Lecture Notes in Mathematics 2195

Léa Blanc-Centi *Editor*

Metrical and Dynamical Aspects in Complex Analysis

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Metrical and Dynamical Aspects in Complex Analysis

Springer

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ISSN 0075-8434 ISSN 1617-9692 (electronic)
Lecture Notes in Mathematics
ISBN 978-3-319-65836-0 ISBN 978-3-319-65837-7 (eBook)
DOI 10.1007/978-3-319-65837-7

Library of Congress Control Number: 2017955541

Mathematics Subject Classification (2010): 32F45, 32T99, 30C65, 53C23, 53B40, 37F99, 51F99

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This is the second volume of the CEMPI subseries, common to Lecture Notes in Mathematics and Lecture Notes in Physics. CEMPI, acronym for "Centre Européen pour les Mathématiques, la Physique et leurs Interactions," is a "Laboratoire d'Excellence" based on the campus of the Université Lille 1. The material in this volume is based on lectures given in Lille during the CNRS's Thematic School "Metrical and dynamical aspects in complex analysis," as part of the 2015 Painlevé-CEMPI Thematic Semester of Analysis.

Complex analysis is by nature at the meeting point of analysis, geometry, and dynamics. The aim of this volume is to reveal the still underexploited connections between complex analysis and metric geometry. Metric geometry provides very powerful tools: some of them have been proving for long their potential in complex analysis, as the Kobayashi metric, with related questions about Kobayashi hyperbolicity, or problems in holomorphic dynamics. Others, as the notion of hyperbolicity in the sense of Gromov, appeared more recently in the understanding of geometrical or dynamical phenomena. In this context, the reader will explore how metrical and dynamical aspects interact in complex geometry and holomorphic dynamics.

The first chapter presents the Kobayashi distance, which will play the leading role in this volume. This distance, introduced by Kobayashi in 1967 for complex manifolds, is one of the most useful (biholomorphically) invariant distances. Written by Marco Abate, professor at the Università di Pisa (Italy), this introductory text also describes several properties and estimates depending on the geometry of the domain.

The second chapter, still written by Marco Abate, deals with the dynamics of holomorphic self-maps of taut manifolds. The proofs rely deeply on the Kobayashi distance to get information about the boundary behavior, in order to obtain a several variables version of the Wolff-Denjoy theorem. This subject has attracted much attention, and the author describes some of his many contributions to the field.

The objective of the third chapter is to introduce the notion of Gromov hyperbolicity and to develop some situations in analysis where this notion is of great help to understand their geometric meaning. Written by Hervé Pajot, professor at the

Institut Fourier, Université de Grenoble (France), it bridges some classical results in complex analysis with a metric phenomenon coming from geometric group theory.

The fourth chapter, entitled "Gromov hyperbolicity of bounded convex domains", is written by Andrew Zimmer, a post-doctoral researcher at the University of Chicago (USA). Very recent results of the author, concerning the Gromov hyperbolicity of the Kobayashi metric, are described carefully. This chapter is in continuity with the previous chapters, since the approach mixes a precise local study of the Kobayashi metric near the boundary with a "large-scale" metric viewpoint. It also draws a parallel with Hilbert geometry and ends with some open questions in the field.

The last two chapters present further applications of the fruitful interplay between analysis and metric geometry. The fifth chapter is an introduction to quasi-conformal geometry, written by Hervé Pajot. It begins with quasi-conformal mappings on the complex plane, which will constitute a very useful introduction for young researchers interested in this area. The second part deals with metric spaces with controlled geometry.

The sixth and last chapter is written by Marco Abate. It gives a (recent) application of the Kobayashi distance to complex functional analysis, more precisely concerning Carleson measures and Toeplitz operators.

The reader, analyst or geometer, shall thus find here a range of metrical tools for complex analysis. The volume offers a unique and accessible overview of the interactions between complex analysis and metric geometry up to the frontiers of recent research.

Villeneuve d'Ascq, France Léa Blanc-Centi
July 2016

Acknowledgements

This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01).

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Acronyms

$\mathrm{Hol}(X, Y)$	Set of all holomorphic maps from X to Y
$\mathrm{Aut}(X)$	Set of automorphisms (invertible holomorphic self-maps) of X
$\\|\cdot\\|$	The standard Euclidean norm
$\mathbb{B}^n$	The unit Euclidean ball of $\mathbb{C}^n$
$\mathbb{B}_r$	The Euclidean ball of radius r
Δ	The unit disc in $\mathbb{C}$
$\langle \cdot , \cdot \rangle$	The canonical Hermitian product on $\mathbb{C}^n$
κ_Δ	The infinitesimal Poincaré metric on Δ
k_Δ	The Poincaré distance on Δ
δ_X	The Lempert function on X
κ_X	The infinitesimal Kobayashi metric on X
k_X	The Kobayashi (pseudo) distance on X
$B_X(z_0, r)$	The Kobayashi ball of center $z_0 \in X$ and radius $r > 0$
$H_{\rho,x}$	The (real) Hessian of the function ρ at point x
T_x	The tangent space at point x
$T_x^{\mathbb{C}}$	The complex tangent space at point x
$L_{\rho,x}$	The Levi form of the function ρ at point x
$d(\cdot, M)$	The usual Euclidean distance from M
diam	The Euclidean diameter
ν	The Lebesgue measure
Fix	Set of fixed points
$\Gamma(f)$	Set of limit maps of f in C^0
$E_{z_0}(x_0, R)$	The small horosphere of center $x_0 \in \partial D$, radius $R > 0$, and pole $z_0 \in D$
$F_{z_0}(x_0, R)$	The large horosphere of center $x_0 \in \partial D$, radius $R > 0$, and pole $z_0 \in D$
$G_{z_0}(x, R, \mathbf{x})$	Sequence horosphere
$(\cdot\\|\cdot)_w$	The Gromov product w.r.t the basepoint w
$\partial_G X$	The Gromov boundary of X
$\mathscr{H}$	The half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$

$\mathscr{H}^{n+1}$	The upper half-space $\{Im(z_{n+1}) > \sum_{j=1}^{n} \lvert z_j\rvert^2\}$ in $\mathbb{C}^{n+1}$
$\mathbb{H}^n$	The Heisenberg group
d_{CC}	The Carnot-Carathéodory distance
B_{CC}	The open ball w.r.t. d_{CC}
d_{CK}	The Cygan-Korànyi distance on $\mathbb{H}$
$\delta_\Omega(p)$	$\inf\{\lVert q-p\rVert : q \in \partial\Omega\}$ where p is in the open set $\Omega \subset \mathbb{C}^d$
$\delta_\Omega(p; v)$	$\inf\{\lVert q-p\rVert : q \in (p+\mathbb{C}\cdot v) \cap \partial\Omega\}$ where p is in the open set $\Omega \subset \mathbb{C}^d$ and $v \in \mathbb{C}^d$
$\mathbb{X}_d$	Set of $\mathbb{C}$-proper convex domains in $\mathbb{C}^d$
$\mathbb{X}_{d,0}$	Set of pairs (Ω, x_0) where $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex open set and $x_0 \in \Omega$
$d_H(A,B)$	The Hausdorff distance between two compact sets A, B in $\mathbb{C}^d$
$d_H^{(R)}(A,B)$	The local Hausdorff semi-norm between two compact sets A, B in $\mathbb{C}^d$, where $R > 0$
$\mathrm{Aff}(\mathbb{C}^d)$	The group of affine automorphisms of $\mathbb{C}^d$
$H_{\mathscr{C}}$	The Hilbert distance
$\mathrm{Aut}_{\mathrm{proj}}$	The real projective automorphism group
ACL	Absolutely continuous on lines
$\Lambda(\Gamma)$	Extremal length of the curve family Γ
mod	The conformal modulus
$\mathrm{BMO}(\Omega)$	Set of bounded mean oscillation functions on Ω
$\mathbb{C}_\infty$	The Riemann sphere
$\mathscr{F}_f$	The Fatou set of f
$\mathscr{J}_f$	The Julia set of f
mod_p	p-modulus $(p \geq 1)$
QC	Quasi-conformal
WQS	Weakly quasisymmetric
QS	quasisymmetric
$\Delta(E,F)$	The relative distance between two continua E and F
LLC	Linearly locally connected
QM	Quasi-Möbius
$L^p(D,\beta)$	Weighted L^p space
$A^p(D)$	Bergman space
$A^p(D,\beta)$	Weighted Bergman space
T_ψ	Toeplitz operator of symbol ψ
$B\mu$	Berezin transform of the finite positive Borel measure μ

Chapter 1
Invariant Distances

Marco Abate

In this chapter we shall define the (invariant) distance we are going to use, and collect some of its main properties we shall need later on. It will not be a comprehensive treatise on the subject; much more informations can be found in, e.g., [2, 17, 24].

Before beginning, let us introduce a couple of notations we shall consistently use.

Definition 1.0.1 Let X and Y be two (finite dimensional) complex manifolds. We shall denote by $\mathrm{Hol}(X, Y)$ the set of all holomorphic maps from X to Y, endowed with the compact-open topology (which coincides with the topology of uniform convergence on compact subsets), so that it becomes a metrizable topological space. Furthermore, we shall denote by $\mathrm{Aut}(X) \subset \mathrm{Hol}(X, X)$ the set of automorphisms, that is invertible holomorphic self-maps, of X. More generally, if X and Y are topological spaces we shall denote by $C^0(X, Y)$ the space of continuous maps from X to Y, again endowed with the compact-open topology.

Definition 1.0.2 We shall denote by $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ the unit disc in the complex plane $\mathbb{C}$, by $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ (where $\|\cdot\|$ is the Euclidean norm) the unit ball in the n-dimensional space $\mathbb{C}^n$, and by $\Delta^n \subset \mathbb{C}^n$ the unit polydisc in $\mathbb{C}^n$. Furthermore, $\langle \cdot , \cdot \rangle$ will denote the canonical Hermitian product on $\mathbb{C}^n$.

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7_1

1.1 The Poincaré Distance

The model for all invariant distances in complex analysis is the Poincaré distance on the unit disc of the complex plane; we shall then start recalling its definitions and main properties (see also Appendix 1).

Definition 1.1.1 The *Poincaré* (or *hyperbolic*) *metric* on Δ is the Hermitian metric whose associated norm is given by

$$\kappa_\Delta(\zeta; v) = \frac{1}{1-|\zeta|^2}|v|$$

for all $\zeta \in \Delta$ and $v \in \mathbb{C} \simeq T_\zeta\Delta$. It is a complete Hermitian metric with constant Gaussian curvature -4.

Definition 1.1.2 The *Poincaré* (or *hyperbolic*) *distance* k_Δ on Δ is the integrated form of the Poincaré metric. It is a complete distance, whose expression is

$$k_\Delta(\zeta_1, \zeta_2) = \tfrac{1}{2}\log \frac{1 + \left|\frac{\zeta_1-\zeta_2}{1-\overline{\zeta_1}\zeta_2}\right|}{1 - \left|\frac{\zeta_1-\zeta_2}{1-\overline{\zeta_1}\zeta_2}\right|} .$$

In particular,

$$k_\Delta(0, \zeta) = \tfrac{1}{2}\log \frac{1+|\zeta|}{1-|\zeta|} .$$

Remark 1.1.3 It is useful to keep in mind that the function

$$t \mapsto \tfrac{1}{2}\log\frac{1+t}{1-t}$$

is the inverse of the hyperbolic tangent $\tanh t = (\mathrm{e}^t - \mathrm{e}^{-t})/(\mathrm{e}^t + \mathrm{e}^{-t})$.

Besides being a metric with constant negative Gaussian curvature, the Poincaré metric strongly reflects the properties of the holomorphic self-maps of the unit disc. For instance, the isometries of the Poincaré metric coincide with the holomorphic or anti-holomorphic automorphisms of Δ (see, e.g., [2, Proposition 1.1.8]):

Proposition 1.1.4 *The group of smooth isometries of the Poincaré metric consists of all holomorphic and anti-holomorphic automorphisms of* Δ.

More importantly, the famous *Schwarz-Pick lemma* says that any holomorphic self-map of Δ is nonexpansive for the Poincaré metric and distance (see, e.g., [2, Theorem 1.1.6]):

Theorem 1.1.5 (Schwarz-Pick lemma) *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be a holomorphic self-map of Δ. Then:*

(i) *we have*

$$\kappa_\Delta\big(f(\zeta); f'(\zeta)v\big) \le \kappa_\Delta(\zeta; v) \tag{1.1}$$

for all $\zeta \in \Delta$ and $v \in \mathbb{C}$. Furthermore, equality holds for some $\zeta \in \Delta$ and $v \in \mathbb{C}^*$ if and only if equality holds for all $\zeta \in \Delta$ and all $v \in \mathbb{C}$ if and only if $f \in \mathrm{Aut}(\Delta)$;

(ii) *we have*

$$k_\Delta\big(f(\zeta_1), f(\zeta_2)\big) \le k_\Delta(\zeta_1, \zeta_2) \tag{1.2}$$

for all $\zeta_1, \zeta_2 \in \Delta$. Furthermore, equality holds for some $\zeta_1 \neq \zeta_2$ if and only if equality holds for all $\zeta_1, \zeta_2 \in \Delta$ if and only if $f \in \mathrm{Aut}(\Delta)$.

In other words, *holomorphic self-maps of the unit disc are automatically 1-Lipschitz, and hence equicontinuous, with respect to the Poincaré distance.*

As an immediate corollary, we can compute the group of automorphisms of Δ, and thus, by Proposition 1.1.4, the group of isometries of the Poincaré metric (see, e.g., [2, Proposition 1.1.2]):

Corollary 1.1.6 *The group $\mathrm{Aut}(\Delta)$ of holomorphic automorphisms of Δ consists in all the functions $\gamma\colon \Delta \to \Delta$ of the form*

$$\gamma(\zeta) = e^{i\theta} \frac{\zeta - \zeta_0}{1 - \overline{\zeta_0}\zeta} \tag{1.3}$$

with $\theta \in \mathbb{R}$ and $\zeta_0 \in \Delta$. In particular, for every pair $\zeta_1, \zeta_2 \in \Delta$ there exists $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(\zeta_1) = 0$ and $\gamma(\zeta_2) \in [0, 1)$.

Remark 1.1.7 More generally, given $\zeta_1, \zeta_2 \in \Delta$ and $\eta \in [0, 1)$, it is not difficult to see that there is $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(\zeta_1) = \eta$ and $\gamma(\zeta_2) \in [0, 1)$ with $\gamma(\zeta_2) \ge \eta$.

A consequence of (1.3) is that all automorphisms of Δ extends continuously to the boundary. It is customary to classify the elements of $\mathrm{Aut}(\Delta)$ according to the number of fixed points in $\overline{\Delta}$:

Definition 1.1.8 An automorphism $\gamma \in \mathrm{Aut}(\Delta) \setminus \{\mathrm{id}_\Delta\}$ is called *elliptic* if it has a unique fixed point in Δ, *parabolic* if it has a unique fixed point in $\partial\Delta$, *hyperbolic* if it has exactly two fixed points in $\partial\Delta$. It is easy to check that these cases are mutually exclusive and exhaustive.

We end this brief introduction to the Poincaré distance by recalling two facts relating its geometry to the Euclidean geometry of the plane (see, e.g., [2, Lemma 1.1.5 and (1.1.11)]):

Proposition 1.1.9 *Let $\zeta_0 \in \Delta$ and $r > 0$. Then the ball $B_\Delta(\zeta_0, r) \subset \Delta$ for the Poincaré distance of center ζ_0 and radius r is the Euclidean ball with center*

$$\frac{1-(\tanh r)^2}{1-(\tanh r)^2|\zeta_0|^2}\zeta_0$$

and radius

$$\frac{(1-|\zeta_0|^2)\tanh r}{1-(\tanh r)^2|\zeta_0|^2} .$$

Proposition 1.1.10 *Let $\zeta_0 = re^{i\theta} \in \Delta$. Then the geodesic for the Poincaré metric connecting 0 to ζ_0 is the Euclidean radius $\sigma\colon [0, k_\Delta(0, \zeta_0)] \to \Delta$ given by*

$$\sigma(t) = (\tanh t)e^{i\theta} .$$

In particular, $k_\Delta\big(0, (\tanh t)e^{i\theta}\big) = |t|$ for all $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

1.2 The Kobayashi Distance in Complex Manifolds

Our next aim is to build on any complex manifold a (pseudo)distance enjoying the main properties of the Poincaré distance; in particular, we would like to preserve the 1-Lipschitz property of holomorphic maps, that is to generalize to several variables Schwarz-Pick lemma. There are several ways for doing this; historically, the first such generalization has been introduced by Carathéodory [11] in 1926, but the most well-known and most useful has been proposed in 1967 by Kobayashi [22, 23]. Here we shall concentrate on the Kobayashi (pseudo)distance; but several other similar metrics and distances have been introduced (see, e.g., [7, 10, 12, 15, 21, 30, 31, 34]; see also [16] for a general context explaining why in a very precise sense the Carathéodory distance is the smallest and the Kobayashi distance is the largest possible invariant distance, and [6] for a different differential geometric approach). Furthermore, we shall discuss only the Kobayashi *distance*; it is possible to define a Kobayashi metric, which is a complex Finsler metric whose integrated form is exactly the Kobayashi distance, see Sect. 4.1. It is also possible to introduce a Kobayashi pseudodistance in complex analytic spaces; again, see [2, 17] and [24] for details and much more.

To define the Kobayashi pseudodistance we first introduce an auxiliary function.

Definition 1.2.1 Let X be a connected complex manifold. The *Lempert function* $\delta_X\colon X \times X \to \mathbb{R}^+ \cup \{+\infty\}$ is defined by

$$\delta_X(z,w) = \inf\big\{k_\Delta(\zeta_0,\zeta_1) \bigm| \exists\varphi \in \mathrm{Hol}(\Delta, X) : \varphi(\zeta_0) = z, \varphi(\zeta_1) = w\big\}$$

for every $z, w \in X$.

Remark 1.2.2 Corollary 1.1.6 yields the following equivalent definition of the Lempert function:

$$\delta_X(z,w) = \inf\bigl\{k_\Delta(0,\zeta) \bigm| \exists\varphi \in \mathrm{Hol}(\Delta, X) : \varphi(0) = z, \varphi(\zeta) = w\bigr\} .$$

The Lempert function in general (but there are exceptions; see Proposition 1.4.7) does not satisfy the triangular inequality (see, e.g., [28] for an example), and so it is not a distance. But this is a problem easily solved:

Definition 1.2.3 Let X be a connected complex manifold. The *Kobayashi (pseudo) distance* $k_X: X \times X \to \mathbb{R}^+$ is the largest (pseudo)distance bounded above by the Lempert function, that is

$$k_X(z,w) = \inf\left\{\sum_{j=1}^{k} \delta_X(z_{j-1}, z_j) \,\middle|\, k \in \mathbb{N}, z_0 = z, z_k = w, z_1, \ldots, z_{k-1} \in X\right\}$$

for all $z, w \in X$.

A few remarks are in order. First of all, it is easy to check that since X is connected then k_X is always finite. Furthermore, it is clearly symmetric, it satisfies the triangle inequality by definition, and $k_X(z,z) = 0$ for all $z \in X$. On the other hand, it might well happen that $k_X(z_0, z_1) = 0$ for two distinct points $z_0 \neq z_1$ of X (it might even happen that $k_X \equiv 0$; see Proposition 1.2.5); so k_X in general is only a pseudodistance. Anyway, the definition clearly implies the following generalization of the Schwarz-Pick lemma:

Theorem 1.2.4 *Let X, Y be two complex manifolds, and $f \in \mathrm{Hol}(X,Y)$. Then*

$$k_Y\bigl(f(z), f(w)\bigr) \le k_X(z,w)$$

for all $z, w \in X$. In particular:

(i) *if X is a submanifold of Y then $k_Y|_{X\times X} \le k_X$;*
(ii) *biholomorphisms are isometries with respect to the Kobayashi pseudodistances.*

A statement like this is the reason why the Kobayashi (pseudo)distance is said to be an *invariant* distance: it is invariant under biholomorphisms.

Using the definition, it is easy to compute the Kobayashi pseudodistance of a few of interesting manifolds (see, e.g., [2, Proposition 2.3.4, Corollaries 2.3.6, 2.3.7]):

Proposition 1.2.5

(i) *The Poincaré distance is the Kobayashi distance of the unit disc Δ.*
(ii) *The Kobayashi distances of $\mathbb{C}^n$ and of the complex projective space $\mathbb{P}^n(\mathbb{C})$ vanish identically.*
(iii) *For every $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \Delta^n$ we have*

$$k_{\Delta^n}(z,w) = \max_{j=1,\ldots,n} \{k_\Delta(z_j, w_j)\} .$$

(iv) *The Kobayashi distance of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ coincides with the classical Bergman distance; in particular, if $O \in \mathbb{C}^n$ is the origin and $z \in \mathbb{B}^n$ then*

$$k_{\mathbb{B}^n}(O,z) = \tfrac{1}{2}\log\frac{1+\|z\|}{1-\|z\|}\;.$$

Remark 1.2.6 As often happens with objects introduced via a general definition, the Kobayashi pseudodistance can seldom be explicitly computed. Besides the cases listed in Proposition 1.2.5, as far as we know there are formulas only for some complex ellipsoids [18], bounded symmetric domains [17], the symmetrized bidisc [5] and a few other scattered examples. On the other hand, it is possible and important to estimate the Kobayashi distance; see Sect. 1.5.

We shall be interested in manifolds where the Kobayashi pseudodistance is a true distance, that is in complex manifolds X such that $k_X(z,w) > 0$ as soon as $z \neq w$.

Definition 1.2.7 A connected complex manifold X is *(Kobayashi) hyperbolic* if k_X is a true distance. In this case, if $z_0 \in X$ and $r > 0$ we shall denote by $B_X(z_0,r)$ the ball for k_X of center z_0 and radius r; we shall call $B_X(z_0,r)$ a *Kobayashi ball*. More generally, if $A \subseteq X$ and $r > 0$ we shall put $B_X(A,r) = \bigcup_{z\in A} B_X(z,r)$.

In hyperbolic manifolds the Kobayashi distance induces the topology of the manifold. More precisely (see, e.g., [2, Proposition 2.3.10]):

Proposition 1.2.8 (Barth, [8]) *A connected complex manifold X is hyperbolic if and only if k_X induces the manifold topology on X.*

To give a first idea of how one can work with the Kobayashi distance, we describe two large classes of examples of hyperbolic manifolds:

Proposition 1.2.9 (Kobayashi, [22, 23])

(i) *A submanifold of a hyperbolic manifold is hyperbolic. In particular, bounded domains in $\mathbb{C}^n$ are hyperbolic.*

(ii) *Let $\pi\colon \tilde{X} \to X$ be a holomorphic covering map. Then X is hyperbolic if and only if $\tilde{X}$ is. In particular, a Riemann surface is hyperbolic if and only if it is Kobayashi hyperbolic.*

Proof (i) The first assertion follows immediately from Theorem 1.2.4.(i). For the second one, we remark that the unit ball $\mathbb{B}^n$ is hyperbolic by Proposition 1.2.5.(iv). Then Theorem 1.2.4.(ii) implies that all balls are hyperbolic; since a bounded domain is contained in a ball, the assertion follows.

(ii) First of all we claim that

$$k_X(z_0,w_0) = \inf\left\{k_{\tilde{X}}(\tilde{z}_0,\tilde{w}) \mid \tilde{w} \in \pi^{-1}(w_0)\right\}\,, \tag{1.4}$$

for any z_0, $w_0 \in X$, where $\tilde{z}_0$ is any element of $\pi^{-1}(z_0)$. Indeed, first of all Theorem 1.2.4 immediately implies that

$$k_X(z_0,w_0) \le \inf\left\{k_{\tilde{X}}(\tilde{z}_0,\tilde{w}) \mid w \in \pi^{-1}(w_0)\right\}\,.$$

Assume now, by contradiction, that there is $\varepsilon > 0$ such that

$$k_X(z_0, w_0) + \varepsilon \leq k_{\widetilde{X}}(\tilde{z}_0, \tilde{w})$$

for all $\tilde{w} \in \pi^{-1}(w_0)$. Choose $z_1, \dots, z_k \in X$ with $z_k = w_0$ such that

$$\sum_{j=1}^{k} \delta_X(z_{j-1}, z_j) < k_X(z_0, w_0) + \varepsilon/2 \,.$$

By Remark 1.2.2, we can find $\varphi_1, \dots, \varphi_k \in \mathrm{Hol}(\Delta, X)$ and $\zeta_1, \dots, \zeta_k \in \Delta$ such that $\varphi_j(0) = z_{j-1}$, $\varphi_j(\zeta_j) = z_j$ for all $j = 1, \dots, k$ and

$$\sum_{j=1}^{k} k_\Delta(0, \zeta_j) < k_X(z_0, w_0) + \varepsilon \,.$$

Let $\tilde{\varphi}_1, \dots, \tilde{\varphi}_k \in \mathrm{Hol}(\Delta, \tilde{X})$ be the liftings of $\varphi_1, \dots, \varphi_k$ chosen so that $\tilde{\varphi}_1(0) = \tilde{z}_0$ and $\tilde{\varphi}_{j+1}(0) = \tilde{\varphi}_j(\zeta_j)$ for $j = 1, \dots, k-1$, and set $\tilde{w}_0 = \tilde{\varphi}_k(\zeta_k) \in \pi^{-1}(w_0)$. Then

$$k_{\tilde{X}}(\tilde{z}_0, \tilde{w}_0) \leq \sum_{j=1}^{k} \delta_{\tilde{X}}\big(\tilde{\varphi}_j(0), \tilde{\varphi}_j(\zeta_j)\big) \leq \sum_{j=1}^{k} k_\Delta(0, \zeta_j) < k_X(z_0, w_0) + \varepsilon \leq k_{\tilde{X}}(\tilde{z}_0, \tilde{w}_0) \,,$$

contradiction.

Having proved (1.4), let us assume that $\tilde{X}$ is hyperbolic. If there are z_0, $w_0 \in X$ such that $k_X(z_0, w_0) = 0$, then for any $\tilde{z}_0 \in \pi^{-1}(z_0)$ there is a sequence $\{\tilde{w}_\nu\} \subset \pi^{-1}(w_0)$ such that $k_{\tilde{X}}(\tilde{z}_0, \tilde{w}_\nu) \to 0$ as $\nu \to +\infty$. Then $\tilde{w}_\nu \to \tilde{z}_0$ (Proposition 1.2.8) and so $\tilde{z}_0 \in \pi^{-1}(w_0)$, that is $z_0 = w_0$.

Conversely, assume X hyperbolic. Suppose $\tilde{z}_0$, $\tilde{w}_0 \in \tilde{X}$ are so that $k_{\tilde{X}}(\tilde{z}_0, \tilde{w}_0) = 0$; then $k_X\big(\pi(\tilde{z}_0), \pi(\tilde{w}_0)\big) = 0$ and so $\pi(\tilde{z}_0) = \pi(\tilde{w}_0) = z_0$. Let $\widetilde{U}$ be a connected neighborhood of $\tilde{z}_0$ such that $\pi|_{\widetilde{U}}$ is a biholomorphism between $\widetilde{U}$ and the (connected component containing z_0 of the) Kobayashi ball $B_X(z_0, \varepsilon)$ of center z_0 and radius $\varepsilon > 0$ small enough; this can be done because of Proposition 1.2.8. Since $k_{\widetilde{X}}(\tilde{z}_0, \tilde{w}_0) = 0$, we can find $\varphi_1, \dots, \varphi_k \in \mathrm{Hol}(\Delta, \tilde{X})$ and $\zeta_1, \dots, \zeta_k \in \Delta$ with $\varphi_1(0) = \tilde{z}_0$, $\varphi_j(\zeta_j) = \varphi_{j+1}(0)$ for $j = 1, \dots, k-1$ and $\varphi_k(\zeta_k) = \tilde{w}_0$ such that

$$\sum_{j=1}^{k} k_\Delta(0, \zeta_j) < \varepsilon \,.$$

Let σ_j be the radial segment in Δ joining 0 to ζ_j; by Proposition 1.1.10 the σ_j are geodesics for the Poincaré metric. The arcs $\varphi_j \circ \sigma_j$ in $\widetilde{X}$ connect to form a continuous curve σ from $\tilde{z}_0$ to $\tilde{w}_0$. Now the maps $\pi \circ \varphi_j \in \mathrm{Hol}(\Delta, X)$ are non-expanding; therefore every point of the curve $\pi \circ \sigma$ should belong to $B_X(z_0, \varepsilon)$. But then σ is contained in $\widetilde{U}$, and this implies $\tilde{z}_0 = \tilde{w}_0$.

The final assertion on Riemann surfaces follows immediately because hyperbolic Riemann surfaces can be characterized as the only Riemann surfaces whose universal covering is the unit disc. □

It is also possible to prove the following (see, e.g., [2, Proposition 2.3.13]):

Proposition 1.2.10 *Let X_1 and X_2 be connected complex manifolds. Then $X_1 \times X_2$ is hyperbolic if and only if both X_1 and X_2 are hyperbolic.*

Remark 1.2.11 The Kobayashi pseudodistance can be useful even when it is degenerate. For instance, the classical Liouville theorem (a bounded entire function is constant) is an immediate consequence, thanks to Theorem 1.2.4, of the vanishing of the Kobayashi pseudodistance of $\mathbb{C}^n$ and the fact that bounded domains are hyperbolic.

A technical fact we shall need later on is the following:

Lemma 1.2.12 *Let X be a hyperbolic manifold, and choose $z_0 \in X$ and $r_1, r_2 > 0$. Then*

$$B_X\big(B_X(z_0, r_1), r_2\big) = B_X(z_0, r_1 + r_2) .$$

Proof The inclusion $B_D\big(B_D(z_0, r_1), r_2\big) \subseteq B_D(z_0, r_1 + r_2)$ follows immediately from the triangular inequality. For the converse, let $z \in B_D(z_0, r_1 + r_2)$, and set $3\varepsilon = r_1 + r_2 - k_X(z_0, z)$. Then there are $\varphi_1, \ldots, \varphi_m \in \mathrm{Hol}(\Delta, X)$ and $\zeta_1, \ldots, \zeta_m \in \Delta$ so that $\varphi_1(0) = z_0$, $\varphi_j(\zeta_j) = \varphi_{j+1}(0)$ for $j = 1, \ldots, m-1$, $\varphi_m(\zeta_m) = z$ and

$$\sum_{j=1}^{m} k_\Delta(0, \zeta_j) < r_1 + r_2 - 2\varepsilon .$$

Let $\mu \leq m$ be the largest integer such that

$$\sum_{j=1}^{\mu-1} k_\Delta(0, \zeta_j) < r_1 - \varepsilon .$$

Let η_μ be the point on the Euclidean radius in Δ passing through $\zeta_{\mu+1}$ (which is a geodesic for the Poincaré distance) such that

$$\sum_{j=1}^{\mu-1} k_\Delta(0, \zeta_j) + k_\Delta(0, \eta_\mu) = r_1 - \varepsilon .$$

If we set $w = \varphi_\mu(\eta_\mu)$, then $k_X(z_0, w) < r_1$ and $k_X(w, z) < r_2$, so that

$$z \in B_D(w, r_2) \subseteq B_D\big(B_D(z_0, r_1), r_2\big) ,$$

and we are done. □

A condition slightly stronger than hyperbolicity is the following:

Definition 1.2.13 A hyperbolic complex manifold X is *complete hyperbolic* if the Kobayashi distance k_X is complete.

Complete hyperbolic manifolds have a topological characterization (see, e.g., [2, Proposition 2.3.17]):

Proposition 1.2.14 *Let X be a hyperbolic manifold. Then X is complete hyperbolic if and only if every closed Kobayashi ball is compact. In particular, compact hyperbolic manifolds are automatically complete hyperbolic.*

Examples of complete hyperbolic manifolds are contained in the following (see, e.g., [2, Propositions 2.3.19 and 2.3.20]):

Proposition 1.2.15

(i) *A homogeneous hyperbolic manifold is complete hyperbolic. In particular, both $\mathbb{B}^n$ and Δ^n are complete hyperbolic.*
(ii) *A closed submanifold of a complete hyperbolic manifold is complete hyperbolic.*
(iii) *The product of two hyperbolic manifolds is complete hyperbolic if and only if both factors are complete hyperbolic.*
(iv) *If $\pi : \tilde{X} \to X$ is a holomorphic covering map, then $\tilde{X}$ is complete hyperbolic if and only if X is complete hyperbolic.*

We shall see more examples of complete hyperbolic manifolds later on (Proposition 1.4.8 and Corollary 1.5.20). We end this subsection recalling the following important fact (see, e.g., [24, Theorem 5.4.2]):

Theorem 1.2.16 *The automorphism group* $\mathrm{Aut}(X)$ *of a hyperbolic manifold X has a natural structure of real Lie group.*

1.3 Taut Manifolds

For our dynamical applications we shall need a class of manifolds which is intermediate between complete hyperbolic and hyperbolic manifolds. To introduce it, we first show that hyperbolicity can be characterized as a precompactness assumption on the space $\mathrm{Hol}(\Delta, X)$.

If X is a topological space, we shall denote by $X^* = X \cup \{\infty\}$ its one-point (or Alexandroff) compactification; see, e.g., [20, p. 150] for details.

Theorem 1.3.1 ([3]) *Let X be a connected complex manifold. Then X is hyperbolic if and only if* $\mathrm{Hol}(\Delta, X)$ *is relatively compact in the space* $C^0(\Delta, X^*)$ *of continuous functions from Δ into the one-point compactification of X. In particular, if X is compact then it is hyperbolic if and only if* $\mathrm{Hol}(\Delta, X)$ *is compact. Finally, if X is hyperbolic then* $\mathrm{Hol}(Y, X)$ *is relatively compact in* $C^0(Y, X^*)$ *for any complex manifold Y.*

If X is hyperbolic and not compact, the closure of $\mathrm{Hol}(\Delta, X)$ in $C^0(\Delta, X^*)$ might contain continuous maps whose image might both contain ∞ and intersect X, exiting thus from the realm of holomorphic maps. Taut manifolds, introduced by Wu [33], are a class of (not necessarily compact) hyperbolic manifolds where this problem does not appear, and (as we shall see) this will be very useful when studying the dynamics of holomorphic self-maps.

Definition 1.3.2 A complex manifold X is *taut* if it is hyperbolic and every map in the closure of $\mathrm{Hol}(\Delta, X)$ in $C^0(\Delta, X^*)$ either is in $\mathrm{Hol}(\Delta, X)$ or is the constant map ∞.

This definition can be rephrased in another way not requiring the one-point compactification.

Definition 1.3.3 Let X and Y be topological spaces. A sequence $\{f_\nu\} \subset C^0(Y, X)$ is *compactly divergent* if for every pair of compacts $H \subseteq Y$ and $K \subseteq X$ there exists $\nu_0 \in \mathbb{N}$ such that $f_\nu(H) \cap K = \emptyset$ for every $\nu \geq \nu_0$. A family $\mathscr{F} \subseteq C^0(Y, X)$ is *normal* if every sequence in $\mathscr{F}$ admits a subsequence which is either uniformly converging on compact subsets or compactly divergent.

By the definition of one-point compactification, a sequence in $C^0(Y, X)$ converges in $C^0(Y, X^*)$ to the constant map ∞ if and only if it is compactly divergent. When X and Y are manifolds (more precisely, when they are Hausdorff, locally compact, connected and second countable topological spaces), a subset in $C^0(Y, X^*)$ is compact if and only if it is sequentially compact; therefore we have obtained the following alternative characterization of taut manifolds:

Corollary 1.3.4 *A connected complex manifold X is taut if and only if the family* $\mathrm{Hol}(\Delta, X)$ *is normal.*

Actually, it is not difficult to prove (see, e.g., [2, Theorem 2.1.2]) that the role of Δ in the definition of taut manifolds is not essential:

Proposition 1.3.5 *Let X be a taut manifold. Then* $\mathrm{Hol}(Y, X)$ *is a normal family for every complex manifold Y.*

It is easy to find examples of hyperbolic manifolds which are not taut:

Example 1.3.6 Let $D = \Delta^2 \setminus \{(0,0)\}$. Since D is a bounded domain in $\mathbb{C}^2$, it is hyperbolic. For $\nu \geq 1$ let $\varphi_\nu \in \mathrm{Hol}(\Delta, D)$ given by $\varphi_\nu(\zeta) = (\zeta, 1/\nu)$. Clearly $\{\varphi_\nu\}$ converges as $\nu \to +\infty$ to the map $\varphi(\zeta) = (\zeta, 0)$, whose image is not contained either in D or in ∂D. In particular, the sequence $\{\varphi_\nu\}$ does not admit a subsequence which is compactly divergent or converging to a map with image in D—and thus D is not taut.

On the other hand, complete hyperbolic manifolds are taut. This is a consequence of the famous Ascoli-Arzelà theorem (see, e.g., [20, p. 233]):

Theorem 1.3.7 (Ascoli-Arzelà theorem) *Let X be a metric space, and Y a locally compact metric space. Then a family $\mathscr{F} \subseteq C^0(Y, X)$ is relatively compact in $C^0(Y, X)$ if and only if the following two conditions are satisfied:*

(i) *$\mathscr{F}$ is equicontinuous;*
(ii) *the set $\mathscr{F}(y) = \{f(y) \mid f \in \mathscr{F}\}$ is relatively compact in X for every $y \in Y$.*

Then:

Proposition 1.3.8 *Every complete hyperbolic manifold is taut.*

Proof Let X be a complete hyperbolic manifold, and $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, X)$ a sequence which is not compactly divergent; we must prove that it admits a subsequence converging in $\mathrm{Hol}(\Delta, X)$.

Up to passing to a subsequence, we can find a pair of compacts $H \subset \Delta$ and $K \subseteq X$ such that $\varphi_\nu(H) \cap K \neq \emptyset$ for all $\nu \in \mathbb{N}$. Fix $\zeta_0 \in H$ and $z_0 \in K$, and set $r = \max\{k_X(z, z_0) \mid z \in K\}$. Then for every $\zeta \in \Delta$ and $\nu \in \mathbb{N}$ we have

$$k_X\bigl(\varphi_\nu(\zeta), z_0\bigr) \le k_X\bigl(\varphi_\nu(\zeta), \varphi_\nu(\zeta_0)\bigr) + k_X\bigl(\varphi_\nu(\zeta_0), z_0\bigr) \le k_\Delta(\zeta, \zeta_0) + r\,.$$

So $\{\varphi_\nu(\zeta)\}$ is contained in the closed Kobayashi ball of center z_0 and radius $k_\Delta(\zeta, \zeta_0) + r$, which is compact since X is complete hyperbolic (Proposition 1.2.14); as a consequence, $\{\varphi_\nu(\zeta)\}$ is relatively compact in X. Furthermore, since X is hyperbolic, the whole family $\mathrm{Hol}(\Delta, X)$ is equicontinuous (it is 1-Lipschitz with respect to the Kobayashi distances); therefore, by the Ascoli-Arzelà theorem, the sequence $\{\varphi_\nu\}$ is relatively compact in $C^0(\Delta, X)$. In particular, it admits a subsequence converging in $C^0(\Delta, X)$; but since, by Weierstrass theorem, $\mathrm{Hol}(\Delta, X)$ is closed in $C^0(\Delta, X)$, the limit belongs to $\mathrm{Hol}(\Delta, X)$, and we are done. □

Thus complete hyperbolic manifolds provide examples of taut manifolds. However, there are taut manifolds which are not complete hyperbolic; an example has been given by Rosay (see [29]). Finally, we have the following equivalent of Proposition 1.2.15 (see, e.g., [2, Lemma 2.1.15]):

Proposition 1.3.9

(i) *A closed submanifold of a taut manifold is taut.*
(ii) *The product of two complex manifolds is taut if and only if both factors are taut.*

Just to give an idea of the usefulness of the taut condition in studying holomorphic self-maps we end this subsection by quoting Wu's generalization of the classical Cartan-Carathéodory and Cartan uniqueness theorems (see, e.g., [2, Theorem 2.1.21 and Corollary 2.1.22]):

Theorem 1.3.10 (Wu, [33]) *Let X be a taut manifold, and let $f \in \mathrm{Hol}(X, X)$ be with a fixed point $z_0 \in X$. Then:*

(i) *the spectrum of* $\mathrm{d}f_{z_0}$ *is contained in* $\overline{\Delta}$*;*
(ii) $|\det \mathrm{d}f_{z_0}| \le 1$*;*
(iii) $|\det \mathrm{d}f_{z_0}| = 1$ *if and only if* $f \in \mathrm{Aut}(X)$*;*
(iv) $\mathrm{d}f_{z_0} = \mathrm{id}$ *if and only if f is the identity map;*
(v) *$T_{z_0}X$ admits a $\mathrm{d}f_{z_0}$-invariant splitting $T_{z_0}X = L_N \oplus L_U$ such that the spectrum of $\mathrm{d}f_{z_0}|_{L_N}$ is contained in Δ, the spectrum of $\mathrm{d}f_{z_0}|_{L_U}$ is contained in $\partial\Delta$, and $\mathrm{d}f_{z_0}|_{L_U}$ is diagonalizable.*

Corollary 1.3.11 (Wu, [33]) *Let X be a taut manifold, and $z_0 \in X$. Then if $f, g \in \mathrm{Aut}(X)$ are such that $f(z_0) = g(z_0)$ and $\mathrm{d}f_{z_0} = \mathrm{d}g_{z_0}$ then $f \equiv g$.*

Proof Apply Theorem 1.3.10.(iv) to $g^{-1} \circ f$. □

1.4 Convex Domains

In the following we shall be particularly interested in two classes of bounded domains in $\mathbb{C}^n$: convex domains and strongly pseudoconvex domains. Consequently, in this and the next section we shall collect some of the main properties of the Kobayashi distance respectively in convex and strongly pseudoconvex domains.

We start with convex domains recalling a few definitions.

Definition 1.4.1 Given $x, y \in \mathbb{C}^n$ let

$$[x, y] = \{sx+(1-s)y \in \mathbb{C}^n \mid s \in [0, 1]\} \text{ and } (x, y) = \{sx+(1-s)y \in \mathbb{C}^n \mid s \in (0, 1)\}$$

denote the *closed,* respectively *open, segment* connecting x and y. A set $D \subseteq \mathbb{C}^n$ is *convex* if $[x, y] \subseteq D$ for all $x, y \in D$; and *strictly convex* if $(x, y) \subseteq D$ for all $x, y \in \overline{D}$. A convex domain not strictly convex will sometimes be called *weakly convex.*

An easy but useful observation (whose proof is left to the reader) is:

Lemma 1.4.2 *Let $D \subset \mathbb{C}^n$ be a convex domain. Then:*

(i) $(z, w) \subset D$ *for all* $z \in D$ *and* $w \in \partial D$*;*
(ii) *if* $x, y \in \partial D$ *then either* $(x, y) \subset \partial D$ *or* $(x, y) \subset D$.

This suggests the following

Definition 1.4.3 Let $D \subset \mathbb{C}^n$ be a convex domain. Given $x \in \partial D$, we put

$$\mathrm{ch}(x) = \{y \in \partial D \mid [x, y] \subset \partial D\} ;$$

we shall say that x is a *strictly convex point* if $\mathrm{ch}(x) = \{x\}$. More generally, given $F \subseteq \partial D$ we put

$$\mathrm{ch}(F) = \bigcup_{x \in F} \mathrm{ch}(x) .$$

A similar construction having a more holomorphic character is the following:

Definition 1.4.4 Let $D \subset \mathbb{C}^n$ be a convex domain. A *complex supporting functional* at $x \in \partial D$ is a $\mathbb{C}$-linear map $L\colon \mathbb{C}^n \to \mathbb{C}$ such that $\mathrm{Re}\, L(z) < \mathrm{Re}\, L(x)$ for all $z \in D$. A *complex supporting hyperplane* at $x \in \partial D$ is an affine complex hyperplane $H \subset \mathbb{C}^n$ of the form $H = x + \ker L$, where L is a complex supporting functional at x (the existence of complex supporting functionals and hyperplanes is guaranteed by the Hahn-Banach theorem). Given $x \in \partial D$, we shall denote by $\mathrm{Ch}(x)$ the intersection of $\overline{D}$ with of all complex supporting hyperplanes at x. Clearly, $\mathrm{Ch}(x)$ is a closed convex set containing x; in particular, $\mathrm{Ch}(x) \subseteq \mathrm{ch}(x)$. If $\mathrm{Ch}(x) = \{x\}$ we say that x

is a *strictly* $\mathbb{C}$*-linearly convex point;* and we say that D is *strictly* $\mathbb{C}$*-linearly convex* if all points of ∂D are strictly $\mathbb{C}$-linearly convex. Finally, if $F \subset \partial D$ we set

$$\mathrm{Ch}(F) = \bigcup_{x \in F} \mathrm{Ch}(x) ;$$

clearly, $\mathrm{Ch}(F) \subseteq \mathrm{ch}(F)$.

Definition 1.4.5 Let $D \subset \mathbb{C}^n$ be a convex domain, $x \in \partial D$ and $L\colon \mathbb{C}^n \to \mathbb{C}$ a complex supporting functional at x. The *weak peak function* associated to L is the function $\psi \in \mathrm{Hol}(D, \Delta)$ given by

$$\psi(z) = \frac{1}{1 - \bigl(L(z) - L(x)\bigr)} .$$

Then ψ extends continuously to $\overline{D}$ with $\psi(\overline{D}) \subseteq \overline{\Delta}$, $\psi(x) = 1$, and $|\psi(z)| < 1$ for all $z \in D$; moreover $y \in \partial D$ is such that $|\psi(y)| = 1$ if and only if $\psi(y) = \psi(x) = 1$, and hence if and only if $L(y) = L(x)$.

Remark 1.4.6 If $x \in \partial D$ is a strictly convex point of a convex domain $D \subset \mathbb{C}^n$ then it is possible to find a complex supporting functional L at x so that $\mathrm{Re}\, L(z) < \mathrm{Re}\, L(x)$ for all $z \in \overline{D} \setminus \{x\}$. In particular, the associated weak peak function $\psi\colon \mathbb{C}^n \to \mathbb{C}$ is a true peak function (see Definition 1.5.17) in the sense that $|\psi(z)| < 1$ for all $z \in \overline{D} \setminus \{x\}$.

We shall now present three propositions showing how the Kobayashi distance is particularly well-behaved in convex domains. The first result, due to Lempert, shows that in convex domains the definition of Kobayashi distance can be simplified:

Proposition 1.4.7 (Lempert, [28]) *Let $D \subset \mathbb{C}^n$ be a convex domain. Then $\delta_D = k_D$.*

Proof First of all, note that $\delta_D(z, w) < +\infty$ for all $z, w \in D$. Indeed, let

$$\Omega = \{\lambda \in \mathbb{C} \mid (1 - \lambda)z + \lambda w \in D\} .$$

Since D is convex, Ω is a convex domain in $\mathbb{C}$ containing 0 and 1. Let $\phi\colon \Delta \to \Omega$ be a biholomorphism such that $\phi(0) = 0$; then the map $\varphi\colon \Delta \to D$ given by

$$\varphi(\zeta) = \bigl(1 - \phi(\zeta)\bigr)z + \phi(\zeta)w$$

is such that $z, w \in \varphi(\Delta)$.

Now, by definition we have $\delta_D(z, w) \geq k_D(z, w)$; to get the reverse inequality it suffices to show that δ_D satisfies the triangular inequality. Take $z_1, z_2, z_3 \in D$ and fix $\varepsilon > 0$. Then there are $\varphi_1, \varphi_2 \in \mathrm{Hol}(\Delta, D)$ and $\zeta_1, \zeta_2 \in \Delta$ such that $\varphi_1(0) = z_1$, $\varphi_1(\zeta_1) = \varphi_2(\zeta_1) = z_2$, $\varphi_2(\zeta_2) = z_3$ and

$$k_\Delta(0, \zeta_1) < \delta_D(z_1, z_2) + \varepsilon \,,$$
$$k_\Delta(\zeta_1, \zeta_2) < \delta_D(z_2, z_3) + \varepsilon \,.$$

Moreover, by Remark 1.1.7 we can assume that ζ_1 and ζ_2 are real, and that $\zeta_2 > \zeta_1 > 0$. Furthermore, up to replacing φ_j by a map φ_j^r defined by $\varphi_j^r(\zeta) = \varphi_j(r\zeta)$ for r close enough to 1, we can also assume that φ_j is defined and continuous on $\overline{\Delta}$ (and this for $j = 1, 2$).

Let $\lambda\colon \mathbb{C} \setminus \{\zeta_1, \zeta_1^{-1}\} \to \mathbb{C}$ be given by

$$\lambda(\zeta) = \frac{(\zeta - \zeta_2)(\zeta - \zeta_2^{-1})}{(\zeta - \zeta_1)(\zeta - \zeta_1^{-1})} \,.$$

Then λ is meromorphic in $\mathbb{C}$, and in a neighborhood of $\overline{\Delta}$ the only pole is the simple pole at ζ_1. Moreover, $\lambda(0) = 1$, $\lambda(\zeta_2) = 0$ and $\lambda(\partial\Delta) \subset [0, 1]$. Then define $\phi\colon \overline{\Delta} \to \mathbb{C}^n$ by

$$\phi(\zeta) = \lambda(\zeta)\varphi_1(\zeta) + \bigl(1 - \lambda(\zeta)\bigr)\varphi_2(\zeta) \,.$$

Since $\varphi_1(\zeta_1) = \varphi_2(\zeta_1)$, it turns out that ϕ is holomorphic on Δ; moreover, $\phi(0) = z_1$, $\phi(\zeta_2) = z_3$ and $\phi(\partial\Delta) \subset \overline{D}$. We claim that this implies that $\phi(\Delta) \subset D$. Indeed, otherwise there would be $\zeta_0 \in \Delta$ such that $\phi(\zeta_0) = x_0 \in \partial D$. Let L be a complex supporting functional at x_0, and ψ the associated weak peak function. Then we would have $|\psi \circ \phi| \le 1$ on $\partial\Delta$ and $|\psi \circ \phi(\zeta_0)| = 1$; thus, by the maximum principle, $|\psi \circ \phi| \equiv 1$, i.e., $\phi(\Delta) \subset \partial D$, whereas $\phi(0) \in D$, contradiction.

So $\phi \in \mathrm{Hol}(\Delta, D)$. In particular, then,

$$\delta_D(z_1, z_3) \le k_\Delta(0, \zeta_2) = k_\Delta(0, \zeta_1) + k_\Delta(\zeta_1, \zeta_2) \le \delta_D(z_1, z_2) + \delta_D(z_2, z_3) + 2\varepsilon \,,$$

and the assertion follows, since ε is arbitrary. □

Bounded convex domains, being bounded, are hyperbolic. But actually more is true:

Proposition 1.4.8 (Harris, [16]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Then D is complete hyperbolic.*

Proof We can assume $O \in D$. By Proposition 1.2.14, it suffices to show that all the closed Kobayashi balls $\overline{B_D(O, r)}$ of center O are compact. Let $\{z_\nu\} \subset \overline{B_D(O, r)}$; we must find a subsequence converging to a point of D. Clearly, we may suppose that $z_\nu \to w_0 \in \overline{D}$ as $\nu \to +\infty$, for D is bounded.

Assume, by contradiction, that $w_0 \in \partial D$, and let $L\colon \mathbb{C}^n \to \mathbb{C}$ be a complex supporting functional at w_0; in particular, $L(w_0) \neq 0$ (because $O \in D$). Set

$H = \{\zeta \in \mathbb{C} \mid \operatorname{Re} L(\zeta w_0) < \operatorname{Re} L(w_0)\}$; clearly H is a half-plane of $\mathbb{C}$, and the linear map $\pi\colon \mathbb{C}^n \to \mathbb{C}$ given by $\pi(z) = L(z)/L(w_0)$ sends D into H. In particular

$$r \geq k_D(0, z_\nu) \geq k_H\big(0, \pi(z_\nu)\big) .$$

Since H is complete hyperbolic, by Proposition 1.2.14 the closed Kobayashi balls in H are compact; therefore, up to a subsequence $\{\pi(z_\nu)\}$ tends to a point of H. On the other hand, $\pi(z_\nu) \to \pi(w_0) = 1 \in \partial H$, and this is a contradiction. □

Remark 1.4.9 There are unbounded convex domains which are not hyperbolic; for instance, $\mathbb{C}^n$ itself. However, unbounded hyperbolic convex domains are automatically complete hyperbolic, because Harris (see [16]) proved that a convex domain is hyperbolic if and only if it is biholomorphic to a bounded convex domain. Furthermore, Barth (see [9]) has shown that an unbounded convex domain is hyperbolic if and only if it does not contain any complex line.

Finally, the convexity is reflected by the shape of Kobayashi balls. To prove this (and also because they will be useful later) we shall need a couple of estimates:

Proposition 1.4.10 ([19, 26, 28]) *Let $D \subset \mathbb{C}^n$ be a convex domain. Then:*

(i) *if $z_1, z_2, w_1, w_2 \in D$ and $s \in [0, 1]$ then*

$$k_D\big(sz_1 + (1-s)z_2, sw_1 + (1-s)w_2\big) \leq \max\{k_D(z_1, w_1), k_D(z_2, w_2)\} ;$$

(ii) *if $z, w \in D$ and $s, t \in [0, 1]$ then*

$$k_D\big(sz + (1-s)w, tz + (1-t)w\big) \leq k_D(z, w) .$$

Proof Let us start by proving (i). Without loss of generality we can assume that $k_D(z_2, w_2) \leq k_D(z_1, w_1)$. Fix $\varepsilon > 0$; by Proposition 1.4.7, there are $\varphi_1, \varphi_2 \in \operatorname{Hol}(\Delta, D)$ and $\zeta_1, \zeta_2 \in \Delta$ such that $\varphi_j(0) = z_j$, $\varphi_j(\zeta_j) = w_j$ and $k_\Delta(0, \zeta_j) < k_D(z_j, w_j) + \varepsilon$, for $j = 1, 2$; moreover, we may assume $0 \leq \zeta_2 \leq \zeta_1 < 1$ and $\zeta_1 > 0$. Define $\psi\colon \Delta \to D$ by

$$\psi(\zeta) = \varphi_2\left(\frac{\zeta_2}{\zeta_1}\zeta\right) ,$$

so that $\psi(0) = z_2$ and $\psi(\zeta_1) = w_2$, and $\phi_s\colon \Delta \to \mathbb{C}^n$ by

$$\phi_s(\zeta) = s\varphi_1(\zeta) + (1-s)\psi(\zeta) .$$

Since D is convex, ϕ_s maps Δ into D; furthermore, $\phi_s(0) = sz_1 + (1-s)z_2$ and $\phi_s(\zeta_1) = sw_1 + (1-s)w_2$. Hence

$$\begin{aligned} k_D\big(sz_1 + (1-s)z_2, sw_1 + (1-s)w_2\big) = {}& k_D\big(\phi_s(0), \phi_s(\zeta_1)\big) \\ & \leq k_\Delta(0, \zeta_1) < k_D(z_1, w_1) + \varepsilon , \end{aligned}$$

and (i) follows because ε is arbitrary.

Given $z_0 \in D$, we obtain a particular case of (i) by setting $z_1 = z_2 = z_0$:

$$k_D\big(z_0, sw_1 + (1-s)w_2\big) \le \max\{k_D(z_0, w_1), k_D(z_0, w_2)\} \tag{1.5}$$

for all $z_0, w_1, w_2 \in D$ and $s \in [0, 1]$.

To prove (ii), put $z_0 = sz + (1-s)w$; then two applications of (1.5) yield

$$\begin{aligned} k_D\big(sz + (1-s)w, tz + (1-t)w\big) &\le \max\big\{k_D\big(sz + (1-s)w, z\big), k_D\big(sz + (1-s)w, w\big)\big\} \\ &\le k_D(z, w) \ , \end{aligned}$$

and we are done. □

Corollary 1.4.11 *Closed Kobayashi balls in a hyperbolic convex domain are compact and convex.*

Proof The compactness follows from Propositions 1.2.14 and 1.4.8 (and Remark 1.4.9 for unbounded hyperbolic convex domains); the convexity follows from (1.5). □

1.5 Strongly Pseudoconvex Domains

Another important class of domains where the Kobayashi distance has been studied in detail is given by strongly pseudoconvex domains. In particular, in strongly pseudoconvex domains it is possible to estimate the Kobayashi distance by means of the Euclidean distance from the boundary.

To recall the definition of strongly pseudoconvex domains, and to fix notations useful later, let us first introduce smoothly bounded domains. For simplicity we shall state the following definitions in $\mathbb{R}^N$, but they can be easily adapted to $\mathbb{C}^n$ by using the standard identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Definition 1.5.1 A domain $D \subset \mathbb{R}^N$ has C^r *boundary* (or is a C^r *domain*), where $r \in \mathbb{N} \cup \{\infty, \omega\}$ (and C^ω means real analytic), if there is a C^r function $\rho\colon \mathbb{R}^N \to \mathbb{R}$ such that:

(a) $D = \{x \in \mathbb{R}^N \mid \rho(x) < 0\}$;
(b) $\partial D = \{x \in \mathbb{R}^N \mid \rho(x) = 0\}$; and
(c) $\operatorname{grad}\rho$ is never vanishing on ∂D.

The function ρ is a *defining function* for D. The *outer unit normal vector* $\mathbf{n}_x$ at x is the unit vector parallel to $-\operatorname{grad}\rho(x)$.

Remark 1.5.2 It is not difficult to check that if ρ_1 is another defining function for a domain D then there is a never vanishing C^r function $\psi\colon \mathbb{R}^N \to \mathbb{R}^+$ such that

$$\rho_1 = \psi\rho \ . \tag{1.6}$$

If $D \subset \mathbb{R}^N$ is a C^r domain with defining function ρ, then ∂D is a C^r manifold embedded in $\mathbb{R}^N$. In particular, for every $x \in \partial D$ the tangent space of ∂D at x can be identified with the kernel of $\mathrm{d}\rho_x$ (which by (1.6) is independent of the chosen defining function). In particular, $T_x(\partial D)$ is just the hyperplane orthogonal to $\mathbf{n}_x$.

Using a defining function it is possible to check when a C^2-domain is convex.

Definition 1.5.3 If $\rho\colon \mathbb{R}^N \to \mathbb{R}$ is a C^2 function, the *Hessian* $H_{\rho,x}$ of ρ at $x \in \mathbb{R}^N$ is the symmetric bilinear form given by

$$H_{\rho,x}(v,w) = \sum_{h,k=1}^{N} \frac{\partial^2 \rho}{\partial x_h \partial x_k}(x)\, v_h w_k$$

for every $v, w \in \mathbb{R}^N$.

The following result is well-known (see, e.g, [25, p. 102]):

Proposition 1.5.4 *A C^2 domain $D \subset \mathbb{R}^N$ is convex if and only if for every $x \in \partial D$ the Hessian $H_{\rho,x}$ is positive semidefinite on $T_x(\partial D)$, where ρ is any defining function for D.*

This suggests the following.

Definition 1.5.5 A C^2 domain $D \subset \mathbb{R}^N$ is *strongly convex* at $x \in \partial D$ if for some (and hence any) C^2 defining function ρ for D the Hessian $H_{\rho,x}$ is positive definite on $T_x(\partial D)$. We say that D is *strongly convex* if it is so at each point of ∂D.

Remark 1.5.6 It is easy to check that strongly convex C^2 domains are strictly convex. Furthermore, it is also possible to prove that every strongly convex domain D has a C^2 defining function ρ such that $H_{\rho,x}$ is positive definite on the whole of $\mathbb{R}^N$ for every $x \in \partial D$ (see, e.g., [25, p. 101]).

Remark 1.5.7 If $D \subset \mathbb{C}^n$ is a convex C^1 domain and $x \in \partial D$ then the unique (up to a positive multiple) complex supporting functional at x is given by $L(z) = \langle z, \mathbf{n}_x \rangle$. In particular, $\mathrm{Ch}(x)$ coincides with the intersection of the associated complex supporting hyperplane with ∂D. But non-smooth points can have more than one complex supporting hyperplanes; this happens for instance in the polydisc.

Let us now move to a more complex setting.

Definition 1.5.8 Let $D \subset \mathbb{C}^n$ be a domain with C^2 boundary and defining function $\rho\colon \mathbb{C}^n \to \mathbb{R}$. The *complex tangent space* $T_x^{\mathbb{C}}(\partial D)$ of ∂D at $x \in \partial D$ is the kernel of $\partial \rho_x$, that is

$$T_x^{\mathbb{C}}(\partial D) = \left\{ v \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(x)\, v_j = 0 \right\} .$$

As usual, $T_x^{\mathbb{C}}(\partial D)$ does not depend on the particular defining function. The *Levi form* $L_{\rho,x}$ of ρ at $x \in \mathbb{C}^n$ is the Hermitian form given by

$$L_{\rho,x}(v,w) = \sum_{h,k=1}^{n} \frac{\partial^2 \rho}{\partial z_h \partial \bar{z}_k}(x)\, v_h \overline{w_k}$$

for every $v, w \in \mathbb{C}^n$.

Definition 1.5.9 A C^2 domain $D \subset \mathbb{C}^n$ is called *strongly pseudoconvex* (respectively, *weakly pseudoconvex*) at a point $x \in \partial D$ if for some (and hence all) C^2 defining function ρ for D the Levi form $L_{\rho,x}$ is positive definite (respectively, weakly positive definite) on $T_x^{\mathbb{C}}(\partial D)$. The domain D is *strongly pseudoconvex* (respectively, *weakly pseudoconvex*) if it is so at each point of ∂D.

Remark 1.5.10 If D is strongly pseudoconvex then there is a defining function ρ for D such that the Levi form $L_{\rho,x}$ is positive definite on $\mathbb{C}^n$ for every $x \in \partial D$ (see, e.g., [25, p. 109]).

Roughly speaking, strongly pseudoconvex domains are locally strongly convex. More precisely, one can prove (see, e.g., [2, Proposition 2.1.13]) the following:

Proposition 1.5.11 *A bounded C^2 domain $D \subset\subset \mathbb{C}^n$ is strongly pseudoconvex if and only if for every $x \in \partial D$ there is a neighborhood $U_x \subset \mathbb{C}^n$ and a biholomorphism $\Phi_x: U_x \to \Phi_x(U_x)$ such that $\Phi_x(U_x \cap D)$ is strongly convex.*

From this one can prove that strongly pseudoconvex domains are taut; but we shall directly prove that they are complete hyperbolic, as a consequence of the boundary estimates we are now going to state.

Definition 1.5.12 If $M \subset \mathbb{C}^n$ is any subset of $\mathbb{C}^n$, we shall denote by $d(\cdot, M): \mathbb{C}^n \to \mathbb{R}^+$ the Euclidean distance from M, defined by

$$d(z,M) = \inf\{\|z - x\| \mid x \in M\} .$$

To give an idea of the kind of estimates we are looking for, we shall prove an easy lemma:

Lemma 1.5.13 *Let $\mathbb{B}_r \subset \mathbb{C}^n$ be the Euclidean ball of radius $r > 0$ in $\mathbb{C}^n$ centered at the origin. Then*

$$\tfrac{1}{2}\log r - \tfrac{1}{2}\log d(z,\partial\mathbb{B}_r) \le k_{\mathbb{B}_r}(O,z) \le \tfrac{1}{2}\log(2r) - \tfrac{1}{2}\log d(z,\partial\mathbb{B}_r)$$

for every $z \in \mathbb{B}_r$.

Proof We have

$$k_{B_r}(O,z) = \tfrac{1}{2}\log\frac{1 + \|z\|/r}{1 - \|z\|/r} ,$$

and $d(z, \partial\mathbb{B}_r) = r - \|z\|$. Then, setting $t = \|z\|/r$, we get

$$\begin{aligned}\tfrac{1}{2}\log r - \tfrac{1}{2}\log d(z, \partial\mathbb{B}_r) = \ & \tfrac{1}{2}\log\frac{1}{1-t} \le \tfrac{1}{2}\log\frac{1+t}{1-t} = k_{B_r}(O, z) \\ & \le \tfrac{1}{2}\log\frac{2}{1-t} = \tfrac{1}{2}\log(2r) - \tfrac{1}{2}\log d(z, \partial\mathbb{B}_r)\ ,\end{aligned}$$

as claimed. □

Thus in the ball the Kobayashi distance from a reference point is comparable with one-half of the logarithm of the Euclidean distance from the boundary. We would like to prove similar estimates in strongly pseudoconvex domains. To do so we need one more definition.

Definition 1.5.14 Let M be a compact C^2-hypersurface of $\mathbb{R}^N$, and fix an unit normal vector field $\mathbf{n}$ on M. We shall say that M has a *tubular neighborhood* of *radius* $\varepsilon > 0$ if the segments $\{x + t\mathbf{n}_x \mid t \in (-\varepsilon, \varepsilon)\}$ are pairwise disjoint, and we set

$$U_\varepsilon = \bigcup_{x\in M}\{x + t\mathbf{n}_x \mid t \in (-\varepsilon, \varepsilon)\}\ .$$

Note that if M has a tubular neighborhood of radius ε, then $d(x + t\mathbf{n}_x, M) = |t|$ for every $t \in (-\varepsilon, \varepsilon)$ and $x \in M$; in particular, U_ε is the union of the Euclidean balls $\mathbb{B}(x, \varepsilon)$ of center $x \in M$ and radius ε.

Remark 1.5.15 A proof of the existence of a tubular neighborhood of radius sufficiently small for any compact C^2-hypersurface of $\mathbb{R}^N$ can be found, e.g., in [27, Theorem 10.19].

And now, we begin proving the estimates. The upper estimate does not even depend on the strong pseudoconvexity:

Theorem 1.5.16 ([1, 32]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded C^2 domain, and $z_0 \in D$. Then there is a constant $c_1 \in \mathbb{R}$ depending only on D and z_0 such that*

$$k_D(z_0, z) \le c_1 - \tfrac{1}{2}\log d(z, \partial D) \tag{1.7}$$

for all $z \in D$.

Proof Since D is a bounded C^2 domain, ∂D admits tubular neighborhoods U_ε of radius $\varepsilon < 1$ small enough. Put

$$c_1 = \sup\bigl\{k_D(z_0, w) \bigm| w \in D \setminus U_{\varepsilon/4}\bigr\} + \max\bigl\{0, \tfrac{1}{2}\log\operatorname{diam}(D)\bigr\}\ ,$$

where $\operatorname{diam}(D)$ is the Euclidean diameter of D.

There are two cases:

(i) $z \in U_{\varepsilon/4} \cap D$. Let $x \in \partial D$ be such that $\|x - z\| = d(z, \partial D)$. Since $U_{\varepsilon/2}$ is a tubular neighborhood of ∂D, there exists $\lambda \in \mathbb{R}$ such that $w = \lambda(x - z) \in \partial U_{\varepsilon/2} \cap D$ and the Euclidean ball $\mathbb{B}$ of center w and radius $\varepsilon/2$ is contained in $U_\varepsilon \cap D$ and tangent to ∂D in x. Therefore Lemma 1.5.13 yields

$$\begin{aligned} k_D(z_0, z) &\leq k_D(z_0, w) + k_D(w, z) \leq k_D(z_0, w) + k_B(w, z) \\ &\leq k_D(z_0, w) + \tfrac{1}{2} \log \varepsilon - \tfrac{1}{2} \log d(z, \partial B) \\ &\leq c_1 - \tfrac{1}{2} \log d(z, \partial D) \ , \end{aligned}$$

because $w \notin U_{\varepsilon/4}$ (and $\varepsilon < 1$).

(ii) $z \in D \setminus U_{\varepsilon/4}$. Then

$$k_D(z_0, z) \leq c_1 - \tfrac{1}{2} \log \operatorname{diam}(D) \leq c_1 - \tfrac{1}{2} \log d(z, \partial D) \ ,$$

because $d(z, \partial D) \leq \operatorname{diam}(D)$, and we are done. □

To prove the more interesting lower estimate, we need to introduce the last definition of this subsection.

Definition 1.5.17 Let $D \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$, and $x \in \partial D$. A *peak function* for D at x is a holomorphic function $\psi \in \mathrm{Hol}(D, \Delta)$ continuous up to the boundary of D such that $\psi(x) = 1$ and $|\psi(z)| < 1$ for all $z \in \overline{D} \setminus \{x\}$.

If $D \subset \mathbb{C}^n$ is strongly convex and $x \in \partial D$ then by Remark 1.4.6 there exists a peak function for D at x. Since a strongly pseudoconvex domain D is locally strongly convex, using Proposition 1.5.11 one can easily build peak functions defined in a neighborhood of a point of the boundary of D. To prove the more interesting lower estimate on the Kobayashi distance we shall need the non-trivial fact that in a strongly pseudoconvex domain it is possible to build a family of *global* peak functions continuously dependent on the point in the boundary:

Theorem 1.5.18 (Graham, [14]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain. Then there exist a neighborhood D' of $\overline{D}$ and a continuous function $\Psi \colon \partial D \times D' \to \mathbb{C}$ such that $\Psi_{x_0} = \Psi(x_0, \cdot)$ is holomorphic in D' and a peak function for D at x_0 for each $x_0 \in \partial D$.*

With this result we can prove

Theorem 1.5.19 ([1, 32]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex C^2 domain, and $z_0 \in D$. Then there is a constant $c_2 \in \mathbb{R}$ depending only on D and z_0 such that*

$$c_2 - \tfrac{1}{2} \log d(z, \partial D) \leq k_D(z_0, z) \tag{1.8}$$

for all $z \in D$.

Proof Let $D' \supset\supset D$ and $\Psi\colon \partial D \times D' \to \mathbb{C}$ be given by Theorem 1.5.18, and define $\phi\colon \partial D \times \Delta \to \mathbb{C}$ by

$$\phi(x,\zeta) = \frac{1-\overline{\Psi(x,z_0)}}{1-\Psi(x,z_0)} \cdot \frac{\zeta - \Psi(x,z_0)}{1-\overline{\Psi(x,z_0)}\zeta} \ . \tag{1.9}$$

Then the map $\Phi(x,z) = \Phi_x(z) = \phi\big(x, \Psi(x,z)\big)$ is defined on a neighborhood $\partial D \times D_0$ of $\partial D \times \overline{D}$ (with $D_0 \subset\subset D'$) and satisfies

(a) Φ is continuous, and Φ_x is a holomorphic peak function for D at x for any $x \in \partial D$;
(b) for every $x \in \partial D$ we have $\Phi_x(z_0) = 0$.

Now set $U_\varepsilon = \bigcup_{x\in\partial D} P(x,\varepsilon)$, where $P(x,\varepsilon)$ is the polydisc of center x and polyradius $(\varepsilon,\dots,\varepsilon)$. The family $\{U_\varepsilon\}$ is a basis for the neighborhoods of ∂D; hence there exists $\varepsilon > 0$ such that $U_\varepsilon \subset\subset D_0$ and U_ε is contained in a tubular neighborhood of ∂D. Then for any $x \in \partial D$ and $z \in P(x,\varepsilon/2)$ the Cauchy estimates yield

$$\begin{aligned} |1-\Phi_x(z)| = |\Phi_x(x) - \Phi_x(z)| &\le \left\| \frac{\partial \Phi_x}{\partial z} \right\|_{P(x,\varepsilon/2)} \|z-x\| \\ &\le \frac{2\sqrt{n}}{\varepsilon} \|\Phi\|_{\partial D \times U_\varepsilon} \|z-x\| = M\|z-x\| \ , \end{aligned}$$

where M is independent of z and x; in these formulas $\|F\|_S$ denotes the supremum of the Euclidean norm of the map F on the set S.

Put $c_2 = -\frac{1}{2}\log M$; note that $c_2 \le \frac{1}{2}\log(\varepsilon/2)$, for $\|\Phi\|_{\partial D\times U_\varepsilon} \ge 1$. Then we again have two cases:

(i) $z \in D \cap U_{\varepsilon/2}$. Choose $x \in \partial D$ so that $d(z,\partial D) = \|z-x\|$. Since $\Phi_x(D) \subset \Delta$ and $\Phi_x(z_0) = 0$, we have

$$k_D(z_0,z) \ge k_\Delta\big(\Phi_x(z_0), \Phi_x(z)\big) \ge \tfrac{1}{2}\log\frac{1}{1-|\Phi_x(z)|} \ .$$

Now,

$$1-|\Phi_x(z)| \le |1-\Phi_x(z)| \le M\|z-x\| = M\, d(z,\partial D) \ ;$$

therefore

$$k_D(z_0,z) \ge -\tfrac{1}{2}\log M - \tfrac{1}{2}\log d(z,\partial D) = c_2 - \tfrac{1}{2}\log d(z,\partial D)$$

as desired.

(ii) $z \in D \setminus U_{\varepsilon/2}$. Then $d(z, \partial D) \geq \varepsilon/2$; hence

$$k_D(z_0, z) \geq 0 \geq \tfrac{1}{2}\log(\varepsilon/2) - \tfrac{1}{2}\log d(z, \partial D) \geq c_2 - \tfrac{1}{2}\log d(z, \partial D) \ ,$$

and we are done. □

A first consequence is the promised:

Corollary 1.5.20 (Graham, [14]) *Every bounded strongly pseudoconvex C^2 domain D is complete hyperbolic.*

Proof Take $z_0 \in D$, $r > 0$ and let $z \in B_D(z_0, r)$. Then (1.8) yields

$$d(z, \partial D) \geq \exp\bigl(2(c_2 - r)\bigr) \ ,$$

where c_2 depends only on z_0. Then $B_D(z_0, r)$ is relatively compact in D, and the assertion follows from Proposition 1.2.14. □

For dynamical applications we shall also need estimates on the Kobayashi distance $k_D(z_1, z_2)$ when both z_1 and z_2 are close to the boundary. The needed estimates were proved by Forstnerič and Rosay (see [13], and [2, Corollary 2.3.55, Theorem 2.3.56]):

Theorem 1.5.21 ([13]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex C^2 domain, and choose two points $x_1, x_2 \in \partial D$ with $x_1 \neq x_2$. Then there exist $\varepsilon_0 > 0$ and $K \in \mathbb{R}$ such that for any $z_1, z_2 \in D$ with $\|z_j - x_j\| < \varepsilon_0$ for $j = 1, 2$ we have*

$$k_D(z_1, z_2) \geq -\tfrac{1}{2}\log d(z_1, \partial D) - \tfrac{1}{2}\log d(z_2, \partial D) + K \ . \tag{1.10}$$

Theorem 1.5.22 ([13]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded C^2 domain and $x_0 \in \partial D$. Then there exist $\varepsilon > 0$ and $C \in \mathbb{R}$ such that for all $z_1, z_2 \in D$ with $\|z_j - x_0\| < \varepsilon$ for $j = 1, 2$ we have*

$$k_D(z_1, z_2) \leq \tfrac{1}{2}\log\left(1 + \frac{\|z_1 - z_2\|}{d(z_1, \partial D)}\right) + \tfrac{1}{2}\log\left(1 + \frac{\|z_1 - z_2\|}{d(z_2, \partial D)}\right) + C \ . \tag{1.11}$$

We end this section by quoting a theorem, that we shall need in Chap. 6, giving a different way of comparing the Kobayashi geometry and the Euclidean geometry of strongly pseudoconvex domains:

Theorem 1.5.23 ([4]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^∞ domain, and $R > 0$. Then there exist $C_R > 0$ depending only on R and D such that*

$$\frac{1}{C_R} d(z_0, \partial D)^{n+1} \leq \nu\bigl(B_D(z_0, R)\bigr) \leq C_R d(z_0, \partial D)^{n+1}$$

for all $z_0 \in D$, where $\nu\bigl(B_D(z_0, R)\bigr)$ denotes the Lebesgue volume of the Kobayashi ball $B_D(z_0, R)$.

Acknowledgements Partially supported by the FIRB 2012 grant “Differential Geometry and Geometric Function Theory”, by the Progetto di Ricerca d’Ateneo 2015 “Sistemi dinamici: logica, analisi complessa e teoria ergodica”, and by GNSAGA-INdAM.

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Chapter 2
Dynamics in Several Complex Variables

Marco Abate

In this chapter we shall describe the dynamics of holomorphic self-maps of taut manifolds, and in particular the dynamics of holomorphic self-maps of convex and strongly pseudoconvex domains. A main tool in this exploration will be provided by the Kobayashi distance.

Definition 2.0.1 Let $f: X \to X$ be a self-map of a set X. Given $k \in \mathbb{N}$, we define the k-th *iterate* f^k of f setting by induction $f^0 = \mathrm{id}_X$, $f^1 = f$ and $f^k = f \circ f^{k-1}$. Given $x \in X$, the *orbit* of x is the set $\{f^k(x) \mid k \in \mathbb{N}\}$.

Studying the dynamics of a self-map f means studying the asymptotic behavior of the sequence $\{f^k\}$ of iterates of f; in particular, in principle one would like to know the behavior of all orbits. In general this is too an ambitious task; but as we shall see it can be achieved for holomorphic self-maps of taut manifolds, because the normality condition prevents the occurrence of chaotic behavior.

The model theorem for this theory is the famous Wolff-Denjoy theorem (for a proof see, e.g., [2, Theorem 1.3.9]):

Theorem 2.0.2 (Wolff-Denjoy, [12, 23]) *Let $f \in \mathrm{Hol}(\Delta, \Delta) \setminus \{\mathrm{id}_\Delta\}$ be a holomorphic self-map of Δ different from the identity. Assume that f is not an elliptic automorphism. Then the sequence of iterates of f converges, uniformly on compact subsets, to a constant map $\tau \in \overline{\Delta}$.*

Definition 2.0.3 Let $f \in \mathrm{Hol}(\Delta, \Delta) \setminus \{\mathrm{id}_\Delta\}$ be a holomorphic self-map of Δ different from the identity and not an elliptic automorphism. Then the point $\tau \in \overline{\Delta}$ whose existence is asserted by Theorem 2.0.2 is the *Wolff point* of f.

Actually, we can even be slightly more precise, introducing a bit of terminology.

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7_2

Definition 2.0.4 Let $f: X \to X$ be a self-map of a set X. A *fixed point* of f is a point $x_0 \in X$ such that $f(x_0) = x_0$. We shall denote by $\mathrm{Fix}(f)$ the set of fixed points of f. More generally, we shall say that $x_0 \in X$ is *periodic* of *period* $p \geq 1$ if $f^p(x_0) = x_0$ and $f^j(x_0) \neq x_0$ for all $j = 1, \ldots, p-1$. We shall say that f is *periodic* of *period* $p \geq 1$ if $f^p = \mathrm{id}_X$, that is if all points are periodic of period (at most) p.

Definition 2.0.5 Let $f: X \to X$ be a continuous self-map of a topological space X. We shall say that a continuous map $g: X \to X$ is a *limit map* of f if there is a subsequence of iterates of f converging to g (uniformly on compact subsets). We shall denote by $\Gamma(f) \subset C^0(X, X)$ the set of limit maps of f. If $\mathrm{id}_X \in \Gamma(f)$ we shall say that f is *pseudoperiodic*.

Example 2.0.6 Let $\gamma_\theta \in \mathrm{Aut}(\Delta)$ be given by $\gamma_\theta(\zeta) = e^{2\pi i\theta}\zeta$. It is easy to check that γ_θ is periodic if $\theta \in \mathbb{Q}$, and it is pseudoperiodic (but not periodic) if $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 2.0.7 Let X and Y be two sets (topological spaces, complex manifolds, etc.). Two self-maps $f: X \to X$ and $g: Y \to Y$ are *conjugate* if there exists a bijection (homeomorphism, biholomorphism, etc.) $\psi: X \to Y$ such that $f = \psi^{-1} \circ g \circ \psi$.

If f and g are conjugate via ψ, we clearly have $f^k = \psi^{-1} \circ g^k \circ \psi$ for all $k \in \mathbb{N}$; therefore f and g share the same dynamical properties.

Example 2.0.8 It is easy to check that any elliptic automorphism of Δ is (biholomorphically) conjugated to one of the automorphisms γ_θ introduced in Example 2.0.6. Therefore an elliptic automorphism of Δ is necessarily periodic or pseudoperiodic.

We can now better specify the content of Theorem 2.0.2 as follows. Take $f \in \mathrm{Hol}(\Delta, \Delta)$ different from the identity. We have two cases: either f has a fixed point $\tau \in \Delta$ or $\mathrm{Fix}(f) = \emptyset$ (notice that, by the Schwarz-Pick lemma and the structure of the automorphisms of Δ, the only holomorphic self-map of Δ with at least two distinct fixed points is the identity). Then:

(a) If $\mathrm{Fix}(f) = \{\tau\}$, then either f is an elliptic automorphism—and hence it is periodic or pseudoperiodic—or the whole sequence of iterates converges to the constant function τ;
(b) if $\mathrm{Fix}(f) = \emptyset$ then there exists a unique point $\tau \in \partial\Delta$ such that the whole sequence of iterates converges to the constant function τ.

So there is a natural dichotomy between self-maps with fixed points and self-maps without fixed points. Our aim is to present a (suitable) generalization of the Wolff-Denjoy theorem to taut manifolds in any (finite) dimension. Even in several variables a natural dichotomy will appear; but it will be slightly different.

2.1 Dynamics in Taut Manifolds

Let X be a taut manifold. Then the whole family $\mathrm{Hol}(X, X)$ is normal; in particular, if $f \in \mathrm{Hol}(X, X)$ the sequence of iterates $\{f^k\}$ is normal. This suggests to subdivide the study of the dynamics of self-maps of X in three tasks:

(a) to study the dynamics of f when the sequence $\{f^k\}$ is not compactly divergent;
(b) to find conditions on f ensuring that the sequence $\{f^k\}$ is not compactly divergent;
(c) to study the dynamics of f when the sequence $\{f^k\}$ is compactly divergent.

So in several variables the natural dichotomy to consider is between maps having a compactly divergent sequence of iterates and maps whose sequence of iterates is not compactly divergent. If f has a fixed point its sequence of iterates cannot be compactly divergent; so this dichotomy has something to do with the dichotomy discussed in the introduction to this section but, as we shall see, in general they are not the same.

In this subsection we shall discuss tasks (a) and (b). To discuss task (c) we shall need a boundary; we shall limit ourselves to discuss (in the next three subsections) the case of bounded (convex or strongly pseudoconvex) domains in $\mathbb{C}^n$.

An useful notion for our discussion is the following

Definition 2.1.1 A *holomorphic retraction* of a complex manifold X is a holomorphic self-map $\rho \in \mathrm{Hol}(X, X)$ such that $\rho^2 = \rho$. In particular, $\rho(X) = \mathrm{Fix}(\rho)$. The image of a holomorphic retraction is a *holomorphic retract.*

The dynamics of holomorphic retraction is trivial: the iteration stops at the second step. On the other hand, it is easy to understand why holomorphic retractions might be important in holomorphic dynamics. Indeed, assume that the sequence of iterates $\{f^k\}$ converges to a map ρ. Then the subsequence $\{f^{2k}\}$ should converge to the same map; but $f^{2k} = f^k \circ f^k$, and thus $\{f^{2k}\}$ converges to $\rho \circ \rho$ too—and hence $\rho^2 = \rho$, that is ρ is a holomorphic retraction.

In dimension one, a holomorphic retraction must be either the identity or a constant map, because of the open mapping theorem and the identity principle. In several variables there is instead plenty of non-trivial holomorphic retractions.

Example 2.1.2 Let $\mathbb{B}^2$ be the unit Euclidean ball in $\mathbb{C}^2$. The power series

$$1 - \sqrt{1-t} = \sum_{k=1}^{\infty} c_k t^k$$

is converging for $|t| < 1$ and has $c_k > 0$ for all $k \geq 1$. Take $g_k \in \mathrm{Hol}(\mathbb{B}^2, \mathbb{C})$ such that $|g_k(z, w)| \leq c_k$ for all $(z, w) \in \mathbb{B}^2$, and define $\phi \in \mathrm{Hol}(\mathbb{B}^2, \Delta)$ by

$$\phi(z, w) = z + \sum_{k=1}^{\infty} g_k(z, w)\, w^{2k}.$$

Then $\rho(z,w) = \big(\phi(z,w),0\big)$ always satisfies $\rho^2 = \rho$, and it is neither constant nor the identity.

On the other hand, holomorphic retracts cannot be wild. This has been proven for the first time by Rossi [22]; here we report a clever proof due to H. Cartan [11]:

Lemma 2.1.3 *Let X be a complex manifold, and $\rho\colon X \to X$ a holomorphic retraction of X. Then the image of ρ is a closed submanifold of X.*

Proof Let $M = \rho(X)$ be the image of ρ, and take $z_0 \in M$. Choose an open neighborhood U of z_0 in X contained in a local chart for X at z_0. Then $V = \rho^{-1}(U) \cap U$ is an open neighborhood of z_0 contained in a local chart such that $\rho(V) \subseteq V$. Therefore without loss of generality we can assume that X is a bounded domain D in $\mathbb{C}^n$.

Set $P = d\rho_{z_0}\colon \mathbb{C}^n \to \mathbb{C}^n$, and define $\varphi\colon D \to \mathbb{C}^n$ by

$$\varphi = \mathrm{id}_D + (2P - \mathrm{id}_D) \circ (\rho - P) .$$

Since $d\varphi_{z_0} = \mathrm{id}$, the map φ defines a local chart in a neighborhood of z_0. Now $P^2 = P$ and $\rho^2 = \rho$; hence

$$\begin{aligned}\varphi \circ \rho &= \rho + (2P - \mathrm{id}_D) \circ \rho^2 - (2P - \mathrm{id}_D) \circ P \circ \rho \\ &= P \circ \rho = P + P \circ (2P - \mathrm{id}_D) \circ (\rho - P) = P \circ \varphi .\end{aligned}$$

Therefore in this local chart ρ becomes linear, and M is a submanifold near z_0. By the arbitrariness of z_0, the assertion follows. □

Having the notion of holomorphic retraction, we can immediately explain why holomorphic dynamics is trivial in compact hyperbolic manifolds (for a proof see, e.g., [2, Theorem 2.4.9]):

Theorem 2.1.4 (Kaup, [17]) *Let X be a compact hyperbolic manifold, and $f \in$ Hol(X,X). Then there is $m \in \mathbb{N}$ such that f^m is a holomorphic retraction.*

So from now on we shall concentrate on non-compact taut manifolds. The basic result describing the dynamics of self-maps whose sequence of iterates is not compactly divergent is the following:

Theorem 2.1.5 (Bedford, [7]; Abate, [1]) *Let X be a taut manifold, and $f \in$ Hol(X,X). Assume that the sequence $\{f^k\}$ of iterates of f is not compactly divergent. Then there exists a unique holomorphic retraction $\rho \in \Gamma(f)$ onto a submanifold M of X such that every limit map $h \in \Gamma(f)$ is of the form*

$$h = \gamma \circ \rho , \tag{2.1}$$

where γ is an automorphism of M. Furthermore, $\varphi = f|_M \in$ Aut(M) and $\Gamma(f)$ is isomorphic to a subgroup of Aut(M), *the closure of $\{\varphi^k\}$ in* Aut(M).

Proof Since the sequence $\{f^k\}$ of iterates is not compactly divergent, it must contain a subsequence $\{f^{k_\nu}\}$ converging to $h \in \mathrm{Hol}(X,X)$. We can also assume that $p_\nu = k_{\nu+1} - k_\nu$ and $q_\nu = p_\nu - k_\nu = k_{\nu+1} - 2k_\nu$ tend to $+\infty$ as $\nu \to +\infty$, and that $\{f^{p_\nu}\}$ and $\{f^{q_\nu}\}$ are either converging or compactly divergent. Now we have

$$\lim_{\nu\to\infty} f^{p_\nu}\big(f^{k_\nu}(z)\big) = \lim_{\nu\to\infty} f^{k_{\nu+1}}(z) = h(z)$$

for all $z \in X$; therefore $\{f^{p_\nu}\}$ cannot be compactly divergent, and thus converges to a map $\rho \in \mathrm{Hol}(X,X)$ such that

$$h \circ \rho = \rho \circ h = h \; . \tag{2.2}$$

Next, for all $z \in X$ we have

$$\lim_{\nu\to\infty} f^{q_\nu}\big(f^{k_\nu}(z)\big) = \lim_{\nu\to\infty} f^{p_\nu}(z) = \rho(z) \; .$$

Hence neither $\{f^{q_\nu}\}$ can be compactly divergent, and thus converges to a map $g \in \mathrm{Hol}(X,X)$ such that

$$g \circ h = h \circ g = \rho \; . \tag{2.3}$$

In particular

$$\rho^2 = \rho \circ \rho = g \circ h \circ \rho = g \circ h = \rho \; ,$$

and ρ is a holomorphic retraction of X onto a submanifold M. Now (2.2) implies $h(X) \subseteq M$. Since $g \circ \rho = \rho \circ g$, we have $g(M) \subseteq M$ and (2.3) yields

$$g \circ h|_M = h \circ g|_M = \mathrm{id}_M \; ;$$

hence $\gamma = h|_M \in \mathrm{Aut}(M)$ and (2.2) becomes (2.1).

Now, let $\{f^{k'_\nu}\}$ be another subsequence of $\{f^k\}$ converging to a map $h' \in \mathrm{Hol}(X,X)$. Arguing as before, we can assume $s_\nu = k'_\nu - k_\nu$ and $t_\nu = k_{\nu+1} - k'_\nu$ are converging to $+\infty$ as $\nu \to +\infty$, and that $\{f^{s_\nu}\}$ and $\{f^{t_\nu}\}$ converge to holomorphic maps $\alpha \in \mathrm{Hol}(X,X)$, respectively $\beta \in \mathrm{Hol}(X,X)$ such that

$$\alpha \circ h = h \circ \alpha = h' \qquad \text{and} \qquad \beta \circ h' = h' \circ \beta = h \; . \tag{2.4}$$

Then $h(X) = h'(X)$, and so M does not depend on the particular converging subsequence.

We now show that ρ itself does not depend on the chosen subsequence. Write $h = \gamma_1 \circ \rho_1$, $h' = \gamma_2 \circ \rho_2$, $\alpha = \gamma_3 \circ \rho_3$ and $\beta = \gamma_4 \circ \rho_4$, where ρ_1, ρ_2, ρ_3 and ρ_4

are holomorphic retractions of X onto M, and γ_1, γ_2, γ_3 and γ_4 are automorphisms of M. Then $h \circ h' = h' \circ h$ and $\alpha \circ \beta = \beta \circ \alpha$ together with (2.4) become

$$\begin{aligned} \gamma_1 \circ \gamma_2 \circ \rho_2 &= \gamma_2 \circ \gamma_1 \circ \rho_1 \ , \\ \gamma_3 \circ \gamma_1 \circ \rho_1 &= \gamma_1 \circ \gamma_3 \circ \rho_3 = \gamma_2 \circ \rho_2 \ , \\ \gamma_4 \circ \gamma_2 \circ \rho_2 &= \gamma_2 \circ \gamma_4 \circ \rho_4 = \gamma_1 \circ \rho_1 \ , \\ \gamma_3 \circ \gamma_4 \circ \rho_4 &= \gamma_4 \circ \gamma_3 \circ \rho_3 \ . \end{aligned} \tag{2.5}$$

Writing ρ_2 in function of ρ_1 using the first and the second equation in (2.5) we find $\gamma_3 = \gamma_1^{-1} \circ \gamma_2$. Writing ρ_1 in function of ρ_2 using the first and the third equation, we get $\gamma_4 = \gamma_2^{-1} \circ \gamma_1$. Hence $\gamma_3 = \gamma_4^{-1}$ and the fourth equation yields $\rho_3 = \rho_4$. But then, using the second and third equation we obtain

$$\rho_2 = \gamma_3^{-1} \circ \gamma_1^{-1} \circ \gamma_2 \circ \rho_2 = \rho_3 = \rho_4 = \gamma_4^{-1} \circ \gamma_2^{-1} \circ \gamma_1 \circ \rho_1 = \rho_1 \ ,$$

as claimed.

Next, from $f \circ \rho = \rho \circ f$ it follows immediately that $f(M) \subseteq M$. Put $\varphi = f|_M$; if $f^{p_\nu} \to \rho$ then $f^{p_\nu+1} \to \varphi \circ \rho$, and thus $\varphi \in \mathrm{Aut}(M)$.

Finally, for each limit point $h = \gamma \circ \rho \in \Gamma(f)$ we have $\gamma^{-1} \circ \rho \in \Gamma(f)$. Indeed fix a subsequence $\{f^{p_\nu}\}$ converging to ρ, and a subsequence $\{f^{k_\nu}\}$ converging to h. As usual, we can assume that $p_\nu - k_\nu \to +\infty$ and $f^{p_\nu - k_\nu} \to h_1 = \gamma_1 \circ \rho$ as $\nu \to +\infty$. Then $h \circ h_1 = \rho = h_1 \circ h$, that is $\gamma_1 = \gamma^{-1}$. Hence the association $h = \gamma \circ \rho \mapsto \gamma$ yields an isomorphism between $\Gamma(f)$ and the subgroup of $\mathrm{Aut}(M)$ obtained as closure of $\{\varphi^k\}$. □

Definition 2.1.6 Let X be a taut manifold and $f \in \mathrm{Hol}(X, X)$ such that the sequence $\{f^k\}$ is not compactly divergent. The manifold M whose existence is asserted in the previous theorem is the *limit manifold* of the map f, and its dimension is the *limit multiplicity* m_f of f; finally, the holomorphic retraction is the *limit retraction* of f.

It is also possible to describe precisely the algebraic structure of the group $\Gamma(f)$, because it is compact. This is a consequence of the following theorem (whose proof generalizes an argument due to Całka [10]), that, among other things, says that if a sequence of iterates is not compactly divergent then it does not contain any compactly divergent subsequence, and thus it is relatively compact in $\mathrm{Hol}(X, X)$:

Theorem 2.1.7 (Abate, [3]) *Let X be a taut manifold, and $f \in \mathrm{Hol}(X, X)$. Then the following assertions are equivalent:*

(i) *the sequence of iterates $\{f^k\}$ is not compactly divergent;*
(ii) *the sequence of iterates $\{f^k\}$ does not contain any compactly divergent subsequence;*
(iii) *$\{f^k\}$ is relatively compact in $\mathrm{Hol}(X, X)$;*
(iv) *the orbit of $z \in X$ is relatively compact in X for all $z \in X$;*
(v) *there exists $z_0 \in X$ whose orbit is relatively compact in X.*

Proof (v)$\Longrightarrow$(ii). Take $H = \{z_0\}$ and $K = \overline{\{f^k(z_0)\}}$. Then H and K are compact and $f^k(H) \cap K \neq \emptyset$ for all $k \in \mathbb{N}$, and so no subsequence of $\{f^k\}$ can be compactly divergent.

(ii)$\Longrightarrow$(iii). Since $\mathrm{Hol}(X, X)$ is a metrizable topological space, if $\{f^k\}$ is not relatively compact then it admits a subsequence $\{f^{k_\nu}\}$ with no converging subsequences. But then, being X taut, $\{f^{k_\nu}\}$ must contain a compactly divergent subsequence, against (ii).

(iii)$\Longrightarrow$(iv). The evaluation map $\mathrm{Hol}(X, X) \times X \to X$ is continuous.

(iv)$\Longrightarrow$(i). Obvious.

(i)$\Longrightarrow$(v). Let M be the limit manifold of f, and let $\varphi = f|_M$. By Theorem 2.1.5 we know that $\varphi \in \mathrm{Aut}(M)$ and that $\mathrm{id}_M \in \Gamma(\varphi)$.

Take $z_0 \in M$; we would like to prove that $C = \{\varphi^k(z_0)\}$ is relatively compact in M (and hence in X). Choose $\varepsilon_0 > 0$ so that $B_M(z_0, \varepsilon_0)$ is relatively compact in M; notice that $\varphi \in \mathrm{Aut}(M)$ implies that $B_M\big(\varphi^k(z_0), \varepsilon_0\big) = \varphi^k\big(B_M(z_0, \varepsilon_0)\big)$ is relatively compact in M for all $k \in \mathbb{N}$. By Lemma 1.2.12 we have

$$\overline{B_M(z_0, \varepsilon_0)} \subseteq B_M\big(B_M(z_0, 7\varepsilon_0/8), \varepsilon_0/4\big) \;;$$

hence there are $w_1, \ldots, w_r \in B_M(z_0, 7\varepsilon_0/8)$ such that

$$\overline{B_M(z_0, \varepsilon_0)} \cap C \subset \bigcup_{j=1}^{r} B_M(w_j, \varepsilon_0/4) \cap C \;,$$

and we can assume that $B_M(w_j, \varepsilon_0/4) \cap C \neq \emptyset$ for $j = 1, \ldots, r$.

For each $j = 1, \ldots, r$ choose $k_j \in \mathbb{N}$ so that $\varphi^{k_j}(z_0) \in B_M(w_j, \varepsilon_0/4)$; then

$$B_M(z_0, \varepsilon_0) \cap C \subset \bigcup_{j=1}^{r} \big[B_M\big(\varphi^{k_j}(z_0), \varepsilon_0/2\big) \cap C\big] \tag{2.6}$$

Since $\mathrm{id}_M \in \Gamma(\varphi)$, the set $I = \big\{k \in \mathbb{N} \;\big|\; k_M\big(\varphi^k(z_0), z_0\big) < \varepsilon_0/2\big\}$ is infinite; therefore we can find $k_0 \in \mathbb{N}$ such that

$$k_0 \geq \max\{1, k_1, \ldots, k_r\} \quad \text{and} \quad k_M\big(\varphi^{k_0}(z_0), z_0\big) < \varepsilon_0/2 \,. \tag{2.7}$$

Put

$$K = \bigcup_{k=1}^{k_0} \overline{B_M\big(\varphi^k(z_0), \varepsilon_0\big)} \;;$$

since, by construction, K is compact, to end the proof it suffices to show that $C \subset K$. Take $h_0 \in I$; since the set I is infinite, it suffices to show that $\varphi^k(z_0) \in K$ for all $0 \leq k \leq h_0$.

Assume, by contradiction, that h_0 is the least element of I such that $\{\varphi^k(z_0) \mid 0 \leq k \leq h_0\}$ is not contained in K. Clearly, $h_0 > k_0$. Moreover, $k_M\big(\varphi^{h_0}(z_0), \varphi^{k_0}(z_0)\big) < \varepsilon_0$ by (2.7); thus

$$k_M\big(\varphi^{h_0-j}(z_0), \varphi^{k_0-j}(z_0)\big) = k_M\big(\varphi^{h_0}(z_0), \varphi^{k_0}(z_0)\big) < \varepsilon_0$$

for every $0 \leq j \leq k_0$. In particular,

$$\varphi^j(z_0) \in K \tag{2.8}$$

for every $j = h_0 - k_0, \dots, h_0$, and $\varphi^{h_0-k_0}(z_0) \in B_D(z_0, \varepsilon_0) \cap C$. By (2.6) we can find $1 \leq l \leq r$ such that $k_M\big(\varphi^{k_l}(z_0), \varphi^{h_0-k_0}(z_0)\big) < \varepsilon_0/2$, and so

$$k_M\big(\varphi^{h_0-k_0-j}(z_0), \varphi^{k_l-j}(z_0)\big) < \varepsilon_0/2 \tag{2.9}$$

for all $0 \leq j \leq \min\{k_l, h_0 - k_0\}$. In particular, if $k_l \geq h_0 - k_0$ then, by (2.6), (2.8) and (2.9) we have $\varphi^j(z_0) \in K$ for all $0 \leq j \leq h_0$, against the choice of h_0. So we must have $k_l < h_0 - k_0$; set $h_1 = h_0 - k_0 - k_l$. By (2.9) we have $h_1 \in I$; therefore, being $h_1 < h_0$, we have $\varphi^j(z_0) \in K$ for all $0 \leq j \leq h_1$. But (2.8) and (2.9) imply that $\varphi^j(z_0) \in K$ for $h_1 \leq j \leq h_0$, and thus we again have a contradiction. □

Corollary 2.1.8 (Abate, [3]) *Let X be a taut manifold, and $f \in \mathrm{Hol}(X, X)$ such that the sequence of iterates is not compactly divergent. Then $\Gamma(f)$ is isomorphic to a compact abelian group $\mathbb{Z}_q \times \mathbb{T}^r$, where $\mathbb{Z}_q$ is the cyclic group of order q and $\mathbb{T}^r$ is the real torus of dimension r.*

Proof Let M be the limit manifold of f, and put $\varphi = f|_M$. By Theorem 2.1.5, $\Gamma(f)$ is isomorphic to the closed subgroup Γ of $\mathrm{Aut}(M)$ generated by φ. We known that $\mathrm{Aut}(M)$ is a Lie group, by Theorem 1.2.16, and that Γ is compact, by Theorem 2.1.7. Moreover it is abelian, being generated by a single element. It is well known that the compact abelian Lie groups are all of the form $A \times \mathbb{T}^r$, where A is a finite abelian group; to conclude it suffices to notice that A must be cyclic, again because Γ is generated by a single element. □

Definition 2.1.9 Let X be a taut manifold, and $f \in \mathrm{Hol}(X, X)$ such that the sequence of iterates is not compactly divergent. Then the numbers q and r introduced in the last corollary are respectively the *limit period* q_f and the *limit rank* r_f of f.

When f has a periodic point $z_0 \in X$ of period $p \geq 1$, it is possible to explicitly compute the limit dimension, the limit period and the limit rank of f using the eigenvalues of $\mathrm{d}f^p_{z_0}$. To do so we need to introduce two notions.

Let $m \in \mathbb{N}$ and $\Theta = (\theta_1, \dots, \theta_m) \in [0, 1)^m$. Up to a permutation, we can assume that $\theta_1, \dots, \theta_{\nu_0} \in \mathbb{Q}$ and $\theta_{\nu_0+1}, \dots, \theta_m \notin \mathbb{Q}$ for some $0 \leq \nu_0 \leq m$ (where $\nu_0 = 0$ means $\Theta \in (\mathbb{R} \setminus \mathbb{Q})^m$ and $\nu_0 = m$ means $\Theta \in \mathbb{Q}^m$).

Let $q_1 \in \mathbb{N}^*$ be the least positive integer such that $q_1\theta_1, \dots, q_1\theta_{\nu_0} \in \mathbb{N}$; if $\nu_0 = 0$ we put $q_1 = 1$. For $i, j \in \{\nu_0 + 1, \dots, m\}$ we shall write $i \sim j$ if and only if $\theta_i - \theta_j \in \mathbb{Q}$. Clearly, $\sim$ is an equivalence relation; furthermore if $i \sim j$ then there is a smallest $q_{ij} \in \mathbb{N}^*$ such that $q_{ij}(\theta_i - \theta_j) \in \mathbb{Z}$. Let $q_2 \in \mathbb{N}^*$ be the least common multiple of $\{q_{ij} \mid i \sim j\}$; we put $q_2 = 1$ if $\nu_0 = m$ or $i \not\sim j$ for all pairs (i, j).

Definition 2.1.10 Let $\Theta = (\theta_1, \ldots, \theta_m) \in [0, 1)^m$. Then the *period* $q(\Theta) \in \mathbb{N}^*$ of Θ is the least common multiple of the numbers q_1 and q_2 introduced above.

Next, for $j = \nu_0 + 1, \ldots, m$ write $\theta'_j = q(\Theta)\theta_j - \lfloor q(\Theta)\theta_j \rfloor$, where $\lfloor s \rfloor$ is the integer part of $s \in \mathbb{R}$. Since

$$\theta'_i = \theta'_j \iff q(\Theta)(\theta_i - \theta_j) \in \mathbb{Z} \iff i \sim j\ ,$$

the set $\Theta' = \{\theta'_{\nu_0+1}, \ldots, \theta'_m\}$ contains as many elements as the number of $\sim$-equivalence classes. If this number is s, put $\Theta' = \{\theta''_1, \ldots, \theta''_s\}$. Write $i \approx j$ if and only if $\theta''_i / \theta''_j \in \mathbb{Q}$ (notice that $0 \notin \Theta'$); clearly $\approx$ is an equivalence relation.

Definition 2.1.11 Let $\Theta = (\theta_1, \ldots, \theta_m) \in [0, 1)^m$. Then the *rank* $r(\Theta) \in \mathbb{N}$ is the number of $\approx$-equivalence classes. If $\nu_0 = m$ then $r(\Theta) = 0$.

If X is a taut manifold and $f \in \mathrm{Hol}(X, X)$ has a fixed point $z_0 \in X$, Theorem 1.3.10 says that all the eigenvalues of $\mathrm{d}f_{z_0}$ belongs to $\overline{\Delta}$. Then we can prove the following:

Theorem 2.1.12 (Abate, [3]) *Let X be a taut manifold of dimension n, and $f \in \mathrm{Hol}(X, X)$ with a periodic point $z_0 \in X$ of period $p \geq 1$. Let $\lambda_1, \ldots, \lambda_n \in \overline{\Delta}$ be the eigenvalues of $\mathrm{d}(f^p)_{z_0}$, listed accordingly to their multiplicity and so that*

$$|\lambda_1| = \cdots = |\lambda_m| = 1 > |\lambda_{m+1}| \geq \cdots \geq |\lambda_n|$$

for a suitable $0 \leq m \leq n$. For $j = 1, \ldots, m$ write $\lambda_j = e^{2\pi i \theta_j}$ with $\theta_j \in [0, 1)$, and set $\Theta = (\theta_1, \ldots, \theta_m)$. Then

$$m_f = m\ , \qquad q_f = p \cdot q(\Theta) \qquad \textit{and} \qquad r_f = r(\Theta)\ .$$

Proof Let us first assume that z_0 is a fixed point, that is $p = 1$. Let M be the limit manifold of f, and $\rho \in \mathrm{Hol}(X, M)$ its limit retraction. As already remarked, by Theorem 1.3.10 the set $\mathrm{sp}(\mathrm{d}f_{z_0})$ of eigenvalues of $\mathrm{d}f_{z_0}$ is contained in $\overline{\Delta}$; furthermore there is a $\mathrm{d}f_{z_0}$-invariant splitting $T_{z_0}X = L_N \oplus L_U$ satisfying the following properties:

(a) $\mathrm{sp}(\mathrm{d}f_{z_0}|_{L_N}) = \mathrm{sp}(\mathrm{d}f_{z_0}) \cap \Delta$ and $\mathrm{sp}(\mathrm{d}f_{z_0}|_{L_U}) = \mathrm{sp}(\mathrm{d}f_{z_0}) \cap \partial\Delta$;
(b) $(\mathrm{d}f_{z_0}|_{L_N})^k \to O$ as $k \to +\infty$;
(c) $\mathrm{d}f_{z_0}|_{L_U}$ is diagonalizable.

Fix a subsequence $\{f^{k_\nu}\}$ converging to ρ; in particular, $(\mathrm{d}f_{z_0})^{k_\nu} \to \mathrm{d}\rho_{z_0}$ as $\nu \to +\infty$. Since the only possible eigenvalues of $\mathrm{d}\rho_{z_0}$ are 0 and 1, properties (b) and (c) imply that $\mathrm{d}\rho_{z_0}|_{L_N} \equiv O$ and $\mathrm{d}\rho_{z_0}|_{L_U} = \mathrm{id}$. In particular, it follows that $L_U = T_{z_0}M$ and $m_f = \dim T_{z_0}M = \dim L_U = m$, as claimed.

Set $\varphi = f|_M \in \mathrm{Aut}(M)$. By Corollary 1.3.11, the map $\gamma \mapsto \mathrm{d}\gamma_{z_0}$ is an isomorphism between the group of automorphisms of M fixing z_0 and a subgroup of linear transformations of $T_{z_0}M$. Therefore, since $\mathrm{d}\varphi_{z_0}$ is diagonalizable by (c),

$\Gamma(\varphi)$, and hence $\Gamma(f)$, is isomorphic to the closed subgroup of $\mathbb{T}^m$ generated by $\Lambda = (\lambda_1, \dots, \lambda_m)$. So we have to prove that this latter subgroup is isomorphic to $\mathbb{Z}_{q(\Theta)} \times \mathbb{T}^{r(\Theta)}$. Since we know beforehand the algebraic structure of this group (it is the product of a cyclic group with a torus), it will suffice to write it as a disjoint union of isomorphic tori; the number of tori will be the limit period of f, and the rank of the tori will be the limit rank of f.

Up to a permutation, we can find integers $0 \le \nu_0 < \nu_1 < \cdots < \nu_s = m$ such that $\theta_1, \dots, \theta_{\nu_0} \in \mathbb{Q}$, and the $\sim$-equivalence classes are

$$\{\theta_{\nu_0+1}, \dots, \theta_{\nu_1}\}, \dots, \{\theta_{\nu_{s-1}+1}, \dots, \theta_m\} .$$

Then, using the notations introduced for defining $q(\Theta)$ and $r(\Theta)$, we have

$$\Lambda^{q(\Theta)} = (1, \dots, 1, \mathrm{e}^{2\pi \mathrm{i}\theta_1''}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_1''}, \mathrm{e}^{2\pi \mathrm{i}\theta_2''}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_2''}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_s''}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_s''}) .$$

This implies that it suffices to show that the subgroup generated by

$$\Lambda_1 = (\mathrm{e}^{2\pi \mathrm{i}\theta_1''}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_s''})$$

in $\mathbb{T}^s$ is isomorphic to $\mathbb{T}^{r(\Theta)}$.

Up to a permutation, we can assume that the $\approx$-equivalence classes are

$$\{\theta_1'', \dots, \theta_{\mu_1}''\}, \dots, \{\theta_{\mu_{r-1}+1}'', \dots, \theta_s''\} ,$$

for suitable $1 \le \mu_1 < \cdots < \mu_r = s$, where $r = r(\Theta)$. Now, by definition of $\approx$ we can find natural numbers $p_j \in \mathbb{N}^*$ for $1 \le j \le s$ such that

$$\mathrm{e}^{2\pi \mathrm{i} p_1 \theta_1''} = \cdots = \mathrm{e}^{2\pi \mathrm{i} p_{\mu_1} \theta_{\mu_1}''} ,$$
$$\vdots$$
$$\mathrm{e}^{2\pi \mathrm{i} p_{\mu_{r-1}+1} \theta_{\mu_{r-1}+1}''} = \cdots = \mathrm{e}^{2\pi \mathrm{i} p_s \theta_s''} ,$$

and no other relations of this kind can be found among $\theta_1'', \dots, \theta_s''$. It follows that $\{\Lambda_1^k\}_{k\in\mathbb{N}}$ is dense in the subgroup of $\mathbb{T}^s$ defined by the equations

$$\lambda_1^{p_1} = \cdots = \lambda^{p_{\mu_1}}, \dots, \lambda_{\mu_{r-1}+1}^{p_{\mu_{t-1}+1}} = \cdots = \lambda_s^{p_s} ,$$

which is isomorphic to $\mathbb{T}^r$, as claimed.

Now assume that z_0 is periodic of period p, and let ρ_f be the limit retraction of f. Since ρ_f is the unique holomorphic retraction in $\Gamma(f)$, and $\Gamma(f^p) \subseteq \Gamma(f)$, it follows that ρ_f is the limit retraction of f^p too. In particular, the limit manifold of f coincides with the limit manifold of f^p, and hence $m_f = m_{f^p} = m$. Finally, $\Gamma(f)/\Gamma(f^p) \equiv \mathbb{Z}_p$, because $f^j(z_0) \neq z_0$ for $1 \le j < p$; hence $\Gamma(f)$ and $\Gamma(f^p)$ have the same connected component at the identity (and hence $r_f = r_{f^p}$), and $q_f = pq_{f^p}$ follows by counting the number of connected components in both groups. □

If $f \in \mathrm{Hol}(X, X)$ has a periodic point then the sequence of iterates is clearly not compactly divergent. The converse is in general false, as shown by the following example:

Example 2.1.13 Let $D \subset\subset \mathbb{C}^2$ be given by

$$D = \left\{(z, w) \in \mathbb{C}^2 \;\middle|\; |z|^2 + |w|^2 + |w|^{-2} < 3\right\} .$$

The domain D is strongly pseudoconvex domain, thus taut, but not simply connected. Given $\theta \in \mathbb{R}$ and $\varepsilon = \pm 1$, define $f \in \mathrm{Hol}(D, D)$ by

$$f(z, w) = (z/2, \mathrm{e}^{2\pi i\theta} w^{\varepsilon}) .$$

Then the sequence of iterates of f is never compactly divergent, but f has no periodic points as soon as $\theta \notin \mathbb{Q}$. Furthermore, the limit manifold of f is the annulus

$$M = \left\{(0, w) \in \mathbb{C}^2 \;\middle|\; |w|^2 + |w|^{-2} < 3\right\} ,$$

the limit retraction is $\rho(z, w) = (0, w)$, and suitably choosing ε and θ we can obtain as $\Gamma(f)$ any compact abelian subgroup of $\mathrm{Aut}(M)$.

It turns out that self-maps without periodic points but whose sequence of iterates is not compactly divergent can exist only when the topology of the manifold is complicated enough. Indeed, using deep results on the actions of real tori on manifolds, it is possible to prove the following

Theorem 2.1.14 (Abate, [3]) *Let X be a taut manifold with finite topological type and such that $H^j(X, \mathbb{Q}) = (0)$ for all odd j. Take $f \in \mathrm{Hol}(X, X)$. Then the sequence of iterates of f is not compactly divergent if and only if f has a periodic point.*

When $X = \Delta$ a consequence of the Wolff-Denjoy theorem is that the sequence of iterates of a self-map $f \in \mathrm{Hol}(\Delta, \Delta)$ is not compactly divergent if and only if f has a fixed point, which is an assumption easier to verify than the existence of periodic points. It turns out that we can generalize this result to convex domains (see also [19] for a different proof):

Theorem 2.1.15 (Abate, [1]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Take $f \in \mathrm{Hol}(D, D)$. Then the sequence of iterates of f is not compactly divergent if and only if f has a fixed point.*

Proof One direction is obvious; conversely, assume that $\{f^k\}$ is not compactly divergent, and let $\rho\colon D \to M$ be the limit retraction. First of all, note that $k_M = k_D|_{M\times M}$. In fact

$$k_D(z_1, z_2) \le k_M(z_1, z_2) = k_M\big(\rho(z_1), \rho(z_2)\big) \le k_D(z_1, z_2)$$

for every $z_1, z_2 \in M$. In particular, a Kobayashi ball in M is nothing but the intersection of a Kobayashi ball of D with M.

Let $\varphi = f|_M$, and denote by Γ the closed subgroup of $\mathrm{Aut}(M)$ generated by φ; we know, by Corollary 2.1.8, that Γ is compact. Take $z_0 \in M$; then the orbit

$$\Gamma(z_0) = \{\gamma(z_0) \mid \gamma \in \Gamma\}$$

is compact and contained in M. Let

$$\mathscr{C} = \left\{\overline{B_D(w,r)} \;\middle|\; w \in M,\ r > 0 \text{ and } \overline{B_D(w,r)} \supset \Gamma(z_0)\right\} .$$

Every $\overline{B_D(w,r)}$ is compact and convex (by Corollary 1.4.11); therefore, $C = \bigcap \mathscr{C}$ is a not empty compact convex subset of D. We claim that $f(C) \subset C$.

Let $z \in C$; we have to show that $f(z) \in \overline{B_D(w,r)}$ for every $w \in M$ and $r > 0$ such that $\overline{B_D(w,r)} \supset \Gamma(z_0)$. Now, $\overline{B_D(\varphi^{-1}(w),r)} \in \mathscr{C}$: in fact

$$\overline{B_D(\varphi^{-1}(w),r)} \cap M = \varphi^{-1}\big(\overline{B_D(w,r)} \cap M\big) \supset \varphi^{-1}\big(\Gamma(z_0)\big) = \Gamma(z_0) .$$

Therefore $z \in \overline{B_D(\varphi^{-1}(w),r)}$ and

$$k_D\big(w, f(z)\big) = k_D\Big(f\big(\varphi^{-1}(w)\big), f(z)\Big) \le k_D\big(\varphi^{-1}(w), z\big) \le r ,$$

that is $f(z) \in \overline{B_D(w,r)}$, as we want.

In conclusion, $f(C) \subset C$; by Brouwer's theorem, f must have a fixed point in C. □

The topology of convex domains is particularly simple: indeed, convex domains are topologically contractible, that is they have a point as (continuous) retract of deformation. Using very deep properties of the Kobayashi distance in strongly pseudoconvex domains, outside of the scope of these notes, Huang has been able to generalize Theorem 2.1.15 to topologically contractible strongly pseudoconvex domains:

Theorem 2.1.16 (Huang, [15]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded topologically contractible strongly pseudoconvex C^3 domain. Take $f \in \mathrm{Hol}(D,D)$. Then the sequence of iterates of f is not compactly divergent if and only if f has a fixed point.*

This might suggest that such a statement might be extended to taut manifolds (or at least to taut domains) topologically contractible. Surprisingly, this is not true:

Theorem 2.1.17 (Abate-Heinzner, [5]) *There exists a bounded domain $D \subset\subset \mathbb{C}^8$ which is taut, homeomorphic to $\mathbb{C}^8$ (and hence topologically contractible), pseudoconvex, and strongly pseudoconvex at all points of ∂D but one, where a finite cyclic group acts without fixed points.*

This completes the discussion of tasks (a) and (b). In the next two subsections we shall describe how it is possible to use the Kobayashi distance to deal with task (c).

2.2 Horospheres and the Wolff-Denjoy Theorem

When $f \in \mathrm{Hol}(\Delta, \Delta)$ has a fixed point $\zeta_0 \in \Delta$, the Wolff-Denjoy theorem is an easy consequence of the Schwarz-Pick lemma. Indeed if f is an automorphism the statement is clear; if it is not an automorphism, then f is a strict contraction of any Kobayashi ball centered at ζ_0, and thus the orbits must converge to the fixed point ζ_0. When f has no fixed points, this argument fails because there are no f-invariant Kobayashi balls. Wolff had the clever idea of replacing Kobayashi balls by a sort of balls "centered" at points in the boundary, the *horocycles,* and he was able to prove the existence of f-invariant horocycles—and thus to complete the proof of the Wolff-Denjoy theorem.

This is the approach we shall follow to prove a several variable version of the Wolff-Denjoy theorem in strongly pseudoconvex domains, using the Kobayashi distance to define a general notion of multidimensional analogue of the horocycles, the horospheres. This notion, introduced in [1], is behind practically all known generalizations of the Wolff-Denjoy theorem; and it has found other applications as well (see, e.g., the survey paper [4] and other chapters in this book).

Definition 2.2.1 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain. Then the *small horosphere* of *center* $x_0 \in \partial D$, *radius* $R > 0$ and *pole* $z_0 \in D$ is the set

$$E_{z_0}(x_0, R) = \left\{z \in D \;\middle|\; \limsup_{w \to x_0}[k_D(z, w) - k_D(z_0, w)] < \tfrac{1}{2}\log R\right\} ;$$

the *large horosphere* of *center* $x_0 \in \partial D$, *radius* $R > 0$ and *pole* $z_0 \in D$ is the set

$$F_{z_0}(x_0, R) = \left\{z \in D \;\middle|\; \liminf_{w \to x_0}[k_D(z, w) - k_D(z_0, w)] < \tfrac{1}{2}\log R\right\} .$$

The rationale behind this definition is the following. A Kobayashi ball of center $w \in D$ and radius r is the set of $z \in D$ such that $k_D(z, w) < r$. If we let w go to a point in the boundary $k_D(z, w)$ goes to infinity (at least when D is complete hyperbolic), and so we cannot use it to define subsets of D. We then renormalize $k_D(z, w)$ by subtracting the distance $k_D(z_0, w)$ from a reference point z_0. By the triangular inequality the difference $k_D(z, w) - k_D(z_0, w)$ is bounded by $k_D(z_0, z)$; thus we can consider the liminf and the limsup as w goes to $x_0 \in \partial D$ (in general, the limit does not exist; an exception is given by strongly convex C^3 domains, see [2, Corollary 2.6.48]), and the sublevels provide some sort of balls centered at points in the boundary.

The following lemma contains a few elementary properties of the horospheres, which are an immediate consequence of the definition (see, e.g., [2, Lemmas 2.4.10 and 2.4.11]):

Lemma 2.2.2 *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain of $\mathbb{C}^n$, and choose $z_0 \in D$ and $x \in \partial D$. Then:*

(i) *for every $R > 0$ we have $E_{z_0}(x, R) \subset F_{z_0}(x, R)$;*

(ii) *for every* $0 < R_1 < R_2$ *we have* $E_{z_0}(x, R_1) \subset E_{z_0}(x, R_2)$ *and* $F_{z_0}(x, R_1) \subset F_{z_0}(x, R_2)$*;*
(iii) *for every* $R > 1$ *we have* $B_D(z_0, \frac{1}{2}\log R) \subset E_{z_0}(x, R)$*;*
(iv) *for every* $R < 1$ *we have* $F_{z_0}(x, R) \cap B_D(z_0, -\frac{1}{2}\log R) = \emptyset$*;*
(v) $\bigcup_{R>0} E_{z_0}(x, R) = \bigcup_{R>0} F_{z_0}(x, R) = D$ *and* $\bigcap_{R>0} E_{z_0}(x, R) = \bigcap_{R>0} F_{z_0}(x, R) = \emptyset$*;*
(vi) *if* $\varphi \in \mathrm{Aut}(D) \cap C^0(\overline{D}, \overline{D})$*, then for every* $R > 0$

$$\varphi\big(E_{z_0}(x, R)\big) = E_{\varphi(z_0)}\big(\varphi(x), R\big) \qquad \textit{and} \qquad \varphi\big(F_{z_0}(x, R)\big) = F_{\varphi(z_0)}\big(\varphi(x), R\big)\ ;$$

(vii) *if* $z_1 \in D$*, set*

$$\tfrac{1}{2}\log L = \limsup_{w\to x} \big[k_D(z_1, w) - k_D(z_0, w)\big]\ .$$

Then for every $R > 0$ *we have* $E_{z_1}(x, R) \subseteq E_{z_0}(x, LR)$ *and* $F_{z_1}(x, R) \subseteq F_{z_0}(x, LR)$*.*

It is also easy to check that the horospheres with pole at the origin in B^n (and thus in Δ) coincide with the classical horospheres:

Lemma 2.2.3 *If* $x \in \partial\mathbb{B}^n$ *and* $R > 0$ *then*

$$E_O(x, R) = F_O(x, r) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{|1 - \langle z, x\rangle|^2}{1 - \|z\|^2} < R \right\}\ .$$

Proof If $z \in \mathbb{B}^n \setminus \{O\}$, let $\gamma_z \colon \mathbb{B}^n \to \mathbb{C}^n$ be given by

$$\gamma_z(w) = \frac{z - P_z(w) - (1 - \|z\|^2)^{1/2}\big(w - P_z(w)\big)}{1 - \langle w, z\rangle}\ , \tag{2.10}$$

where $P_z(w) = \frac{\langle w,z\rangle}{\langle z,z\rangle} z$ is the orthogonal projection on $\mathbb{C}z$; we shall also put $\gamma_O = \mathrm{id}_{B^n}$. It is easy to check that $\gamma_z(z) = O$, that $\gamma_z(\mathbb{B}^n) \subseteq \mathbb{B}^n$ and that $\gamma_z \circ \gamma_z = \mathrm{id}_{\mathbb{B}^n}$; in particular, $\gamma_z \in \mathrm{Aut}(\mathbb{B}^n)$. Furthermore,

$$1 - \|\gamma_z(w)\|^2 = \frac{(1 - \|z\|^2)(1 - \|w\|^2)}{|1 - \langle w, z\rangle|^2}\ .$$

Therefore for all $w \in \mathbb{B}^n$ we get

$$\begin{aligned} k_{\mathbb{B}^n}(z, w) - k_{\mathbb{B}^n}(O, w) &= k_{\mathbb{B}^n}\big(O, \gamma_z(w)\big) - k_{\mathbb{B}^n}(O, w) \\ &= \tfrac{1}{2}\log\left(\frac{1 + \|\gamma_z(w)\|}{1 + \|w\|} \cdot \frac{1 - \|w\|}{1 - \|\gamma_z(w)\|}\right) \\ &= \log\frac{1 + \|\gamma_z(w)\|}{1 + \|w\|} + \tfrac{1}{2}\log\frac{|1 - \langle w, z\rangle|^2}{1 - \|z\|^2}\ . \end{aligned}$$

Letting $w \to x$ we get the assertion, because $\|\gamma_z(x)\| = 1$. □

Thus in $\mathbb{B}^n$ small and large horospheres coincide. Furthermore, the horospheres with pole at the origin are ellipsoids tangent to $\partial\mathbb{B}^n$ in x, because an easy computation yields

$$E_O(x,R) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{\|P_x(z) - (1-r)x\|^2}{r^2} + \frac{\|z - P_x(z)\|^2}{r} < 1 \right\} ,$$

where $r = R/(1+R)$. In particular if $\tau \in \partial\Delta$ we have

$$E_0(\tau,R) = \left\{ \zeta \in \Delta \;\middle|\; |\zeta - (1-r)\tau|^2 < r^2 \right\} ,$$

and so a horocycle is an Euclidean disk internally tangent to $\partial\Delta$ in τ.

Another domain where we can explicitly compute the horospheres is the polydisc; in this case large and small horospheres are actually different (see, e.g., [2, Proposition 2.4.12]):

Proposition 2.2.4 *Let $x \in \partial\Delta^n$ and $R > 0$. Then*

$$E_O(x,R) = \left\{ z \in \Delta^n \;\middle|\; \max_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\} ;$$

$$F_O(x,R) = \left\{ z \in \Delta^n \;\middle|\; \min_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\} .$$

The key in the proof of the classical Wolff-Denjoy theorem is the

Theorem 2.2.5 (Wolff's Lemma, [23]) *Let $f \in \mathrm{Hol}(\Delta,\Delta)$ without fixed points. Then there exists a unique $\tau \in \partial\Delta$ such that*

$$f\big(E_0(\tau,R)\big) \subseteq E_0(\tau,R) \tag{2.11}$$

for all $R > 0$.

Proof For the uniqueness, assume that (2.11) holds for two distinct points τ, $\tau_1 \in \partial\Delta$. Then we can construct two horocycles, one centered at τ and the other centered at τ_1, tangent to each other at a point of Δ. By (2.11) this point would be a fixed point of f, contradiction.

For the existence, pick a sequence $\{r_\nu\} \subset (0,1)$ with $r_\nu \to 1$, and set $f_\nu = r_\nu f$. Then $f_\nu(\Delta)$ is relatively compact in Δ; by Brouwer's theorem each f_ν has a fixed point $\eta_\nu \in \Delta$. Up to a subsequence, we can assume $\eta_\nu \to \tau \in \overline{\Delta}$. If τ were in Δ, we would have

$$f(\tau) = \lim_{\nu\to\infty} f_\nu(\eta_\nu) = \lim_{\nu\to\infty} \eta_\nu = \tau ,$$

which is impossible; therefore $\tau \in \partial\Delta$.

Now, by the Schwarz-Pick lemma we have $k_\Delta\big(f_\nu(\zeta),\eta_\nu\big) \le k_\Delta(\zeta,\eta_\nu)$ for all $\zeta \in \Delta$; recalling the formula for the Poincaré distance we get

$$1-\left|\frac{f_\nu(\zeta)-\eta_\nu}{1-\overline{\eta_\nu}f_\nu(\zeta)}\right|^2 \ge 1-\left|\frac{\zeta-\eta_\nu}{1-\overline{\eta_\nu}\zeta}\right|^2 ,$$

or, equivalently,

$$\frac{|1-\overline{\eta_\nu}f_\nu(\zeta)|^2}{1-|f_\nu(\zeta)|^2} \le \frac{|1-\overline{\eta_\nu}\zeta|^2}{1-|\zeta|^2} .$$

Taking the limit as $\nu \to \infty$ we get

$$\frac{|1-\overline{\tau}f(\zeta)|^2}{1-|f(\zeta)|^2} \le \frac{|1-\overline{\tau}\zeta|^2}{1-|\zeta|^2} ,$$

and the assertion follows. □

With this result it is easy to conclude the proof of the Wolff-Denjoy theorem. Indeed, if $f \in \mathrm{Hol}(\Delta,\Delta)$ has no fixed points we already know that the sequence of iterates is compactly divergent, which means that the image of any limit h of a converging subsequence is contained in $\partial\Delta$. By the maximum principle, the map h must be constant; and by Wolff's lemma this constant must be contained in $\overline{E_0(\tau,R)} \cap \partial\Delta = \{\tau\}$. So every converging subsequence of $\{f^k\}$ must converge to the constant τ; and this is equivalent to saying that the whole sequence of iterates converges to the constant map τ.

Remark 2.2.6 Let me make more explicit the final argument used here, because we are going to use it often. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain; in particular, it is (hyperbolic and) relatively compact inside an Euclidean ball $\mathbb{B}$, which is complete hyperbolic and hence taut. Take now $f \in \mathrm{Hol}(D,D)$. Since $\mathrm{Hol}(D,D) \subset \mathrm{Hol}(D,\mathbb{B})$, the sequence of iterates $\{f^k\}$ is normal in $\mathrm{Hol}(D,\mathbb{B})$; but since D is relatively compact in $\mathbb{B}$, it cannot contain subsequences compactly divergent in $\mathbb{B}$. Therefore $\{f^k\}$ is relatively compact in $\mathrm{Hol}(D,\mathbb{B})$; and since the latter is a metrizable topological space, to prove that $\{f^k\}$ converges in $\mathrm{Hol}(D,\mathbb{B})$ it suffices to prove that all converging subsequences of $\{f^k\}$ converge to the same limit (whose image will be contained in $\overline{D}$, clearly).

The proof of the Wolff-Denjoy theorem we described is based on two ingredients: the existence of a f-invariant horocycle, and the fact that a horocycle touches the boundary in exactly one point. To generalize this argument to several variables we need an analogous of Theorem 2.2.5 for our multidimensional horospheres, and then we need to know how the horospheres touch the boundary.

There exist several multidimensional versions of Wolff's lemma; we shall present three of them (Theorems 2.2.10, 2.4.2 and 2.4.17). To state the first one we need a definition.

Definition 2.2.7 Let $D \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$. We say that D has *simple boundary* if every $\varphi \in \mathrm{Hol}(\Delta,\mathbb{C}^n)$ such that $\varphi(\Delta) \subseteq \overline{D}$ and $\varphi(\Delta) \cap \partial D \ne \emptyset$ is constant.

Remark 2.2.8 It is easy to prove (see, e.g., [2, Proposition 2.1.4]) that if D has simple boundary and Y is any complex manifold then every $f \in \mathrm{Hol}(Y, \mathbb{C}^n)$ such that $f(Y) \subseteq \overline{D}$ and $f(Y) \cap \partial D \neq \emptyset$ is constant.

Remark 2.2.9 By the maximum principle, every domain $D \subset \mathbb{C}^n$ admitting a peak function at each point of its boundary is simple. For instance, strongly pseudoconvex domain (Theorem 1.5.18) and (not necessarily smooth) strictly convex domains (Remark 1.4.6) have simple boundary.

Then we are able to prove the following

Theorem 2.2.10 (Abate, [3]) *Let $D \subset\subset \mathbb{C}^n$ be a complete hyperbolic bounded domain with simple boundary, and take $f \in \mathrm{Hol}(D, D)$ with compactly divergent sequence of iterates. Fix $z_0 \in D$. Then there exists $x_0 \in \partial D$ such that*

$$f^p\big(E_{z_0}(x_0, R)\big) \subseteq F_{z_0}(x_0, R)$$

for all $p \in \mathbb{N}$ and $R > 0$.

Proof Since D is complete hyperbolic and $\{f^k\}$ is compactly divergent, we have $k_D\big(z_0, f^k(z_0)\big) \to +\infty$ as $k \to +\infty$. Given $\nu \in \mathbb{N}$, let k_ν be the largest k such that $k_D\big(z_0, f^k(z_0)\big) \leq \nu$. In particular for every $p > 0$ we have

$$k_D\big(z_0, f^{k_\nu}(z_0)\big) \leq \nu < k_D\big(z_0, f^{k_\nu+p}(z_0)\big) . \tag{2.12}$$

Since D is bounded, up to a subsequence we can assume that $\{f^{k_\nu}\}$ converges to a holomorphic $h \in \mathrm{Hol}(D, \mathbb{C}^n)$. But $\{f^k\}$ is compactly divergent; therefore $h(D) \subset \partial D$ and so $h \equiv x_0 \in \partial D$, because D has simple boundary (see Remark 2.2.8).

Put $w_\nu = f^{k_\nu}(z_0)$. We have $w_\nu \to x_0$; as a consequence for every $p > 0$ we have $f^p(w_\nu) = f^{k_\nu}\big(f^p(z_0)\big) \to x_0$ and

$$\limsup_{\nu \to +\infty} \big[k_D(z_0, w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \leq 0$$

by (2.12). Take $z \in E_{z_0}(x_0, R)$; then we have

$$\begin{aligned}
\liminf_{w \to x_0} \big[k_D\big(f^p(z), w\big) - k_D(z_0, w)\big] &\leq \liminf_{\nu \to +\infty} \big[k_D\big(f^p(z), f^p(w_\nu)\big) - k_D\big(z_0, f^p(w_\nu)\big)\big] \\
&\leq \liminf_{\nu \to +\infty} \big[k_D(z, w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \\
&\leq \limsup_{\nu \to +\infty} \big[k_D(z, w_\nu) - k_D(z_0, w_\nu)\big] \\
&\qquad + \limsup_{\nu \to +\infty} \big[k_D(z_0, w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \\
&\leq \limsup_{\nu \to +\infty} \big[k_D(z, w_\nu) - k_D(z_0, w_\nu)\big] < \tfrac{1}{2} \log R ,
\end{aligned}$$

that is $f^p(z) \in F_{z_0}(x_0, R)$, and we are done. □

The next step consists in determining how the large horospheres touch the boundary. The main tools here are the boundary estimates seen in Sect. 1.5:

Theorem 2.2.11 (Abate, [1]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then*

$$\overline{E_{z_0}(x_0,R)} \cap \partial D = \overline{F_{z_0}(x_0,R)} \cap \partial D = \{x_0\}$$

for every $z_0 \in D$, $x_0 \in \partial D$ and $R > 0$.

Proof We begin by proving that x_0 belongs to the closure of $E_{z_0}(x_0,R)$. Let $\varepsilon > 0$ be given by Theorem 1.5.22; then, recalling Theorem 1.5.19, for every $z, w \in D$ with $\|z - x_0\|, \|w - x_0\| < \varepsilon$ we have

$$k_D(z,w) - k_D(z_0,w) \leq \frac{1}{2}\log\left(1 + \frac{\|z-w\|}{d(z,\partial D)}\right) + \frac{1}{2}\log\left[d(w,\partial D) + \|z-w\|\right] + K\ ,$$

for a suitable constant $K \in \mathbb{R}$ depending only on x_0 and z_0. In particular, as soon as $\|z - x\| < \varepsilon$ we get

$$\limsup_{w\to x}[k_D(z,w)-k_D(z_0,w)] \leq \frac{1}{2}\log\left(1+\frac{\|z-x\|}{d(z,\partial D)}\right)+\frac{1}{2}\log\|z-x\|+K\ . \tag{2.13}$$

So if we take a sequence $\{z_\nu\} \subset D$ converging to x_0 so that $\{\|z_\nu - x_0\|/d(z_\nu,\partial D)\}$ is bounded (for instance, a sequence converging non-tangentially to x_0), then for every $R > 0$ we have $z_\nu \in E_{z_0}(x_0,R)$ eventually, and thus $x_0 \in \overline{E_{z_0}(x_0,R)}$.

To conclude the proof, we have to show that x_0 is the only boundary point belonging to the closure of $F_{z_0}(x_0,R)$. Suppose, by contradiction, that there exists $y \in \partial D \cap \overline{F_{z_0}(x_0,R)}$ with $y \neq x_0$; then we can find a sequence $\{z_\mu\} \subset F_{z_0}(x_0,R)$ with $z_\mu \to y$.

Theorem 1.5.21 provides us with $\varepsilon > 0$ and $K \in \mathbb{R}$ associated to the pair (x_0,y); we may assume $\|z_\mu - y\| < \varepsilon$ for all $\mu \in \mathbb{N}$. Since $z_\mu \in F_{z_0}(x_0,R)$, we have

$$\liminf_{w\to x}\left[k_D(z_\mu,w) - k_D(z_0,w)\right] < \frac{1}{2}\log R$$

for every $\mu \in \mathbb{N}$; therefore for each $\mu \in \mathbb{N}$ we can find a sequence $\{w_{\mu\nu}\} \subset D$ such that $\lim\limits_{\nu\to\infty} w_{\mu\nu} = x_0$ and

$$\lim_{\nu\to\infty}\left[k_D(z_\mu,w_{\mu\nu}) - k_D(z_0,w_{\mu\nu})\right] < \frac{1}{2}\log R\ .$$

Moreover, we can assume $\|w_{\mu\nu} - x\| < \varepsilon$ and $k_D(z_\mu,w_{\mu\nu}) - k_D(z_0,w_{\mu\nu}) < \frac{1}{2}\log R$ for all $\mu, \nu \in \mathbb{N}$.

By Theorem 1.5.21 for all $\mu, \nu \in \mathbb{N}$ we have

$$\begin{aligned}\frac{1}{2}\log R &> k_D(z_\mu,w_{\mu\nu}) - k_D(z_0,w_{\mu\nu})\\ &\geq -\frac{1}{2}\log d(z_\mu,\partial D) - \frac{1}{2}\log d(w_{\mu\nu},\partial D) - k_D(z_0,w_{\mu\nu}) - K\ .\end{aligned}$$

On the other hand, Theorem 1.5.16 yields $c_1 > 0$ (independent of $w_{\mu\nu}$) such that

$$k_D(z_0, w_{\mu\nu}) \le c_1 - \tfrac{1}{2} \log d(w_{\mu\nu}, \partial D)$$

for every μ, $\nu \in \mathbb{N}$. Therefore

$$\tfrac{1}{2} \log R > -\tfrac{1}{2} \log d(z_\mu, \partial D) - K - c_1$$

for every $\mu \in \mathbb{N}$, and, letting μ go to infinity, we get a contradiction. □

We are then able to prove a Wolff-Denjoy theorem for strongly pseudoconvex domains:

Theorem 2.2.12 (Abate, [3]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain. Take $f \in \mathrm{Hol}(D, D)$ with compactly divergent sequence of iterates. Then $\{f^k\}$ converges to a constant map $x_0 \in \partial D$.*

Proof Fix $z_0 \in D$, and let $x_0 \in \partial D$ be given by Theorem 2.2.10. Since D is bounded, it suffices to prove that every subsequence of $\{f^k\}$ converging in $\mathrm{Hol}(D, \mathbb{C}^n)$ actually converges to the constant map x_0.

Let $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ be the limit of a subsequence of iterates. Since $\{f^k\}$ is compactly divergent, we must have $h(D) \subset \partial D$. Hence Theorem 2.2.10 implies that

$$h\big(E_{z_0}(x_0, R)\big) \subseteq \overline{F_{z_0}(x_0, R)} \cap \partial D$$

for any $R > 0$; since (Theorem 2.2.11) $\overline{F_{z_0}(x_0, R)} \cap \partial D = \{x_0\}$ we get $h \equiv x_0$, and we are done. □

Remark 2.2.13 The proof of Theorem 2.2.12 shows that we can get such a statement in any complete hyperbolic domain with simple boundary satisfying Theorem 2.2.11; and the proof of the latter theorem shows that what is actually needed are suitable estimates on the boundary behavior of the Kobayashi distance. Using this remark, it is possible to extend Theorem 2.2.12 to some classes of weakly pseudoconvex domains; see, e.g., Ren-Zhang [21] and Khanh-Thu [18].

2.3 Strictly Convex Domains

The proof of Theorem 2.2.12 described in the previous subsection depends in an essential way on the fact that the boundary of the domain D is of class at least C^2. Recently, Budzyńska [8] (see also [9]) found a way to prove Theorem 2.2.12 in *strictly convex* domains without any assumption on the smoothness of the boundary; in this subsection we shall describe a simplified approach due to Abate and Raissy [6].

The result which is going to replace Theorem 2.2.11 is the following:

Proposition 2.3.1 *Let $D \subset \mathbb{C}^n$ be a hyperbolic convex domain, $z_0 \in D$, $R > 0$ and $x \in \partial D$. Then we have $[x, z] \subset \overline{F_{z_0}(x, R)}$ for all $z \in \overline{F_{z_0}(x, R)}$. Furthermore,*

$$x \in \bigcap_{R>0} \overline{F_{z_0}(x, R)} \subseteq \mathrm{ch}(x) \; . \tag{2.14}$$

In particular, if x is a strictly convex point then $\bigcap_{R>0} \overline{F_{z_0}(x, R)} = \{x\}$.

Proof Given $z \in F_{z_0}(x, R)$, choose a sequence $\{w_\nu\} \subset D$ converging to x and such that the limit of $k_D(z, w_\nu) - k_D(z_0, w_\nu)$ exists and is less than $\frac{1}{2}\log R$. Given $0 < s < 1$, let $h_\nu^s \colon D \to D$ be defined by

$$h_\nu^s(w) = sw + (1-s)w_\nu$$

for every $w \in D$; then $h_\nu^s(w_\nu) = w_\nu$. In particular,

$$\limsup_{\nu \to +\infty} \left[k_D\big(h_\nu^s(z), w_\nu\big) - k_D(z_0, w_\nu)\right] \leq \lim_{\nu \to +\infty} \left[k_D(z, w_\nu) - k_D(z_0, w_\nu)\right] < \tfrac{1}{2}\log R \; .$$

Furthermore we have

$$\left|k_D\big(sz + (1-s)x, w_\nu\big) - k_D\big(h_\nu^s(z), w_\nu\big)\right| \leq k_D\big(sz + (1-s)w_\nu, sz + (1-s)x\big) \to 0$$

as $\nu \to +\infty$. Therefore

$$\begin{aligned}
\liminf_{w \to x} &\left[k_D\big(sz + (1-s)x, w\big) - k_D(z_0, w)\right] \\
&\leq \limsup_{\nu \to +\infty} \left[k_D\big(sz + (1-s)x, w_\nu\big) - k_D(z_0, w_\nu)\right] \\
&\leq \limsup_{\nu \to +\infty} \left[k_D\big(h_\nu^s(z), w_\nu\big) - k_D(z_0, w_\nu)\right] \\
&\qquad + \lim_{\nu \to +\infty} \left[k_D\big(sz + (1-s)x, w_\nu\big) - k_D\big(h_\nu^s(z), w_\nu\big)\right] \\
&< \tfrac{1}{2}\log R \; ,
\end{aligned}$$

and thus $sz + (1-s)x \in F_{z_0}(x, R)$. Letting $s \to 0$ we also get $x \in \overline{F_{z_0}(x, R)}$, and we have proved the first assertion for $z \in F_{z_0}(x, R)$. If $z \in \partial F_{z_0}(x, R)$, it suffices to apply what we have just proved to a sequence in $F_{z_0}(x, R)$ approaching z.

In particular we have thus shown that $x \in \bigcap_{R>0} \overline{F_{z_0}(x, R)}$. Moreover this intersection is contained in ∂D, by Lemma 2.2.2. Take $y \in \bigcap_{R>0} \overline{F_{z_0}(x, R)}$ different from x. Then the whole segment $[x, y]$ must be contained in the intersection, and thus in ∂D; hence $y \in \mathrm{ch}(x)$, and we are done. □

We can now prove a Wolff-Denjoy theorem in strictly convex domains without any assumption on the regularity of the boundary:

Theorem 2.3.2 (Budzyńska, [8]; Abate-Raissy, [6]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded strictly convex domain, and take $f \in \mathrm{Hol}(D, D)$ without fixed points. Then the sequence of iterates $\{f^k\}$ converges to a constant map $x \in \partial D$.*

Proof Fix $z_0 \in D$, and let $x \in \partial D$ be given by Theorem 2.2.10, that can be applied because strictly convex domains are complete hyperbolic (by Proposition 1.4.8) and have simple boundary (by Remark 2.2.9). So, since D is bounded, it suffices to prove that every converging subsequence of $\{f^k\}$ converges to the constant map x.

Assume that $\{f^{k_\nu}\}$ converges to a holomorphic map $h \in \mathrm{Hol}(D, \mathbb{C}^n)$. Clearly, $h(D) \subset \overline{D}$; since the sequence of iterates is compactly divergent (Theorem 2.1.15), we have $h(D) \subset \partial D$; since D has simple boundary, it follows that $h \equiv y \in \partial D$. So we have to prove that $y = x$.

Take $R > 0$, and choose $z \in E_{z_0}(x, R)$. Then Theorem 2.2.10 yields $y = h(z) \in \overline{F_{z_0}(x, R)} \cap \partial D$. Since this holds for all $R > 0$ we get $y \in \bigcap_{R>0} \overline{F_{z_0}(x, R)}$, and Proposition 2.3.1 yields the assertion. □

2.4 Weakly Convex Domains

The approach leading to Theorem 2.3.2 actually yields results for weakly convex domains too, even though we cannot expect in general the convergence to a constant map.

Example 2.4.1 Let $f \in \mathrm{Hol}(\Delta^2, \Delta^2)$ be given by

$$f(z, w) = \left(\frac{z + 1/2}{1 + z/2}, w\right) .$$

Then it is easy to check that the sequence of iterates of f converges to the non-constant map $h(z, w) = (1, w)$.

The first observation is that we have a version of Theorem 2.2.10 valid in all convex domains, without the requirement of simple boundary:

Theorem 2.4.2 ([1]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and take a map $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x \in \partial D$ such that*

$$f^k\big(E_{z_0}(x, R)\big) \subset F_{z_0}(x, R)$$

for every $z_0 \in D$, $R > 0$ and $k \in \mathbb{N}$.

Proof Without loss of generality we can assume that $O \in D$. For $\nu > 0$ let $f_\nu \in \mathrm{Hol}(D, D)$ be given by

$$f_\nu(z) = \left(1 - \frac{1}{\nu}\right) f(z) ;$$

then $f_\nu(D)$ is relatively compact in D and $f_\nu \to f$ as $\nu \to +\infty$. By Brouwer's theorem, every f_ν has a fixed point $w_\nu \in D$. Up to a subsequence, we may assume that $\{w_\nu\}$ converges to a point $x \in \overline{D}$. If $x \in D$, then

$$f(x) = \lim_{\nu\to\infty} f_\nu(w_\nu) = \lim_{\nu\to\infty} w_\nu = x\ ,$$

impossible; therefore $x \in \partial D$.

Now fix $z \in E_{z_0}(x,R)$ and $k \in \mathbb{N}$. We have

$$\left|k_D\bigl(f_\nu^k(z), w_\nu\bigr) - k_D\bigl(f^k(z), w_\nu\bigr)\right| \le k_D\bigl(f_\nu^k(z), f^k(z)\bigr) \longrightarrow 0$$

as $\nu \to +\infty$. Since w_ν is a fixed point of f_ν^k for every $k \in \mathbb{N}$, we then get

$$\begin{aligned}
\liminf_{w\to x}\bigl[k_D(f^k(z), w) - k_D(z_0, w)\bigr] &\le \liminf_{\nu\to+\infty}\bigl[k_D(f^k(z), w_\nu) - k_D(z_0, w_\nu)\bigr] \\
&\le \limsup_{\nu\to+\infty}\bigl[k_D\bigl(f_\nu^k(z), w_\nu\bigr) - k_D(z_0, w_\nu)\bigr] \\
&\qquad + \lim_{\nu\to+\infty}\bigl[k_D\bigl(f^k(z), w_\nu\bigr) - k_D\bigl(f_\nu^k(z), w_\nu\bigr)\bigr] \\
&\le \limsup_{\nu\to+\infty}\bigl[k_D\bigl(z, w_\nu\bigr) - k_D(z_0, w_\nu)\bigr] \\
&\le \limsup_{w\to x}\bigl[k_D\bigl(z, w\bigr) - k_D(z_0, w)\bigr] < \tfrac{1}{2}\log R\ ,
\end{aligned}$$

and $f^k(z) \in F_{z_0}(x,R)$. □

When D has C^2 boundary this is enough to get a sensible Wolff-Denjoy theorem, because of the following result:

Proposition 2.4.3 ([6]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain with C^2 boundary, and $x \in \partial D$. Then for every $z_0 \in D$ and $R > 0$ we have*

$$\overline{F_{z_0}(x,R)} \cap \partial D \subseteq \mathrm{Ch}(x)\ .$$

In particular, if x is a strictly $\mathbb{C}$-linearly convex point then $\overline{F_{z_0}(x,R)} \cap \partial D = \{x\}$.

To simplify subsequent statements, let us introduce a definition.

Definition 2.4.4 Let $D \subset \mathbb{C}^n$ be a hyperbolic convex domain, and $f \in \mathrm{Hol}(D,D)$ without fixed points. The *target set* of f is defined as

$$T(f) = \bigcup_h h(D) \subseteq \partial D\ ,$$

where the union is taken with respect to all the holomorphic maps $h \in \mathrm{Hol}(D,\mathbb{C}^n)$ obtained as limit of a subsequence of iterates of f. We have $T(f) \subseteq \partial D$ because the sequence of iterates $\{f^k\}$ is compactly divergent.

As a consequence of Proposition 2.4.3 we get:

Corollary 2.4.5 ([6]) *Let $D \subset\subset \mathbb{C}^n$ be a C^2 bounded convex domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x_0 \in \partial D$ such that*

$$T(f) \subseteq \mathrm{Ch}(x_0) .$$

In particular, if D is strictly $\mathbb{C}$-linearly convex then the sequence of iterates $\{f^k\}$ converges to the constant map x_0.

Proof Let $x_0 \in \partial D$ be given by Theorem 2.4.2, and fix $z_0 \in D$. Given $z \in D$, choose $R > 0$ such that $z \in E_{z_0}(x_0, R)$. If $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ is the limit of a subsequence of iterates then Theorem 2.4.2 and Proposition 2.4.3 yield

$$h(z) \in \overline{Fz_0(x, R)} \cap \partial D \subset \mathrm{Ch}(x_0) ,$$

and we are done. □

Remark 2.4.6 Zimmer [24] has proved Corollary 2.4.5 for bounded convex domains with $C^{1,\alpha}$ boundary. We conjecture that it should hold for strictly $\mathbb{C}$-linearly convex domains without smoothness assumptions on the boundary.

Let us now drop any smoothness or strict convexity condition on the boundary. In this general context, an useful result is the following:

Lemma 2.4.7 *Let $D \subset \mathbb{C}^n$ be a convex domain. Then for every connected complex manifold X and every holomorphic map $h: X \to \mathbb{C}^n$ such that $h(X) \subset \overline{D}$ and $h(X) \cap \partial D \neq \emptyset$ we have*

$$h(X) \subseteq \bigcap_{z \in X} \mathrm{Ch}\big(h(z)\big) \subseteq \partial D .$$

Proof Take $x_0 = h(z_0) \in h(X) \cap \partial D$, and let ψ be the weak peak function associated to a complex supporting functional L at x_0. Then $\psi \circ h$ is a holomorphic function with modulus bounded by 1 and such that $\psi \circ h(z_0) = 1$; by the maximum principle we have $\psi \circ h \equiv 1$, and hence $L \circ h \equiv L(x_0)$. In particular, $h(X) \subseteq \partial D$.

Since this holds for all complex supporting hyperplanes at x_0 we have shown that $h(X) \subseteq \mathrm{Ch}\big(h(z_0)\big)$; but since we know that $h(X) \subseteq \partial D$ we can repeat the argument for any $z_0 \in X$, and we are done. □

We can then prove a weak Wolff-Denjoy theorem:

Proposition 2.4.8 *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists $x \in \partial D$ such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{R>0} \mathrm{Ch}\big(\overline{F_{z_0}(x, R)} \cap \partial D\big) . \tag{2.15}$$

Proof Let $x \in \partial D$ be given by Theorem 2.4.2. Choose $z_0 \in D$ and $R > 0$, and take $z \in E_{z_0}(x, R)$. Let $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ be obtained as limit of a subsequence of iterates of f. Arguing as usual we know that $h(D) \subseteq \partial D$; therefore Theorem 2.4.2 yields $h(z) \in \overline{F_{z_0}(x, R)} \cap \partial D$. Then Lemma 2.4.7 yields

$$h(D) \subseteq \mathrm{Ch}\big(h(z)\big) \subseteq \mathrm{Ch}\big(\overline{F_{z_0}(x, R)} \cap \partial D\big) .$$

Since z_0 and R are arbitrary, we get the assertion. □

Remark 2.4.9 Using Lemma 2.2.2 it is easy to check that the intersection in (2.15) is independent of the choice of $z_0 \in D$.

Unfortunately, large horospheres can be too large. For instance, take $(\tau_1, \tau_2) \in \partial\Delta \times \partial\Delta$. Then Proposition 2.2.4 says that the horosphere of center (τ_1, τ_2) in the bidisk are given by

$$F_O\big((\tau_1, \tau_2), R\big) = E_0(\tau_1, R) \times \Delta \cup \Delta \times E_0(\tau_2, R) ,$$

where $E_0(\tau, R)$ is the horocycle of center $\tau \in \partial\Delta$ and radius $R > 0$ in the unit disk Δ, and a not difficult computation shows that

$$\mathrm{Ch}\big(\overline{F_O\big((\tau_1, \tau_2), R\big)} \cap \partial\Delta^2\big) = \partial\Delta^2 ,$$

making the statement of Proposition 2.4.8 irrelevant. So to get an effective statement we need to replace large horospheres with smaller sets.

Small horospheres might be too small; as shown by Frosini [13], there are holomorphic self-maps of the polydisk with no invariant small horospheres. We thus need another kind of horospheres, defined by Kapeluszny, Kuczumow and Reich [16], and studied in detail by Budzyńska [8]. To introduce them we begin with a definition:

Definition 2.4.10 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $z_0 \in D$. A sequence $\mathbf{x} = \{x_\nu\} \subset D$ converging to $x \in \partial D$ is a *horosphere sequence* at x if the limit of $k_D(z, x_\nu) - k_D(z_0, x_\nu)$ as $\nu \to +\infty$ exists for all $z \in D$.

Remark 2.4.11 It is easy to see that the notion of horosphere sequence does not depend on the point z_0.

Horosphere sequences always exist. This follows from a topological lemma:

Lemma 2.4.12 ([20]) *Let (X, d) be a separable metric space, and for each $\nu \in \mathbb{N}$ let $a_\nu : X \to \mathbb{R}$ be a 1-Lipschitz map, i.e., $|a_\nu(x) - a_\nu(y)| \le d(x, y)$ for all $x, y \in X$. If for each $x \in X$ the sequence $\{a_\nu(x)\}$ is bounded, then there exists a subsequence $\{a_{\nu_j}\}$ of $\{a_\nu\}$ such that $\lim_{j\to\infty} a_{\nu_j}(x)$ exists for each $x \in X$.*

Proof Take a countable sequence $\{x_j\}_{j\in\mathbb{N}} \subset X$ dense in X. Clearly, the sequence $\{a_\nu(x_0)\} \subset \mathbb{R}$ admits a convergent subsequence $\{a_{\nu,0}(x_0)\}$. Analogously, the sequence $\{a_{\nu,0}(x_1)\}$ admits a convergent subsequence $\{a_{\nu,1}(x_1)\}$. Proceeding in this way, we get a countable family of subsequences $\{a_{\nu,k}\}$ of the sequence $\{a_\nu\}$ such that for each $k \in \mathbb{N}$ the limit $\lim_{\nu\to\infty} a_{\nu,k}(x_j)$ exists for $j = 0, \ldots, k$. We claim that setting $a_{\nu_j} = a_{j,j}$ the subsequence $\{a_{\nu_j}\}$ is as desired. Indeed, given $x \in X$ and $\varepsilon > 0$ we can find x_h such that $d(x, x_h) < \varepsilon/2$, and then we have

$$\begin{aligned} 0 &\leq \limsup_{j\to\infty} a_{\nu_j}(x) - \liminf_{j\to\infty} a_{\nu_j}(x) \\ &= \Big[\limsup_{j\to\infty} \big(a_{\nu_j}(x) - a_{\nu_j}(x_h)\big) + \lim_{j\to\infty} a_{\nu_j}(x_h)\Big] \\ &\qquad -\Big[\liminf_{j\to\infty} \big(a_{\nu_j}(x) - a_{\nu_j}(x_h)\big) + \lim_{j\to\infty} a_{\nu_j}(x_h)\Big] \\ &\leq 2d(x, x_h) < \varepsilon\ . \end{aligned}$$

Since ε was arbitrary, it follows that the limit $\lim_{j\to\infty} a_{\nu_j}(x)$ exists, as required. □

Then:

Proposition 2.4.13 ([9]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $x \in \partial D$. Then every sequence $\{x_\nu\} \subset D$ converging to x contains a subsequence which is a horosphere sequence at x.*

Proof Let $X = D \times D$ be endowed with the distance

$$d\big((z_1, w_1), (z_2, w_2)\big) = k_D(z_1, z_2) + k_D(w_1, w_2)$$

for all $z_1, z_2, w_1, w_2 \in D$.

Define $a_\nu \colon X \to \mathbb{R}$ by setting $a_\nu(z, w) = k_D(w, x_\nu) - k_D(z, x_\nu)$. The triangular inequality shows that each a_ν is 1-Lipschitz, and for each $(z, w) \in X$ the sequence $\{a_\nu(z, w)\}$ is bounded by $k_D(z, w)$. Lemma 2.4.12 then yields a subsequence $\{x_{\nu_j}\}$ such that $\lim_{j\to\infty} a_{\nu_j}(z, w)$ exists for all $z, w \in D$, and this exactly means that $\{x_{\nu_j}\}$ is a horosphere sequence. □

We can now introduce a new kind of horospheres.

Definition 2.4.14 Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Given $z_0 \in D$, let $\mathbf{x} = \{x_\nu\}$ be a horosphere sequence at $x \in \partial D$, and take $R > 0$. Then the *sequence horosphere* $G_{z_0}(x, R, \mathbf{x})$ is defined as

$$G_{z_0}(x, R, \mathbf{x}) = \Big\{z \in D \,\Big|\, \lim_{\nu\to+\infty} \big[k_D(z, x_\nu) - k_D(z_0, x_\nu)\big] < \tfrac{1}{2}\log R\Big\}\ .$$

The basic properties of sequence horospheres are contained in the following:

Proposition 2.4.15 ([8, 9, 16]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Fix $z_0 \in D$, and let $\mathbf{x} = \{x_\nu\} \subset D$ be a horosphere sequence at $x \in \partial D$. Then:*

(i) $E_{z_0}(x, R) \subseteq G_{z_0}(x, R, \mathbf{x}) \subseteq F_{z_0}(x, R)$ *for all* $R > 0$*;*
(ii) $G_{z_0}(x, R, \mathbf{x})$ *is nonempty and convex for all* $R > 0$*;*
(iii) $\overline{G_{z_0}(x, R_1, \mathbf{x})} \cap D \subset G_{z_0}(x, R_2, \mathbf{x})$ *for all* $0 < R_1 < R_2$*;*
(iv) $B_D(z_0, \frac{1}{2}\log R) \subset G_{z_0}(x, R, \mathbf{x})$ *for all* $R > 1$*;*

(v) $B_D(z_0, -\frac{1}{2}\log R) \cap G_{z_0}(x, R, \mathbf{x}) = \emptyset$ *for all* $0 < R < 1$*;*
(vi) $\bigcup_{R>0} G_{z_0}(x, R, \mathbf{x}) = D$ *and* $\bigcap_{R>0} G_{z_0}(x, R, \mathbf{x}) = \emptyset$.

Remark 2.4.16 If $\mathbf{x}$ is a horosphere sequence at $x \in \partial D$ then it is not difficult to check that the family $\{G_z(x, 1, \mathbf{x})\}_{z\in D}$ and the family $\{G_{z_0}(x, R, \mathbf{x})\}_{R>0}$ with $z_0 \in D$ given, coincide.

Then we have the following version of Theorem 2.2.5:

Theorem 2.4.17 ([6, 8]) *Let* $D \subset\subset \mathbb{C}^n$ *be a convex domain, and let* $f \in \mathrm{Hol}(D, D)$ *without fixed points. Then there exists* $x \in \partial D$ *and a horosphere sequence* $\mathbf{x}$ *at* x *such that*

$$f\big(G_{z_0}(x, R, \mathbf{x})\big) \subseteq G_{z_0}(x, R, \mathbf{x})$$

for every $z_0 \in D$ *and* $R > 0$.

Proof As in the proof of Theorem 2.4.2, for $\nu > 0$ put $f_\nu = (1-1/\nu)f \in \mathrm{Hol}(D, D)$; then $f_\nu \to f$ as $\nu \to +\infty$, each f_ν has a fixed point $x_\nu \in D$, and up to a subsequence we can assume that $x_\nu \to x \in \partial D$. Furthermore, by Proposition 2.4.13 up to a subsequence we can also assume that $\mathbf{x} = \{x_\nu\}$ is a horosphere sequence at x.

Now, for every $z \in D$ we have

$$\big|k_D\big(f(z), x_\nu\big) - k_D\big(f_\nu(z), x_\nu\big)\big| \le k_D\big(f_\nu(z), f(z)\big) \to 0$$

as $\nu \to +\infty$. Therefore if $z \in G_{z_0}(x, R, \mathbf{x})$ we get

$$\begin{aligned}
\lim_{\nu\to+\infty} &\big[k_D\big(f(z), x_\nu\big) - k_D(z_0, x_\nu)\big] \\
&\le \lim_{\nu\to+\infty} \big[k_D\big(f_\nu(z), x_\nu\big) - k_D(z_0, x_\nu)\big] \\
&\qquad + \limsup_{\nu\to+\infty} \big[k_D\big(f(z), x_\nu\big) - k_D\big(f_\nu(z), x_\nu\big)\big] \\
&\le \lim_{\nu\to+\infty} \big[k_D(z, x_\nu) - k_D(z_0, x_\nu)\big] < \tfrac{1}{2}\log R
\end{aligned}$$

because $f_\nu(x_\nu) = x_\nu$ for all $\nu \in \mathbb{N}$, and we are done. □

Putting everything together we can prove the following Wolff-Denjoy theorem for (not necessarily strictly or smooth) convex domains:

Theorem 2.4.18 ([6]) *Let* $D \subset\subset \mathbb{C}^n$ *be a bounded convex domain, and* $f \in \mathrm{Hol}(D, D)$ *without fixed points. Then there exist* $x \in \partial D$ *and a horosphere sequence* $\mathbf{x}$ *at* x *such that for any* $z_0 \in D$ *we have*

$$T(f) \subseteq \bigcap_{z\in D} \mathrm{Ch}\big(\overline{G_z(x, 1, \mathbf{x})} \cap \partial D\big) = \bigcap_{R>0} \mathrm{Ch}\big(\overline{G_{z_0}(x, R, \mathbf{x})} \cap \partial D\big)\,.$$

Proof The equality of the intersections is a consequence of Remark 2.4.16. Then the assertion follows from Theorem 2.4.17 and Lemma 2.4.7 as in the proof of Proposition 2.4.8. □

To show that this statement is actually better than Proposition 2.4.8 let us consider the case of the polydisc.

Lemma 2.4.19 *Let* $\mathbf{x} = \{x_\nu\} \subset \Delta^n$ *be a horosphere sequence converging to* $\xi = (\xi_1, \ldots, \xi_n) \in \partial\Delta^n$. *Then for every* $1 \le j \le n$ *such that* $|\xi_j| = 1$ *the limit*

$$\alpha_j := \lim_{\nu\to+\infty} \min_h \left\{ \frac{1 - |(x_\nu)_h|^2}{1 - |(x_\nu)_j|^2} \right\} \le 1 \tag{2.16}$$

exists, and we have

$$G_O(\xi, R, \mathbf{x}) = \left\{ z \in \Delta^n \;\middle|\; \max_j \left\{ \alpha_j \frac{|\xi_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |\xi_j| = 1 \right\} < R \right\} = \prod_{j=1}^n E_j\,,$$

where

$$E_j = \begin{cases} \Delta & \text{if } |\xi_j| < 1, \\ E_0(\xi_j, R/\alpha_j) & \text{if } |\xi_j| = 1. \end{cases}$$

Proof Given $z = (z_1, \ldots, z_n) \in \Delta^n$, let $\gamma_z \in \mathrm{Aut}(\Delta^n)$ be defined by

$$\gamma_z(w) = \left(\frac{w_1 - z_1}{1 - \overline{z_1} w_1}, \ldots, \frac{w_n - z_n}{1 - \overline{z_n} w_n} \right)\,,$$

so that $\gamma_z(z) = O$. Then

$$k_{\Delta^n}(z, x_\nu) - k_{\Delta^n}(O, x_\nu) = k_{\Delta^n}\big(O, \gamma_z(x_\nu)\big) - k_{\Delta^n}(O, x_\nu)\,.$$

Now, writing $|||z||| = \max_j\{|z_j|\}$ we have

$$k_{\Delta^n}(O, z) = \max_j\{k_\Delta(0, z_j)\} = \max_j \left\{ \tfrac{1}{2} \log \frac{1 + |z_j|}{1 - |z_j|} \right\} = \tfrac{1}{2} \log \frac{1 + |||z|||}{1 - |||z|||}\,,$$

and hence

$$k_{\Delta^n}(z, x_\nu) - k_{\Delta^n}(O, x_\nu) = \log\left(\frac{1 + |||\gamma_z(x_\nu)|||}{1 + |||x_\nu|||} \right) + \tfrac{1}{2} \log\left(\frac{1 - |||x_\nu|||^2}{1 - |||\gamma_z(x_\nu)|||^2} \right).$$

Since $|||\gamma_z(\xi)||| = |||\xi||| = 1$, we just have to study the behavior of the second term, that we know has a limit as $\nu \to +\infty$ because $\mathbf{x}$ is a horosphere sequence. Now

$$1 - |||x_\nu|||^2 = \min_h \{1 - |(x_\nu)_h|^2\};$$

$$1 - |||\gamma_z(x_\nu)|||^2 = \min_j \left\{ \frac{1 - |z_j|^2}{|1 - \overline{z_j}(x_\nu)_j|^2} (1 - |(x_\nu)_j|^2) \right\} .$$

Therefore

$$\frac{1 - |||x_\nu|||^2}{1 - |||\gamma_z(x_\nu)|||^2} = \max_j \min_h \left\{ \frac{1 - |(x_\nu)_h|^2}{1 - |(x_\nu)_j|^2} \cdot \frac{|1 - \overline{z_j}(x_\nu)_j|^2}{1 - |z_j|^2} \right\} .$$

Taking the limit as $\nu \to +\infty$ we get

$$\lim_{\nu \to +\infty} \frac{1 - |||x_\nu|||^2}{1 - |||\gamma_z(x_\nu)|||^2} = \max_j \left\{ \frac{|1 - z_j\overline{\xi_j}|^2}{1 - |z_j|^2} \lim_{\nu \to +\infty} \min_h \left\{ \frac{1 - |(x_\nu)_h|^2}{1 - |(x_\nu)_j|^2} \right\} \right\} . \tag{2.17}$$

In particular, we have shown that the limit in (2.16) exists, and it is bounded by 1 (it suffices to take $h = j$). Furthermore, if $|\xi_j| < 1$ then $\alpha_j = 0$; so (2.17) becomes

$$\lim_{\nu \to +\infty} \frac{1 - |||x_\nu|||^2}{1 - |||\gamma_z(x_\nu)|||^2} = \max \left\{ \alpha_j \frac{|1 - z_j\overline{\xi_j}|^2}{1 - |z_j|^2} \;\middle|\; |\xi_j| = 1 \right\} ,$$

and the lemma follows. □

Now, a not too difficult computation shows that

$$\mathrm{Ch}(\xi) = \bigcap_{|\xi_j|=1} \{\eta \in \partial\Delta^n \mid \eta_j = \xi_j\}$$

for all $\xi \in \partial\Delta^n$. As a consequence,

$$\mathrm{Ch}\big(\overline{G_O(\xi, R, \mathbf{x})} \cap \partial\Delta^n\big) = \bigcup_{j=1}^n \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta} ,$$

where

$$C_j(\xi) = \begin{cases} \{\xi_j\} & \text{if } |\xi_j| = 1, \\ \partial\Delta & \text{if } |\xi_j| < 1. \end{cases}$$

Notice that the right-hand sides do not depend either on R or on the horosphere sequence $\mathbf{x}$, but only on ξ.

So Theorem 2.4.18 in the polydisc assumes the following form:

Corollary 2.4.20 *Let $f \in \mathrm{Hol}(\Delta^n, \Delta^n)$ be without fixed points. Then there exists $\xi \in \partial\Delta^n$ such that*

$$T(f) \subseteq \bigcup_{j=1}^{n} \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta}\,. \tag{2.18}$$

Roughly speaking, this is the best one can do, in the sense that while it might be true (for instance in the bidisk; see Theorem 2.4.21 below) that the image of a limit point of the sequence of iterates of f is always contained in just one of the sets appearing in the right-hand side of (2.18), it is impossible to determine a priori in which one it is contained on the basis of the point ξ only; it is necessary to know something more about the map f. Indeed, Hervé has proved the following:

Theorem 2.4.21 (Hervé, [14]) *Let $F = (f, g)\colon \Delta^2 \to \Delta^2$ be a holomorphic self-map of the bidisc, and write $f_w = f(\cdot, w)$ and $g_z = g(z, \cdot)$. Assume that F has no fixed points in Δ^2. Then one and only one of the following cases occurs:*

(i) *if $g(z, w) \equiv w$ (respectively, $f(z, w) \equiv z$) then the sequence of iterates of F converges uniformly on compact sets to $h(z, w) = (\sigma, w)$, where σ is the common Wolff point of the f_w's (respectively, to $h(z, w) = (z, \tau)$, where τ is the common Wolff point of the g_z's);*

(ii) *if $\mathrm{Fix}(f_w) = \emptyset$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \{y(z)\} \subset \Delta$ for all $z \in \Delta$ (respectively, if $\mathrm{Fix}(f_w) = \{x(w)\}$ and $\mathrm{Fix}(g_z) = \emptyset$) then $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$, where $\sigma \in \partial\Delta$ is the common Wolff point of the f_w's (respectively, $T(f) \subseteq \overline{\Delta} \times \{\tau\}$, where τ is the common Wolff point of the g_z's);*

(iii) *if $\mathrm{Fix}(f_w) = \emptyset$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \emptyset$ for all $z \in \Delta$ then either $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$ or $T(f) \subseteq \overline{\Delta} \times \{\tau\}$, where $\sigma \in \partial\Delta$ is the common Wolff point of the f_w's, and $\tau \in \partial\Delta$ is the common Wolff point of the g_z;*

(iv) *if $\mathrm{Fix}(f_w) = \{x(w)\} \subset \Delta$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \{y(z)\} \subset \Delta$ for all $z \in \Delta$ then there are $\sigma, \tau \in \partial D$ such that the sequence of iterates converges to the constant map (σ, τ).*

All four cases can occur: see [14] for the relevant examples.

Acknowledgements Partially supported by the FIRB 2012 grant "Differential Geometry and Geometric Function Theory", by the Progetto di Ricerca d'Ateneo 2015 "Sistemi dinamici: logica, analisi complessa e teoria ergodica", and by GNSAGA-INdAM.

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Chapter 3
Gromov Hyperbolic Spaces and Applications to Complex Analysis

Hervé Pajot

The goal of this chapter is to explain some connections between hyperbolicity in the sense of Gromov and complex analysis/geometry. For this, we first give a short presentation of the theory of Gromov hyperbolic spaces and their boundaries. Then, we will see that the Heisenberg group can be seen as the boundary at infinity of the complex hyperbolic space. This fact will be used in Chap. 5 to give an idea of the proof of the celebrated Mostow rigidity Theorem in this setting. In the last section, we will explain why strongly pseudoconvex domains equipped with their Kobayashi distance are hyperbolic in the sense of Gromov. As an application of the general theory of Gromov hyperbolic spaces, we get a result about the extension of biholomorphic maps in this setting. Note that the case of the Gromov hyperbolicity of more general domains is discussed in Chap. 4.

3.1 Hyperbolicity in the Sense of Gromov

Let (X, d) be a metric space and fix a basepoint $w \in X$. In this case, the triple (X, d, w) is called a *pointed metric space*. The *Gromov product* is defined by

$$(x|y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)) \, .$$

whenever x, y are in X. By the triangle inequality, $0 \leq (x|y)_w \leq \min(d(x,w), d(y,w))$.

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7_3

Definition 3.1.1 Let $\delta \geq 0$. A metric space (X, d) is *δ-hyperbolic*, or *δ-product-hyperbolic*, if for any $x, y, z, w \in X$,

$$(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - \delta\,.$$

We say that (X, d) is *Gromov-hyperbolic* if (X, d) is δ-hyperbolic for some $\delta \geq 0$. Note that all these definitions of hyperbolicity do not depend on the basepoint $w \in X$.

Remark 3.1.2 The Euclidean space $\mathbb{R}^2$ is not hyperbolic. The case of the Poincaré disk and of manifolds with negative sectional curvature will be discussed later.

Let (X, d, w) be a pointed hyperbolic metric space and assume it is *proper* (that is, closed balls are compact).

Definition 3.1.3 A sequence (x_n) in X tends to infinity if $\liminf_{m,n\to\infty}(x_m|x_n)_w = \infty$. The *visual boundary* $\partial_G X$ of X is the set of diverging sequences modulo the equivalence relation $(x_n) \sim (y_n)$ iff $(x_n|y_n)_w$ tends to infinity (or equivalently the classes of equivalence for this relation).

Note again that the last definition does not depend on the basepoint $w \in X$.

The Gromov product extends to infinity in such a way that the ultrametric inequality remains true. To see this, set $(\xi|\eta)_w = \sup \liminf_{i\to+\infty}(x_i|y_i)_w$ whenever $\xi, \eta \in \partial_G X$ and where the supremum is taken over all divergent sequences (x_i) and (y_i) representing ξ and η respectively.

Definition 3.1.4 A *visual metric* (seen from w) of parameter $\varepsilon > 0$ is a distance δ on $\partial_G X$ such that there exist a constant $C > 0$ and a basepoint $w \in X$ so that

$$C^{-1}e^{-\varepsilon(\xi|\eta)_w} \leq \delta(\xi, \eta) \leq Ce^{-\varepsilon(\xi|\eta)_w}$$

whenever $\xi, \eta \in \partial_G X$.

There always exist visual metrics provided $\varepsilon > 0$ is small enough. Note that two visual metrics are Hölder equivalent.

In the Poincaré disc, triangles are thinner than Euclidean triangles (see Appendix 1). This motivates an alternative definition of hyperbolicity in the case of geodesic spaces.

Definition 3.1.5 Let (X, d) be a metric space. If x and y are two points in X, a *geodesic segment*, denoted by $[x, y]$ (even if such segment is not unique in general), is a curve given by an isometric map $\gamma : [a, b] \to X$ (that is the length of γ is equal to $d(x, y)$) with $\gamma(a) = x$ and $\gamma(b) = y$. The metric space (X, d) is a *geodesic space* if for any x, any y in X, there exists a geodesic segment from x to y.

A *geodesic triangle* is just given by $[x, y] \cup [y, z] \cup [z, x]$ where x, y and z are in X. The geodesic space (X, d) is δ-hyperbolic iff every geodesic triangle of X is δ-thin, that is $d(w, [y, z] \cup [z, x]) \leq \delta$ whenever $x, y\, z \in X$ and $w \in [x, y]$. The boundary $\partial_G X$ could be defined as the set of equivalence classes of geodesic rays $r : [0, \infty) \to X$ with the relation $r \sim r'$ if $\sup\{d(r(t), r'(t)), t \in [0, \infty)\} < \infty$. In this setting, the Gromov product has a nice geometric interpretation. If $w \in X$ and $\xi, \eta \in \partial_G X$, $(\xi|\eta)_w$ is comparable to $d(w, (\xi, \eta))$ where (ξ, η) is a geodesic curve in X joining ξ

and η. Hence, the visual distance $\delta(\xi,\eta)$ is small if the geodesics with endpoints ξ and η are closed a long time. Theorem 4.4.10 gives a precise statement regarding the two definitions of Gromov hyperbolicity, in the general setting and in the geodesic setting.

Example 3.1.6 The Poincaré disc Δ is δ-hyperbolic with $\delta = \frac{1}{2}\log 3$. Its boundary at infinity could be identified with the unit circle S^1. The usual distance on the unit circle is visual in the previous sense: if $z, w \in \Delta$,

$$(z|w)_0 = -\frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{(1+|w|)(1+|z|)}.|1-\bar{z}w|\right)$$

and thus $|\xi-\eta| = 2e^{-(\xi|\eta)_0}$ whenever $\xi,\eta \in S^1 = \partial_G\Delta$. More generally, the standard unit sphere can be seen as the boundary of the real hyperbolic space (if this one is defined by using the model of the unit ball of $\mathbb{R}^n$).

Example 3.1.7 Any (complete) Riemannian manifold M with negative sectional curvature is Gromov hyperbolic. Hence, the notion of Gromov hyperbolicity is related to the sectional curvature (and not to the Ricci curvature). The conformal structure of the boundary at infinity of M is not well understood.

Example 3.1.8 The Heisenberg group is the boundary at infinity of the hyperbolic complex space (in the model of the unit ball of $\mathbb{C}^n$). This point will be discussed in the next section. This example is interesting since the proof of the hyperbolicity of strongly pesudoconvex domains (see the last section) starts by equip the boundary of such domains by its Carnot-Carathéodory distance (as in the case of the Heisenberg group).

Remark 3.1.9 Let (Z,δ) be a complete bounded metric space. Then, Z can be seen as the boundary at infinity of a hyperbolic metric space X. To see this, set $X = Z\times[0,\mathrm{diam}Z]$ and $d_{BS}((x,h),(x',h')) = 2\log\left(\frac{\delta(x,x')+\max(h,h')}{\sqrt{hh'}}\right)$ whenever $x, x' \in X$, $h, h' \in [0,\mathrm{diam}X]$. Then, (X,d_{BS}) is hyperbolic and we can identify $\partial_G X$ with Z. Moreover, δ is a visual distance. The formula given d_{BS} is inspired by other definitions of hyperbolic metrics, for instance the Poincaré metric on Δ. The proof of the hyperbolicity in the sense of Gromov of strictly pseudoconvex domains follows from a similar construction (see the last section of this chapter).

The Gromov hyperbolicity is a large scale property of X. To capture this knowledge, one is led to the notion of quasi-isometry, which was introduced by G. Margulis.

Definition 3.1.10 Let (X,d_X) and (Y,d_Y) be two metric spaces. A map $f : X \to Y$ is a (C,D)-*quasi-isometric embedding* if there exist constants $C \geq 1$ and $D \geq 0$ so that

$$C^{-1}d_X(x,x') - D \leq d_Y(f(x),f(x')) \leq Cd_X(x,x') + D$$

whenever $x, x' \in X$. Note that such a map is not necessary continuous.

If in addition, for any $y \in Y$, there exists $x \in X$ so that $d_Y(y, f(x)) \leq C$, then f is called a (C, D)-*quasi-isometry* and the spaces X and Y are *quasi-isometric*. Moreover, we say that $f : X \to Y$ is a *rough isometry* if f is a $(1, D)$-quasi-isometry for some $D \geq 0$. We will use these notions in Chap. 5.

A *quasi-geodesic* is the image of an interval under a quasi-isometric embedding.

A very important fact is the quasi-isometric invariance of the Gromov hyperbolicity.

Theorem 3.1.11 *Let X and Y be two geodesic spaces and let $f : X \to Y$ be a (C, D)-quasi-isometric embedding. If X is δ-hyperbolic, then Y is δ'-hyperbolic with δ' depending on δ, C and D.*

The proof is based on this fundamental property of Gromov hyperbolic spaces.

Theorem 3.1.12 *There exists a constant $\lambda = \lambda(C, D, \delta)$ so that any (C, D)-quasi-geodesic segment in a δ-hyperbolic space X is at Hausdorff distance less that λ from any geodesic segment joining its endpoints.*

A (C, D)-quasi-geodesic segment in X is the image of a (C, D)-quasi-isometry $\gamma : [a, b] \subset \mathbb{R} \to X$. Using the previous theorem, we can characterize the Gromov hyperbolicity by using quasi-geodesics instead of geodesics (that is by replacing geodesic triangles by quasi-geodesic triangles).

We finish this section by introducing the notion of $CAT(-1)$-spaces. For this purpose, we adopt for the Poincaré metric a new convention, by setting

$$\tilde{\kappa}_\Delta(\zeta; v) = \frac{2}{1 - |\zeta|^2} |v|$$

With this convention, the Poincaré disc has curvature -1 (instead of -4 as in Appendix 1) and is δ-hyperbolic with $\delta = \log 3$.

Let (X, d) be a geodesic (proper) space. Let $[x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle in X. A triangle comparison in the Poincaré disc Δ is then given by three points x', y' and $z' \in \Delta$ so that $d(x, y) = \tilde{k}_\Delta(x', y')$, $d(y, z) = \tilde{k}_\Delta(y', z')$ and $d(z, x) = \tilde{k}_\Delta(z', x')$.

Definition 3.1.13 We say that X is a $CAT(-1)$-*space* if $d(x, [y, z]) \leq \tilde{k}_\Delta(x', [y', z'])$ for any geodesic triangle with vertices x, y and z in X and for any associated comparison triangle in Δ with vertices x', y' and z'.

It is well known that a $CAT(-1)$-space is δ-hyperbolic with $\delta = \log 3$. The meaning of this definition is that geodesic triangles in X are thinner than geodesic triangles in Δ which is the model space with curvature -1.

We can define the notion of $CAT(0)$-space by comparing geodesic triangles in X and triangles in the Euclidean plane (which is the model of space with zero curvature). More generally, we can define the notion of $CAT(K)$ for some K by comparing triangles in X with triangles in the model space of (sectional) curvature K.

3.2 The Heisenberg Group as Boundary of the Complex Hyperbolic Space

We first recall the definition of the Heisenberg group. This one appears in several domains in mathematics : CR geometry, analysis on Lie groups, control theory, sub-riemannian geometry, complex analysis, ... It can be seen also as a model of the functional structure of the mammalian visual cortex ! Since these notes correspond to lectures given during a spring school in complex analysis, we will present the Heisenberg group from the complex analysis viewpoint.

Let $\mathbb{B} = \{w = (w_1, \dots, w_{n+1}) \in \mathbb{C}^{n+1}, \sum_{i=1}^{n+1} |w_i|^2 < 1\}$ in $\mathbb{C}^{n+1}$. The starting point of this section is the study of the group of biholomorphic self-mappings $\mathrm{Aut}(\mathbb{B})$ of $\mathbb{B}$. For our purpose, it is more convenient to consider $\mathrm{Aut}(\mathscr{H})$ where $\mathscr{H}$ is an "upper half-space". In the one-dimensional case, the unit disc Δ is biholomorphically equivalent to the upper half-plane $\{z = x+iy; y > 0\}$. In higher dimensions, we will consider the upper half-space $\mathscr{H} = \{z \in \mathbb{C}^{n+1}; Im(z_{n+1}) > \sum_{j=1}^{n} |z_j|^2\}$ which is biholomorphically equivalent to $\mathbb{B} = \{w \in \mathbb{C}^{n+1}; \sum_{j=1}^{n+1} |w_j|^2 < 1\}$ by considering

$$w_{n+1} = \frac{i - z_{n+1}}{i + z_{n+1}} \text{ and for } k = 1, \dots, n, w_k = \frac{2iz_k}{i + z_{n+1}}$$

or equivalently

$$z_{n+1} = i\frac{1 - w_{n+1}}{1 + w_{n+1}} \text{ and for } k = 1, \dots, n, z_k = \frac{w_k}{1 + w_{n+1}}$$

Note that this equivalence extends also to the boundaries of the domains, that is $\partial\mathscr{H} = \{z \in \mathbb{C}^{n+1}; Im(z_{n+1}) = \sum_{j=1}^{n} |z_j|^2\}$ and $\partial\mathbb{B} = \{z \in \mathbb{C}^{n+1}; \sum_{j=1}^{n+1} |z_j|^2 = 1\}$ (except for the south pole $(0, \dots, 0, 1)$ of $\partial\mathbb{B}$ which should be seen as the image of the "point at infinity" of $\partial\mathscr{H}$). If we set $r(z) = Im(z_{n+1}) - |z'|^2$ for $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$, then $\mathscr{H} = \{z, r(z) > 0\}$ and $\partial\mathscr{H} = \{r(z) = 0\}$. So, r is the defining function of $\mathscr{H}$.

There is a natural isomorphism between $\mathrm{Aut}(\mathbb{B})$ and $\mathrm{Aut}(\mathscr{H})$ and we will study the last one. To do this, we recall that $\mathrm{Aut}(\mathscr{H})$ has an Iwasawa decomposition: $\mathrm{Aut}(\mathscr{H}) = K.A.N$ where K is a compact subgroup, A an abelian subgroup and N a nilpotent subgroup of $\mathrm{Aut}(\mathscr{H})$. If we use the notation $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$ (where $z' \in \mathbb{C}^n$ and $z_{n+1} \in \mathbb{C}$), the abelian part of the decomposition is given by the dilation $\delta_s(z) = (sz', s^2 z_{n+1})$ (the factor 2 in s^2 is explained by the definition of $\mathscr{H}$) and the compact part is given by rotations $u(z) = (u(z'), z_{n+1})$ where u is an unitary linear transformation of $\mathbb{C}^n$ (that is given by a matrix whose rows and columns form a Hermitian orthonormal basis of $\mathbb{C}^n$ and that has determinant 1). The nilpotent part is associated to the Heisenberg group:

Definition 3.2.1 The *Heisenberg group* is as a set $\mathbb{C}^n \times \mathbb{R} = \{[\xi, t]; \xi \in \mathbb{C}^n, t \in \mathbb{R}\}$. The law is given by $[\xi, t].[\eta, s] = [\xi + \eta, t + s + 2Im(\xi.\overline{\eta})]$ where $\xi.\eta = \sum_{i=1}^{n} \xi_i\overline{\eta_i}$.

It is easy to check that $\mathbb{C}^n \times \mathbb{R}$ equipped with this law is a nonabelian group whose identity is $[0, 0]$ and where the inverse is given by $[\xi, t]^{-1} = [-\xi, -t]$. We will denote this group by $\mathbb{H}^n$. To each $[\xi, t] \in \mathbb{H}^n$, we associate the (holomorphic) affine transformation :

$$\phi_{\xi,t} : (z', z_{n+1}) \to (z' + \xi, z_{n+1} + t + 2iz'.\overline{\xi} + i|\xi|^2). \tag{3.1}$$

Since $|z'+\xi|-|z|^2 = Im(i(z'.\overline{\xi}+|\xi|^2))$, this mapping preserves the defining function $r(z) = Im(z_{n+1}) - |z'|^2$ and so maps $\mathscr{H}$ to $\mathscr{H}$ and its boundary $\partial\mathscr{H}$ to itself. In fact, (3.1) defines an action of $\mathbb{H}^n$ on $\mathscr{H}$. Indeed, if we compose the mappings associated by (3.1) to $[\xi, t]$ and $[\eta, s]$, we get the mapping associated to the element $[\xi, t].[\eta, s]$ of $\mathbb{H}^n$. Thus, we get a realization of $\mathbb{H}^n$ as the group of affine holomorphic bijections of $\mathscr{H}$. Note also that mappings given by (3.1) have a transitive action on $\partial\mathscr{H}$ (that is for every two points of $\partial\mathscr{H}$, there is exactly one element of $\mathbb{H}^n$ that maps the first to the second). In particular, the mapping associated to $[\xi, t]$ is the unique mapping that maps $(0, 0)$ to $(\xi, t + i|\xi|^2)$ and we can identified $\mathbb{H}^n$ with $\partial\mathscr{H}$ via its action on the origin.

We now describe the structure of the Heisenberg group as a (real) Lie group. Set for any $1 \leq j \leq n$

$$\tau_{2j-1}(s) = [(0, \dots, s + i0, \dots, 0), 0]$$

$$\tau_{2j}(s) = [(0, \dots, 0 + is, \dots, 0), 0]$$

$$\tau_{2n+1}(s) = [0, s].$$

We now differentiate functions by using these one-group parameters of directions to get the following vector fields:

$$X_j f([\xi, t]) = \frac{d}{ds} f([\xi, t].\tau_{2j-1}(s))|_{s=0} = \left(\frac{\partial f}{\partial x_j} + 2y_j \frac{\partial f}{\partial t}\right)[\xi, t]$$

$$Y_j f([\xi, t]) = \frac{d}{ds} f([\xi, t].\tau_{2j}(s))|_{s=0} = \left(\frac{\partial f}{\partial y_j} - 2x_j \frac{\partial f}{\partial t}\right)[\xi, t]$$

$$Tf([\xi, t]) = \frac{d}{ds} f([\xi, t].\tau_{2n+1}(s))|_{s=0} = \frac{\partial f}{\partial t} f([\xi, t]).$$

It is not difficult to see that all commutators of these vector fields are trivial unless $[X_j, Y_j] = -4T$ for any $j = 1, \dots, n$. Recall that $[X_j, Y_j] = X_j Y_j - Y_j X_j$. So the Lie algebra of $\mathbb{H}^n$ is generated by the X_j, Y_j and their commutators $[X_j, Y_j]$. This implies that for any vector fields A, B and C, we get $[[A, B], C] = 0$. Thus the vector fields of the Heisenberg group form a nilpotent Lie algebra of step one.

Remark 3.2.2 The previous results show that the Heisenberg group could be also seen as a CR manifold. We will not discuss this point of view in these notes.

Remark 3.2.3 Consider $n = 1$ for simplicity. Set $X = \partial_{x_1} - \frac{1}{2}x_2\partial_{x_3}$, $Y = \partial_{x_2} + \frac{1}{2}x_1\partial_{x_3}$ and $T = \partial_{x_3}$. Then, (X, Y, T) is also a basis of the Heisenberg algebra. Consider $\omega = dx_3 - 1/2(x_1dx_2 - x_2dx_1)$. Then, $\omega \wedge d\omega = -dx_1 \wedge dx_2 \wedge dx_3$, so ω is a contact form. The horizontal distribution is given by the kernel of ω which admits (X, Y) as a basis.

A natural measure on $\mathbb{H}^n$ is given by the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. Since the left translations of $\mathbb{H}^n$ on itself are affine when considered as mappings of $\mathbb{C}^n \times \mathbb{R}$ and their linear parts have determinant 1, the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ is left-invariant (and for the same reason right-invariant) and so is the Haar measure of the Lie group $\mathbb{H}^n$. We will denote by $|A|$ the measure of a subset A in $\mathbb{H}^n$.

We now construct a distance on $\mathbb{H}^n$ called the *Carnot-Carathéodory distance*. First, an absolutely continuous curve $\gamma : [0, 1] \to \mathbb{H}^n$ is said to be *horizontal* if for almost every $\tau \in [0, 1]$, there exist measurable functions a_j, b_j so that $\gamma'(\tau) = \sum_{j=1}^n (a_jX_j(\gamma(\tau)) + b_jY_j(\gamma(\tau)))$. Given two points $[\xi, t]$ and $[\eta, s]$ in $\mathbb{H}^n$, there always exists an horizontal curve joining them.

Definition 3.2.4 We set

$$d_{CC}([\xi, t], [\eta, s]) = \inf_{\gamma} l(\gamma)$$

where the infimum is taken over all horizontal curves between $[\xi, t]$ and $[\eta, s]$ and where $l(\gamma) = \int_0^1 \left(\sum_{j=1}^n (a_j(\tau)^2 + b_j(\tau)^2)\right)^{1/2} d\tau$ is the length of the curve γ. We denote by $B_{CC}([\xi, t], r)$ the (open) ball with center $[\xi, t] \in \mathbb{H}$ and radius $r > 0$ with respect to d_{CC}.

We have the following properties of d_{CC}:

- The topology defined by d_{CC} coincides with the Euclidean topology. Thus, the topological dimension of $(\mathbb{H}^n, d_{CC})$ is $2n + 1$.
- The Carnot-Carathéodory distance is left invariant, that is $d_{CC}(T_g[\xi, t], T_g[\eta, s]) = d_{CC}([\xi, t], [\eta, s])$ where T_g is a translation, that is $T_g[\xi, t] = g.[\xi, t]$ for a $g \in \mathbb{H}^n$.
- For any $k > 0$, $d_{CC}(\delta_k[\xi, t], \delta_k[\eta, s]) = k^{2n+2}d_{CC}([\xi, t], [\eta, s])$ where $\delta_k[\xi, t] = [k\xi, k^2t]$. The (δ_k) form the natural group of dilations of the Heisenberg group $\mathbb{H}^n$.

It follows that for any $[\xi, t] \in \mathbb{H}^n$ and any $r > 0$, we have

$$|B_{CC}([\xi, t], r)| = |B_{CC}(0_{\mathbb{H}^n}, r)| = r^{2n+2}|B_{CC}(0_{\mathbb{H}^n}, 1)|.$$

This implies that the measure of a ball of radius r is like r^{2n+2} and that the Hausdorff dimension of $(\mathbb{H}^n, d_{CC})$ is $2n + 2$.

Remark 3.2.5 The sub-Riemannian structure of $\mathbb{H}^n$ is given by the horizontal distribution, but it is possible to approximate it by Riemannian structures. For simplicity, we consider the case $n = 1$. Let $\gamma : [0,1] \to \mathbb{H}^1$ be a smooth curve. Write $\gamma'(t) = a_1(t)X_1(\gamma(t)) + a_2(t)Y_1(\gamma(t)) + a_3(t)T(\gamma(t))$. For any $L > 0$, consider the Riemannian length $l_L(\gamma) = \int_0^1 (a_1(t)^2 + a_2(t)^2 + La_3(t)^2)^{1/2}\mathrm{d}t$ and the Riemannian distance associated. When L tends to $+\infty$, we penalize the T-direction and we obtain the Carnot-Carathéodory distance which can be well approximated by Riemannian metrics by this method (in the sense of the Hausdorff-Gromov distance).

We now explain why the Heisenberg group can be seen as the boundary at infinity of the hyperbolic complex space. First, note that another interesting distance on the Heisenberg group is the Cygan-Korànyi distance which is defined as follows.

Definition 3.2.6 Let the *Cygan-Korànyi gauge* be $||[\xi, t]||_{CK} = (||\xi||^4 + |t|^2)^{1/4}$ for any $[\xi, t] \in \mathbb{H}^n$. Here, $||\xi||$ is the usual norm in $\mathbb{C}^n$. Then, set $d_{CK}([\xi, t], [\eta, s]) = ||[\eta, s]^{-1}.[\xi, t]||_{CK}$.

It is an exercise to show that d_{CK} is a distance on $\mathbb{H}^n$ which is (bilipschitz) equivalent to d_{CC}. We would like to see the Heisenberg group (with its Cygan-Koranyi distance) as the boundary at infinity of the complex hyperbolic space.

The first step is to see the complex hyperbolic space $H^{n+1}_{\mathbb{C}}$ as a Gromov hyperbolic space. For simplicity of the presentation and to avoid some technical computations, we discuss only the case $n = 1$.

Consider the Bergman space associated to a domain $\Omega \subset \mathbb{C}^2$:

$$A^2(\Omega) = \{f \text{ holomorphic on } \Omega; ||f||_{L^2(\Omega)} < \infty\}$$

and consider the evaluation operator T_z at $z \in \Omega$, that is $T_z : A^2(\Omega) \to \mathbb{C}$ given by $T_z(f) = f(z)$. By the maximum principle, for any compact set $K \subset \Omega$, there exists a constant $C_K \geq 0$ so that $\sup_K |f| \leq C_K ||f||_{L^2(\Omega)}$. Hence, T_z is bounded and then, by the Riesz representation theorem, there exists a function $K(z, \xi)$ (called the *Bergman kernel* or reproducing kernel) so that $T_z(f) = f(z) = \int_\Omega K(z, \xi) f(\xi)\mathrm{d}\xi$ for any $f \in A^2(\Omega)$. The *Bergman metric* is thus the quadratic form whose coefficients are given by $b_{i,j}(z) = \partial_{z_i}\partial_{\overline{z_j}} \log(K(z, z))$. Then, the *Bergman distance* is given by $d_B(z, w) = \inf l_B(\Gamma)$ where the infimum is taken over all (smooth) curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = z$, $\gamma(1) = w$ and where $l_B(\gamma) = \int_0^1 \left(\sum_{i,j} b_{i,j}(\gamma(s))\gamma_i'(s)\gamma_j'(s)\right)^{1/2} \mathrm{d}s$. As for the Kobayashi metric, a crucial point is that a biholomorphism $f : \Omega \to \Omega'$ between two domains equipped with their Bergman metrics is an isometry. In the special $\Omega = \mathbb{B}$, we can use the symmetries of the unit ball to get

$$K(z, \xi) = 2/\pi^2\,(1- < z, \xi >)^{-3}$$

$$b_{i,j}(z) = 3(1 - |z|^2)^{-2}((1 - |z|^2)\delta_{ij} + \overline{z_i} z_j)$$

$$d_B(z,w) = \frac{\sqrt{3}}{2}\log\left(\frac{|1-<z,w>|+\sqrt{|w-z|^2+|<z,w>|^2-|z|^2|w|^2}}{|1-<z,w>|-\sqrt{|w-z|^2+|<z,w>|^2-|z|^2|w|^2}}\right).$$

Here $<z,w>=\overline{z_1}w_1+\overline{z_2}w_2$ and δ_{ij} is the Kronecker symbol. The complex hyperbolic plane is just the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ equipped with the Bergman distance.

Remark 3.2.7 As for the real hyperbolic plane, there are several models for the hyperbolic complex plane $H^2_{\mathbb{C}}$. For instance, we can consider the unil ball $\mathbb{B}$ of $\mathbb{C}^2$ equipped with the distance function

$$d_H(z,w) = arccosh\left(\frac{1-<z,w>}{\sqrt{(1-|z|^2)(1-|w|^2)}}\right)$$

and we get a Gromov hyperbolic space. It turns out that the Bergman metric and the hyperbolic metric d_H are comparable. More precisely, for any z, $w \in \mathbb{B}$, $d_H(z,w) = (1/\sqrt{3})d_B(z,w)$.

We would like to see the Heisenberg group (with its Cygan-Koranyi distance) as the boundary at infinity of $H^2_{\mathbb{C}}$. For this, we first estimate the Gromow product for z, $w \in \mathbb{B}$ (with basepoint the origin 0):

$$\exp(-(z|w)_0) = \left(\frac{|1-<z,w>|+\sqrt{|1-<z,w>|^2-(1-|z|^2)(1-|w|^2)}}{(1+|z|)(1+|w|)}\right)^{1/2}.$$

The boundary $\partial_G\mathbb{B}$ could be identified with the geometric boundary $\partial\mathbb{B}$ which could be identified with $\mathbb{H}^1$ by using the stereographic projection (see below). Moreover, for ξ, $\eta \in \partial_G H^2_{\mathbb{C}}$, we get

$$\lim_{z\to\xi, w\to\eta}\exp(-(z|w)_0) = \sqrt{\frac{|1-<\xi,\eta>|}{2}} = 1/2\sqrt{||\xi-\eta|^2-2iIm(<\xi,\eta>)|}.$$

It turns out that using a good model for the hyperbolic complex plane and invariance of the Gromov product by isometries, we can prove that the Cygan-Korànyi distance is comparable to $\exp(-(\xi|\eta)_0)$.

Remark 3.2.8 We have seen previously that the unit ball $\mathbb{B}$ is conformally equivalent to the upper space $\mathscr{H}^2 = \{(z_1,z_2); Im(z_2)-|z_1|^2>0\}$ by considering $C(z_1,z_2) = \left(\frac{iz_1}{1+z_2}, i\frac{1-z_2}{1+z_2}\right)$. Recall also that $\mathbb{H}^1$ could be identified with $\partial\mathscr{H}^2$. It is possible to identify explicitly $\partial\mathbb{B}$ and $\mathbb{H}^1$ by considering $\Pi\circ C$ where $\Pi(z_1,z_2) = (z_1, \frac{1}{4}Re(z_2))$ for any $(z_1,z_2)\in\mathscr{H}^2$. The map $\Pi\circ C$ could be seen as a generalisation of the classical stereographic projection.

As we will see in Chap. 5, the fact that the Heisenberg group is the boundary of the complex hyperbolic spaces is crucial when you like to prove Mostow Rigidity Theorem by using the theory of quasiconformal mappings.

3.3 Gromov Hyperbolicity of Strongly Pseudoconvex Domains

In this section, we assume that the bounded C^2 domain $\Omega \subset \mathbb{C}^n$ is *strongly pseudoconvex* (see definition 1.5.9). We would like to prove that Ω equipped with its Kobayashi metric is hyperbolic in the sense of Gromov.

Remark 3.3.1 The notion of Gromov hyperbolicity is purely metric and is different from the notion of hyperbolicity in the sense of Kobayashi (as discussed in the first chapter). However, there exist some formal relationships between these two notions. For instance, if X is a complex space and if there exists a length function F with holomorphic sectional curvature K_F bounded above by -1 everywhere, then X is Kobayashi hyperbolic.

We define the Carnot-Carathéodory metric d_{CC} on $\partial\Omega$ as follows (compare with the construction of the Carnot-Carathéodory distance on the Heisenberg group): here the horizontal space is the complex tangent subspace, that is $H_p(\partial\Omega) := T_p^{\mathbb{C}}(\partial\Omega)$.

A (piecewise) C^1-curve $\gamma : [0,1] \to \partial\Omega$ is said to be horizontal if $\gamma'(t) \in H_{\gamma(t)}\partial\Omega$ (whenever $\gamma'(t)$ exists). The key point is that the strict pseudoconvexity implies that $\partial\Omega$ is connected. Moreover, any pair of points x and $y \in \partial\Omega$ can be joined by an horizontal curve. Hence, we can define for $x, y \in \partial\Omega$

$$d_{CC}(x,y) = \inf \rho\text{-length}(\gamma)$$

where ρ-length $(\gamma) = \int_0^1 L_{\rho,\gamma(t)}(\gamma'(t))^{1/2}\mathrm{d}t$ and where the infimum is taken over all horizontal curve $\gamma : [0,1] \to \partial\Omega$ joining x and y (that is $\gamma(0) = x$ and $\gamma(1) = y$).

The main point is that in this setting the Kobayashi metric is Gromov hyperbolic. More precisely, we have the

Theorem 3.3.2 *Let $\Omega \in \mathbb{C}^n$ $(n \geq 2)$ be a bounded, strictly pseudoconvex domain with C^2 boundary $\partial\Omega$. Then, Ω equipped with the Kobayashi distance k_Ω is hyperbolic in the sense of Gromov. Moreover, $\partial_G\Omega$ could be identified with $\partial\Omega$ and the Carnot-Carathéodory distance d_{CC} is in the conformal gauge of (Ω, k_Ω).*

Roughly speaking, this means that d_{CC} is quasiconformally equivalent to a visual metric. Precise definitions are given in Chap. 5. The strategy of proof is quite natural.

Step 1: Equip $\partial\Omega$ with the Carnot-Carathéodory distance d_{CC}. By analogy of the case of the hyperbolic complex space, we can expect that d_{CC} is a visual metric on $\partial\Omega$ ...if we think that Ω with the Kobayashi distance is Gromov hyperbolic. As we mentioned previously, any (compact) metric space could be seen as boundary at infinity of a Gromov hyperbolic space. Thus, we consider for $x, y \in \Omega$,

$$d_{BS}(x,y) = 2\log\left(\frac{d_{CC}(\pi(x),\pi(y)) + \max(h(x),h(y)))}{\sqrt{h(x)h'y)}}\right),$$

where $\delta(x) = d(x, \partial\Omega), h(x) = \delta(x)^{1/2}$ and $\pi(x) \in \partial\Omega$ satisfies $|x - \pi(x)| = \delta(x)$. The distance d_{BS} is just a modification of the previous one. Note that since $\partial\Omega$ is C^2, the map $x \mapsto \pi(x)$ is well defined if x is closed enough from $\partial\Omega$. The ambiguity of the definition of π is just a technical problem that we forget in the rest of the discussion.

Step 2. By construction, Ω with the distance d_{BS} is Gromov-hyperbolic. We can conclude if we can prove that the metric spaces (Ω, k_Ω) and (Ω, d_{BS}) are quasi-isometric by Theorem 3.1.11.

This proof illustrates the fact that "a Gromov hyperbolic space is determined by its boundary". Let $p \in \partial\Omega$. We can consider the splitting of the tangent space at p: $\mathbb{C}^n = H_p\partial\Omega \oplus N_p\partial\Omega$ where $N_p\partial\Omega$ is the complex one-dimensional subspace of $\mathbb{C}^n$ orthogonal to $H_p\partial\Omega$. Thus, any tangent vector Z could be uniquely written as $Z = Z_H + Z_N$ where $Z_H \in H_p\partial\Omega$ and $Z_N \in N_p\partial\Omega$. The main step of the proof is to show there exist $\varepsilon > 0$, $s > 0$, $C_1 > 0$, $C_2 \geq 1$ so that the infinitesimal Kobayashi metric κ_Ω satisfies

$$(1 - C_1\delta^s(x))\left(\frac{|Z_N|^2}{4\delta^2(x)} + C_2^{-1}\frac{L_{\rho,\pi(x)}(Z_H)}{\delta(x)}\right)^{1/2}$$
$$\leq \kappa_\Omega(x; Z) \leq (1 + C_1\delta^s(x))\left(\frac{|Z_N|^2}{4\delta^2(x)} + C_2\frac{L_{\rho,\pi(x)}(Z_H)}{\delta(x)}\right)^{1/2}$$

whenever $x \in \Omega$ is in a neighborhood of $\partial\Omega$ and $Z = Z_H + Z_N \in \mathbb{C}^n$ where the splitting is at $p = \pi(x)$. To do this, as in the case of the classical Heisenberg group, the distance could be approximated by a sequence of Riemannian metrics. The second step of the proof is to show that for any pseudo distance function F that satisfies the previous estimate, there exists $D \geq 0$ so that $d_{BS}(x, y) - D \leq d_F(x, y) \leq d_{BS}(x, y) + D$ for any x, any $y \in \Omega$. In particular, in the case of the Kobayashi metric, this estimates gives a good control of the behaviour of $k_\Omega(x, y)$ for all the possible x and $y \in \Omega$.

If $\Omega \subset \mathbb{C}^n$ is a strictly pseudoconvex domain (with C^2 boundary) equipped with its Kobayashi metric, then it is Gromov hyperbolic, and any biholomorphism is an isometry with respect to the Kobayashi pseudo-distance. We thus get a variant of a theorem of Fefferman about the extension of biholomorphic maps between smooth strictly pseudoconvex domains:

Theorem 3.3.3 *Let Ω_1, $\Omega_2 \in \mathbb{C}^n$ $(n \geq 2)$ be strictly pseudoconvex domains with C^2-boundary and let $f : \Omega_1 \to \Omega_2$ be a biholomorphism. Then f has a continuous extension $\overline{f} : \overline{\Omega_1} \to \overline{\Omega_2}$ and the induced boundary map $\tilde{f} : \partial\Omega_1 \to \partial\Omega_2$ is bilipschitz with respect to the Carnot-Carathéodory distances.*

The point is that this extension comes for free by the general theory of Gromov hyperbolic spaces (See Theorem 5.3.4). However, this does not give the optimal regularity of the induced map.

3.4 Notes

A complete introduction to spaces with non-negative curvature is [4], see also [8]. Reference [5] provides a good overview of metric geometry, in particular spaces with curvature bounded below or above. A very nice introduction to the Heisenberg group is given in [6] where the reader will find a complete proof that the Heisenberg group could be seen as the boundary at infinity of the hyperbolic complex space. References [10] (the last two chapters) and [9] explain the use of the Heisenberg groups in harmonic analysis and complex analysis. A more exhaustive presentation of the Sub-Riemannian geometry is in [2] (See also in the same book the paper by M. Gromov which contains a lot of interesting informations about sub-Riemannian spaces but is hard to read). Proofs of the Gromov hyperbolicity of the metric d_{BS} and of the Kobayashi metric on strictly pseudoconvex domains of $\mathbb{C}^n$ could be found in the original papers [3] and [1] respectively. The proof of theorem 3.3.3 is given in [1]. Fefferman's result is in [7] where the proof is based on the study of geodesics for the Bergman metric (but not in relation with metric geometry and some curvature estimates).

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Chapter 4
Gromov Hyperbolicity of Bounded Convex Domains

Andrew Zimmer

It is well known that the unit ball endowed with the Kobayashi metric is isometric to complex hyperbolic space and in particular is an example of a negatively curved Riemannian manifold. One would then suspect that when $\Omega \subset \mathbb{C}^d$ is a domain close to the unit ball, then the Kobayashi metric on Ω should be negatively curved (in some sense). Unfortunately, for general domains the Kobayashi metric is no longer Riemannian and thus will no longer have curvature in a local sense. Instead one can ask if the Kobayashi metric satisfies a coarse notion of negative curvature from geometric group theory called Gromov hyperbolicity.

Gromov hyperbolic metric spaces have been intensively studied and have a number of remarkable properties. Thus it seems natural to determine the domains for which the Kobayashi metric is Gromov hyperbolic and then to use the theory of such metric spaces to prove new results in several complex variables.

The first major result in this direction is Theorem 3.3.2 due to Balogh and Bonk [3]. This theorem was later extended to strongly pseudoconvex domains in almost complex manifolds [12, 13, 16]. In these arguments, one establishes Gromov hyperbolicity by using very precise estimates for the Kobayashi infinestimal metric. In particular, if $\Omega \subset \mathbb{C}^d$ is a strongly pseudoconvex domain and $x \in \Omega$ is close to the boundary, then there is a unique point $\xi \in \partial\Omega$ closest to x. If $\nu(\xi)$ is the inward pointing normal line at ξ and $T_\xi^{\mathbb{C}}\partial\Omega$ is the complex tangent space of $\partial\Omega$ at ξ then we have the following estimates for the Kobayashi metric at x:

$$\kappa_\Omega(x;v) \approx \frac{\|v\|}{\delta_\Omega(x)} \text{ if } v \in \mathbb{C}\cdot\nu(\xi) \quad \text{and} \quad \kappa_\Omega(x;v) \approx \frac{\|v\|}{\delta_\Omega(x)^{1/2}} \text{ if } v \in T_\xi^{\mathbb{C}}\partial\Omega.$$

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7_4

Here $\delta_\Omega(x)$ is the distance from x to $\partial\Omega$ (which in this case is $\|x - \xi\|$). The fact that the metric behaves identically in every complex tangential direction seems to be key for these arguments.

Given Theorem 3.3.2 it is natural to ask about the case of general finite type domains:

Question 4.0.1 ([3, Section 6]) Is the Kobayashi metric Gromov hyperbolic for a weakly pseudoconvex domain with finite type in the sense of D'Angelo?

Unfortunately, not much is known about the Kobayashi metric on a general weakly pseudoconvex domains of finite type so this problem currently seems out of reach. For instance, it is unknown whether or not the Kobayashi metric is Cauchy complete for domains of this type. So at this point it seems natural to impose additional constraints on the domain Ω such as convex, $\mathbb{C}$-convex, h-extendible, etc.

Amongst the set of convex domains with smooth boundary we recently characterized the domains which are Gromov hyperbolic:

Theorem 4.0.2 ([60]) *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. Then (Ω, k_Ω) is Gromov hyperbolic if and only if $\partial\Omega$ has finite type.*

For convex domains of finite type there are good estimates for the infinitesimal Kobayashi metric for points near the boundary [2, Proposition 1.6] but the metric no longer grows identically in every complex tangential direction which makes the types of arguments used in the strongly pseudoconvex case very difficult to implement. Instead the main strategy of the proof of Theorem 4.0.2 is to study the orbit of convex sets under the group $\mathrm{Aff}(\mathbb{C}^d)$ of affine automorphisms of $\mathbb{C}^d$.

In particular, let $\mathbb{X}_{d,0}$ be the set of pairs (Ω, x) where $\Omega \subset \mathbb{C}^d$ is a convex domain that does not contain a complex affine line and $x \in \Omega$. By studying the closure of $\mathrm{Aff}(\mathbb{C}^d)$-orbits in $\mathbb{X}_{d,0}$ it is possible to establish necessary and sufficient conditions for the Gromov hyperbolicity of the Kobayashi metric. This approach is motivated by Benoist's recent work on the Hilbert metric [9]. It is also related to the scaling methods of Pinchuk [52] and Frankel [24–26] (for an overview see [42]).

The main purpose of this chapter is to sketch the proof of Theorem 4.0.2. In particular Sects. 4.5 and 4.6 are devoted to the describing the proof of Theorem 4.0.2. We will also provide:

1. detailed proofs of some well known estimates for the Kobayashi metric and distance on convex domains (Sect. 4.2),
2. a detailed description of a natural topology on the space of convex domains and work of Frankel (Sect. 4.3),
3. a sketch of an alternative proof of Theorem 3.3.2 (Sect. 4.7),
4. an introduction of the Hilbert metric, its basic properties, its connection to the Kobayashi metric, and the work of Benoist (Sect. 4.7),
5. some open conjectures and questions (Sect. 4.8).

4.1 Preliminaries

Given a domain $\Omega \subset \mathbb{C}^d$ the *(infinitesimal) Kobayashi metric* is the pseudo-Finsler metric

$$\kappa_\Omega(x; v) = \inf\{|\xi| : f \in \mathrm{Hol}(\Delta, \Omega),\ f(0) = x,\ df_0(\xi) = v\}\,.$$

By a result of Royden [54, Proposition 3] the Kobayashi metric is an upper semicontinuous function on $\Omega \times \mathbb{C}^d$. In particular if $\sigma : [a, b] \to \Omega$ is an absolutely continuous curve (as a map $[a, b] \to \mathbb{C}^d$), then the function

$$t \in [a, b] \to \kappa_\Omega(\sigma(t); \sigma'(t))$$

is integrable and we can define the *length of* σ to be

$$\ell_\Omega(\sigma) = \int_a^b k_\Omega(\sigma(t); \sigma'(t))\mathrm{d}t.$$

One can then define the *Kobayashi pseudo-distance* to be

$$k_\Omega(x, y) = \inf\{\ell_\Omega(\sigma) : \sigma\colon [a, b] \to \Omega \text{ is absolutely continuous, with } \sigma(a) = x,\ \text{and } \sigma(b) = y\}\,.$$

This definition is equivalent to the standard Definition 1.2.3 of k_Ω via analytic chains, see [56, Theorem 3.1].

Example 4.1.1 As a consequence of the Schwarz lemma, we obtain as in Definition 1.1.1

$$\kappa_\Delta(x; v) = \frac{|v|}{1 - |z|^2}$$

and hence

$$k_\Delta(z, w) = \tanh^{-1}\left(\frac{|z - w|}{|1 - \bar{z}w|}\right)$$

Then, if $\mathscr{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ we see that

$$\kappa_{\mathscr{H}}(z; v) = \frac{|v|}{2\,\mathrm{Im}(z)}$$

and

$$k_{\mathscr{H}}(z,w) = \frac{1}{2}\operatorname{arcosh}\left(1 + \frac{|z-w|^2}{2\operatorname{Im}(z)\operatorname{Im}(w)}\right).$$

Given a domain $\Omega \subset \mathbb{C}^d$ and an interval $I \subset \mathbb{R}$, a curve $\sigma : I \to \Omega$ is called a *geodesic* if

$$k_\Omega(\sigma(s), \sigma(t)) = |t-s|$$

for all $t, s \in I$. For reasonable domains, geodesics have nice properties:

Proposition 4.1.2 *Suppose $\Omega \subset \mathbb{C}^d$ is a domain and (Ω, k_Ω) is Cauchy complete. Then for every two points $x, y \in \Omega$ there exists a geodesic $\sigma : [a,b] \to \Omega$ so that $\sigma(a) = x$ and $\sigma(b) = y$. Moreover, if $\sigma : I \to \Omega$ is a geodesic, then σ is absolutely continuous (as a map $I \to \mathbb{C}^d$) and*

$$\kappa_\Omega(\sigma(t); \sigma'(t)) = 1$$

for almost every $t \in I$.

A detailed proof of this Proposition can be found in [15, Proposition 4.6].

Given a metric space (X, d), the length of a continuous curve $\sigma : [a,b] \to X$ is defined to be

$$L_d(\sigma) = \sup\left\{\sum_{i=1}^n d(\sigma(t_{i-1}), \sigma(t_i)) : a = t_0 < t_2 < \cdots < t_n = b\right\}.$$

Then the induced metric d_I on X is defined as in Appendix A.1 to be

$$d_I(x,y) = \inf\{L_d(\sigma) : \sigma : [a,b] \to X \text{ is continuous}, \sigma(a) = x, \text{ and } \sigma(b) = y\}.$$

When $d_I = d$, the metric space (X, d) is called a *length metric space*. When the Kobayashi pseudo-distance is actually a distance, then the metric space (Ω, k_Ω) is a length metric space (by construction). For such metric spaces we have the following characterization of Cauchy completeness:

Theorem 4.1.3 (Hopf–Rinow) *Suppose (X, d) is a length metric space. Then the following are equivalent:*

1. *(X, d) is a proper metric space; that is, every bounded set is relatively compact.*
2. *(X, d) is Cauchy complete and locally compact.*

For a proof see, for instance, Proposition 3.7 and Corollary 3.8 in Chapter I of [18]. When k_Ω is a distance on $\Omega \subset \mathbb{C}^d$ the Kobayashi distance generates the standard topology on Ω and so the metric space (Ω, k_Ω) is locally compact. In particular we obtain:

Proposition 4.1.4 *Suppose k_Ω is a distance on $\Omega \subset \mathbb{C}^d$. Then the following are equivalent:*

1. *(Ω, k_Ω) is a proper metric space; that is, every bounded set is relatively compact.*
2. *(Ω, k_Ω) is Cauchy complete.*

Given a function $f : \mathbb{C} \to \mathbb{R}$ with $f(0) = 0$ let $\nu(f)$ denote the *order of vanishing* of f at 0, that is

$$\nu(f) = \sup\left\{ n : \lim_{z \to 0} |z|^{1-n} |f(z)| = 0 \right\}.$$

Definition 4.1.5 Suppose that $\Omega = \{z \in \mathbb{C}^d : r(z) < 0\}$ where r is a C^∞ function with $\nabla r \neq 0$ near $\partial\Omega$. We say that a point $x \in \partial\Omega$ has *finite line type L* if

$$\sup\{\nu(r \circ \ell) | \ell : \mathbb{C} \to \mathbb{C}^d \text{ is a non-trivial affine map and } \ell(0) = x\} = L.$$

Notice that $\nu(r \circ \ell) \geq 2$ if and only if $\ell(\mathbb{C})$ is tangent to Ω. McNeal [47] proved that if Ω is convex then $x \in \partial\Omega$ has finite line type if and only if it has finite type in the sense of D'Angelo (see also [17]). In this paper, we say a convex domain Ω with C^∞ boundary has *finite line type L* if the line type of all $x \in \partial\Omega$ is at most L and this bound is realized at some boundary point. Finite line type has the following geometric consequence:

Proposition 4.1.6 *Suppose Ω is a convex domain and $\partial\Omega$ is C^L and has finite line type L near some $\xi \in \partial\Omega$. Then there exists a neighborhood U of ξ and a $C > 0$ such that, for all $p \in U \cap \Omega$ and $v \in \mathbb{C}^d$ nonzero,*

$$\delta_\Omega(p; v) \leq C \delta_\Omega(p)^{1/L}$$

where $\delta_\Omega(p) = \inf\{\|q - p\| : q \in \partial\Omega\}$ *and*

$$\delta_\Omega(p; v) = \inf\{\|q - p\| : q \in (p + \mathbb{C} \cdot v) \cap \partial\Omega\}.$$

4.2 The Kobayashi Metric and Distance on Convex Domains

In this section we discuss some well known estimates for the Kobayashi metric and distance on convex domains. One of the most important applications of these estimates is the following theorem of Barth:

Theorem 4.2.1 ([4]) *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Then the following are equivalent:*

1. *(Ω, k_Ω) is a Cauchy complete metric space,*
2. *Ω does not contain a complex affine line.*

Motivated by Theorem 4.2.1 we make the following definition:

Definition 4.2.2 A convex domain $\Omega \subset \mathbb{C}^d$ is called $\mathbb{C}$-*proper* if Ω does not contain a complex affine line.

Using these estimates we will also establish a basic connection between the geometry of the boundary and the behavior of the Kobayashi distance:

Proposition 4.2.3 *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Assume that $x_m, y_n \in \Omega$, $x_m \to \xi \in \partial\Omega$, $y_n \to \eta \in \partial\Omega$, and*

$$\liminf_{m,n\to\infty} k_\Omega(x_m, y_n) < \infty.$$

Then either $\xi = \eta$ or there exists a complex line L so that ξ and η are contained in the interior of $\partial\Omega \cap L$ in L.

By considering affine maps of the unit disk into a domain Ω one immediately obtains the following upper bound on the infinitesimal Kobayashi metric:

Lemma 4.2.4 *Suppose $\Omega \subset \mathbb{C}^d$ is a domain. Then*

$$\kappa_\Omega(x; v) \leq \frac{\|v\|}{\delta_\Omega(x; v)}$$

for any $x \in \Omega$ and $v \in \mathbb{C}^d$.

For convex domains we can use supporting real hyperplanes to obtain a lower bound:

Lemma 4.2.5 *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Then*

$$\frac{\|v\|}{2\delta_\Omega(x; v)} \leq \kappa_\Omega(x; v)$$

for any $x \in \Omega$ and $v \in \mathbb{C}^d$.

This result is originally due to Graham [33, Theorem 5] but proofs can also be found in [6, Theorem 4.1] and [26, Theorem 2.2]).

Proof Let $L := x + \mathbb{C}\, v$ and $\xi \in L \setminus \Omega \cap L$ such that $\|\xi - x\| = \delta_\Omega(x; v)$. Let H be a real hyperplane through ξ which does not intersect Ω. By rotating and translating we may assume $\xi = 0$, $x = (x_1, 0, \dots, 0)$, $H = \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) = 0\}$, and $\Omega \subset \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) > 0\}$. With this choice of normalization $v = (v_1, 0, \dots, 0)$ for some $v_1 \in \mathbb{C}$.

Then if $P : \mathbb{C}^d \to \mathbb{C}$ is the projection onto the first component we have

$$\kappa_\Omega(x; v) \geq \kappa_{P(\Omega)}(x_1; v_1) \geq \kappa_{\mathscr{H}}(x_1; v_1) = \frac{|v_1|}{2\operatorname{Im}(x_1)} \geq \frac{|v_1|}{2\,|x_1|}.$$

Since $|x_1| = \|\xi - x\| = \delta_\Omega(x; v)$ and $|v_1| = \|v\|$ this completes the proof.

Essentially the same argument provides a lower bound on the Kobayashi distance:

Lemma 4.2.6 *Suppose $\Omega \subset \mathbb{C}^d$ is an open convex set and $x, y \in \Omega$. If L is the complex line containing x, y and $\xi \in L \setminus L \cap \Omega$ then*

$$\frac{1}{2} \operatorname{arcosh}\left(1 + \frac{\|x-y\|^2}{2\,\|x-\xi\|\,\|y-\xi\|}\right) \leq k_\Omega(x, y).$$

In particular,

$$\frac{1}{2}\left|\log\left(\frac{\|x-\xi\|}{\|y-\xi\|}\right)\right| \leq k_\Omega(x, y).$$

Proof The second assertion follows from the first since

$$\begin{aligned}\operatorname{arcosh}\left(1 + \frac{\|x-y\|^2}{2\,\|x-\xi\|\,\|y-\xi\|}\right) &\geq \operatorname{arcosh}\left(1 + \frac{(\|x-\xi\| - \|y-\xi\|)^2}{2\,\|x-\xi\|\,\|y-\xi\|}\right)\\ &= \operatorname{arcosh}\left(\frac{\|x-\xi\|}{2\,\|y-\xi\|} + \frac{\|y-\xi\|}{2\,\|x-\xi\|}\right) = \left|\log\left(\frac{\|x-\xi\|}{\|y-\xi\|}\right)\right|.\end{aligned}$$

To prove the first assertion, notice that since x, y, ξ are all co-linear both sides of the desired inequality are invariant under affine transformations, in particular we can replace Ω by $A\Omega$ for some affine map A. Now let H be a real hyperplane through ξ which does not intersect Ω. Using an affine transformation we may assume $\xi = 0$, $x = (x_1, 0, \dots, 0)$, $y = (y_1, 0, \dots, 0)$, $H = \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) = 0\}$, and $\Omega \subset \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) > 0\}$.

Then if $P : \mathbb{C}^d \to \mathbb{C}$ is the projection onto the first coordinate, we have

$$\begin{aligned}k_\Omega(x, y) \geq k_{P(\Omega)}(x_1, y_1) \geq k_{\mathscr{H}}(x_1, y_1) &= \frac{1}{2} \operatorname{arcosh}\left(1 + \frac{|x_1 - y_1|^2}{2 \operatorname{Im}(x_1) \operatorname{Im}(y_1)}\right)\\ &\geq \frac{1}{2} \operatorname{arcosh}\left(1 + \frac{(|x_1| - |y_1|)^2}{2\,|x_1|\,|y_1|}\right).\end{aligned}$$

Since $\|x-y\| = |x_1 - y_1|$, $\|x-\xi\| = |x_1|$, and $\|y-\xi\| = |y_1|$ the lemma follows.

For a bounded convex domain, Lemma 4.2.4 can be used to establish the following upper bound of the Kobayashi distance:

Lemma 4.2.7 *If $\Omega \subset \mathbb{C}^d$ is a bounded convex domain and $x_0 \in \Omega$, then there exists $C, \alpha > 0$ so that for all $x \in \Omega$,*

$$k_\Omega(x_0, x) \leq C + \alpha \log \frac{1}{\delta_\Omega(x)}$$

In some of the arguments below it will be helpful to have the following "uniform" version of the above Observation:

Lemma 4.2.8 *For any $R, \epsilon > 0$ there exists $C, \alpha > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a convex domain, $x_0 \in \Omega$, and $\delta_\Omega(x_0) \in [\epsilon, 1/\epsilon]$, then for all $x \in \mathbb{B}_R(x_0)$,*

$$k_\Omega(x, x_0) \leq C + \alpha \log\left(\frac{1}{\delta_\Omega(x)}\right)$$

Proof Fix $x \in \mathbb{B}_R(x_0)$ and define

$$\xi = x + \frac{\delta_\Omega(x)}{\|x - x_0\|}(x - x_0) \in \overline{\Omega}.$$

Then consider the curve $\sigma(t) = \xi + e^{-t}(x_0 - \xi)$. Notice that $\sigma(t^*) = x$ when

$$t^* = \log\left(\|x - x_0\| + \delta_\Omega(x)\right) + \log\frac{1}{\delta_\Omega(x)}.$$

Since $\delta_\Omega(x) \leq \delta_\Omega(x_0) + R \leq 1/\epsilon + R$ we have

$$t^* \leq \log\left(2R + 1/\epsilon\right) + \log\frac{1}{\delta_\Omega(x)}.$$

Now, since Ω contains the convex hull of $\mathbb{B}_\epsilon(x_0)$ and ξ, we see that

$$\delta_\Omega(\sigma(t)) \geq \epsilon e^{-t}.$$

Moreover,

$$\left\|\sigma'(t)\right\| = e^{-t}\left\|x_0 - \xi\right\| \leq e^{-t}\left(\|x_0 - x\| + \delta_\Omega(x)\right) \leq e^{-t}\left(2R + 1/\epsilon\right).$$

So

$$\begin{aligned}
k_\Omega(x, x_0) &\leq \int_0^{t^*} \kappa_\Omega(\sigma(t); \sigma'(t))\mathrm{d}t \leq \int_0^{t^*} \frac{e^{-t}\left\|x_0 - \xi\right\|}{\delta_\Omega(\sigma(t))}\mathrm{d}t \leq \int_0^{t^*} (2R/\epsilon + 1/\epsilon^2)\mathrm{d}t \\
&= (2R/\epsilon + 1/\epsilon^2)t^* \\
&\leq (2R/\epsilon + 1/\epsilon^2)\log\left(2R + 1/\epsilon\right) + (2R/\epsilon + 1/\epsilon^2)\log\frac{1}{\delta_\Omega(x)}.
\end{aligned}$$

Proof (of Theorem 4.2.1) If Ω contains an entire complex affine line L, then clearly $k_\Omega(x, y) = 0$ when $x, y \in L$. So we see that (1) implies (2).

Now suppose that Ω does not contain a complex affine line, we claim that (Ω, k_Ω) is a Cauchy complete metric space. Notice that (Ω, k_Ω) is a metric space by Lemma 4.2.6. Then by Proposition 4.1.4, it is enough to show that closed metric balls are compact. So fix some $x_0 \in \Omega$ and $R > 0$, we claim that the closed Kobayashi ball

$$\overline{B_\Omega(x_0, R)} = \{x \in \Omega : K_\Omega(x, x_0) \leq R\}$$

is compact. So suppose that $x_n \in \overline{B_\Omega(x_0, R)}$. After passing to a subsequence we can assume that x_n converges to ξ in $\overline{\Omega} \cup \{\infty\}$. If $\xi \in \Omega$ then, since the Kobayashi distance is continuous, $\xi \in \overline{B_\Omega(x_0, R)}$.

So suppose that $\xi \in \partial\Omega \cup \{\infty\}$. Let L_n be the complex line containing x and x_n. After passing to a subsequence we can assume that the sequence L_n converges to a complex line L (if $\xi \neq \infty$, then passing to a subsequence is unnecessary).

First consider the case when $\xi \in \partial\Omega$. Then, since ξ is contained in a real supporting hyperplane, we see that there exists $\xi_n \in \partial\Omega \cap L_n$ so that $\xi_n \to \xi$. But then

$$R \geq \limsup_{n\to\infty} k_\Omega(x_0, x_n) \geq \limsup_{n\to\infty} \frac{1}{2} \log \frac{\|x_0 - \xi_n\|}{\|x_n - \xi_n\|} = \infty$$

and we have a contradiction.

Next consider the case when $\xi = \infty$. Then there exists some $\eta \in L \cap \partial\Omega$. Then, since η is contained in a real supporting hyperplane, we see that there exists $\eta_n \in \partial\Omega \cap L_n$ so that $\eta_n \to \eta$. But then

$$R \geq \limsup_{n\to\infty} k_\Omega(x_0, x_n) \geq \limsup_{n\to\infty} \frac{1}{2} \log \frac{\|x_n - \eta_n\|}{\|x_0 - \eta_n\|} = \infty$$

and we have a contradiction.

Proof (of Proposition 4.2.3) Assume that $\xi \neq \eta$. By passing to subsequences we may suppose that there exists $M < \infty$ such that $k_\Omega(x_n, y_n) < M$ for all $n \in \mathbb{N}$. For each n, let L_n be the complex affine line containing x_n and y_n. Let

$$\epsilon_n = \min\{\|\xi - x_n\| : \xi \in L_n \setminus \Omega \cap L_n\}$$

and $\xi_n \in L_n \setminus \Omega \cap L_n$ be a point closest to x_n. Then by Lemma 4.2.6

$$\begin{aligned} M &\geq \limsup_{n\to\infty} k_\Omega(x_n, y_n) \geq \limsup_{n\to\infty} \frac{1}{4} \log \frac{\|y_n - \xi_n\|}{\|x_n - \xi_n\|} \\ &\geq \limsup_{n\to\infty} \frac{1}{4} \log \frac{\|y_n - x_n\| - \epsilon_n}{\epsilon_n} \\ &\geq \limsup_{n\to\infty} \frac{1}{4} \log \frac{\|\xi - \eta\| - \epsilon_n}{\epsilon_n}. \end{aligned}$$

Since $\xi \neq \eta$ there exists an $\epsilon > 0$ such that $\mathbb{B}_\epsilon(x_n) \cap L_n \subset L_n \cap \Omega$ for all n sufficiently large. Which implies that ξ is in the interior of $\overline{\Omega} \cap L$ in L. The same argument applies to η.

4.3 The Space of Convex Domains

Let $\mathbb{X}_d$ be the set of $\mathbb{C}$-proper convex domains in $\mathbb{C}^d$ and let $\mathbb{X}_{d,0}$ be the set of pairs (Ω, x_0) where $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex open set and $x_0 \in \Omega$.

Given a set $A \subset \mathbb{C}^d$, let $\mathcal{N}_\epsilon(A)$ denote the *ϵ-neighborhood of A* with respect to the Euclidean distance. The *Hausdorff distance* between two compact sets A, B is given by

$$d_H(A,B) = \inf \{\epsilon > 0 : A \subset \mathcal{N}_\epsilon(B) \text{ and } B \subset \mathcal{N}_\epsilon(A)\}.$$

Equivalently,

$$d_H(A,B) = \max \left\{ \sup_{a\in A} \inf_{b \in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\| \right\}.$$

The Hausdorff distance is a complete metric on the space of compact sets in $\mathbb{C}^d$.

The space of all closed convex sets in $\mathbb{C}^d$ can be given a topology from the local Hausdorff semi-norms. For $R > 0$ and a set $A \subset \mathbb{C}^d$ let $A^{(R)} := A \cap \mathbb{B}_R(0)$. Then define the *local Hausdorff semi-norms* by

$$d_H^{(R)}(A,B) := d_H(A^{(R)}, B^{(R)}).$$

Since an open convex set is completely determined by its closure, we say a sequence of open convex sets A_n converges in the local Hausdorff topology to an open convex set A if there exists some $R_0 \geq 0$ so that

$$\lim_{n\to\infty} d_H^{(R)}(\overline{A}_n, \overline{A}) = 0$$

for all $R \geq R_0$.

Finally we introduce a topology on $\mathbb{X}_d$ and $\mathbb{X}_{d,0}$ using the local Hausdorff topology:

1. A sequence Ω_n converges to Ω_∞ in $\mathbb{X}_d$ if $\Omega_n \to \Omega_\infty$ in the local Hausdorff topology.
2. A sequence (Ω_n, x_n) converges to $(\Omega_\infty, x_\infty)$ in $\mathbb{X}_{d,0}$ if $\Omega_n \to \Omega_\infty$ in the local Hausdorff topology and $x_n \to x_\infty$.

Unsurprisingly, the Kobayashi distance is continuous with respect to the local Hausdorff topology.

Theorem 4.3.1 *Suppose Ω_n converges to Ω in $\mathbb{X}_d$. Then*

$$k_\Omega(x,y) = \lim_{n\to\infty} k_{\Omega_n}(x,y)$$

for all $x, y \in \Omega$ uniformly on compact sets of $\Omega \times \Omega$.

See [60, Theorem 4.1] for a detailed argument.

The group of affine automorphisms $\mathrm{Aff}(\mathbb{C}^d)$ of $\mathbb{C}^d$ acts (in the obvious way) on $\mathbb{X}_d$ and $\mathbb{X}_{d,0}$. Remarkably, this action is co-compact:

Theorem 4.3.2 (Frankel [24]) *The group $\mathrm{Aff}(\mathbb{C}^d)$ acts co-compactly on $\mathbb{X}_{d,0}$, that is there exists a compact set $K \subset \mathbb{X}_{d,0}$ so that $\mathrm{Aff}(\mathbb{C}^d) \cdot K = \mathbb{X}_{d,0}$.*

We will sketch the proof of Theorem 4.3.2 below, but we will first describe an application. As an immediate corollary to Theorem 4.3.2 we have:

Corollary 4.3.3 *Suppose $f : \mathbb{X}_{d,0} \to \mathbb{R}_{>0}$ is a function which is continuous in the topology of $\mathbb{X}_{d,0}$ and affine invariant, that is*

$$f(A(\Omega, x)) = f(\Omega, x)$$

for any $A \in \mathrm{Aff}(\mathbb{C}^d)$ and $(\Omega, x) \in \mathbb{X}_{d,0}$. Then there exists $c_d, C_d > 0$ so that

$$c_d \le f \le C_d.$$

This corollary says that naturally defined objects on convex sets are naturally comparable. For instance using language from [46, Section 9.2], we call a continuous map F which associates to each $(\Omega, x) \in \mathbb{X}_{d,0}$ a norm $F_\Omega(x; \cdot)$ on $T_x\Omega = \mathbb{C}^d$ a *natural affine metric*. Given a natural affine metric, we can define the length of an absolutely continuous curve $\sigma : [a, b] \to \Omega$ by

$$\ell_\Omega^F(\sigma) = \int_a^b F_\Omega(\sigma(t); \sigma'(t))\mathrm{d}t$$

and then a distance by

$$d_\Omega^F(x,y) = \inf\left\{\ell_\Omega^F(\sigma) : \sigma\colon [a,b] \to \Omega \text{ is absolutely continuous, } \text{with } \sigma(a) = x, \text{ and } \sigma(b) = y\right\}.$$

The Kobayashi metric is an example of a natural affine metric and using Frankel's co-compactness theorem we immediately have the following:

Corollary 4.3.4 *Suppose F is a natural affine metric. Then there exists $C_d > 0$ so that*

$$\frac{1}{C_d} F_\Omega(x;\cdot) \le \kappa_\Omega(x;\cdot) \le C_d F_\Omega(x;\cdot)$$

for all $(\Omega, x) \in \mathbb{X}_{d,0}$. *In particular,*

$$\frac{1}{C_d} d_\Omega^F(x, y) \leq k_\Omega(x, y) \leq C_d d_\Omega^F(x, y)$$

for all $\Omega \in \mathbb{X}_d$ *and* $x, y \in \Omega$.

See [24] for more results of this nature and [46, Section 9.2] for similar results in the real projective setting.

We begin the proof of Theorem 4.3.2 with a simple lemma:

Lemma 4.3.5 *Let* $e_1, \ldots, e_d$ *be the standard basis of* $\mathbb{C}^d$ *and for* $1 \leq i \leq d$ *define the complex* $(d - i - 1)$*-plane* P_i *by*

$$P_i = \left\{ e_i + \sum_{j=1}^{d-i} z_j e_{i+j} : z_1, \ldots, z_{d-i} \in \mathbb{C} \right\}.$$

Let $K \subset \mathbb{X}_{d,0}$ *be the set of pairs* $(\Omega, 0)$ *where*

1. $\mathbb{D}\, e_i \subset \Omega$ *for* $1 \leq i \leq d$,
2. $P_i \cap \Omega = \emptyset$ *for* $1 \leq i \leq d$.

Then K *is compact in* $\mathbb{X}_{d,0}$.

Proof Suppose $(\Omega_n, 0)$ is a subsequence in K. By passing to a subsequence we can assume that $\overline{\Omega}_{n_k}$ converges in the local Hausdorff topology to some closed convex set $\mathscr{C} \subset \mathbb{C}^d$. Then $\mathbb{D}\, e_i \subset \mathscr{C}$ for $1 \leq i \leq d$ and so $\mathscr{C}$ has non-empty interior. Let Ω be the interior of $\mathscr{C}$. We claim that Ω is $\mathbb{C}$-proper and is in K. Assuming, for the moment, this claim we see that $(\Omega_{n_k}, 0)$ converges to $(\Omega, 0)$ in $\mathbb{X}_{d,0}$. Since $(\Omega_n, 0)$ is an arbitrary subsequence in K we then see that K is compact.

We now prove that Ω is $\mathbb{C}$-proper and is in K. Now for each n and $1 \leq i \leq d$ there exists a real hyperplane $H_{i,n}$ so that $P_i \subset H_{i,n}$ and $H_{i,n} \cap \Omega_n = \emptyset$. By passing to a subsequence, we can suppose that $H_{i,n}$ converges to some real hyperplane H_i. Now by construction $P_i \subset H_i$ and $H_i \cap \Omega = \emptyset$. Now for each $1 \leq i \leq d$ there exists $v_i = (v_{i,1}, \ldots, v_{i,d})$ so that

$$H_i = \{z \in \mathbb{C}^d : \operatorname{Re} \langle v_i, z \rangle = 1\}.$$

Since $P_i \subset H_i$, we see that $v_{i,j} = 0$ for $j > i$ and $v_{i,i} = 1$. Thus $v_1, \ldots, v_d$ forms a basis for $\mathbb{C}^d$.

Now suppose that L is a complex line, we claim that L is not contained in Ω. Fix $a, b \in \mathbb{C}^d$ so that

$$L = \{b + az : z \in \mathbb{C}\}.$$

Then, since $v_1, \dots, v_d$ is a basis of $\mathbb{C}^d$, there exists $1 \leq i \leq d$ so that $\langle v_i, a \rangle \neq 0$. But then $L \cap H_i \neq \emptyset$ and so L is not contained in Ω. Since L was an arbitrary complex line, we see that Ω is $\mathbb{C}$-proper.

Proof (Proof of Theorem 4.3.2) Suppose that $(\Omega, x) \in \mathbb{X}_{d,0}$ and let K be the compact set defined in Lemma 4.3.5. We will show that there exists $A \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A(\Omega, x) \in K$. Let $T \in \mathrm{Aff}(\mathbb{C}^d)$ be the translation $T(z) = z - x$. Then $T(\Omega, x) = (T\Omega, 0)$.

We next pick points $\xi_1, \dots, \xi_d \in \partial T\Omega$ as follows: first let ξ_1 be a point in $\partial T\Omega$ closest to 0. Then assuming $\xi_1, \dots, \xi_k$ have already been selected, let V_k be the maximal complex subspace through 0 orthogonal to the lines $\{\mathbb{R} \cdot \xi_i : 1 \leq i \leq k\}$. Then let ξ_{k+1} be a point in $V_k \cap \partial T\Omega$ closest to 0.

Once $\xi_1, \dots, \xi_d$ have been selected let $\tau_i = \|\xi_i\|$ for $1 \leq i \leq d$. Next let $\Lambda \in \mathrm{GL}_d(\mathbb{C})$ be the linear map

$$\begin{pmatrix} \tau_1^{-1} & & \\ & \ddots & \\ & & \tau_d^{-1} \end{pmatrix},$$

and let U be the unitary map so that

$$\Lambda U(\xi_i) = e_i.$$

Notice that if $\Omega' = (\Lambda U T)\Omega$, then $\mathbb{D}\, e_i \subset \Omega'$ for all $1 \leq i \leq d$.

Now since ξ_i is a point in $V_k \cap \partial T\Omega$ closest point to 0, we see that the complex plane

$$P_i = \left\{ e_i + \sum_{j=i+1}^{d} z_j e_j : z_{i+1}, \dots, z_d \in \mathbb{C} \right\}$$

does not intersect Ω'. So $(\Lambda U T)(\Omega, x) \in K$.

We end this section with one more application of Frankel's compactness theorem.

Theorem 4.3.6 ([25, 41]) *Suppose $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex domain. If there exists $\varphi_n \in \mathrm{Aut}(\Omega)$, $x \in \Omega$, and $\xi \in \partial\Omega$ so that $\varphi_n x \to \xi$ and $\partial\Omega$ is C^2 at ξ, then* $\mathrm{Aut}(\Omega)$ *contains a one-parameter group.*

Here is a sketch of the proof: fix the compact set $K \subset \mathbb{X}_{d,0}$ from the proof of Theorem 4.3.2 and consider the pairs $(\Omega, \varphi_n x)$. Let $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ be the affine automorphism constructed in the proof of Theorem 4.3.2 so that $A_n(\Omega, \varphi_n x) \in K$. Now since $K \subset \mathbb{X}_{d,0}$ is compact we can pass to a subsequence so that $A_n(\Omega_n, \varphi_n x)$ converges to some $(\Omega_\infty, x_\infty)$ in $\mathbb{X}_{d,0}$. Since the map $A_n \varphi_n : \Omega \to A_n\Omega$ is an isometry with respect to the Kobayashi metric, using Theorem 4.3.1 one can show that $A_n \varphi_n$ converges locally uniformly to a bi-holomorphic map $F : \Omega \to \Omega_\infty$. Then, using the fact that ξ is a C^2 point of $\partial\Omega$, one can show that Ω_∞ contains the real line $\{te_1 : t \in \mathbb{R}\}$. Since Ω_∞ is convex we then see that $\{z + te_1 : t \in \mathbb{R}\} \subset \Omega_\infty$ if $z \in \Omega_\infty$. Hence $\mathrm{Aut}(\Omega_\infty)$ contains the one-parameter group $\{u_t\}_{t\in\mathbb{R}}$ given by

$$u_t(z) = z + te_1.$$

Since Ω is bi-holomorphic to Ω_∞ we see that $\mathrm{Aut}(\Omega)$ contains a one-parameter group.

Theorem 4.3.6 plays a key role in the proof of the following theorem of Frankel:

Theorem 4.3.7 ([25]) *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain and there exists $\Gamma \leq \mathrm{Aut}(\Omega)$ a discrete group which acts properly, freely, and co-compactly on Ω. Then Ω is a bounded symmetric domain.*

The idea is to first use Theorem 4.3.6 to first show that $\mathrm{Aut}(\Omega)$ is non-discrete. Then Frankel argues that $\mathrm{Aut}_0(\Omega)$, the connected component of the identity in $\mathrm{Aut}(\Omega)$, is a semi-simple Lie group which acts transitively on Ω. The argument is the second step was later generalized in [27, 49] and is related to some rigidity results in the Riemannian setting, see [21, 22, 55].

4.4 Finite Type is Necessary

In this section we sketch the proof of:

Theorem 4.4.1 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. If (Ω, k_Ω) is Gromov hyperbolic, then $\partial\Omega$ has finite type.*

The proof has three main steps:

Step 1 Show that complex affine discs in the boundary are an obstruction to Gromov hyperbolicity, more precisely: If $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex domain and $\partial\Omega$ contains a non-trivial complex affine disc, then (Ω, k_Ω) is not Gromov hyperbolic.

Step 2 Show that δ-hyperbolicity is a closed condition in $\mathbb{X}_d$, more precisely: If Ω_n converges to Ω_∞ in $\mathbb{X}_d$ and each (Ω_n, k_{Ω_n}) is δ-hyperbolic, then $(\Omega_\infty, k_{\Omega_\infty})$ is Gromov hyperbolic.

Step 3 Show that zooming in on a point of infinite type produces an affine disc in the boundary, more precisely: If $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and $x \in \partial\Omega$ has infinite type, then there exists a sequence of affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A_n\Omega$ converges to some Ω_∞ in $\mathbb{X}_d$ and $\partial\Omega_\infty$ contains a non-trivial complex affine disc.

Combining the three steps proves Theorem 4.4.1. We should emphasize that this approach avoids the need to establish estimates for the Kobayashi distance near a point of infinite type.

We begin with the sketch the proof of:

Proposition 4.4.2 *If $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex domain and $\partial\Omega$ contains a non-trivial complex affine disc, then (Ω, k_Ω) is not Gromov hyperbolic.*

Remark 4.4.3

1. Proposition 4.4.2 was proven when Ω is bounded and $\partial\Omega$ is C^∞ in [31], when $\partial\Omega$ is $C^{1,1}$ and $d = 2$ in [50], and in full generality in [60].
2. By [28], if Ω is a convex set then $\partial\Omega$ contains a non-trivial complex affine disc if and only if $\partial\Omega$ contains a non-trivial holomorphic disc.

The proof of Proposition 4.4.2 is based on three ideas:

1. In a Gromov hyperbolic geodesic metric space, quasi-geodesics triangles are thin.
2. When $\Omega \subset \mathbb{C}^d$ is convex, straight lines can be parametrized as quasi-geodesics in (Ω, k_Ω).
3. If $\partial\Omega$ contains a non-trivial complex affine disc, then a quasi-geodesic triangle consisting of straight lines with one side parallel to this disc is fat.

Here are the details of each step:

Definition 4.4.4

1. Suppose (X, d) is a metric space, $I \subset \mathbb{R}$ is an interval, $A \geq 1$, and $B \geq 0$. A map $\sigma : I \to X$ is called a (A, B)*-quasi-geodesic* if
$$\frac{1}{A}|t - s| - B \leq d(\sigma(s), \sigma(t)) \leq A|t - s| + B$$
for all $s, t \in I$.
2. A (A, B)*-quasi-geodesic triangle* in a metric space (X, d) is a choice of three points in X and (A, B)-quasi-geodesic segments connecting these points. A (A, B)-quasi-geodesic triangle is said to be *M-thin* if any point on any of the sides of the triangle is within distance M of the other two sides.

Lemma 4.4.5 *For any $A \geq 1$, $B \geq 0$, and $\delta \geq 0$ there exists $M > 0$ such that: if (X, d) is δ-hyperbolic, then every (A, B)-quasi-geodesic triangle is M-thin.*

Proof This follows from the fact that quasi-geodesics are always shadowed by actual geodesics, see for instance [19, Theorem 1.3.2].

Lemma 4.4.6 *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Assume $x \in \Omega$ and $\xi \in \partial\Omega$ are such that $\delta_\Omega(x; \xi - x) \geq \epsilon$ and $\|x - \xi\| \leq R$ for some $\epsilon, R > 0$. If*
$$\sigma(t) = \xi + e^{-2t}(x - \xi)$$

then

$$|t_1 - t_2| \leq K_\Omega(\sigma(t_1), \sigma(t_2)) \leq (2R/\epsilon)\,|t_1 - t_2|$$

for all $t_1, t_2 \geq 0$. *In particular, the line segment* $[x, \xi)$ *can be parametrized to be an* $(2R/\epsilon, 0)$*-quasi-geodesic in* (Ω, k_Ω).

Remark 4.4.7 This lemma not only says that every bounded line segment in Ω can be parametrized to be a (A, B)-quasi-geodesic, but also that the parameters A, B can be chosen to depend on simple geometric quantities.

Proof First notice that Lemma 4.2.6 immediately implies that

$$|t_1 - t_2| \leq k_\Omega(\sigma(t_1), \sigma(t_2))$$

for all $t_1, t_2 \geq 0$.

Now Ω contains the convex hull of ξ and $\mathbb{B}_\epsilon(x) \cap L$ where L is the complex line containing x and ξ. This implies that

$$\delta_\Omega(\sigma(t); \sigma(t)) \geq \epsilon e^{-2t}$$

for all $t \geq 0$. Then when $t_2 \geq t_1 \geq 0$, Lemma 4.2.4 implies that

$$\begin{aligned} k_\Omega(\sigma(t_1), \sigma(t_2)) &\leq \int_{t_1}^{t_2} \kappa_\Omega(\sigma(t); \sigma'(t))\mathrm{d}t \leq \int_{t_1}^{t_2} \frac{\|\sigma'(t)\|}{\delta_\Omega(\sigma(t); \sigma(t))}\mathrm{d}t \\ &\leq \int_{t_1}^{t_2} \frac{2\,\|x - \xi\|}{\epsilon}\mathrm{d}t \leq (2R/\epsilon)\,|t_2 - t_1|\,. \end{aligned}$$

With these two lemmas we can sketch the proof of Proposition 4.4.2 (complete details can be found in [60, Theorem 3.1]):

Proof (Proof of Proposition 4.4.2) Suppose for a contradiction that (Ω, k_Ω) is Gromov hyperbolic.

Fix a point $x_0 \in \Omega$ and a complex line L so that $L \cap \partial\Omega$ has non-empty interior U in L. Fix a point $\xi \in$ U. Since Ω is $\mathbb{C}$-proper, $\partial\,\mathrm{U} \neq \emptyset$. So fix a point $\eta \in \partial\,\mathrm{U}$. Next consider the curves $\sigma_\xi, \sigma_\eta : \mathbb{R}_{\geq 0} \to \Omega$ given by $\sigma_\xi(t) = \xi + e^{-2t}(x_0 - \xi)$ and $\sigma_\eta(t) = \eta + e^{-t}(x_0 - \eta)$. For $T > 0$ let ℓ_T be the real line

$$\{s\sigma_\xi(T) + (1 - s)\sigma_\eta(T) : s \in \mathbb{R}\}.$$

Since $\eta \in \partial\,\mathrm{U}$, for T large there exists some $z_T \in \partial\Omega \cap \ell_T$ so that we have the ordering $\sigma_x(T)$, $\sigma_y(T)$, z_T along ℓ_T. Notice, that because $\eta \in \partial\,\mathrm{U}$, we have

$$\lim_{T\to\infty} z_T = \eta.$$

Finally consider the curve $\gamma_T : \mathbb{R}_{\geq 0} \to \Omega$ given by $\gamma_T(t) = \sigma_\xi(T) + e^{-t}(z_T - \sigma_\eta(T))$.

Using Lemma 4.4.6 there exists $A \geq 1$ so that the curves σ_x, σ_y, and γ_T are all $(A, 0)$-quasi-geodesics. Then there exists $M > 0$ so that every $(A, 0)$-quasi-geodesic triangle in (Ω, k_Ω) is M-thin.

Now using Proposition 4.2.3 and the fact that $\eta \in \partial \mathrm{U}$, one can find $t_0 > 0$ so that

$$M < k_\Omega(\sigma_\xi(t_0), \sigma_\eta).$$

Next, using Lemma 4.2.6, we can find $T_0 > t_0$ so that

$$M < k_\Omega(\sigma_\xi(t_0), \gamma_{T_0}).$$

But then the $(A, 0)$-quasi-geodesic triangle with vertices x_0, $\sigma_\xi(T_0)$, $\sigma_\eta(T_0)$ and sides σ_ξ, σ_η, and γ_{T_0} (restricted to appropriate intervals) is not M-thin. So we have a contradiction.

We now show that δ-hyperbolicity is a closed condition in $\mathbb{X}_d$.

Proposition 4.4.8 *If Ω_n converges to some Ω_∞ in $\mathbb{X}_d$ and each (Ω_n, k_{Ω_n}) is δ-hyperbolic, then $(\Omega_\infty, k_{\Omega_\infty})$ is Gromov hyperbolic.*

The proof will use the formulation of Gromov hyperbolicity using the Gromov product. We recall that, for a metric space (X, d), the *Gromov product* of three points $o, y, z \in X$ is defined to be:

$$(x|y)_o = \frac{1}{2}(d(o, x) + d(o, y) - d(x, y)).$$

Using the Gromov product it is possible to give the following definition of Gromov hyperbolicity.

Definition 4.4.9 A metric space (X, d) is called δ-*product-hyperbolic* if

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for all $o, x, y, z \in X$.

This notion of hyperbolicity is essentially equivalent to the definition in terms of thin triangles, in particular the proof of Proposition III.H.1.22 in [18] implies:

Theorem 4.4.10 *There exists $c_1, c_2 > 1$ so that:*

1. *If (X, d) is a proper geodesic δ-hyperbolic metric space then (X, d) is $c_1\delta$-product-hyperbolic.*
2. *If (X, d) is a proper geodesic δ-product-hyperbolic metric space then (X, d) is $c_2\delta$-hyperbolic.*

Remark 4.4.11 One advantage of the product definition of hyperbolicity is that the definition make sense even if the metric space (X, d) is not geodesic.

Proof *(Proof of Proposition 4.4.8)* Since each (Ω_n, k_{Ω_n}) is $c_1\delta$-product-hyperbolic using Theorem 4.3.1 we see that $(\Omega_\infty, k_{\Omega_\infty})$ is $c_1\delta$-product-hyperbolic. So $(\Omega_\infty, k_{\Omega_\infty})$ is $c_1c_2\delta$-hyperbolic and hence Gromov hyperbolic.

The final step in the proof of Theorem 4.4.1 is proving the following:

Proposition 4.4.12 *If $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and $x \in \partial\Omega$ has infinite type, then there exists a sequence of affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A_n\Omega$ converges to some Ω_∞ in $\mathbb{X}_d$ and $\partial\Omega_\infty$ contains a non-trivial complex affine disc.*

The proof is a fairly straightforward application of Taylor's theorem applied to a defining function for $\partial\Omega$ near a point of infinite type, see [60, Proposition 6.1] for details.

Proof *(of Theorem 4.4.1)* Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. Assume for a contradiction that (Ω, k_Ω) is Gromov hyperbolic and $\partial\Omega$ contains a point of infinite type. Then by Proposition 4.4.12, there exists a sequence of affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A_n\Omega$ converges to some Ω_∞ in $\mathbb{X}_d$ and $\partial\Omega_\infty$ contains a non-trivial complex affine disc. Now the map A_n induces an isometry (Ω, k_Ω) to $(A_n\Omega, k_{A_n\Omega})$. So there exists $\delta > 0$ so that each $(A_n\Omega, k_{A_n\Omega})$ is δ-hyperbolic. Then Proposition 4.4.8 implies that $(\Omega_\infty, k_{\Omega_\infty})$ is Gromov hyperbolic. But this contradicts Proposition 4.4.2 since $\partial\Omega_\infty$ contains a non-trivial complex affine disc.

4.5 Finite Type is Sufficient

In this section we sketch the proof of:

Theorem 4.5.1 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. If $\partial\Omega$ has finite type, then (Ω, k_Ω) is Gromov hyperbolic.*

4.5.1 The Special Case of the Unit Ball

To motivate the proof of Theorem 4.5.4 we begin by proving that the Kobayashi metric on the unit ball $\mathbb{B} \subset \mathbb{C}^d$ is Gromov hyperbolic. There are many ways to do this, but the proof we will present only relies on the following four basic properties of the metric space $(\mathbb{B}, k_\mathbb{B})$:

1. (Symmetry) For any $x \in \mathbb{B}$ there exists $\varphi \in \mathrm{Aut}(\mathbb{B})$ so that $\varphi(x) = 0$.
2. (Well behaved geodesics) If $\sigma : \mathbb{R} \to \mathbb{B}$ is a geodesic, then the limits

$$\lim_{t\to\infty} \sigma(t) \text{ and } \lim_{t\to-\infty} \sigma(t)$$

both exist in $\partial\mathbb{B}$ and are distinct.

3. (Limits of geodesics) Suppose that $x_0 \in \mathbb{B}$ and a sequence $y_n \in \mathbb{B}$ converges to a point $\xi \in \partial\mathbb{B}$. If $\sigma_n : [0, T_n] \to \mathbb{B}$ is a geodesic with $\sigma_n(0) = x_0$ and $\sigma_n(T_n) = y_n$ then σ_n converges locally uniformly to a geodesic $\sigma : [0, \infty) \to \mathbb{B}$ and

$$\lim_{t\to\infty} \sigma(t) = \xi.$$

4. (Visibility) Suppose that $x_n, y_n \in \mathbb{B}$, $x_n \to \xi \in \partial\mathbb{B}$, $y_n \to \eta \in \partial\mathbb{B}$, and $\xi \neq \eta$. If $\sigma_n : [a_n, b_n] \to \mathbb{B}$ is a geodesic with $\sigma_n(a_n) = x_n$ and $\sigma_n(b_n) = y_n$, then there exists $T_n \in [a_n, b_n]$ so that $\sigma_n(\cdot + T_n)$ converges locally uniformly to a geodesic $\sigma : \mathbb{R} \to \mathbb{B}$.

Remark 4.5.2 These are not a minimal set of properties. In particular, Property 4 implies Property 3 and also the existence part of Property 2.

Theorem 4.5.3 $(\mathbb{B}, k_{\mathbb{B}})$ *is Gromov hyperbolic.*

Proof Suppose for a contradiction that $(\mathbb{B}, k_{\mathbb{B}})$ is not Gromov hyperbolic. Then there exists points $x_n, y_n, z_n \in \mathbb{B}$, geodesic segments $\sigma_{x_ny_n}, \sigma_{y_nz_n}, \sigma_{z_nx_n}$ joining them, and a point u_n in the image of $\sigma_{x_ny_n}$ such that

$$k_{\mathbb{B}}(u_n, \sigma_{y_nz_n} \cup \sigma_{z_nx_n}) > n.$$

Using Property 1 of $\mathbb{B}$ we can assume that $u_n = 0$.

Next, by passing to a subsequence, we can assume that

$$x_n, y_n, z_n \to x_\infty, y_\infty, z_\infty \in \overline{\mathbb{B}}.$$

Since $k_{\mathbb{B}}(u_n, \{x_n, y_n, z_n\}) \geq k_{\mathbb{B}}(u_n, \sigma_{y_nz_n} \cup \sigma_{z_nx_n}) > n$ we see that $x_\infty, y_\infty, z_\infty \in \partial\mathbb{B}$.

We can parametrize each $\sigma_{x_ny_n}$ so that $\sigma_{x_ny_n}(0) = 0$. Then using Property 3, the geodesics $\sigma_{x_ny_n}$ converge locally uniformly to a geodesic $\sigma : \mathbb{R} \to \mathbb{B}$ with

$$\lim_{t\to-\infty} \sigma(t) = x_\infty \text{ and } \lim_{t\to\infty} \sigma(t) = y_\infty.$$

By Property 2, we must have $x_\infty \neq y_\infty$. Thus, after possibly relabeling, we may assume that $z_\infty \neq x_\infty$.

Then using Property 4 we can assume that $\sigma_{x_nz_n}$ converges locally uniformly to a geodesic $\gamma : \mathbb{R} \to \mathbb{B}$. But then

$$k_{\mathbb{B}}(0, \gamma(0)) = \lim_{n\to\infty} k_{\mathbb{B}}(0, \sigma_{x_nz_n}(0)) \geq \lim_{n\to\infty} k_{\mathbb{B}}(0, \sigma_{x_nz_n}) = \infty.$$

So we have a contradiction and thus $(\mathbb{B}, k_{\mathbb{B}})$ must be Gromov hyperbolic.

4.5.2 The General Case

The fact that $\mathrm{Aut}(\mathbb{B})$ acts transitively on $\mathbb{B}$ plays a major role in the proof of Theorem 4.5.3. Unfortunately, for a general convex domain Ω it is suspected that the group $\mathrm{Aut}(\Omega)$ will be quite small (see for instance [35] and the references therein). However, the proof that $(\mathbb{B}, k_{\mathbb{B}})$ is Gromov hyperbolic can be adapted to general convex domains if we replace the action of $\mathrm{Aut}(\mathbb{B})$ on $\mathbb{B}$ by the action of $\mathrm{Aff}(\mathbb{C}^d)$ on $\mathbb{X}_{d,0}$.

Briefly delaying definitions we will establish the following sufficient condition for the Kobayashi metric to be Gromov hyperbolic:

Theorem 4.5.4 *Suppose Ω is a $\mathbb{C}$-proper convex domain. Assume for any sequence $u_n \in \Omega$ there exists $n_k \to \infty$ and affine maps $A_k \in \mathrm{Aff}(\mathbb{C}^d)$ so that*

1. *$A_k(\Omega, u_{n_k})$ converges to some $(\Omega_\infty, u_\infty)$ in $\mathbb{X}_{d,0}$,*
2. *geodesics in $(\Omega_\infty, k_{\Omega_\infty})$ are well behaved,*
3. *$A_k\Omega$ is a visibility sequence.*

Then (Ω, k_Ω) is Gromov hyperbolic.

We now define "well behaved geodesics" and "visibility sequence." Given a curve $\sigma : \mathbb{R} \to \mathbb{C}^d$ define the backward and forward accumulation sets as

$$\sigma(\infty) := \left\{z \in \mathbb{C}^d \cup \{\infty\} : \text{ there exists } t_n \to \infty \text{ with } \sigma(t_n) \to z\right\}$$

and

$$\sigma(-\infty) := \left\{z \in \mathbb{C}^d \cup \{\infty\} : \text{ there exists } t_n \to -\infty \text{ with } \sigma(t_n) \to z\right\}.$$

Definition 4.5.5 Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. We say *geodesics in (Ω, k_Ω) are well-behaved* if for every geodesic $\sigma : \mathbb{R} \to \Omega$ we have

$$\sigma(\infty) \cap \sigma(-\infty) = \emptyset.$$

Definition 4.5.6 Suppose Ω_n converges to Ω_∞ in $\mathbb{X}_d$. We say Ω_n is a *visibility sequence* if for every sequence $\sigma_n : [a_n, b_n] \to \Omega_n$ of geodesics with $\sigma_n(a_n) \to \xi \in \partial\Omega_\infty \cup \{\infty\}$, $\sigma_n(b_n) \to \eta \in \partial\Omega_\infty \cup \{\infty\}$, and $\xi \neq \eta$ there exists $n_k \to \infty$ and $T_k \in [a_{n_k}, b_{n_k}]$ so that $\sigma_{n_k}(\cdot + T_k)$ converges locally uniformly to a geodesic $\sigma : \mathbb{R} \to \Omega_\infty$.

Remark 4.5.7 Definition 4.5.5 is a weaker version of Property 2 for the unit ball. In the case of general convex domains, we do not know that that the limit of geodesic lines exist which leads us to consider the forward and backward accumulation sets. Also, there exist convex domains where two points are joined by many different geodesics and hence it is necessary to pass to a subsequence n_k in Definition 4.5.6.

Before starting the proof of Theorem 4.5.4 we will show that limits of geodesics in a visibility sequences satisfy a natural analogue of Property 3 for the unit ball:

Lemma 4.5.8 *Assume that Ω_n is a visibility sequence converging to some Ω_∞ in $\mathbb{X}_d$. Suppose that $x_0 \in \Omega_\infty$ and a sequence $y_n \in \Omega_n$ converges to a point $\xi \in \partial\Omega_\infty \cup \{\infty\}$. If $\sigma_n : [0, T_n] \to \Omega_n$ is a geodesic with $\sigma_n(0) = x_0$ and $\sigma_n(T_n) = y_n$ then there exists $n_k \to \infty$ so that σ_{n_k} converges locally uniformly to a geodesic $\sigma : [0, \infty) \to \Omega_\infty$ and*

$$\lim_{t\to\infty} \sigma(t) = \xi.$$

Proof Using Theorem 4.3.1 we can pass to a subsequence so that σ_{n_k} converges locally uniformly to a geodesic $\sigma : [0, \infty) \to \Omega_\infty$.

Now suppose for a contradiction that

$$\lim_{t\to\infty} \sigma(t) \neq \xi.$$

Then there exists $s_m \to \infty$ so that $\sigma(s_m) \to \eta$ and $\eta \neq \xi$. Since σ_n converges locally uniformly to σ we can pick s_n' so that $\sigma_n(s_n') \to \eta$. Since $\eta \in \partial\Omega_\infty \cup \{\infty\}$ we see that $s_n' \to \infty$.

Now let $\gamma_n = \sigma_n|_{[s_n', T_n]}$. Since Ω_n is a visibility sequence in $\mathbb{X}_d$ we can pass to another subsequence and find $S_n \in [s_n', T_n]$ so that the geodesics $\gamma_n(\cdot + S_n)$ converges locally uniformly to a geodesic $\gamma : \mathbb{R} \to \Omega_\infty$. But then

$$\begin{aligned} k_{\Omega_\infty}(\gamma(0), \sigma(0)) &= \lim_{n\to\infty} k_{\Omega_n}(\gamma_n(S_n), \sigma_n(0)) = \lim_{n\to\infty} k_{\Omega_n}(\sigma_n(S_n), \sigma_n(0)) \\ &= \lim_{n\to\infty} S_n = \infty. \end{aligned}$$

So we have a contradiction.

Proof (of Theorem 4.5.4) Suppose for a contradiction that (Ω, k_Ω) is not Gromov hyperbolic. Then there exists points $x_n, y_n, z_n \in \Omega$, geodesic segments $\sigma_{x_n y_n}$, $\sigma_{y_n z_n}$, $\sigma_{z_n x_n}$ joining them, and a point u_n in the image of $\sigma_{x_n y_n}$ such that

$$k_\Omega(u_n, \sigma_{y_n z_n} \cup \sigma_{z_n x_n}) > n.$$

Now, we can pass to a subsequence and find affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ so that

1. $A_n(\Omega, u_n)$ converges to some $(\Omega_\infty, u_\infty)$ in $\mathbb{X}_d$,
2. geodesics in $(\Omega_\infty, k_{\Omega_\infty})$ are well behaved,
3. $A_n\Omega$ is a visibility sequence.

Next, by passing to a subsequence, we can assume that

$$A_n x_n, A_n y_n, A_n z_n \to x_\infty, y_\infty, z_\infty \in \overline{\Omega}_\infty \cup \{\infty\}.$$

Since

$$k_{A\Omega_n}(Au_n, \{Ax_n, Ay_n, Az_n\}) = k_{\Omega_n}(u_n, \{x_n, y_n, z_n\}) \geq k_{\Omega_n}(u_n, \sigma_{y_n z_n} \cup \sigma_{z_n x_n}) > n$$

we see that $x_\infty, y_\infty, z_\infty \in \partial\Omega_\infty \cup \{\infty\}$.

Using Theorem 4.3.1 and passing to a subsequence we can assume that $A_n \sigma_{x_n y_n}$ converge locally uniformly to a geodesic $\sigma : \mathbb{R} \to \Omega_\infty$. Then using Lemma 4.5.8 we have that

$$\lim_{t\to-\infty} \sigma(t) = x_\infty \text{ and } \lim_{t\to\infty} \sigma(t) = y_\infty.$$

Since geodesics in $(\Omega_\infty, k_{\Omega_\infty})$ are well behaved, we must have $x_\infty \neq y_\infty$. Thus, after possibly relabeling, we may assume that $z_\infty \neq x_\infty$.

Then using the fact that $A_n\Omega$ is a visibility sequence we can assume that $A_n\sigma_{x_n z_n}$ converges locally uniformly to a geodesic $\gamma : \mathbb{R} \to \Omega_\infty$. But then

$$k_{\Omega_\infty}(u_\infty, \gamma(0)) = \lim_{n\to\infty} k_{A_n\Omega}(A_n u_n, A_n \sigma_{x_n z_n}(0)) \geq \lim_{n\to\infty} k_\Omega(u_n, \sigma_{x_n z_n}) = \infty.$$

So we have a contradiction and thus (Ω, k_Ω) must be Gromov hyperbolic.

4.5.3 Rescaling Convex Domains of Finite Type

In the context of studying bounded convex domains of finite type with non-compact automorphism groups, Bedford and Pinchuk [6] and later Gaussier [29] proved results about the action of $\mathrm{Aff}(\mathbb{C}^d)$ on convex domains of finite type. Using their arguments it is possible to establish the following:

Theorem 4.5.9 *Suppose $\Omega \subset \mathbb{C}^{d+1}$ is a convex domain such that $\partial\Omega$ is C^L and has finite line type L near some $\xi \in \partial\Omega$. If $u_n \in \Omega$ is a sequence converging to ξ, then there exists $n_k \to \infty$ and affine maps $A_k \in \mathrm{Aff}(\mathbb{C}^d)$ such that*

1. *$A_k\Omega$ converges in the local Hausdorff topology to a $\mathbb{C}$-proper convex domain Ω_∞ of the form:*

$$\Omega_\infty = \{(z_0, z_1 \ldots, z_d) \in \mathbb{C}^d : \mathrm{Re}(z_0) > P(z_1, z_2, \ldots, z_d)\}$$

where P is a non-negative non-degenerate convex polynomial with $P(0) = 0$,

2. *$A_k u_{n_k} \to u_\infty \in \Omega_\infty$,*
3. *If $x_k \in \Omega$ and $\liminf_{k\to\infty} \|x_k - \xi\| > 0$ then $\lim_{n\to\infty} A_k x_k = \infty$, and*

4. for any $R > 0$ there exists $C = C(R) > 0$ and $N = N(R) > 0$ such that

$$\delta_{A_k\Omega}(p; v) \leq C\delta_{A_k\Omega}(p)^{1/L}$$

for all $k > N$, $p \in \mathbb{B}_R(0) \cap A_k\Omega$, and $v \in \mathbb{C}^d$ non-zero.

For a detailed proof see [60, Theorem 10.1].

4.5.4 Visibility Sequences

In this subsection we describe how the estimate

$$\delta_{A_k\Omega}(p; v) \leq C\delta_{A_k\Omega}(p)^{1/L}$$

in the statement of Theorem 4.5.9 implies that the rescaled domains $A_k\Omega$ form a visibility sequence.

There are a number of visibility type results in the literature for both complex geodesics and (real) geodesics. Chang, Hu, and Lee proved that complex geodesics in a bounded strongly convex domain satisfy a visibility condition (see [20, Section 2]). Shortly after, Mercer [48] extended these results to *L-convex domains*, that is convex domains $\Omega \subset \mathbb{C}^d$ where there exists $C > 0$ such that

$$\delta_\Omega(p; v) \leq C\delta_\Omega(p)^{1/L} \tag{4.1}$$

for all $p \in \Omega$ and $v \in \mathbb{C}^d$ non-zero. Every strongly convex set is 2-convex. Karlsson [38] proved a visibility result for geodesics only assuming $\partial\Omega$ had $C^{1,\alpha}$ boundary, the metric space (Ω, k_Ω) was Cauchy complete, and the Kobayashi metric obeyed the estimate in (4.1). Recently, visibility results for both real and complex geodesics have been established for domains which do not satisfy the estimate in (4.1), see [14, 15, 61].

In [60], we adapted Mercer's argument to prove a visibility result for sequences of geodesic lines $\sigma_n : \mathbb{R} \to \Omega_n$ when Ω_n is a sequence of convex sets which converges in $\mathbb{X}_d$ and satisfies a uniform L-convex property. The proof in [60] was quite complicated: first a visibility result for complex geodesics was established and then this was used to establish a visibility result for geodesics. In this subsection we provide simpler proof of this result using an argument from [15].

Proposition 4.5.10 *Suppose Ω_n converges to Ω in $\mathbb{X}_d$. Assume for any $R > 0$ there exists $C = C(R) > 0$, $N = N(R) > 0$, and $L = L(R) > 0$ such that*

$$\delta_{\Omega_n}(p; v) \leq C\delta_{\Omega_n}(p)^{1/L}$$

for all $n > N$, $p \in B_R(0) \cap \Omega_n$, and $v \in \mathbb{C}^d$ non-zero.

If $\sigma_n : [a_n, b_n] \to \Omega_n$ is a sequence of geodesics such that $\sigma_n(a_n) \to \xi \in \partial\Omega_\infty \cup \{\infty\}$, $\sigma_n(b_n) \to \eta \in \partial\Omega_\infty \cup \{\infty\}$, and $\xi \neq \eta$ then exists $n_k \to \infty$ and $T_k \in [a_{n_k}, b_{n_k}]$ so that $\sigma_{n_k}(\cdot + T_k)$ converges locally uniformly to a geodesic $\sigma : \mathbb{R} \to \Omega_\infty$.

The following argument is the proof of [15, Theorem 1.4] taken essentially verbatim.

Proof Since $\xi \neq \eta$ at least one must be finite. So (after possibly relabeling) we can fix $R > 0$ so that $\xi \in \partial\Omega \cap \mathbb{B}_R(0)$ and there exists $b_n' \in [a_n, b_n]$ so that $\sigma_n([a_n, b_n']) \subset \mathbb{B}_R(0)$, $\sigma_n(b_n') \to \eta' \in \overline{\Omega}$, and $\xi \neq \eta'$. Fix $C, N, L > 0$ so that

$$\delta_{\Omega_n}(p; v) \leq C\delta_{\Omega_n}(p)^{1/L}$$

for all $n > N$, $p \in \mathbb{B}_R(0) \cap \Omega_n$, and $v \in \mathbb{C}^d$ non-zero.

By reparametrizing each σ_n we can assume, in addition, that $a_n \leq 0 \leq b_n'$ and

$$\delta_{\Omega_n}(\sigma_n(0)) = \max\{\delta_{\Omega_n}(\sigma_n(t)) : t \in [a_n, b_n']\}.$$

Then by passing to a subsequence we can assume $a_n \to a \in [-\infty, 0]$ and $b_n' \to b \in [0, \infty]$.

Since Ω_n is $\mathbb{C}$-proper, (Ω, k_Ω) is Cauchy complete and so by Proposition 4.1.2 we have:

$$\kappa_\Omega(\sigma_n(t); \sigma_n'(t)) = 1$$

for almost every $t \in [a_n, b_n]$. This implies that

$$\left\|\sigma_n'(t)\right\| = \frac{1}{k_\Omega\left(\sigma_n(t); \frac{1}{\left\|\sigma_n'(t)\right\|}\sigma_n'(t)\right)} \leq 2\delta_\Omega(\sigma_n(t); \sigma_n')) \leq (2C)\delta_{\Omega_n}(\sigma_n(t))^{1/L}$$

for almost every $t \in [a_n, b_n]$ and $n > N$.

Since each Ω_n is convex,

$$\frac{\|v\|}{2\delta_{\Omega_n}(x; v)} \leq \kappa_{\Omega_n}(x; v)$$

for all $x \in \Omega_n$ and $v \in \mathbb{C}^d$ nonzero. Since $\Omega_n \to \Omega$ there exists $M > 0$ so that

$$\delta_{\Omega_n}(x; v) \leq M$$

for all $x \in \mathbb{B}_R(0) \cap \Omega_n$ and $v \in \mathbb{C}^d$ non-zero. Then

$$\frac{\|v\|}{2M} \leq k_{\Omega_n}(x; v)$$

for all $x \in \mathbb{B}_R(0) \cap \Omega_n$ and $v \in \mathbb{C}^d$. In particular,

$$\left\| \sigma_n'(t) \right\| = \frac{1}{k_\Omega\left(\sigma_n(t); \frac{1}{\left\|\sigma_n'(t)\right\|}\sigma'(t)\right)} \leq 2M$$

for almost every $t \in [a_n, b_n]$. So each $\sigma_n|_{[a_n, b_n']}$ is $2M$-Lipschitz (with respect to the Euclidean distance) and by passing to a subsequence we can assume $\sigma_n|_{[a_n, b_n']}$ converges locally uniformly on (a, b) to a curve $\sigma : (a, b) \to \overline{\Omega}$ (we restrict to the open interval because a could be $-\infty$ and b could be ∞). Notice that $a \neq b$ because each σ_n is $2M$-Lipschitz and so

$$0 < \left\| \xi - \eta' \right\| = \lim_{n\to\infty} \left\| \sigma_n(a_n) - \sigma_n(b_n') \right\| \leq 2M \left| b - a \right|.$$

Claim 1 $\sigma : (a, b) \to \overline{\Omega}$ is a constant map.

Proof By construction

$$\delta_{\Omega_n}(\sigma_n(t)) \leq \delta_{\Omega_n}(\sigma_n(0))$$

for $t \in [a_n, b_n']$ and so $\delta_{\Omega_n}(\sigma_n(t)) \to 0$ uniformly. But then if $u \leq w$ and $u, w \in (a, b)$

$$\begin{aligned} \left\| \sigma(u) - \sigma(w) \right\| &= \lim_{n\to\infty} \left\| \sigma_n(u) - \sigma_n(w) \right\| \leq \limsup_{n\to\infty} \int_u^w \left\| \sigma_n'(t) \right\| \mathrm{d}t \\ &\leq \limsup_{n\to\infty} \int_u^w C\delta_{\Omega_n}(\sigma_n(t))^{1/L} \mathrm{d}t = 0. \end{aligned}$$

Thus σ is constant. ◀

We will establish a contradiction by proving the following:

Claim 2 $\sigma : (a, b) \to \overline{\Omega}$ is not a constant map.

Proof Fix $x_0 \in \Omega$. Then by Lemma 4.2.8 there exists $C, \alpha > 0$ so that

$$k_{\Omega_n}(x, x_0) \leq C + \alpha \log \frac{1}{\delta_{\Omega_n}(x)}$$

for all n sufficiently large and $x \in \mathbb{B}_R(0) \cap \Omega_n$. Therefore for n sufficiently large and $t \in [a_n, b_n']$ we have

$$\begin{aligned} |t| \leq k_{\Omega_n}(\sigma_n(0), \sigma_n(t)) &\leq k_{\Omega_n}(\sigma_n(0), x_0) + k_\Omega(x_0, \sigma_n(t)) \\ &\leq 2C + \alpha \log \frac{1}{\delta_{\Omega_n}(\sigma_n(0))\delta_{\Omega_n}(\sigma_n(t))}. \end{aligned}$$

Thus for n sufficiently large and $t \in [a_n, b_n']$ we have

$$\delta_{\Omega_n}(\sigma_n(t)) \leq \sqrt{\delta_{\Omega_n}(\sigma_n(0))\delta_{\Omega_n}(\sigma_n(t))} \leq Ae^{-B|t|}$$

where $A = e^{(2C+\kappa)/(2\alpha)}$ and $B = 1/(2\alpha\lambda)$.

Thus for almost every $t \in [a_n, b_n']$ we have

$$\left\|\sigma_n'(t)\right\| \leq C\delta_{\Omega_n}(\sigma_n(t))^{1/L} \leq C_1 e^{-r|t|}$$

where $C_1 = CA$ and $r = B/L$.

Now fix $a', b' \in (a, b)$ so that

$$\epsilon := \lim_{n\to\infty} \left\|\sigma_n(b_n') - \sigma_n(a_n)\right\| > \int_a^{a'} C_1 e^{-r|t|}\mathrm{d}t + \int_{b'}^{b} C_1 e^{-r|t|}\mathrm{d}t.$$

Then

$$\begin{aligned}
\left\|\sigma(b') - \sigma(a')\right\| &= \lim_{n\to\infty} \left\|\sigma_n(b') - \sigma_n(a')\right\| \\
&\geq \lim_{n\to\infty} \left(\left\|\sigma_n(b_n') - \sigma_n(a_n)\right\| - \left\|\sigma_n(b_n') - \sigma_n(b')\right\| - \left\|\sigma_n(a') - \sigma_n(a_n)\right\| \right) \\
&\geq \epsilon - \limsup_{n\to\infty} \int_{b'}^{b_n} \left\|\sigma_n'(t)\right\| \mathrm{d}t - \limsup_{n\to\infty} \int_a^{a'} \left\|\sigma_n'(t)\right\| \mathrm{d}t \\
&\geq \epsilon - \limsup_{n\to\infty} \int_{b'}^{b_n} C_1 e^{-r|t|}\mathrm{d}t - \limsup_{n\to\infty} \int_{a_n}^{a'} C_1 e^{-r|t|}\mathrm{d}t > 0.
\end{aligned}$$

Thus $\sigma : (a, b) \to \overline{\Omega}$ is non-constant. ◀

The above contradicts Claim 1.

4.5.5 Well Behaved Geodesics

Theorem 4.5.11 *Suppose $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex domain and every point in $\partial\Omega$ has finite line type. If $\sigma : \mathbb{R} \to \Omega$ is a geodesic, then the limits*

$$\lim_{t\to\infty} \sigma(t) \text{ and } \lim_{t\to-\infty} \sigma(t)$$

both exist in $\partial\Omega \cup \{\infty\}$. Moreover, if one of the limits is finite, then they are distinct.

Proof By Proposition 4.1.6, for any $R > 0$ there exists $C = C(R) > 0$ and $L = L(R) > 0$ such that

$$\delta_\Omega(p; v) \leq C\delta_\Omega(p)^{1/L}$$

for all $p \in \mathbb{B}_R(0) \cap \Omega$ and $v \in \mathbb{C}^d$ non-zero. It then follows from Proposition 4.5.10 that the constant sequence $\Omega_n = \Omega$ is a visibility sequence. So by Lemma 4.5.8 the limits $\lim_{t\to\infty} \sigma(t)$ and $\lim_{t\to-\infty} \sigma(t)$ both exist in $\partial\Omega \cup \{\infty\}$.

Now suppose for a contradiction that $\lim_{t\to\infty} \sigma(t) = \lim_{t\to-\infty} \sigma(t) = \xi \in \partial\Omega$. By hypothesis, $\partial\Omega$ has finite line type at ξ. Now let $\boldsymbol{\nu}(\xi)$ be the inward pointing normal line at ξ and fix a sequence $t_n \searrow 0$. Then we can apply Theorem 4.5.9 to the sequence $(\Omega, \xi + t_n\boldsymbol{\nu}(\xi)(\xi))$. After passing to a subsequence, Theorem 4.5.9 implies the existence of affine maps A_n so that

1. $A_n\Omega$ converges to some Ω_∞ in $\mathbb{X}_d$,
2. $A_n\Omega$ is a visibility sequence,
3. $A_n\sigma(0) \to \infty$.

In addition, since $\xi + t_n\boldsymbol{\nu}(\xi)(\xi)$ converges to ξ non-tangentially, the proof of Theorem 4.5.9 implies that we can assume that $A_n\xi = 0$. Now since $\lim_{t\to\infty} \sigma(t) = \lim_{t\to-\infty} \sigma(t) = \xi$ we can pick $b_n \to \infty$ and $a_n \to -\infty$ so that

$$\lim_{n\to\infty} A_n\sigma(b_n) = \lim_{n\to\infty} A_n\sigma(a_n) = 0.$$

Next consider the sequence of geodesics $\gamma_n = A_n\sigma|_{[0,b_n]}$ and $\sigma_n = A_n\sigma_{[a_n,0]}$ mapping into $A_n\Omega$. Since $A_n\Omega$ is a visibility sequence there exists $s_n \in [0, b_n]$ and $t_n \in [a_n, 0]$ so that $A_n\sigma(s_n) \to p \in \Omega_\infty$ and $A_n\sigma(t_n) \to q \in \Omega_\infty$. Since $A_n\sigma(0) \to \infty$ we see that $s_n \to \infty$ and $t_n \to -\infty$. Then

$$\begin{aligned} k_{\Omega_\infty}(p, q) &= \lim_{n\to\infty} k_{A_n\Omega}(A_n\sigma(s_n), A_n\sigma(t_n)) = \lim_{n\to\infty} k_\Omega(\sigma(s_n), \sigma(t_n)) \\ &= \lim_{n\to\infty} s_n - t_n = \infty. \end{aligned}$$

So we have a contradiction.

As an immediate corollary we obtain that geodesics are well behaved on bounded convex domains with finite type:

Corollary 4.5.12 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and $\partial\Omega$ has finite type. If $\sigma : \mathbb{R} \to \Omega$ is a geodesic, then the limits $\lim_{t\to\infty} \sigma(t)$ and $\lim_{t\to-\infty} \sigma(t)$ both exist in $\partial\Omega$ and are distinct.*

We can also use Theorem 4.5.11 to prove that geodesics are well behaved on polynomial domains:

Corollary 4.5.13 *Suppose $\Omega \subset \mathbb{C}^d$ is a domain of the form*

$$\Omega = \{(z_0, z_1 \dots, z_d) \in \mathbb{C}^{d+1} : \mathrm{Re}(z_0) > P(z_1, z_2, \dots, z_d)\}$$

where $P : \mathbb{C}^d \to \mathbb{R}$ is a non-negative, non-degenerate, convex polynomial with $P(0) = 0$.

If $\sigma : \mathbb{R} \to \Omega$ is a geodesic, then $\lim_{t\to\infty} \sigma(t)$ and $\lim_{t\to-\infty} \sigma(t)$ both exist in $\partial\Omega \cup \{\infty\}$ and are distinct.

Proof Since every point in $\partial\Omega$ has finite line type, Theorem 4.5.11 implies that $\lim_{t\to\infty}\sigma(t)$ and $\lim_{t\to-\infty}\sigma(t)$ both exist in $\partial\Omega\cup\{\infty\}$. Moreover, if one of the limits is finite then they are distinct.

So suppose for a contradiction that $\lim_{t\to\infty}\sigma(t)=\lim_{t\to-\infty}\sigma(t)=\infty$.

The idea of the proof is to use a version of Theorem 4.5.9 for polynomial domains where the sequence u_n is allowed to be unbounded. The key step is defining a multi-type at ∞ for such domains, see [60, Section 12] for details. In particular, by [60, Proposition 12.1] we can pick a sequence of **linear** maps $A_n \in \mathrm{GL}_{d+1}(\mathbb{C})$ so that

1. $A_n\Omega$ converges to some Ω_∞ in $\mathbb{X}_{d+1}$,
2. $A_n\Omega$ is a visibility sequence,
3. $\lim_{n\to\infty}\|A_n\|=0$.

Now part (3) implies that $A_n\sigma(0)\to 0$. Since $\lim_{t\to\infty}\sigma(t)=\lim_{t\to-\infty}\sigma(t)=\infty$ we can pick $b_n\to\infty$ and $a_n\to-\infty$ so that

$$\lim_{n\to\infty}A_n\sigma(b_n)=\lim_{n\to\infty}A_n\sigma(a_n)=\infty.$$

Next consider the sequence of geodesics $\gamma_n=A_n\sigma|_{[0,b_n]}$ and $\sigma_n=A_n\sigma_{[a_n,0]}$ mapping into $A_n\Omega$. Since $A_n\Omega$ is a visibility sequence there exists $s_n\in[0,b_n]$ and $t_n\in[a_n,0]$ so that $A_n\sigma(s_n)\to p\in\Omega_\infty$ and $A_n\sigma(t_n)\to q\in\Omega_\infty$. Since $A_n\sigma(0)\to\infty$ we see that $s_n\to\infty$ and $t_n\to-\infty$. Then

$$\begin{aligned}k_{\Omega_\infty}(p,q)&=\lim_{n\to\infty}k_{A_n\Omega}(A_n\sigma(s_n),A_n\sigma(t_n))=\lim_{n\to\infty}k_\Omega(\sigma(s_n),\sigma(t_n))\\&=\lim_{n\to\infty}s_n-t_n=\infty.\end{aligned}$$

So we have a contradiction.

4.5.6 Finite Type Implies Gromov Hyperbolic

Suppose $\Omega\subset\mathbb{C}^d$ is a bounded convex domain with finite type. To show that (Ω,k_Ω) is Gromov hyperbolic, it is enough to verify the hypothesis of Theorem 4.5.4. So let $u_n\in\Omega$ be a sequence. By passing to a subsequence we can assume that $u_n\to\xi\in\overline{\Omega}$.

If $\xi\in\Omega$ then $A_n=\mathrm{Id}$ satisfies the conditions in Theorem 4.5.4:

1. Clearly $\mathrm{Id}(\Omega,u_n)=(\Omega,u_n)$ converges to (Ω,ξ) in $\mathbb{X}_{d,0}$.
2. By Corollary 4.5.12, geodesics in (Ω,k_Ω) are well behaved.
3. By Proposition 4.1.6 and Proposition 4.5.10, the constant sequence Ω is a visibility sequence.

Next consider the case in which $\xi\in\partial\Omega$. Using Theorem 4.5.9, Proposition 4.5.10, and Corollary 4.5.13 we can pass to a subsequence and find $A_n\in\mathrm{Aff}(\mathbb{C}^d)$ so that

1. $A_n(\Omega, u_n)$ converges to some $(\Omega_\infty, u_\infty)$ in $\mathbb{X}_{d,0}$,
2. geodesics in $(\Omega_\infty, k_{\Omega_\infty})$ are well behaved,
3. $A_n\Omega$ is a visibility sequence.

In either case we can find affine maps A_n satisfying the hypothesis of Theorem 4.5.4. Hence (Ω, k_Ω) is Gromov hyperbolic.

4.6 Strongly Pseudoconvex Domains

In this section we show that the proof of Theorem 4.5.1 can be adapted to prove that the Kobayashi metric on a strongly pseudoconvex domain is Gromov hyperbolic.

Theorem 4.6.1 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. Then (Ω, k_Ω) is Gromov hyperbolic.*

It is unclear if the argument we present is simpler than the original argument of Balogh and Bonk. However, it does adapt to a wider class of domains. In particular, the argument in this section can also be used to show:

Theorem 4.6.2 *Suppose Ω is locally convexifiable and has finite type in the sense of D'Angelo. Then (Ω, k_Ω) is Gromov hyperbolic.*

See [60, Section 14] for details.

4.6.1 Estimates

In this subsection we recall some well-known estimates for the Kobayashi metric and distance on a strongly pseudoconvex domain.

Theorem 4.6.3 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain.*

1. *There exists $C > 0$ so that*

$$C\frac{\|v\|}{\delta_\Omega(x)^{1/2}} \leq \kappa_\Omega(x; v)$$

for all $x \in \Omega$ and $v \in \mathbb{C}^d$ nonzero.

2. *For any $x_0 \in \Omega$ there exists $C > 0$ so that*

$$-C + \frac{1}{2}\log\frac{1}{\delta_\Omega(x)} \leq k_\Omega(x, x_0) \leq C + \frac{1}{2}\log\frac{1}{\delta_\Omega(x)}$$

for all $x \in \Omega$.

3. *If $\xi, \eta \in \partial\Omega$ are distinct, then there exists neighborhoods V, U of ξ, η and a constant $C > 0$ so that*

$$k_\Omega(x, y) \geq -C + \frac{1}{2}\log\frac{1}{\delta_\Omega(x)} + \frac{1}{2}\log\frac{1}{\delta_\Omega(y)}.$$

for all $x \in U \cap \Omega$ and $y \in V \cap \Omega$.

4. *If $\xi \in \partial\Omega$, then there exists a neighborhood U of ξ and a constant $C > 0$ so that*

$$k_\Omega(x, y) \leq C + \frac{1}{2}\log\frac{\|x-y\|}{\delta_\Omega(x)} + \frac{1}{2}\log\frac{\|x-y\|}{\delta_\Omega(y)}$$

for all $x, y \in U \cap \Omega$.

5. *If $\xi \in \partial\Omega$ and V_2 is a neighborhood of ξ, then there exists a neighborhood $V_1 \Subset V_2$ of ξ and a constant $c > 0$ so that*

$$\kappa_\Omega(x, v) \leq \kappa_{V_2\cap\Omega}(x; v) \leq e^{c\delta_\Omega(x)}\kappa_\Omega(x; v)$$

for all $x \in V_1 \cap \Omega$ and $v \in \mathbb{C}^d$.

All these estimates follow from the results in Section 2 of [23].

4.6.2 Behavior of Geodesics

Proposition 4.6.4 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. Assume that $x_n, y_n \in \Omega$, $x_n \to \xi \in \partial\Omega$, $y_n \to \eta \in \partial\Omega$, and $\xi \neq \eta$. If $\sigma_n : [a_n, b_n] \to \Omega$ is a geodesic with $\sigma_n(a_n) = x_n$ and $\sigma_n(b_n) = y_n$, then there exists $T_n \in [a_n, b_n]$ so that $\sigma_n(\cdot + T_n)$ converges locally uniformly to a geodesic $\sigma : \mathbb{R} \to \Omega$.*

This follows immediately from [38, Lemma 36]. Alternatively, one can prove Proposition 4.6.4 by simply repeating the argument in the proof of Proposition 4.5.10 and using the estimates in Theorem 4.6.3 when necessary.

Arguing as in Lemma 4.5.8, we can use Proposition 4.6.4 to establish:

Corollary 4.6.5 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. Assume that $x_0 \in \Omega$ and a sequence $y_n \in \Omega$ converges to a point $\xi \in \partial\Omega$. If $\sigma_n : [0, T_n] \to \Omega$ is a geodesic with $\sigma_n(0) = x_0$ and $\sigma_n(T_n) = y_n$, then after passing to a subsequence σ_n converges locally uniformly to a geodesic $\sigma : [0, \infty) \to \Omega$ and $\lim_{t\to\infty}\sigma(t) = \xi$.*

Proposition 4.6.6 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. If $\sigma : \mathbb{R} \to \Omega$ is a geodesic, then the limits $\lim_{t\to-\infty}\sigma(t)$ and $\lim_{t\to\infty}\sigma(t)$ both exist in $\partial\Omega$ and are distinct.*

Proof The proof of Theorem 4.5.11 can be adapted to this situation, but we will provide another argument using the Gromov product. For three points $o, x, y \in \Omega$

the Gromov product $(x|y)_o$ is

$$(x|y)_o := \frac{1}{2}\left(k_\Omega(x,o) + k_\Omega(o,y) - k_\Omega(x,y)\right).$$

Using the estimates in Theorem 4.6.3:

1. If $\xi \in \partial\Omega$ then $\liminf_{x,y\to\xi}(x|y)_o = \infty$.
2. If $\xi, \eta \in \partial\Omega$ and $\xi \neq \eta$, then $\limsup_{x\to\xi, y\to\eta}(x|y)_o < \infty$.

Now if $\sigma : \mathbb{R} \to \Omega$ is a geodesic then

$$\lim_{s,t\to\infty}(\sigma(s)|\sigma(t))_{\sigma(0)} = \infty \quad \text{and} \quad \lim_{t\to\infty, s\to-\infty}(\sigma(s)|\sigma(t))_{\sigma(0)} = 0.$$

So the lemma follows.

4.6.3 Localization

Theorem 4.6.7 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain and $\xi \in \partial\Omega$. Then there exist neighborhoods $V_1 \Subset V_2 \Subset V_3$ of ξ, a holomorphic map $\Phi : V_3 \to \mathbb{C}^d$, and some $\lambda > 1$ so that:*

1. *Φ is a bi-holomorphism onto its image, $\mathscr{C} := \Phi(V_3 \cap \Omega)$ is convex set, and $\Phi(\xi)$ is a strongly convex point of $\mathscr{C}$, that is $\partial\Omega$ is C^2 and had finite line type 2 near ξ.*
2. *For all $x \in V_2 \cap \Omega$ and $v \in \mathbb{C}^d$*

$$\kappa_\Omega(x,v) \leq \kappa_{\Omega\cap V_3}(x;v) \leq \lambda\kappa_\Omega(x;v).$$

3. *For all $x, y \in V_2 \cap \Omega$*

$$k_\Omega(x,y) \leq k_{V_3\cap\Omega}(x,y) \leq k_\Omega(x,y) + \lambda.$$

4. *If $x, y \in V_1 \cap \Omega$ and $\sigma : [a,b] \to \Omega$ is a geodesic with $\sigma(a) = x$ and $\sigma(b) = y$ then $\sigma([a,b]) \subset V_2$.*

Proof (Sketch of proof) Since Ω is strongly pseudoconvex, there exist a neighborhood V_3 of ξ and a holomorphic map $\Phi : V_3 \to \mathbb{C}^d$ which is a bi-holomorphism onto its image, $\mathscr{C} := \Phi(V_3 \cap \Omega)$ is convex set, and $\Phi(\xi)$ is a strongly convex point of $\mathscr{C}$ (see for instance [1, Proposition 2.1.13]).

Now by part (5) of Theorem 4.6.3, there exists $c > 0$ and a neighborhood U_2 of ξ so that $U_2 \Subset V_3$ and

$$\kappa_{V_3\cap\Omega}(x;v) \leq e^{c\delta_\Omega(x)}\kappa_\Omega(x;v)$$

for all $x \in U_2 \cap \Omega$.

We claim that there exists a neighborhood V_2 of ξ so that $V_2 \Subset U_2$ and if $x, y \in V_2 \cap \Omega$ and $\sigma : [a, b] \to \Omega$ is a geodesic with $\sigma(a) = x$ and $\sigma(b) = y$ then $\sigma([a, b]) \subset U_2$. Suppose not, then for any $n > 0$ there exists $x_n, y_n \in \mathbb{B}_{1/n}(\xi)$, a geodesic $\sigma : [a_n, b_n] \to \Omega$ with $\sigma(a_n) = x_n$ and $\sigma(b_n) = y_n$, and some $t_n \in [a_n, b_n]$ so that $\sigma(t_n) \in \Omega \setminus U_2$. Now, by passing to a subsequence we can suppose that $\sigma(t_n) \to \eta \in \overline{\Omega} \setminus U_2$. If $\eta \in \Omega$, we obtain a contradiction by considering parts (2) and (4) in Theorem 4.6.3. And if $\eta \in \partial\Omega$, we obtain a contradiction by considering parts (3) and (4) in Theorem 4.6.3. In either case we have a contradiction and hence there exists a neighborhood V_2 of ξ so that $V_2 \Subset U_2$ and if $x, y \in V_2 \cap \Omega$ and $\sigma : [a, b] \to \Omega$ is a geodesic with $\sigma(a) = x$ and $\sigma(b) = y$ then $\sigma([a, b]) \subset U_2$.

Repeating the above argument, we can find a neighborhood V_1 of ξ so that $V_1 \Subset V_2$ and if $x, y \in V_1 \cap \Omega$ and $\sigma : [a, b] \to \Omega$ is a geodesic with $\sigma(a) = x$ and $\sigma(b) = y$ then $\sigma([a, b]) \subset V_2$.

Finally, we sketch the proof of part (3). Suppose that $x, y \in V_2 \cap \Omega$. Let $\sigma : [a, b] \to \Omega$ be a geodesic with $\sigma(a) = x$ and $\sigma(b) = y$. Then by our choices, $\sigma([a, b]) \subset U_2$. Now by Proposition 4.1.2, we have $\kappa_\Omega(\sigma(t); \sigma'(t)) = 1$ for almost every $t \in [a, b]$. And so

$$k_{V_3 \cap \Omega}(\sigma(t); \sigma'(t)) \leq e^{c\delta_\Omega(\sigma(t))}$$

for almost every $t \in [a, b]$. Then arguing as in the proof of Proposition 4.5.10 there exists $A, r > 0$ (which can be chosen to be independent of x, y) so that

$$\delta_\Omega(\sigma(t)) \leq Ae^{-r|t|}.$$

Then

$$\begin{aligned} k_{V_3 \cap \Omega}(x, y) &\leq \int_a^b \kappa_{V_3 \cap \Omega}(\sigma(t); \sigma'(t))\mathrm{d}t \leq \int_a^b e^{cAe^{-r|t|}}\mathrm{d}t \leq \int_a^b 1 + cAe^{e^{cA}}e^{-r|t|}\mathrm{d}t \\ &= b - a + \int_a^b cAe^{e^{cA}}e^{-r|t|}dt \\ &\leq k_\Omega(x, y) + \int_{-\infty}^{\infty} cAe^{e^{cA}}e^{-r|t|}\mathrm{d}t \end{aligned}$$

where we used the fact that $e^x \leq 1 + e^R x$ for $x \in [0, R]$. Thus we can pick $\lambda > 1$ so that part (2) and part (3) are satisfied.

4.6.4 Visibility of Almost-Geodesics

Theorem 4.6.7 allows us to reduce to the convex setting, but there is a cost: with the notation of the theorem, if $\sigma : [a, b] \to \Omega$ is a geodesic with $\sigma(a), \sigma(b) \in V_1$ then σ

will not be a geodesic in $V_3 \cap \Omega$. This causes us to consider a larger class of almost length minimizing curves:

Definition 4.6.8 Suppose $\Omega \subset \mathbb{C}^d$ is a domain, $I \subset \mathbb{R}$ is an interval, and $\lambda \geq 1$. A curve $\sigma : I \to \Omega$ is called an λ-*almost-geodesic* if σ is absolutely continuous (as a map $I \to \mathbb{C}^d$), $\kappa_\Omega(\sigma(t); \sigma'(t)) \leq \lambda$ for almost every $t \in I$, and

$$|t-s| - \lambda \leq k_\Omega(\sigma(s), \sigma(t)) \leq |t-s| + \lambda$$

for all $s, t \in I$.

Repeating the proof of Proposition 4.5.10 almost verbatim implies the following more general result:

Proposition 4.6.9 *Suppose Ω_n converges to Ω in $\mathbb{X}_d$. Assume for any $R > 0$ there exists $C = C(R) > 0$, $N = N(R) > 0$, and $L = L(R) > 0$ such that*

$$\delta_{\Omega_n}(p; v) \leq C\delta_{\Omega_n}(p)^{1/L}$$

for all $n > N$, $p \in \mathbb{B}_R(0) \cap \Omega_n$, and $v \in \mathbb{C}^d$ non-zero.

If $\sigma_n : [a_n, b_n] \to \Omega_n$ is a sequence of λ-almost-geodesics such that $\sigma_n(a_n) \to \xi \in \partial\Omega_\infty \cup \{\infty\}$, $\sigma_n(b_n) \to \eta \in \partial\Omega_\infty \cup \{\infty\}$, and $\xi \neq \eta$ then exists $n_k \to \infty$ and $T_k \in [a_{n_k}, b_{n_k}]$ so that $\sigma_{n_k}(\cdot + T_k)$ converges locally uniformly to an λ-almost-geodesic $\sigma : \mathbb{R} \to \Omega_\infty$.

Arguing as in Lemma 4.5.8, we can use Proposition 4.6.9 to establish:

Lemma 4.6.10 *Suppose Ω_n converges to Ω in $\mathbb{X}_d$. Assume for any $R > 0$ there exists $C = C(R) > 0$, $N = N(R) > 0$, and $L = L(R) > 0$ such that*

$$\delta_{\Omega_n}(p; v) \leq C\delta_{\Omega_n}(p)^{1/L}$$

for all $n > N$, $p \in \mathbb{B}_R(0) \cap \Omega_n$, and $v \in \mathbb{C}^d$ non-zero.

Suppose that $x_0 \in \Omega_\infty$ and a sequence $y_n \in \Omega_n$ converges to a point $\xi \in \partial\Omega_\infty \cup \{\infty\}$. If $\sigma_n : [0, T_n] \to \Omega_n$ is an λ-almost-geodesic with $\sigma_n(0) = x_0$ and $\sigma_n(T_n) = y_n$ then there exists $n_k \to \infty$ so that σ_{n_k} converges locally uniformly to an λ-almost-geodesic $\sigma : [0, \infty) \to \Omega_\infty$ and

$$\lim_{t\to\infty} \sigma(t) = \xi.$$

Finally arguing as in the proof of Corollary 4.5.13 one can show:

Corollary 4.6.11 *Suppose $\Omega \subset \mathbb{C}^d$ is a domain of the form*

$$\Omega = \{(z_0, z_1 \ldots, z_d) \in \mathbb{C}^{d+1} : \mathrm{Re}(z_0) > P(z_1, z_2, \ldots, z_d)\}$$

where $P : \mathbb{C}^d \to \mathbb{R}$ is a non-negative, non-degenerate, convex polynomial with $P(0) = 0$.

If $\sigma : \mathbb{R} \to \Omega$ *is a* λ*-almost-geodesic, then* $\lim_{t\to\infty} \sigma(t)$ *and* $\lim_{t\to-\infty} \sigma(t)$ *both exist in* $\partial\Omega \cup \{\infty\}$ *and are distinct.*

4.6.5 Proof of Theorem 4.6.1

Suppose for a contradiction that (Ω, k_Ω) is not Gromov hyperbolic. Then there exists points $x_n, y_n, z_n \in \Omega$, geodesic segments $\sigma_{x_n y_n}, \sigma_{y_n z_n}, \sigma_{z_n x_n}$ joining them, and a point u_n in the image of $\sigma_{x_n y_n}$ such that

$$k_\Omega(u_n, \sigma_{y_n z_n} \cup \sigma_{z_n x_n}) > n.$$

After passing to a subsequence we can assume that

$$u_n, x_n, y_n, z_n \to u_\infty, x_\infty, y_\infty, z_\infty \in \overline{\Omega}.$$

Case 1 Suppose that $u_\infty \in \Omega$. Since

$$k_\Omega(u_n, \{x_n, y_n, z_n\}) \geq k_\Omega(u_n, \sigma_{y_n z_n} \cup \sigma_{z_n x_n}) > n.$$

we see that $x_\infty, y_\infty, z_\infty \in \partial\Omega$. Now we can assume that $\sigma_{x_n y_n}(0) = u_n$ and pass to a subsequence so that $\sigma_{x_n y_n}$ converges locally uniformly to a geodesic $\sigma : \mathbb{R} \to \Omega$. By Corollary 4.6.5

$$\lim_{t\to\infty} \sigma(t) = x_\infty \text{ and } \lim_{t\to-\infty} \sigma(t) = y_\infty.$$

By Proposition 4.6.6, we must have $x_\infty \neq y_\infty$. Thus, after possibly relabeling, we may assume that $z_\infty \neq x_\infty$.

Then using Proposition 4.6.4 we can assume that $\sigma_{x_n z_n}$ converges locally uniformly to a geodesic $\gamma : \mathbb{R} \to \Omega$. But then

$$k_\Omega(u_\infty, \gamma(0)) = \lim_{n\to\infty} k_\Omega(u_n, \sigma_{x_n z_n}(0)) \geq \lim_{n\to\infty} k_\Omega(u_n, \sigma_{x_n z_n}) = \infty.$$

So we have a contradiction.

Case 2 Suppose that $u_\infty \in \partial\Omega$. Fix neighborhoods $V_1 \Subset V_2 \Subset V_3$ of u_∞, a holomorphic map $\Phi : U \to \mathbb{C}^d$, and some $\lambda > 1$ as in Theorem 4.6.7. Let $\mathscr{C} := \Phi(V_3 \cap \Omega)$ and consider the sequence $\widehat{u}_n = \Phi(u_n)$.

Now $\widehat{u}_n \to \xi := \Phi(u_\infty)$ and $\partial\mathscr{C}$ is strongly convex at ξ (in particular it is C^2 has a finite line type 2 at ξ). Thus by Theorem 4.5.9 after passing to a subsequence there exists affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ so that:

1. $A_n(\mathscr{C}, \widehat{u}_n)$ converges to some $(\mathscr{C}_\infty, \widehat{u}_\infty)$ in $\mathbb{X}_{d,0}$,
2. $\mathscr{C}_\infty$ is a polynomial domain,
3. for any $R > 0$ there exists $C = C(R) > 0$ and $N = N(R) > 0$ such that

$$\delta_{A_k\mathscr{C}}(x; v) \leq C\delta_{A_k\mathscr{C}}(x)^{1/2}$$

for all $k > N$, $x \in \mathbb{B}_R(0) \cap A_k\mathscr{C}$, and $v \in \mathbb{C}^d$ non-zero.

Remark 4.6.12 In this special case above, ξ is a strongly convex point and so the proof of Theorem 4.5.9 is substantially easier. Moreover the limiting domain will be the hyperboloid model of complex hyperbolic d-space:

$$\mathscr{H}_d = \left\{ (z_1, \ldots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > \sum_{i=2}^{d} |z_i|^2 \right\}.$$

Then standard facts about complex hyperbolic space imply that λ-almost-geodesics are well behaved in $(\mathscr{H}_d, K_{\mathscr{H}_d})$. So in the case of strongly pseudoconvex domains, much of the technicalities can be avoided.

At this point the rest of the proof closely follows the proof of Theorem 4.5.4, however there is some extra work to do based on the fact that x_n, y_n, z_n may not be in V_3.

Special Case Assume $x_\infty = y_\infty = z_\infty = u_\infty$.

Proof By passing to a subsequence we may suppose that $x_n, y_n, z_n \in V_1$ for all n. Then passing to another subsequence we can suppose that $A_n\Phi_\xi(x_n) \to \widehat{x}_\infty \in \overline{\mathbb{C}^d}$, $A_n\Phi_\xi(y_n) \to \widehat{y}_\infty \in \overline{\mathbb{C}^d}$, and $A_n\Phi_\xi(z_n) \to \widehat{z}_\infty \in \overline{\mathbb{C}^d}$.

Since $x_n, y_n, z_n \in V_1$ we see that

$$\sigma_{x_ny_n}, \sigma_{y_nz_n}, \sigma_{z_nx_n} \subset V_2$$

hence

$$\widehat{\sigma}_{x_ny_n} := \Phi_\xi(\sigma_{x_ny_n}), \quad \widehat{\sigma}_{y_nz_n} := \Phi_\xi(\sigma_{y_nz_n}), \quad \widehat{\sigma}_{z_nx_n} := \Phi_\xi(\sigma_{z_nx_n})$$

are all λ-almost-geodesics in $(\mathscr{C}, k_{\mathscr{C}})$.

Now suppose $\widehat{\sigma}_{x_ny_n} : [a_n, b_n] \to \mathscr{C}$ is parametrized so that $\widehat{\sigma}_{x_ny_n}(0) = \widehat{u}_n$. Then we can pass to a subsequence so that $A_n\widehat{\sigma}_{x_ny_n}$ converges locally uniformly to an λ-almost-geodesic $\widehat{\sigma} : \mathbb{R} \to \mathscr{C}_\infty$. By Lemma 4.6.10

$$\lim_{t\to-\infty} \widehat{\sigma}(t) = \lim_{n\to\infty} A_n\Phi_\xi(x_n) = \widehat{x}_\infty$$

and

$$\lim_{t \to +\infty} \widehat{\sigma}(t) = \lim_{n \to \infty} A_n \Phi_\xi(y_n) = \widehat{y}_\infty.$$

Since geodesics in $\mathscr{C}_\infty$ are well behaved, we see that $\widehat{x}_\infty \neq \widehat{y}_\infty$. So by relabeling we can suppose that $\widehat{x}_\infty \neq \widehat{z}_\infty$. Then, by Proposition 4.6.9, there exists a parametrization of $A_n \widehat{\sigma}_{x_n z_n}$ which converges locally uniformly to an λ-almost-geodesic $\widehat{\gamma} : \mathbb{R} \to \mathscr{C}_\infty$. But then

$$\begin{aligned} k_{\mathscr{C}_\infty}(\widehat{u}_\infty, \widehat{\gamma}(0)) &= \lim_{n \to \infty} k_{A_n \mathscr{C}}(A_n \widehat{u}_n, A_n \widehat{\sigma}_{x_n z_n}(0)) = \lim_{n \to \infty} k_{\mathscr{C}}(\widehat{u}_n, \widehat{\sigma}_{x_n z_n}(0)) \\ &= \lim_{n \to \infty} k_{U \cap \Omega}(u_n, \sigma_{x_n z_n}(0)) \geq \lim_{n \to \infty} k_\Omega(u_n, \sigma_{x_n z_n}) = \infty \end{aligned}$$

which is a contradiction.

We now prove the general case. Suppose $\sigma_{x_n y_n} : [a_n, b_n] \to \Omega$ is parametrized so that $\sigma_{x_n y_n}(0) = u_n$. Let

$$a_n' = \inf\{t \in [a_n, b_n] : \sigma_{x_n y_n}([t, 0]) \subset V_2\}$$

and

$$b_n' = \sup\{t \in [a_n, b_n] : \sigma_{x_n y_n}([0, t]) \subset V_2\}.$$

Since $u_n \to u_\infty$, by Theorem 4.6.3 we have that $a_n' \to -\infty$ and $b_n' \to +\infty$. Also

$$\widehat{\sigma}_n := (A_n \circ \Phi_\xi \circ \sigma_{x_n y_n})|_{[a_n', b_n']}$$

is an λ-almost-geodesic in $A_n \mathscr{C}$. Hence we may pass to a subsequence such that $\widehat{\sigma}_n$ converges locally uniformly to λ-almost-geodesic $\widehat{\sigma} : \mathbb{R} \to \mathscr{C}_\infty$. By passing to a subsequence we may assume that $\lim_{n \to \infty} \widehat{\sigma}_n(a_n') = \widehat{x}_\infty$ and $\lim_{n \to \infty} \widehat{\sigma}_n(b_n') = \widehat{y}_\infty$ for some $\widehat{x}_\infty, \widehat{y}_\infty \in \overline{\mathbb{C}^d}$.

The points x_∞, y_∞ and $\widehat{x}_\infty, \widehat{y}_\infty$ have the following relationship:

Lemma 4.6.13 *If $x_\infty \neq u_\infty$ then $\widehat{x}_\infty = \infty$. Likewise, if $y_\infty \neq u_\infty$ then $\widehat{y}_\infty = \infty$.*

Proof If $x_\infty \neq u_\infty$ then

$$\liminf_{n \to \infty} \left\| \sigma_{x_n y_n}(a_n') - u_n \right\| > 0.$$

So $\widehat{x}_\infty = \lim_{n \to \infty} A_n \sigma_{x_n y_n}(a_n') = \infty$ by part (3) of Theorem 4.5.9. The y case is identical.

Now by Lemma 4.6.10

$$\lim_{t \to -\infty} \widehat{\sigma}(t) = \lim_{n \to \infty} \widehat{\sigma}_n(a_n') = \widehat{x}_\infty$$

and

$$\lim_{t \to +\infty} \widehat{\sigma}(t) = \lim_{n \to \infty} \widehat{\sigma}_n(b_n') = \widehat{y}_\infty.$$

Hence $\widehat{x}_\infty \neq \widehat{y}_\infty$ by Corollary 4.6.11. So by relabeling we may assume that $\widehat{x}_\infty \neq \infty$. This implies, by Lemma 4.6.13, that $x_\infty = \xi$. So by passing to subsequence we can suppose that $x_n \in V_1$ for all n. Then $a_n' = a_n$ for all n and

$$\lim_{n \to \infty} A_n \Phi(x_n) = \lim_{n \to \infty} \widehat{\sigma}_n(a_n') = \widehat{x}_\infty.$$

Now suppose $\sigma_{x_n z_n} : [0, T_n] \to \Omega$ is parametrized so that $\sigma_{x_n z_n}(0) = x_n$. Let

$$T_n' = \sup\{t \in [0, T_n] : \sigma_{x_n z_n}([0, t]) \subset V_2\},$$

then

$$\widehat{\gamma}_n := (A_n \circ \Phi \circ \sigma_{x_n z_n})|_{[0, T_n']}$$

is an λ-almost-geodesic in $A_n \mathscr{C}$. By passing to a subsequence we may assume that

$$\lim_{n \to \infty} \widehat{\gamma}_n(T_n') = \widehat{z}_\infty$$

for some $\widehat{z}_\infty \in \overline{\mathbb{C}^d}$.

If $\widehat{z}_\infty \neq \widehat{x}_\infty$ then, by Proposition 4.6.9, there exists some $\alpha_n \in [0, T_n']$ such that $t \to \widehat{\gamma}_n(t + \alpha_n)$ converges to a λ-almost-geodesic $\widehat{\gamma} : \mathbb{R} \to \mathscr{C}_\infty$. But then

$$\begin{aligned} k_{\mathscr{C}_\infty}(\widehat{u}_\infty, \widehat{\gamma}(0)) &= \lim_{n \to \infty} k_{A_n \mathscr{C}}(A_n \widehat{u}_n, \widehat{\gamma}_n(0)) = \lim_{n \to \infty} k_{\Omega \cap V_3}(u_n, \sigma_{x_n z_n}(\alpha_n)) \\ &\geq \lim_{n \to \infty} k_\Omega(u_n, \sigma_{x_n z_n}) = \infty \end{aligned}$$

which is a contradiction.

It remains to consider the case where $\widehat{z}_\infty = \widehat{x}_\infty$. Then since $\widehat{z}_\infty \neq \infty$ arguing as in Lemma 4.6.13 shows that $z_\infty = \xi$. So by passing to a subsequence we can suppose that $z_n \in V_1$ for all n. Then $T_n' = T_n$. So

$$\lim_{n \to \infty} A_n \Phi(z_n) = \lim_{n \to \infty} \widehat{\gamma}_n(T_n') = \widehat{z}_\infty.$$

Suppose $\sigma_{z_n y_n} : [0, S_n] \to \Omega$ is parametrized so that $\sigma_{z_n y_n}(0) = z_n$. Let

$$S_n' = \sup\{s \in [0, S_n] : \sigma_{z_n y_n}([0, s]) \subset V_2\}.$$

Then

$$\widehat{\eta}_n := (A_n \circ \Phi \circ \sigma_{y_n z_n})|_{[0,S_n']}$$

is an λ-almost-geodesic in $A_n \mathscr{C}$. By passing to a subsequence we may assume that

$$\lim_{n\to\infty} \widehat{\eta}_n(S_n') = \widehat{w}_\infty \in \overline{\mathbb{C}^d}$$

for some $\widehat{w}_\infty \in \overline{\mathbb{C}^d}$.

If $\widehat{z}_\infty = \widehat{w}_\infty$ then $\widehat{w}_\infty \neq \infty$ and hence arguing as in Lemma 4.6.13 shows that $y_\infty = \xi$. But then we are in the Special Case.

If $\widehat{z}_\infty \neq \widehat{w}_\infty$ then, by Proposition 4.6.9, there exists some $\beta_n \in [0, S_n']$ such that $t \to \widehat{\eta}_n(t + \beta_n)$ converges to an λ-almost-geodesic $\widehat{\eta} : \mathbb{R} \to \mathscr{C}_\infty$. But then

$$\begin{aligned} k_{\mathscr{C}}(\widehat{u}_\infty, \widehat{\eta}(0)) &= \lim_{n\to\infty} k_{A_n \mathscr{C}}(A_n \widehat{u}_n, \widehat{\eta}_n(0)) = \lim_{n\to\infty} k_{\Omega\cap V_3}(u_n, \sigma_{y_n z_n}(\beta_n)) \\ &\geq \lim_{n\to\infty} k_{\Omega}(u_n, \sigma_{y_n z_n}) = \infty \end{aligned}$$

which is a contradiction.

Thus (Ω, k_Ω) is Gromov hyperbolic.

4.7 The Hilbert Metric

In this section we describe the Hilbert metric, its connections to the Kobayashi metric, and some important properties. Connections between the Hilbert and Kobayashi metric are further discussed in [45] and for a detailed account of recent developments in the theory of the Hilbert metric we refer the reader to the *Handbook of Hilbert geometry* [51].

Definition 4.7.1 An open set $\mathscr{C} \subset \mathbb{R}^d$ is called *properly convex* if it is convex (its intersection with every real line is connected) and does not contain any real affine lines.

Suppose $\mathscr{C}$ is properly convex. Given two points $x, y \in \mathscr{C}$ let ℓ_{xy} be a real line containing x and y. Then the *Hilbert distance* between them is defined to be

$$H_{\mathscr{C}}(x, y) = \log \frac{|y-a|\,|x-b|}{|x-a|\,|y-b|}$$

where $\{a, b\} = \partial\mathscr{C} \cap \ell_{xy}$ and we have the ordering a, x, y, b along ℓ_{xy}. Remarkably this formula yields a distance on $\mathscr{C}$:

Theorem 4.7.2 *Suppose $\mathscr{C}$ is a properly convex domain. Then $(\mathscr{C}, H_{\mathscr{C}})$ is a Cauchy complete geodesic metric space.*

The Hilbert metric has a number of important properties, including:

1. Essentially by definition, straight lines can always be parameterized as geodesics in $(\mathscr{C}, H_{\mathscr{C}})$. However, when $\mathscr{C}$ is not strictly convex (that is, when $\partial\mathscr{C}$ contains line segments) there can exist geodesics which are not straight lines.
2. If $\mathbb{B} \subset \mathbb{R}^d$ is the unit ball, then $(\mathbb{B}, H_{\mathbb{B}})$ is the Klein-Beltrami model of real hyperbolic d-space.

Now given a properly convex domain $\mathscr{C} \subset \mathbb{R}^d$ define the *real projective automorphism group* $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ to the group of diffeomorphisms $f : \mathscr{C} \to \mathscr{C}$ such that

$$f(x) = \frac{Ax + b}{\ell(x)}$$

where A is a d-by-d matrix, $b \in \mathbb{R}^d$, and $\ell : \mathbb{R}^d \to \mathbb{R}$ is an affine map. This group is called the real projective automorphism group for the following reason: if we identify $\mathbb{R}^d$ with an affine chart of $\mathbb{P}(\mathbb{R}^{d+1})$ then $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ can be identified with the group

$$\{\varphi \in \mathrm{PGL}_{d+1}(\mathbb{R}) : \varphi(\mathscr{C}) = \mathscr{C}\}.$$

Using the fact that real projective maps send straight lines to straight lines one can show the following:

Theorem 4.7.3 *Suppose $\mathscr{C} \subset \mathbb{R}^d$ is a properly convex domain. Then $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ acts by isometries on $(\mathscr{C}, H_{\mathscr{C}})$.*

Due to the restrictive nature of the maps being considered, one might expect that the group $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ will be quite small, but in fact there are many examples of bounded convex domains $\mathscr{C} \subset \mathbb{R}^d$ where $\mathscr{C}$ is non-homogeneous but $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ is very large. In particular, one has the following result:

Theorem 4.7.4 *For any $d \geq 2$ there exists a bounded convex domain $\mathscr{C} \subset \mathbb{R}^d$ where $\partial\mathscr{C}$ is C^1, $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ does not act transitively on $\mathscr{C}$, and there exists a discrete group $\Gamma \leq \mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ which acts freely, properly, and co-compactly on $\mathscr{C}$.*

Here are two sources of examples:

1. Again let $\mathbb{B} \subset \mathbb{R}^d$ be the unit ball. Johnson-Millson [36] proved that certain co-compact lattices in $\Gamma \leq \mathrm{Aut}(\mathbb{B})$ have non-trivial deformations as subgroups of $\mathrm{PGL}_{d+1}(\mathbb{R})$. A general result of Koszul [44] about geometric structures then implies that when Γ' is a deformation sufficiently close to Γ there exists a convex set $\mathscr{C}$ close to the unit ball $\mathbb{B}$ where Γ' act co-compactly (see [8, Section 1.3] for $d > 2$ and [32] for $d = 2$).
2. For any dimension $d \geq 4$, Gromov and Thurston [34] have constructed families of Riemannian manifolds M_n whose sectional curvature converges to -1 but have

no Riemannian metric of constant negative curvature. Kapovich [37] has shown that some of the these infinite families can be realized as quotients $\Gamma \backslash \mathscr{C}$ where $\mathscr{C}$ is a properly convex set and $\Gamma \leq \mathrm{Aut}_{\mathrm{proj}}(\Omega)$ is a discrete group which acts freely, properly, and co-compactly.

These examples lead to the following definition:

Definition 4.7.5 A properly convex domain $\mathscr{C} \subset \mathbb{R}^d$ is called a *convex divisible domain* if there exists a discrete group $\Gamma \leq \mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ which acts freely, properly, and co-compactly on $\mathscr{C}$.

For more details about these special type of domains see the survey papers by Benoist [10] and Quint [53].

The existence of non-homogeneous convex divisible domains in the real projective setting makes Frankel's rigidity theorem (see Theorem 4.3.7 above) seem even more remarkable. But, as the examples above indicate, the rigidity coming from the fact that complex lines have two real dimensions outweighs the flexibility obtained by considering maps defined by general power series (instead of a special type of rational maps as in the real projective case).

For domains $\mathscr{O} \subset \mathbb{R}^d$ which are not convex, Kobayashi [43] constructed two metrics using projective maps to and from the unit interval: the Carathéodory and Kobayashi metric.

For two open sets $\mathscr{O}_1 \subset \mathbb{R}^{d_1}$ and $\mathscr{O}_2 \subset \mathbb{R}^{d_2}$ let $\mathrm{Proj}(\mathscr{O}_1, \mathscr{O}_2)$ be the space of maps $f : \mathscr{O}_1 \to \mathscr{O}_2$ such that

$$f(x) = \frac{Ax + b}{\ell(x)}$$

where A is a d_2-by-d_1 matrix, $b \in \mathbb{R}^{d_2}$, and $\ell : \mathbb{R}^{d_1} \to \mathbb{R}$ is an affine map.

Remark 4.7.6 If we identify each $\mathbb{R}^{d_i}$ with an affine chart of $\mathbb{P}(\mathbb{R}^{d_i+1})$ then $\mathrm{Proj}(\mathscr{O}_1, \mathscr{O}_2)$ is exactly the set of maps $f : \mathscr{O}_1 \to \mathscr{O}_2$ so that $f = T|_{\mathscr{O}_1}$ for some $T \in \mathbb{P}(\mathrm{Lin}(\mathbb{R}^{d_1+1}, \mathbb{R}^{d_2+1}))$ with $\ker T \cap \mathscr{O}_1 = \emptyset$.

Next let $I = (-1, 1)$ be the unit interval in $\mathbb{R}$ and let H_I be the Hilbert metric on I. For a domain $\mathscr{O} \subset \mathbb{R}^d$ define the two quantities:

$$C^{\mathbb{P}}_{\mathscr{O}}(x, y) := \sup \{H_I(f(x), f(y)) : f \in \mathrm{Proj}(\Omega, I)\},$$

and

$$L^{\mathbb{P}}_{\mathscr{O}}(x, y) := \inf \{H_I(u, w) : f \in \mathrm{Proj}(I, \Omega) \text{ with} f(u) = x \text{ and} f(w) = y\}.$$

The function $C^{\mathbb{P}}_{\mathscr{O}}$ always satisfies the triangle inequality, but $L^{\mathbb{P}}_{\mathscr{O}}$ may not. So we introduce:

$$K^{\mathbb{P}}_{\mathscr{O}}(x, y) := \inf \left\{ \sum_{i=0}^{N} L_{\mathscr{O}}(x_i, x_{i+1}) : N > 0, x = x_0, x_1, \dots, x_{N+1} = y \in \mathscr{O} \right\}.$$

Kobayashi then proved:

Theorem 4.7.7 ([43]) *Suppose $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain. Then $C^{\mathbb{P}}_{\mathcal{O}}$ and $K^{\mathbb{P}}_{\mathcal{O}}$ are* $\mathrm{Aut}_{\mathrm{proj}}(\mathcal{O})$*-invariant metrics on $\mathcal{O}$. Moreover, if $\mathcal{O}$ is convex then $C^{\mathbb{P}}_{\mathcal{O}} = K^{\mathbb{P}}_{\mathcal{O}} = H_{\mathcal{O}}$.*

This shows that the Hilbert metric is a real projective analogue of the Kobayashi metric.

Now we change our perspective slightly and consider domains in real projective space $\mathbb{P}(\mathbb{R}^{d+1})$.

Definition 4.7.8 A domain $\mathscr{C} \subset \mathbb{P}(\mathbb{R}^{d+1})$ is called *properly convex* if for every real projective line $\ell \subset \mathbb{P}(\mathbb{R}^{d+1})$ the intersection $\ell \cap \mathscr{C}$ is connected or empty and $\overline{\ell \cap \mathscr{C}} \neq \ell$.

If $\mathscr{C} \subset \mathbb{P}(\mathbb{R}^{d+1})$ is properly convex, then it is contained as a bounded set in some affine chart and it is convex in any affine chart which contains it.

Now let $\mathbb{Y}_{d,0}$ be the set of pairs (Ω, x) where $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is properly convex and $x \in \Omega$. We can endow $\mathbb{Y}_{d,0}$ with a topology where $(\Omega_n, x_n) \to (\Omega, x)$ if $\Omega_n \to \Omega$ in the Hausdorff topology (obtained by fixing some Riemannian metric on $\mathbb{P}(\mathbb{R}^{d+1})$) and $x_n \to x$. Now the group $\mathrm{PGL}_{d+1}(\mathbb{R})$ acts on $\mathbb{Y}_{d,0}$ and Benzécri [11] proved the following compactness theorem:

Theorem 4.7.9 (Benzécri) *The group* $\mathrm{PGL}_{d+1}(\mathbb{R})$ *acts co-compactly on* $\mathbb{Y}_{d,0}$.

A detailed proof and some applications can also be found in [46]. Benzécri's Theorem is a real projective analogue of Frankel's compactness theorem, see Theorem 4.3.2. Notice that Frankel's theorem considers the local Hausdorff topology and the action of the affine group while Benzécri's theorem consider the Hausdorff topology and the action of the entire affine group. The cause of this difference is the fact that complex projective transformations do not preserve convexity and hence in the complex case one is forced to only consider affine transformations.

Let us discuss some results concerning the Gromov hyperbolicity of the Hilbert metric. In particular, we will survey some results of Benoist [9] and discuss their connections to the arguments presented in the previous sections.

Karlsson and Noskov appear to be the first to study the Gromov hyperbolicity of the Hilbert metric. Among other things, they established some obstructions to Gromov hyperbolicity:

Theorem 4.7.10 ([39]) *If $\Omega \subset \mathbb{R}^d$ is a bounded convex domain and (Ω, H_Ω) is Gromov hyperbolic, then $\partial\Omega$ is a C^1 hypersurface and Ω is strictly convex (that is $\partial\Omega$ does not contain any line segments).*

Remark 4.7.11 Notice that $\partial\Omega$ being C^1 is a equivalent to there being a unique supporting real hyperplane through each boundary point and Ω being strictly convex is equivalent to each supporting real hyperplane intersecting $\partial\Omega$ at exactly one point.

Next Benoist presented several necessary and sufficient conditions in terms of orbit closures in the action of $\mathrm{PGL}_{d+1}(\mathbb{R})$ on $\mathbb{Y}_d$.

Theorem 4.7.12 ([9, Proposition 1.6]) *Suppose $\mathscr{C} \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a properly convex domain. Then the following are equivalent:*

1. *$(\mathscr{C}, H_{\mathscr{C}})$ is Gromov hyperbolic,*
2. *Every $\mathscr{C}_\infty$ in $\overline{\mathrm{PGL}_{d+1}(\mathbb{R}) \cdot \mathscr{C}} \cap \mathbb{Y}_d$ is strictly convex,*
3. *Every $\mathscr{C}_\infty$ in $\overline{\mathrm{PGL}_{d+1}(\mathbb{R}) \cdot \mathscr{C}} \cap \mathbb{Y}_d$ has C^1 boundary.*

Benoist's proof of Theorem 4.7.12 motivates the approach to studying the Gromov hyperbolicity of the Kobayashi metric taken in [60]. However the geometry of the Kobayashi metric introduces a number of technicalities not present for the Hilbert metric. The most fundamental being the behavior of geodesics: for a strictly convex domain in real projective space every two points are joined by a unique geodesic and this geodesic is a parameterization of the real projective line joining them. This fact allows one to easily take limits and understand their behavior. With the Kobayashi metric geodesics are fairly mysterious and it is rather involved to establish results like Proposition 4.5.10.

Returning to our discussion of the Hilbert metric, using Theorem 4.7.12 (and several other results) Benoist completely characterized the convex domains which have Gromov hyperbolic Hilbert metric in terms of the derivatives of local defining functions. In particular:

Definition 4.7.13 Suppose $U \subset \mathbb{R}^D$ is an open set and $F : U \to \mathbb{R}$ is a C^1 function. Then for $x, x + h \in U$ define

$$D_x(h) := F(x + h) - F(x) - F'(x) \cdot h.$$

Then F is said to be *quasi-symmetric convex* if there exists $H \geq 1$ so that

$$D_x(h) \leq H D_x(-h)$$

whenever $x, x + h, x - h \in U$.

Definition 4.7.14 Suppose $\mathscr{C} \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a properly convex domain with C^1 boundary. Then $\mathscr{C}$ is said to be *quasi-symmetric convex* if the boundary is locally the graph of a quasi-symmetric function.

Then:

Theorem 4.7.15 ([9, Theorem 1.4]) *Suppose $\mathscr{C} \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a properly convex domain. Then the following are equivalent:*

1. *$(\mathscr{C}, H_{\mathscr{C}})$ is Gromov hyperbolic,*
2. *$\mathscr{C}$ is quasi-symmetric convex.*

One might hope that a similar characterization will hold for the Kobayashi metric, but one would be severely disappointed. In particular, since every bounded convex domain in $\mathbb{C}$ will have Gromov hyperbolic Kobayashi metric (it is biholomorphic to the disc), the boundary of convex domains with Gromov hyperbolic Kobayashi metric need not be C^1.

4.8 Some Problems

We end this chapter with some open problems and questions.

4.8.1 Obstructions to Gromov Hyperbolicity

A fundamental question is if "flatness" of the boundary forms an obstruction to Gromov hyperbolicity. In particular:

Question 4.8.1 Suppose that Ω is a bounded weakly pseudoconvex domain with C^∞ boundary. If (Ω, k_Ω) is Gromov hyperbolic, can there exist a non-constant holomorphic map $\varphi : \Delta \to \partial\Omega$? More generally: if (Ω, k_Ω) is Gromov hyperbolic, can $\partial\Omega$ have a boundary point of infinite type?

Remark 4.8.2 Without any regularity assumption, the answer to the first question is no: There exist bounded domains $\Omega \subset \mathbb{C}^d$ whose Kobayashi metric is Cauchy complete and Gromov hyperbolic but whose boundary contains a complex affine ball of dimension $d - 1$, see [59].

This question seems out of reach, but there are a number of special cases that could be tractable. For instance, one could consider the class of domains where each boundary point has some sort of holomorphic support function. Along these lines, we recently proved the following:

Theorem 4.8.3 ([59]) *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded $\mathbb{C}$-convex open set, $\partial\Omega$ is a C^1 hypersurface, and (Ω, k_Ω) is Gromov hyperbolic. If $\Delta \subset \mathbb{C}$ is the unit disc, then every holomorphic map $\varphi : \Delta \to \partial\Omega$ is constant.*

4.8.2 Sufficient Conditions for Gromov Hyperbolicity

The most natural question is:

Question 4.8.4 Suppose that Ω is a bounded weakly pseudoconvex domain of finite type. Is (Ω, k_Ω) Gromov hyperbolic?

Not much is known about the Kobayashi metric on weakly pseudoconvex domains of finite type so this problem seems currently out of reach. Moreover, it seems necessary to prove the following first:

Problem 4.8.5 Suppose that Ω is a bounded weakly pseudoconvex domain of finite type. Show that (Ω, k_Ω) is a Cauchy complete metric space.

However, there is at least one special case that seem tractable:

Problem 4.8.6 Suppose that $\Omega \subset \mathbb{C}^2$ is a bounded weakly pseudoconvex domain of finite type. Show that (Ω, k_Ω) is Gromov hyperbolic.

Here it is known that the Kobayashi metric is Cauchy complete (see [58]). Moreover, there has been success in implementing rescaling arguments for non-convex domains in $\mathbb{C}^2$, see [5, 7].

It also seems natural to try and extend Theorem 4.0.2 to a characterization amongst the set of all convex domains (and not just the ones with C^∞ boundary):

Problem 4.8.7 Characterize the convex domains $\Omega \subset \mathbb{C}^d$ where the Kobayashi metric is Gromov hyperbolic.

This problem also seems out of reach, but a natural fist step would be to try and reduce to the dimension two case:

Problem 4.8.8 Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Then the following are equivalent:

1. (Ω, k_Ω) is Gromov hyperbolic,
2. $(\Omega \cap P, k_{\Omega \cap P})$ is Gromov hyperbolic for every complex affine 2-plane P.

Another natural subproblem is the following:

Question 4.8.9 Is there a natural complex analogue of Benoist's definition of quasi-symmetric convexity so that every bounded "complex quasi-symmetric convex domain" is Gromov hyperbolic?

4.8.3 Other Notions of Non-positive Curvature

There are several notation of non-positive curvature for metric space, however we suspect the Kobayhashi metric rarely satisfies these conditions:

Problem 4.8.10 Suppose that $\Omega \subset \mathbb{C}^d$ is a domains where (Ω, k_Ω) is Cauchy complete. If (Ω, k_Ω) is non-positively curved in the sense of Busemann, then Ω is bi-holomorphic to the unit ball.

This is true for the Hilbert metric [40].

4.8.4 The Automorphism Group of Convex Domains

It is generally believed that the bi-holomorphism group of convex domains is very small. Probably the most natural generalization of Frankel's rigidity theorem is the following:

Question 4.8.11 Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain and there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts freely and properly so that the quotient manifold $\Gamma \backslash \Omega$ has finite volume. Is Ω a bounded symmetric domain?

Remark 4.8.12 Notice that Corollary 4.3.3 implies that the quotient having finite volume with respect to one "natural" notion of volume implies it has finite volume with respect to any "natural" notion of volume.

Another natural question is the following:

Problem 4.8.13 Characterize the bounded convex domains $\Omega \subset \mathbb{C}^d$ where $\partial\Omega$ has C^∞ boundary and $\mathrm{Aut}(\Omega)$ is non-compact.

There are many partial answers to the problem, see [61] and the references therein.

It also seems natural to try and understand the relationship between the biholomorphism group and the isometry group:

Question 4.8.14 If Ω is a bounded convex domain, does $\mathrm{Aut}(\Omega)$ have finite index in $\mathrm{Isom}(\Omega, k_\Omega)$, the group of isometries of (Ω, k_Ω)?

This is known to be true when Ω is strongly convex and has C^3 boundary [30]. It is also known that when $\mathscr{C} \subset \mathbb{R}^d$ is a properly convex domain, then $\mathrm{Aut}_{\mathrm{proj}}(\mathscr{C})$ has finite index in $\mathrm{Isom}(\mathscr{C}, H_{\mathscr{C}})$, see [57].

If X is a proper CAT(0) metric space, then one characterize isometries of X as being elliptic, parabolic, or hyperbolic based on their dynamics. In seems natural to ask if one can characterize the elements of $\mathrm{Aut}(\Omega)$ via their action on Ω.

Problem 4.8.15 For a bounded convex domain $\Omega \subset \mathbb{C}^d$, define what it means for an element $\varphi \in \mathrm{Aut}(\Omega)$ to be hyperbolic, elliptic, or parabolic via their dynamics on Ω.

This was done when Ω has $C^{1,\alpha}$ boundary in [61].

Acknowledgements This material is based upon work supported by the National Science Foundation under Grant Number NSF 1400919.

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Chapter 5
Quasi-Conformal Mappings

Hervé Pajot

In this chapter, we first give a brief overview of the classical theory of quasiconformal mappings in the complex plane and then we explain how to extend it in general metric spaces (under geometric assumptions). Applications to complex dynamics and to (complex) hyperbolic geometry are also discussed.

5.1 Introduction to Quasiconformal Geometry in the Complex Plane

5.1.1 Quasiconformal Mappings

There are different possible definitions of quasiconformality in the complex plane. The basic idea is that f is quasiconformal if f maps infinitesimal circles to ellipses whose eccentricity is uniformly bounded (see Appendix A.4). We present here some of them, starting with the analytic definitions. The reader should have in mind that we would like to extend all these definitions in abstract spaces (and for some of them, this is not obvious, see the remarks below).

Let f be an orientation preserving homeomorphism in $\mathbb{C}$.

- **(A1)** f belongs to the Sobolev space $W^{1,2}_{loc}(\mathbb{C})$ and there exists $\mu \in L^\infty(\mathbb{C})$ with $||\mu||_\infty < 1$ so that $\partial_{\bar{z}} f = \mu \partial_z f$ in the sense of distributions (that is f satisfies the Beltrami equation, see the next section for more details about this). Recall that the space of distributions is the topological dual of C^∞ functions with compact support and that the derivative of a distribution T in direction θ is $\partial_\theta T$ given

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7_5

by $< \partial_\theta T, \phi >= - < T, \partial_\theta \phi >$ whenever ϕ is a C^∞ function with compact support. Here, $\partial_\theta \phi$ is the classical derivative (in the direction θ). The Sobolev space $W^{1,2}_{loc}(\mathbb{C})$ is the space of locally integrable functions with partial derivatives in $L^2_{loc}(\mathbb{C})$.

- **(A2)** f is ACL (*absolutely continuous on lines*) and there exists $K \geq 0$ so that

$$\max_{|\xi|=1} |df_z(\xi)| \leq K|Jf(z)| \tag{5.1}$$

for almost every z. Here, df_z is the differential of f at $z \in \mathbb{C}$ and $Jf(z)$ is the determinant of the Jacobian matrix of f at $z \in \mathbb{C}$. We recall that $f : \mathbb{C} \to \mathbb{C}$ is ACL if f is continuous and if for any rectangle $[a, b] \times [c, d]$, for almost every $x \in [a, b]$ (respectively almost any $y \in [c, d]$), the map $f_x : y \in [c, d] \to f(x + iy)$ (respectively $f_y : x \in [a, b] \to f(x + iy)$) is absolutely continuous in the usual sense. By a result of Gehring-Lehto, any ACL homeomorphism $f : \mathbb{C} \to \mathbb{C}$ is almost everywhere differentiable. Hence, the previous estimate (5.1) makes sense almost everywhere.

Note that the first challenge to extend these definitions in general spaces is to give the right definitions of Jacobians, derivatives, Sobolev spaces, and so on in this setting.

- **(G1)** There exists $K > 0$ so that for any family Γ of rectifiable curves in $\mathbb{C}$, we have

$$K^{-1}\Lambda(\Gamma) \leq \Lambda(f\Gamma) \leq K\Lambda(\Gamma).$$

Here $f\Gamma$ is the set of curves $\{f \circ \gamma, \gamma \in \Gamma\}$ and $\Lambda(\Gamma)$ is the extremal length of the curve family Γ which is a conformal invariant. We now give the definition of the extremal length and for this, we say that a Borel function $\rho : \mathbb{C} \to \mathbb{R}^+$ is admissible if $A(\rho) = \int\int_{\mathbb{R}^2} \rho^2 \mathrm{d}x\mathrm{d}y \neq 0, \infty$. For such ρ, the ρ-length of $\gamma \in \Gamma$ is $L_\rho(\gamma) = \int_\gamma \rho|\mathrm{d}z|$ if ρ is measurable along γ and $L_\rho(\gamma) = +\infty$ otherwise. Finally, we set $L(\rho) = \inf_{\gamma\in\Gamma} L_\rho(\gamma)$ and $\Lambda(\Gamma) = \sup_\rho L(\rho)^2/A(\rho)$ where the supremum is taken over all admissible functions ρ.

For instance, if Γ is the set of curves joining the circle $\partial\Delta(0, r_1)$ to the circle $\partial\Delta(0, r_2)$ (where $r_1 < r_2$), then $\Lambda(\Gamma) = (1/2\pi)\log(r_1/r_2)$ is the extremal length of an annulus r_2/r_1.

- **(G2)** There exists $K > 0$ so that for any family Γ of curves in $\mathbb{C}$, we have

$$K^{-1}\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f\Gamma) \leq K\mathrm{mod}(\Gamma).$$

We say that a Borel function $\rho : \mathbb{C} \to \overline{\mathbb{R}^+}$ is *admissible for* Γ if for any (locally) rectifiable curve $\gamma \in \Gamma$, $\int_\gamma \rho \mathrm{d}s \geq 1$. Then, the *conformal modulus* is given by

$\text{mod}(\Gamma) = \inf_{\rho} \int_{\mathbb{C}} \rho^2 \text{d}s$ where the infimum is taken over all admissible functions ρ. The fact that the conformal modulus is a conformal invariant will be proved later in the Riemannian setting.

If Γ is the set of curves joining the circle $\partial\Delta(0, r_1)$ to the circle $\partial\Delta(0, r_2)$ (where $r_1 < r_2$), then the conformal modulus of an annulus r_2/r_1 is $\text{mod}(\Gamma) = 2\pi(\log(r_1/r_2))^{-1}$, that is $\text{mod}(\Gamma) = \Lambda(\Gamma)^{-1}$. This last equality is a general fact, so $(G1) \iff (G2)$. We will discuss more generally the notion of modulus later in the general setting of metric spaces (whereas the notion of extremal length does not make sense in metric spaces).

- **(QS1)** There exists a continuous increasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ so that $\eta(0) = 0$ and if $|z - w_1| \leq t|z - w_2|$ for some $t \geq 0$ and any z, w_1, w_2 in $\mathbb{C}$, then $|f(z) - f(w_1)| \leq \eta(t)|f(z) - f(w_2)|$. This condition is equivalent to the following weaker condition: There exists a constant $K \geq 1$ so that $|f(z) - f(y)| \leq K|f(z) - f(x)|$ whenever $|z - y| \leq |z - x|$.
- **(QS2)** There exists $H \geq 0$ so that for any $z \in \mathbb{C}$ and any $r > 0$, $L_f(x, r) \leq Hl_f(x, r)$ where $L_f(z, r) = \sup\{|f(z) - f(w)|; |z - w| \leq r|\}$ and $l_f(x, r) = \inf\{|f(w) - f(z)|; |z - w| \geq r\}$.
- **(M1)** There exists $H \geq 0$ so that for any $z \in \mathbb{C}$, $\limsup_{r\to 0} L_f(z, r)/l_f(z, r) \leq H$. Such homeomorphisms with this property are usually said to be quasiconformal.
- **(M2)** There exists $H \geq 0$ so that for any $z \in \mathbb{C}$, $\liminf_{r\to 0} L_f(z, r)/l_f(z, r) \leq H$.

(A1) and (A2) (respectively (G1) and (G2)) (respectively (QS1) and (QS2)) (respectively (M1) and (M2)) are analytic (respectively geometric) (respectively quasisymmetric) (respectively metric) definitions of quasiconformality.

Theorem 5.1.1 *Let $f : \mathbb{C} \to \mathbb{C}$ be an orientation preserving homeomorphism. Then, the conditions (A1), (A2), (G1), (G2), (QS1), (QS2), (M1) and (M2) are quantitatively equivalent (and in this case, we say that f is quasiconformal).*

It is obvious from (G1) or (G2) that f^{-1} is quasiconformal if f is so. The proof of these equivalences was a long story. The original proofs use classical tools from real analysis (Lebesgue differentiation theorem, the Hardy-Littlewood maximal function, ...) but also arguments from complex analysis. A similar statement is also true in Euclidean spaces in higher dimension but this needs other arguments. A key point is that the definition of quasiconformality (M1) is local whereas the metric definition of quasisymmetry (QS2) is global. The implication (QS2) $\Longrightarrow$ (M1) is obvious, but the proof of the converse was one of the major challenge in the theory. The ubiquity of quasiconformal mappings explains that they are useful in several areas of mathematics. We will explain in Sect. 5.1.3 a very famous application to complex dynamics. Note that the implication $(M2) \Longrightarrow (M1)$ was discovered by Heinonen and Koskela in their study of quasiconformal mappings ... in the Heisenberg group. The equivalence of these definitions (except (G1) and (M2)) in the setting of Carnot groups and in metric spaces with controlled geometry was also a "tour de force". We will come later to this point.

5.1.2 The Beltrami Equation and the Measurable Riemann Mapping Theorem

Let μ be a bounded measurable function on a domain Ω of the complex plane (or of the Riemann sphere $\mathbb{C}_\infty$). We assume that $||\mu||_\infty < 1$. The Beltrami equation is the partial differential equation given by

$$f_{\bar{z}} = \mu f_z$$

In the case where μ is identically zero, we recover the Cauchy-Riemann equation. The next result is a generalization of the classical Riemann theorem.

Theorem 5.1.2 *For any $\mu \in L^\infty(\mathbb{C})$ with $||\mu||_\infty < \infty$, there exists a unique quasiconformal homeomorphism f^μ that extends to $\mathbb{C}_\infty$ which fixes 0, 1 and ∞, and is solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$.*

There are several proofs of this result. For most of them, the first step is to consider the case where μ is smooth with compact support for instance and then to conclude by approximation argument. The proof by Douady is based on Fourier analysis in the complex plane. The first proof is due to Morrey, whereas the use of the Calderon-Zygmund theory of singular integral operators for the Beltrami equation was initiated by Bojarski. Roughly speaking, singular integral operators are of the form

$$Tf(z) = \int K(w,z)f(w)\mathrm{d}w$$

where the kernel K has some controlled singularities on the diagonal $z = w$. A typical example is the Hilbert operator H on $\mathbb{R}$ which is formally given by $Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y}\mathrm{d}y$ which is a convolution-type operator. Under reasonable conditions, Calderòn and Zygmund proved that if T extends as a bounded operator on L^p with $1 < p < \infty$, the same is true for any $1 < q < \infty$. The case $p = 2$ plays an important role because it could be studied by classical Fourier analysis. The $T(1)$ theorem of David-Journé and the $T(b)$-theorem of David-Journé-Semmes provide useful criterions of L^2-boundedness. The operator considered by Ahlfors-Bers in their proof is given formally by $Tf(z) = 1/\pi \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\mathrm{d}w$. It turns out that in this case the operator can be well defined as a principal value (for smooth function f):

$$Tf(z) = 1/\pi \lim_{\varepsilon\to 0} \int_{|z-w|\geq\varepsilon} \frac{f(w)}{(z-w)^2}\mathrm{d}w.$$

What happens in the case $||\mu||_\infty = 1$? A solution of the Beltrami equation is not quasiconformal, but is BMO-quasiconformal. An interesting point is that the solution of the Beltrami equation in this case is not in general in the Sobolev space $W^{1,2}$. Recall that a function g is in BMO(Ω) where Ω is a domain in $\mathbb{C}$ (BMO means Bounded Mean Oscillation) if there exists $C \geq 0$ so that for any disc D with $\overline{D} \subset \Omega$, we have $(*) \frac{1}{\nu(D)} \int_D |f - f_D| \mathrm{d}z \leq C$ where $\nu(D)$ is the Lebesgue measure of D and f_D is the mean value of f on D: $f_D = \fint_D f(z)dz = \frac{1}{\nu(D)} \int_D f(z)\mathrm{d}z$. The best constant C so that $(*)$ holds is the BMO norm of f, denoted by $||f||_{BMO}$. The space BMO is bigger than L^∞ and is in fact the dual of L^1. In higher dimensions, the space BMO($\mathbb{R}^n$) can be defined in a similar way and a famous result of C. Fefferman states that BMO($\mathbb{R}^n$) is the dual of the Hardy space $H^1(\mathbb{R}^n)$. We say that the homeomorphism f is BMO-*quasiconformal* on Ω if f is ACL on Ω and if there exists $\phi : \Omega \to [1, +\infty[$ in BMO so that $K_f = \frac{1 + |f|}{1 - |f|} \leq \phi$ almost everywhere.

There is an analog of the measurable Riemann mapping theorem in this setting.

Theorem 5.1.3 *Let Ω be a domain in $\mathbb{C}$ and let $\mu \in L^\infty(\Omega)$ with $||\mu||_\infty = 1$. Assume furthermore that there exists $\phi \in$ BMO(Ω) so that $\frac{1 + |\mu|}{1 - |\mu|} \leq \phi$ almost everywhere in Ω. Then, there exists a* BMO-*quasiconformal homeomorphism f on Ω so that $\partial_{\bar{z}} f = \mu \partial_f f$. Moreover, if g is another solution of this generalized Beltrami equation on Ω, $f \circ g^{-1}$ is holomorphic.*

A first version of this result is due to G. David who considered Beltrami coefficients μ which satisfy the so-called logarithmic condition, that is there exist positive constants C and α such that for ε small enough,

$$\nu(z; |\mu(z)| > 1 - \varepsilon) \leq C e^{-\alpha/\varepsilon}$$

where ν is the Lebesgue measure. This condition gives a control of the size of the bad set, that is the set of z so that $|\mu(z)|$ is closed to 1. The relationship between the logarithmic condition and the BMO condition comes from the John-Niremberg inequality that we recall. If f is a function in BMO($\mathbb{R}^n$), there exists positive constants C_1, C_2 so that

$$\nu(\{x \in Q, |f - f_Q| > \lambda\}) \leq C_1 \exp\left(-C_2 \frac{\lambda}{||f||_{\mathrm{BMO}}}\right) \nu(Q).$$

whenever Q is a cube in $\mathbb{R}^n$ and f_Q denotes the mean value of f on Q, that is $f_Q = \frac{1}{\nu(Q)} \int_Q f(x)dx$. In fact, the previous estimate characterizes the space BMO($\mathbb{R}^n$).

As we will see in the next subsection, the measurable Riemann mapping theorems (5.1.2 and 5.1.3) have nice applications in complex dynamics.

5.1.3 An Application to Complex Dynamics: The Non-Wandering Theorem of Sullivan

Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function on the Riemann sphere $\mathbb{C}_\infty$. The orbit of $z_0 \in \mathbb{C}_\infty$ is the collection of points $f^n(z_0)$ where for any $n \in \mathbb{N}, f^n = f \circ \ldots \circ f$ n times (f^n is the n-th iteration of f). A classical problem in complex dynamics is to study the nature of the orbits. The *Fatou set* of f denoted by $\mathscr{F}_f$ is the set of points z_0 such that there exists a neighborhood U_{z_0} of z_0 where the sequence (f^n) is a normal family (recall that this means that any sequence in (f^n) has a subsequence which converges uniformly on compact subsets of U_{z_0}). The complement of the Fatou set is the *Julia set* $\mathscr{J}_f = \mathbb{C}_\infty \setminus \mathscr{F}_f$. The key point is that the orbit of z_0 in the Fatou set is stable under perturbations of z_0 whereas the dynamical behaviour of the orbit of z_0 in the Julia set is sensible dependent of the initial data. The Fatou set is an open totally invariant set, in the sense that $f(\mathscr{F}_f) = f^{-1}(\mathscr{F}_f) = \mathscr{F}_f$. This implies that the Julia set is a totally invariant closed set.

Definition 5.1.4 The point z_0 is said to be *periodic* if there exists $p \in \mathbb{N}^*$ so that $f^p(z_0) = z_0$. The set of points $\{z_0, \ldots, z_p\}$ is called a *cycle*. Here, $z_j = f^j(z_0)$. The *multiplicity* λ of this cycle is defined by

$$\lambda = (f^p)'(z_0) = f'(z_0)f'(z_1)\ldots f'(z_{p-1}).$$

When $|\lambda| > 1$, the cycle is said to be *repelling*. If $|\lambda| < 1$, the cycle is said to be *attracting* (and *super-attracting* in the special case $\lambda = 0$).

All the repelling cycles are contained in the Julia set $\mathscr{J}_f$. Attracting cycles are contained in the Fatou set $\mathscr{F}_f$. To each point z_i of an attracting cycle, we can associate its *basin of attraction* given by $A(z_i) = \{w \in \mathbb{C}_\infty; \lim_{n\to+\infty} f^{np}(w) = z_i\}$ which is contained in the Fatou set. We denote by $A^*(z_i)$ the connected component of $A(z_i)$ that contains z_i. More generally, a Fatou component is a connected component of the Fatou set. A key point is that the image by f of a Fatou component Ω is again a Fatou component Ω'. Moreover, $f : \Omega \to \Omega'$ is proper. This implies that if Ω_1, Ω_2 are two Fatou components of f with $f(\Omega_1) \cap \Omega_2 \neq \emptyset$ then $f(\Omega_1) = \Omega_2$.

Consider now a Fatou component Ω of f and set for any $n \in \mathbb{N}$, $\Omega_n = f^n(\Omega)$. There are two cases.

- Ω is a *preperiodic component* , that is $\Omega_{m+p} = \Omega_m$ for some $m \geq 0$ and some $p \geq 1$ (the case $m = 0$ corresponds to the periodic case).
- Ω is a *wandering component*, that is $\Omega_m \neq \Omega_p$ for any $m \neq p$.

The classification of preperiodic components is known from the work of Fatou and Cremer. We can now state the beautiful result of Sullivan which is related to the existence of wandering components.

Theorem 5.1.5 *The Fatou set of a rational function of degree $d \geq 2$ does not have a wandering component.*

We will not give the complete proof of the result which is quite long and complicated. But we would like to illustrate how quasiconformal mappings are useful. Let f be rational function on the Riemann sphere $\mathbb{C}_\infty$.

Definition 5.1.6 A *quasiconformal deformation of* f is a rational function $\tilde{f}$ of the form $\tilde{f} = h \circ f \circ h^{-1}$ where h is a quasiconformal homeomorphism of $\mathbb{C}_\infty$.

Assume that Ω is a wandering component of f. To construct quasiconformal deformations, the idea is to solve the Beltrami equation $h_{\bar{z}} = \mu h_z$ where $\mu \in L^\infty(\Omega)$ satisfies $||\mu||_\infty < 1$ and should be chosen carefully. It turns out that it is possible to construct a space of quasiconformal deformations (with degree d) with arbitrary high dimension. This is impossible since the space of rational functions of degree d could be identified (locally) to $\mathbb{C}^{2d}$.

5.2 Quasiconformal Mappings in Metric Spaces

5.2.1 Modulus of a Curve Family

We would like to extend the notion of modulus of a curve family (as considered in the definition (G2) of quasiconformality in the complex plane) to metric spaces. Let (X, d) be a metric space. We assume that (X, d) is equipped with an outer measure μ. Furthermore, we assume that μ is a Borel measure (that is Borel sets are μ-measurable) and that $0 < \mu(B) < \infty$ for any ball B in X. A curve γ in X is given by a continuous map $\gamma : I \to X$ where I is an interval in $\mathbb{R}$. Most of the time, we identify the map γ with its image $\gamma(I)$. In the case where $I = [a, b]$ is a closed interval, we define the length $l(\gamma)$ of γ by

$$l(\gamma) = \sup \sum_{j=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

where the supremum is taken over all subdivisions $t_0 = a < t_1 \ldots. < t_N = b$ of $[a, b]$. If I is unbounded or not closed, we set $l(\gamma) = \sup l(\gamma_{|J})$ where the supremum is taken over all closed bounded subintervals J of I. We say that γ is a rectifiable curve if its length is finite. In a similar way, we say that $\gamma : I \to X$ is locally rectifiable if its restriction to any closed subinterval J of I is rectifiable. If γ is a rectifiable curve in X, we can write (in an unique way) $\gamma = \gamma_s \circ s_\gamma$ where $s_\gamma : I \to [0, l(\gamma)]$ is the length function of γ and $\gamma_s : [0, l(\gamma)] \to X$ is a 1-lipschitz function called the arc length parametrization of γ. In this case, if $f : X \to [0, +\infty]$ is a nonnegative Borel function, we set

$$\int_\gamma f ds = \int_0^{l(\gamma)} f \circ \gamma_s(t) dt.$$

If the curve γ is only locally rectifiable, we set

$$\int_\gamma f ds = \sup_{\tilde{\gamma}} \int_{\tilde{\gamma}} f ds$$

where the supremum is taken over all rectifiable subcurves $\tilde{\gamma}$ of γ.

We come back to the modulus of curves and we first discuss the Euclidean/ Riemannian case. Let (M, g) be a Riemannian manifold of dimension n (for instance, $M = \mathbb{R}^n$ equipped with its Euclidean structure). We denote by vol_g its Riemannian volume (that is the Lebesgue measure in $\mathbb{R}^n$).

Definition 5.2.1 If Γ is a curve family in M, the *conformal modulus of* Γ is defined by

$$\mathrm{mod}(\Gamma) = \inf \int_M \rho^n \mathrm{d}vol_g$$

where the infimum is taken over all Borel functions $\rho : M \to [0, +\infty]$ such that $\int_\gamma \rho(s)\mathrm{d}s \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$. Such a ρ is called *admissible for the curve family* Γ.

The key point is that the conformal modulus is conformally invariant.

Theorem 5.2.2 *Let M, N be two Riemannian manifolds of dimension n. If $f : M \to N$ is conformal, then* $\mathrm{mod}(f\Gamma) = \mathrm{mod}(\Gamma)$ *for all curve family Γ in M, where $f\Gamma = \{f(\gamma), \gamma \in \Gamma\}$.*

Proof Let σ be an admissible function for $f\Gamma$, and let us set

$$\rho = (\sigma \circ f) \cdot \|df\| .$$

If $\gamma \in \Gamma$ is (locally) rectifiable, it follows from the change of variables provided by f that

$$\int_\gamma \rho \mathrm{d}s = \int_\gamma (\sigma \circ f) \cdot \|df\| \mathrm{d}s = \int_{f(\gamma)} \sigma \mathrm{d}s$$

so that ρ is admissible as well.

We note that since f is conformal, $\|df\|^n = |Jf|$ at all points (this can be seen as the definition of conformality). Therefore,

$$\begin{aligned}\int_M \rho^n \mathrm{d}vol_g &= \int_M (\sigma \circ f)^n \|df\|^n \mathrm{d}vol_g \\ &= \int_M (\sigma \circ f)^n Jf \mathrm{d}vol_g \\ &= \int_N \sigma^n \mathrm{d}vol_g\end{aligned}$$

Hence

$$\mathrm{mod}(\Gamma) \leq \int_M \rho^n \mathrm{d}vol_g = \int_N \sigma^n \mathrm{d}vol_g$$

so that $\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f\Gamma)$ and the reverse inequality is obtained by symmetry.

Example 5.2.3 We now give a standard computation of modulus in the Euclidean case for the ring. Let Γ be the family of curves in $\mathbb{R}^n$ joining the two boundary components of the annular region $\mathbb{B}(x_0, R) \setminus \overline{\mathbb{B}(x_0, r)} = \{x; r < |x - x_0| < R\}$ (where $x_0 \in \mathbb{R}^n$, $0 < r < 2R$). Then, if we denote by ω_{n-1} the area of the unit sphere S^{n-1} in $\mathbb{R}^n$,

$$\mathrm{mod}(\Gamma) = \omega_{n-1} \left(\log \left(\frac{R}{r} \right) \right)^{1-n}.$$

Proof By conformal invariance of the modulus, we can assume that $x_0 = 0$. We first check that the function defined by $\rho(x) = (\log(R/r))^{-1} |x|^{-1}$ if $r < |x| < R$ and $\rho(x) = 0$ elsewhere is admissible for Γ. To see this, consider $\gamma : [a, b] \to X$ a rectifiable curve of Γ and denote by $[\gamma(a), \gamma(b)]$ the Euclidean (closed) segment joining $\gamma(a)$ and $\gamma(b)$. Since ρ is radial, we get

$$\begin{aligned}\int_\gamma \rho(s) \mathrm{d}s &\geq \int_{[\gamma(a),\gamma(b)]} \rho(s) \mathrm{d}s \\ &\geq \int_r^R (\log(R/r))^{-1} \frac{\mathrm{d}t}{t} \\ &= 1.\end{aligned}$$

This implies (by using a change in polar coordinates) that

$$\begin{aligned}\operatorname{mod}(\Gamma) &\le \int_{\mathbb{B}(0,R)\setminus\mathbb{B}(0,r)} \rho(x)^n \mathrm{d}x \\ &= (\log(R/r))^{-n} \int_{S^{n-1}} \left(\int_r^R \frac{\mathrm{d}t}{t}\right) \mathrm{d}w \\ &= \omega_{n-1} \left(\log(R/r)\right)^{1-n}.\end{aligned}$$

To prove the reverse inequality, consider an admissible function ρ for Γ. Without loss of generality, we can assume that $\rho(x) = 0$ if $|x| \ge R$ or if $|x| \le r$. For any $w \in S^{n-1}$, we have

$$\begin{aligned}1 &\le \int_r^R \rho(tw)\mathrm{d}t \text{ (since } \rho \text{ is admissible)} \\ &= \int_r^R \rho(tw) t^{\frac{n-1}{n}} t^{\frac{1-n}{n}} \mathrm{d}t \\ &\le \left(\int_r^R \rho(tw)^n t^{n-1} \mathrm{d}t\right)^{1/n} \left(\int_r^R t^{-1}\mathrm{d}t\right)^{\frac{n-1}{n}} \quad \text{(by Hölder inequality)} \\ &\le \left(\int_r^R \rho(tw)^n t^{n-1} \mathrm{d}t\right)^{1/n} \left(\log(R/r)\right)^{\frac{n-1}{n}}.\end{aligned}$$

Hence,

$$\int_r^R \rho(tw)^n t^{n-1} \mathrm{d}t \ge \left(\log(R/r)\right)^{1-n}.$$

This implies

$$\begin{aligned}\int_{\mathbb{B}(0,R)\setminus\mathbb{B}(0,r)} \rho^n(x)\mathrm{d}x &= \int_{S^{n-1}} \left(\int_r^R \rho(tw)^n t^{n-1}\mathrm{d}t\right) \mathrm{d}w \\ &\ge \omega_{n-1} \left(\log(R/r)\right)^{1-n}.\end{aligned}$$

So, $\operatorname{mod}(\Gamma) \ge \omega_{n-1} \left(\log(R/r)\right)^{1-n}$ and the proof is complete.

We now consider the general case.

Definition 5.2.4 Let (X, d, μ) be a metric measure space. Let Γ be a curve family in X and let $p \ge 1$. We define the *p-modulus of* Γ by

$$\operatorname{mod}_p(\Gamma) = \inf \int_X \rho^p \mathrm{d}\mu$$

where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ such that $\int_\gamma \rho(s)\mathrm{d}s \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$. Such a ρ is called *admissible for the curve family* Γ.

Remark 5.2.5 The p-modulus of the family Γ of all curves that are not locally rectifiable is zero, since any function $\rho : X \to [0, +\infty]$ is admissible for Γ.

Remark 5.2.6 If Γ contains a constant curve, then there is no admissible function for Γ, and (by convention) $\mathrm{mod}_p(\Gamma) = +\infty$.

Theorem 5.2.7 *Let* (X, d, μ) *be a metric measure space and let* $p \geq 1$.

(i) $\mathrm{mod}_p(\emptyset) = 0$.
(ii) *If* Γ_1, Γ_2 *are two curve families in* X *with* $\Gamma_1 \subset \Gamma_2$, *then* $\mathrm{mod}_p(\Gamma_1) \leq \mathrm{mod}_p(\Gamma_2)$.
(iii) *If* $(\Gamma_i)_{i\in\mathbb{N}}$ *is a countable collection of curve families in* X, *then* $\mathrm{mod}_p(\cup_i \Gamma_i) \leq \sum_i \mathrm{mod}_p(\Gamma_i)$.
(iv) *If* Γ, $\tilde{\Gamma}$ *are two curve families in* X *such that each curve* $\gamma \in \Gamma$ *has a subcurve* $\tilde{\gamma} \in \tilde{\Gamma}$, *then* $\mathrm{mod}_p(\Gamma) \leq \mathrm{mod}_p(\tilde{\Gamma})$.

Note that properties (i), (ii) and (iii) imply that the p-modulus is an outer measure on the set of all curves in X (but in general there is no nontrivial measurable family of curves !). In the case of a Riemannian manifold of dimension n (for instance in $\mathbb{R}^n$), the conformal modulus corresponds to the case $p = n$. But, in general metric spaces, it is more convenient to consider a family of moduli.

Proof The proofs of (i) and (ii) are left to the reader as (easy) exercises. The proof of (iv) follows from the fact that, if ρ is admissible for $\tilde{\Gamma}$, then ρ is admissible for Γ. We now prove (iii). Without loss of generality, we can assume that $\mathrm{mod}_p(\Gamma_i) < \infty$ for any $i \in \mathbb{N}$ (otherwise the conclusion is clear). Fix $\varepsilon > 0$. For any $i \in \mathbb{N}$, choose ρ_i an admissible function for Γ_i so that $\int_X \rho_i^p \mathrm{d}\mu \leq \mathrm{mod}_p(\Gamma_i) + \varepsilon/2^i$, and set $\rho = \left(\sum_{i\in\mathbb{N}} \rho_i^p\right)^{1/p}$. Since $\rho \geq \rho_i$ for any $i \in \mathbb{N}$, ρ is admissible for $\cup_i \Gamma_i$, thus

$$\begin{aligned}
\mathrm{mod}_p(\cup_i \Gamma_i) &\leq \int_X \rho^p \mathrm{d}\mu \\
&\leq \sum_i \int_X \rho_i^p \mathrm{d}\mu \\
&\leq \sum_i \mathrm{mod}_p(\Gamma_i) + \varepsilon \sum_i 1/2^i \\
&= \sum_i \mathrm{mod}_p(\Gamma_i) + 2\varepsilon.
\end{aligned}$$

Since the last inequality is true for any $\varepsilon > 0$, we conclude by taking $\varepsilon \to 0$.

We now investigate the example of the modulus of a ring, in the general case. Assume that (X, d, μ) is a metric measure space such that there exist a constant

$C_0 \geq 0$ and an exponent $Q > 1$ with $\mu(B(x,R) \leq C_0 R^Q$ whenever $x \in X$, $R > 0$. In some sense, Q will play the role of the dimension and we will consider the Q-modulus.

Lemma 5.2.8 *Let Γ be the family of curves joining $\overline{B(x_0,r)}$ to $X \setminus B(x_0/R)$ (where $x_0 \in X$, $0 < r < 2R$). Then there exists a constant $C \geq 0$ (depending only on C_0 and on Q) so that*

$$\operatorname{mod}_Q(\Gamma) \leq C\left(\log\left(\frac{R}{r}\right)\right)^{1-Q}.$$

In general metric spaces, we can not expect to have a lower bound (as in the Riemannian case). Spaces for which such a bound exists will be defined later (they are called Loewner spaces). Note also that this modulus estimate implies that the Q-modulus of a family of curves passing through a given point is zero.

Proof As previously, set $\rho(x) = (\log(R/r))^{-1}\, d(x_0,x)^{-1}$ for $r < d(x_0,x) < R$ and $\rho(x) = 0$ elsewhere. Then, ρ is admissible for Γ and if we denote by N the least integer such that $2^{N+1}r \geq R$, we have

$$\begin{aligned}
\operatorname{mod}_Q(\Gamma) &\leq \int_X \rho^Q \mathrm{d}\mu \leq (\log(R/r))^{-Q} \sum_{j=0}^{N} \int_{\{2^j r \leq d(x_0,x) \leq 2^{j+1} r\}} d(x_0,x)^{-Q} \mathrm{d}\mu \\
&\leq (\log(R/r))^{-Q} \sum_{j=0}^{N} (2^j r)^{-Q} \mu(B(x_0, 2^{j+1} r)) \\
&\leq (\log(R/r))^{-Q} \sum_{j=0}^{N} (2^j r)^{-Q} (C_0 (2^{j+1} r)^Q) \\
&\leq 2^Q C_0 (N+1)\, (\log(R/r))^{-Q} \\
&\leq 10 \cdot 2^Q C_0\, (\log(R/r))^{1-Q} \text{ (since } 2^N r < R).
\end{aligned}$$

5.2.2 Quasi-Conformal Homeomorphisms

Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be two metric measure spaces and let $f : X \to Y$ be a homeomorphism. For any $x \in X$, any $R > 0$, set

$$\begin{aligned}
L_f(x,R) &= \sup\{d_Y(f(x), f(y)); d_X(x,y) \leq R\}, \\
l_f(x,R) &= \inf\{d_Y(f(x), f(y)); d_X(x,y) \geq R\}, \\
H_f(x,R) &= \frac{L_f(x,R)}{l_f(x,R)}.
\end{aligned}$$

Definition 5.2.9 We say that f is

- *quasiconformal* (QC) if there exists $H \geq 1$ so that $\limsup_{R \to 0} H_f(x, R) \leq H$ for any $x \in X$;
- *weakly quasisymmetric* (WQS) if there exists $H \geq 1$ so that $H_f(x, R) \leq H$ for any $x \in X$ and any $R > 0$.

Roughly speaking, QC-homeomorphisms (respectively WQS-homeomorphisms) distort the shape of infinitesimal balls (respectively every balls) by a uniformly bounded amount. It is clear that (WQS) $\Rightarrow$ (QC).

Another way to define quasiconformality is to use the "3 points condition" of Tukia and Väisälä:

Definition 5.2.10 The homeomorphism f is *quasisymmetric* (QS) if there exists a homeomorphism $\eta_f = \eta : [0, +\infty) \to [0, +\infty)$ so that

$$d_X(x, a) \leq t d_X(x, b) \Rightarrow d_Y(f(x), f(a)) \leq \eta(t) d_Y(f(x), f(b))$$

whenever $a, b\, x \in X$ and $t > 0$.

Example 5.2.11 Any bilipschitz homeomorphism (with constant L) is QS (and in this case $\eta(t) = L^2 t$). Recall that $f : (X, d_X) \to (Y, d_Y)$ is bilipschitz if there exists $L > 0$ so that $L^{-1} d_X(x, x') \leq d_Y(f(x), f(x')) \leq L d_X(x, x')$ whenever $x, x' \in X$.

Remark 5.2.12 The conditions (QC), (WQS) and (QS) correspond respectively to the conditions (M1), (QS2) and (QS1) given previously in the complex plane. We follow here the (almost) usual terminology in the general setting of metric spaces.

It is easy to see that the inverse of a QS-homeomorphism is QS ($\eta_{f^{-1}}(t) = 1/\eta^{-1}(t^{-1})$) and that the composition of two QS-homeomorphims is QS ($\eta_{f \circ g} = \eta_f \circ \eta_g$). Note that these properties are not obvious for QC-homeomorphisms or WQS-homeomorphisms. The implication (QS) $\Rightarrow$ (WQS) is always true whereas the implication (WQS) $\Rightarrow$ (QS) holds if μ_X and μ_Y are doubling. Recall that a measure μ on a metric space (X, d) is *doubling* if there exists $C_{DV} \geq 1$ so that $\mu(B(x, 2r)) \leq C_{DV} \mu(B(x, r))$ whenever $x \in X$ and $R > 0$.

5.2.3 Loewner Spaces

We now introduce spaces with controlled geometry, that are spaces for which we have an upper bound for the modulus of some curve families. Throughout all this subsection, we assume that for all $x, y \in X$, there exists a rectifiable curve joining x and y. We start with some notations. Let E and F be two non degenerate continua in X. *Non degenerate* means that E and F contain more than one point and we recall that a *continuum* in X is a compact connected subset of X. We denote by $\Gamma(E, F)$

the family of all curves joining E and F in X. We set $\mathrm{mod}_p(E,F) = \mathrm{mod}_p(\Gamma(E,F))$. Finally, we denote by $\Delta(E,F)$ the relative distance between E and F, that is

$$\Delta(E,F) = \frac{d(E,F)}{\min(\mathrm{diam}E, \mathrm{diam}F)}.$$

Definition 5.2.13 Let $Q > 1$ (this assumption is important since $\mathbb{R}$ is not a Loewner space). We say that X (or (X,d,μ)) satisfies the *Loewner property of exponent Q* if there exists a function $\phi :]0,+\infty[\to]0,+\infty[$ so that

$$\mathrm{mod}_Q(E,F) \geq \phi(\Delta(E,F))$$

for any disjoint non degenerate continua E and F in X.
Moreover, if μ is Q-regular, ϕ could be chosen so that

$$\phi(t) \sim \log(1/t) \ \text{ when } t \to 0,$$

$$\phi(t) \sim (\log t)^{1-Q} \ \text{ when } t \to +\infty.$$

Recall that a measure μ on X is *Q-regular* if there exists a constant $C_{AR} > 0$ so that

$$C_{AR}^{-1} R^Q \leq \mu(B(x,R)) \leq C_{AR} R^Q$$

whenever $x \in X$, $R \in]0, \mathrm{diam}X[$. In particular, an Ahlfors-regular measure μ is doubling (this notion is defined in the previous subsection). The Hausdorff dimension of a metric space (X,d) equipped with a Q-Ahlfors measure is Q. In these notes, a Q-Loewner space is a metric measure space (X,d,μ) so that X satisfies the Loewner condition of exponent Q and μ is Q-regular. **This terminology is not exactly the usual one**.

Since $Q > 1$, $\mathrm{mod}_Q(E,F)$ is big if $\Delta(E,F)$ is small, that is if E and F are closed, or if E and F have a huge diameter. A typical example of a Loewner space is the n-Euclidean space ($n > 1$). Other examples include Heisenberg/Carnot groups, (noncompact, complete) Riemannian manifolds with nonnegative Ricci curvature and maximal growth. This means that the Riemannian volume vol_g of the manifold (M,g) of dimension n satisfies for all $x \in M$ and $R > 0$, $vol_g(B(x,R)) \geq C^{-1}R^n$ for some positive constant C. Note that by the classical comparison Theorem of Bishop-Gromov, we have also $vol_g(B(x,R)) \leq CR^n$ if M has nonnegative Ricci curvature. Hence, a (noncompact complete) manifold with nonnegative Ricci curvature is a Loewner space if and only if its Riemannian volume vol_g is Ahlfors-regular. Moreover, for any $Q > 1$, there exist Loewner spaces with dimension Q (Laakso spaces, Bourdon-Pajot spaces). The case of the Heisenberg groups and of Riemannian manifolds will be discussed with more details later.

Loewner spaces have the nice following geometric properties.

Theorem 5.2.14 *Let (X, d, μ) be a Q-Loewner space (with $Q > 1$). Then,*

- *X is* linearly locally connected *(LLC). This means that there exists a constant $C > 0$ so that for any $x \in X$, any $r > 0$,*
  1. *Every pair of point in $B(x, r)$ can be joined (by a continuum) in $B(x, Cr)$;*
  2. *Every pair of point in $X \setminus B(x, r)$ can be joined (by a continuum) in $X \setminus B(x, r/C)$.*
- *X is* quasiconvex*, that is there exists a constant $C \geq 0$ such that any pair of points $(x, y) \in X \times X$ can be joined by a curve γ in X whose length is less than $C\dot{d}(x, y)$.*

We now give a characterization of Loewner spaces in terms of Poincaré inequalities. We need first some definitions.

Definition 5.2.15 Let $u : X \to \mathbb{R}$ be a continuous function. We say that $\rho : X \to \overline{\mathbb{R}^+}$ is *an upper gradient of u* if

$$|u(x) - u(y)| \leq \int_\gamma \rho(s)\mathrm{d}s$$

for all $x, y \in X$ and any rectifiable curve γ joining x and y.

Example 5.2.16 If X is the Euclidean space $\mathbb{R}^n$ (or a smooth manifold) and if $u : \mathbb{R}^n \to \mathbb{R}$ is smooth, then $|\nabla u|$ is an upper gradient of u. Moreover, if ρ is another upper gradient of u, then $|\nabla u| \leq \rho$ almost everywhere.

Example 5.2.17 Assume that $u : X \to \mathbb{R}$ is a Lipschitz function (with Lipschitz constant L) and that X is rectifiably connected (that is every pair of points can be joined by a rectifiable curve). Then, $\rho(x) = L$ and $\rho(x) = \liminf_{r\to 0} \sup_{d(x,y)=r} \dfrac{|u(x) - u(y)|}{r}$ are upper gradients of u.

The first claim is obvious. We now prove the second claim. Recall that we assume that given x and y in X, there exists a rectifiable curve in X joining these two points. Let γ be a (locally) rectifiable curve in X parameterized (locally) by arclength. Then the restriction u_γ of u to γ (that is $u \circ \gamma$) is also Lipschitz and so is differentiable almost everywhere. It follows that if γ is a rectifiable curve joining x to y,

$$|u(x) - u(y)| \leq \int_\gamma |u'_\gamma(s)|\mathrm{d}s.$$

Thus, we have to prove that

$$|u'_\gamma(s)| \leq \liminf_{r\to 0} \sup_{d(x,y)=r} \frac{|u(x) - u(y)|}{r}$$

whenever $u'_\gamma(s)$ exists. Fix such a s_0 and assume that $u'_\gamma(s_0) \neq 0$ (otherwise, the previous inequality is obvious). Hence, if s is sufficiently closed to s_0, $d(\gamma(s), \gamma(s_0)) \neq 0$. Moreover, for all r small enough, there exists a smaller $s(r)$ so that $d(\gamma(s(r)), \gamma(s_0)) = r \leq |s(r) - s_0|$. Note that $s(r) \to 0$ when $r \to 0$. Now we have

$$\frac{|u_\gamma(s(r)) - u_\gamma(s_0)|}{s(r) - s_0} \leq \sup_{d(z,\gamma(s_0)=r} \frac{|u(z) - u(\gamma(s_0))|}{r}.$$

Since $u'_\gamma(s_0)$ exists, we get

$$\begin{aligned}|u'_\gamma(s_0)| &\leq \liminf_{r\to 0} \frac{|u_\gamma(s(r)) - u_\gamma(s_0)|}{s(r) - s_0} \\ &\leq \liminf_{r\to 0} \sup_{d(z,\gamma(s_0))=r} \frac{|u(z) - u(\gamma(s_0))|}{r}\end{aligned}$$

and the claim is proved. Note that this implies that the other pointwise Lipschitz constants $\rho(x) = \liminf_{r\to 0} \sup_{d(x,y)\leq r} \frac{|u(x) - u(y)|}{r}$ and $\rho(x) = \limsup_{r\to 0} \sup_{d(x,y)\leq r} \frac{|u(x) - u(y)|}{r}$ are also upper gradients. This illustrates the fact that in general an upper gradient is not unique.

Remark 5.2.18 By definition, $\rho = +\infty$ is an upper gradient for any continuous function $u : X \to \mathbb{R}$. This implies that an upper gradient always exists.

Let $p \geq 1$. If B is a ball, we denote by $g_B = -\!\!\!\!\!\!\int_B g \mathrm{d}\mu = \frac{1}{\mu(B)} \int_B g \mathrm{d}\mu$ the mean value of a Borel function $g : X \to \mathbb{R}$ on B. We say that the metric measure space (X, d, μ) *supports a weak* $(1, p)$*-Poincaré inequality* if there exist constant $C \geq 0$ and $\tau \geq 1$ so that

$$\int_B |u - u_B| \mathrm{d}\mu \leq C \mathrm{diam} B \left(-\!\!\!\!\!\!\int_{\tau B} \rho^p \mathrm{d}\mu \right)^{1/p} \tag{5.2}$$

whenever B is a ball in X, $u : X \to \mathbb{R}$ is a continuous function and $\rho : X \to \mathbb{R}^+$ is an upper gradient of u in B. Here, τB denotes the ball with the same center as B but whose radius is τ times the radius of B. If (5.2) is satisfied with $\tau = 1$, we say that (X, d, μ) supports a $(1, p)$-Poincaré inequality.

Remark 5.2.19 By Hölder inequality, if (X, d, μ) supports a $(1, 1)$-Poincaré inequality, then (X, d, μ) supports a $(1, p)$-Poincaré inequality for any $p > 1$. Thus, the $(1, 1)$-Poincaré inequality is the strongest one.

Remark 5.2.20 In metric spaces which satisfy a chain condition, the weak-$(1, p)$-Poincaré inequality implies the $(1, p)$-Poincaré inequality (See [6] for the definition and a proof). For instance, the chain condition is satisfied in geodesic metric spaces. Thus, most of the time, we will omit the term "weak".

Remark 5.2.21 If X is a proper and quasiconvex metric space equipped with a doubling measure, then X admits a Poincaré inequality if and only if X admits a Poincaré inequality only for bounded Lipschitz functions (and if and only if X admits a Poincaré inequality for Borel functions).

The connection between the Loewner estimate and the Poincaré inequality is given in the next result.

Theorem 5.2.22 *Let (X, d, μ) be a proper, Q-regular metric measure space. Then, X is a Q-Loewner space if and only if X supports a (weak) $(1, Q)$-Poincaré inequality.*

This criterion is very useful to check that a given space is a Loewner space. We now give two applications.

Theorem 5.2.23 *The Heisenberg group equipped with its Carnot-Carathéodory distance and its left invariant Haar measure is a $(2n+2)$-Loewner space.*

Proof First, we already mentioned (see chap. 3) that the Haar measure is $2n+2$-regular. So to conclude, we have to prove the Poincaré inequality. Let u be a smooth function on the Heisenberg group and let ρ be a weak gradient of u. For any $z \in \mathbb{H}^n$, set $|z| = d_{CC}(0, z)$ and choose a geodesic path $\gamma_z : [0, |z|] \to \mathbb{H}^n$ from 0 to z. Note that $s \mapsto x.\gamma_z(s)$ is a shortest path that joins x to $x.z$ (if $x \neq 0$). Hence, by the definition of the upper gradient, we have

$$|u(x) - u(x.z)| \le \int_0^{|z|} \rho(x.\gamma_z(s))\mathrm{d}s.$$

To simplify the notation, we denote by $\mathrm{d}x$ the integration with respect to the Haar measure and by $|A|$ the Haar measure of $A \subset \mathbb{H}^n$. Now, we use the left invariance of the measure to get (with $y = x.z$)

$$\begin{aligned}\int_B |u(x) - u_B|\mathrm{d}x &\le \frac{1}{|B|}\int_B\int_B |u(x) - u(y)|\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{|B|}\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \chi_B(x)\chi_B(x.z)|u(x) - u(x.z)|\mathrm{d}x\mathrm{d}z \\ &\le \frac{1}{|B|}\int_{\mathbb{H}^n}\int_{\mathbb{H}^n}\int_0^{|z|} \chi_B(x)\chi_B(x.z)\rho(x.\gamma_z(s))\mathrm{d}s\mathrm{d}x\mathrm{d}z.\end{aligned}$$

Here, χ_A denotes the characteristic function of A. By using the right invariance of the Haar measure, we get (with $\xi = x.\gamma_z(s)$)

$$\int_{\mathbb{H}^n} \chi_B(x)\chi_B(x.z)\rho(x.\gamma_z(s))\mathrm{d}x = \int_{\mathbb{H}^n} \chi_{B.\gamma_z(s)}(\xi)\chi_{B.z^{-1}.\gamma_z(s)}(\xi)\rho(\xi)\mathrm{d}\xi.$$

Assume that $\chi_{B.\gamma_z(s)}(\xi)\chi_{B.z^{-1}.\gamma_z(s)}(\xi) \neq 0$. Then, there exist x and y in B so that $\xi = x.\gamma_z(s) = y.z^{-1}.\gamma_z(s)$. Note that $z = x^{-1}.y$ is in $2B$ since $d_{CC}(0, z) = d_{CC}(0, x^{-1}.y) = d_{CC}(x, y) \le 2r$ where r is the radius of B. Moreover, $\xi = x\gamma_{x^{-1}.y}(s)$

is on a geodesic path from x to y and so $d_{CC}(x,\xi)+d_{CC}(y,\xi)=d_{CC}(x,y)\leq 2r$. This implies that $d_{CC}(x,\xi)\leq r$ or $d_{CC}(y,\xi)\leq r$. Hence, by the triangle inequality, $\xi\in 2B$. It follows that

$$\int_{\mathbb{H}^n}\chi_{B.\gamma_z(s)}(\xi)\chi_{B.z^{-1}.\gamma_z(s)}(\xi)\rho(\xi)\mathrm{d}\xi\leq\chi_{2B}(z)\int_{2B}\rho(\xi)\mathrm{d}\xi,$$

and then

$$\begin{aligned}\int_B|u(x)-u_B|\mathrm{d}x&\leq\frac{1}{|B|}\int_{\mathbb{H}^n}\int_0^{|z|}\chi_{2B}(z)\int_{2B}\rho(\xi)\mathrm{d}\xi\mathrm{d}s\mathrm{d}z\\&\leq\frac{1}{|B|}\int_{2B}\int_{2B}|z|\rho(\xi)\mathrm{d}\xi\mathrm{d}z\\&\leq C\mathrm{diam}(B)\int_{2B}\rho(z)\mathrm{d}z.\end{aligned}$$

Another application is

Theorem 5.2.24 *Let (M,g) be a noncompact, complete Riemannian manifold of dimension n. We assume that M has nonnegative Ricci curvature. Then, M is a Loewner space if and only if the Riemannian volume vol_g of M is n-Ahlfors regular.*

Proof We have just to prove that a manifold with nonnegative curvature supports a Poincaré inequality. For simplicity, we denote by μ the Riemannian measure on M. For any pair of points x and y in X, we choose one minimizing geodesic $\gamma_{x,y}$ joining x to y. We assume that $\gamma_{x,y}:[0,d(x,y)]\to M$ is the arc length parametrization of this geodesic. The geodesic joining y to x is then given by $\gamma_{y,x}(t)=\gamma_{x,y}(t-d(x,y))$ for any $t\in[0,d(x,y)]$. This means that the choice of the geodesic $\gamma_{x,y}$ implies the choice of the geodesic $\gamma_{y,x}$.

Now, we fix a ball B in X of radius r. Consider u and ρ as in the definition of the Poincaré inequality. If x and y are in B, then by definition of the upper gradient,

$$|u(x)-u(y)|\leq\int_{\gamma_{x,y}}\rho(s)ds=\int_0^{d(x,y)}\rho(\gamma_{x,y}(t))dt.$$

Thus,

$$\begin{aligned}\int_B|u-u_B|d\mu&\leq\frac{1}{\mu(B)}\int_B\int_B|u(x)-u(y)|d\mu(x)d\mu(y)\\&\leq\frac{1}{\mu(B)}\int_B\int_B\left(\int_0^{d(x,y)}\rho(\gamma_{x,y}(t))dt\right)d\mu(x)d\mu(y).\end{aligned}$$

Write now by using the definitions of $\gamma_{x,y}$ and $\gamma_{y,x}$

$$\int_0^{d(x,y)} \rho(\gamma_{x,y}(t))dt = \int_0^{d(x,y)/2} \rho(\gamma_{x,y}(t))dt + \int_{d(x,y)/2}^{d(x,y)} \rho(\gamma_{x,y}(t))dt$$
$$= \int_{d(x,y)/2}^{d(x,y)} \rho(\gamma_{y,x}(t))dt + \int_{d(x,y)/2}^{d(x,y)} \rho(\gamma_{x,y}(t))dt.$$

Hence, by symmetry in x and y, we get

$$\int_B |u - u_B| d\mu \leq \frac{2}{\mu(B)} \int_B \int_B \left(\int_{d(x,y)/2}^{d(x,y)} \rho(\gamma_{x,y}(t))dt \right) d\mu(x)d\mu(y).$$

Denote now by $\phi_{x,t}(y) = \gamma_{x,y}(t)$. Let $J_{x,t}(y)$ the volume derivative of $\phi_{x,t}(y)$, that is

$$J_{x,t}(y) = \lim_{r\to 0} \frac{\mu(\phi_{x,t}(B(y,r))}{\mu(B(y,r))}.$$

From the proof of the Bishop-Gromov comparison Theorem, we get that there exists an absolute constant $C > 0$ such that $J_{x,t}(y) \geq C$ for any $x \in B$, any $y \in B$, any $t \in [d(x,y)/2, d(x,y)]$. In fact, $C = 2^{-n+1}$ where n is the dimension of the manifold. Hence, we have

$$\int_B \int_B \left(\int_{d(x,y)/2}^{d(x,y)} \rho(\gamma_{x,y}(t))dt \right) d\mu(x)d\mu(y) = \int_B \int_B \int_{d(x,y)/2}^{d(x,y)} \rho(\phi_{x,t}(y))dt d\mu(x)d\mu(y)$$
$$\leq C^{-1} \int_B \int_B \int_{d(x,y)/2}^{d(x,y)} \rho(\phi_{x,t}(y))J_{x,t}(y)dt d\mu(x)d\mu(y)$$
$$\leq C^{-1} \int_B \int_B \int_0^{\mathrm{diam} B} \rho(\phi_{x,t}(y))J_{x,t}(y)dt d\mu(x)d\mu(y)$$
$$\leq C^{-1} \int_0^{\mathrm{diam} B} \int_B (\int_B \rho(\phi_{x,t}(y))J_{x,t}(y)d\mu(y))d\mu(x)dt$$
$$\leq C^{-1} \int_0^{\mathrm{diam} B} \int_B (\int_{\phi_{x,y}(B)} \rho(z)d\mu(z))d\mu(x)dt$$
$$\leq C^{-1} \int_0^{\mathrm{diam} B} \int_B (\int_{2B} \rho(z)d\mu(z))d\mu(x)dt$$
$$\leq C^{-1}\mathrm{diam}(B)\mu(B) \int_{2B} \rho(z)d\mu(z).$$

From this, we can easily conclude that

$$\int_B |u - u_B| d\mu \leq 2C^{-1} r \int_{2B} \rho(z) d\mu(z).$$

Since μ is doubling by the Bishop-Gromov comparison Theorem, we can conclude and get the Poincaré inequality (with the mean values).

In Loewner spaces, the previous definitions of quasiconformality are equivalent.

Theorem 5.2.25 *Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be two (bounded) Q-Loewner spaces ($Q > 1$) and let $f : X \to Y$ be a homeomorphism. The following assertions are equivalent:*

(i) *f is QC;*
(ii) *f is WQS;*
(iii) *f is QS;*
(iv) *There exists a constant $C > 0$ so that*

$$C^{-1}\mathrm{mod}_Q(\Gamma) \leq \mathrm{mod}_Q(f\Gamma) \leq C\mathrm{mod}_Q(\Gamma)$$

for all curve family Γ in X.

Moreover, if these conditions hold, f is absolutely continuous in measure and is absolutely continuous on Q-almost every curve in X (that is, the Q-modulus of the family of curves on which f is not absolutely continuous is zero).

The last condition could be seen as a metric analog of the property ACL. The conditions (i), (ii) and (iii) are metric/quasisymmetric definitions of quasiconformality in metric spaces whereas (iv) is a geometric one (and corresponds to (G2) in the complex plane). There is also an analytic definition (which is equivalent to (i), (ii) (iii) and (iv) in Loewner spaces) in terms of Sobolev spaces between metric measure spaces. This last definition is not discussed in these notes. Hence, the previous theorem gives an analog of Theorem 5.1.1 in metric spaces with controlled geometry. Recall that the complex plane is a Loewner space.

5.2.4 Quasi-Möbius Maps

We now give another definition of quasiconformality which is very useful in geometric group theory and in hyperbolic geometry. We say that a homeomorphism $f : X \to Y$ is *quasi-Möbius* (QM) if there exists a homeomorphism $\eta_f = \eta : [0, +\infty) \to [0, \infty)$ such that, for any distinct points x_1, x_2, x_3 and x_4 in X,

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4])$$

holds, here $[x_1, x_2, x_3, x_4]$ denotes the metric crossratio defined by

$$[x_1, x_2, x_3, x_4] = \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

Theorem 5.2.26 *Let X, Y be two metric spaces and $f : X \to Y$ be a homeomorphism.*

(i) *A quasi-Möbius map is uniformly locally quasisymmetric.*
(ii) *A quasisymmetric map is quasi-Möbius.*
(iii) *Let $f : X \to Y$ be a quasi-Möbius map. If X and Y are unbounded, then f is quasisymmetric if and only if $f(x)$ tends to infinity when x tends to infinity. If X and Y are bounded, and if for three points $z_1, z_2, z_3 \in X$, we have $d(z_i, z_j) \geq \operatorname{diam} X/\lambda$ and $|f(z_i) - f(z_j)| \geq \operatorname{diam} Y/\lambda$ for some $\lambda > 0$, then f is η-quasisymmetric, where η only depends λ and on the distortion of crossratios.*

Before proving this result, we give a geometric interpretation of the crossratio following Bonk and Kleiner:

Lemma 5.2.27 *Let X be a metric space. If x_1, x_2, x_3, x_4 are four distinct points in X, we define*

$$\langle x_1, x_2, x_3, x_4 \rangle = \frac{\min\{d(x_1, x_3), d(x_2, x_4)\}}{\min\{d(x_1, x_4), d(x_2, x_3)\}}.$$

Then

$$\langle x_1, x_2, x_3, x_4 \rangle \leq \eta_0([x_1, x_2, x_3, x_4]) \quad \textit{and} \quad [x_1, x_2, x_3, x_4] \leq \eta_1(\langle x_1, x_2, x_3, x_4 \rangle)$$

where

$$\eta_0(t) = t + \sqrt{t^2 + t} \quad \textit{and} \quad \eta_1(t) = t(2 + t).$$

Proof To simplify the notation, we will set $|x - y| = d(x, y)$ for $x, y \in X$. Let us assume that $|x_1 - x_2| \leq |x_3 - x_4|$; then

$$\begin{cases} |x_1 - x_3| \leq |x_1 - x_2| + |x_2 - x_4| + |x_4 - x_3| \leq 2|x_4 - x_3| + |x_2 - x_4|; \\ |x_2 - x_4| \leq |x_2 - x_1| + |x_1 - x_3| + |x_3 - x_4| \leq 2|x_4 - x_3| + |x_1 - x_3|. \end{cases}$$

Hence

$$\max\{|x_1 - x_3|, |x_2 - x_4|\} \le 2|x_4 - x_3| + \min\{|x_1 - x_3|, |x_2 - x_4|\}$$

$$\le \left(2 + \frac{\min\{|x_1 - x_3|, |x_2 - x_4|\}}{|x_3 - x_4|}\right) |x_3 - x_4|$$

$$\le \left(2 + \frac{1}{\langle x_1, x_2, x_3, x_4\rangle}\right) |x_3 - x_4| .$$

Therefore,

$$[x_1, x_2, x_3, x_4] \ge \left(2 + \frac{1}{\langle x_1, x_2, x_3, x_4\rangle}\right)^{-1} \frac{|x_1 - x_2|}{\min\{|x_1 - x_3|, |x_2 - x_4|\}}$$

$$\ge \langle x_1, x_2, x_3, x_4\rangle \left(2 + \frac{1}{\langle x_1, x_2, x_3, x_4\rangle}\right)^{-1} .$$

By inverting the function of $\langle x_1, x_2, x_3, x_4\rangle$, we obtain

$$\langle x_1, x_2, x_3, x_4\rangle \le \eta_0([x_1, x_2, x_3, x_4]) .$$

Observe that

$$\eta_0(t) \le 3 \max\{t, \sqrt{t}\} .$$

By permuting points, we have

$$[x_1, x_3, x_2, x_4] \ge \langle x_1, x_3, x_2, x_4\rangle \left(2 + \frac{1}{\langle x_1, x_3, x_2, x_4\rangle}\right)^{-1}$$

and taking the inverse, it follows that

$$[x_1, x_2, x_3, x_4] \le \eta_1(\langle x_1, x_2, x_3, x_4\rangle) .$$

Proof (Theorem 5.2.26)

(i) Let $x \in X$ and pick $x' \ne x$. By continuity of f, we may find $r > 0$ such that $r \le |x - x'|/4$ and $\operatorname{diam} f(B(x, r)) \le |f(x) - f(x')|/4$. Using $\langle\cdot\rangle$ for points (x_1, x_2, x_3, x') with $x_j \in B(x, r), j = 1, 2, 3$, we see at once that a quasi-Möbius map is uniformly quasisymmetric on $B(x, r)$.

(ii) Let us prove that if f is η-quasisymmetric, then

$$\langle f(x_1), f(x_2), f(x_3), f(x_4)\rangle \le \eta(\langle x_1, x_2, x_3, x_4\rangle) .$$

Let us assume that $|x_1 - x_2| \leq |x_3 - x_4|$. We have

$$\min\{|f(x_1) - f(x_2)|, |f(x_3) - f(x_4)|\} \leq |f(x_1) - f(x_2)|.$$

On the other hand, observe that

$$\min\{|f(x_1) - f(x_3)|, |f(x_2) - f(x_4)|\} = |f(x_i) - f(x_j)|$$

where we can choose $i \in \{1, 2\}$.

Thus,

$$\begin{aligned} \frac{|f(x_1) - f(x_2)|}{|f(x_i) - f(x_j)|} &\leq \eta\left(\frac{|x_1 - x_2|}{|x_i - x_j|}\right) \\ &\leq \eta\left(\frac{|x_1 - x_2|}{\min\{|x_1 - x_3||x_2 - x_4|\}}\right) \\ &\leq \eta(\langle x_1, x_2, x_3, x_4\rangle)\,. \end{aligned}$$

Lemma 5.2.27 enables us to conclude.

(iii) If X and Y are unbounded, then f can be quasisymmetric only if $f(x)$ tends to infinity with x: if x_1 and x_2 are fixed an η-quasisymmetric map satisfies

$$|f(x) - f(x_1)| \leq \eta\left(\frac{|x - x_1|}{|x_2 - x_1|}\right)|f(x_2) - f(x_1)|$$

which implies that $f(x)$ remains bounded if x is; and similarly, $f(x)$ has to go infinity when x does since

$$|f(x_2) - f(x_1)| \leq \eta\left(\frac{|x_2 - x_1|}{|x - x_1|}\right)|f(x) - f(x_1)|\,.$$

In this case, we let x_4 go infinity and we obtain that f is η-quasisymmetric.

Otherwise, for any $x \in X$, there are z_i, z_j such that $|z_i - x| \geq \operatorname{diam} X/2\lambda$ and $|z_j - x| \geq \operatorname{diam} X/2\lambda$. Similarly, there are z_k, z_m such that $|f(z_k) - f(x)| \geq \operatorname{diam} Y/2\lambda$ and $|f(z_m) - f(x)| \geq \operatorname{diam} Y/2\lambda$.

Let x_1, x_2, x_3. Choose z_i so that $|z_i - x_2| \geq \operatorname{diam} X/2\lambda$ and $|f(z_i) - f(x_3)| \geq \operatorname{diam} Y/2\lambda$.

Hence,

$$\begin{aligned} \frac{|f(x_1) - f(x_2)|}{|f(x_1) - f(x_3)|} &\leq \frac{|f(x_2) - f(z_i)|}{|f(x_3) - f(z_i)|}\eta([x_1, x_2, x_3, z_i]) \\ &\leq 2\lambda\eta\left(2\lambda\frac{|x_1 - x_2|}{|x_1 - x_3|}\right)\,. \end{aligned}$$

Remark 5.2.28 If x_1, x_2, x_3 and x_4 are four distinct points in X, *J. Ferrand's crossratio* is defined by

$$[x_1, x_2, x_3, x_4] = \inf_{E,F} \text{mod}_Q(E, F)$$

where the infimum is taken over all the (non degenerate disjoint) continua E and F so that x_1, $x_2 \in E$ and x_3, $x_4 \in F$. Then, any quasiconformal homeomorphism quasi-preserves J. Ferrand's crossratio in Q-Loewner spaces.

5.3 Back to Hyperbolic Geometry

Gromov spaces and other related notions are defined in Sect. 3.1.

Definition 5.3.1 Let (X, d) be a metric space. The *conformal gauge* $\mathscr{C}(X, d)$ of (X, d) is the set of distances δ on X such that the identity map $Id : (X, d) \to (X, \delta)$ is QS. The *conformal dimension* of (X, d) is then $\text{C}dim(X, d) = \inf \text{H}dim(X, \delta)$ where the infimum is taken over all distances δ in the conformal gauge of (X, d) and Hdim denotes the Hausdorff dimension.

If $\delta \in \mathscr{C}(x, d)$ is so that (X, δ) equipped with its Hausdorff measure is a Q-Loewner space, then $\text{C}dim(X, d) = Q$.

Example 5.3.2 The conformal dimension of the Euclidean $\mathbb{R}^n$ is n and is attained by the Euclidean metric whereas the conformal dimension of the Heisenberg group $\mathbb{H}^n$ is $2n + 2$ and is attained by the Carnot-Carathéodory distance.

Remark 5.3.3 There are examples of metric spaces for which the conformal dimension is not attained.

In the particular case where $X = \partial_G Z$ with Z a Gromov hyperbolic space, the conformal gauge of X (or of Z) is the conformal gauge associated to a visual metric. This definition does not depend on the choice of this visual metric: indeed, if d_1 and d_2 are visual distances based at w_1 and w_2 and of parameter ε_1 and ε_2, then $Id : (\partial Z, d_1) \to (\partial Z, d_2)$ is quasisymmetric. The following result relates the quasiconformal geometry with the Gromov hyperbolicity:

Theorem 5.3.4 *Let Z_1, Z_2 be Gromov hyperbolic spaces and let $F : Z_1 \to Z_2$ be a (C, D)-quasi-isometry. Then, F extends continuously as a homeomorphism $f : \partial_G Z_1 \to \partial_G Z_2$ (equipped with some visual metrics).*

Moreover, if d_1 and d_2 are visual metrics of parameter $\varepsilon_X, \varepsilon_Y$, then there exists $\alpha = \alpha(C, D, \varepsilon_X/\varepsilon_Y) > 0$ and $\beta = \beta(C, D, \varepsilon_X/\varepsilon_Y) \geq 1$ such that f is η-quasimöbius, with $\eta(t) = \alpha \cdot \max\{t^\beta, t^{1/\beta}\}$. If $\varepsilon_X = \varepsilon_Y$, and F is a rough isometry, then the map f is bilipschitz (and if F is an isometry, then one may choose $\beta = 1$).

Hence, the conformal dimension is a quasi-isometric invariant of Gromov hyperbolic spaces.

Remark 5.3.5 If the Gromov hyperbolic spaces Z_1 and Z_2 admit a cocompact group of isometries, any QS-homeomorphism $f : \partial_G Z_1 \to \partial_G Z_2$ is induced by a quasi-isometry $F : Z_1 \to Z_2$.

To study the rigidity of a Gromov hyperbolic space Z, it could be useful to find a distance δ on ∂Z so that the conformal dimension of Z is attained for δ. Indeed, this distance will give a good conformal structure on ∂Z (for instance, a structure of Loewner space). As an example, we conclude this chapter with the famous rigidity theorem of Mostow for the complex hyperbolic space.

Theorem 5.3.6 *Let Γ_1, Γ_2 be two lattices in the complex hyperbolic space $H^n_{\mathbb{C}}$. If Γ_1 and Γ_2 are isomorphic, then $M_{\Gamma_1} = H^n_{\mathbb{C}}/\Gamma_1$ and $M_{\Gamma_2} = H^n_{\mathbb{C}}/\Gamma_1$ are isometric, in particular are conformally equivalent.*

Recall that a lattice Γ in $H^n_{\mathbb{C}}$ is a discrete subgroup of the group of isometries of $H^n_{\mathbb{C}}$ so that $M_\Gamma = H^n_{\mathbb{C}}/\Gamma$ is compact.

We now describe the main ideas of the proof. Consider an isomorphism $\phi : \Gamma_1 \to \Gamma_2$. Then, ϕ induces a quasi-isometry $F : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ which is equivariant. Recall that the boundary $\partial_G H^n_{\mathbb{C}}$ could be identified with the Heisenberg group $\mathbb{H}^n$ equipped with the Carnot-Carathéodory distance (for which the conformal dimension is attained), see Sect. 3.2. Then, F has a quasiconformal extension on the boundary $f : \mathbb{H}^n \to \mathbb{H}^n$. Since $\mathbb{H}^n$ is a Loewner space, f is absolutely continuous along almost every rectifiable curves γ (and hence is differential along almost every curve γ). By Pansu's version of the Rademacher theorem, it turns out that f is differentiable almost everywhere. Moreover, it can be shown by an ergodic argument that f is 1-quasiconformal. By an analog of the classical Liouville theorem, f is the boundary value of an isometry of $H^n_{\mathbb{C}}$ which is equivariant and therefore descends to an isomety between M_{Γ_1} and M_{Γ_2}. To avoid confusion, we recall that the Liouville theorem we mention states that any conformal map on the Euclidean n-sphere is the boundary value of an isometry of the real hyperbolic $n+1$-space.

5.4 Notes

Classical references on quasiconformal mappings in the complex plane include [1] or [10], and in higher dimension [14] or [17]. For a discussion of the Beltrami equation, see [12] or [8]. The presentation we follow is due to P. Haissinsky (unpublished). A complete proof of the wandering theorem could be found in [2]. The best introduction to analysis in metric spaces (and in particular quasiconformal mappings) is [6]. The theory of quasiconformal mappings in spaces with controlled geometry was first developed in [7]. Poincaré inequalities (with examples and applications) are discussed in [5] and in [15]. In particular, proofs of the Poincaré inequalities in the Heisenberg group and in manifolds with nonnegative curvature are taken from this book. More general versions of theorem 5.3.6 are in [11] and in [13]. A general discussion of rigidity theorems in connection with quasiconformal geometry and analysis in metric spaces can be found in [3, 4] and [9]. For more informations about BMO spaces and singular integral operators, see [16].

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Chapter 6
Carleson Measures and Toeplitz Operators

Marco Abate

In this last chapter we shall describe a completely different application of the Kobayashi distance to complex analysis. To describe the problem we need a few definitions.

Definition 6.0.1 We shall denote by ν the Lebesgue measure in $\mathbb{C}^n$. If $D \subset\subset \mathbb{C}^n$ is a bounded domain and $1 \le p \le \infty$, we shall denote by $L^p(D)$ the usual space of measurable p-integrable complex-valued functions on D, with the norm

$$\|f\|_p = \left[\int_D |f(z)|^p \, \mathrm{d}\nu(z)\right]^{1/p}$$

if $1 \le p < \infty$, while $\|f\|_\infty$ will be the essential supremum of $|f|$ in D. Given $\beta \in \mathbb{R}$, we shall also consider the *weighted L^p-spaces* $L^p(D, \beta)$, which are the L^p spaces with respect to the measure $\delta^\beta \nu$, where $\delta\colon D \to \mathbb{R}^+$ is the Euclidean distance from the boundary: $\delta(z) = d(z, \partial D)$. The norm in $L^p(D, \beta)$ is given by

$$\|f\|_{p,\beta} = \left[\int_D |f(z)|^p \delta(z)^\beta \, \mathrm{d}\nu(z)\right]^{1/p}$$

for $1 \le p < \infty$, and by $\|f\|_{\infty,\beta} = \|f\delta^\beta\|_\infty$ for $p = \infty$.

Definition 6.0.2 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$, and $1 \le p \le \infty$. The *Bergman space* $A^p(D)$ is the Banach space $A^p(D) = L^p(D) \cap \mathrm{Hol}(D, \mathbb{C})$ endowed with the norm $\|\cdot\|_p$. More generally, given $\beta \in \mathbb{R}$ the *weighted Bergman space* $A^p(D, \beta)$ is the Banach space $A^p(D, \beta) = L^p(D, \beta) \cap \mathrm{Hol}(D, \mathbb{C})$ endowed with the norm $\|\cdot\|_{p,\beta}$.

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7_6

The Bergman space $A^2(D)$ is a Hilbert space; this allows us to introduce one of the most studied objects in complex analysis.

Definition 6.0.3 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. The *Bergman projection* is the orthogonal projection $P: L^2(D) \to A^2(D)$.

It is a classical fact (see, e.g., [11, Sect. 1.4] for proofs) that the Bergman projection is an integral operator: it exists a function $K: D \times D \to \mathbb{C}$ such that

$$Pf(z) = \int_D K(z,w)f(w)\,\mathrm{d}\nu(w) \tag{6.1}$$

for all $f \in L^2(D)$. It turns out that K is holomorphic in the first argument, $K(w,z) = \overline{K(z,w)}$ for all $z, w \in D$, and it is a *reproducing kernel* for $A^2(D)$ in the sense that

$$f(z) = \int_D K(z,w)f(w)\,\mathrm{d}\nu(w)$$

for all $f \in A^2(D)$.

Definition 6.0.4 Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. The function $K: D \times D \to \mathbb{C}$ satisfying (6.1) is the *Bergman kernel* of D.

Remark 6.0.5 It is not difficult to show (see again, e.g., [11, Sect. 1.4]) that $K(\cdot, w) \in A^2(D)$ for all $w \in D$, and that

$$\|K(\cdot,w)\|_2^2 = K(w,w) > 0 .$$

In case $D = \mathbb{B}$ the unit ball, the explicit formula is given in Sect. 3.2.

A classical result in complex analysis says that in strongly pseudoconvex domains the Bergman projection can be extended to all L^p spaces:

Theorem 6.0.6 (Phong and Stein [16]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and $1 \le p \le \infty$. Then the formula (6.1) defines a continuous operator P from $L^p(D)$ to $A^p(D)$. Furthermore, for any $r > p$ there is $f \in L^p(D)$ such that $Pf \notin A^r(D)$.*

Recently, Čučković and McNeal posed the following question: does there exist a natural operator, somewhat akin to the Bergman projection, mapping $L^p(D)$ into $A^r(D)$ for some $r > p$? To answer this question, they considered Toeplitz operators.

Definition 6.0.7 Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Given a measurable function $\psi: D \to \mathbb{C}$, the *multiplication operator* of *symbol* ψ is simply defined by $M_\psi(f) = \psi f$. Given $1 \le p \le \infty$, a symbol ψ is *p-admissible* if M_ψ sends $L^p(D)$ into itself; for instance, a $\psi \in L^\infty(D)$ is p-admissible for all p. If ψ is p-admissible, the *Toeplitz operator* $T_\psi: L^p(D) \to A^p(D)$ of *symbol* ψ is defined by $T_\psi = P \circ M_\psi$, that is

$$T_\psi(f)(z) = P(\psi f)(z) = \int_D K(z,w)f(w)\psi(w)\,\mathrm{d}\nu(w) .$$

Remark 6.0.8 More generally, if A is a Banach algebra, $B \subset A$ is a Banach subspace, $P: A \to B$ is a projection and $\psi \in A$, the Toeplitz operator T_ψ of symbol ψ is defined by $T_\psi(f) = P(\psi f)$. Toeplitz operators are a much studied topic in functional analysis; see, e.g., [18].

Then Čučković and McNeal were able to prove the following result:

Theorem 6.0.9 (Čučković and McNeal [5]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. If $1 < p < \infty$ and $0 \le \beta < n+1$ are such that*

$$\frac{n+1}{n+1-\beta} < \frac{p}{p-1} \tag{6.2}$$

then the Toeplitz operator T_{δ^β} maps continuously $L^p(D)$ in $A^{p+G}(D)$, where

$$G = \frac{p^2}{\frac{n+1}{\beta} - p} \ .$$

Čučković and McNeal also asked whether the gain G in integrability is optimal; they were able to positively answer to this question only for $n = 1$. The positive answer in higher dimension has been given by Abate et al. [2], as a corollary of their study of a larger class of Toeplitz operators on strongly pseudoconvex domains. This study, putting into play another important notion in complex analysis, the one of Carleson measure, used as essential tool the Kobayashi distance; in the next couple of sections we shall describe the gist of their results.

6.1 Definitions and Results

In this subsection and the next D will always be a bounded strongly pseudoconvex domain with C^∞ boundary. We believe that the results might be generalized to other classes of domains with C^∞ boundary (e.g., finite type domains), and possibly to domains with less smooth boundary, but we will not pursue this subject here.

Let us introduce the main player in this area.

Definition 6.1.1 Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and μ a finite positive Borel measure on D. Then the *Toeplitz operator* T_μ of *symbol* μ is defined by

$$T_\mu(f)(z) = \int_D K(z,w) f(w)\, \mathrm{d}\mu(w) \ ,$$

where K is the Bergman kernel of D.

For instance, if ψ is an admissible symbol then the Toeplitz operator T_ψ introduced in Definition 6.0.7 is the Toeplitz operator $T_{\psi\nu}$ according to Definition 6.1.1.

In Definition 6.1.1 we did not specify domain and/or range of the Toeplitz operator T_μ because the main point of the theory we are going to discuss is exactly to link properties of the measure μ with domain and range of T_μ.

Toeplitz operators associated to measures have been extensively studied on the unit disc Δ and on the unit ball $\mathbb{B}^n$ (see, e.g., [10, 12, 13, 19] and references therein); but [2] has been one of the first papers studying them in strongly pseudoconvex domains.

The kind of measure we shall be interested in is described in the following

Definition 6.1.2 Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, A a Banach space of complex-valued functions on D, and $1 \le p \le \infty$. We shall say that a finite positive Borel measure μ on D is a *p-Carleson measure* for A if A embeds continuously into $L^p(\mu)$, that is if there exists $C > 0$ such that

$$\int_D |f(z)|\,\mathrm{d}\mu(z) \le C\|f\|_A^p$$

for all $f \in A$, where $\|\cdot\|_A$ is the norm in A.

Remark 6.1.3 When the inclusion $A \hookrightarrow L^p(\mu)$ is compact, μ is called *vanishing Carleson measure*. Here we shall discuss vanishing Carleson measures only in the remarks.

Carleson measures for the Hardy spaces $H^p(\Delta)$ were introduced by Carleson [3] to solve the famous corona problem. We shall be interested in Carleson measures for the weighted Bergman spaces $A^p(D,\beta)$; they have been studied by many authors when $D = \Delta$ or $D = \mathbb{B}^n$ (see, e.g., [6, 14, 19] and references therein), but more rarely when D is a strongly pseudoconvex domain (see, e.g., [4, 8] and [1]).

The main point here is to give a geometric characterization of which measures are Carleson. To this aim we introduce the following definition, bringing into play the Kobayashi distance.

Definition 6.1.4 Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and $\theta > 0$. We shall say that a finite positive Borel measure μ on D is θ*-Carleson* if there exists $r > 0$ and $C_r > 0$ such that

$$\mu\big(B_D(z_0,r)\big) \le C_r \nu\big(B_D(z_0,r)\big)^\theta \tag{6.3}$$

for all $z_0 \in D$. We shall see that if (6.3) holds for some $r > 0$ then it holds for all $r > 0$.

Remark 6.1.5 There is a parallel vanishing notion: we say that μ is *vanishing* θ*-Carleson* if there exists $r > 0$ such that

$$\lim_{z_0 \to \partial D} \frac{\mu\big(B_D(z_0,r)\big)}{\nu\big(B_D(z_0,r)\big)^\theta} = 0\,.$$

For later use, we recall two more definitions.

Definition 6.1.6 Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Given $w \in D$, the *normalized Bergman kernel* in w is given by

$$k_w(z) = \frac{K(z,w)}{\sqrt{K(w,w)}} \,.$$

Remark 6.0.5 shows that $k_w \in A^2(D)$ and $\|k_w\|_2 = 1$ for all $w \in D$.

Definition 6.1.7 Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and μ a finite positive Borel measure on D. The *Berezin transform* of μ is the function $B\mu\colon D \to \mathbb{R}^+$ defined by

$$B\mu(z) = \int_D |k_z(w)|^2 \, \mathrm{d}\mu(w) \,.$$

Again, part of the theory will describe when the Berezin transform of a measure is actually defined.

We can now state the main results obtained in [2]:

Theorem 6.1.8 (Abate et al. [2]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, $1 < p < r < \infty$ and μ a finite positive Borel measure on D. Then T_μ maps $A^p(D)$ into $A^r(D)$ if and only if μ is a p-Carleson measure for $A^p\big(D, (n+1)(\frac{1}{p} - \frac{1}{r})\big)$.*

Theorem 6.1.9 (Abate et al. [2]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, $1 < p < \infty$ and $\theta \in \big(1 - \frac{1}{n+1}, 2\big)$. Then a finite positive Borel measure μ on D is a p-Carleson measure for $A^p\big(D, (n+1)(\theta - 1)\big)$ if and only if μ is a θ-Carleson measure.*

Theorem 6.1.10 (Abate et al. [2]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and $\theta > 0$. Then a finite positive Borel measure μ on D is θ-Carleson if and only the Berezin transform $B\mu$ exists and $\delta^{(n+1)(1-\theta)} B\mu \in L^\infty(D)$.*

Remark 6.1.11 This is just a small selection of the results contained in [2]. There one can find statements also for $p = 1$ or $p = \infty$, for other values of θ, and on the mapping properties of Toeplitz operators on weighted Bergman spaces. Furthermore, there it is also shown that T_μ is a compact operator from $A^p(D)$ into $A^r(D)$ if and only if μ is a vanishing p-Carleson measure for $A^p\big(D, (n+1)(\frac{1}{p} - \frac{1}{r})\big)$; that μ is a vanishing p-Carleson measure for $A^p\big(D, (n+1)(\theta - 1)\big)$ if and only if μ is a vanishing θ-Carleson measure; and that μ is a vanishing θ-Carleson measure if and only if $\delta^{(n+1)(1-\theta)}(z) B\mu(z) \to 0$ as $z \to \partial D$.

Remark 6.1.12 The condition "p-Carleson" is independent of any radius $r > 0$, while the condition "θ-Carleson" does not depend on p. Theorem 6.1.9 thus implies that if μ satisfies (6.3) for some $r > 0$ then it satisfies the same condition (with possibly different constants) for all $r > 0$; and that if μ is p-Carleson for $A^p\big(D, (n+1)$

$(\theta-1)\big)$ for some $1 < p < \infty$ then it is p-Carleson for $A^p\big(D, (n+1)(\theta-1)\big)$ for all $1 < p < \infty$.

In the next subsection we shall describe the proofs; we end this subsection showing why these results give a positive answer to the question raised by Čučković and McNeal.

Assume that T_{δ^β} maps $L^p(D)$ (and hence $A^p(D)$) into $A^{p+G}(D)$. By Theorem 6.1.8 $\delta^\beta\mu$ must be a p-Carleson measure for $A^p\big(D, (n+1)(\frac{1}{p}-\frac{1}{p+G})\big)$. By Theorem 6.1.9 this can happen if and only if $\delta^\beta\nu$ is a θ-Carleson measure, where

$$\theta = 1 + \frac{1}{p} - \frac{1}{p+G}\;; \tag{6.4}$$

notice that $1 \le \theta < 2$ because $p > 1$ and $G \ge 0$. So we need to understand when $\delta^\beta\nu$ is θ-Carleson. For this we need the following

Lemma 6.1.13 *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary, Then there exists $C > 0$ such that for every $z_0 \in D$ and $r > 0$ one has*

$$\forall z \in B_D(z_0, r) \quad C\mathrm{e}^{2r}\,\delta(z_0) \ge \delta(z) \ge \frac{\mathrm{e}^{-2r}}{C}\delta(z_0)\;.$$

Proof Let us fix $w_0 \in D$. Then Theorems 1.5.16 and 1.5.19 yield $c_0, C_0 > 0$ such that

$$\begin{aligned} c_0 - \tfrac{1}{2}\log\delta(z) &\le k_D(w_0, z) \le k_D(z_0, z) + k_D(z_0, w_0) \\ &\le r + C_0 - \tfrac{1}{2}\log\delta(z_0)\;, \end{aligned}$$

for all $z \in B_D(z_0, r)$, and hence

$$e^{2(c_0-C_0)}\delta(z_0) \le \mathrm{e}^{2r}\delta(z)\;.$$

The left-hand inequality is obtained in the same way reversing the roles of z_0 and z. □

Corollary 6.1.14 *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary, Given $\beta > 0$, put $\nu_\beta = \delta^\beta\nu$. Then ν_β is θ-Carleson if and only if $\beta \ge (n+1)(\theta-1)$.*

Proof Using Lemma 6.1.13 we find that

$$\begin{aligned} \frac{\mathrm{e}^{-2r}}{C}\delta(z_0)^\beta\nu\big(B_D(z_0, r)\big) \le \nu_\beta\big(B_D(z_0, r)\big) &= \int_{B_D(z_0,r)} \delta(z)^\beta\,\mathrm{d}\nu(z) \\ &\le C\mathrm{e}^{2r}\delta(z_0)^\beta\nu\big(B_D(z_0, r)\big) \end{aligned}$$

for all $z_0 \in D$. Therefore ν_β is θ-Carleson if and only if

$$\delta(z_0)^\beta \leq C_1 \nu\big(B_D(z_0,r)\big)^{\theta-1}$$

for some $C_1 > 0$. Recalling Theorem 1.5.23 we see that this is equivalent to requiring $\beta \geq (n+1)(\theta-1)$, and we are done. □

In our case, θ is given by (6.4); therefore $\beta \geq (n+1)(\theta-1)$ if and only if

$$\beta \geq (n+1)\left(\frac{1}{p} - \frac{1}{p+G}\right) .$$

Rewriting this in term of G we get

$$G \leq \frac{p^2}{\frac{n+1}{\beta} - p} ,$$

proving that the exponent in Theorem 6.0.9 is the best possible, as claimed. Furthermore, $G > 0$ if and only if

$$\frac{\beta}{n+1} < \frac{1}{p} \Leftrightarrow 1 - \frac{\beta}{n+1} > 1 - \frac{1}{p} \Leftrightarrow \frac{n+1}{n+1-\beta} < \frac{p}{p-1} ,$$

and we have also recovered condition (6.2) of Theorem 6.0.9.

Corollary 6.1.14 provides examples of θ-Carleson measures. A completely different class of examples is provided by Dirac masses distributed along uniformly discrete sequences.

Definition 6.1.15 Let (X,d) be a metric space. A sequence $\Gamma = \{x_j\} \subset X$ is *uniformly discrete* if there exists $\varepsilon > 0$ such that $d(x_j,x_k) \geq \varepsilon$ for all $j \neq k$.

Then it is possible to prove the following result:

Theorem 6.1.16 ([2]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞ boundary, considered as a metric space with the Kobayashi distance, and choose $1 - \frac{1}{n+1} < \theta < 2$. Let $\Gamma = \{z_j\}_{j\in\mathbb{N}}$ be a sequence in D. Then Γ is a finite union of uniformly discrete sequences if and only if $\sum_j \delta(z_j)^{(n+1)\theta}\delta_{z_j}$ is a θ-Carleson measure, where δ_{z_j} is the Dirac measure in z_j.*

6.2 Proofs

In this section we shall prove Theorems 6.1.8, 6.1.9 and 6.1.10. To do so we shall need a few technical facts on the Bergman kernel and on the Kobayashi distance. To simplify statements and proofs, let us introduce the following notation.

Definition 6.2.1 Let $D \subset \mathbb{C}^n$ be a domain. Given two non-negative functions $f, g: D \to \mathbb{R}^+$ we shall write $f \preceq g$ or $g \succeq f$ to say that there is $C > 0$ such that $f(z) \le Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc.).

The first technical fact we shall need is an integral estimate on the Bergman kernel:

Theorem 6.2.2 ([2, 12, 15]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Take $p > 0$ and $\beta > -1$. Then*

$$\int_D |K(w, z_0)|^p \delta(w)^\beta \, d\nu(w) \preceq \begin{cases} \delta(z_0)^{\beta-(n+1)(p-1)} & \text{if } -1 < \beta < (n+1)(p-1), \\ |\log \delta(z_0)| & \text{if } \beta = (n+1)(p-1), \\ 1 & \text{if } \beta > (n+1)(p-1), \end{cases}$$

for all $z_0 \in D$.

In particular, we have the following estimates on the weighted norms of the Bergman kernel and of the normalized Bergman kernel (see, e.g., [2]):

Corollary 6.2.3 *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Take $p > 1$ and $-1 < \beta < (n+1)(p-1)$. Then*

$$\|K(\cdot, z_0)\|_{p,\beta} \preceq \delta(z_0)^{\frac{\beta}{p} - \frac{n+1}{p'}} \quad \text{and} \quad \|k_{z_0}\|_{p,\beta} \preceq \delta(z_0)^{\frac{n+1}{2} + \frac{\beta}{p} - \frac{n+1}{p'}}$$

for all $z_0 \in D$, where $p' > 1$ is the conjugate exponent of p.

We shall also need a statement relating the Bergman kernel with Kobayashi balls.

Lemma 6.2.4 ([1, 12]) *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Given $r > 0$ there is $\delta_r > 0$ such that if $\delta(z_0) < \delta_r$ then*

$$\forall\, z \in B_D(z_0, r) \qquad \min\{|K(z, z_0)|, |k_{z_0}(z)|^2\} \succeq \delta(z_0)^{-(n+1)} .$$

Remark 6.2.5 Notice that Lemma 6.2.4 implies the well-known estimate

$$K(z_0, z_0) \succeq \delta(z_0)^{-(n+1)} ,$$

which is valid for all $z_0 \in D$.

The next three lemmas involve instead the Kobayashi distance only.

Lemma 6.2.6 ([1]) *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex bounded domain with C^2 boundary. Then for every $0 < r < R$ there exist $m \in \mathbb{N}$ and a sequence $\{z_k\} \subset D$ of points such that $D = \bigcup_{k=0}^\infty B_D(z_k, r)$ and no point of D belongs to more than m of the balls $B_D(z_k, R)$.*

Proof Let $\{B_j\}_{j\in\mathbb{N}}$ be a sequence of Kobayashi balls of radius $r/3$ covering D. We can extract a subsequence $\{\Delta_k = B_D(z_k, r/3)\}_{k\in\mathbb{N}}$ of disjoint balls in the following way: set $\Delta_1 = B_1$. Suppose we have already chosen $\Delta_1, \dots, \Delta_l$. We define Δ_{l+1} as the first ball in the sequence $\{B_j\}$ which is disjoint from $\Delta_1 \cup \cdots \cup \Delta_l$. In particular, by construction every B_j must intersect at least one Δ_k.

We now claim that $\{B_D(z_k, r)\}_{k\in\mathbb{N}}$ is a covering of D. Indeed, let $z \in D$. Since $\{B_j\}_{j\in\mathbb{N}}$ is a covering of D, there is $j_0 \in \mathbb{N}$ so that $z \in B_{j_0}$. As remarked above, we get $k_0 \in \mathbb{N}$ so that $B_{j_0} \cap \Delta_{k_0} \neq \emptyset$. Take $w \in B_{j_0} \cap \Delta_{k_0}$. Then

$$k_D(z, z_{k_0}) \le k_D(z, w) + k_D(w, z_{k_0}) < r ,$$

and $z \in B_D(z_{k_0}, r)$.

To conclude the proof we have to show that there is $m = m_r \in \mathbb{N}$ so that each point $z \in D$ belongs to at most m of the balls $B_D(z_k, R)$. Put $R_1 = R + r/3$. Since $z \in B_D(z_k, R)$ is equivalent to $z_k \in B_D(z, R)$, we have that $z \in B_D(z_k, R)$ implies $B_D(z_k, r/3) \subseteq B_D(z, R_1)$. Furthermore, Theorem 1.5.23 and Lemma 6.1.13 yield

$$\nu\big(B_D(z_k, r/3)\big) \succeq \delta(z_k)^{n+1} \succeq \delta(z)^{n+1}$$

when $z_k \in B_D(z, R)$. Therefore, since the balls $B_D(z_k, r/3)$ are pairwise disjoint, using again Theorem 1.5.23 we get

$$\operatorname{card}\{k \in \mathbb{N} \mid z \in B_D(z_k, R)\} \le \frac{\nu\big(B_D(z, R_1)\big)}{\nu\big(B_D(z_k, r/3)\big)} \preceq 1 ,$$

and we are done. □

Lemma 6.2.7 ([1]) *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex bounded domain with C^2 boundary, and $r > 0$. Then*

$$\chi(z_0) \preceq \frac{1}{\nu\big(B_D(z_0, r)\big)} \int_{B_D(z_0,r)} \chi \, d\nu$$

for all $z_0 \in D$ and all non-negative plurisubharmonic functions $\chi\colon D \to \mathbf{R}^+$.

Proof Let us first prove the statement when D is an Euclidean ball $\mathbb{B}$ of radius $R > 0$. Without loss of generality we can assume that $\mathbb{B}$ is centered at the origin. Fix $z_0 \in \mathbb{B}$, let $\gamma_{z_0/R} \in \mathrm{Aut}(\mathbb{B}^n)$ be given by (2.10), and let $\Phi_{z_0}\colon \mathbb{B}^n \to \mathbb{B}$ be defined by $\Phi_{z_0} = R\gamma_{z_0/R}$; in particular, Φ_{z_0} is a biholomorphism with $\Phi_{z_0}(O) = z_0$, and thus $\Phi_{z_0}\big(B_{\mathbb{B}^n}(O, \hat{r})\big) = B_{\mathbb{B}}(z_0, \hat{r})$. Furthermore (see [17, Theorem 2.2.6])

$$|\mathrm{Jac}_{\mathbb{R}} \Phi_{z_0}(z)| = R^{2n} \left(\frac{R^2 - \|z_0\|^2}{|R - \langle z, z_0\rangle|^2} \right)^{n+1} \ge \frac{R^{n-1}}{4^{n+1}} d(z_0, \partial\mathbb{B})^{n+1} ,$$

where $\mathrm{Jac}_{\mathbb{R}}\Phi_{z_0}$ denotes the (real) Jacobian determinant of Φ_{z_0}. It follows that

$$\begin{aligned}\int_{B_{\mathbb{B}}(z_0,r)} \chi \, \mathrm{d}\nu &= \int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0}) |\mathrm{Jac}_{\mathbb{R}}\, \Phi_{z_0}| \, \mathrm{d}\nu \\ &\geq \frac{R^{n-1}}{4^{n+1}} d(z_0, \partial\mathbb{B})^{n+1} \int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0}) \, \mathrm{d}\nu \ .\end{aligned}$$

Using [17, 1.4.3 and 1.4.7.(1)] we obtain

$$\int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0}) \, \mathrm{d}\nu = 2n \int_{\partial\mathbb{B}^n} \mathrm{d}\sigma(x) \frac{1}{2\pi} \int_0^{\tanh r} \int_0^{2\pi} \chi \circ \Phi_{z_0}(t e^{\mathrm{i}\theta} x) t^{2n-1} \mathrm{d}t \, \mathrm{d}\theta \ ,$$

where σ is the area measure on $\partial\mathbb{B}^n$ normalized so that $\sigma(\partial\mathbb{B}^n) = 1$. Now, $\zeta \mapsto \chi \circ \Phi_{z_0}(\zeta x)$ is subharmonic on $(\tanh r)\Delta = \{|\zeta| < \tanh r\} \subset \mathbb{C}$ for any $x \in \partial\mathbb{B}^n$, since Φ_{z_0} is holomorphic and χ is plurisubharmonic. Therefore [9, Theorem 1.6.3] yields

$$\frac{1}{2\pi} \int_0^{\tanh r} \int_0^{2\pi} \chi \circ \Phi_{z_0}(t e^{\mathrm{i}\theta} x) t^{2n-1} \mathrm{d}t \, \mathrm{d}\theta \geq \chi(z_0) \int_0^{\tanh r} t^{2n-1} \, \mathrm{d}t = \frac{1}{2n} (\tanh r)^{2n} \chi(z_0) \ .$$

So

$$\int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0}) \, \mathrm{d}\nu \geq (\tanh r)^{2n} \chi(z_0) \ ,$$

and the assertion follows from Theorem 1.5.23.

Now let D be a generic strongly pseudoconvex domain. Since D has C^2 boundary, there exists a radius $\varepsilon > 0$ such that for every $x \in \partial D$ the euclidean ball $\mathbb{B}_x(\varepsilon)$ of radius ε internally tangent to ∂D at x is contained in D.

Let $z_0 \in D$. If $\delta(z_0) < \varepsilon$, let $x \in \partial D$ be such that $\delta(z_0) = \|z_0 - x\|$; in particular, z_0 belongs to the ball $\mathbb{B}_x(\varepsilon) \subset D$. If $\delta(z_0) \geq \varepsilon$, let $\mathbb{B} \subset D$ be the Euclidean ball of center z_0 and radius $\delta(z_0)$. In both cases we have $\delta(z_0) = d(z_0, \partial\mathbb{B})$; moreover, the decreasing property of the Kobayashi distance yields $B_D(z_0, r) \supseteq B_{\mathbb{B}}(z_0, r)$ for all $r > 0$.

Let χ be a non-negative plurisubharmonic function. Then Theorem 1.5.23 and the assertion for a ball imply

$$\begin{aligned}\int_{B_D(z_0,r)} \chi \, d\nu &\geq \int_{B_{\mathbb{B}}(z_0,r)} \chi \, d\nu \succeq \nu\big(B_B(z_0, r)\big) \chi(z_0) \\ &\succeq d(z_0, \partial\mathbb{B})^{n+1} \chi(z_0) = \delta(z_0)^{n+1} \chi(z_0) \\ &\succeq \nu\big(B_D(z_0, r)\big) \chi(z_0) \ ,\end{aligned}$$

and we are done. □

Lemma 6.2.8 ([1]) *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex bounded domain with C^2 boundary. Given $0 < r < R$ we have*

$$\forall z_0 \in D \quad \forall z \in B_D(z_0, r) \qquad \chi(z) \preceq \frac{1}{\nu\big(B_D(z_0, r)\big)} \int_{B_D(z_0,R)} \chi \, \mathrm{d}\nu$$

for every nonnegative plurisubharmonic function $\chi \colon D \to \mathbb{R}^+$.

Proof Let $r_1 = R - r$; by the triangle inequality, $z \in B_D(z_0, r)$ yields $B_D(z, r_1) \subseteq B_D(z_0, R)$. Lemma 6.2.7 then implies

$$\begin{aligned}\chi(z) &\preceq \frac{1}{\nu(B_D(z, r_1))} \int_{B_D(z,r_1)} \chi \, \mathrm{d}\nu \\ &\le \frac{1}{\nu(B_D(z, r_1))} \int_{B_D(z_0,R)} \chi \, \mathrm{d}\nu = \frac{\nu(B_D(z_0, r))}{\nu(B_D(z, r_1))} \cdot \frac{1}{\nu(B_D(z_0, r))} \int_{B_D(z_0,R)} \chi \, \mathrm{d}\nu\end{aligned}$$

for all $z \in B_D(z_0, r)$. Now Theorem 1.5.23 and Lemma 6.1.13 yield

$$\frac{\nu(B_D(z_0, r))}{\nu(B_D(z, r_1))} \preceq 1$$

for all $z \in B_D(z_0, r)$, and so

$$\chi(z) \preceq \frac{1}{\nu\big(B_D(z_0, r)\big)} \int_{B_D(z_0,R)} \chi \, \mathrm{d}\nu$$

as claimed. □

Finally, the linking between the Berezin transform and Toeplitz operators is given by the following

Lemma 6.2.9 *Let μ be a finite positive Borel measure on a bounded domain $D \subset\subset \mathbb{C}^n$. Then*

$$B\mu(z) = \int_D (T_\mu k_z)(w) \overline{k_z(w)} \, \mathrm{d}\nu(w) \tag{6.5}$$

for all $z \in D$.

Proof Indeed using Fubini's theorem and the reproducing property of the Bergman kernel we have

$$\begin{aligned}B\mu(z) &= \int_D \frac{|K(x, z)|^2}{K(z, z)} \, \mathrm{d}\mu(x) \\ &= \int_D \frac{K(x, z)}{K(z, z)} K(z, x) \, \mathrm{d}\mu(x)\end{aligned}$$

$$
\begin{aligned}
&= \int_D \frac{K(x,z)}{K(z,z)} \left(\int_D K(w,x)K(z,w)\, \mathrm{d}\nu(w) \right) \mathrm{d}\mu(x) \\
&= \int_D \left(\int_D \frac{K(x,z)}{\sqrt{K(z,z)}} K(w,x)\, \mathrm{d}\mu(x) \right) \frac{\overline{K(w,z)}}{\sqrt{K(z,z)}}\, \mathrm{d}\nu(w) \\
&= \int_D \left(\int_D K(w,x)k_z(x)\, \mathrm{d}\mu(x) \right) \overline{k_z(w)}\, \mathrm{d}\nu(w) \\
&= \int_D (T_\mu k_z)(w)\overline{k_z(w)}\, \mathrm{d}\nu(w) ,
\end{aligned}
$$

as claimed. □

We can now prove Theorems 6.1.8, 6.1.9 and 6.1.10.

Proof (of Theorem 6.1.9) Assume that μ is a p-Carleson measure for $A^p\big(D, (n+1)(\theta-1)\big)$, and fix $r > 0$; we need to prove that $\mu\big(B_D(z_0, r)\big) \preceq \nu\big(B_D(z_0, r)\big)^\theta$ for all $z_0 \in D$.

First of all, it suffices to prove the assertion for z_0 close to the boundary, because both μ and ν are finite measures. So we can assume $\delta(z_0) < \delta_r$, where δ_r is given by Lemma 6.2.4. Since, by Corollary 6.2.3, $k_{z_0}^2 \in A^p\big(D, (n+1)(\theta-1)\big)$, we have

$$
\begin{aligned}
\frac{1}{\delta(z_0)^{(n+1)p}} \mu\big(B_D(z_0,r)\big) &\preceq \int_{B_D(z_0,r)} |k_{z_0}(w)|^{2p}\, \mathrm{d}\mu(w) \le \int_D |k_{z_0}(w)|^{2p}\, \mathrm{d}\mu(w) \\
&\preceq \int_D |k_{z_0}(w)|^{2p} \delta(w)^{(n+1)(\theta-1)}\, \mathrm{d}\nu(w) \\
&\preceq \delta(z_0)^{(n+1)p} \int_D |K(w,z_0)|^{2p} \delta(w)^{(n+1)(\theta-1)}\, \mathrm{d}\nu(w) \\
&\preceq \delta(z_0)^{(n+1)(\theta-p)}
\end{aligned}
$$

by Theorem 6.2.2, that we can apply because $1 - \frac{1}{n+1} < \theta < 2$. Recalling Theorem 1.5.23 we see that μ is θ-Carleson.

Conversely, assume that μ is θ-Carleson for some $r > 0$, and let $\{z_k\}$ be the sequence given by Lemma 6.2.6. Take $f \in A^p\big(D, (n+1)(\theta-1)\big)$. First of all

$$
\int_D |f(z)|^p\, \mathrm{d}\mu(z) \le \sum_{k\in\mathbb{N}} \int_{B_D(z_k,r)} |f(z)|^p\, \mathrm{d}\mu(z) .
$$

Choose $R > r$. Since $|f|^p$ is plurisubharmonic, by Lemma 6.2.8 we get

$$
\begin{aligned}
\int_{B_D(z_k,r)} |f(z)|^p\, \mathrm{d}\mu(z) &\preceq \frac{1}{\nu\big(B_D(z_k,r)\big)} \int_{B_D(z_k,r)} \left[\int_{B_D(z_k,R)} |f(w)|^p\, \mathrm{d}\nu(w) \right] \mathrm{d}\mu(z) \\
&\preceq \nu\big(B_D(z_k,r)\big)^{\theta-1} \int_{B_D(z_k,R)} |f(w)|^p\, \mathrm{d}\nu(w)
\end{aligned}
$$

because μ is θ-Carleson. Recalling Theorem 1.5.23 and Lemma 6.1.13 we get

$$\int_{B_D(z_k,r)} |f(z)|^p \, \mathrm{d}\mu(z) \preceq \delta(z_k)^{(n+1)(\theta-1)} \int_{B_D(z_k,R)} |f(w)|^p \, \mathrm{d}\nu(w)$$
$$\preceq \int_{B_D(z_k,R)} |f(w)|^p \delta(w)^{(n+1)(\theta-1)} \, \mathrm{d}\nu(w) \, .$$

Since, by Lemma 6.2.6, there is $m \in \mathbb{N}$ such that at most m of the balls $B_D(z_k, R)$ intersect, we get

$$\int_D |f(z)|^p \, \mathrm{d}\mu(z) \preceq \int_D |f(w)|^p \delta(w)^{(n+1)(\theta-1)} \, \mathrm{d}\nu(w) \, ,$$

and so we have proved that μ is p-Carleson for $A^p\big(D, (n+1)(\theta-1)\big)$. □

We explicitly remark that the proof of the implication "θ-Carleson implies p-Carleson for $A^p\big(D, (n+1)(\theta-1)\big)$" works for all $\theta > 0$, and actually gives the following

Corollary 6.2.10 *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^2 boundary, $\theta > 0$, and μ a θ-Carleson measure on D. Then*

$$\int_D \chi(z) \, d\mu(z) \preceq \int_D \chi(w) \delta(w)^{(n+1)(\theta-1)} \, \mathrm{d}\nu(w)$$

for all nonnegative plurisubharmonic functions $\chi : D \to \mathbb{R}^+$ such that $\chi \in L^p\big(D, (n+1)(\theta-1)\big)$.

Now we prove the equivalence between θ-Carleson and the condition on the Berezin transform.

Proof (of Theorem 6.1.10) Let us first assume that μ is θ-Carleson. By Theorem 6.1.9 we know that μ is 2-Carleson for $A^2\big(D, (n+1)(\theta-1)\big)$. Fix $z_0 \in D$. Then Corollary 6.2.3 yields

$$B\mu(z_0) = \int_D |k_{z_0}(w)|^2 \, \mathrm{d}\mu(w) \preceq \|k_{z_0}\|^2_{2,(n+1)(\theta-1)} \preceq \delta(z_0)^{(n+1)(\theta-1)} \, ,$$

as required.

Conversely, assume that $\delta^{(n+1)(1-\theta)} B\mu \in L^\infty(D)$, and fix $r > 0$. Then Lemma 6.2.4 yields

$$\delta(z_0)^{(n+1)(\theta-1)} \succeq B\mu(z_0) = \int_D |k_{z_0}(w)|^2 \, \mathrm{d}\mu(w) \geq \int_{B_D(z_0,r)} |k_{z_0}(w)|^2 \, \mathrm{d}\mu(w)$$
$$\succeq \frac{1}{\delta(z_0)^{n+1}} \mu\big(B_D(z_0, r)\big)$$

as soon as $\delta(z_0) < \delta_r$, where $\delta_r > 0$ is given by Lemma 6.2.4. Recalling Theorem 1.5.23 we get

$$\mu\big(B_D(z_0,r)\big) \preceq \delta(z_0)^{(n+1)\theta} \preceq \nu\big(B_D(z_0,r)\big)^\theta ,$$

and the assertion follows when $\delta(z_0) < \delta_r$. When $\delta(z_0) \ge \delta_r$ we have

$$\mu\big(B_D(z_0,r)\big) \le \mu(D) \preceq \delta_r^{(n+1)\theta} \le \delta(z_0)^{(n+1)\theta} \preceq \nu\big(B_D(z_0,r)\big)^\theta$$

because μ is a finite measure, and we are done. □

For the last proof we need a final

Lemma 6.2.11 *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^2 boundary, and $\theta, \eta \in \mathbb{R}$. Then a finite positive Borel measure μ is θ-Carleson if and only if $\delta^\eta\mu$ is $(\theta + \frac{\eta}{n+1})$-Carleson.*

Proof Assume μ is θ-Carleson, set $\mu_\eta = \delta^\eta\mu$, and choose $r > 0$. Then Theorem 1.5.23 and Lemma 6.1.13 yield

$$\begin{aligned}\mu_\eta\big(B_D(z_0,r)\big) &= \int_{B_D(z_0,r)} \delta(w)^\eta\, d\mu(w) \preceq \delta(z_0)^\eta \mu\big(B_D(z_0,r)\big)\\ &\preceq \delta(z_0)^\eta \nu\big(B_D(z_0,r)\big)^\theta \preceq \nu\big(B_D(z_0,r)\big)^{\theta+\frac{\eta}{n+1}} ,\end{aligned}$$

and so μ_η is $\big(\theta + \frac{\eta}{n+1}\big)$-Carleson. Since $\mu = (\mu_\eta)_{-\eta}$, the converse follows too. □

And at last we have reached the

Proof (of Theorem 6.1.8) Let us assume that T_μ maps $A^p(D)$ continuously into $A^r(D)$, and let r' be the conjugate exponent of r. Since, by Corollary 6.2.3, $k_{z_0} \in A^q(D)$ for all $q > 1$, applying Hölder estimate to (6.5) and using twice Corollary 6.2.3 we get

$$\begin{aligned}B\mu(z_0) &\le \|T_\mu k_{z_0}\|_r \|k_{z_0}\|_{r'} \preceq \|k_{z_0}\|_p \|k_{z_0}\|_{r'}\\ &\preceq \delta(z_0)^{(n+1)(1-\frac{1}{p'}-\frac{1}{r})} = \delta(z_0)^{(n+1)(\frac{1}{p}-\frac{1}{r})} ,\end{aligned}$$

where p' is the conjugate exponent of p. By Theorem 6.1.10 it follows that μ is $\left(1 + \frac{1}{p} - \frac{1}{r}\right)$-Carleson, and Theorem 6.1.9 yields that μ is p-Carleson for $A^p\big(D, (n+1)(\frac{1}{p} - \frac{1}{r})\big)$ as claimed.

Conversely, assume that μ is p-Carleson for $A^p\big(D, (n+1)(\frac{1}{p} - \frac{1}{r})\big)$; we must prove that T_μ maps continuously $A^p(D)$ into $A^r(D)$. Put $\theta = 1 + \frac{1}{p} - \frac{1}{r}$. Choose $s \in (p, r)$ such that

$$\frac{\theta}{p'} < \frac{1}{s'} < \frac{\theta}{p'} + \frac{1}{(n+1)r} , \tag{6.6}$$

where s' be its conjugate exponent of s; this can be done because $p' \geq s' \geq r'$ and

$$\frac{\theta}{p'} < \frac{1}{r'} \; .$$

Take $f \in A^p(D)$; since $|K(z,\cdot)|^{p'/s'}$ is plurisubharmonic and belongs to $L^p\big(D, (n+1)(\theta-1)\big)$ applying the Hölder inequality, Corollary 6.2.10 and Theorem 6.2.2 (recalling that $\theta < p'/s'$) we get

$$\begin{aligned}
|T_\mu f(z)| &\leq \int_D |K(z,w)||f(w)|\,\mathrm{d}\mu(w)\\
&\leq \left[\int_D |K(z,w)|^{p/s}|f(w)|^p\,\mathrm{d}\mu(w)\right]^{1/p}\left[\int_D |K(z,w)|^{p'/s'}\,\mathrm{d}\mu(w)\right]^{1/p'}\\
&\preceq \left[\int_D |K(z,w)|^{p/s}|f(w)|^p\,\mathrm{d}\mu(w)\right]^{1/p}\\
&\qquad\times\left[\int_D |K(z,w)|^{p'/s'}\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)\right]^{1/p'}\\
&\preceq \left[\int_D |K(z,w)|^{p/s}|f(w)|^p\,\mathrm{d}\mu(w)\right]^{1/p}\delta(z)^{(n+1)\frac{1}{p'}(\theta-\frac{p'}{s'})}\;.
\end{aligned}$$

Applying the classical Minkowski integral inequality (see, e.g., [7, 6.19] for a proof)

$$\left[\int_D\left[\int_D |F(z,w)|^p\,\mathrm{d}\mu(w)\right]^{r/p}\mathrm{d}\nu(z)\right]^{1/r} \leq \left[\int_D\left[\int_D |F(z,w)|^r\,\mathrm{d}\nu(z)\right]^{p/r}\mathrm{d}\mu(w)\right]^{1/p}$$

we get

$$\begin{aligned}
\|T_\mu f\|_r^p &\preceq \left[\int_D\left[\int_D |K(z,w)^{p/s}|f(w)|^p\delta(z)^{(n+1)\frac{p}{p'}(\theta-\frac{p'}{s'})}\,\mathrm{d}\mu(w)\right]^{r/p}\mathrm{d}\nu(z)\right]^{p/r}\\
&\leq \int_D |f(w)|^p\left[\int_D |K(z,w)|^{r/s}\delta(z)^{(n+1)\frac{r}{p'}(\theta-\frac{p'}{s'})}\,\mathrm{d}\nu(z)\right]^{p/r}\mathrm{d}\mu(w)\;.
\end{aligned}$$

To estimate the integral between square brackets we need to know that

$$-1 < (n+1)\frac{r}{p'}\left(\theta-\frac{p'}{s'}\right) < (n+1)\left(\frac{r}{s}-1\right)\;.$$

The left-hand inequality is equivalent to the right-hand inequality in (6.6), and thus it is satisfied by assumption. The right-hand inequality is equivalent to

$$\frac{\theta}{p'} - \frac{1}{s'} < \frac{1}{s} - \frac{1}{r} \iff \frac{\theta}{p'} < 1 - \frac{1}{r} .$$

Recalling the definition of θ we see that this is equivalent to

$$\frac{1}{p'}\left(1 + \frac{1}{p} - \frac{1}{r}\right) < 1 - \frac{1}{r} \iff \frac{1}{p'} < 1 - \frac{1}{r} ,$$

which is true because $p < r$. So we can apply Theorem 6.2.2 and we get

$$\begin{aligned} \|T_\mu f\|_r^p &\preceq \int_D |f(w)|^p \delta(w)^{(n+1)p\left[\frac{1}{p'}(\theta-1)+\frac{1}{r}-\frac{1}{p}\right]} \mathrm{d}\mu(w) \\ &= \int_D |f(w)|^p \delta(w)^{-(n+1)(\theta-1)}\, \mathrm{d}\mu(w) \\ &\preceq \|f\|_p^p , \end{aligned}$$

where in the last step we applied Theorem 6.1.9 to $\delta^{-(n+1)(\theta-1)}\mu$, which is 1-Carleson (Lemma 6.2.11) and hence p-Carleson for $A^p(D)$, and we are done. □

Acknowledgements Partially supported by the FIRB 2012 grant "Differential Geometry and Geometric Function Theory", by the Progetto di Ricerca d'Ateneo 2015 "Sistemi dinamici: logica, analisi complessa e teoria ergodica", and by GNSAGA-INdAM.

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Appendix A
Geometric Analysis in One Complex Variable

Hervé Pajot

A.1 Metrics and Curvature

In this section, we recall basic notions of differential geometry in the particular case of the complex plane. We first recall that if $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve (for instance, C^1 or Lipschitz), its Euclidean length is given by $l_{eucl}(\gamma) = \int_a^b ||\gamma'(t)||\mathrm{d}t$ where $||\gamma'(t)||$ is the Euclidean norm of the tangent vector $\gamma'(t)$.

Let Ω be a domain in the complex plane $\mathbb{C}$. A metric ρ on Ω is a continuous function so that $\rho(z) \geq 0$ for any $z \in \Omega$ and ρ is twice continuously differentiable on $\{z \in \Omega; \rho(z) > 0\}$. Most of the time, we will assume that $\rho(z) > 0$ everywhere. If $z \in \Omega$ and $\xi \in \mathbb{C}$ (ξ should be seen as a vector, for instance in the tangent space of z), we set $||\xi||_{z,\rho} = \rho(z).||\xi||$ where $||\xi||$ denotes the Euclidean length of ξ.

If $\gamma : [a, b] \to \Omega$ is a continuously differentiable path, then the length of γ with respect to the metric ρ is defined by

$$l_\rho(\gamma) = \int_a^b ||\gamma'(t)||_{\rho,\gamma(t)}\mathrm{d}t.$$

The associated distance between $z, z' \in \Omega$ is given by

$$d_\rho(z, z') = \inf l_\rho(\gamma)$$

where the infimum is taken over all the continuously differentiable paths γ : $[a, b] \to \Omega$ so that $\gamma(a) = z$ and $\gamma(b) = z$.

We now go to the notion of isometry in the sense of differential geometry.

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L. Blanc-Centi (ed.), *Metrical and Dynamical Aspects in Complex Analysis*,
Lecture Notes in Mathematics 2195, DOI 10.1007/978-3-319-65837-7

Definition A.1.1 Let Ω_1 and Ω_2 be two domains in the complex plane $\mathbb{C}$ equipped with metrics ρ_1 and ρ_2 respectively. Then, a one-to-one continuously differentiable map $f : \Omega_1 \to \Omega_2$ is an *isometry* from (Ω_1, ρ_1) to (Ω_2, ρ_2) if $f^*\rho_2(z) = \rho_1(z)$ for any $z \in \Omega_1$. Here, $f^*\rho_2(z)$ is the pull-back of the metric ρ_2 under f and is defined by $f^*\rho_2(z) = \rho_2(f(z)) \left|\frac{\partial f}{\partial z}\right|$.

We leave to the reader to check that $f^*\rho_2(z)$ is a metric on Ω_1. The connection with the classical notion of isometry is given by the next result.

Proposition A.1.2 *Let Ω_1 and Ω_2 be two domains in the complex plane $\mathbb{C}$ equipped with metrics ρ_1 and ρ_2 respectively. If $f : \Omega_1 \to \Omega_2$ is holomorphic and is an isometry in the previous sense, then $d_{\rho_1}(z, z') = d_{\rho_2}(f(z), f(z'))$ whenever $z, z' \in \Omega_1$.*

Proof We have to prove that if $\gamma : [a, b] \to \Omega_1$ is a C^1-curve so is $f_*\gamma = f \circ \gamma$ and $l_{\rho_1}(\gamma) = l_{\rho_2}(f*\gamma)$. Note that by definition and by the usual chain rule, we have

$$
\begin{aligned}
l_{\rho_2}(f_*\gamma) &= \int_a^b ||(f_*\gamma)'(t)||_{\rho_2, f_*\gamma(t)} \mathrm{d}t \\
&= \int_a^b ||\frac{\partial f}{\partial z}(\gamma(t)).\gamma'(t)||_{\rho_2, f_*\gamma(t)} \mathrm{d}t \\
&= \int_a^b \left|\frac{\partial f}{\partial z}(\gamma(t))\right| .||\gamma'(t)||.\rho_2(f(\gamma(t))\mathrm{d}t \\
&= \int_a^b ||\gamma'(t)||_{f^*\rho_2, \gamma(t)} \mathrm{d}t \\
&= \int_a^b ||\gamma'(t)||_{\rho_1, \gamma(t)} \mathrm{d}t (\text{since } f \text{ is an isometry}) \\
&= l_{\rho_1}(\gamma).
\end{aligned}
$$

The last notion we need is the curvature of a metric.

Definition A.1.3 If ρ is a metric on the domain $\Omega \subset \mathbb{C}$ then its *curvature* at $z \in \Omega$ is defined by

$$
K_{\Omega,\rho}(z) = -\frac{\Delta \log \rho(z)}{\rho(z)^2}.
$$

Note that the curvature has singularities where $\rho(z) = 0$. The curvature is conformally invariant in the following sense.

Proposition A.1.4 *Let Ω_1 and Ω_2 be two domains in the complex plane and let $f : \Omega_1 \to \Omega_2$ be a conformal map. If ρ_2 is a metric on Ω_2, then for any $z \in \Omega_1$, $K_{\Omega_1, f^*\rho_2}(z) = K_{\Omega_2, \rho_2}(z)$.*

Proof Direct computations give

$$\begin{aligned} K_{\Omega_1, f^*\rho_2} &= \frac{-\Delta \log(\rho_2(f(z)).|f'(z)|)}{(\rho_2(f(z)).(f'(z)))^2} \\ &= \frac{-\Delta \log(\rho_2(f(z)) - \Delta \log(h'(z)|)}{(\rho_2(f(z)).(f'(z)))^2} \\ &= \frac{-\Delta \log(\rho_2|(f(z))|f'(z)|^2}{(\rho_2(f(z)).(f'(z)))^2} \\ &= \frac{-\Delta \log \rho_2(f(z))}{\rho_2(f(z))^2} \\ &= K_{\Omega_2,\rho_2}(f(z)). \end{aligned}$$

Example A.1.5 The more basic example is the case of the Euclidean metric: $\rho_{eucl}(z) = 1$, so the curvature of this metric is zero everywhere. Thus, the Euclidean metric is a flat metric (which is what we can expect !).

A.2 The Schwarz-Pick Lemma and the Poincaré Metric

The main goal of this section is to define the Poincaré metric on the unit disc $\Delta = \{z \in \mathbb{C}; |z| < 1\}$ of $\mathbb{C}$ and to show a conformally invariant version of the Schwarz lemma which is useful to study isometries and geodesics for this metric.

We first recall the classical version of Schwarz lemma.

Theorem A.2.1 (Schwarz) *Let f be an analytic function in Δ. Assume that $|f(z)| \leq 1$ for any $z \in \Delta$ and that $f(0) = 0$. Then $|f(z)| \leq |z|$ for any $z \in \Delta$. Furthermore, if equality holds at some $z_0 \in \Delta$, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ so that $f(z) = \lambda z$ for any $z \in \Delta$.*

The proof which is based on the maximum principle could be found in any textbook in complex analysis. If we take $z \to 0$, we get the

Corollary A.2.2 *Let f be an analytic function in Δ. Assume that $|f(z)| \leq 1$ for any $z \in \Delta$ and that $f(0) = 0$. Then, $|f'(0)| \leq 1$, with equality if and only if $f(z) = \lambda z$ with $|\lambda| = 1$.*

A classical application of the Schwarz lemma is the characterization of conformal self-maps of the unit disc.

Theorem A.2.3 *The conformal self-maps of the unit disc Δ are precisely the fractional linear transformation of the form (for any $z \in \Delta$)*

$$\gamma_{a,\theta}(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}$$

where a is a complex number with $|a| = 1$ and $0 \leq \theta \leq 2\pi$.

For the convenience of the reader, we give a proof even if the material is quite standard.

Proof

1) First, note that $\gamma_{a,\theta}$ is a conformal self-map of Δ. So we have to prove that all conformal self-maps of Δ have this form.
2) Assume that $g : \Delta \to \Delta$ is conformal with $g(0) = 0$. We can apply the Schwarz lemma to g to get $|g(z)| \leq |z|$ for any $z \in \Delta$ and to g^{-1} to get $|z| \leq |g(z)|$ for any $z \in \Delta$. Hence, $g(z)/z$ has constant modulus (equal to 1). This implies that $g(z)/z$ is constant and g is a rotation (that is g is of the form $g(z) = e^{i\theta} z$ for some $0 \leq \phi \leq 2\pi$).
3) Let $g : \Delta \to \mathbb{R}$ is a conformal map. Set $a = g^{-1}(0)$ and $h(z) = \dfrac{z-a}{1-\overline{a}z}$. Then $g \circ h^{-1}$ is conformal and $(g \circ h^{-1})(0) = g(a) = 0$. By point 2, this implies that there exists $0 \leq \phi \leq 2\pi$ so that $(g \circ h^{-1})(w) = e^{i\theta} w$ for any $w \in \Delta$. By applying the previous equality to $w = h(z)$ for any $z \in \Delta$, we get $g(z) = e^{i\theta} h(z)$ and the proof is complete.

We now give a version of Schwarz lemma which is invariant under conformal self-maps of Δ.

Theorem A.2.4 (Schwarz-Pick) *If $f : \Delta \to \Delta$ is analytic, then for any $z \in \Delta$,*

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}. \tag{A.1}$$

If f is conformal then equality in (A.1) holds everywhere in Δ. Otherwise, the inequality is strict for any $z \in \Delta$.

Proof Fix $z_0 \in \Delta$ and set $w_0 = f(z_0)$. We now consider the conformal self-maps g and h mapping 0 to z_0 and w_0 to 0 respectively, that is

$$g(z) = \frac{z + z_0}{1 + \overline{z_0} z} \text{ and } g(w) = \frac{w - w_0}{1 - \overline{w_0} w}.$$

Then, $h \circ f \circ g(0) = 0$ and we can apply the previous corollary to get

$$|(h \circ f \circ g)'(0)| = |h'(w_0) f'(z_0) g'(0)| \leq 1. \tag{A.2}$$

Since $g'(0) = 1 - |z_0|^2$ and $h'(w_0) = 1/(1 - |w_0|^2)$, we get (A.1).

If $f : \Delta \to \Delta$ is conformal, then $h \circ f \circ g$ is also conformal and we get equality in (A.2) and hence in (A.1). Conversely, assume that $f : \Delta \to \Delta$ is an analytic function such that equality (A.1) holds at some point z_0. The previous computations show that we have also equality in (A.2) and we can easily conclude by using the corollary above.

We now give a geometric interpretation of the Schwarz-Pick lemma that leads to the definition of the Poincaré metric. Let $w = f(z)$ be a conformal self-map of the unit disc Δ. Then, by the Schwarz-Pick lemma, we get $\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1-|w|^2}{1-|z|^2}$, that is in differential form $\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}$. This means that for any smooth curve γ, we have

$$\int_{f\circ\gamma} \frac{|\mathrm{d}w|}{1-|w|^2} = \int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2}.$$

Thus, if we set $\rho_P(z) = \frac{1}{1-|z|^2}$, we get a metric which is invariant under conformal self-maps of the unit disc Δ. This metric is usually called the *Poincaré metric*, and we denote by k_Δ the associated distance. The Poincaré metric has curvature -4. Indeed, $-\Delta \log \rho_P(z) = \Delta \log(1-|z|^2) = 4(\partial/\partial z)(\partial/\partial \bar{z}) \log(1-|z|^2)$. By writing $|z|^2 = z.\bar{z}$, we easily get $-\Delta \log \rho_P(z) = -4/(1-|z|^2)^2 = -4(\rho_P(z))^2$ and thus $K_{\rho_P}(z) = -4$ for any $z \in \Delta$.

Let γ be the path defined by $\gamma(t) = t$ for $0 \leq t \leq 1-\varepsilon$. Then,

$$l_{\rho_P}(\gamma) = \int_0^{1-\varepsilon} \frac{|\gamma'(t)|}{1+|\gamma(t)|^2}\mathrm{d}t = \int_0^{1-\varepsilon} \frac{1}{1-t^2}\mathrm{d}t = \frac{1}{2}\ln\left(\frac{2-\varepsilon}{\varepsilon}\right).$$

In particular, note that $\lim_{\varepsilon\to 0} l_{\rho_P}(\gamma) = +\infty$. Let α be another curve joining 0 and $1-\varepsilon$: $\alpha(t) = t + iw(t)$ for $0 \leq t \leq 1-\varepsilon$ so that $\alpha(0) = 0$ and $\alpha(1-\varepsilon) = 1-\varepsilon$. Then,

$$l_{\rho_P}(\alpha) = \int_0^{1-\varepsilon} \frac{1}{1-t^2-(w(t))^2}(1+(w'(t))^2)^{1/2}\mathrm{d}t \geq \int_0^{1-\varepsilon} \frac{1}{1-t^2}\mathrm{d}t.$$

This suggest that straightlines should be geodesics for the Poincaré metric. We now give a precise statement.

Proposition A.2.5 *For any distinct points z_0, z_1 in the unit disc Δ, there is a unique shortest path in Δ from z_0 to z_1 in the hyperbolic metric, that is the arc of circle passing through z_0 and z_1 that is orthogonal to the unit circle. In particular, if $z_0 = 0$, the shortest path is just the segment* $[0, z_1)$.

In particular, the last statement implies by basic computations that $k_\Delta(0,z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)$. Thus, if $|z| \to 1$, then $k_\Delta(0,z)$ tends to infinity.

Proof Let f be a conformal self-map of Δ so that $f(z_0) = 0$. By multiplying by a constant with modulus 1, we can also assume that $f(z_1) = r > 0$. Recall that conformal mappings preserve the crossratio. Hence, f maps circles orthogonal to the unit circle onto circles orthogonal to the unit circle. Moreover f preserves the hyperbolic length as we have seen previously. So it is enough to prove that the segment $[0, r]$ is the unique geodesic for the Poincaré metric from 0 to r. To see this, consider a C^1 path $\alpha : [0, 1] \to \Delta$ joining 0 and r. Then, $\gamma(t) = Re(\alpha(t)) = x(t)$ (if we set $\alpha(t) = x(t) + iy(t)$) defines a path in Δ from 0 to r contained in the real axis and we have

$$\int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2} = \int_0^1 \frac{|\mathrm{d}x(t)|}{1-|x(t)|^2} \leq \int_0^1 \frac{|\mathrm{d}x(t)|}{1-|\alpha(t)|^2} \leq \int_\alpha \frac{|\mathrm{d}z|}{1-|z|^2}.$$

If $y(t) \neq 0$ for some t, the previous inequality is strict and the path γ is strictly shorter (with respect to the Poincaré metric) than the path α. To get a geodesic, we have to assume that γ is nondecreasing and the proof is complete.

We can now give a geometric interpretation of the fact that the curvature of the Poincaré metric is negative. A geodesic triangle in Δ is an area bounded by three hyperbolic geodesics (arcs of circle that are orthogonal to the unit circle). It is not difficult to see that the sum of the angles in a geodesic triangle of Δ is less that π which is exactly the sum of angles in an Euclidean triangle. Hence, geodesic triangles in Δ are thinner that Euclidean triangles. The notion of Gromov hyperbolicity is based on this observation.

Theorem A.2.6 *Every analytic function $f : \Delta \to \Delta$ is a contraction for the Poincaré distance:*

$$k_\Delta(f(z_0), f(z_1)) \leq k_\Delta(z_0, z_1) \textit{ for any } z_0, z_1 \in \Delta.$$

Furthermore, the inequality is strict if $z_0 \neq z_1$, unless f is a conformal self-map of the unit disc Δ. In this case, f is an isometry for the Poincaré distance, in the sense that it preserves the Poincaré distance:

$$k_\Delta(f(z_0), f(z_1)) = k_\Delta(z_0, z_1) \textit{ for any } z_0, z_1 \in \Delta.$$

Proof Take a geodesic curve in Δ joining z_0 and z_1 (as given by the previous proposition). Then $f \circ \alpha$ is a curve (in general not a geodesic !) from $f(z_0)$ to $f(z_1)$. By the definition of the Poincaré distance and by the Schwarz-Pick lemma, we easily get

$$k_\Delta(f(z_0), f(z_1)) \leq \int_{f\circ\gamma} \frac{|\mathrm{d}w|}{1-|w|^2} = \int_\gamma \frac{|f'(z)||\mathrm{d}z|}{1-|f(z)|^2} \leq \int_\gamma \frac{|\mathrm{d}z|}{1-|z|^2} = k_\Delta(z_0, z_1).$$

The case of equality is the same as in the Schwarz-Pick lemma.

As an application of the previous theorem, we have the following description of the Poincaré distance.

Proposition A.2.7 *If z_0 and z_1 are in the unit disc Δ, then the Poincaré distance between these points is given by*

$$k_\Delta(z_0, z_1) = \tfrac{1}{2} \log \left(\frac{1 + \left| \frac{z_0 - z_1}{1 - \overline{z_0} z_1} \right|}{1 - \left| \frac{z_0 - z_1}{1 - \overline{z_0} z_1} \right|} \right).$$

In particular, $k_\Delta(0,z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)$.

Proof We have already see how to compute $k_\Delta(0,z)$. In the general case, we use this formula and we consider the Möbius transformation $\phi(z) = \dfrac{z-z_0}{1-\overline{z_0}z}$. Then, by the previous theorem, we get

$$k_\Delta(z_0,z_1) = k_\Delta(\phi(z_0),\phi(z_1)) = k_\Delta(0,\phi(z_1)).$$

By using the invariance of d_P under rotations, we get $k_\Delta(z_0,z_1) = k_\Delta(0,|\phi(z_1|)$ and we can easily conclude.

We already mentioned that the limit of $k_\Delta(0,z)$ is infinite when $|z| \to 1$. Thus, the distance between the origin 0 of the unit disc Δ and the unit circle is infinite! This motivates the notion of boundary at infinity (see Sect. 3.1 for a more precise definition in the general setting). The boundary at infinity $\partial_\infty\Delta$ is the set of geodesic rays starting from the origin 0, that is by the previous proposition, $\partial_\infty\Delta$ is the set of "lines" of the form $L_\theta = \{re^{i\theta}; 0 \le r < 1\}$. Of course, this boundary could be identified with the unit disc $\{z \in \mathbb{C}; |z| = 1\}$.

Remark A.2.8 It is not so difficult to check that the topology induced by the Poincaré distance is the Euclidean topology and that the unit disc equipped with the Poincaré distance is a complete metric space.
There are other interesting invariant metrics in the complex plane.

- The *Carathéodory metric* is given at a point z in a domain Ω by

$$\rho_C^\Omega(z) = \sup|f'(z)|$$

  where the supremum is taken over all holomorphic functions $f : \Omega \to \Delta$ with $f(z) = 0$. Note that this definition is related to the optimisation problem in the classical proof of the Riemann mapping theorem.
- The *Kobayashi metric* is given at a point z in a domain Ω by

$$\rho_K^\Omega(z) = \inf|1/f'(z)|$$

  where the infimum is taken over all holomorphic functions $f : \Omega \to \Delta$ with $f(0) = z$.

In some sense, the Kobayashi metric is obtained from the Carathéodory metric by duality.

Theorem A.2.9 *In the case of the unit disc (that is $\Omega = \Delta$), the Carathéordory metric and the Kobayashi metric coincide with the Poincaré metric.*

In particular, this implies that the Kobayashi metric has curvature -4. We conclude this section with a version of Schwarz-Pick lemma in higher dimensions.

Theorem A.2.10 *Let X and Y be two Kähler manifolds with the same dimension. Denote by g_Y and g_X their Riemannian metrics. Assume furthermore that Y is compact and that $Ricc_{g_Y} \geq -g_Y$ and $Ricc_{g_X} \leq -g_X$ everywhere on Y and on X. Then any holomorphic map $f : Y \to X$ satisfies $|Jacf(y)| \leq 1$ for any $y \in Y$ (where $Jacf(y)$ denotes the Jacobian of f at y). Moreover, if we have equality at some $y \in Y$, then the tangent map $d_y f$ is an isometry (that is preserves the metrics).*

A.3 Curvature and Complex Analysis in the Complex Plane

We start with Ahlfors's version of the Schwarz lemma. First, we need a notation. Let $\Delta(0, r)$ be the open disc with center 0 and radius $0 < r \leq 1$. For $A > 0$, we define the metric ρ_r^A by

$$\rho_r^A(z) = \frac{2r}{\sqrt{A}(r^2 - |z|^2)}.$$

It is not difficult to check that the curvature of this metric is $-A$.

Theorem A.3.1 *Let Ω a domain in the complex plane. We assume that this domain is equipped with a metric ρ whose curvature is bounded above by a negative constant $-B$. Then, every holomorphic function $f : \Delta(0, r) \to \Omega$ satisfies for any $z \in \Delta(0, r)$*

$$f^*\rho(z) \leq \frac{\sqrt{A}}{\sqrt{B}} \rho_r^A(z).$$

In particular, if we consider the case $r = 1$ and we assume that the unit disc is equipped with the Poincaré metric ρ_P, we have $f^*\rho \leq \rho_P(z)$ for any $z \in \Delta$ whenever $f : \Delta \to \Omega$ is holomorphic and curvature of the metric ρ on Ω does not exceed -1. This property is sometimes called the distance decreasing property of the Poincaré metric.

Proof For any $0 < s < r$, we set $v_s = \dfrac{\sqrt{B} f^*\rho}{\sqrt{A}\rho_s^A}$. Note first that v_s has a maximum M_s which is attained at some point $z_s \in \Delta(0, s)$. This follows from the fact that v_s is continuous, nonnegative on $\Delta(0, s)$ and that $\lim_{|z| \to s} v_s = 0$. If we prove that $M_s \leq 1$, we can easily conclude by taking $s \to r$. We can assume that $f^*\rho(z_s) > 0$ (otherwise $v_s = 0$ on $D(0, s)$) and hence $K_{f^*\rho}$ is defined at z_s. Since $\log v_s$ has a maximum at z_s, we have by using the bounds on the curvatures,

$$\begin{aligned} 0 \geq \Delta \log v_s(z_s) &= \Delta \log f^*\rho(z_s) - \Delta \log \rho_s^A(z_s) \\ &= -K_{f^*\rho}(z_s)(f^*\rho(z_s))^2 + K_{\rho_s^A}(\rho_s(z_s))^2 \\ &\geq Bf^*\rho(z_s)^2 - A(\rho_s^A(z_s))^2 \end{aligned}$$

This gives $0 \geq B\left(f^*\rho(z_s)^2 - \frac{A}{B}\rho_s^A(z_s)^2\right)$. Hence $M_s \leq 1$ for any $0 < s < r$ and the proof is complete.

Let Ω be a domain in the complex plane $\mathbb{C}$.

Definition A.3.2 We say that Ω is *hyperbolic in the sense of Brody* if any entire function $f : \mathbb{C} \to \Omega$ is constant.

This notion is related to the more classical of hyperbolicity in the sense of Kobayashi. The next result states that domains that can be equipped with a metric whose curvature is negative are hyperbolic in the sense of Brody.

Theorem A.3.3 *Let Ω be an open set equipped with a metric ρ. We assume that there exists a positive constant B such that its curvature $K_\rho(z) \leq -B < 0$ for all $z \in \Omega$. Then, any entire function $f : \mathbb{C} \to \Omega$ must be constant.*

Proof For $r > 0$, we consider the Euclidean disc $\Delta(0, r)$ equipped with the metric ρ_r^A for some fixed A and the restriction f_r of f to this disc $\Delta(0, r)$. For any fixed z and any $r > |z|$, we have by the previous theorem $f^*\rho(z) \leq \frac{\sqrt{A}}{\sqrt{B}}\rho_r^A(z)$. If we take $r \to 0$, we get $f^*\rho \leq 0$ and hence $f^*\rho = 0$. This is possible only if $f'(z) = 0$. Thus, f is constant (since z is arbitrary).

As an application using the Poincaré metric, we get the classical Liouville Theorem: if f is a bounded entire function, we can assume that the range of f is inside the unit disc Δ which is equipped with the Poincaré metric. The curvature of this metric is -1, so we conclude by using the previous theorem that f is constant. Another easy application is Picard's little theorem:

Theorem A.3.4 (Picard) *Let $f : \mathbb{C} \to \Omega$ be an entire function taking its values in a open set Ω. If $\mathbb{C} \setminus \Omega$ contains at least two points, then f must be constant.*

Proof We have to prove that Ω could be equipped with a metric ρ whose curvature K_ρ satisfies $K_\rho(z) \leq -B < 0$ for some positive constant B and for all $z \in \Omega$. Without loss of generality, we can assume that the omitted points are 0 and 1. Set

$$\rho(z) = \left(\frac{(1 + |z|^{1/3})^{1/2}}{|z|^{5/6}}\right)\left(\frac{(1 + |z-1|^{1/3})^{1/2}}{|z-1|^{5/6}}\right).$$

Straightforward computations give

$$K_\rho(z) = -1/18\left(\frac{|z-1|^{5/3}}{(1 + |z|^{1/3})^3(1 + |z-1|^{1/3})} + \frac{|z|^{5/3}}{(1 + |z|^{1/3})(1 + |z-1|^{1/3})^3}\right)$$

Note now that

(i) $K_\rho(z) < 0$ for all $z \neq 0, 1$;
(ii) $\lim_{z\to 0} K_\rho(z) = -1/36$;
(iii) $\lim_{z\to 1} K_\rho(z) = -1/36$;
(iv) $\lim_{z\to 0\infty} K_\rho(z) = -\infty$;

These conditions imply easily that K_ρ is bounded above by a negative constant.

A.4 A First Approach of Quasiconformal Mappings

The starting point of the theory of quasiconformal mappings is supposed to be the problem of Grötzsch which can be formulated as follows. Take a square Q and a rectangle (not a square) R. There is no conformal mapping from Q to R which maps vertices on vertices. The question of Grötzsch was to ask for the most nearly conformal mapping of this kind. For this, we have to define the dilation of a map.

Let Ω_1, Ω_2 be open sets in $\mathbb{C}$, and $f : \Omega_1 \to \Omega_2$ be a C^1 mapping. As usual, we use the following notation:

$$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),\ f_{\overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

The differential of f is an affine transformation that maps circles about the origin into similar ellipses.

Definition A.4.1 The ratio of the major to the minor axis is given by the *dilation of f at z* :

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1$$

and the *complex dilation* is given by $\mu_f = \dfrac{f_{\overline{z}}}{f_z}$.

The connection with the dilation is given by $D_f = \dfrac{1 + |\mu_f|}{1 - |\mu_f|}$ or similarly $|\mu_f| = \dfrac{D_f - 1}{D_f + 1}$. Note that $\mu_f < 1$, and that if $\mu_f = 0$ then f satisfies the Cauchy-Riemann equation $f_{\overline{z}} = 0$. The complex dilation is related to the Beltrami equation. Following Grötzsch, we say that:

Definition A.4.2 f is *K-quasiconformal* for some $K \geq 1$ iff $D_f \leq K$ everywhere.

We now give the solution of the problem of Grötzsch. Assume that R (respectively R') is a rectangle with sidelength a and b (respectively a' and b'). We suppose for instance that $(a/b) \leq (a'/b')$. Then, the quasiconformal mapping f with the least dilatation (which maps a-sides to a'-sides and b-sides to b'-sides) is the affine transformation:

$$f(z) = 1/2\left(\frac{a'}{a} + \frac{b'}{b}\right)z + 1/2\left(\frac{a'}{a} - \frac{b'}{b}\right)\bar{z}$$

Set $m = a/b$ and $m' = a'/b'$. Then, there exists a K-quasiconformal mapping f of R on R' so that $1/K \leq m/m' \leq K$.

Definition A.4.3 The *modulus* of the rectangle R with sidelength a and b is the quotient a/b. A *quadrilateral* Q is a closed Jordan curve with two disjoint closed arcs (that are called the b-sides). Its modulus a/b is given by a conformal mappings on a rectangle of sidelength a/b that preserves the b-sides.

This suggests the following definition, generalizing Definition A.4.2 with no regularity properties assumptions on f:

Definition A.4.4 A sense-preserving homeomorphism $f : \Omega_1 \to \Omega_2$ is said K-*quasiconformal* if for all quadrilaterals Q so that $\overline{Q} \subset \Omega_1$, $\frac{1}{K}m(Q) \leq m(f(Q)) \leq Km(Q)$.

Note that if f is a 1-quasiconformal mapping, then f is a conformal mapping. It is not difficult to see that this definition coincides with the previous one in the special case $f \in C^1$. Note also that the composite of two quasiconformal mappings is quasiconformal and the inverse f^{-1} of a quasiconformal mapping is also quasiconformal.

We consider now the case $\Omega_1 = \Omega_2 = H$ where H is the upper half-plane of $\mathbb{C}$, that is $H = \{x + iy; y > 0\}$. It is well known that H and Δ are conformally equivalent (consider $\psi(z) = i\frac{1-z}{1+z}$ for $z \in \Delta$ or $\phi(\xi) = \frac{\xi - i}{\xi + i}$ for any $\xi \in H$). We equip H with the *hyperbolic metric* $\rho_H(z) = 1/y$ where $z = x + iy$. We will denote by d_H the associated distance. The boundary at infinity of H could be identified with $\mathbb{R}$. Note also that $\mathbb{R}$ acts simply transitively as the group of translations on the boundary of H. We say that a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfies a *M-condition* if $1/M \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M$ for any $x, y\ t \in \mathbb{R}$ (such a map will also be called *quasisymmetric*).

Theorem A.4.5

1) *The boundary values of a K-quasiconformal mapping $f : H \to H$ satisfy a M-condition with $M = M(K)$.*
2) *Conversely, every homeomorphism $h : \mathbb{R} \to \mathbb{R}$ which satisfies a M-condition could be extended as a K-quasiconformal mapping $\phi : H \to H$ with $K = K(M)$.*

The construction of the extension ϕ is explicit. It turns out that ϕ is a *quasi-isometry* for the hyperbolic metric on H, that is there exist constants $C, D \geq 0$ so that

$$C^{-1}d_H(x, y) - D \leq d_H(\phi(x), \phi(y)) \leq Cd_H(x, y) + D$$

whenever $x, y \in H$ (In fact, in our case, $D = 0$). Other versions of these results in general hyperbolic spaces are very useful to prove rigidity theorems (for instance Mostow rigidity type theorems).

A.5 Notes

A good introduction to the Schwarz lemma and its connection with hyperbolic geometry is [2] (Chap. IX). The use of the curvature of metrics in complex analysis is inspired by [3] where the reader will also find geometric proofs of the Montel theorem for normal families and the big Picard theorem. The book of Ahlfors [1] gives the basis of the theory of quasiconformal mappings (in particular, the Grötzsch problem). A proof of Theorem A.2.10 is provided in [4] where this result is used to get rigidity results.

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