactly one prime ideal in V lying over P only if there is exactly one prime ideal in *V* lying over *P*.  $\square$ <br>This proposition has a simpler dual form in the FCP almost-Prüfer case.

**Proposition 4.26** *Let*  $R \subseteq S$  *be an FCP almost-Prüfer extension and*  $T \subseteq S$  *subertension of*  $R \subseteq S$  *Set*  $U := T \cap \widetilde{R}$   $U' := T' \cap \widetilde{R}$   $V := T\widetilde{R}$  and  $V' := T\widetilde{R}$ **Proposition 4.26** Let  $R \subseteq S$  be an FCP almost-Prüfer extension and  $T \subseteq T'$  a This proposition has a simpler dual form in the FCP almost-Prüfer case.<br>**Proposition 4.26** Let  $R \subseteq S$  be an FCP almost-Prüfer extension and  $T \subseteq T'$  a subextension of  $R \subseteq S$ . Set  $U := T \cap \widetilde{R}$ ,  $U' := T' \cap \widetilde{R}$ ,  $V := T\widetilde{$ *The following statements hold:*

*(1)*  $T \subseteq T'$  is integral (and minimal) if and only if  $U = U'$  (and  $V \subset V'$  is minimal).<br>*(2)*  $T \subset T'$  is Prifer (and minimal) if and only if  $V = V'$  (and  $U \subset U'$  is minimal). (2)  $T \subseteq T'$  *is Prüfer (and minimal) if and only if*  $V = V'$  *(and*  $U \subset U'$  *is minimal).* 

*Proof* In view of Proposition [4.21,](#page-19-2) *T* (resp. *T'*) is the integral closure of *U* (resp. *U'*) in *V* (resp. *V*<sup> $\prime$ </sup>). The result is gotten by localizing at the elements of  $MSupp_U(V'/U)$ and using Proposition [4.16.](#page-17-0)  $\Box$ 

<span id="page-23-0"></span>**Lemma 4.27** *Let*  $R \subseteq S$  *be an FCP almost-Prüfer extension and*  $U \in [R, R]$ ,  $V \in \overline{R}$  *SI Then II*  $\subset V$  *has FCP and is almost-Prüfer. The same result holds when*  $[R, S]$ . Then  $U \subseteq V$  has FCP and is almost-Prüfer. The same result holds when  $U \in [R, \widetilde{R}]$  and  $V \in [\widetilde{R}, S]$ **Lemma 4.27** Let  $R \subseteq S$ <br>  $[\overline{R}, S]$ . Then  $U \subseteq V$  has<br>  $U \in [R, \widetilde{R}]$  and  $V \in [\widetilde{R}, S]$  $U \in [R, R]$  and  $V \in [R, S]$ .

*Proof* Assume first that  $U \in [R, R]$  and  $V \in [R, S]$ . Obviously,  $U \subseteq V$  has FCP and  $\overline{R}$  is the integral closure of  $U$  in  $V$  Proposition 4.16 entails that Supp.  $(\overline{R}/R)$ and  $\overline{R}$  is the integral closure of *U* in *V*. Proposition [4.16](#page-17-0) entails that Supp<sub>*R*</sub> $(\overline{R}/R) \cap$  $\text{Supp}_R(S/\overline{R}) = \emptyset$ . We claim that  $\text{Supp}_U(\overline{R}/U) \cap \text{Supp}_U(V/\overline{R}) = \emptyset$ . Deny and let  $Q \in \text{Supp}_{U}(\overline{R}/U) \cap \text{Supp}_{U}(\overline{V}/\overline{R})$ . Then,  $\overline{R}_{Q} \neq U_{Q}$ ,  $V_{Q}$ . If  $P := Q \cap R$ , we get that  $\overline{R}_P \neq U_P$ ,  $V_P$ , giving  $\overline{R}_P \neq R_P$ ,  $S_P$ , a contradiction. Another use of Proposition [4.16](#page-17-0) shows that  $U \subseteq V$  is almost-Prüfer. The second result is obvious. shows that  $U \subseteq V$  is almost-Prüfer. The second result is obvious.

**Theorem 4.28** *Let*  $R \subseteq S$  *be an FCP almost-Prüfer extension and*  $T \subseteq T'$  *a subertension of*  $R \subseteq S$  *Set*  $U := T \cap \overline{R}$  *and*  $V' := T' \overline{R}$  *Let* W be the Prüfer  $subextension$  of  $R \subseteq S$ . Set  $U := T \cap R$  and  $V' := T'R$ . Let W be the Prüfer<br>bull of  $U \subseteq V'$ . Then W is also the Prüfer hull of  $T \subseteq T'$  and  $T \subseteq T'$  is an ECP *hull of U*  $\subseteq$  *V'*. Then, *W* is also the Prüfer hull of  $T \subseteq T'$  and  $T \subseteq T'$  is an FCP almost-Prüfer extension *almost-Prüfer extension.*

*Proof* By Lemma [4.27,](#page-23-0) we get that  $U \subseteq V'$  is an FCP almost-Prüfer extension. Let  $\widetilde{T}$  be the Prijfer bull of  $T \subseteq T'$  Since  $U \subseteq T$  and  $T \subseteq \widetilde{T}$  are Prijfer so is  $U \subseteq \widetilde{T}$  and  $T \subseteq T'$ *The textension.*<br> *T* be the Prüfer extension. Let  $\widetilde{T}$  be the Prüfer hull of  $T \subseteq T'$ . Since  $U \subseteq T$  and  $T \subseteq \widetilde{T}$  are Prüfer, so is  $U \subseteq \widetilde{T}$  and  $\widetilde{T} \subseteq V'$  gives that  $\widetilde{T} \subseteq W$ . Then  $T \subseteq W$  is Prüfer as a su *T*  $\subseteq$  *V'* gives that *T*  $\subseteq$  *W*. Then, *T*  $\subseteq$  *W* is Prüfer as a subextension of *U*  $\subseteq$  *W*.<br>Moreover in view of Proposition 4.17 *W* is the least *I*-subalgebra of *V' V* gives the Prüfer hull of  $\subseteq V'$  gives that  $\widetilde{T} \subseteq V'$  gives that  $\widetilde{T} \subseteq V'$  Moreover in view

Moreover, in view of Proposition  $4.17$ , *W* is the least *U*-subalgebra of *V'* over which *V'* is integral. Since  $T' \subseteq V'$  is integral, we get that  $W \subseteq T'$ , so that  $W \in T'$  with  $W \subset T'$  integral as a subextension of  $W \subset V'$ . It follows that *W* is also  $[T, T']$ , with  $W \subseteq T'$  integral as a subextension of  $W \subseteq V'$ . It follows that *W* is also<br>the Prijfer bull of  $T \subseteq T'$  and  $T \subseteq T'$  is an ECP almost-Prijfer extension the Prüfer hull of  $T \subseteq T'$  and  $T \subseteq T'$  is an FCP almost-Prüfer extension.  $\square$ 

*Remark 4.29* The result of this theorem may not hold if the FCP hypothesis is lacking. Take the example of [\[13,](#page-28-0) Remark 2.9(c)], where  $R \subseteq S \subseteq T$  is almost-<br>Prijfer  $R \subset S$  Prijfer  $S \subset T$  integral and  $R \subset T$  has not ECP Here  $(R, M)$  is a Prüfer,  $R \subseteq S$  Prüfer,  $S \subseteq T$  integral and  $R \subseteq T$  has not FCP. Here,  $(R, M)$  is a

one-dimensional valuation domain with quotient field *S* and  $T = S[X]/(X^2) = S[x]$ .<br>Set  $R' := R[x]$ . Then  $R'$  is local, with  $Snec(R') = \{P' := Rx \mid M' := M + Rx\}$ . Set  $R' := R[x]$ . Then,  $R'$  is local, with Spec $(R') = \{P' := Rx, M' := M + Rx\}$ .<br>By the characterization of a Prijfer extension in Proposition 1.2.(3)  $R' = \widetilde{R'}$  but By the characterization of a Prüfer extension in Proposition [1.2](#page-3-1) (3),  $R' = R'$ , but  $R' \subset T$  is not integral, so that  $R' \subset T$  is not almost-Prüfer  $R' \subset T$  is not integral, so that  $R' \subset T$  is not almost-Prüfer.

#### <span id="page-24-0"></span>**5 Fibers of Quasi-Prüfer Extensions**

We intend to complete some results of Ayache-Dobbs [\[5\]](#page-28-4). We begin by recalling some features about quasi-finite ring morphisms. A ring morphism  $R \rightarrow S$  is called quasi-finite by [\[39\]](#page-29-12) if it is of finite type and  $\kappa(P) \to \kappa(P) \otimes_R S$  is finite (as a  $\kappa(P)$ vector space), for each  $P \in \text{Spec}(R)$  [\[39,](#page-29-12) Proposition 3, p. 40].

<span id="page-24-2"></span>**Proposition 5.1** *A ring morphism of finite type is incomparable if and only if it is quasi-finite and, if and only if its fibers are finite.*

*Proof* Use  $[40,$  Corollary 1.8] and the above definition.  $\Box$ 

<span id="page-24-1"></span>**Theorem 5.2** An extension  $R \subseteq S$  is quasi-Prüfer if and only if  $R \subseteq T$  is quasi-<br>finite (equivalently, has finite fibers) for each  $T \in \{R, S\}$  such that  $T$  is of finite type *finite (equivalently, has finite fibers) for each*  $T \in [R, S]$  *such that T is of finite type*<br>over R if and only if  $R \subset T$  has integral fiber morphisms for each  $T \in [R, S]$ *over R, if and only if*  $R \subseteq T$  *has integral fiber morphisms for each*  $T \in [R, S]$ *.* 

*Proof* Clearly,  $R \subseteq S$  is an INC-pair implies the condition by Proposition [5.1.](#page-24-2) To prove the converse, write  $T \in [R, S]$  as the union of its finite type *R*-subalgebras prove the converse, write  $T \in [R, S]$  as the union of its finite type *R*-subalgebras  $T \in \text{Now}$  let  $Q \subset Q'$  be prime ideals of  $T$  lying over a prime ideal  $P$  of  $R$  and set *T*<sup>a</sup>. Now let  $Q \subseteq Q'$  be prime ideals of *T*, lying over a prime ideal *P* of *R* and set  $Q \nightharpoonup Q \cap T$  and  $Q' \nightharpoonup Q' \cap T$  if  $R \subset T$  is quasi-finite then  $Q = Q'$  so that  $Q_{\alpha} := Q \cap T_{\alpha}$  and  $Q'_{\alpha} := Q' \cap T_{\alpha}$ . If  $R \subseteq T_{\alpha}$  is quasi-finite, then  $Q_{\alpha} = Q'_{\alpha}$ , so that  $Q - Q'$  and then  $R \subseteq T$  is incomparable. The last statement is Proposition 3.8  $Q = Q'$  and then  $R \subseteq T$  is incomparable. The last statement is Proposition [3.8.](#page-8-3)  $\Box$ 

**Corollary 5.3** *An integrally closed extension is Prüfer if and only if each of its*  $sub extensions R \subseteq T$  *of finite type has finite fibers.* 

*Proof* It is enough to observe that the fibers of a (flat) epimorphism have a cardinal  $\leq$  1, because an epimorphism is spectrally injective.  $\Box$ <br>An extension  $R \subset S$  is called *strongly affine* if each of its subextensions  $R \subset T$ 

An extension  $R \subseteq S$  is called *strongly affine* if each of its subextensions  $R \subseteq T$ <br>of finite type. The above considerations show that in this case  $R \subseteq S$  is quasiis of finite type. The above considerations show that in this case  $R \subseteq S$  is quasi-<br>Prijfer if and only if each of its subextensions has finite fibers. For example, an ECP Prüfer if and only if each of its subextensions has finite fibers. For example, an FCP extension is strongly affine and quasi-Prüfer. We are also interested in extensions  $R \subseteq S$  that are not necessarily strongly affine, whose subextensions have finite fibers fibers.

<span id="page-24-3"></span>Next lemma will be useful, its proof is obvious.

**Lemma 5.4** *Let*  $R \subseteq S$  *be an extension and*  $T \in [R, S]$ *.* 

- *(1)* If  $T \subseteq S$  is spectrally injective and  $R \subseteq T$  has finite fibers, then  $R \subseteq S$  has finite fibers *fibers.*
- *(2)* If  $R \subseteq T$  is spectrally injective, then  $T \subseteq S$  has finite fibers if and only if  $R \subseteq S$  has finite fibers *has finite fibers.*

*Remark 5.5* Let  $R \subseteq R$ <br>extension  $T := \widetilde{R} \subset$ *Remark* 5.5 Let  $R \subseteq S$  be an almost-Prüfer extension, such that the integral Extension *T* :=  $\overline{R}$   $\subseteq$  *S* be an almost-Prüfer extension, such that the integral extension *T* :=  $\overline{R}$   $\subseteq$  *S* has finite fibers and let *P*  $\in$  Spec(*R*). The study of the finiteness of Fibers (*P*) can be finiteness of Fib<sub>R<sub>*S</sub>*</sub>(*P*) can be reduced as follows. As  $R \subseteq S$  is an epimorphism, because it is Prijfer it is spectrally injective (see Scholium A). The hypotheses of</sub> because it is Prüfer, it is spectrally injective (see Scholium A). The hypotheses of Proposition [4.5](#page-14-0) hold. We examine three cases. In case  $(R : \overline{R}) \nsubseteq P$ , it is well known that  $R_P = (\overline{R})_P$  so that  $|\text{Fib}_{RS}(P)| = 1$ , because  $\overline{R} \rightarrow S$  is spectrally injective. Suppose now that  $(R : \overline{R}) = P$ . From  $(R : \overline{R}) = (T : S) \cap R$ , we deduce that *P* is lain over by some  $Q \in \text{Spec}(T)$  and then  $\text{Fib}_{R\overline{R}}(P) \cong \text{Fib}_{T,S}(Q)$ . The conclusion follows as above. Thus the remaining case is  $(R : \overline{R}) \subset P$  and we can assume that  $PT = T$  for if not  $\text{Fib}_{R,\overline{R}}(P) \cong \text{Fib}_{T,S}(Q)$  for some  $Q \in \text{Spec}(T)$  by Scholium A (1).<br>**Proposition 5.6** *Let*  $R \subseteq S$  *be an almost-Prü*  $PT = T$  for if not Fib<sub>R</sub> $\overline{R}(P) \cong$  Fib<sub>*T*,*S*</sub>(*Q*) for some  $Q \in$  Spec(*T*) by Scholium A (1).

**Proposition 5.6** *Let R*  $\subseteq$  *morphisms and*  $(\widetilde{R}_P \cdot S_P)$ **Proposition 5.6** Let  $R \subseteq S$  be an almost-Prüfer extension. If  $\widetilde{R} \subseteq S$  has finite fiber *PT* = *T* for if not Fib<sub>R,R</sub>(*P*)  $\cong$  Fib<sub>T,S</sub>(*Q*) for some *Q*  $\in$  Spec(*T*) by Scholium A (1).<br>**Proposition 5.6** *Let*  $R \subseteq S$  *be an almost-Prüfer extension. If*  $\widetilde{R} \subseteq S$  *has finite fiber morphisms and*  $(\wid$  $R \subseteq R$  and  $R \subseteq S$  have finite fibers.

*Proof* The Prüfer closure commutes with the localization at prime ideals by Proposition [4.12.](#page-16-1) We set  $T := \widetilde{R}$ . Let *P* be a prime ideal of *R* and  $\varphi : R \to R_P$  the canonical morphism. We clearly have  $Fib_{R_{\mu}}(P) = {}^a\varphi(Fib_{R_{R_{\mu},p}}(PR_P))$ . Therefore, we can localize the data at *P* and we can assume that *R* is local.

In case  $(T : S) = T$ , we get a factorization  $R \to \overline{R} \to T$ . Since  $R \to T$  is Prüfer so is  $R \to \overline{R}$  and it follows that  $R = \overline{R}$  because a Prüfer extension is integrally closed.

From Proposition [1.2](#page-3-1) applied to  $R \subseteq T$ , we get that there is some  $\mathfrak{P} \in \text{Spec}(R)$ <br>th that  $T = R_{\mathfrak{D}} \cdot R/\mathfrak{N}$  is a valuation ring with quotient field  $T/\mathfrak{N}$  and  $\mathfrak{N} = \mathfrak{N}T$ . It such that  $T = R_{\Re}$ ,  $R/\Re$  is a valuation ring with quotient field  $T/\Re$  and  $\Re = \Re T$ . It follows that  $(T : S) = \mathfrak{P}T = \mathfrak{P} \subseteq R$ , and hence  $(T : S) = (T : S) \cap R = (R : R)$ .<br>We have therefore a pushout diagram by Theorem 4.6; We have therefore a pushout diagram by Theorem [4.6:](#page-14-1)

$$
R' := R/\mathfrak{P} \longrightarrow \overline{R}/\mathfrak{P} := \overline{R'}
$$

$$
\downarrow \qquad \qquad \downarrow
$$

$$
T' := T/\mathfrak{P} \longrightarrow S/\mathfrak{P} := S'
$$

where  $R/\mathfrak{P}$  is a valuation domain,  $T/\mathfrak{P}$  is its quotient field, and  $\overline{R}/\mathfrak{P} \rightarrow S/\mathfrak{P}$  is Prüfer by [\[25,](#page-28-1) Proposition 5.8, p. 52].

Because  $\overline{R}^{\prime} \rightarrow S^{\prime}$  is injective and a flat epimorphism, there is a bijective map  $\text{Min}(S') \rightarrow \text{Min}(R')$ . But  $T' \rightarrow S'$  is the fiber at  $\mathfrak{P}$  of  $T \rightarrow S$  and is therefore<br>finite. Therefore,  $\text{Min}(S')$  is a finite set  $\{N, \dots, N\}$  of maximal ideals lying over finite. Therefore,  $\text{Min}(S')$  is a finite set  $\{N_1, \ldots, N_n\}$  of maximal ideals lying over<br>the minimal prime ideals *{M}*  $\qquad \qquad M$  } of  $\overline{R'}$  lying over 0 in  $R'$ . We infer from the minimal prime ideals  $\{M_1, \ldots, M_n\}$  of *R'* lying over 0 in *R'*. We infer from<br>Lemma 3.7 that  $\overline{R^{\prime}}/M \rightarrow S^{\prime}/N$  is Prijfer, whence integrally closed. Therefore Lemma [3.7](#page-8-0) that  $R'/M_i \rightarrow S'/N_i$  is Prüfer, whence integrally closed. Therefore,  $\overline{R}/M_i$  is an integral domain and the integral closure of  $R'$  in  $S'/N_i$ . Any maximal  $R'/M_i$  is an integral domain and the integral closure of  $R'$  in  $S'/N_i$ . Any maximal ideal *M* of  $\overline{R}$ <sup>*i*</sup> contains some  $M_i$ . To conclude it is enough to use a result of Gilmer [\[19,](#page-28-25) Corollary 20.3] because the number of maximal ideals in  $R'/M_i$  is less than the separable degree of the extension of fields  $T' \subseteq S'$  $/N_i$ .

#### *Remark 5.7*

- Remark 5.7<br>(1) Suppose that  $(\widetilde{R} : S)$  is a maximal ideal of  $\widetilde{R}$ . We clearly have  $(\widetilde{R} : S)_P \subseteq (\widetilde{R}_P : S_P)$ <br> $S_P$  and the hypotheses on  $(\widetilde{R} : S)$  of the above proposition hold *Separanceries*  $S$ / *Separaceries is a maximal ideal of*  $\widetilde{R}$ . We clearly have  $\widetilde{R}(S_P)$  and the hypotheses on  $(\widetilde{R}:S)$  of the above proposition hold.<br>In case  $\widetilde{R} \subset S$  is a tower of finitely many integral (1) Suppose that<br>  $S_P$ ) and the<br>
(2) In case  $\widetilde{R} \subseteq$ <br>
with  $M =$
- $\subseteq$  *S* is a tower of finitely many integral minimal extensions  $R_{i-1} \subseteq R_i$ 1  $\succeq$ <br>wh Suppose that  $(K : S)$  is a maximal ideal of  $K$ . We clearly have  $(K : S)_P \subseteq (K_P : S_P)$  and the hypotheses on  $(\widetilde{R} : S)$  of the above proposition hold.<br>In case  $\widetilde{R} \subseteq S$  is a tower of finitely many integral minimal extension *N<sub>P</sub>*) and the hypotheses on  $(K : S)$  of the above proposition hold.<br>
In case  $\widetilde{R} \subseteq S$  is a tower of finitely many integral minimal extensions  $R_{i-1}$ <br>
with  $M_i = (R_{i-1} : R_i)$ , then  $\text{Supp}_{\widetilde{R}}(S/\widetilde{R}) = \{N_1, \ldots, N_n\$  $\subseteq$  *S* is integral minimal and the above result may apply. This generalizes the Ayachewith  $M_i = (R_{i-1} : R_i)$ , then Supp<sub>R</sub>(S/*F*<br>  $N_i = M_i \cap \overline{R}$ . If the ideals  $N_i$  are differe<br>
integral minimal and the above result ma<br>
Dobbs result [\[5,](#page-28-4) Lemma 3.6], where  $\overline{R} \subseteq$ Dobbs result [5, Lemma 3.6], where  $\widetilde{R} \subseteq S$  is supposed to be integral minimal.

<span id="page-26-0"></span>**Theorem 5.8** *Let*  $R \subseteq S$  *be a quasi-Prüfer ring extension. The following three* conditions are equivalent: *conditions are equivalent:*

- (1)  $R \subseteq S$  has finite fibers.<br>(2)  $R \subseteq \overline{R}$  has finite fibers.
- (2)  $R \subseteq R$  has finite fibers.<br>(3) Fach extension  $R \subset T$
- *(3)* Each extension  $R \subseteq T$ , where  $T \in [R, S]$  has finite fibers.

*Proof* (1)  $\Leftrightarrow$  (2) Let *P*  $\in$  Spec(*R*) and the morphisms  $\kappa(P) \rightarrow \kappa(P) \otimes_R R \rightarrow$  $\kappa(P) \otimes_R S$ . The first (second) morphism is integral (a flat epimorphism) because deduced by base change from the integral morphism  $R \to \overline{R}$  (the flat epimorphism  $\overline{R}$   $\rightarrow$  *S*). Therefore, the ring  $\kappa(P) \otimes_R \overline{R}$  is zero-dimensional, so that the second morphism is surjective by Scholium A (2). Set  $A := \kappa(P) \otimes_R \overline{R}$  and  $B := \kappa(P) \otimes_R S$ , we thus have a module finite flat ring morphism  $A \rightarrow B$ . Hence,  $A_Q \rightarrow B_Q$  is free for each  $Q \in \text{Spec}(A)$  [\[16,](#page-28-26) Proposition 9] and  $B_Q \neq 0$  because it contains  $\kappa(P) \neq 0$ . Therefore,  $A_0 \rightarrow B_0$  is injective and it follows that  $A \cong B$  giving (1)  $\Leftrightarrow$  (2).

(2)  $\Rightarrow$  (3) Suppose that *R*  $\subseteq$  *R* has finite fibers and let *T*  $\in$  [*R*, *S*], then *R*  $\subseteq$  *RT*<br>a flat enimorphism by Proposition 4.5(1) and so is  $\kappa(P) \otimes_R \overline{R} \rightarrow \kappa(P) \otimes_R \overline{R}T$ is a flat epimorphism by Proposition [4.5\(](#page-14-0)1) and so is  $\kappa(P) \otimes_R \overline{R} \to \kappa(P) \otimes_R \overline{R}T$ . Since  $Spec(\kappa(P) \otimes_R RT) \rightarrow Spec(\kappa(P) \otimes_R R)$  is injective,  $R \subseteq RT$  has finite fibers.<br>Now  $R \subseteq T$  has finite fibers because  $T \subseteq \overline{RT}$  is integral and is therefore spectrally Now  $R \subseteq T$  has finite fibers because  $T \subseteq RT$  is integral and is therefore spectrally surjective surjective.

 $(3) \Rightarrow (1)$  is obvious.

*Remark* 5.9 Actually, the statement (1)  $\Leftrightarrow$  (2) is valid if we only suppose that  $R \subseteq S$  is a flat epimorphism. But this equivalence fails in case  $R \subseteq S$  is not a flat<br>epimorphism as we can see in the following example. Let R be an integral domain epimorphism as we can see in the following example. Let *R* be an integral domain with quotient field K and integral closure  $\overline{R}$  such that  $R \subset \overline{R}$  is a minimal extension. Then *R*  $\subset$  *R* has finite fibers. Consider the polynomial ring *S* := *K*[X]. It follows that  $\overline{R}$  is also the integral closure of *R* in *S*. Moreover *K*  $\subset$  *S* and then *R*  $\subset$  *S* have that  $\overline{R}$  is also the integral closure of  $R$  in *S*. Moreover,  $K \subset S$  and then  $R \subset S$  have not finite fibers. Actually,  $K \subset S$  and  $\overline{R} \subset S$  are not flat epimorphisms.

Next result contains [\[5,](#page-28-4) Lemma 3.6], gotten after a long proof.

**Corollary 5.10** *Let*  $R \subseteq S$  *be an almost-Prüfer extension. Then*  $R \subseteq S$  *has finite fibers if and only if*  $R \subseteq S$  *has finite fibers* Next result contains [5, Lemma 3.6], gotten after a long proof.<br> **Corollary 5.10** Let  $R \subseteq S$  be an almost-Prüfer extension. Then  $R \subseteq S$  has finite fibers if and only if  $\overline{R} \subseteq \overline{R}$  has finite fibers, and if and only

*Proof* By Theorem [5.8](#page-26-0) the first equivalence is clear. The second is a consequence of Lemma [5.4\(](#page-24-3)2).  $\Box$ <br>The following result is then clear and obviates any need to assume FCP or FMC.

 $\Box$ 

**Theorem 5.11** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension with finite fibers, then*  $R \subseteq T$  *has finite fibers for each*  $T \in [R, S]$ 

<span id="page-27-0"></span>*has finite fibers for each*  $T \in [R, S]$ .<br> **Corollary 5.12** If  $R \subseteq S$  is quasifibers for each  $T \in [R, S]$  and  $R \subseteq S$ . **Corollary 5.12** *If*  $R \subseteq S$  *is quasi-finite and quasi-Prüfer, then*  $R \subseteq T$  *has finite fibers for each*  $T \in [R \times S]$  *and*  $\overline{R} \subseteq S$  *is module finite* fibers for each  $T \in [R, S]$  and  $R \subseteq S$  is module finite.

*Proof* By the Zariski Main Theorem, there is a factorization  $R \subseteq F \subseteq R \subseteq F$  is module finite and  $F \subseteq S$  is a flat enimorphism [39] Corollaire *Proof* By the Zariski Main Theorem, there is a factorization  $R \subseteq F \subseteq S$  where  $R \subseteq F$  is module finite and  $F \subseteq S$  is a flat epimorphism [\[39,](#page-29-12) Corollaire 2, p. 42].<br>To conclude, we use Scholium A in the rest of the proof. The man  $\widetilde{R} \otimes_R F \to S$  is *Proof* By the Zariski Main Theorem, there is a factorization  $R \subseteq F \subseteq S$  where  $R \subseteq F$  is module finite and  $F \subseteq S$  is a flat epimorphism [39, Corollaire 2, p. 42].<br>To conclude, we use Scholium A in the rest of the proof. Th *Froof* By the Zariski Main Theorem, there is a factorization  $R \subseteq F \subseteq S$  where  $R \subseteq F$  is module finite and  $F \subseteq S$  is a flat epimorphism [39, Corollaire 2, p. 42].<br>To conclude, we use Scholium A in the rest of the proof. Th To conclude, we use Scholium A in the rest of the proof. The map  $\overline{R} \otimes_R F \to S$  is injective because  $F \to \overline{R} \otimes_R F$  is a flat epimorphism and is surjective, since it is integral and a flat epimorphism because  $\overline{R} \$ 

**Corollary 5.13** An FMC extension  $R \subseteq S$  is such that  $R \subseteq T$  has finite fibers for each  $T \in [R \ S]$  $\text{each } T \in [R, S].$ 

*Proof* Such an extension is quasi-finite and quasi-Prüfer. Then use Corollary [5.12.](#page-27-0)

[\[5,](#page-28-4) Example 4.7] exhibits some FMC extension  $R \subseteq S$ , such that  $R \subseteq \overline{R}$  has not <br>P. Actually  $\overline{R}$  is an infinite (maximal) chain FCP. Actually, [R, R] is an infinite (maximal) chain.

**Proposition 5.14** *Let*  $R \subseteq S$  *be a quasi-Prüfer extension such that*  $R \subseteq R$  *has finite* there and R is semi-local T hen T is semi-local for each  $T \in [R \mid S]$ *fibers and R is semi-local. Then T is semi-local for each*  $T \in [R, S]$ *.*<br>  $-$ 

*Proof* Obviously *R* is semi-local. From the tower  $R \subseteq TR \subseteq S$  we deduce that  $\overline{R} \subset TR$  is Prifer. It follows that  $T\overline{R}$  is semi-local [5, I emma 2.5 (f)]. As  $T \subset T\overline{R}$  is  $R \subseteq TR$  is Prüfer. It follows that *TR* is semi-local [\[5,](#page-28-4) Lemma 2.5 (f)]. As  $T \subseteq TR$  is integral we get that *T* is semi-local

integral, we get that *T* is semi-local.  $\square$ <br>The next proposition gives a kind of converse, but, before, we rewrite [\[4,](#page-28-11) Theorem 3.10] proved in the integral domains context, which holds in a more general context.

<span id="page-27-1"></span>**Theorem 5.15** *Let*  $R \subseteq S$  *be an integrally closed extension with*  $R$  *semi-local. The following three conditions are equivalent: following three conditions are equivalent:*

- *(1)*  $R \subseteq S$  *is a Prüfer extension.*<br>*(2)*  $|Max(T)| < |Max(R)|$  for extension.
- (2)  $|\text{Max}(T)| \leq |\text{Max}(R)|$  *for each*  $T \in [R, S]$ .<br>(3) *Each*  $T \in [R, S]$  *is a semi-local ring*
- (3) Each  $T \in [R, S]$  is a semi-local ring.

*Proof* It is enough to mimic the proof of [\[4,](#page-28-11) Theorem 3.10] which is still valid for an arbitrary integrally closed extension of rings  $R \subseteq S$ . Indeed,  $R \subseteq S$  is a Prüfer<br>extension if and only if  $(R, S)$  is a residually algebraic pair such that  $R \subseteq S$  is an extension if and only if  $(R, S)$  is a residually algebraic pair such that  $R \subseteq S$  is an integrally closed extension by Theorem 2.3 and Definition 2.1 integrally closed extension by Theorem  $2.3$  and Definition  $2.1$ .

**Proposition 5.16** *Let*  $R \subseteq S$  *be an extension with*  $R$  *semi-local. Then*  $R \subseteq S$  *is anasi-Prifer if and only if*  $T$  *is semi-local for each*  $T \in [R \ S]$ *quasi-Prüfer if and only if T is semi-local for each T*  $\in$   $[R, S]$ *.* 

*Proof* If  $R \subseteq S$  is quasi-Prüfer,  $R \subseteq S$  is Prüfer. Let  $T \in [R, S]$  and set  $T' := TR$ , so that  $T \subset T'$  is integral, and  $\overline{R} \subset T'$  is Prüfer (and then a normal pair). It follows so that  $T \subseteq T'$  is integral, and  $R \subseteq T'$  is Prüfer (and then a normal pair). It follows<br>from [5] I emma 2.5 (f)] that  $T'$  is semi-local, and so is T from  $[5, Lemma 2.5 (f)]$  $[5, Lemma 2.5 (f)]$  that  $T'$  is semi-local, and so is  $T$ .

If *T* is semi-local for each  $T \in [R, S]$ , so is any  $T \in [R, S]$ . Then,  $R \subseteq S$  is Prüfer<br>Theorem 5.15 and  $R \subseteq S$  is quasi-Prüfer by Theorem [5.15](#page-27-1) and  $R \subseteq S$  is quasi-Prüfer. **Acknowledgements** The authors wish to thank the referee whose many suggestions improved the presentation of the paper.

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