Advances in Intelligent Systems and Computing 559

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Uncertainty and Imprecision in Decision Making and Decision Support: Cross-Fertilization, New Models, and Applications

Selected Papers from BOS-2016 and IWIFSGN-2016 held on October 12–14, 2016 in Warsaw, Poland

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Preface

This volume is composed of selected papers from two important conferences held on October 12–14, 2016, in Warsaw, Poland: Fourteenth National Conference of Operational and Systems Research, BOS-2016, one of premiere conferences in the field of operational and systems research not only in Poland but also at the European level, and Fifteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, IWIFSGN-2016, one of premiere conferences on fuzzy logic, notably on extensions of the traditional fuzzy sets, and also comprising a considerable part on the generalized nets (GNs), a powerful extension of the traditional Petri net paradigm. The first collocated conferences of the above type were held two years earlier, i.e., BOS-2014 and IWIFSGN-2014, also including the collocation with the IEEE Intelligent Systems IEEE IS'2014 conference, and this solution received a very positive reaction of the research community. That is why the collocation of BOS-2016 and IWIFSGN-2016 was continued in 2016 too.

Among many reasons for a positive opinion on such a collocation, one can certainly point out the following ones. First, the scope of the BOS conferences covers all kinds of problems related to systems modeling, systems analysis, broadly perceived operational research, notably optimization, decision making, and decision support, to just mention a few. In all these areas virtually all models used have to take into account not only the traditionally meant uncertainty, but also imprecision of information. That is, in addition to traditional probabilistic and statistical tools and techniques, the use of fuzzy set-based ones can be relevant. Even more so, the use of some extensions of the classic concept of a fuzzy set can be very useful. The use of the intuitionistic fuzzy sets, which are at the core of the IWIFSGN conferences, may be here a very good example.

One can therefore clearly see that the two conference series, and—to be more specific—the two conferences, BOS-2016 and IWIFSGN-2016, have been perfect venues for the exchange of ideas, cross-fertilization, and mutual inspiration. And, indeed, this has proved to happen what can also be seen from the papers contained in this volume.

The volume is composed of some parts that cover the main areas and challenges related to the above rationale and philosophy. The first part, “Issues in the Representation and Processing of Uncertain Information,” contains papers that concern issues and problems of a more general interest, notably related to uncertainty and imprecision.

Jolanta Jarnicka and Zbigniew Nahorski (“Estimation of Means in a Bivariate Discrete-Time Process”) consider a discrete-timenon-stationary stochastic process that is a sum of two other processes. Assuming that a data matrix of its realizations is given, they try to estimate and then analyze the mean values of the component processes as functions of time. Both the existence and uniqueness of a solution to this problem are dealt with. The authors propose an algorithm for the estimation of the mean values. The method is applied to the analysis and processing of uncertainty in National Inventory Reports (NIRs) on the emission of greenhouse gases (GHG) which is provided annually by the cosignatories to the UNFCCC and its Kyoto Protocol. Each report contains data on the GHG emission from a given year and also some revisions of past data, recalculated due to new or improved knowledge and methodology. Uncertainty is clearly present, and should be dealt with, when the GHG emissions are quantified. The method proposed can help to better deal with the inaccuracy and imprecision in processing the rough data. Applications for the data for Poland and some selected EU-15 countries are shown.

Ewa Straszecka (“On Fuzzy Focal Elements Combining”) considers some aspects of the Dempster–Shafer theory of evidence, namely the case of its extension meant to include fuzzy focal elements. To be more specific, the author deals with the problem of a combination of knowledge by means of the conjunction of data-driven membership functions. In this study, emphasis is on medical knowledge, represented by focal elements and the basic probability assignment. Influence of the combination on the basic probability assignment is provided. Examples of applications of the method proposed for the transfer of medical knowledge are shown.

Daniela Kluvancová (“On the conditional expectation on Kôpka’s D-posets”) discusses some relevant extension of the MV-algebras, which are known to be a very important structure in many-valued logic, both in theory and in applications. In particular, probability theory on the MV-algebras, especially on the MV-algebras with product, has proven to be very useful. In this contribution, the author proceeds further to some generalization of the MV-algebras, the D-posets with product, which are called Kôpka’s D-posets, and defines the conditional probability on Kôpka’s D-posets, as well as a version of the conditional expectation on Kôpka’s D-posets.

Jaroslav Považan (“Strong Law of Large Numbers on D-poset”) deals with the laws of large numbers, notably the strong law of large numbers, which is a very important result in the standard probability theory. The author obtains a more general version of the strong law of large numbers by changing the Σ-algebra by an algebraic structure called the D-poset introduced by Chovanec and Kôpka.

Leszek Klukowski (“Estimation of Trees on the Basis of Pairwise Comparisons with Random Errors”) is concerned with the estimation of the trees on the basis of multiple pairwise comparisons, with random errors. The estimators proposed are based on the idea of the nearest adjoining order. Two types of trees are examined: non-directed and directed. The approach is similar to the estimation of a preference relation with incomparable elements on the basis of binary comparisons. The estimates are obtained by solving a discrete optimization problem. The trees dealt with can be applied to the modeling of many phenomena and problems such as biological evolution, decision problems.

The second part of the volume, "Issues in the Representation and Processing of Imprecise Information," is concerned mostly with the imprecision of information and contains papers on both the traditional fuzzy sets and their extensions, notably the intuitionistic fuzzy sets, interval-valued fuzzy sets, etc.

Krassimir T. Atanassov ("On Extended Intuitionistic Fuzzy Index Matrices with Elements Being Predicates") extends his original concept of an index matrix and notably an extended intuitionistic index matrix, by assuming that their elements are predicates. The author defines some basic concepts and operations related to the newly introduced ideas.

Katarína Čunderlíková and Beloslav Riečan ("On Two Formulations of the Representation Theorem for an IF-state") are concerned with the representation and processing of an IF-state which is represented by some classical Kolmogorovian probabilities. The authors show that two formulations of an IF-state known from the literature representation are equivalent. The results can be very important for intuitionistic fuzzy set-based extensions of probability theory.

Andrzej Piegat and Karina Tomaszewska ("Optimal Representation (ORD) Method of Intuitionistic Fuzzy Defuzzification") propose a novel method of intuitionistic fuzzy defuzzification, called the Optimal Representation Defuzzification (ORD). In the method, in the first step, the membership and non-membership functions of the system's inputs and output are transformed into the membership functions of the interval type-2 fuzzy sets. Then, the inference process is run to determine activation degrees of the output fuzzy sets (conclusions of the rules). Subsequently, for all activated membership functions, one fuzzy set optimally representing them is determined. Finally, one crisp value, optimally representing this fuzzy set, is found as a defuzzification result. In the ORD method, each rule is treated as a local expert system. For illustration, the use of the method in an intuitionistic fuzzy controller of the fan speed of a room heater is shown.

Wojciech T. Dobrosielski, Jacek M. Czerniak, Janusz Szczepański, and Hubert Zarzycki ("Two New Defuzzification Methods Useful for Different Fuzzy Arithmetics") present new algorithms for the defuzzification block which is the final process of the fuzzy controller (fuzzy control system) in which a defuzzified value is to be applied as control to a given object. The new methods presented are based on the well-known, since the ancient times, Golden Ratio (GR) rule and the so-called Mandala Factor (MF), which is based on the interpretation of a drawing technique used in Asia consisting in arranging pictures of color sand grains. In the Tibetan Buddhism, this technique is known as Mandala, a symbol of perfection and harmony. The methods proposed are compared with other methods used in the defuzzification process, including the weight-averaging method, the centroid, and the mean of maxima. The discussion proceeds in the context of the ordered fuzzy number (OFN) theory which also makes it possible to make use of the trend of a given phenomenon. A special property of the proposed methods is their sensitivity to the order of the OFN numbers used.

Paweł Drygaś and Anna Król ("Two Constructions of Ordinal Sums of Fuzzy Implications") discuss the important problem related to the fuzzy implications. They deal with ordinal sums of the fuzzy implications and propose two new generalized methods. The generalizations make it possible to consider summands on intervals of different types: open, closed, or half-open. The authors show the sufficiency properties of fuzzy implications as summands for obtaining a fuzzy implication.

The third part, "Novel Tools and Techniques in Modeling, Decision Making and Decision Support: Theory and Applications," has a stronger operational and systems research orientation and includes contributions mostly on various aspects of decision making, notably in multicriteria and gaming contexts, but also with uncertainty and imprecision playing a relevant role. Some applications in these and related areas are also shown.

Krzysztof Dyczkowski, Anna Stachowiak, and Maciej Wygralak ("A Decision-Making Model in an IVFS Environment Based on Sigma f-Count Cardinality") present a new approach to decision making when information, possibly incomplete, is provided by many sources. The proposed method is based on the scalar cardinality (sigma f-count) for the IVFSs (interval-valued fuzzy sets). First, a general algorithm is introduced, and next, an application for supporting medical decisions in the ovarian tumor differentiation (based on multiple diagnostic models) is presented and discussed.

Lech Kruś ("On Computer-Based Support in Noncooperative Multicriteria Games") deals with noncooperative games, in which each player has some number of criteria measuring his or her payoff. A decision support system is considered as a computer-based tool that allows the players to make an analysis of the conflict situation, taking into account their preferences. The analysis can be done using an interactive, learning procedure utilizing methods of multicriteria optimization. An algorithm supporting the analysis of payoffs in the multicriteria game and derivation of the best response strategies, satisfying preferences of the players, is proposed. The reference point approach based on the respective achievement functions is used in the interactive procedure in which payoffs of the players are calculated closely to their preferences. A parametric characterization of the multicriteria gains representing preferences of the players is used, and relations among equilibria in the multicriteria games and the respective classic games are discussed.

Ignacy Kaliszewski, Janusz Miroforidis, and Dmitry Podkopaev ("Multiple Criteria Decision Making and Multiobjective Optimization – A Toolbox") present an integrated approach to solving multiple criteria decision problems in the sequence of intelligence, modeling, choice, and review phases, often with iterations, to identify the most preferred decision variant. The approach taken is human-centric, with the (human) user taking the final decision to be a sole and sovereign actor in the decision-making process. To ensure generality, no assumption about the decision maker's preferences or behavior is made, neither about the underlying formal model. The intended goal of the approach is to lower the cognitive barrier related to an unsupported use of multicriteria methodologies in a day-to-day practice. Some successful applications are shown.

Urszula Bentkowska, Józef Drewniak, Paweł Drygaś, Anna Król, and Ewa Rak ("Dominance of Binary Operations on Posets") deal with a dominance property of binary algebraic operations on a partially ordered set, and not—as traditionally—of the direct inequality or inclusion. Such a dominance is strictly connected with a generalized distributivity of the operations. Consequences of the bisymmetry assumption and the existence of a neutral element for the operations are presented. Some results known for operations on the unit interval are generalized to the case of a partially ordered set or a lattice.

Evdokia Sotirova, Sotir Sotirov, Lilia Atanassova, and Krassimir Atanassov ("Game Method for Modeling with Intuitionistic Fuzzy Rules") are concerned with the game method for modeling (GMM) introduced in the mid-1970s as a modification of Conway's game of life (CGL). The authors extend the traditional approaches by introducing intuitionistic fuzzy estimations for the rules of the GMM. An example of a relevant application for predicting the forest dynamics is shown.

Łukasz Apiecionek, Hubert Zarzycki, Jacek M. Czerniak, Wojciech T. Dobrosielski, and Dawid Ewald ("The Cellular Automata Theory with Fuzzy Numbers in Simulation of Real Fires in Buildings") consider the use of cellular automata extended with fuzzy numbers for the simulation and testing building conditions with respect to fire. The tests performed on real accidents showed that using some extension of the fuzzy numbers could give a realistic simulation of human evacuation. The authors analyze real accidents and prove that the method proposed is very effective and efficient in particular in the cases of building renovations or temporary unavailability of escape routes.

Desislava Vankova, Sotir Sotirov, and Lyubka Doukovska ("An Application of Neural Network to Health-Related Quality of Life Process with Intuitionistic Fuzzy Estimation") consider a very important problem of the quality of life (QoL) assessment in the context of health care. For the modeling, the authors use a neural network model and employ intuitionistic fuzzy estimations to enrich the possibilities to represent information.

Jacek M. Czerniak, Hubert Zarzycki, Wojciech T. Dobrosielski, and Janusz Szczepański ("Application of OFN Notation in the Fuzzy Observation of WIG20 Index Trend for the Period 2008–2016") discuss an important problem of seeking patterns in trends expressed in a linguistic form. The linguistic variables are assumed to take on their values as a result of the calculations in the ordered fuzzy number (OFN) notation. First, the fuzzification of the source data is performed. Daily quotations (min, max, the opening value, the closing value, and the direction of change) are interpreted as a single OFN number, representing five different parameters in a single number. A dedicated computer program for performing the pattern search is implemented and tested on data on the main index of the Warsaw Stock Exchange, i.e., WIG20, for the period 2008–2016.

The fourth part, “Advanced IT/ICT Applications,” is focused on one of the most important areas of science and technology that is decisive for the progress in virtually all fields in the present-day world, i.e., information technology or, maybe more generally, information and communication technology.

Włodzimierz Ogryczak, Tomasz Śliwiński, Jarosław Hurkała, Mariusz Kaleta, Piotr Pałka, and Bartosz Kozłowski (“Large-scale Periodic Routing Problems for Supporting Planning of Mobile Personnel Tasks”) discuss issues related to the implementation of a decision support system for large-scale periodic time-dependent vehicle routing and scheduling problems with complex constraints supporting the planning and management of mobile personnel tasks (sales representatives and others). The case of complex, non-uniform constraints, related to the frequency, time windows, working times, etc., is discussed. Fast adaptive procedures for operational rescheduling of plans in the presence of various disturbances are discussed. Five individual solution quality indicators with respect to a single personnel member are considered.

Grzegorz Zalewski and Włodzimierz Ogryczak (“Network Dimensioning with Minimum Unfairness Cost for the Efficiency”) consider network dimensioning, a specific kind of optimization problems. Basically, the main goal in this task is to ensure a connection between given pairs of nodes (source-target) with a possibly high efficiency. When each pair (demand) brings different revenue, the problem of blocking less attractive demands can occur. Usually, this situation is caused by not including any fairness criterion into the optimization model and thus optimizing only the total (revenue) efficiency of the system. Another complication is the fact of inverse proportionality of these criteria. In this paper, an optimization model has been examined which takes into account a fairness criterion and minimizes the loss of system efficiency. It may also be understood as optimizing the ratio of fairness degree to the mean of the traffic flow in the network. For the implementation of the model, the CPLEX package is used, and an example problem is solved. The approach is evaluated by using, e.g., the standard deviation, kurtosis, and the Gini coefficient.

Rafał Bieńkowski, Krzysztof Leśniewski, and Weronika Radziszewska (“Spatial Data Analysis in Archaeology: Computer-Aided Selection of Priority Location for Archaeological Survey”) describe a GIS-based method to help experts selecting the most probable locations of settlements inhabited by the people in Crete under the Venetian rule (1204–1669). The method ranks a possible location in a designated area on the basis of some criteria, geographic and anthropogenic, but also considering natural obstacles, e.g., characteristic features of the terrain. This multicriteria problem is solved using a modification of the Bellman–Ford algorithm for each criterion and then combining them using the weighted sum. An example on real data is provided.

Łukasz Apiecionek, Jacek M. Czerniak, Wojciech T. Dobrosielski, and Dawid Ewald (“New Proposal of Fuzzy Observation of DDoS Attack”) show a potential use of the implementation of fuzzy observation for discovering and protecting a computer network from the Distributed Denial of Service (DDoS) attacks. Such attacks can block Web servers and could be started from any place in the network.

Some real experimental results are presented. A developed network and DDoS attack tool are used for collecting IP packets during an attack, and then, some extension to fuzzy logic is implemented and used for discovering an attack.

Stefka Fidanova, Vassia Atanassova, and Olympia Roeva ("Ant Colony Optimization Application to GPS Surveying Problems: InterCriteria Analysis") consider the use of the ant colony optimization (ACO), which has been applied successfully to solve many hard combinatorial optimization problems, to provide near-optimal solutions for the Global Positioning System (GSP) surveying problem. In designing a Global Positioning System (GPS) surveying network, a given set of earth points must be observed consecutively (schedule). The cost of the schedule is the sum of time needed to go from one point to another. The problem is to search for the best order in which this observation is executed, minimizing the cost of the schedule. The authors use the InterCriteria Analysis (ICrA) for the results obtained to examine some relations between the considered GSPs and the ACO algorithm performance.

Marcin Woźniak, Michał Terlecki, Piotr Brażkiewicz, Krzysztof Wosiński, Adam Baszyński, Tomasz Gromacki, Michał Iwicki, and Paweł Splitt ("Ontology Usage for Database Conversion in Practical Solution for Military Systems – Case Study") present ADTA (Automated Data Transformation and Aggregation) system to achieve interoperability between systems which use different data exchange protocol versions and different database structures. This mechanism is modeled using semantic transformation. In this paper, an ADTA mechanism is presented and some practical results of test case solutions are described.

Hubert Zarzycki, Wojciech T. Dobrosielski, Dawid Ewald, and Łukasz Apiecionek ("Effective Search of Proteins in the DNA Code") discuss an important issue that is related to the processing of large amounts of data associated with the genome. The statistics of gene content of the human genome is demonstrated accompanied by important information about how the proteins are stored in the DNA code, and RNA transcription, splicing, and protein translation processes are performed. This is followed by the description of a practical algorithm for searching protein and pattern matching in the DNA code for given complete genome and protein data.

Urszula Bentkowska and Barbara Pękala ("Generalized Reciprocity Property for Interval-Valued Fuzzy Setting in Some Aspect of Social Network") discuss interval-valued fuzzy relations, notably introduce the fuzzy negation-based reciprocity property, and examine the connection of this property with the weak transitivity and some equivalence relation for the interval-valued fuzzy relations, as well as the preservation of such a reciprocity by some operators. The authors present an algorithm to find the best alternative in decision-making problem with the use of their new reciprocity concept. An example from the area of social network analysis is provided.

The last, fifth part, "Generalized Nets: Theory and Applications," is concerned with the use of tools and techniques of the Generalized Nets, a substantial extension of the Petri nets, which can provide powerful tools and techniques for the analysis and solution of many kinds of discrete event type problems.

Stanislav Simeonov, Vassia Atanassova, Evdokia Sotirova, Neli Simeonova, and Todor Kostadinov (“Generalized Net of a Centralized Embedded System”) discuss some issues related to embedded controlling computer systems in which multiple servicing subsystems can process multiple tasks. The problem arises from an optimal centralized distribution of the tasks among the multiple subsystems. This problem is solved by designing a generalized net (GN) model to organize the behavior of these servicing devices and the performance of the various tasks assigned to them in parallel.

Veselina Bureva, Stanislav Popov, Evdokia Sotirova, and Krassimir T. Atanassov (“Generalized Net of MapReduce Computational Model”) consider some issues related to the well-known MapReduce paradigm, a programming model for parallel processing of large volumes of data in a distributed environment, which can be of much use while dealing with various big data type problems. MapReduce is applied in the clusters of commodity machines. The workflow of the MapReduce computational model is constructed using the tools and techniques of generalized nets (GNs).

Todor Petkov, Sotir Sotirov, and Stanislav Popov (“Generalized Net Model of Optimization of the Self-Organizing Map Learning Algorithm”) describe the optimization of the algorithm of self-organizing map neural network. The proposed algorithm is employed during the learning trial. The authors take into consideration the number of epochs so that their number needs to be decreased. In order to do that, for each epoch the distance from each cluster unit to all training vectors is measured. If the total distance is the same as the distance estimated from the previous epoch, then it is assumed that the network is trained and the learning trial stops. The process of optimization is described with the generalized net.

Lenko Erbakanov, Krassimir T. Atanassov, and Sotir Sotirov (“Generalized Net Model of Synchronous Binary Counter”) propose a new generalized net (GN) model of a synchronous binary counter which is one of very important and basic logical circuit. The time delays are considered by using tools and techniques of the theory of generalized nets (GNs).

Simeon Ribagin, Krassimir T. Atanassov, Olympia Roeva, and Tania Pencheva (“Generalized Net Model of Adolescent Idiopathic Scoliosis Diagnosing”) discuss the diagnosis of one of the main postural disorders found in the population, the idiopathic scoliosis, affecting 2% to 4% of adolescent population. The authors present the use of generalized nets (GNs) to model the diagnosis process for a timely detection of adolescent idiopathic scoliosis and its categorization.

We hope that the inspiring and interesting contributions, included in this volume, will be of much interest and use for a wide research community.

We wish to thank the contributors for their great works, as well as other participants of the BOS-2016 and IWIFSGN-2016 conferences, whose contributions appeared in different proceedings, for their active participation, the vivid discussions, eagerness to exchange and share new ideas, and friendly atmosphere. Special thanks are due to anonymous referees whose deep and constructive remarks and suggestions have helped to greatly improve the quality and clarity of contributions.

And last but not least, we wish to thank Dr. Tom Ditzinger, Dr. Leontina di Cecco, and Mr. Holger Schaepe for their dedication and help to implement and finish this important publication project on time, while maintaining the highest publication standards.

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Advanced IT/ICT Applications

Generalized Nets: Theory and Applications

Issues in the Representation and Processing of Uncertain Information

Estimation of Means in a Bivariate Discrete-Time Process

Jolanta Jarnicka[1,2(✉)] and Zbigniew Nahorski[1,3]

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Abstract. We consider a discrete-time non-stationary stochastic process being a sum of two other processes. Given a data matrix of its realizations, we aim to estimate and then analyze the mean values of the component processes as functions of time. Both existence and uniqueness of a solution to this problem are investigated. An algorithm for estimating the mean values is proposed. The method is applied to analyze the uncertainty in National Inventory Reports (NIR) on greenhouse gases (GHG) emission, provided annually by cosignatories to the UNFCCC and its Kyoto Protocol. Each report contains data on GHG emission from a given year and revisions of past data, recalculated due to improved knowledge and methodology. However, it has to also deal with uncertainty, present whether GHG emissions are quantified. The method proposed can be used as an attempt to improve inaccuracy and imprecision in processing the rough data in time. The results are presented for Poland, and a few selected EU-15 countries.

1 Introduction

Analysis of non-stationary stochastic processes is quite a challenge, even though a lot of effort has been made to elaborate possibly general methods to deal with them. In particular, application of a parametric approach heavily depends on proper modeling of parameter variability in time, while a nonparametric one requires rather extensive empirical material to provide meaningful results. In this paper we focus on a specific non-stationary discrete-time process, which consists of a sum of two other processes and propose a method to estimate time-varying mean values of the two component processes. The estimation is performed from several realizations of the aggregated process.

The problem discussed here arose when analyzing the greenhouse gases (GHG) time series, reported annually by parties to the UNFCCC and its Kyoto Protocol. Preparing a report in a given year, countries also revise data provided in earlier reports, due to more precise knowledge and improved methodology.

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The question was therefore posed in [1] (see also [4]), if these revisions can be used for assessing changes in reports uncertainty over time. This was in a way extension of the problem considered in [6], where a country uncertainty was estimated from a single sequence of yearly emissions.

The problem was up to now attacked from two directions. In the early report [1] analysis was conducted over years, i.e. all revised values in a given year were considered consecutively. This direction was also applied in [7]. Another approach was based on analyzing revisions performed in consecutive years, as in [2,3,5], and hence generalizing the method considered in [6] for multiple revisions, and resulting in a time sequence of uncertainty estimates. In this paper analysis over years and over revisions is done simultaneously, using a new approach. As compared to earlier publications, we restrict only to estimation of the mean values. We focus on mathematical analysis of the problem and the solution algorithm.

Emissions vary in time. To assess uncertainty, deviations from the most recently revised data series are considered. This can be done directly, assuming that the most adequate information comes from the last report, or by de-trending the 'best' time series first, using smoothing splines, to shave out random effects in the data. The latter approach is considered in the sequel. From the statistical point of view, it means that the results obtained are conditional on the last available report, i.e. on data from 2014.

In Sect. 2 we present the idea of interpreting the data and state the estimation problem. Existence and uniqueness of a solution to that problem are analyzed in Sect. 3. Section 4 describes a method of estimating parameters and Sect. 5 contains the results of its application to the NIR data on CO_2 emissions without LULUCF for Poland, and a few selected EU-15 countries. Summary and conclusions are given in Sect. 6.

2 Problem Statement

Let us consider a discrete-time stochastic process and its realizations, denoted by S_y^t, where y identifies a realization, and t denotes the time. Let us assume that, realizations S_y^t are not equally long, i.e. they have the same start time t_0, but various end times y, where $y < Y$, and Y denotes the end time of the longest realization considered. The structure of realizations is graphically presented in Table 1.

We call shortly the 'realization y' a realization which ends at time y. The index t determines the place of an element in realization S_y^t, and is called in the sequel the 'position t'. In particular, end time y of the realization y is on the position y. We allow for the lack of some intermediate realizations of the process, as well as the lack of some data points in available realizations, assume however some restrictions. For simplicity we consider the following notation. By $\mathcal{T}_y$ we denote a set of indices t of non-missing entries in the realization y. By $\mathcal{Y}_t$ we denote a set of indices y of non-missing entries in the position t from all existing realizations. The set $\mathcal{Y}$ consists of all indices y of existing realization, and the

Table 1. Indexing the realization data.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	S_y^{t-1}	S_y^t	S_y^{t+1}	$\ldots$	S_y^y	none	$\ldots$	none
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	S_Y^{t-1}	S_Y^t	S_Y^{t+1}	$\ldots$	S_Y^y	S_Y^{y+1}	$\ldots$	S_Y^Y

set $\mathcal{T}$ – of all indices t, for which there exists a non-missing entry in at least one realization, i.e. $\mathcal{T} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y$.

The process considered is the sum of two other processes, such that

$$S_y^t = V^t + H_y, \tag{1}$$

where V^t and H_y are (possibly dependent) random variables with finite first and second moments. The assumed data structure is illustrated in Table 2.

Table 2. The assumed data structure.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	$V^{t-1}+H_y$	V^t+H_y	$V^{t+1}+H_y$	$\ldots$	V^y+H_y	none	$\ldots$	none
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	$V^{t-1}+H_Y$	V^t+H_Y	$V^{t+1}+H_Y$	$\ldots$	V^y+H_Y	$V^{y+1}+H_Y$	$\ldots$	V^Y+H_Y

This data structure is motivated by analysis of GHG inventories. An inventory consists of sums of emissions from all atom sources. According to the UNFCCC guidelines, the emission of the atom source E_i is calculated as the product of the source activity A_i, and the source emission factor EF_i

$$E_i = EF_i \cdot A_i.$$

Assuming small deviations ΔEF_i and ΔA_i of both variables, the relative error of the emission can be expressed as the sum of relative errors of the emission factor and the activity

$$\tfrac{\Delta E_i}{E_i} = \tfrac{\Delta EF_i}{EF_i} + \tfrac{\Delta A_i}{A_i}.$$

Thus we consider here two types of uncertainty, that related to preparing the rough data for reporting, called in the sequel *the data gathering errors*, and the one related to compilation of the final inventory from the rough data, called *the data processing errors*.

To simplify the notation, we put $E(H_y) \stackrel{\text{def}}{=} \overline{H}_y$, and $E(V^t) \stackrel{\text{def}}{=} \overline{V}^t$. The problem can be stated as follows.

Problem P1
Given realizations $\{S_y^t; t \in \mathcal{T}_y, y \in \mathcal{Y}\}$ of discrete-time process (1), estimate the mean values $m_t = \overline{V}^t$, $t \in \mathcal{T}$ and $m_y = \overline{H}_y$, $y \in \mathcal{Y}$.

3 Existence and Uniqueness of a Solution

It is quite obvious that, a solution to the Problem P1 may not exist, due to too many missing entries. Therefore we assume that, an additional condition, which we call the *Existence Condition* (EC), is satisfied.

Existence Condition EC1
Let S_y^t, be realizations of (1). Then the following conditions are met

(i) for all $y \in \mathcal{Y} - Y$ there exists S_Y^y,
(ii) for all $t \notin \{y : y \in \mathcal{Y}\}$ and for all $y \in \mathcal{Y} - Y$ there exists S_y^t.

EC1 requires that, in addition to the endpoints of realizations, the indicated entries are non-missing. Condition (i) guaranties the existence of all entries of the longest realization, and (ii) assumes the existence of at least one entry for all intermediate positions t in any realization except the longest one.

Remark 1. EC1 is sufficient but not necessary. If EC1 is not satisfied, existence of a solution to P1 can be shown for some practical cases. In particular, if either (i) or (ii) is not satisfied, a partial solution exists.

Observe that, decomposition (1) is not unique. Indeed, for any $C \in \mathbb{R}$ we get

$$S_y^t = (V^t + C) + (H_y - C) = (V')^t + (H')_y. \tag{2}$$

Since $\overline{V}^t$ and $\overline{H}_y$ depend on arbitrary constant C, (2) results in infinitely many solutions to P1. An additional condition is therefore requred. We call it the *Uniqueness Condition* (UC).

Uniqueness condition UC1
Let H_Y be a random variable, such that $\overline{H}_Y < \infty$. Then

$$\overline{H}_Y = 0. \tag{3}$$

The assumption in (3) was motivated by the application considered in Sect. 5.

Remark 2. Specifying (3), UC1 determines the value of a constant C in (2), provided that decomposition (1) exists.

Corollary 1. *Under* EC1 *and* UC1, *determination of mean values $m_t = \overline{V}^t$ and $m_y = \overline{H}_y$ from S_y^t is unique.*

4 Estimation Method

Consider all realizations S_y^t, satisfying EC1 and UC1. We subtract $\overline{V}^t$ from all non-missing S_y^t, $t \in \mathcal{T}$, $y \in \mathcal{Y}$. Then $\overline{H}_y$ can then be estimated as arithmetic means of the differences $S_y^t - \overline{V}^t$, $t \in \mathcal{T}_y$. Similarly, $\overline{V}^t$ can be estimated as arithmetic means of $S_y^t - \overline{H}_y$ over appropriate $y \in \mathcal{Y}_t$. Having subtracted both means $\overline{V}^t$ and $\overline{H}_y$ from each non-missing entry S_y^t, all arithmetic means calculated for each realization and each position in the residual structure should be equal zero. This motivates the following iterative algorithm.

Algorithm 1. Estimating mean values $\overline{H}_y$ and $\overline{V}^t$.

$\Delta S_y^{t,(0)} \leftarrow S_y^t$
$k \leftarrow 0$
repeat ▷ Iterate
  $k \leftarrow k+1$
  for all $y \in \mathcal{Y}$ **do**
    calculate the means $\overline{H}_y^{(k)}$ from $\Delta S_y^{t,(k)}$, $t \in \mathcal{T}_y$
    $\Delta S_y^{t,(k)} \leftarrow \Delta S_y^{t,(k-1)} - \overline{H}_y^{(k)}$
  end for
  $k \leftarrow k+1$
  for all $t \in \mathcal{T}$ **do**
    calculate the means $\overline{V}^{t,(k)}$
    $\Delta S_y^{t,(k)} \leftarrow \Delta S_y^{t,(k-1)} - \overline{V}^{t,(k)}$
  end for
until $|\overline{H}_y^{(k)}|, y \in \mathcal{Y}$ and $|\overline{V}^{t,(k)}|, t \in \mathcal{T}$ are small enough
for all $y \in \mathcal{Y}$ **do** ▷ Initial estimates
  $\tilde{\overline{H}}_y \leftarrow \sum_{\text{odd } k} \overline{H}_y^{(k)}$
end for
for all $t \in \mathcal{T}$ **do**
  $\tilde{\overline{V}}^t \leftarrow \sum_{\text{even } k} \overline{V}^{t,(k)}$
end for
for all $y \in \mathcal{Y}$ **do** ▷ Final estimates
  $\hat{\overline{H}}_y \leftarrow \tilde{\overline{H}}_y - \tilde{\overline{H}}_Y$
end for
for all $t \in \mathcal{T}$ **do**
  $\hat{\overline{V}}^t \leftarrow \tilde{\overline{V}}^t + \tilde{\overline{H}}_Y$
end for

5 Application

According to the UNFCCC and its Kyoto Protocol, each of the cosignatories is obliged to provide annual data on GHG emissions. The National Inventory

Reports (NIR) submitted, are publicly available on the UNFCCC website. Each report contains data on emissions from a given year and revisions of past data (back to 1990, considered a base year for emission inventories), recalculated using improved knowledge and methodology. But it also has to deal with uncertainty from two sources – related to data gathering, i.e. caused by missing, inaccurate, or non-representative data, and related to processing of the data, e.g. inadequate models or wrongly specified parameters. The question arises, whether it is possible to compare and organize data on GHG emissions, as well as to model time evolution of inventory uncertainties, taking into account all the data revised in consecutive years.

For this purpose, the method developed above is used. The NIR data on CO_2 emissions without LULUCF (Land-use, Land-use-change, and Forestry) for Poland and three EU-15 countries: Austria, the UK, and Belgium, were analyzed. These data span from 1990 to 2014, with revisions made every year from 2001 to 2014. It is easy to notice that, they follow data structure presented in Table 1, as well as satisfy EC1. Figure 1 presents the trajectories of the last emission data reported (the longest realizations, for the year $Y = 2014$) with fitted smoothing splines.

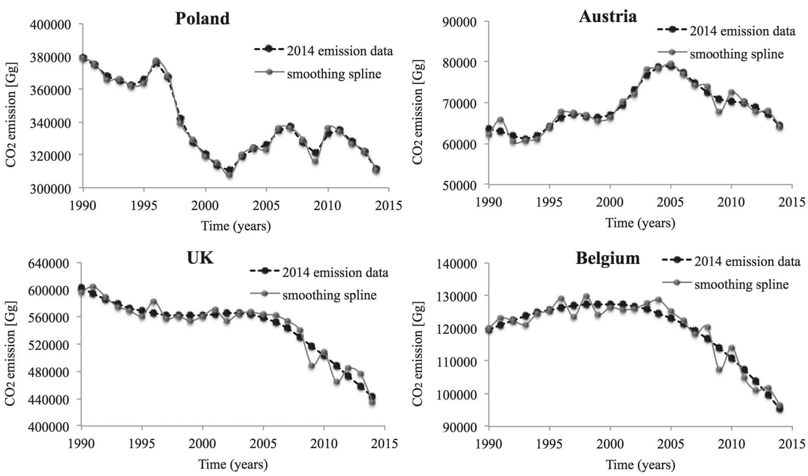

Fig. 1. CO_2 emissions without LULUCF from 2014 along with the smoothing splines fitted.

The splines were used to de-trend the last emission data series. For each of the countries considered, the data matrix was then obtained by subtracting the smoothing splines from all revisions S_y^t, for $y < Y$. The converted matrices satisfy EC1, so the Algorithm 1 from Sect. 4 was applied. After six iterations, estimates of mean values $\overline{V}^t$ for positions t from 1990 to 2014, and of mean values $\overline{H}_y$ for revisions y from 2001 to 2014 were obtained. The trajectories of estimated $\overline{V}^t$, i.e. of the data gathering errors, are depicted in Fig. 2 (left panels). The estimates of $\overline{H}_y$, i.e. of the data processing errors, are presented in the right panels of Fig. 2.

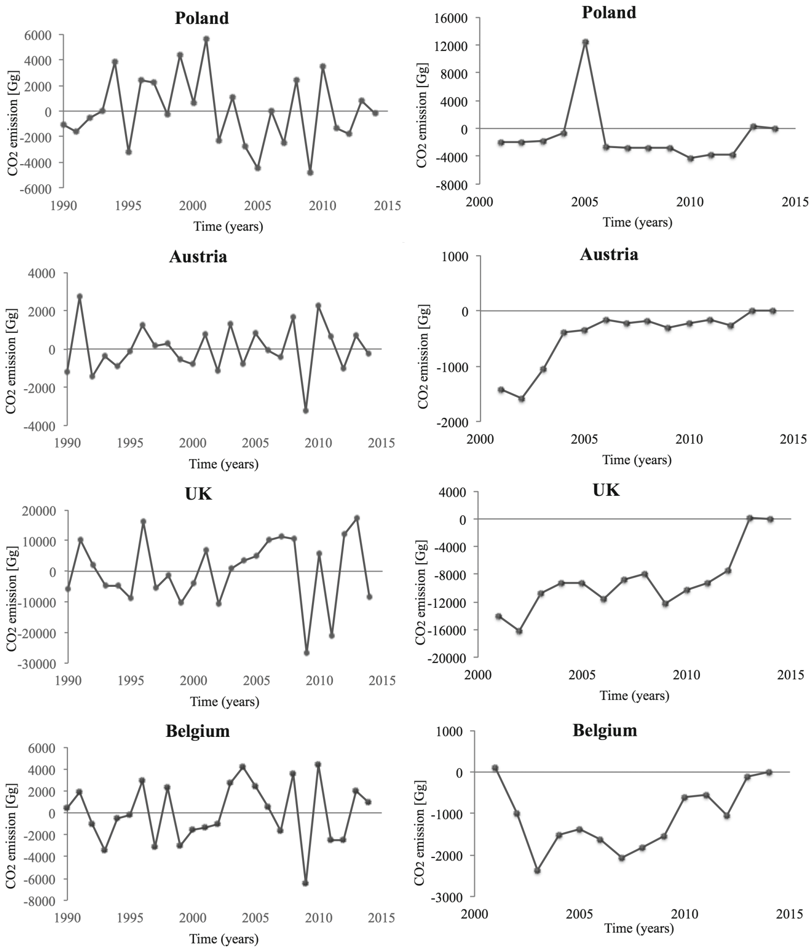

Fig. 2. Estimated mean values of the data gathering errors, for positions (left panel) and of the data processing errors, for revisions (right panel).

While the estimates of $\overline{H}_y$ (for revisions) form reasonably smooth curves (except the year 2005 for Poland), the estimates of $\overline{V}^t$ for positions seem to be purely random. Due to the uniqueness condition UC1, it holds $\sum_{t\in\mathcal{T}_Y} V^t = 0$, which is quite well satisfied for $\hat{\overline{V}}^t$ in all four cases (0.53 for Poland, 0.56 for the UK, 0.44 for Austria, and 0.64 for Belgium). Hence, no learning feature was observed in gathering the rough data. Some learning could be attributed to the process of processing the rough data, which is particularly visible in the case of Austria and the UK.

One can ask the question on the speed of convergence in Algorithm 1. It could be observed that, the values $m_y^{(k)}$ and $m_t^{(k)}$ in consecutive iterations converged quickly to zero. After six iterations the estimates of $m_y^{(k)}$ were less than 0.1 and of $m_t^{(k)}$ practically equaled zero. Hence, only a few iterations provided sufficient accuracy. The rate of convergence can be illustrated in terms of the Euclidean norms. Logarithms of the Euclidean norms of the vectors $\mathrm{m}^{(2k-1)} = [m_y^{(2k-1)}, \ldots, m_y^{(2k-1)}]^T$ and $\mathrm{m}^{(2k)} = [m_t^{(2k)}, \ldots, m_t^{(2k)}]^T$ are approximately linear in k (see Fig. 3), which suggests that the convergence of the norms may be exponential.

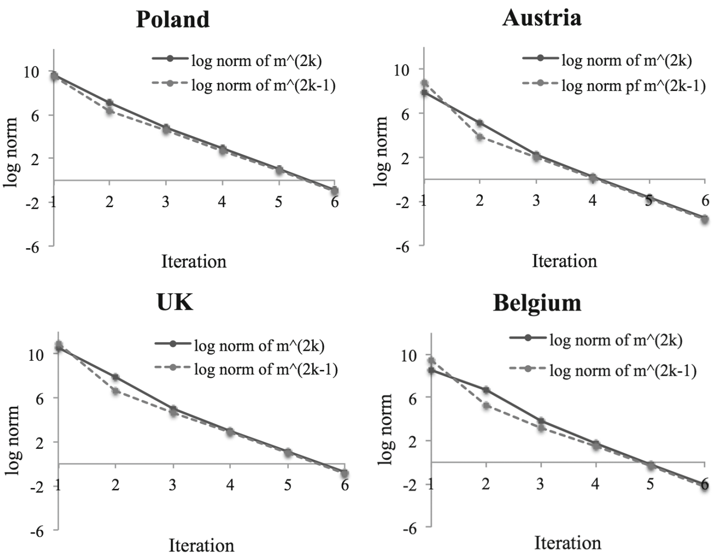

Fig. 3. The logarithms of $|\mathrm{m}_y^{(2k)}|$ and $|\mathrm{m}_y^{(2k-1)}|$ for Poland, Austria, UK, and Belgium.

6 Conclusions

We presented a method of estimating mean values in a structured discrete-time stochastic process. The problem is motivated by a real application coming from analysis of the National Inventory Reports on GHG emissions, containing data from a given year and revisions of past data. We aimed at estimating the uncertainty, taking into account all data revised in consecutive years. For this, we calculated deviations of revisions from the smoothing spline fitted to the latest data reported, treating the deviations obtained as realizations of a non-stationary process. We focused on estimation of the mean values of these deviations. It is assumed that uncertainties related to data gathering and processing, enter additively into the combined data, which is well motivated by the way the uncertainties appear in the inventory reports. Moreover, it is assumed that, only the latter uncertainty is affected during preparation of new revisions, while the former ones are only attributed to the year when the emissions were inventoried for the first time.

The problem was analyzed from mathematical point of view. It was pointed that a solution may not exist in general, and if it does, it is not unique. Conditions for existence and uniqueness of a solution were introduced, and the algorithm for solving the problem was proposed. The results of its application to the NIR data for Poland, Austria, UK, and Belgium, were presented. The method worked well and converged to a solution close enough to the optimal one in only few iterations. Revisions of past data are quite common in economic temporal records. Hence, the problem presented may have much wider applications.

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On Fuzzy Focal Elements Combining

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Abstract. The Dempster-Shafer theory can be extended for fuzzy focal elements. When knowledge from different sources is combined, membership functions of the elements should be also joined. The paper suggests a combination of knowledge by means of the conjunction of data-driven membership functions. A discussion on an influence of the combination on the basic probability assignment is provided. The method can be helpful for medical knowledge transfer.

Keywords: Dempster-Shafer theory · Fuzzy sets · Diagnosis support

1 Introduction

The Dempster-Shafer theory of evidence (DST) [1,5] may be extended for fuzzy focal elements [6]. The extension consists in defining focal elements by means of membership functions (mfs) that correspond to linguistic values. For instance if a variable is a 'glucose level' then the mf 'high' can be used in a premise of a diagnostic rule and this mf represents the medical symptom. A conclusions of the rule is a diagnose, for example 'diabetes'. Thus, a set of focal elements is determined for each diagnosis. The basic probability assignment (bpa) which is next defined becomes an evaluation of the rule weights. Afterwards, the belief [1,5] can be calculated for all considered diagnoses and compared to chose the diagnosis of the greatest belief value as the final conclusion. Such a use of the DST is convenient in medical diagnosis support, but fuzzy focal elements create not only new opportunities, but also new problems to solve. One of them is the combination of assignments.

Medical knowledge represented by focal elements and the bpa should be subjected to combination from different sources, i.e. an expert and a training database or from two experts, as well as from two databases [7]. The classical DST combination refers only to bpas, while mfs require an individual treatment [7]. The present paper suggests an approach for the fuzzy focal elements combination and makes an attempt to solve several related problems. Firstly, a conjunction is proposed for the combination. Secondly, a dependence between a similarity of combined focal elements and changes in resulting bpa are studied. Since similarity factors for a comparison of mfs and bpas are necessary, a choice of the factors

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as well as an evaluation criteria for the combination effectiveness are proposed. Conclusions are driven as a result of multiple numerical simulations, which are close to a simplified task of medical diagnosis support and face usual difficulties of medical knowledge transfer. Theoretical background for the extension of the DST for fuzzy focal elements is given in [6,7] and other works of the author. Because of the limited length of this paper, only the most necessary information is provided. For the same reason experiments are shortly summarized. Yet, every interested reader can easily build and investigate the proposed diagnostic model.

2 DST for Fuzzy Focal Elements

The bpa for fuzzy focal elements [6] is defined as:

$$m(f) = 0, \sum_{\substack{s_i \in S, i=1,\ldots,n \\ \eta_i > \eta_{BPA}}} m(s_i) = 1. \tag{1}$$

where η_{BPA} is the minimal level of precision for which a symptom is considered as carrying information. The symptom is defined by means of the $\mu(x)$ mf and η is the actual precision of a symptom found for the x^* data case, i.e. $\eta = \mu(x^*)$. For a premise including n symptoms $\eta = \min_{j=1,\ldots,n}(\mu_j(x_j^*)$. The bpa can be found from data when mfs for symptoms are determined. To this end, the η_{BPA} threshold is assumed and a number of data cases that fit mfs better than the threshold ($\eta \geq \eta_{BPA}$) is counted. After normalization of the numbers for all symptoms the (1) conditions hold true [7].

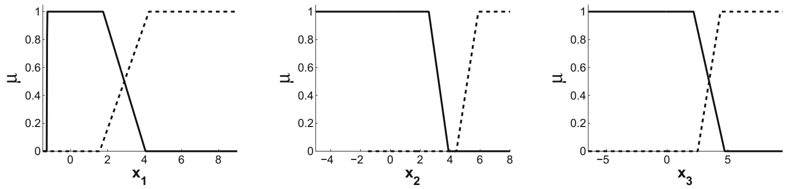

Fig. 1. Mfs for two competitive diagnoses: D_1 (solid line) and D_2 (dashed line).

Shapes of mfs are based on quartiles and intersection points of the training data distributions [6]. The quartile indicates the point for which $\mu(x) = 1$, while the intersection point determines the point of half membership ($\mu(x) = 0.5$). The intersection point is the x value for which theoretical distributions of x for two competitive diagnoses crosses [6]. When quartiles and the intersection point do not correspond each other then quartiles make the mfs with 'steep' slopes, e.g. with 100 coefficient gradient. The Fig. 1 shows mfs for two diagnoses (D_1, D_2), each of them considering symptoms as low and high values of the variables x_1, x_2 and x_3. Training data distributions for x_2 are incompatible, so mfs shapes for this variable are 'steep'.

3 A Model of a Diagnosis

Let us assume that the diagnosis is based on three symptoms: X_1, X_2 and X_3. The symptoms are numerical variables which low values are assigned to the D_1 diagnosis and high values to the D_2. The symptoms X_2 and X_3 are correlated, since disorders of the human body are often related. Thus, four rules can be formulated for D_k, $k = 1, 2$:

$s_j^{(k)}$: IF X_j is $A_j^{(k)}$ then D_k, $j = 1, 2, 3$,
$s_4^{(k)}$: IF X_2 is $A_2^{(k)}$ and X_3 is $A_3^{(k)}$ then D_k.

The linguistic values $A_j^{(k)}$, $j = 1, 2, 3$, for D_1 are represented by the 'low' mf $\mu^{(1)}(x_j)$, and for D_2 by the 'high' mf $\mu^{(2)}(x_j)$. Mfs are data driven using simulated data of numbers generated from the normal distribution: one-dimensional for X_1 and two-dimensional for X_2 and X_3. In the present model it is assumed that X_1–X_3 have different distributions for the D_1 diagnosis and the same distribution for the D_2. Obviously, parameters of the distributions differ and each training data case is independently simulated. The bpas are calculated assuming $\eta_{bpa} = 0.5$. Bpas for different data sets are compared.

Similarity of bpas can be determined concerning a cardinality of focal elements. To this end, the Jaccard index matrix [2] is defined:

$$J^{(k)}(s_i, s_j) = \frac{\left|s_i^{(k)} \cap s_j^{(k)}\right|}{\left|s_i^{(k)} \cup s_j^{(k)}\right|}, i, j = 1, 2, 3; k = 1, 2. \tag{2}$$

The index matrix for the proposed focal elements and the both diagnoses is:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0.5 & 0.5 & 1 \end{bmatrix}. \tag{3}$$

Let us assume that the diagnostic knowledge comes from two sources of information, particularly, the mfs originate from two populations. A combination of them could be the conjunction, since a cautious approach is presumed in medicine [7]. Let us use the simplest conjunction which is the minimum of mfs. Now, it should be investigated how differences in training data may influence variability of bpas. Various diagnostic situations are considered as it is described in the next section. Mfs of two sets will be compared, then combined and the result will be compared to the mf obtained for the data of both sets put into the one set. The bpas change along with mfs, so they will be compared in the same way. The comparisons should show how resistant are bpas for changes of knowledge, or – on the other hand, if we can tell irrelevant data from the changes of mfs or bpas.

4 Simulated Data

Five samples of data are simulated, each of them includes 100 data sets. Every data set contain 400 cases, i.e. 200 for each of two diagnoses. Each data sample is generated for different parameters of the normal distribution $N(\bar{x}, \sigma)$, where x - mean, σ) - variance. When correlated data are necessary, data of two-dimensional distribution of the mentioned mean and variance are generated. Normality of data is verified by the Matlab® Liliefors test. The correlation coefficient maintains $r \geq 0.5$, except for $N(10, 5)$ sample, for which $r \geq 0.2$. In the present paper the following samples are used:

Sample 1 – is the 'mixed sample'. Its single set includes 100 cases from distribution 1 and 100 from distribution 2, for each diagnosis. Distribution 1 for D_1 is: N(1,1) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 1$ and variances $\sigma_2 = 2$, $\sigma_3 = 3$ for x_2 and x_3. Distribution 2 for D_1 is: N(1,2) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 1$ and variances $\sigma_2 = 3$, $\sigma_3 = 4$ for x_2 and x_3. Distribution 1 for D_2 is: N(5,1) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 5$ and variances $\sigma_2 = \sigma_3 = 1$ for x_2 and x_3. Distribution 2 for D_2 is: N(5,2) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 5$ and variances $\sigma_2 = \sigma_3 = 2$ for x_2 and x_3. This sample should simulate combining knowledge from two similar, but not identical populations, for instance social groups of different living conditions or habits.

Sample 2 – is another 'mixed sample'. A data set includes 200 data cases from one distribution for the D_1 and twice 100 cases from various distributions for the D_2. Distribution for D_1 is: N(1,2) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 1$ and variances $\sigma_2 = 3$, $\sigma_3 = 4$ for x_2 and x_3. Distribution 1 for D_2 is: N(5,2) for x_1 and two-dimensional normal distribution of the same means and variances for x_2 and x_3. Distribution 2 for D_2 is: N(10,5) for $x_1$and two-dimensional normal distribution of the same means and variances for x_2 and x_3. This sample may simulate situation when irrelevant data are attached to the training set.

Sample 3 – is a 'uniform sample', its sets include 200 data cases from one distribution for the D_1 diagnosis and the same number from another distribution for the D_2. Distribution for D_1 is: N(1,1) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 1$ and variances $\sigma_2 = 2$, $\sigma_3 = 3$ for x_2 and x_3. Distribution for D_2 is: N(5,1) for x_1, two-dimensional normal distribution of the same means and variances for x_2 and x_3. Consistent knowledge from two sources is simulated by this sample. Variances are small and means are not close to each other which means that symptoms are quite significant and the diagnosis is easy.

Sample 4 – is the second 'uniform sample', with greater variances. Its structure is analogical to Sample 3. Distribution for D_1 is: N(1,2) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 1$ and variances $\sigma_2 = 3$, $\sigma_3 = 4$ for x_2 and x_3. Distribution for D_2 is: N(5,2) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 5$ and variances $\sigma_2 = \sigma_3 = 2$ for x_2 and x_3. The sample simulates consistent knowledge when symptoms are less significant and the diagnosis is more difficult.

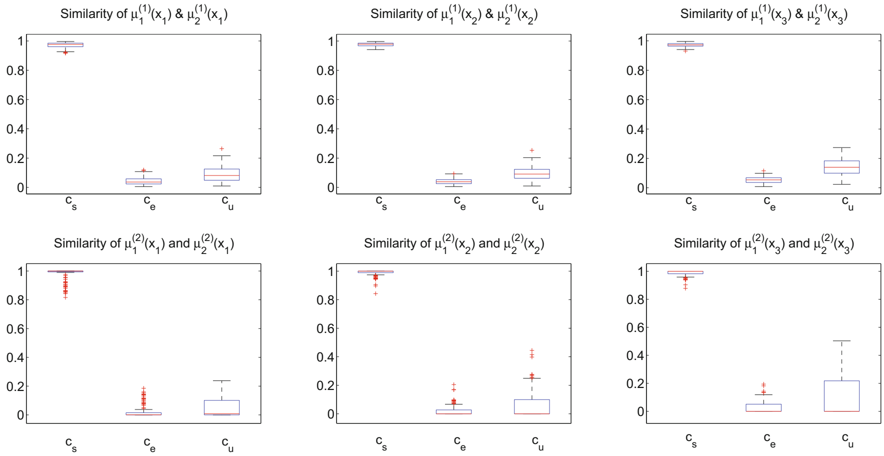

Fig. 2. A comparison of similarity factors for mfs.

Sample 5 – is the third 'uniform sample', yet the D_2 symptoms are very ambiguous with great variance and a distant mean. Its structure is the same as two previous samples. Distribution for D_1 is the same as for the Sample 4. Distribution for D_2 is: N(10,5) for x_1, two-dimensional normal distribution of means $\bar{x}_2 = \bar{x}_3 = 10$ and variances $\sigma_2 = \sigma_3 = 5$ for x_2 and x_3. The sample simulate a diagnosis when symptoms of D_1 and D_2 have very different characteristics.

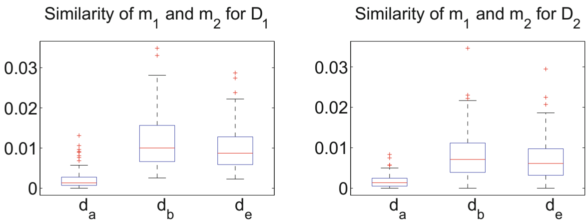

Fig. 3. A comparison of similarity factors for bpas.

5 Similarity Factors

Numerical experiments on mfs and bpas must be preceded by the choice of proper similarity factors. Let us first discuss factors of mfs comparison. The simplest is the maximum absolute distance:

$$c_u(\mu_1, \mu_2) = \max_x(|\mu_1(x) - \mu_2(x)|), \tag{4}$$

that is numerically calculated for multiple x points (e.g. $n = 300$). The distance represents rather a difference than a similarity, but the minimal distance cannot

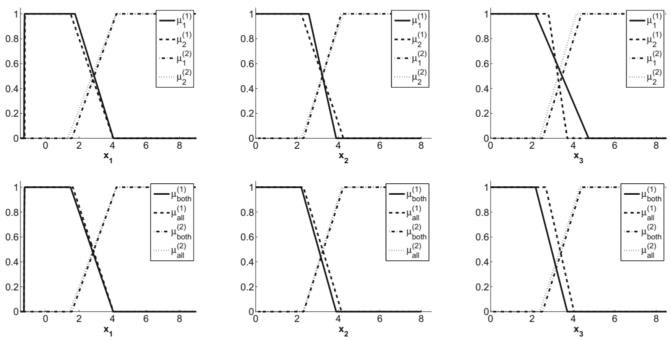

Fig. 4. Change of shape of mfs for two data sets of Sample 3.

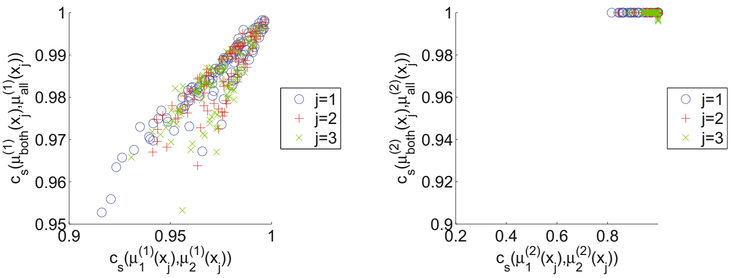

Fig. 5. Values of c_s for Sample 1,3,4 and 5.

be used for trapezoid mfs, as there must be a point in which compared mfs both equal 1 (see Fig. 4). The Euclidean distance is also applicable:

$$c_e(\mu_1, \mu_2) = \sqrt{\sum_{i=1}^{n} (\mu_1(x_i) - \mu_2(x_i))^2}. \tag{5}$$

Another possible similarity factor is [4]:

$$c_s(\mu_1, \mu_2) = \left| \min_x(\mu_1(x), \mu_2(x)) \right| + \left| \max_x(\mu_1(x), \mu_2(x)) \right| - 1, \tag{6}$$

calculated for the same points as (4) and (5).

In the Fig. 2 means, standard deviations, value intervals and outliers for similarity factors c_s, c_e and c_u for mfs of variables x_1, x_2 and x_3 and two diagnoses are presented. The mfs for the D_1 are determined as in sample 2 and for the D_2 according to the second distribution of this sample. The diagram is obtained for 100 data sets. The factors behave similarly. It is shown that for great variance (for the N(10,5) distribution) outliers of similarity factors appear. For the D_1 the factors c_s and c_e have small standard deviations. The former has greater values, so it is chosen for further research of the mf comparison.

Secondly, factors of evaluation the bpas similarity should be selected. Several factors can be used. The simplest is the distance [8]:

$$d_a(m_1, m_2) = \min(|\mathbf{m_1} - \mathbf{m_2}|). \tag{7}$$

The Euclidean distance can be also used:

$$d_e(m_1, m_2) = \sqrt{(\mathbf{m_1} - \mathbf{m_2})'(\mathbf{m_1} - \mathbf{m_2})}. \tag{8}$$

The distance with the Jaccard index (2), (3) evaluates focal elements [3]:

$$d_b(m_1, m_2) = \sqrt{(\mathbf{m_1} - \mathbf{m_2})'\mathbf{B}(\mathbf{m_1} - \mathbf{m_2})}. \tag{9}$$

These distances have proved to be useful for combination of bpas [8], hence they are also applied for the present evaluation. Results of their performance tests are in the Fig. 3. Means, standard deviations, value intervals and outliers for the factors are presented. They are obtained for a comparison of two bpas (m_1 and m_2) calculated for the same data as mfs. The worst differentiation among various data sets is given by d_a, while d_b and d_e are comparable. It can be expected that if mf shapes are more sophisticated, d_b could work better than d_e. Thus, the d_b factor (9) will be used further on.

Table 1. Similarity μ_1 and μ_2

	Diagnosis D_1				Diagnosis D_2		
smp	par	$c_s(x_1)$	$c_s(x_2)$	$c_s(x_3)$	$c_s(x_1)$	$c_s(x_2)$	$c_s(x_3)$
1	$\bar{x}$	0.9339	0.9544	0.9456	0.9987	0.9968	1.0000
	std	0.0521	0.0364	0.0349	0.0117	0.0321	0.0000
2	$\bar{x}$	0.1738	0.1857	0.1954	0.2015	0.2300	0.2300
	std	0.3183	0.3402	0.3580	0.3711	0.4208	0.4208
3	$\bar{x}$	0.9555	0.9672	0.9568	0.9915	0.9978	0.9835
	std	0.0490	0.0308	0.0383	0.0664	0.0152	0.0580
4	$\bar{x}$	0.9473	0.9535	0.9505	1.0000	1.0000	0.9976
	std	0.0441	0.0393	0.0397	0.0000	0.0000	0.0243
5	$\bar{x}$	0.9721	0.9759	0.9718	0.9751	0.9880	0.9878
	std	0.0187	0.0137	0.0133	0.0479	0.0248	0.0220

6 Similarity of Focal Elements and Its Influence on Probability Assignment

The shapes of mfs influence the bpa, since it is calculated using frequency of occurrence of training data cases. However, the dependence is not straightforward as the slope of the mf concerns an interval of a variable domain in which cases fall less frequently (the tail of the distribution) and one bpa value is influenced by the other focal elements. The Fig. 4 illustrates mfs $\mu_1^{(k)}(x_j)$ and $\mu_2^{(k)}(x_j)$ $k = 1, 2,\ j = 1, 2, 3$, obtained using two data sets of 100 elements each, $\mu_{both}^{(k)}$ found as minimum of the mfs and $\mu_{all}^{(k)}$ determined for the joined cases (200 elements). Differences between mfs of the individual samples are easy to notice, while between μ_{both} and μ_{all} are less significant. This observation allows to suppose that the conjunction can properly combine information given by mfs.

Table 2. Similarity μ_{all} and μ_{both}

	Diagnosis D_1				Diagnosis D_2		
smp	par	$c_s(x_1)$	$c_s(x_2)$	$c_s(x_3)$	$c_s(x_1)$	$c_s(x_2)$	$c_s(x_3)$
1	$\bar{x}$	0.9670	0.9769	0.9718	0.9999	1.0000	1.0000
	std	0.0286	0.0190	0.0208	0.0010	0.0000	0.0000
2	$\bar{x}$	0.1820	0.2226	0.2257	0.2171	0.2216	0.2293
	std	0.3344	0.4079	0.4130	0.3985	0.4057	0.4196
3	$\bar{x}$	0.9802	0.9809	0.9749	1.0000	1.0000	0.9998
	std	0.0159	0.0175	0.0230	0.0000	0.0000	0.0013
4	$\bar{x}$	0.9736	0.9773	0.9742	0.9998	1.0000	1.0000
	std	0.0219	0.0213	0.0226	0.0022	0.0000	0.0000
5	$\bar{x}$	0.9848	0.9856	0.9826	1.0000	1.0000	0.9998
	std	0.0093	0.0077	0.0079	0.0000	0.0001	0.0007

The Table 1 show mean values ($\bar{x}$) and standard deviations (std) of $c_s(\mu_1^{(1)}(x_j),\ \mu_2^{(1)}(x_j)),\ j = 1, 2, 3$, calculated for D_1 and D_2 for 100 sets of data. The c_s values for the D_2 are very close to 1, since the variance of distribution for this diagnosis is quite low. The Sample 2 is an exception - this sample is simulated for the distribution of high variance and its c_s is low. In Fig. 5 the $c_s(\mu_1^{(k)}(x_j), \mu_2^{(k)}(x_j))$ vs. $c_s(\mu_{both}^{(k)}(x_j), \mu_{all}^{(k)}(x_j)),\ k = 1, 2,\ j = 1, 2, 3$, are depicted (except for Sample 2). In the left diagram it is noticeable, that a dependence between the two c_s is linear or better, i.e. the mf after combining ($\mu_{both}^{(1)}$) is at least as similar to the mf $\mu_{all}^{(1)}$ as $\mu_1^{(1)}$ to $\mu_2^{(1)}$. Few different values of $c_s(\mu_1^{(2)}(x_j), \mu_2^{(2)}(x_j))$ and $c_s(\mu_{both}^{(2)}(x_j), \mu_{all}^{(2)}(x_j))$ are obtained. Points in the right diagram represent multiple values and because of low sample variance the similarity is even better. Complete calculation results of $c_s(\mu_{both}^{(k)}(x_j), \mu_{all}^{(k)}(x_j))$, $k = 1, 2,\ j = 1, 2, 3$ are in Table 2. They confirm that similarity of mfs after combination is high for uniform mfs or mfs with low data variability. The standard deviation of c_s for the mfs is low. The opposite is observed for Sample 2.

The divergence between mfs influences differences of bpas, but a relation is not straightforward. The Table 3 presents $d_b(m_{both}, m_{all})$ related to $d_b(m_1, m_2)$. The mean values of the d_b are not very different, which may suggest that the bpa is quite resistant to changes of knowledge. It is noticeable that the standard deviation of $d_b(m_{both}, m_{all})$ is low for the uniform samples and high for the mixed samples. Generally, this parameter for $d_b(m_{both}, m_{all})$ is lower in comparison to $d_b(m_1, m_2)$ for uniform samples and higher for mixed samples. The only exception is the standard deviation for D_2 and sample 1, but for this sample the mixed distributions do not vary much. The c_s and d_b values are not comparable, since they are completely different measures. Yet, if we study the $\bar{x}$/std coefficient for the measures using values from Tables 1, 2 and 3, we see that this coefficient is similar for sample 1, 3, 4 and 5 and very different for sample 2. Thus, the bpa is resistant for small changes of data samples, but it is changed significantly when irrelevant data are introduced.

Table 3. Similarity of bpas

		$d_b(m_1, m_2)$		$d_b(m_{both}, m_{all})$	
		D_1	D_2	D_1	D_2
1	$\bar{x}$	0.0135	0.0738	0.0082	0.0292
	std	0.0140	0.1438	0.0154	0.0718
2	$\bar{x}$	0.0135	0.0107	0.0137	0.0067
	std	0.0071	0.0066	0.0265	0.0137
3	$\bar{x}$	0.0171	0.0567	0.0115	0.0120
	std	0.0117	0.1229	0.0070	0.0162
4	$\bar{x}$	0.0119	0.0499	0.0134	0.0105
	std	0.0120	0.0926	0.0086	0.0074
5	$\bar{x}$	0.0117	0.0080	0.0045	0.0082
	std	0.0070	0.0058	0.0023	0.0040

7 Conclusions

In the paper a method of fuzzy focal elements combination is suggested. Tests performed on simulated data show that the conjunction of the elements defined as minimum of mfs is the proper operation of joining knowledge represented by the same linguistic values, but different fuzzy sets. Data simulated for the normal distribution confirm that mfs after combining provide functions more similar to the functions found for unified data, than the mfs to each other before combining. If the combined mfs originate from consistent sources of data, also the standard deviation of the bpa determined for combined focal elements is low. It indicates that the generalization of knowledge is correct. On the contrary, the great standard deviation of the resulting assignment appears when combined mfs derive from differently distributed data. This means that the focal elements and the assignment are sufficiently sensitive to a change of training data.

Three diagnostic situations were modeled to test the combination characteristics. The first was combining knowledge from the same source with different training data sets. The tests resulted with similar mfs and the similarity factor of basic probability values of low standard deviation. Thus, the method do not spoil the basic assignment when the sources of information are consistent.

The second was joining knowledge from various, but not very different sources. The combination outputs were similar mf and the similarity factor of bpa with the standard deviation not very different from the previous test. Hence, the method allows for some generalization of knowledge.

The third situation was modeled by different distributions and resulted with less similar mfs and the bpa similarity factors of high standard deviation. Therefore, knowledge from diverse sources should not be combined and fuzzy focal elements as well as the bpa should be build separately for each individual population. Moreover, if an assignment values after combination show high variance, it should be suspected that the combination was unjustified.

The proposed method of knowledge combination and its features presented for simulated data may facilitate medical knowledge transfer when diagnostic rules remain the same, but populations disclose slightly different characteristics.

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On the Conditional Expectation on Kôpka's D-posets

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Abstract. MV-algebras are very important structure in many-valued logic. Probability theory on MV-algebras, especially on MV-algebras with product [6] has proven very useful. In this contribution we deal with one of the generalization of MV-algebras, D-posets with product called Kôpka's D-posets. We define conditional probability on Kôpka's D-poset and introduce a version of conditional expectation on Kôpka's D-posets.

Keywords: D-poset · Conditional probability · Conditional expectation

1 Introduction

D-posets [3] and equivalently effect algebras [2] and A-posets [8] are three structures with important role in quantum theory. Similarly as in the case of MV-algebras, probability theory can be build on D-posets. Conditional probabilities on a classical measurable space are studied in several different ways. In [7] was indtroducted the notion of conditional probability on Kôpka's D-posets. In this paper we define one of the possible version of conditional expectation.

2 Basic Notions

The basic term of this article is a D-poset. D-poset is an algebraic structure equivalent to the effect algebra and it was introducted by F. Kôpka and F. Chovanec in [3].

Definition 1. *An algebraic structure $D = (D, \ominus, \leq, 0_D, 1_D)$ is a D-poset D if $\leq$ is a partial ordering on D with the least element 0_D and the greatest element 1_D, $\ominus : D \times D \to D$ is a partial binary operation and there holds*

- *$b \ominus a$ is defined iff $a \leq b$,*
- *if $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) \leq b$,*
- *if $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

$a, b, c \in D$.

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Example 1. Let X be a nonempty set and $\mathcal{S}$ be the set of all subsets of X.

- $A \leq B$ iff $A \subseteq B$,
- $\emptyset$ is the least element, X is the greatest element, $A^c = X - A$,
- $B - A$ is defined iff $A \subseteq B$ and $B - A$ is a relative component of A with respect to B.

The structure $(\mathcal{S}, \subseteq, \cup, \emptyset, X)$ is a D-poset.

Example 2. Let X be a nonempty set, $\mathcal{F}$ is a the set of all real functions $f : X \to [0, 1]$.

- Let $\leq$ is the partial ordering on $\mathcal{F}$ such that for $f, g \in D$ holds $f \leq q$ iff $f(x) \leq g(x)$ for all $x \in X$.
- The least element is a function identically equal to 0 and the greatest element is a function identically equal to 1.
- Let $f \leq q$. Then $(f - g)(x)$ is defined as f(x)-g(x) for all $x \in X$.

The structure $(\mathcal{F}, -, \leq, 0, 1)$ is a D-poset.

An another important term for our purposes is an Kôpka's D-poset introducted by F. Kôpka in [4]. Kôpka's D-poset is a D-poset with binary operation called product.

Definition 2. *Let $(D, \ominus, \leq, 0_D, 1_D)$ be a D-poset. Then Kôpka's D-poset is an algebraic structure $(D, \ominus, \leq, *, 0_D, 1_D)$, such that $* : D \times D \to D$ is a commutative and associative binary operation satisfying the following properties:*

- $a * 1_D = a$,
- *if* $a \leq b$ *then* $a * c \leq b * c$,
- $a \ominus (a * b) \leq 1_D \ominus b$,

$a, b, c \in D$.

Example 3. Let X be a nonempty set and $\mathcal{S}$ be the set of all subsets of X. The structure $(\mathcal{S}, \subseteq, \cup, \emptyset, X)$ is an D-poset.

An operation $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is defined by $A \star B = A \cap B$. This structure is a Kôpka's D-poset.

Example 4. Let X be a nonempty set, $\mathcal{F}$ is a the set of all real functions $f : X \to [0, 1]$. Then $\mathcal{F}$, $\leq$ and "$-$" form a D-poset.

Let $* : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be defined by $(f * g)(x) = f(x) * g(x)$ for all $x \in X$. Then $(F, \leq, -, 1, 0, *)$ is a Kôpka's D-poset.

Definition 3. *Kôpka's D-poset is called continuous if for every elements $a_n, a, b \in D$ holds:*

$$a_n \nearrow a \Longrightarrow b * a_n \nearrow b * a.$$

Definition 4. *A state on an D-poset D is a mapping* $m : D \to [0, 1]$

1. $m(1) = 1$, *m(0)=0,*
2. $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$, $\forall a_n, a \in A$,
3. *if* $a \leq b$ *then m(a-b)=m(a)-m(b).*

Definition 5. *Let* $\mathcal{B}(\mathbb{R})$ *be a Borel* σ*-algebra and D be a D-poset. An observable on an D-poset D is any mapping* $x : \mathcal{B}(\mathbb{R}) \to D$ *satisfying the following properties:*

1. $A_n \nearrow \mathbb{R} \Longrightarrow x(A_n) \nearrow 1_D$,
2. $A_n \searrow \emptyset \Longrightarrow x(A_n) \searrow 0_D$,
3. $A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A)$.

Theorem 1. *Let D be a D-poset,* $x : \mathcal{B}(\mathbb{R}) \to A$ *be an observable,* $m : A \to [0, 1]$ *be a state. The mapping* $F : \mathcal{B}(\mathbb{R}) \to [0, 1]$ *defined by*

$$F(A) = m_x(A) = m(x(A)), \quad A \in \mathcal{B}(\mathbb{R})$$

is a probability measure

Proof. First, it is clear that

$$F(\mathbb{R}) = m(x(\mathbb{R})) = m(1_D) = 1$$

and F is continuous. Now we can prove additivity of F. Let $A, B \in \mathcal{B}(\mathbb{R})$ and $A \subseteq B$. Then

$$F(A - B) = m(x(A - B)) = m(x(A) \ominus x(B)) = m(x(A)) - m(x(B)).$$

Proved subtractivity is equivalent to additivity.

Definition 6. *Let D be a D-poset, m be a state on D, x be an observable on D and* $F : \mathcal{B}(\mathbb{R}) \to [0, 1]$ *be a mapping such that* $F(A) = m(x(A))$ *for every* $A \in \mathcal{B}(\mathbb{R})$*. An observable* $x : \mathcal{B}(\mathbb{R}) \to D$ *is called integrable if there exists*

$$E(x) = \int_{\mathbb{R}} t dF(t).$$

Observable x is called observable with dispersion if

$$\sigma^2 = \int_{\mathbb{R}} (t - E(x))^2 dF(t) = \int_{\mathbb{R}} t^2 dF(t) - E(x)^2$$

exists.

3 Conditional Probability

Consider the classical probability space $(\Omega, \mathcal{S}, P)$ where Ω is a nonempty set, $\mathcal{S}$ is the set of all subsets of Ω and $P : \mathcal{S} \to [0,1]$ is probability measure. Let $\mathcal{S}_0 \subset \mathcal{S}$ be a σ-algebra, $B \in \mathcal{S}$. The conditional probability $P(B|\mathcal{S}_0) : \Omega \to [0,1]$ is a real function such that each element $\omega \in A_i$, $A_i \in \mathcal{S}_0$ is assigned a value $P(B|A_i)(\omega)$. Moreover the function $P(B|\mathcal{S}_0)$ must fulfill following conditions:

- $P(B|\mathcal{S}_0)$ is $\mathcal{S}_0$-measurable,
- for all $E \in \mathcal{S}_0$ there holds:

$$\int_E P(B|\mathcal{S}_0)dP = P(B \cap E).$$

Let $\xi : \Omega \to \mathbb{R}$ be a random variable. Family of all pre-images of ξ

$$\{\xi^{-1}(B); B \in \mathcal{B}(\mathbb{R})\}$$

is a σ-algebra. Therefore we can consider $S_0 = \{\xi^{-1}(B); B \in \mathcal{B}(\mathbb{R})\}$ and

$$\int_E P(B|\xi)dP = P(B \cap E),$$

where $E = \xi^{-1}(F)$, $F \in \mathcal{B}(\mathbb{R})$. Define a mapping $P_\xi : \mathcal{B}(\mathbb{R}) \to [0,1]$ such that for every $C \in \mathcal{B}(\mathbb{R})$, $P_\xi(C) = P(\xi^{-1}(C))$.

Then we can transform the previous equation to

$$P(B \cap \xi^{-1}(F)) = \int_{\xi^{-1}(F)} g \circ \xi dP = \int_F g dP_\xi$$

for each $F \in \mathcal{B}(\mathbb{R})$. Analogously can be defined conditional probability on Kôpka's D-posets.

Lemma 1. *Let D be an Kôpka's D-poset, $a \in D$ is an element of D, $m : D \to [0,1]$ be a state and $x : \mathcal{B}(\mathbb{R}) \to D$ be an observable. Let $F : \mathcal{B}(\mathbb{R}) \to [0,1]$ be a mapping such that $F(B) = m(x(B))$ for each $B \in \mathcal{B}(\mathbb{R})$. Define a mapping $G_a : \mathcal{B}(\mathbb{R}) \to [0,1]$ by*

$$G_a(A) = m(a * x(A)), \quad \textit{for any } A \in \mathcal{B}(\mathbb{R})$$

Then there holds

$$G_a(A) \le F(A) \ \textit{ for each } A \in \mathcal{B}(\mathbb{R}).$$

Proof. Let $A \in \mathcal{B}(\mathbb{R})$. For all elements a of D-poset D there holds $a \le 1_D$ and $a * x(A) \le \ 1_D * x(A) = x(a)$. Hence

$$G_a(A) = m(a * x(A)) \le m(x(A)) = F(A).$$

Theorem 2. *Let D be a Kôpka's D-poset, a is an element of D, $m : D \to [0,1]$ be a state, $x : \mathcal{B}(\mathbb{R}) \to D$ be an observable and F, $G_a : \mathcal{B}(\mathbb{R}) \to [0,1]$ be two probability measures such that*

$$F(A) = m(x(A))$$

and

$$G_a(A) = m(a * x(At)),$$

for each $A \in \mathcal{B}(\mathbb{R})$.
There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_A f dF = G_a(A),$$

for every $A \in \mathcal{B}(\mathbb{R})$.

Proof. Let $A \in \mathcal{B}(\mathbb{R})$ and assume $F(A) = 0$. Then

$$G_a(A) = m(a * x(A)) \leq m(x(A)) = F(A) = 0.$$

Hence if $F(A) = 0$ then $G_a(A) = 0$ so G_a is absolutely continuous according to the measure F. By the Radon-Nikodym theorem there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_A f dF = G_a(A).$$

for any $A \in \mathcal{B}(\mathbb{R})$.

Definition 7. *Let D be a Kôpka's D-poset, $a \in D$, $m : D \to [0,1]$ be a state and $x : \mathcal{B}(\mathbb{R}) \to D$ be an observable. A Borel measurable function $p(a|x) : \mathbb{R} \to \mathbb{R}$ is called a conditional probability on D if satisfies*

$$\int_A p(a|x) dF = G_a(A)$$

for each $A \in \mathcal{B}(\mathbb{R})$.

Theorem 3. *Let D be a continuous Kôpka's D-poset with product, $a \in D$, $m : D \to [0,1]$ be a state, $x : \mathcal{B}(\mathbb{R}) \to D$ be an observable. For the conditional probability $p(a|x) : \mathbb{R} \to \mathbb{R}$ hold properties:*

1. *$p(0_D|x) = 0$ and $p(1_D|x) = 1$ F-almost everywhere*
2. *$0 \leq p(0_D|x) \leq 1$ F-almost everywhere*
3. *if $a_n \nearrow a$ then $p(a_n|x) \nearrow p(a|x)$.*

Proof. 1. Let $A \in \mathcal{B}(\mathbb{R})$. A conditional probability is defined as a function $p(a|x)$ such that

$$\int_A p(a|x)dF = G_a(A).$$

Let $a = 0_D$. Then

$$\int_B 0dF = 0 = m(0_D) = m(0_D * x(A)) = G_a(A).$$

Hence

$$p(0_D|x) = 0.$$

Now let $a = 1_D$. Then

$$\int_A 1dF = F(A) = m(x(A)) = m(1_D \bullet x(A)) = G_a(A).$$

From the definition:

$$p(1_D|x) = 1.$$

2. Let $E = \{t \in \mathbb{R}; p(a|x) < 0\}$.
If $F(E) = m(x(E)) > 0$ then

$$G_a(E) = m(a * x(E)) = \int_E p(a|x)dF < 0$$

Recall that G_a is a probability measure so previous inequality is not possible. On the other hand, let $K = \{t \in \mathbb{R}; p(a|x) > 1\}$. If $F(K) > 0$ then there holds

$$G_a(K) = m(a * x(K)) = \int_K p(a|x)dF > 1$$

and it is a contradiction with the definition of the probability measure.
3. Let $a_n \nearrow a$ and $A \in \mathcal{B}(\mathbb{R})$. Then

$$a_n * x(A) \nearrow a * x(A) \text{ and } m(a_n * x(A)) \nearrow m(a * x(A)).$$

Hence

$$\begin{aligned}\int_A \lim_{n\to\infty} p(a_n|x)dF &= \lim_{n\to\infty} \int_A p(a_n|x)dF = \lim_{n\to\infty} G_{a_n}(A)\\ &= \lim_{n\to\infty} m(a_n * x(A)) = m(a * x(A)) = \int_A p(a|x)dF.\end{aligned}$$

4 Conditional Expectation

We start by the classical probability space $(\Omega, \mathcal{S}, P)$. Denote by $\mathcal{S}_0$ a σ-algebra such that $\mathcal{S}_0 \subset \mathcal{S}$ and let $\xi : \Omega \to \mathbb{R}$ be a random variable. A conditional expectation is defined as a $\mathcal{S}_0$-measurable function $E(\xi|\mathcal{S}_0) : \Omega \to \mathbb{R}$ which satisfies

$$\int_C E(\xi|\mathcal{S}_0)dP = \int_C \xi dP$$

for each $C \in \mathcal{S}_0$.

Let $\eta : \Omega \to \mathbb{R}$ be a random variable. If we use previous notation where $S_0 = \{\eta^{-1}(B); B \in \mathcal{B}(\mathbb{R})\}$ we can rewrite the definition of conditional expectation by the following way

$$\int_B E(\xi|\eta)dP_\eta = \int_{\eta^{-1}(B)} \xi dP.$$

Similarly, let $x, y : \mathcal{B}(\mathbb{R}) \to D$ be two observables. For each $A \in \mathcal{B}(\mathbb{R})$ we can define indefinite integral of x over $y(A)$:

$$\int_{y(A)} xdm = \int_{\mathbb{R}} tdG_{y(A)}(t)$$

where

$$G_{y(B)}(C) = m(y(B) * x(C)), \quad B, C \in \mathcal{B}(\mathbb{R}).$$

Lemma 2. *Let D be a Kôpka's D-poset, $m : D \to [0, 1]$ be a state, $x, y : \mathcal{B}(\mathbb{R}) \to D$ be two observables and x be integrable. Define F, $G_{y(A)} : \mathcal{B}(\mathbb{R}) \to [0, 1]$ as two probability measures such that*

$$F(B) = m(y(B))$$

and

$$G_{y(A)}(B) = m(y(A) * x(B)).$$

for $A, B \in \mathcal{B}(\mathbb{R})$.
Then there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_A f dF = \int_{y(A)} xdm = \int_{\mathbb{R}} tdG_{y(A)}(t).$$

Proof. Let D be a Kôpka's D-poset. Let $A \in \mathcal{B}(\mathbb{R})$ and denote $\mu(A) = F(A) = m(y(A)) = 0$. Then $G_{y(A)}(B) = m(y(A) * x(B)) \leq m(y(A)) = 0$ for every $B \in \mathcal{B}(\mathbb{R})$ and

$$\nu(A) = \int_{y(A)} xdm = \int_{\mathbb{R}} tdG_{y(A)}(t) = 0.$$

Therefore the measure ν is absolutely continuous according to μ and from Radon-Nikodym theorem follows that there exists a function f such that

$$\int_A f dF = \int_{y(A)} x dm = \int_{\mathbb{R}} t dG_{y(A)}(t).$$

Definition 8. *Let D be a Kôpka's D-poset, $m : D \to [0,1]$ be a state, $x, y : \mathcal{B}(\mathbb{R}) \to D$ be observables and x be integrable. Define F, $G_{y(A)} : \mathcal{B}(\mathbb{R}) \to [0,1]$ as two probability measures such that*

$$F(B) = m(y(B))$$

and

$$G_{y(A)}(B) = m(y(A) * x(B)).$$

for $A, B \in \mathcal{B}(\mathbb{R})$. The function $E(x|y) : \mathbb{R} \to \mathbb{R}$ such that

$$\int_A E(x|y) dF = \int_{y(A)} x dm = \int_{\mathbb{R}} t dG_{y(A)}(t)$$

for each $A \in \mathcal{B}(\mathbb{R})$.

is called conditional expectation on Kôpka's D-poset.

5 Conclusion

In this paper we introducted just basic notions of probability theory. Of course it may be extended to the other structures, for example A-posets or effect algebras and it can be applied to IF sets.

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Strong Law of Large Numbers on D-posets

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Abstract. Laws of large numbers are very important results in the standard probability theory. Especially the Strong Law of Large Numbers is one of the most important version of them. In this contribution, we get more general version of the Strong Law of Large Numbers, in which we change $\sigma-$algebra by an algebraic structure called D-poset, which was introduced by F. Chovanec and F. Kôpka in [2].

Keywords: Strong Law of Large Numbers · D-poset · State · Observable

1 Introduction

In this contribution, we present more general version of the Strong Law of Large Numbers, in which we change $\sigma-$algebra by an algebraic structure called D-poset, which was introduced by F. Chovanec and F. Kôpka in [2]. As a reaction to D-posets, A-posets resp. effect algebras were some years later introduced by V. Skřivánek and R. Frič in [9] resp. by M. K. Bennet and D.J. Foullis in [1].

In the standard probability theory the Strong Law of Large Numbers tell us, that if we have sequence $(\xi_n)_{n=1}^{\infty}$ of independent random variables, then sequence $\left(\frac{1}{n}\sum\limits_{i=1}^{n}\xi_i\right)_{n=1}^{\infty}$ converges almost everywhere to a constant c.

In this contribution we change the triplet $(\Omega, \mathscr{S}, P)$ by a pair (D, m), where D is an D-poset and m is a state on D.

2 Preliminaries

The following definition of D-poset was introduced in [2]:

Definition 1. *Ordered quintuplet* $\mathbb{D} = (D, \leq, 0_D, 1_D, \ominus)$ *is called a* ***D-poset*** *iff*

1. $\leq$ *is a partial ordering on* D *with the least element* 0_D *and greatest element* 1_D,
2. $\ominus : D \times D \to D$ *is a partial operation on* D,
3. $(\exists\,(a \ominus b)) \Leftrightarrow (a \leq b)\,,$

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4. $a \le b \Rightarrow (((b \ominus a) \le b)$ *and* $b \ominus (b \ominus a) = a)$,
5. $a \le b \le c \Rightarrow (c \ominus b \le c \ominus a$ *and* $(c \ominus a) \ominus (c \ominus b) = b \ominus a)$.

By Standard Strong Law of Large Numbers we will understand the following theorem:

Theorem 1. *Let $(\Omega, \mathscr{S}, P)$ be a probability space. Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of independent identically distributed integrable random variables. Then*

$$\frac{\sum\limits_{i=1}^{n} \xi_i}{n} \overset{P-a.e.}{\rightarrow} E(\xi_1).$$

By Standard Kolmogorov Strong Law of Large Numbers we will understand the following theorem:

Theorem 2. *Let $(\Omega, \mathscr{S}, P)$ be a probability space. Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of independent random variables with finite dispersion. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\mathrm{Var}(\xi_k)}{k^2} < \infty \Rightarrow \frac{\sum\limits_{k=1}^{n} (\xi_i - E(x_i))}{n} \overset{P-a.e.}{\rightarrow} 0.$$

The proofs of above theorems can be found e.g. in [7].

3 States on D-posets

Definition 2. *Let $(D, \le, 0_D, 1_D, \ominus)$ be a D-poset.*
A mapping $m : D \to [0, 1]$ is called a ***state*** *iff:*

- $m(1_D) = 1$,
- $\forall a_n, a \in D \quad a_n \nearrow a \Rightarrow m(a_n) \nearrow m(a)$,
- $\forall a, b \in D \quad a \le b \Rightarrow m(b \ominus a) = m(b) - m(a)$.

Definition 3. *Let $(D, \le, 0_D, 1_D, \ominus)$ be a D-poset. $\mathscr{B}(\mathbb{R})$ be the $\sigma-$algebra of Borel sets of $\mathbb{R}$. A mapping $x : \mathscr{B}(\mathbb{R}) \to D$ is called an* ***observable*** *iff*

- $x(\mathbb{R}) = 1_D$,
- $\forall B_n, B \in \mathscr{B}(\mathbb{R}) \quad B_n \nearrow B \Rightarrow x(B_n) \nearrow x(B)$,
- $\forall B, C \in \mathscr{B}(\mathbb{R}) \quad B \subset C \Rightarrow x(C \setminus B) = x(C) \ominus x(B)$.

Definition 4. *Let $(D, \le, 0_D, 1_D, \ominus)$ be a D-poset. Let $(x_n)_{n=1}^{\infty}$ be a sequence of observables. We say that $(x_n)_{n=1}^{\infty}$ is* ***independent*** *iff for arbitrary positive integer n, there exists n-dimensional observable*

$$h_n : \mathscr{B}(\mathbb{R}^n) \to D,$$

such that for arbitrary Borel sets B_i:

$$m(h_n (B_1 \times \cdots \times B_n)) = \prod_{i=1}^{n} m(x_i(B_i)).$$

Definition 5. *Let $(D, \leq, 0_D, 1_D, \ominus)$ be a D-poset. Let $(x_n)_{n=1}^{\infty}$ be an independent sequence of observables. We say that $(y_n)_{n=1}^{\infty}$ is **the average sequence for $(\mathbf{x_n})_{\mathbf{n=1}}^{\infty}$ normalized by sequence** $(c_n)_{n=1}^{\infty}$ iff*

$$y_n = h_n\big(g_n^{-1}\big),$$

where

$$g_n(u_1, u_2, \cdots, u_n) = \frac{\sum\limits_{i=1}^{n} (u_i - c_i)}{n}.$$

Definition 6. *Let $(D, \leq, 0_D, 1_D, \ominus)$ be a D-poset. Let $y_1, y_2, \cdots, y_n, \cdots$ be a sequence of observables, $a \in \mathbb{R}$. We say that $(y_n)_{n=1}^{\infty}$ **converges m-almost everywhere to a** iff:*

$$\lim_{i\to\infty} \lim_{k\to\infty} \lim_{j\to\infty} m\left(\bigwedge_{n=k}^{j} y_n\left(\left(a - \frac{1}{i}, a + \frac{1}{i}\right)\right)\right) = 1.$$

In that case, we write

$$y_n \longrightarrow a \ m\text{-}a.e.$$

Definition 7. *Let $(D, \leq, 0_D, 1_D, \ominus)$ be a D-poset. Let x be an observable. Let m be a state. Put $m_x(t) = m(x((-\infty, t)))$. We say that x is **observable with dispersion** iff*

$$E(x) = \int_{\mathbb{R}} t \mathrm{d} m_x(t)$$

$$\mathrm{Var}(x) = \int_{\mathbb{R}} (t - E(x))^2 \,\mathrm{d} m_x(t)$$

are finite.

*Real numbers $E(x)$ resp. $\mathrm{Var}(x)$ are then called the **expected value** or **dispersion** of the observable x resp.*

4 Strong Law of Large Numbers on D-posets

Theorem 3. *Let $(D, \leq, 0_D, 1_D, \ominus)$ be a D-poset. Let m be a state. Let $(x_i)_{i=1}^{\infty}$ be an independent identically distributed sequence of integrable observables. Let $(y_n)_{n=1}^{\infty}$ be the average sequence for $(x_n)_{n=1}^{\infty}$, i.e. $y_n = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$. Then*

$$y_n \to E(x_1) \ m\text{-}a.e.$$

Proof. Before proving the Strong Law of Large Numbers, we show how the problem can be transform to the $\sigma-$algebra of families of real sequences of real numbers.

Let $Q_n : \mathscr{B}(\mathbb{R}^n) \to [0,1]$ be the mapping defined by the relationship:

$$\forall B \in \mathscr{B}(\mathbb{R}^n) \qquad Q_n(B) = m(h_n(B)).$$

Since

$$\begin{aligned} 0 \leq m \leq 1 \Rightarrow 0 \leq Q_n \leq 1, \\ Q_n(\emptyset) = m(h_n(\emptyset)) = m(0_A) = 0, \\ Q_n(\mathbb{R}^n) = m(h_n(\mathbb{R}^n)) = m(1_A) = 1, \end{aligned}$$

and for arbitrary disjoint sets $B \supset C \in \mathscr{B}(\mathbb{R}^n)$

$$\begin{aligned} Q_n(B \setminus C) &= m(h_n(B \setminus C)) = m(h_n(B) \ominus h_n(C)) \\ &= m(h_n(B)) - m(h_n(C)) = Q_n(B) - Q_n(C), \end{aligned}$$

Q_n is subtractive. Consider arbitrary $B_i \, (i \in \mathbb{N}), B \in \mathscr{B}(\mathbb{R}^n)$ such that $B_i \nearrow B$. Then $Q_n(B_i) = m(h_n(B_i)) \nearrow m(h_n(B)) = Q_n(B)$. From this, it follows that Q_n is a probability.

Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences. For arbitrary positive integer n we define **n-th projection**

$$\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$$

by the relationship:

$$\pi_n((u_i)_{i=1}^{\infty}) = (u_1, u_2, \cdots, u_n).$$

Similarly for $n \leq k$ we define

$$\pi_{k,n} : (u_1, \cdots, u_n, \cdots, u_k) \mapsto (u_1, \cdots, u_n).$$

Preimages $\pi_n^{-1}(B)$ of sets $B \in \mathscr{B}(\mathbb{R}^n)$ are called **cylinders**.

It can be shown that for arbitrary positive integers k, n

$$Q_{n+k}\Big(\pi_{n+k,n}^{-1}(B)\Big) = Q_n(B).$$

It suffices to show that

$$\begin{aligned} &m\Big(h_{n+k}\Big(\pi_{n+k,n}^{-1}(B_1 \times \cdots \times B_n)\Big)\Big) \\ &= m(h_{n+k}(B_1 \times \cdots \times B_n \times \mathbb{R} \times \cdots \times \mathbb{R})) = m(h_n(B_1 \times \cdots \times B_n)). \end{aligned}$$

From the Kolmogorov consistency theorem it follows that there is the unique probability P on $\sigma(\mathscr{C})$, where $\mathscr{C}$ is the system of all cylinders, such that for any $C = \pi_n^{-1}(B)$ it holds

$$P(C) = Q_n(B) = m(h_n(B)).$$

Let $\left(\mathbb{R}^{\mathbb{N}}, \sigma(\mathscr{C}), P\right)$ be the probability space from the above construction. Define random variables $\xi_n((u_i)_{i=1}^{\infty}) = u_n$. The variables ξ_i are independent, since

$$P\left(\bigcap_{i=1}^{n} \xi_i^{-1}(B_i)\right) = m(h_n(B_1 \times \cdots \times B_i)) = \prod_{i=1}^{n} m(x_i(B_i)) = \prod_{i=1}^{n} P\big(\xi_i^{-1}(B_i)\big).$$

Moreover

$$P\big(\xi_i^{-1}((-\infty, t))\big) = m(x_i((-\infty, t))) = m(x_j((-\infty, t))) = P\big(\xi_j^{-1}((-\infty, t))\big)$$

and

$$E(\xi_1) = \int_{\mathbb{R}^{\mathbb{N}}} \xi_1 \mathrm{d}P = \int_{\mathbb{R}} t \mathrm{d}m_{x_1}(t) = E(x_1).$$

Hence ξ_n fulfills the conditions for the standard strong law of numbers. Let $\eta_n = \frac{1}{n} \sum\limits_{i=1}^{n} \xi_i$. Then

$$\forall B \in \mathscr{B}(\mathbb{R}) \quad P\big(\eta_n^{-1}(B)\big) = m\big(h_n\big(g_n^{-1}(B)\big)\big) = m(y_n(B)),$$

because if we put $g_n((u_1, \cdots, u_n)) = \frac{\sum\limits_{i=1}^{n} u_i}{n}$, then

$$P\big(\eta_n^{-1}(B)\big) = P\big(\pi_n^{-1}\big(g_n^{-1}(B)\big)\big) = m\big(h_n\big(g_n^{-1}(B)\big)\big) = m(y_n(B)).$$

In [6] it is proved that

$$m\left(\bigwedge_{n=k}^{j} y_n\left(\left(a - \frac{1}{i}, a + \frac{1}{i}\right)\right)\right) \geq P\left(\bigcap_{n=k}^{j} \eta_n^{-1}\left(\left(a - \frac{1}{i}, a + \frac{1}{i}\right)\right)\right)$$

holds for arbitrary $a \in \mathbb{R}, i, k, j \in \mathbb{N}, j \geq k$. From this it follows, that

$$\eta_n \longrightarrow a \,\text{P-a.e.} \Rightarrow y_n \longrightarrow a \text{ m-a.e.}$$

It suffices to use the Standard Strong Law of Large Numbers. $\square$

5 Kolmogorov Strong Law of Large Numbers on D-posets

Theorem 4. *Let $(D, \leq, 0_D, 1_D, \ominus)$ be a D-poset. Let m be a state. Let $(x_i)_{i=1}^{\infty}$ be an independent sequence of observables with dispersion. Let*

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{\operatorname{Var}(x_k)}{k^2} < \infty.$$

Let $(y_n)_{n=1}^{\infty}$ be the average sequence for $(x_n)_{n=1}^{\infty}$ normalized by the sequence $(E(x_n))_{n=1}^{\infty}$, i.e. $y_n = \frac{\sum\limits_{i=1}^{n} (x_i - E())}{n}$ Then

$$y_n \to 0 \; m\text{-a.e.}$$

Proof. Proof is the same as the proof of the theorem 3. We must only prove that the random variables $\xi_n : (u_i)_{i=1}^{\infty} \mapsto u_n$ have dispersion and $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\mathrm{Var}(\xi_k)}{k^2} < \infty$. But

$$\mathrm{Var}(\xi_i) = \int_{\mathbb{R}^{\mathbb{N}}} (\xi_i - E(\xi_i))^2 \,\mathrm{d}P$$

$$= \int_{\mathbb{R}} (t - E(\xi_i))^2 \,\mathrm{d}P\big(\xi_1^{-1}((-\infty, t))\big) = \int_{\mathbb{R}} (t - E(x_i))^2 \,\mathrm{d}m_{x_i}(t) = \mathrm{Var}(x_i).$$

It suffices to use Standard Kolmogorov Strong Law of Large Numbers. □

6 Conclusion

In this contribution we presented the version of Strong Law of Large Numbers on D-poset. There are some other structures which are equivalent with D-poset. For example A-posets [2] and effect-algebras [1]. For these structures can be Strong Laws of large numbers reformulated.

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Estimation of Trees on the Basis of Pairwise Comparisons with Random Errors

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Abstract. The estimators of the trees on the basis of multiple pairwise comparisons, with random errors, are proposed in the paper. The estimators are based on the idea of the nearest adjoining order (see Slater 1961; Klukowski 2011). Two kinds of trees are examined: non-directed and directed. The approach is similar to estimation of the preference relation with incomparable elements on the basis of binary comparisons. The estimates are obtained on the basis of discrete optimization problems; their properties are similar to those for the preference relation. Such the trees can be applied to modelling of many phenomena, e.g. biological evolution, decision problems, etc.

Keywords: Estimation of trees · Non-directed and directed trees · Pairwise comparisons with random errors

1 Introduction

The problems of estimation of the relations of: preference (complete and with partial order), equivalence and tolerance, on the basis of multiple pairwise comparisons with random errors, has been examined in Klukowski (1994, 2011 Chaps. 7–11, 2014a, b). The same approach can be applied to the trees – non-directed and directed; they are more general objects than the relations mentioned. The non-directed tree can be defined as a graph – non-directed, acyclic and complete; in other words: it doesn't exist a path (sequence of edges) from a fixed node (element) to this node and each pair of nodes is connected with a path. The directed tree can be defined as a graph directed, acyclic and complete. The directed graph has a root (initial node), paths leading in one direction and leafs (final nodes of a tree).

The problem of estimation of a tree, non-directed or directed, can be expressed as follows:

- it is given a finite set of nodes (elements) with unknown paths (system of edges);
- instead of system of edges, it is known a set of pairwise comparisons, which evaluate unknown paths, with random errors; any comparison states existence or non-existence of a connection between two elements – in the case of non-directed tree connection means an edge in the case of directed tree a connection mean a path and its direction;

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– a random error means that a result of any comparison can be true or not with a probability satisfying some weak assumptions; any pair is compared N times ($N \geq 1$), all comparisons are assumed independent in stochastic way;
– the form of a tree, i.e. the system of its paths, has to be determined (estimated) on the basis of the set of pairwise comparisons characterized above.

The idea of estimation consists in minimization of differences between the form of a tree, expressed in appropriate way, and a given set of pairwise comparisons with random errors. The estimates are obtained as the optimal solutions of the discrete programming problems, defined below; the number of solutions can exceed one.

The approach rested on the statistical paradigm provides the properties of estimates and the possibility of verification of the results obtained. The main property is consistency, for the number of comparisons N (for each pair) converging to infinity, under non-restricted assumptions about comparison errors. In general it is assumed that probability of correct comparison is greater than ½ and that multiple comparisons of each pair are independent random variables. The estimators can be also applied in the case of unknown distributions of comparison errors, which have to satisfy the assumptions made.

The idea of the estimators was introduced firstly by Slater (1961) - for the case of single, binary comparisons and the complete preference relation; some other ideas, in the area of pairwise comparisons, have been presented in: David (1988), Bradley (1984), Flinger and Verducci (1993), Gordon (1999).

The paper consists of four sections. The second section presents the definitions, notations and assumptions about comparison errors. The next sections consider the form of estimators, for both kinds of trees, and their properties. The last section summarizes the results. The Appendix presents proofs of some relationships determining properties of the estimators proposed.

2 Definitions, Notations and Assumptions About Comparisons Errors

2.1 Definitions and Notations

The problem of estimation of the non-directed tree on the basis of pairwise comparisons can be stated as follows.

We are given a finite set of elements $\mathbf{X} = \{x_1, \ldots, x_m\}$ ($3 \leq m < \infty$). The elements of the set $\mathbf{X}$ (nodes) are connected with edges generating a non-directed tree (non-directed, acyclic and complete). Each pair of elements (x_i, x_j) can have an edge or not; thus the set of pairs of indices:

$$R_m = \{<i,j> \mid i = 1, \ldots, m-1, j = i+1, \ldots, m\} \tag{1}$$

can be divided into two disjoint subsets – the first one I_o include pairs connected with an edge, the second one I_v pairs not connected with an edge, $R_m = I_o \cup I_v$. Any pair $<i, j>$ is not ordered, i.e. is the same, as $<j, i>$.

The (non-directed) tree can be expressed with a use of values $T_v(x_i, x_j)$ ($<i,j> \in R_m$), indicating existence or non-existence of an edge:

$$T_v(x_i, x_j) = \begin{cases} 1 \text{ } if \text{ } x_i \text{ } and \text{ } x_j \text{ } are\ connected\ with\ an\ edge, \\ 0 \text{ } if \text{ } x_i \text{ } and \text{ } x_j \text{ } are\ not\ connected\ with\ an\ edge. \end{cases} \tag{2}$$

The values $T_v(x_i, x_j)$ define the non-directed tree in the unique way.

The similar considerations relate to the directed tree (directed, acyclic and complete). Such the tree can be expressed with a use of values $T_d(x_i, x_j)$ ($<i,j> \in R_m$), indicating existence or non-existence of a path between elements (nodes) and direction of the path:

$$T_d(x_i, x_j) = \begin{cases} -1 \text{ } if\ there\ exists\ a\ path\ from \text{ } x_i \text{ } to \text{ } x_j, \\ 1 \text{ } if\ there\ exists\ a\ path\ from \text{ } x_j \text{ } to \text{ } x_i, \\ 2 \text{ } if\ there\ not\ exists\ a\ path\ between \text{ } x_i \text{ } and \text{ } x_j \text{ } . \end{cases} \tag{3}$$

The set of indices R_m can be expressed as the alternative of the subsets $R_m = I_{\pm 1} \cup I_v$, where: $I_{\pm 1}$ includes indices of pairs of elements connected with a path and I_v includes indices of non-connected pairs; any pair of indices $<i,j> \in I_{\pm 1}$ of connected elements is ordered, i.e. shows that the direction of a path between x_i and x_j. Of course $T_d(x_i, x_j) = -T_d(x_j, x_i)$ for $<i,j> \in I_{\pm 1}$.

The values $T_d(x_i, x_j)$ define the directed tree in the unique way.

Examples

The non-directed tree - the set $\mathbf{X} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with pairs connected with an edge: (x_1, x_2), (x_1, x_3), (x_3, x_4), (x_3, x_5), (x_3, x_6); the sets I_0 and I_v assume the forms:

$I_o = \{<1, 2>, <1, 3>, <3, 4>, <3, 5>, <3, 6>\}$,
$I_v = \{<1, 4>, <1, 5>, <1, 6>, <2, 3>, <2, 4>, <2, 5>, <2, 6>, <4, 5>, <4, 6>, <5, 6>\}$.

The values $T_v(x_i, x_j)$ assume the form:

$$T_v(x_i, x_j) = \begin{bmatrix} \times & 1 & 1 & 0 & 0 & 0 \\ & \times & 0 & 0 & 0 & 0 \\ & & \times & 1 & 1 & 1 \\ & & & \times & 0 & 0 \\ & & & & \times & 0 \\ & & & & & \times \end{bmatrix}$$

The directed tree the set $\mathbf{X} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with the following order: x_1 precedes x_2, x_1 precedes x_3, x_2 incomparable with x_3, x_3 precedes x_4, x_3 precedes x_5, x_3

precedes x_6, x_4 incomparable with x_5, x_4 incomparable with x_6, x_5 incomparable with x_6. The sets $I_{\pm 1}$ and I_v assume the forms:

$$\begin{matrix} I_{\pm 1} = \{<1,\ 2>,\ <1,\ 3>,\ <1,\ 4>,\ <1,\ 5>,\ <1,\ 6>,\ <3,\ 4>,\ <3,\ 5>,\ <3,\ 6>\}, \\ I_v = \{<2,\ 3>,\ <2,\ 4>,\ <2,\ 5>,\ <2,\ 6>,\ <4,\ 5>,\ <4,\ 6>,\ <5,\ 6>\} \end{matrix}.$$

The values $T_d(x_i, x_j)$:

$$T_d(x_i, x_j) = \begin{bmatrix} \times & -1 & -1 & -1 & -1 & -1 \\ & \times & 2 & 2 & 2 & 2 \\ & & \times & -1 & -1 & -1 \\ & & & \times & 2 & 2 \\ & & & & \times & 2 \\ & & & & & \times \end{bmatrix}$$

2.2 Assumptions About Distributions of Comparisons Errors

The form of both types of the trees, expressed by $T_v(x_i, x_j)$ or $T_d(x_i, x_j)$, has to be determined (estimated) on the basis of N $(N \geq 1)$ comparisons of each pair (x_i, x_j) $(<i,j> \in R_m)$, evaluating the values $T_v(x_i, x_j)$ or $T_d(x_i, x_j)$, disturbed by random errors. The comparisons evaluating $T_v(x_i, x_j)$ and $T_d(x_i, x_j)$ will be denoted – respectively - $g_k^{(v)}(x_i, x_j)$ and $g_k^{(d)}(x_i, x_j)$ $(k = 1, \ldots, N)$. The comparison errors - respectively $\phi_k^{(v)*}(x_i, x_j)$ and $\phi_k^{(d)*}(x_i, x_j)$ - can be expressed in the following form:

$$\phi_k^{(v)*}(x_i, x_j) = \begin{cases} 0 \text{ if } g_k^{(v)}(x_i, x_j) \text{ and } T_v(x_i, x_j) \text{ are the same,} \\ 1 \text{ if } g_k^{(v)}(x_i, x_j) \text{ and } T_v(x_i, x_j) \text{ are not the same,} \end{cases} \tag{4}$$

$$\phi_k^{(d)*}(x_i, x_j) = \begin{cases} 0 \text{ if } g_k^{(d)}(x_i, x_j) \text{ and } T_d(x_i, x_j) \text{ are the same,} \\ 1 \text{ if } g_k^{(d)}(x_i, x_j) \text{ and } T_d(x_i, x_j) \text{ are not the same.} \end{cases} \tag{5}$$

The distributions of comparison errors have to satisfy the following assumptions.

A1. Any comparison $g_k^{(\upsilon)}(x_i, x_j)$ $(\upsilon \in \{v,\ d\};\ k = 1, \ldots, N;\ \ <i,j> \in R_m)$ is an evaluation of the value $T_\upsilon(x_i, x_j)$; the probabilities of errors $P(\phi_k^{(\upsilon)*}(x_i, x_j) = l)$ $(\upsilon \in \{v,\ d\},\ \ l \in \{0,\ 1\})$ have to satisfy the following assumptions:

$$P(\phi_k^{(\upsilon)*}(x_i, x_j) = 0) \geq 1 - \delta_\upsilon \quad (\delta_\upsilon \in (0,\ 1/2)), \tag{6}$$

$$P(\phi_k^{(\upsilon)*}(x_i, x_j) = 0) + P(\phi_k^{(\upsilon)*}(x_i, x_j) = 1) = 1, \tag{7}$$

A2. The comparisons: $g_k^{(\upsilon)}(x_i, x_j)$ $(k = 1, \ldots, N;\ \ <i,j> \in R_m)$ are independent random variables.

The assumptions about comparisons errors reflect the following facts. The probability of a correct comparison is greater than incorrect one (assumptions (6), (7)). The comparisons errors are independent in the stochastic way. The assumption can be relaxed in such a way that (multiple) comparisons of the same pair are independent and comparisons of pairs comprising different elements are independent.

The random variables $\phi_k^{(v)}(x_i, x_j)$ and $\phi_k^{(d)}(x_i, x_j)$, corresponding to any tree (non-directed or directed) - denoted, respectively, by $t_v(x_i, x_j)$ and $t_d(x_i, x_j)$, expressing differences between relation form and comparisons, assume the following form:

$$\phi_k^{(\upsilon)}(x_i, x_j) = \begin{cases} 0 \; if \;\; g_k^{(\upsilon)}(x_i, x_j) \; and \;\; t_\upsilon(x_i, x_j) \; are \; the \; same, \\ 1 \; if \;\; g_k^{(\upsilon)}(x_i, x_j) \; and \;\; t_\upsilon(x_i, x_j) \; are \; not \; the \; same. \end{cases} \tag{8}$$

3 Estimation Problems and Properties of Estimates

The idea of the nearest adjoining order estimators is to minimize the differences between the comparisons, on the one hand, and the relation, expressed in a "compatible" way, on the other hand. Thus, the estimates $\hat{T}_v(x_i, x_j)$ or $\hat{T}_d(x_i, x_j)$ $(<i,j> \; \in R_m)$ are the optimal solutions of the discrete programming problems – respectively:

$$\min_{F_X^{(v)}} \left\{ \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \phi_k^{(v)}\left(x_i, x_j\right) \right\}, \tag{9}$$

$$\min_{F_X^{(d)}} \left\{ \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \phi_k^{(d)}\left(x_i, x_j\right) \right\}, \tag{10}$$

where:

$F_X^{(\upsilon)}$ $(\upsilon \in \{v, \; d\}$ - feasible set, i.e. family of all trees (non-directed or directed) determined on the set **X**.

In the earlier works of the author Klukowski (2011, 2012, 2014a, b), about estimation of the relations form (equivalence, tolerance and preference) it was proved that the estimators based on the optimal solutions of the problems (9), (10) have good statistical properties, especially, they are consistent, as $N \to \infty$. Moreover, the speed of the convergence can be determined - it is of the exponential type. The precision of estimates can be evaluated with the use of simulation approach. The proofs of the consistency are based on the following facts.

Firstly, the expected values of the random variables:

$$W_v^* = \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \phi_k^{(v)*}(x_i, x_j), \tag{11}$$

$$W_d^* = \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \phi_k^{(d)*}(x_i, x_j), \tag{12}$$

expressing the differences between the comparisons and the actual tree, ($T_v(x_i, x_j)$ or $T_d(x_i, x_j)$ $(<i, j> \ \in R_m)$), are lower than the expected values of the variables:

$$\tilde{W}_v = \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \tilde{\phi}_k^{(v)}(x_i, x_j), \tag{13}$$

$$\tilde{W}_d = \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \tilde{\phi}_k^{(d)}(x_i, x_j), \tag{14}$$

expressing differences between comparisons and any other relation, denoted by $\tilde{T}_v(x_i, x_j)$ or $\tilde{T}_d(x_i, x_j)$.

Secondly, the variances of these variables, i.e.: $Var(\frac{1}{N} W_v^*)$, $Var(\frac{1}{N} W_d^*)$, $Var(\frac{1}{N}\tilde{W}_v)$, $Var(\frac{1}{N}\tilde{W}_d)$, converge to zero, as $N \to \infty$.

Thirdly, the probabilities: $P(W_b^* < \tilde{W}_b)$ and $P(W_\mu^* < \tilde{W}_\mu)$ converge to one, as $N \to \infty$; the speed of convergence is determined by the exponential subtrahend. These relationships can be formulated shortly in the following

Theorem. *The following relationships hold true*:

$$E(W_v^*) < E(\tilde{W}_v), \tag{15}$$

$$E(W_d^*) < E(\tilde{W}_d), \tag{16}$$

$$\lim_{N\to\infty} Var(\frac{1}{N} W_v^*) = 0, \ \lim_{N\to\infty} Var(\frac{1}{N}\tilde{W}_v) = 0, \tag{17}$$

$$\lim_{N\to\infty} Var(\frac{1}{N} W_d^*) = 0, \ \lim_{N\to\infty} Var(\frac{1}{N}\tilde{W}_d) = 0, \tag{18}$$

$$P(W_v^* < \tilde{W}_v) \geq 1 - \exp\left\{-2N(1/2 - \delta_v)^2\right\}, \tag{19}$$

$$P(W_d^* < \tilde{W}_d) \geq 1 - \exp\left\{-2N(1/2 - \delta_d)^2\right\}. \tag{20}$$

Proofs of the relationships (15)–(20) are similar to the case of the relations mentioned (see Klukowski 1994, 2011 Chaps. 7, 8, 2014a, b), their idea is presented in the Appendix.

The relationships (15)–(20) are the theoretical basis for establishing the estimators $\hat{T}_v(x_i, x_j)$ and $\hat{T}_d(x_i, x_j)$ - they indicate consistency. This is so, because the random variables $\frac{1}{N} W_v^*$ or $\frac{1}{N} W_d^*$, corresponding to the actual trees, have minimal expected values in the family $F_X^{(v)}$ or $F_X^{(d)}$ and variances converging to zero. The optimal solutions of the problems (9) and (10), determining trees with minimal values of differences with respect to comparisons, converge to "true" trees with probability converging to one. The simulation experiments, concerning the preference relation (see Klukowski 2011, Chap. 9), show that for $\delta_\upsilon \leq 0,1$ and $N \geq 3$ the frequency of correct estimates exceeds 75%. Moreover, estimation errors, i.e.:

$$\sum_{<i,j> \in R_m} \left| \hat{T}_\upsilon(x_i, x_j) - T_\upsilon(x_i, x_j) \right| \qquad (\upsilon \in \{v,\ d\}),$$

are close to zero. The values of N equal 7 provides frequency of correct results exceeding 95%.

Known properties of the estimates allow verification the results of estimation, i.e. checking if the tree is true model of the data, with the use of statistical tests (see Klukowski 2011, Chap. 10, Gordon 1999, Chap. 7).

The approach can be applied also in the case of unknown probabilities of comparison errors; it is necessary to satisfy the condition $\delta_\upsilon < 1/2$. In such a case, and the number of N at least several, the values of δ_υ $(\upsilon \in \{v, d\})$ can be estimated. The precision of the estimates of the relations can be determined with the use of simulation approach, in a similar way, as in Klukowski 2011, Chap. 9.

Minimization of the problems (9), (10) is not an easy to solve. For a low number m of elements of the set $\mathbf{X}$, i.e. several, the minimization can be performed simply by complete enumeration. For the moderate values of m, i.e. $m \leq 50$ the problem can be solved with the use of known discrete programming algorithms. They are similar to the algorithms presented in Hansen et al. (1994), for the equivalence relation, and in David (1988), for the preference relation. In the case of m > 50 and multiple comparisons (N > 1), heuristic algorithms are necessary.

4 Concluding Remarks

The paper presents the estimators of the trees, non-directed and directed, which are based on the pairwise comparisons, in the binary form, disturbed by random errors. They have similar properties to the estimators of the relations of: equivalence, tolerance and preference (also including incomparable elements); in particular – consistency and speed of convergence. The statistical properties together with possibility of verification of estimates produce results which are trustworthy and reliable.

Appendix

The idea of the proof of the Theorem (relationships (15)–(20)).

The proof of the inequality (15), i.e. $E(W_\upsilon^*) < E(\tilde{W}_\upsilon)$ is simple: the expected value of the difference $W_\upsilon^* - \tilde{W}_\upsilon$ assumes the form:

$$
\begin{aligned}
&E(W_\upsilon^* - \tilde{W}_\upsilon) \\
&= E\left(\sum_{<i,j> \in R_m} \sum_{k=1}^{N} \phi_k^{(\upsilon)*}(x_i, x_j) - \sum_{<i,j> \in R_m} \sum_{k=1}^{N} \tilde{\phi}_k^{(\upsilon)}(x_i, x_j) \right) \\
&= \sum_{k=1}^{N} \sum_{<i,j> \in R_m} E\left(\phi_k^{(\upsilon)*}(x_i, x_j) - \tilde{\phi}_k^{(\upsilon)}(x_i, x_j) \right).
\end{aligned}
\tag{A1}
$$

It is clear that each component $E\left(\phi_k^{(\upsilon)*}(x_i, x_j) - \tilde{\phi}_k^{(\upsilon)}(x_i, x_j)\right)$ can be either zero or negative; the value of zero corresponds to the case $T_b(x_i, x_j) = \tilde{T}_b(x_i, x_j)$, negative – to the case of $T_b(x_i, x_j) \neq \tilde{T}_b(x_i, x_j)$. The negative value results from the fact that any correct comparison, i.e. $\phi_k^{(\upsilon)*}(x_i, x_j) = 0$, indicates $\tilde{\phi}_k^{(\upsilon)}(x_i, x_j) = 1$ and probability of the event equals δ_υ, i.e. is greater than ½. The inequality $T_b(x_i, x_j) \neq \tilde{T}_b(x_i, x_j)$ indicates existence of negative components. These facts is sufficient for proving the inequality (15).

The proof of the inequality (16) is similar.

The proof of the inequality (17) is obvious: the variable W_v^* is the sum of N iid. random variables $\sum_{<i,j> \in R_m} \phi_k^{(\upsilon)*}(x_i, x_j)$ $(k = 1, \ldots, N)$ with finite expected value and variance. Therefore the variance of the variable $1/N\, W_v^*$ converges to zero. The convergence to zero of variances of the variables $1/N\tilde{W}_v$, $1/NW_d^*$, $1/N\tilde{W}_d$ of the remaining random variables is proved in similar way.

The inequalities (19) and (20) can be proved on the basis of Hoeffding's (1963) inequality for a sum of independent, binary random variables. The inequality applied in the case under consideration assumes the form:

$$
P\left(\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} E(Y_k) \geq Nt\right) \leq \exp(-2N_t^2), \tag{$*$}
$$

where:

Y_k $(k = 1, \ldots, N)$ iid. random variables, satisfying:

$P\ (0 \leq Y_k \leq 1) = 1,\ E(Y_k) < 1/2,$
t – positive constant.

The inequality (*) can be applied to the random variables $\sum_{k=1}^{N} \sum_{<i,j> \in R_m} \left(\phi_{\upsilon k}^*(x_i, x_j) - \tilde{\phi}_{\upsilon k}(x_i, x_j)\right)$ $(\upsilon \in \{b, \mu\})$, after a following transformations:

$$
\begin{aligned}
&P\left(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} (\phi_{\upsilon k}^*(x_i, x_j) - \tilde{\phi}_{\upsilon k}(x_i, x_j)) < 0\right) \\
&= 1 - P\left(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} (\phi_{\upsilon k}^*(x_i, x_j) - \tilde{\phi}_{\upsilon k}(x_i, x_j)) \geq 0\right).
\end{aligned}
$$

Moreover:

$$\begin{aligned} & P(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} (\phi^*_{\upsilon k}(x_i,x_j) \ - \tilde{\phi}_{\upsilon k}(x_i,x_j)) \geq 0) \\ & = P(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} (2\phi^*_{\upsilon k}(x_i,x_j) \ - 1) \geq 0). \end{aligned} \tag{A2}$$

Assuming that the sum $\sum_{<i,j> \in R_m} (2\phi^*_{\upsilon k}(x_i,x_j) \ - 1)$ includes τ non-zero components, the inequality (A2) can be transformed in the following way:

$$\begin{aligned} & P\left(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} \left(2\phi^*_{\upsilon k}\left(x_i,x_j\right) - 1\right) \geq 0\right) \\ & = P\left(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} \phi^*_{\upsilon k}\left(x_i,x_j\right) \geq N\tau/2\right) \\ & = P\left(\sum_{k=1}^{N} \sum_{<i,j> \in R_m} \phi^*_{\upsilon k}\left(x_i,x_j\right) N\tau\delta_\upsilon \geq N\tau/2 - N\tau\delta_\upsilon\right) \\ & = P\left(\frac{1}{\tau}\sum_{k=1}^{N} \sum_{<i,j> \in R_m} \phi^*_{\upsilon k}\left(x_i,x_j\right) - N\delta_\upsilon \geq N\left(\frac{1}{2} - \delta_\upsilon\right)\right). \end{aligned} \tag{A3}$$

The last expression in (A3) can be evaluated on the basis of Hoeffding inequality:

$$\begin{aligned} & P(\tfrac{1}{\tau}\sum_{k=1}^{N} \sum_{<i,j> \in R_m} \phi^*_{\upsilon k}\left(x_i,x_j\right) \ - N\,\delta_\upsilon \geq N(\tfrac{1}{2} - \delta_\upsilon)) \\ & \leq \exp\left(-2N(\tfrac{1}{2} - \delta_\upsilon)^2\right). \end{aligned} \tag{A4}$$

The evaluation (A4) is equivalent to the inequality (19).

The inequality (20) can be proved in a similar way.

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Issues in the Representation and Processing of Imprecise Information

On Extended Intuitionistic Fuzzy Index Matrices with Elements Being Predicates

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Abstract. In this paper, the concepts of an index matrix with elements being predicates and of an extended intuitionistic fuzzy index matrix with elements being predicates are introduced. Some operations, relations and operators over these mew types of matrices are defined. Some properties of these concepts are discussed and statements are formulated.

Keywords: Extended intuitionistic fuzzy index matrix · Index matrix · Intuitionistic fuzziness

1 Introduction

Here, as a continuation of [1,3–8], we discuss the concepts of an Extended Intuitionistic Fuzzy Index Matrix (EIFIM), Index Matrix with Elements being Variables (IMEV), Index Matrix with Elements being Predicates (IMEP) and Extended Intuitionistic Fuzzy Index Matrix with Elements being Predicates (EIFIMEP), and introduce over them new operations from quantifier type.

2 Basic Definition

Firstly, we give some remarks on Intuitionistic Fuzzy Sets (IFSs, see, e.g., [2,5]) and especially, of their partial case, Intuitionistic Fuzzy Pairs (IFPs; see [11]). The IFP is an object in the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process and which conponents (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$.

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First, in [11] we defined the relations

$$\begin{array}{lll} x < y & \text{iff} & a < c \text{ and } b > d \\ x \leq y & \text{iff} & a \leq c \text{ and } b \geq d \\ x > y & \text{iff} & a > c \text{ and } b < d \\ x \geq y & \text{iff} & a \geq c \text{ and } b \leq d \\ x = y & \text{iff} & a = c \text{ and } b = d \end{array}$$

Second, we defined two examples for definitions of operations "conjunction" and "disjunction":

$$x \vee_1 y = \langle \max(a, c)), \min(b, d) \rangle,$$
$$x \wedge_1 y = \langle \min(a, c), \max(b, d) \rangle,$$
$$x \vee_2 y = \langle a + c - a.c, b.d \rangle,$$
$$x \wedge_2 y = \langle a.c, b + d - b.d \rangle.$$

Third, following [8], the definition of an Extended IM is proposed.

Let $\mathcal{I}$ be again a fixed set of indices,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n \rangle | (\forall j : 1 \leq j \leq n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1 \leq n \leq \infty} \mathcal{I}^n.$$

Let everywhere below $\mathcal{X}$ be a fixed set of some objects. In the particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, etc.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be given.

We call the object $[K, L, \{a_{k_i,l_j}\}]$ with index sets K and L ($K, L \subset \mathcal{I}^*$) and elements from set $\mathcal{X}$, "Extended IM" (EIM) if it is defined in the form (see, [8]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_j} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1} & \ldots & a_{k_i,l_j} & \ldots & a_{k_i,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_j} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, for $1 \leq i \leq m$, and $1 \leq j \leq n$: $a_{k_i,l_j} \in \mathcal{X}$.

When elements a_{k_i,l_j} are some variables, propositions or formulas, we obtain an EIM with elements from the respective type. Then, we can define the evaluation function V that juxtaposes to this IM a new one with elements – Intuitionistic Fuzzy pairs (IFPs) $\langle \mu, \nu \rangle$, where $\mu, \nu, \mu + \nu \in [0, 1]$. The new IM, called Intuitionistic Fuzzy IM (IFIM), contains the evaluations of the variables, propositions, etc., i.e., it has the form

$$V([K, L, \{a_{k_i,l_j}\}]) = [K, L, \{V(a_{k_i,l_j})\}] = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$= \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $V(a_{k_i,l_j}) = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$ and $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$.

The Extended IFIM (EIFIM) is defined by:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1],$$

$$\alpha_1^k, \beta_1^k, \alpha_1^k + \beta_1^k \in [0, 1],$$
$$\alpha_1^l, \beta_1^l, \alpha_1^l + \beta_1^l \in [0, 1],$$

and here and below

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \leq i \leq m\},$$

$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \leq j \leq n\}.$$

3 Main Results

3.1 Basic Definitions

Let x be a variable, obtaining (finite number of) values $a_1, a_2, ..., a_n$ and let $P(x)$ be a predicate with a variable x. Let V be an evaluation function, defined by

$$V(P(x)) = \langle \mu(P(x)), \nu(P(x)) \rangle,$$

where $\mu(P(x))$ and $\nu(P(x))$ are the degrees of validity and non-validity of P.

The IF-interpretations of the (intuitionistic fuzzy) quantifiers *for all* ($\forall$) and *there exists* ($\exists$) are introduced in [10,12] by

$$V(\exists x P(x)) = \langle \max_{y \in E} \mu(P(y)), \min_{y \in E} \nu(P(y)) \rangle, \tag{1}$$

$$V(\forall x P(x)) = \langle \min_{y \in E} \mu(P(y)), \max_{y \in E} \nu(P(y)) \rangle. \tag{2}$$

From classical logic it is well known that for each predicate P with argument x having interpretations $a_1, a_2, ..., a_n$ (finite number), it holds that

$$V(\forall x P(x)) = V(P(a_1) \wedge P(a_2) \wedge ... \wedge P(a_n)), \tag{3}$$

$$V(\exists x P(x)) = V(P(a_1) \vee P(a_2) \vee ... \vee P(a_n)). \tag{4}$$

Here, for the first time we will introduce "Index Matrices with Elements being Predicates" (IMEP). This IM-modification is an analogue of the IM with function-type of elements, introduced in [9].

Let the set of all used functions be $\mathcal{P}$. The research over IMEPs develops in the following two cases:

- each predicate of set $\mathcal{P}$ has one argument and it is exactly x (i.e., it is not possible that one of the predicates has argument x and another predicate has argument y) – let us mark the set of these predicates by $\mathcal{P}_x^1$;
- each predicate of set $\mathcal{P}$ has one argument, but that argument might be different for the different predicates or the different predicates of set $\mathcal{P}$ have different numbers of arguments.

Let us use predicates from $\mathcal{F}_x^1$ or from $\mathcal{P}$, but only from one of both types. The IMEP has the form

$$[K, L, \{P_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & P_{k_1,l_1} & P_{k_1,l_2} & \dots & P_{k_1,l_n} \\ k_2 & P_{k_2,l_1} & P_{k_2,l_2} & \dots & P_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & P_{k_m,l_1} & P_{k_m,l_2} & \dots & P_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, For $1 \le i \le m$, and $1 \le j \le n$: $P_{k_i,l_j} \in \mathcal{P}_x^1$ or $P_{k_i,l_j} \in \mathcal{P}$, but all these are elements from exactly one of both sets.

In the more general case EIFIMEP has the form:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & P_{k_1,l_1} & \ldots & P_{k_1,l_j} & \ldots & P_{k_1,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & P_{k_i,l_1} & \ldots & P_{k_i,l_j} & \ldots & P_{k_i,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & P_{k_m,l_1} & \ldots & P_{k_m,l_j} & \ldots & P_{k_m,l_n} \end{array},$$

where K^* and L^* have the above forms.

In the next two subsections, we introduce operations and relations, defined over EIFIMEP, but they can be transformed easily for the simpler case of IMEP.

3.2 Operations over EIFIMEP

In this Section a colision arises between symbol P for a predicate, used above and symbol P for an index set, used in [8]. By this reasom, below, the predicates will be noted by symbols f, g and h (with the respective indices).

Let

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K : (\alpha_i^k < \alpha_i^p) \ \& \ (\beta_i^k > \beta_i^p)).$$

$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K : (\alpha_i^k \le \alpha_i^p) \ \& \ (\beta_i^k \ge \beta_i^p)).$$

Following [6,8], for the EIFIMEPs $A = [K^*, L^*, \{f_{k_i,l_j}\}], B = [P^*, Q^*, \{g_{p_r,q_s}\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other specific ones. Let $\circ \in \{\vee_1, \wedge_1, ...\}$.

(a) addition-$(\circ)$

$$A \oplus_{(\circ)} B = [T^*, V^*, \{h_{t_u,v_w}\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \max(\alpha_i^k, \alpha_r^p), & \text{if } t_u \in K \cap P \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w \in L - Q \\ \\ \beta_s^q, & \text{if } t_w \in Q - L \\ \\ \min(\beta_j^l, \beta_s^q), & \text{if } t_w \in L \cap Q \end{cases},$$

and

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ f_{k_i,l_j} \circ g_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ false, & \text{otherwise} \end{cases}$$

(b) termwise multiplication-$(\circ)$

$$A \otimes_{(\circ)} B = [T^*, V^*, \{h_{t_u,v_w}\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cap P\},$$

$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cap Q\},$$

$$\alpha_u^t = \circ(\alpha_i^k, \alpha_r^p), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta_w^v = *(\beta_j^l, \beta_s^q), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$h_{t_u,v_w} = f_{k_i,l_j} \circ g_{p_r,q_s}.$$

(c) multiplication-$(\circ, *)$

$$A \odot_{(\circ,*)} B = [T^*, V^*, \langle h_{t_u,v_w}\}],$$

where

$$T^* = (K \cup (P - L))^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cup (P - L)\},$$

$$V^* = (Q \cup (L - P))^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in Q \cup (L - P)\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u = k_i \in K \\ \\ \alpha_r^p, & \text{if } t_u = p_r \in P - L \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - P \\ \\ \beta_s^q, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \underset{l_j = p_r \in L \cap P}{\circ} (f_{k_i,l_j} * g_{p_r,q_s})), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ false, & \text{otherwise} \end{cases}$$

(d) structural subtraction $A \ominus B = [T^*, V^*, \{h_{t_u,v_w}\}]$, where

$$T^* = (K - P)^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K - P\},$$

$$V^* = (L - Q)^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L - Q\},$$

for the set–theoretic subtraction operation and

$$\alpha_u^t = \alpha_i^k, \text{ for } t_u = k_i \in K - P,$$

$$\beta_w^v = \beta_j^l, \text{ for } v_w = l_j \in L - Q$$

and

$$h_{t_u,v_w} = f_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

(e) negation of an EIFIM $\neg A = [T^*, V^*, \{\neg f_{k_i,l_j}\}]$, where $\neg$ is one of the negations, described in [5,8].

(f) termwise subtraction-$(\circ)$

$$A -_{(\circ)} B = A \oplus_{(\circ)} \neg B,$$

where operation negation $\neg$ is one of the negations described in [5,8].

3.3 Relations over an IMEP

Let $\kappa(P)$ be the number of the different values that the variable of predicate P can obtain.

Let the two IMEPs $A = [K, L, \{f_{k,l}\}]$ and $B = [P, Q, \{g_{p,q}\}]$ be given. In this subsection, for brevity, we write K and L instead of K^* and K^*. We introduce the following definitions where $\subset$ and $\subseteq$ denote the relations "*strong inclusion*" and "*weak inclusion*".

Definition 1.a: The strict relation "inclusion about dimension", when the IMEP-elements of both matrices are elements exactly of $\mathcal{P}_x^1$, is

$$A \subset_d B \text{ iff } (((K \subset P) \;\&\; (L \subset Q)) \;\vee\; ((K \subseteq P) \;\&\; (L \subset Q))$$
$$\vee\; ((K \subset P) \;\&\; (L \subseteq Q))) \;\&\; (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})(V(f_{k,l}(a)) = V(g_{k,l}(a))).$$

Definition 1.b: The strict relation "inclusion about dimension", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subset_d B \text{ iff } (((K \subset P) \ \& \ (L \subset Q)) \ \vee \ ((K \subseteq P) \ \& \ (L \subset Q))$$

$$\vee \ ((K \subset P) \ \& \ (L \subseteq Q))) \ \& \ (\forall k \in K)(\forall l \in L)(\kappa(f_{k,l}) = \kappa(g_{k,l}))$$

$$\& \ \ (\forall a_1, ..., a_{\kappa(f_{k,l})} \in \mathcal{X})(V(f_{k,l}(a_1, ..., a_{\kappa(f_{k,l})})) = V(g_{k,l}(a_1, ..., a_{\kappa(f_{k,l})}))).$$

Definition 2.a: The non-strict relation "inclusion about dimension", when the IMEP-elements of both matrices are elements of $\mathcal{P}_x^1$, is

$$A \subseteq_d B \text{ iff } (K \subseteq P) \ \& \ (L \subseteq Q) \ \& \ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(V(f_{k,l}(a)) = V(g_{k,l}(a))).$$

Definition 2.b: The non-strict relation "inclusion about dimension", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subseteq_d B \text{ iff } (K \subseteq P) \ \& \ (L \subseteq Q) \ \& \ (\forall k \in K)(\forall l \in L)(\kappa(f_{k,l}) = \kappa(g_{k,l}))$$

$$\& \ \ (\forall a_1, ..., a_{\kappa(f_{k,l})} \in \mathcal{X})(V(f_{k,l}(a_1, ..., a_{\kappa(f_{k,l})})) = V(g_{k,l}(a_1, ..., a_{\kappa(f_{k,l})}))).$$

Definition 3.a: The strict relation "inclusion about value", when the IMEP-elements of both matrices are elements of $\mathcal{P}_x^1$, is

$$A \subset_v B \text{ iff } (K = P) \ \& \ (L = Q) \ \& \ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(V(f_{k,l}(a)) < V(g_{k,l}(a))).$$

Definition 3.b: The strict relation "inclusion about value", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subset_v B \text{ iff } (K = P) \ \& \ (L = Q) \ \& \ (\forall k \in K)(\forall l \in L)(\kappa(f_{k,l}) = \kappa(g_{k,l}))$$

$$\& \ \ (\forall a_1, ..., a_{\kappa(f_{k,l})} \in \mathcal{X})(V(f_{k,l}(a_1, ..., a_{\kappa(f_{k,l})})) < V(g_{k,l}(a_1, ..., a_{\kappa(f_{k,l})}))).$$

Definition 4.a: The non-strict relation "inclusion about value", when the IMEP-elements of both matrices are elements of $\mathcal{P}_x^1$, is

$$A \subseteq_v B \text{ iff } (K = P) \ \& \ (L = Q) \ \& \ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(V(f_{k,l}(a)) \leq V(g_{k,l}(a))).$$

Definition 4.b: The non-strict relation "inclusion about value", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subseteq_v B \text{ iff } (K = P) \ \& \ (L = Q) \ \& \ (\forall k \in K)(\forall l \in L)(\kappa(f_{k,l}) = \kappa(g_{k,l}))$$

$$\& \ \ (\forall a_1, ..., a_{\kappa(f_{k,l})} \in \mathcal{X})(V(f_{k,l}(a_1, ..., a_{\kappa(f_{k,l})})) \leq V(g_{k,l}(a_1, ..., a_{\kappa(f_{k,l})}))).$$

Definition 5.a: The strict relation "inclusion", when the IMEP-elements of both matrices are elements of $\mathcal{P}_x^1$, is

$$A \subset B \text{ iff } (((K \subset P) \,\&\, (L \subset Q)) \ \vee \ ((K \subseteq P) \,\&\, (L \subset Q))$$
$$\vee\ ((K \subset P) \,\&\, (L \subseteq Q))) \,\&\, (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})(V(f_{k,l}(a)) < V(g_{k,l}(a))).$$

Definition 5.b: The strict relation "inclusion", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subset B \text{ iff } (((K \subset P) \,\&\, (L \subset Q)) \ \vee \ ((K \subseteq P) \,\&\, (L \subset Q))$$
$$\vee\ ((K \subset P) \,\&\, (L \subseteq Q))) \,\&\, (\forall k \in K)(\forall l \in L)(\kappa(f_{k,l}) = \kappa(g_{k,l}))$$
$$\&\ \ (\forall a_1, ..., a_{\kappa(f_{k,l})} \in \mathcal{X})(V(f_{k,l}(a_1, ..., a_{\kappa(f_{k,l})})) < V(g_{k,l}(a_1, ..., a_{\kappa(f_{k,l})}))).$$

Definition 6.a: The non-strict relation "inclusion", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subseteq B \text{ iff } (K \subseteq P) \,\&\, (L \subseteq Q) \,\&\, (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$
$$(V(f_{k,l}(a)) \leq V(g_{k,l}(a))).$$

Definition 6.b: The non-strict relation "inclusion", when the IMEP-elements of both matrices are elements not only of $\mathcal{P}_x^1$, is

$$A \subseteq B \text{ iff } (K \subseteq P) \,\&\, (L \subseteq Q) \,\&\, (\forall k \in K)(\forall l \in L)(\kappa(f_{k,l}) = \kappa(g_{k,l}))$$
$$\&\ \ (\forall a_1, ..., a_{\kappa(f_{k,l})} \in \mathcal{X})(V(f_{k,l}(a_1, ..., a_{\kappa(f_{k,l})})) < V(g_{k,l}(a_1, ..., a_{\kappa(f_{k,l})}))).$$

It can be directly seen that for every two IMs A and B,

(a) if $A \subset_d B$, then $A \subseteq_d B$;
(b) if $A \subset_v B$, then $A \subseteq_v B$;
(c) if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
(d) if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

Four new operations are introduced, that are analogous to the operations over IMs. For simplicity, these operations are defined over IMEP, but their definitions can be transformed for the EIFIMEP case, too.

It is important to note that the hierarhical operators over IM cannot be introduced over IMEP.

3.4 Operations "reduction" over an IMEP

First, we introduce operations $(k, \perp)$- and $(\perp, l)$-reduction of a given IM $A = [K, L, \{f_{k_i,l_j}\}]$:

$$A_{(k,\perp)} = [K - \{k\}, L, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \ \text{ for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L,$$

and

$$A_{(\perp,l)} = [K, L - \{l\}, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \text{ for } t_u = k_i \in K \text{ and } v_w = l_j \in L - \{l\}.$$

Second, we define

$$A_{(k,l)} = (A_{(k,\perp)})_{(\perp,l)} = (A_{(\perp,l)})_{(k,\perp)},$$

i.e.,

$$A_{(k,l)} = [K - \{k\}, L - \{l\}, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \text{ for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L - \{l\}.$$

Theorem 1. For every IMEP A and for every $k_1, k_2 \in K$, $l_1, l_2 \in L$,

$$(A_{(k_1,l_1)})_{(k_2,l_2)} = (A_{(k_2,l_2)})_{(k_1,l_1)}.$$

Third, let $P = \{k_1, k_2, ..., k_s\} \subseteq K$ and $Q = \{q_1, q_2, ..., q_t\} \subseteq L$. Finally, we define the following three operations:

$$A_{(P,l)} = (...((A_{(k_1,l)})_{(k_2,l)})...)_{(k_s,l)},$$

$$A_{(k,Q)} = (...((A_{(k,l_1)})_{(k,l_2)})...)_{(k,l_t)},$$

$$A_{(P,Q)} = (...((A_{(p_1,Q)})_{(p_2,Q)})...)_{(p_s,Q)} = (...((A_{(P,q_1)})_{(P,q_2)})...)_{(P,q_t)}.$$

Obviously,

$$A_{(K,L)} = I_{\emptyset},$$

$$A_{(\emptyset,\emptyset)} = A.$$

Theorem 2. For every two IMs $A = [K, L, \{f_{k_i,l_j}\}], B = [P, Q, \{g_{p_r,q_s}\}]$:

$$A \subseteq_d B \text{ iff } A = B_{(P-K,Q-L)}.$$

Proof. Let $A \subseteq_d B$. Therefore, $K \subseteq P$ and $L \subseteq Q$ and for every $k \in K, l \in L$, for every $a \in X$: $f_{k,l}(a) = g_{k,l}(a)$. From the definition,

$$B_{(P-K,Q-L)} = (...((B_{(p_1,q_1)})_{(p_1,q_2)})...)_{(p_r,q_s)},$$

where $p_1, p_2, ..., p_r \in P - K$, i.e., $p_1, p_2, ..., p_r \in P$, and $p_1, p_2, ..., p_r \notin K$, and $q_1, q_2, ..., q_s \in Q - L$, i.e., $q_1, q_2, ..., q_s \in Q$, and $q_1, q_2, ..., q_s \notin L$. Therefore,

$$B_{(P-K,Q-L)} = [P - (P - K), Q - (Q - L), \{g_{k,l}\}] = [K, L, \{g_{k,l}\}] = [K, L, \{f_{k,l}\}] = A,$$

because by definition the elements of the two IMs, which are indexed by equal symbols, coincide.

For the opposite direction we obtain that if $A = B_{(P-K,Q-L)}$, then

$$A = B_{(P-K,Q-L)} \subseteq_d B_{\emptyset,\emptyset} = B.$$

3.5 Operation "projection" over an IM

Let $M \subseteq K$ and $N \subseteq L$. Then,

$$pr_{M,N}A = [M, N, \{g_{k_i,l_j}\}],$$

where

$$(\forall k_i \in M)(\forall l_j \in N)(g_{k_i,l_j} = f_{k_i,l_j}).$$

Theorem 3. For every IMEP A and sets $M_1 \subseteq M_2 \subseteq K$ and $N_1 \subseteq N_2 \subseteq L$ the equality

$$pr_{M_1,N_1}pr_{M_2,N_2}A = pr_{M_1,N_1}A$$

holds.

3.6 "Inflating operation" over an IM

We can define "inflating operation" that is defined for index sets $K \subset P \subset \mathcal{I}$ and $L \subset Q \subset \mathcal{I}$ by

$${}^{(P,Q)}A =^{(P,Q)} [K, L, \{a_{k_i,l_j}\}] = [P, Q, \{b_{p_r,q_s}\}],$$

where

$$b_{p_r,q_s} = \begin{cases} a_{k_i,l_j}, & \text{if } p_r = k_i \in K \text{ and } q_s = l_j \in L \\ \perp, & \text{otherwise} \end{cases}$$

3.7 Operation "substitution" over an IM

Let IM $A = [K, L, \{f_{k,l}\}]$ be given.

First, local substitution over the IM is defined for the couples of indices (p, k) and/or (q, l), respectively, by

$$\left[\frac{p}{k}\right] A = [(K - \{k\}) \cup \{p\}, L, \{f_{k,l}\}],$$

$$\left[\frac{q}{l}\right] A = [K, (L - \{l\}) \cup \{q\}, \{f_{k,l}\}],$$

Second,

$$\left[\frac{p}{k}\,\frac{q}{l}\right] A = \left[\frac{p}{k}\right]\left[\frac{q}{l}\right] A,$$

i.e.

$$\left[\frac{p}{k}\,\frac{q}{l}\right] A = [(K - \{k\}) \cup \{p\}, (L - \{l\}) \cup \{q\}, \{f_{k,l}\}].$$

Obviously, for the above indices k, l, p, q:

$$\left[\frac{k}{p}\right]\left(\left[\frac{p}{k}\right] A\right) = \left[\frac{l}{q}\right]\left(\left[\frac{q}{l}\right] A\right) = \left[\frac{k}{p}\,\frac{l}{q}\right]\left(\left[\frac{p}{k}\,\frac{q}{l}\right] A\right) = A,$$

Let the sets of indices $P = \{p_1, p_2, ..., p_m\}$, $Q = \{q_1, q_2, ..., q_n\}$ be given.

Third, for them we sequentially define:

$$\left[\frac{P}{K}\right] A = \left[\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_n}{k_n}\right] A,$$

$$\left[\frac{Q}{L}\right] A = (\left[\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}\right] A),$$

$$\left[\frac{K}{P}\frac{Q}{L}\right] A = \left[\frac{P}{K}\right]\left[\frac{Q}{L}\right] A,$$

i.e.,

$$\left[\frac{P}{K}\frac{Q}{L}\right] A = \left[\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_m}{k_m}\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}\right] A = [P, Q, \{f_{k,l}\}]$$

Obviously, for the sets K, L, P, Q:

$$\left[\frac{K}{P}\right](\left[\frac{P}{K}\right] A) = \left[\frac{L}{Q}\right](\left[\frac{Q}{L}\right] A) = \left[\frac{K}{P}\frac{L}{Q}\right](\left[\frac{P}{K}\frac{Q}{L}\right] A) = A.$$

Theorem 4. For every four sets of indices P_1, P_2, Q_1, Q_2

$$\left[\frac{P_2}{P_1}\frac{Q_2}{Q_1}\right]\left[\frac{P_1}{K}\frac{Q_1}{L}\right] A = \left[\frac{P_2}{K}\frac{Q_2}{L}\right] A.$$

3.8 Operations over IMEPs and IMs

Let the IM $A = [K, L, \{a_{k_i,l_j}\}]$, where a_{k_i,l_j} are values of the variables and IMEP $B = [P, Q, \{f_{p_r,q_s}\}]$ be given. Then

(a) $B \bigoplus A = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} V(f_{p_r,q_s}(a_{k_i,l_j})), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \perp, & \text{otherwise} \end{cases}.$$

(b) $B \bigotimes A = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = V(f_{p_r,q_s}(a_{k_i,l_j})),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Let the IM $A = [K, L, \{\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle\}]$, for the natural number $n \geq 2$, where $a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}$ are values of the variables and IMEP $B = [P, Q, \{f_{p_r,q_s}\}]$, where $f_{p_r,q_s} \in \mathcal{P}$. Then

(c) $B \bigoplus A = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} V(f_{p_r,q_s}(\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle)), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \perp, & \text{otherwise} \end{cases}.$$

(d) $B \bigotimes A = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = V(f_{p_r,q_s}(\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle)),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Therefore, in a result of these operations, we obtain a new IM with values of the predicates (from IM B) over the values of their variables (from IM A).

3.9 Aggregation Operations over IMEPs and EIFIMEPs

Let the EIFIMEP A be given. Let $(\circ, *) \in \{(\max, \min), (\min, \max), ...\}$ and $\bullet \in \{\vee_1, \wedge_1, ...\}$. Then

$((\circ, *); \bullet)$**-row-aggregation** operator is defined by

$$\rho(A, k_0)_{((\circ,*);\bullet)} = \begin{array}{c|ccc} & l_1, \langle \alpha^l_1, \beta^l_1\rangle & \ldots & l_n, \langle \alpha^l_n, \beta^l_n\rangle \\ \hline k_0, \langle \underset{i=1}{\overset{m}{\circ}} \alpha^k_i, \underset{i=1}{\overset{m}{*}} \beta^k_i\rangle & \underset{i=1}{\overset{m}{\bullet}} f_{k_i,l_1} & \ldots & \underset{i=1}{\overset{m}{\bullet}} f_{k_i,l_n} \end{array}$$

$((\circ, *); \bullet)$**-column-aggregation**

$$\sigma_(A, l_0)((\circ, *); \bullet) = \begin{array}{c|c} & l_0, \langle \underset{i=1}{\overset{m}{\circ}} \alpha^l_j, \underset{i=1}{\overset{m}{*}} \beta^l_j\rangle \\ \hline k_1, \langle \alpha^k_1, \beta^k_1\rangle & \underset{j=1}{\overset{m}{\bullet}} f_{k_1,l_j} \\ \vdots & \vdots \\ k_i, \langle \alpha^k_i, \beta^k_i\rangle & \underset{j=1}{\overset{m}{\bullet}} f_{k_i,l_j} \\ \vdots & \vdots \\ k_m, \langle \alpha^k_m, \beta^k_m\rangle & \underset{j=1}{\overset{m}{\bullet}} f_{k_m,l_j} \end{array},$$

Obviously, these operators oved IMEP give the same IM, but the indices do not have intuitionistic fuzzy evaluations:

$(\perp; \bullet)$**-row-aggregation** operator is defined by

$$\rho(A, k_0)_{(\perp;\bullet)} = \begin{array}{c|ccc} & l_1 & \ldots & l_n \\ \hline k_0 & \underset{i=1}{\overset{m}{\bullet}} f_{k_i,l_1} & \ldots & \underset{i=1}{\overset{m}{\bullet}} f_{k_i,l_n} \end{array}$$

$(\perp; \bullet)$-column-aggregation

$$\sigma(A, l_0)_{(\perp;\bullet)} = \begin{array}{c|c} & l_0 \\ \hline k_1 & \underset{j=1}{\overset{m}{\bullet}} f_{k_1,l_j} \\ \vdots & \vdots \\ k_i & \underset{j=1}{\overset{m}{\bullet}} f_{k_i,l_j} \\ \vdots & \vdots \\ k_m & \underset{j=1}{\overset{m}{\bullet}} f_{k_m,l_j} \end{array},$$

3.10 Intuitionistic Fuzzy Quantifiers and IMEP

First, we give the simplest IM-interpretation of these constructions. Let us have the IM

$$x = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline x & a_1 & a_2 & \dots & a_n \end{array}$$

that corresponds to the values that x can obtain. For brevity, here we use x as the name of the IM, as well as the name of the index for the row. Then

$$P(x) \equiv P\left(\begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline x & a_1 & a_2 & \dots & a_n \end{array} \right) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline P(x) & P(a_1) & P(a_2) & \dots & P(a_n) \end{array}$$

Of course, we can use more complex IM-forms, as IFIM or EIFIM, too, but we prefer the simplest form as more illustrative.

Below, let us denote the lack of a parameter by the symbol $\perp$.

Using the $(\circ, *)$-row-aggregation operation, we obtain

$$\rho(P(x), l_0)_{(\perp,\wedge)} = \begin{array}{c|c} & l_0 \\ \hline P(x) & P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n) \end{array} = \begin{array}{c|c} & l_0 \\ \hline P(x) & V(\forall x P(x)) \end{array}.$$

By analogy, we obtain

$$\rho(P(x), l_0)_{(\perp,\vee)} = \begin{array}{c|c} & l_0 \\ \hline P(x) & P(a_1) \vee P(a_2) \vee \dots \vee P(a_n) \end{array} = \begin{array}{c|c} & l_0 \\ \hline P(x) & V(\exists x P(x)) \end{array}.$$

If for the above example, we like to work with the intuitionistic fuzzy evaluations of values of x, we must use IFIM. In this case, we have the IMs in the forms

$$P(x) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline P(x) & \langle \mu(P(a_1)), \nu(P(a_1)) \rangle & \langle \mu(P(a_2)), \nu(P(a_2)) \rangle & \dots & \langle \mu(P(a_n)), \nu(P(a_n)) \rangle \end{array},$$

$$\rho(P(x), l_0)_{(\bot,\wedge)} = \begin{array}{c|c} & l_0 \\ \hline P(x) & \langle \min_{1\le i\le n} \mu(P(a_i)), \max_{1\le i\le n} \nu(P(a_i))\rangle \end{array},$$

$$\rho(P(x), l_0)_{(\bot,\vee)} = \begin{array}{c|c} & l_0 \\ \hline P(x) & \langle \max_{1\le i\le n} \mu(P(a_i)), \min_{1\le i\le n} \nu(P(a_i))\rangle \end{array}.$$

In the most complex case, the indices also have intuitionistic fuzzy evaluations – e.g., for the above case, we will have:

$$P(x) = \begin{array}{c|cccc} & l_1, \langle\alpha_1^l, \beta_1^l\rangle & l_2, \langle\alpha_2^l, \beta_2^l\rangle & \dots & l_n, \langle\alpha_n^l, \beta_n^l\rangle \\ \hline P(x) & \langle\mu(P(a_1)), \nu(P(a_1))\rangle & \langle\mu(P(a_2)), \nu(P(a_2))\rangle & ... & \langle\mu(P(a_n)), \nu(P(a_n))\rangle \end{array}.$$

For example, if we have n experts $E_1, ..., E_n$ who evaluated some object or process O by evaluations $\langle\mu_{O,E_1}, \nu_{O,E_1}\rangle$, ..., $\langle\mu_{O,E_n}, \nu_{O,E_n}\rangle$, we can construct an EIFIM that has the above form:

$$P(x) = \begin{array}{c|cccc} & E_1, \langle\alpha_1^E, \beta_1^E\rangle & E_2, \langle\alpha_2^E, \beta_2^E\rangle & \dots & E_n, \langle\alpha_n^E, \beta_n^E\rangle \\ \hline P(x) & \langle\mu_{O,E_1}, \nu_{O,E_1}\rangle & \langle\mu_{O,E_2}, \nu_{O,E_2}\rangle & \dots & \langle\mu_{O,E_n}, \nu_{O,E_n}\rangle \end{array},$$

where the IFP $\langle\alpha_i^E, \beta_i^E\rangle$ can be interpreted, e.g., as the score of the i-th expert E_i.

4 Conclusion

In [8], two-, three- and more dimensional IMs are discussed. Here, we discussed only the two-dimensional case. In a next research, we will discuss the three-dimensional case, where essentially more complex situation arises. For example, the operation multiplication for the case of two-dimensional IM is one, while for three-dimensional IM there are six operations of multiplication. Other forms of quantifiers will be discussed as well.

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On Two Formulations of the Representation Theorem for an IF-state

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Abstract. We know two representation theorems for IF-state [2,6]. They represent IF-state by some classical Kolmogorovian probabilities. Of course, they must be equivalent, but the formulations correspond with the constructions of the probabilities.

Keywords: IF-set · IF-state · Representation theorem · IF-probability

1 Basic Notions

Recall the definition of IF-sets and some operations with them.

Definition 1. *Let Ω be a nonempty set. An IF-sets $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

If $\mathbf{A}, \mathbf{B}$ are IF-sets, then we define

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0)),$$
$$\mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1))$$

and partial ordering is given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$$

The following terminology is probably inspired by quantum theory. Therefore we speak about states instead of probabilities.

Definition 2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of the subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

Definition 3. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;
(ii) *if* $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, *then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = m(\mathbf{A}) + m(\mathbf{B})$;
(iii) *if* $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

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2 Butnariu - Klement Formulation

Now we shall present the first solution of a problem of Radko Mesiar, i.e. find all IF-states on Ω.

Theorem 1. *To each IF-state* $\mathbf{m} : \mathcal{F} \to [0,1]$ *there exists exactly one probability measure* $P : \mathcal{S} \to [0,1]$ *and exactly one* $\alpha \in [0,1]$ *such that*

$$\mathbf{m}(\mathbf{A}) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

Proof. Now we use method, which was based in the proof the Butnariu-Klement theorem (see [1]). Let denote

$$P(A) = \mathbf{m}\big((\chi_A, 1-\chi_A)\big).$$

It can be proved that the mapping $P : \mathcal{S} \to [0,1]$ is the probability measure (see [1,7]). If $(\Omega, \mathcal{S}, P)$ is a probability space and $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, then we put

$$\psi((\mu_A, \nu_A)) = \Big(\int_\Omega \mu_A dP, \int_\Omega \nu_A dP\Big).$$

Then evidently $\psi : \mathcal{F} \to \triangle$, where

$$\triangle = \{(x,y) \in R \times R; x \geq 0, y \geq 0, x + y \leq 1\}$$

(see [5]).

Using the method of proof in [6], then the IF-state $\mathbf{m}$ can be found in the form

$$\mathbf{m}(\mathbf{A}) = f\bigg(\Big(\int_\Omega \mu_A dP, \int_\Omega \nu_A dP\Big)\bigg) = f \circ \psi(\mathbf{A}).$$

There $f : \triangle \to [0,1]$ is a mapping such that

$$f((x,y)) = (1-\alpha)x + \alpha(1-y).$$

Therefore

$$\mathbf{m}(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big)$$

for any $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

If β, Q is another pair, such that

$$\mathbf{m}(A) = (1-\beta)\int_\Omega \mu_A dQ + \beta\Big(1 - \int_\Omega \nu_A dQ\Big),$$

then

$$(1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big) = \mathbf{m}(A) = (1-\beta)\int_\Omega \mu_A dQ + \beta\Big(1 - \int_\Omega \nu_A dQ\Big).$$

Put $\mu_A = \nu_A = 0_\Omega$. Then $\int_\Omega 0_\Omega dP = 0$, $\int_\Omega 0_\Omega dQ = 0$, hence $\alpha = \beta$. Therefore

$$(1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big) = (1-\alpha)\int_\Omega \mu_A dQ + \alpha\Big(1 - \int_\Omega \nu_A dQ\Big).$$

If we put $\mu_A = \chi_A$, $\nu_A = 0_\Omega$, then we obtain

$$(1-\alpha)\int_\Omega \chi_A dP + \alpha = (1-\alpha)\int_\Omega \chi_A dQ + \alpha,$$
$$(1-\alpha)\int_\Omega \chi_A dP = (1-\alpha)\int_\Omega \chi_A dQ.$$

If $\alpha \neq 1$, then we obtain

$$P(A) = \int_\Omega \chi_A dP = \int_\Omega \chi_A dQ = Q(A)$$

for each $A \in \mathcal{S}$. Hence $P = Q$.

Let $\alpha = 1$, then

$$1 - \int_\Omega \nu_A dP = 1 - \int_\Omega \nu_A dQ.$$

Put $\nu_A = \chi_A$, then

$$1 - \int_\Omega \chi_A dP = 1 - \int_\Omega \chi_A dQ,$$
$$1 - P(A) = 1 - Q(A)$$

for each $A \in \mathcal{S}$. Hence $P = Q$. □

3 Ciungu - Riečan Formulation

In [2] Ciungu and Riečan presented a proof of representation theorem without the Butnariu-Klement theorem.

Theorem 2. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exist some probability measures $P, Q : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$\mathbf{m}(\mathbf{A}) = \int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega (\mu_A + \nu_A) dQ\Big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

Theorem 3. *Theorem 2 implies Theorem 1.*

Proof. Put $Q = P$, then

$$\begin{aligned}\mathbf{m}(\mathbf{A}) &= \int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega (\mu_A + \nu_A) dP\Big)\\ &= \int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \mu_A dP - \int_\Omega \nu_A dP\Big)\\ &= (1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big),\end{aligned}$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$. □

Now we show that also Theorem 1 implies Theorem 2.

Theorem 4. *Let $P, Q, R : \mathcal{S} \to [0,1]$ be some probability measures and $\alpha, \beta \in [0,1]$ such that*

$$(1-\beta)\int_\Omega \mu_A dR + \beta\Big(1 - \int_\Omega \nu_A dR\Big) = \int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega (\mu_A + \nu_A) dQ\Big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$. Then $\alpha = \beta$ and $P = Q = R$.

Proof. First put $\mu_A = \nu_A = 0_\Omega$. Then $\int_\Omega 0_\Omega dR = \int_\Omega 0_\Omega dP = \int_\Omega 0_\Omega dQ = 0$, hence $\beta = \alpha$. Therefore

$$(1-\alpha)\int_\Omega \mu_A dR + \alpha\Big(1 - \int_\Omega \nu_A dR\Big) = \int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega (\mu_A + \nu_A) dQ\Big).$$

If we put $\mu_A = 0_\Omega$, $\nu_A = \chi_A$, then we obtain

$$\alpha\Big(1 - \int_\Omega \chi_A dR\Big) = \alpha\Big(1 - \int_\Omega \chi_A dQ\Big),$$
$$\alpha(1 - R(A)) = \alpha(1 - Q(A)),$$

for each $A \in \mathcal{S}$. Therefore, if $\alpha \neq 0$, then we obtain

$$R(A) = Q(A)$$

for each $A \in \mathcal{S}$. Hence $R = Q$.

Let $\alpha = 0$, then

$$\int_\Omega \mu_A dR = \int_\Omega \mu_A dP.$$

Put $\mu_A = \chi_A$, then

$$\int_\Omega \chi_A dR = \int_\Omega \chi_A dP,$$
$$R(A) = P(A)$$

for each $A \in \mathcal{S}$. Hence $R = P$. Then $Q = R = P$. □

4 Grzegorzewski - Mrówka Formulation

Consider a probability space $(\Omega, \mathcal{S}, P)$. Then in [3] the IF-probability $\mathbf{P}(\mathbf{A})$ of an IF-event $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$ has been defined as a compact interval by the equality

$$\mathbf{P}(\mathbf{A}) = \left[\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP \right].$$

Let $\mathcal{J}$ be the family of all compact intervals. Then the mapping $\mathbf{P} : \mathcal{F} \to \mathcal{J}$ can be defined axiomatically similarly as in [4].

Definition 4. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{P} : \mathcal{F} \to \mathcal{J}$ is called an IF-probability, if the following conditions hold:*

(i) $\mathbf{P}((1_\Omega, 0_\Omega)) = [1,1]$, $\mathbf{P}((0_\Omega, 1_\Omega)) = [0,0]$;
(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, *then* $\mathbf{P}(\mathbf{A} \oplus \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B})$;
(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, *then* $\mathbf{P}(\mathbf{A}_n) \nearrow \mathbf{P}(\mathbf{A})$.
(Recall that $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ *means that* $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$, *but* $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$ *means* $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$.*)*

Very well known is the following assertion.

Theorem 5. *Let* $\mathbf{P} : \mathcal{F} \to \mathcal{J}$ *be an IF-probability,* $\mathbf{P}(\mathbf{A}) = [\mathbf{P}^\flat(\mathbf{A}), \mathbf{P}^\sharp(\mathbf{A})]$, $\mathbf{P}^\flat(\mathbf{A}) \le \mathbf{P}^\sharp(\mathbf{A})$ *for each* $\mathbf{A} \in \mathcal{F}$. *Then* $\mathbf{P}^\flat : \mathcal{F} \to [0,1]$, $\mathbf{P}^\sharp : \mathcal{F} \to [0,1]$ *are IF-states.*

Of course, we know the general form of IF-states. Therefore the following theorem is evident.

Theorem 6. *Let* $\mathbf{P} : \mathcal{F} \to \mathcal{J}$ *be an IF-probability. Then there exists some Kolmogorov probabilities* $P^\flat : \mathcal{S} \to [0,1]$, $P^\sharp : \mathcal{S} \to [0,1]$ *and* $\alpha^\flat, \alpha^\sharp \in [0,1]$ *such that*

$$\mathbf{P}^\flat(\mathbf{A}) = (1 - \alpha^\flat) \int_\Omega \mu_A dP^\flat + \alpha^\flat \left(1 - \int_\Omega \nu_A dP^\flat \right),$$

$$\mathbf{P}^\sharp(\mathbf{A}) = (1 - \alpha^\sharp) \int_\Omega \mu_A dP^\sharp + \alpha^\sharp \left(1 - \int_\Omega \nu_A dP^\sharp \right),$$

where

$$\alpha^\flat \le \alpha^\sharp$$

and

$$\alpha^\flat (1 - P^\flat(A)) \le \alpha^\sharp (1 - P^\sharp(A)),$$

for each $A \in \mathcal{S}$.

Proof. Let $\mathbf{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $\mathbf{P}(\mathbf{A}) = [\mathbf{P}^\flat(\mathbf{A}), \mathbf{P}^\sharp(\mathbf{A})]$, $\mathbf{P}^\flat(\mathbf{A}) \le \mathbf{P}^\sharp(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{F}$. Then $\mathbf{P}^\flat$, $\mathbf{P}^\sharp$ can be expressed in the form

$$\mathbf{P}^\flat(\mathbf{A}) = (1-\alpha^\flat)\int_\Omega \mu_A dP^\flat + \alpha^\flat\left(1 - \int_\Omega \nu_A dP^\flat\right),$$
$$\mathbf{P}^\sharp(\mathbf{A}) = (1-\alpha^\sharp)\int_\Omega \mu_A dP^\sharp + \alpha^\sharp\left(1 - \int_\Omega \nu_A dP^\sharp\right).$$

Hence

$$(1-\alpha^\flat)\int_\Omega \mu_A dP^\flat + \alpha^\flat\left(1 - \int_\Omega \nu_A dP^\flat\right) \le (1-\alpha^\sharp)\int_\Omega \mu_A dP^\sharp + \alpha^\sharp\left(1 - \int_\Omega \nu_A dP^\sharp\right).$$

Put $\mu_A = \nu_A = 0_\Omega$. Then $\int_\Omega 0_\Omega dP^\flat = \int_\Omega 0_\Omega dP^\sharp = 0$, hence $\alpha^\flat \le \alpha^\sharp$. Moreover $\mu_A = 0_\Omega$, $\nu_A = \chi_A$ implies

$$\alpha^\flat\left(1 - \int_\Omega \chi_A dP^\flat\right) \le \alpha^\sharp\left(1 - \int_\Omega \chi_A dP^\sharp\right),$$
$$\alpha^\flat(1 - P^\flat(A)) \le \alpha^\sharp(1 - P^\sharp(A)),$$

for each $A \in \mathcal{S}$. □

5 Conclusion

Representation theorems belong to the best results in mathematical theories and their applications. In the paper we have shown that two formulations of IF-state representation are equivalent. They can be used for obtaining further results for intuitionistic fuzzy sets probability theory.

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Optimal Representation (ORD) Method of Intuitionistic Fuzzy Defuzzification

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Abstract. The paper presents a novel method of intuitionistic fuzzy defuzzification that was called Optimal Representation Defuzzification or shortly ORD-method. In the method, in the first step membership and non-membership functions of inputs and of the system output are transformed in interval Type 2 membership functions. Then the inference process is realized to determine activation degrees of the output fuzzy sets (rules' conclusions). Next, for all activated MFs one fuzzy set optimally representing them is determined. And, in the end, one crisp value optimally representing this set is found as defuzzification result. In the ORD method each of rules is treated as a local system expert and activation degree of the rule as coefficient of its competence in the inference process. The approach used in the ORD-method is considerably different from the approach of Mamdani inference. To facilitate the ORD-method understanding in the paper it was explained on the example of intuitionistic fuzzy controller of the fan speed of a room heater.

Keywords: Intuitionistic fuzzy defuzzification · Intuitionistic fuzzy sets · Fuzzy inference systems · Defuzzification

1 Introduction

Notion of intuitionistic fuzzy sets (IF-set, IFS) has been introduced by K.T. Atanassov in 1983 in [5]. An IF-set A in the domain set X is a mathematical object defined as in (1).

$$A = \{ <x, \mu_A(x), \nu_A(x)> | x \in X\} \tag{1}$$

X is a non-empty fixed set and $A \in X$. The function μ_A called membership function defines the degree of membership $\mu_A(x)$ of the element $x \in X$ to the set A and the function ν_A called the non-membership function defines the degree of non-membership $\nu_A(x)$ of the element $x \in X$. In the case of an IF-set the restriction (2) must be hold.

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \forall x \in X \tag{2}$$

Because of (2) the hesitation function $\pi_A(x)$ can be defined (3).

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$$\pi_A(x) = 1 - [\mu_A(x) + v_A(x)] \quad (3)$$

The hesitation function $\pi_A(x)$ expresses uncertainty of membership function and can also be called uncertainty function.

Systems of intuitionistic logic are more and more used to solve practical problems. Examples can be their applications to technical plants' control such as heater fans [1], washing machines [2], economic system as corporate bankruptcy prediction [8]. One of important processes realized by logical inference machines is defuzzification [1, 12] (Fig. 1.).

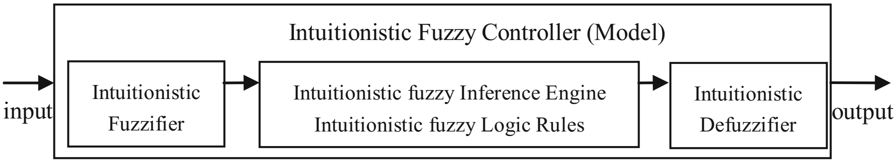

Fig. 1. Structure of the intuitionistic fuzzy controller or IF-model.

At present there exist defuzzification methods for IFS-inference systems of Mamdani type [4] and Takagi-Sugani type [1]. Author of [3] proposes to call defuzzification of IF-sets "crispification" and introduces 3 methods of it (intuitionistic version of Center of Area (COA), Mean of Maximum (MOM) and BADD defuzzification) based on the difference of membership and non-membership functions. This paper will present a method called "optimal representation defuzzification method (ORD-method)" that according authors' knowledge is new. The method consists of two general steps.

Step 1. Determine one fuzzy set that optimally represents activated (fired) fuzzy sets of rule conclusions.
Step 2. Determine one crisp value optimally representing the fuzzy set achieved in Step 1.

Both defuzzification steps will be presented on an example of heater fans' control taken from [1]. Authors of [1] have used in this paper a defuzzification method based on Takagi-Sugani formula. However, it seems that they made some error that negatively influenced the fuzzy inference and the result accuracy. Therefore in Chapter 2 apart from the defuzzification process, the full correctly realized inference process will be shown.

2 Optimal Representation Defuzzification Method in Application to the Intuitionistic Fuzzy Logic Control of Heater Fans

In this problem input variable x is a room temperature in Fahrenheit degrees with domain $x \in [43, 79]$ [°F]. On the basis of the room temperature x the fuzzy logic (FL) controller determines its output variable y (heater fan-speed) expressed in rpm (revolutions pro minute). Domain of the speed is $y \in [0, 100]$ [rpm]. The specific task described in the paper [1] is determining the fan speed (FS) corresponding to the room temperature $x = 70$ [°F]. To linguistic evaluation of the room temperature following linguistic values were used: *cold, cool, warm, hot*. Membership functions of these values are given by (4) and are visualized in Fig. 2.

$$\underline{\mu}_{cold}(x) = \begin{cases} \frac{55-x}{55-44}, & for\ x \in [43, 55] \\ 0, & else \end{cases} \quad \underline{\mu}_{cool}(x) = \begin{cases} \frac{67-x}{67-55}, & for\ x \in [55, 67] \\ 0, & else \end{cases}$$
$$\underline{\mu}_{warm}(x) = \begin{cases} \frac{x-55}{67-55}, & for\ x \in [55, 67] \\ \frac{79-x}{79-67}, & for\ x \in [67, 79] \\ 0, & else \end{cases} \quad \underline{\mu}_{hot}(x) = \begin{cases} \frac{x-67}{79-67}, & for\ x \in [67, 79] \\ 0, & else \end{cases} \tag{4}$$

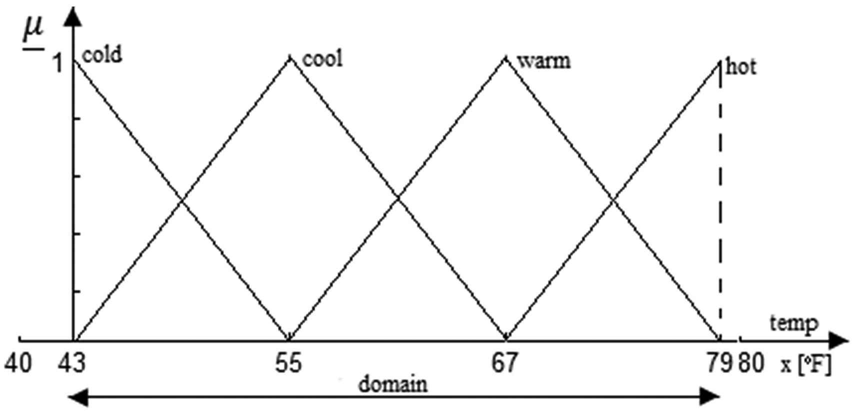

Fig. 2. Membership functions of fuzzy sets A_i ($i = 1 - 4$) of the room temperature.

The membership functions were underlined ($\underline{\mu}$) because they have sense of lower MFs in the later used MFs Type 2.

Non-membership functions of temperature are given by (5).

$$\nu_{cold}(x) = \begin{cases} \frac{x-43}{60-43}, & for\ x \in [43, 60] \\ 1, & else \end{cases} \quad \nu_{cool}(x) = \begin{cases} \frac{55-x}{55-43}, & for\ x \in [43, 55] \\ \frac{x-55}{72-55}, & for\ x \in [55, 72] \\ 1, & else \end{cases}$$
$$\nu_{warm}(x) = \begin{cases} \frac{67-x}{67-50}, & for\ x \in [50, 67] \\ \frac{x-67}{79-67}, & for\ x \in [67, 79] \\ 1, & else \end{cases} \quad \nu_{hot}(x) = \begin{cases} \frac{79-x}{79-62}, & for\ x \in [62, 79] \\ 1, & else \end{cases} \tag{5}$$

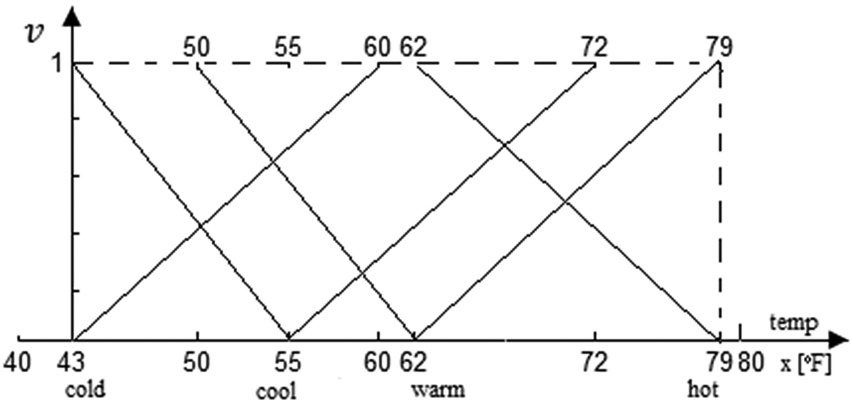

Fig. 3. Non-membership functions of temperature.

The non-membership functions (NMF-functions) are shown in Fig. 3.

Original MFs of the output y (fan speed) taken from [1] are given by (6) and are shown in Fig. 4.

$$\underline{\mu}_{zero}(y) = \begin{cases} \frac{25-y}{25-0}, & for\ y \in [0, 25] \\ 0, & else \end{cases} \quad \underline{\mu}_{low}(y) = \begin{cases} \frac{y-0}{25-0}, & for\ y \in [0, 25] \\ \frac{75-y}{75-25}, & for\ y \in [25, 75] \\ 0, & else \end{cases} \tag{6}$$

$$\underline{\mu}_{medium}(y) = \begin{cases} \frac{y-25}{75-25}, & for\ y \in [25, 75] \\ \frac{100-y}{100-75}, & for\ y \in [75, 100] \\ 0, & else \end{cases} \quad \underline{\mu}_{high}(y) = \begin{cases} \frac{y-75}{100-75}, & for\ y \in [75, 100] \\ 0, & else \end{cases}$$

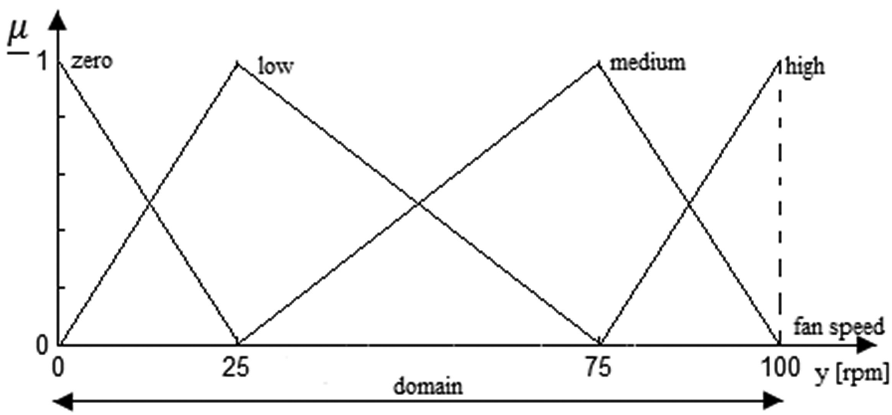

Fig. 4. Original membership functions [1] of the controller output y *fan speed* [rpm].

The original non-membership functions [1] of the fan speed are given by (7) and shown in Fig. 5.

$$v_{zero}(y) = \begin{cases} \frac{y-0}{30-0}, & for\ y \in [0,30] \\ 1, & else \end{cases} \quad v_{low}(y) = \begin{cases} \frac{25-y}{25-0}, & for\ y \in [0,25] \\ \frac{y-25}{80-25}, & for\ y \in [25,80] \\ 1, & else \end{cases} \tag{7}$$

$$v_{medium}(y) = \begin{cases} \frac{75-y}{75-20}, & for\ y \in [20,75] \\ \frac{y-75}{100-75}, & for\ y \in [75,100] \\ 1, & else \end{cases} \quad v_{high}(y) = \begin{cases} \frac{100-y}{100-70}, & for\ y \in [70,100] \\ 1, & else \end{cases}$$

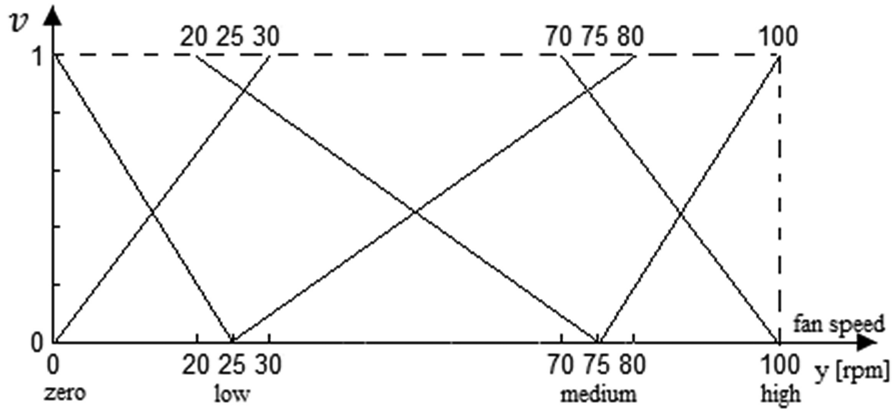

Fig. 5. Original non-membership functions of the fan speed without extended border functions taken from [1].

MFs and NMFs with non-extended border functions cannot be used in the case of certain defuzzification methods as e.g. center of gravity (COG) because as defuzzification result the minimal and maximal output value never can be achieved. In the case of the fan speed controller the speed of 0-rpm and 100 rpm never could be achieved. The maximal achievable speed could only be equal to the y_{COG} position of the COG of the border set *high* speed, 91.67 and not 100 rpm. To prevent such effect of the controller domain-narrowing the border fuzzy sets have to be symmetrically extended outside the real domain as shown in Figs. 6 and 7.

Rule base of the fan speed controller [1] consist of 4 rules (8).

$$\begin{aligned} &R1: \text{ IF } (temperature\ is\ cold) \text{ THEN } (fan\ speed\ is\ high) \\ &R2: \text{ IF } (temperature\ is\ cool) \text{ THEN } (fan\ speed\ is\ medium) \\ &R3: \text{ IF } (temperature\ is\ warm) \text{ THEN } (fan\ speed\ is\ low) \\ &R4: \text{ IF } (temperature\ is\ hot) \text{ THEN } (fan\ speed\ is\ zero) \end{aligned} \tag{8}$$

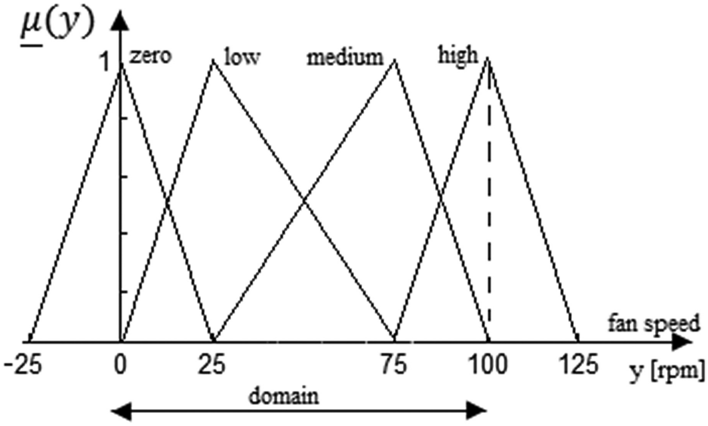

Fig. 6. Membership functions of the controller output y (fan speed) with extended border MFs *zero* and *high*.

To enable understanding of the Optimal Representation Defuzzification (ORD) method, the method will be partitioned in steps.

1. Determine upper MFs $\bar{\mu}(x)$ for fuzzy sets of the input x and output y of the controller.
2. Determine MFs Type 2 for the input x and output y of the controller.
3. Determine activation degrees $Act(B_k)$ of the output sets B_k activated (fired) by the considered input x-value in the inference process.
4. Determine one fuzzy set Type 2 B_R^{opt} optimally representing activated output sets B_k.
5. Determine the left and right border embedded set of the representing set B_R^{opt}, calculate positions of their centers of gravity and determine interval of possible positions of COGs.
6. Determine the representative COG of the interval of possible COGs.

Step 1.

If the original MF $\mu_A(x)$ of F-set A will be called *lower* MF and denoted as $\underline{\mu}_A(x)$ then formula (3) takes form (9).

$$\pi_A(x) = 1 - [\underline{\mu}_A(x) + \nu_A(x)] \tag{9}$$

Hesitancy $\pi_A(x)$ means uncertainty of membership of element x to set A. The upper limit $\bar{\mu}_A(x)$ of this membership can be calculated with (10).

$$\bar{\mu}_A(x) = \underline{\mu}_A(x) + \pi_A(x) = 1 - \nu_A(x) \tag{10}$$

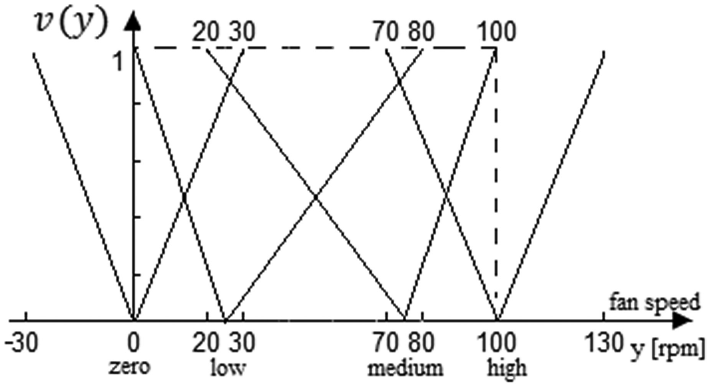

Fig. 7. Non-membership functions of the controller output y (fan speed) with symmetrically extended border NMFs *zero* and *high*.

On the basis of the original (lower) MF $\underline{\mu}_A(x)$ and of formula (10) the original MF and NMF $v(x)$ can be aggregated in one MF of Type 2 $\mu_A^{\mathrm{Type2}}(x)$. The possibility of such aggregation is described in [7, 11]. It is also discussed by creator of IFSs Atanassov in [6]. Interval Type 2 fuzzy sets are described e.g. in [8, 9].

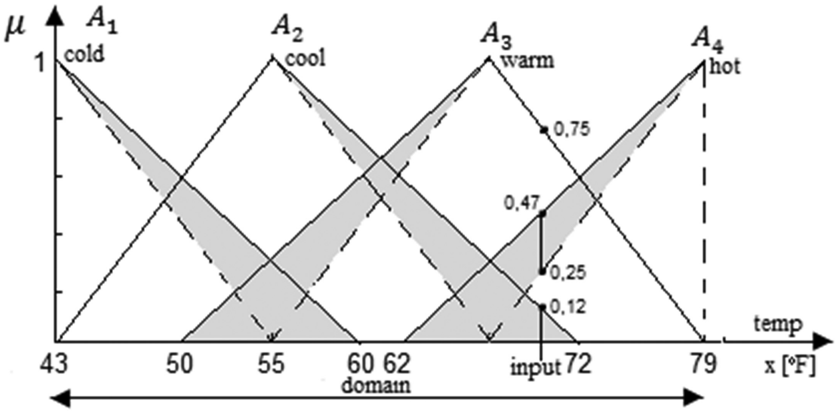

Fig. 8. Membership functions of Interval-Type 2 (IT2) of the controller input x achieved after aggregation of the MFs $\underline{\mu}_{Ai}(x)$ and NMFs $v_{\mathrm{Ai}}(x)$ of Type 1 and visualization of the considered input value $x = 70$ [°F].

Figure 8 shows interval Type 2 fuzzy sets of the input x achieved as the aggregation result.

Figure 9 shows IMF Type 2 of the output y achieved in the result of aggregation.

As can be seen in Fig. 8 membership of input x = 70 [°F] to 3 MFs *cool*, *warm*, *hot* is non-zero one. It means that in the inference process 3 sets of the output y B_1 = *zero*, B_2 = *low* and B_3 = *medium* fan speed were activated:

$$Act(B_1) \in [0.25, 0.47],\ Act(B_2) = 0.75,\ Act(B_3) \in [0, 0.12].$$

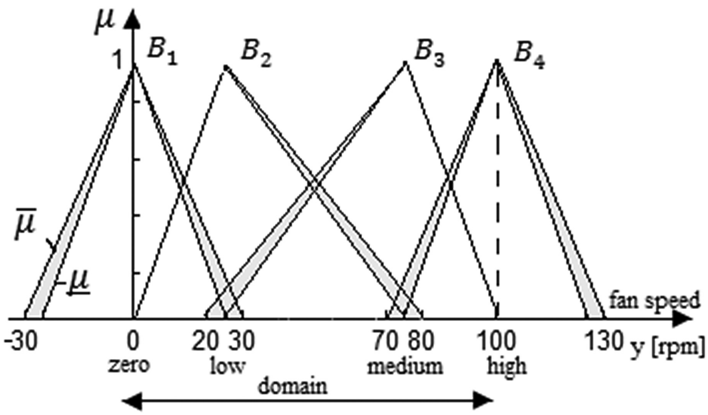

Fig. 9. Membership functions Interval Type 2 (IT2) of the controller output y (fan speed) achieved as result of aggregation of MFs $\underline{\mu}_{Bj}(y)$ and NMs $v_{Bj}(y)$ of Type 1.

To prevent increasing of the granularity level of calculations and calculations' burden for uncertain, interval activations optimal crisp representations should be determined. If as optimality criterion minimization of integral of square errors is assumed then as the optimal representation $Act(B_j)$ the mean value $Act^{opt} = 0.5$ $(Act_{max} - Act_{min})$ is achieved. Hence, optimal representations of particular activation intervals are:

$$Act^{opt}(B_1) = 0.355,\ Act^{opt}(B_2) = 0.750,\ Act^{opt}(B_3) = 0.060.$$

Because the activation sum exceeds 1, for calculation convenience simplicity, it should be normalized to 1 according to (12).

$$Act^{optN} = \frac{Act^{opt}(B_k)}{\sum_{j=1}^{3} Act^{opt}(B_j)},\ k = 1 - 3 \tag{12}$$

After the normalization following activation values are achieved:

$$Act^{optN}(B_1) = 0.30472, \; Act^{optN}(B_2) = 0.64378, \; Act^{optN}(B_3) = 0.05150.$$

In the ORD-method particular rules from the rule base are treated as experts of their local inputs' sub-domains. E.g. Rule 4: IF (*temperature is hot*) THEN (*fan speed is zero*) is expert that "knows" what fan speed should be set if the temperature x belongs to the input sub-domain classified as *hot*. Fuzzy set B_R^{opt} optimally represents all 3 activated output sets B_k, $k = 1 - 3$, it symbolically can be expressed by (13).

$$B_R^{opt} = \sum\nolimits_{k=1}^{3} Act^{optN}(B_k) \cdot B_k \tag{13}$$

In the practice, MFs Type 2 optimally representing activated output fuzzy sets can be determined in a simple way on the basis of formula (14), where parameters b_i ($i = 1 - 5$) are parameters determining the triangle, interval Type 2 membership function, Fig. 10.

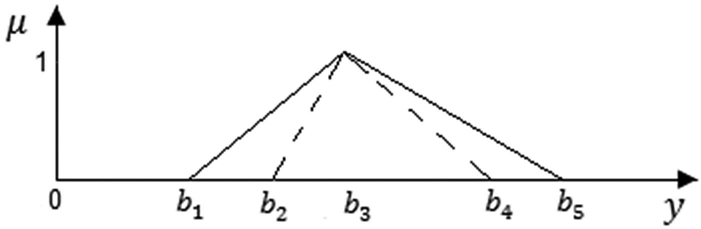

Fig. 10. Parameters b_i (i = 1−5), determining triangle IFP Type 2.

$$b_{Ri} = \sum\nolimits_{k=1}^{3} b_i(B_k) \cdot Act^{optN}(B_k), \; i = 1 - 5 \tag{14}$$

As example, calculation of the parameter b_{R1} of the representing MF is given by (15).

$$b_{R1} = -30 \cdot 0.30472 + 0 \cdot 0.64378 + 20 \cdot 0.05150 = -8.1116 \tag{15}$$

Values of other representative parameters are:

$$b_{R2} = -6.3305, \; b_{R3} = 19.9570, \; b_{R4} = 61.05150, \; b_{R5} = 65.7940.$$

Figure 11a shows 3 output sets B_k activated by the input x = [70° F] and the set B_R^{opt} optimally (according to MSSE-criterion) representing them.

It is easy to notice that the set B_R^{opt} to the largest degree is similar to the set B_2 = *low* because this set has the greatest activation among all output sets. The representative set B_R^{opt} from Fig. 11b is a much more simpler representation form of the 3 partly activated

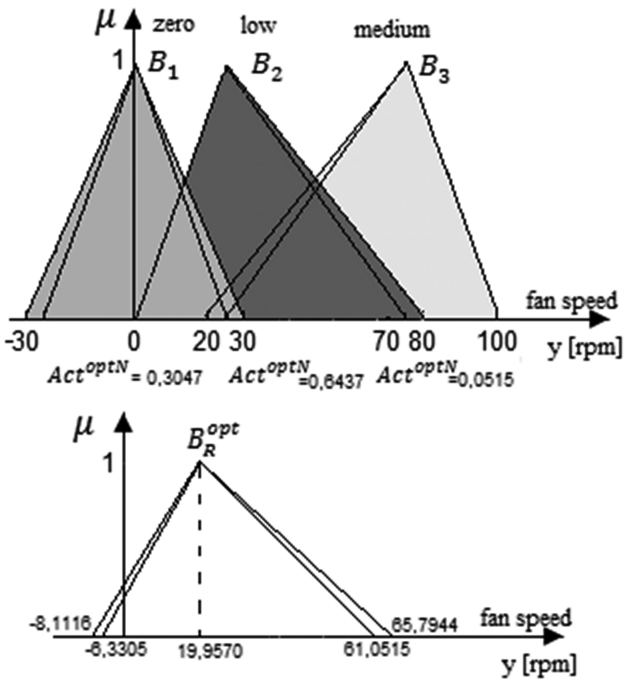

Fig. 11. Partly activated 3 output sets B_1 = *zero*, B_2 = *low* and B_3 = *medium* fan speed (Fig. 11a) and single IFS Type2 B_R^{opt} optimally them representing (Fig. 11b).

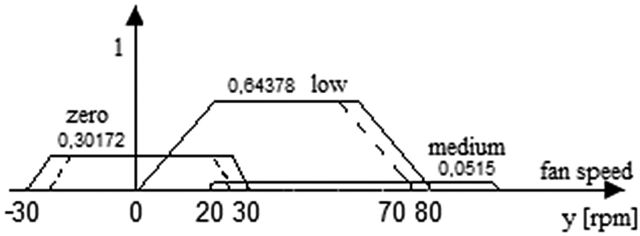

Fig. 12. Inference result in the considered problem achieved with the commonly used Mamdani-inference method.

output FSs shown in Fig. 11a than the complicated form achieved in the result of traditional Mamdani defuzzification shown in Fig. 12.

In the defuzzification process a single crisp output value should be determined that optimally represents the fuzzy set achieved as inference result. If the conventional center of gravity (COG) defuzzification is used then calculating the crisp representing value in Fig. 12 is much more complicated than in the case of the triangle fuzzy set in Fig. 11b. In this case left and right border embedded FSs Type 1 should be determined, as shown in Fig. 13.

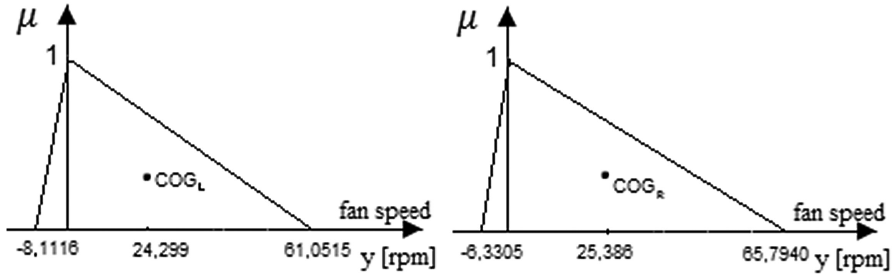

Fig. 13. Border left embedded FS Type 1 (Fig. 13a) and border right embedded FS Type 1 of the representing IFS Type 2 B_R^{opt} from Fig. 11b and their centers of gravity.

Center of gravity y_{COG} is such y^*-value which minimizes criterion in the form of integral of square distances $(y - y^*)^2$ of all possible y-values weighted by their membership. Hence, it is the optimal crisp representation of the representative fuzzy set B_R^{opt}. As can be seen in Fig. 13 position y_{COG} of the center of gravity is uncertain $y_{\mathrm{COG}}\left[y_{\mathrm{COGL},}y_{\mathrm{COGR}}\right] = [24.299, 25.386]$. Value $y_{\mathrm{COG}}^{\mathrm{opt}}$ that optimally, according to MSSE-criterion represents the interval [24.299, 25.386], is the mean value $y_{\mathrm{COG}}^{\mathrm{opt}} =$ 24.8425 [rpm]. It should be mentioned that authors of [1] applying non-extended MFs of border FS of the controller output and Takagi-Sugani defuzzification method achieved after defuzzification smaller result $y = 22.08$ [rpm].

3 Conclusions

The paper presented a method of intuitionistic fuzzy defuzzification in which, in the first step membership and non-membership functions of inputs and output are aggregated in interval MFs Type 2. Then, in the inference process activated (fired) output fuzzy sets are detected and their activation degrees are determined. Next, a single fuzzy set is found that optimally represents the activated fuzzy sets of the system output and a crisp value that optimally represents this fuzzy set. This crisp value is the defuzzification result. The ORD method is simple and intuitively convincing. The achieved defuzzification results can lay in the full, non-narrowed domain of the system output. The method decreases the computational burden in comparison with the commonly used COG-defuzzification method.

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Two New Defuzzification Methods Useful for Different Fuzzy Arithmetics

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Abstract. One of the many reasons why a human searches for new solutions is inspiration for innovation perceived in natural processes and phenomena. The authors of the paper present new algorithms for the defuzzyfication block which is the final process of the fuzzy controller (fuzzy control systems), for which a defuzzyfied value controls a given object. The presented new methods are: **Golden Ratio (GR)** and **Mandala Factor (MF)**. The first of them uses the ancient Golden Ratio rule which is known, among others, from the Fibonacci sequence. The second proposal is based on the interpretation of drawing technique used in Asia, consisting in arranging pictures of color sand grains. In Tibetan Buddhism this technique is known as Mandala, a symbol of perfection and harmony. The interpretation of the perfection symbol and the Golden Ratio method in this paper has been referenced to other methods used in the defuzzyfication process, including weight averange method, centroid and mean of maxima. The scene for solutions presented here is provided by the ordered fuzzy numbers (OFN) theory which allows to use both the trend of a given phenomena, as well as more precisely wield mathematical methods. A special property of the proposed methods is their sensitivity to the OFN number order. This means that MF and GR operators applied to the numbers of the same shape but of opposite orders would result in different defuzzyfication values. The paragraph relating to discussion on the research includes a comparison of the existing defuzzyfication operators as regards the sensitivity to order.

Keywords: Fuzzy logic · Ordered fuzzy numbers · OFN · Defuzzification

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1 Introduction

The idea of fuzzy sets [41] and related studies resulted in wide use of their applications in the world around us. The effect of fuzzy sets concept shows mainly by the use of those solutions in the areas of control systems, robotics and decision-making systems. Among many other applications, one can observe that solutions are used in the medicine image processing systems or speech recognition. The essence of that solution consists mainly in the possibility to express imprecise problems. Here, the existing, conventional mathematical models that operate in binary "yes-no" pattern have been redefined to intermediate elements. Considering fuzzy control models of any application area, one can conclude that the system is a classic block containing inputs, to which input signals are fed from the sensors and outputs that control a given actuating device. The internal structure of fuzzy controller is shown in Fig. 1, which presents such operations as: fuzzyfication, inference, defuzzyfication.

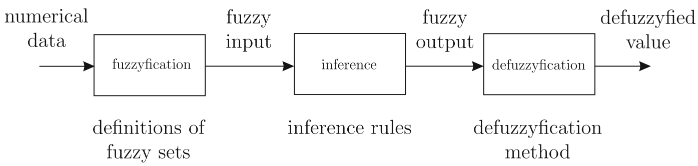


Fig. 1. The structure of fuzzy controller

In the first phase, the fuzzyfication operation is carried out, which represents the degree of membership in individual fuzzy sets. The conclusion of the phase of inference from the input degrees of membership is the calculated resulting membership function. Both the fuzzyfication and the inference operation contain a number of specific elements. More detailed discussion of the presented information can be found in the literature [1–4, 6, 8–19, 21–23, 25, 27–29, 31, 33, 34, 36, 39, 40, 42]. The final operation of the system takes place in the defuzzyfication block, which gives defuzzyfied value at the output. This value will be a product of the method used for the resulting membership function, which allows tuning of the actuating element as desired. This transformation is hereinafter referred to as defuzzyfication and at the same time will provide basic background for further research.

In specific approach, defuzzyfication process comes down to the use of appropriate methods. The methods allow to reduce the fuzzy set to an individual defuzzyfied value. In research carried out so far main defuzzyfication methods included those related to maxima (FOM, LOM, MOM, RCOM) or the area (COG, COA). Depending on the studied problem and assumptions as regards operation of the analyzed system, there is a need to develop new methods that would respond to new trends in fuzzy logic. In the particular case of interest associated with the trend of observed phenomena, those research directions concern ordered fuzzy numbers presented in studies [5].

The term "fuzzy set" formulated by L. Zadeh in the middle of 1960-ties, as part of the theory of fuzzy sets [41], preceded by the study of J. Lukaszewicz [32] where former propositional calculus based on binary logic was presented as multi-value calculus. This has contributed to development of derivative theories, broadening that subject.

Many scientists believe that the father of the fuzzy logic is a Pole, JanŁukasiewicz [32]. In 1920, he published his first paper on 3-valued logic. It was used by L. Zadeh to develop the fuzzy sets theory [41]. However, it was applied in practice only 10 years later, in 1975. Then E.H. Mamdanii described and built the first control system. Until that time computers processed numeric values only. Introduction of the fuzzy sets theory enabled processing of linguistic data. As a result, approximate inference systems were developed.

Fuzzy logic is also widely studied by Polish scientists. Polish professors have also greatly contributed to the development of fuzzy sets theory, e.g. prof. E. Czogała, prof. J. Kacprzyk and prof. W. Pedrycz and many other groups and societies collaborating with them. They became part of the fuzzy logic classics. Another prematurely died Polish scientific authority in fuzzy logic domain must also be mentioned here, i.e. Witold Kosiński, professor of Kazimierz Wielki University in Bydgoszcz and Polish-Japan School of Computer Technology in Warsaw who broadened this theory by ordered fuzzy numbers (OFN). Algebra of fuzzy numbers presented by him has been used in the controllers ensuring correct operation of equipment and technological processes.

2 Main Terms and Definitions

As defined by Lotfia Zadeh, fuzzy set is:

Definition 1. *Fuzzy set A, in a certain area X, is the set of pairs:*

$$A = \{(x, \mu(x))\} \ \forall x \in X \tag{1}$$

where: $\mu(x)$ membership function assigning to each element $\forall x \in X$ (of the assumed area of consideration of X) its degree of membership to set A, whereas:

$$\mu_A : X \rightarrow [0,1] \, thus \, \mu_A(x) \in [0,1] \tag{2}$$

Definition 2. *The support for fuzzy set A in X is the non-fuzzy set supp(A), which we define as:*

$$supp(A) = \{x : \mu_A(x) > 0, \, x \in X\} \tag{3}$$

Definition 3. *The height of a fuzzy set A in X determines the maximum value accepted by membership function $\mu_A(x)$ in the whole X set, as follows:*

$$h(A) = height(A) = hgt(A) = sup_{x \in X}(\mu(x)) \tag{4}$$

Definition 4. *We define the core of fuzzy set A in X as:*

$$core(A) = \{x : \mu_A(x) = 1, \, x \in X\} \tag{5}$$

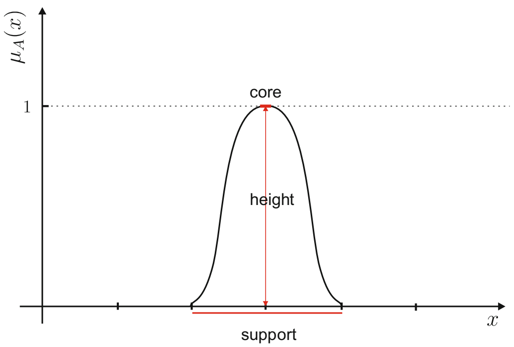


Fig. 2. Fuzzy set including a carrier, height and core

A graphic interpretation of a fuzzy set, along with the respective Definitions 2, 3 and 4 are shown in Fig. 2.

The essence of an Ordered Fuzzy Number is discussed in the introduction to this paper. Re-definition of classic fuzzy sets, where, according to Zadeh, it is an organized pair, has widened the definition by an organized pair of functions. OFN is defined as follows:

Definition 5.

$$A = (x_{up}, x_{down}) \tag{6}$$

where: $x_{up}, x_{down} : [0, 1] \to R$ *are continuous functions.*

These functions are called: up part and down part, respectively, where both those parts are connected by a constant function equal 1. Order of a fuzzy number is its arrangement so that up part is the beginning of OFN while down part is the end of this number.

Interpretation of the ordered fuzzy number is shown in Fig. 3, where an example of OFN is referred to a classic fuzzy number. The defuzzyfication process, as the last step in the three-step model of fuzzy control, converts fuzzy set into a single real (defuzzyfied) value, on which the membership function is defined. The following expression describes defuzzyfication in a formal way, i.e.:

Definition 6.

$$W = \{f : X \to [0, 1]\} \to X \tag{7}$$

where: W *defuzzification operator,* f *is the membership function and* X *the universe, on which membership functions are defined.*

The process can be characterized on the basis of the properties, which are more desirable for a particular system. Considering the type of the system, one can distinguish a fuzzy inference system, for which such property like e.g. processing power, is less important than for diffuse control system, for which the processing power is an important parameter. The study [35,37] introduced criteria of deffuzification operators for classic fuzzy numbers, on the basis of which

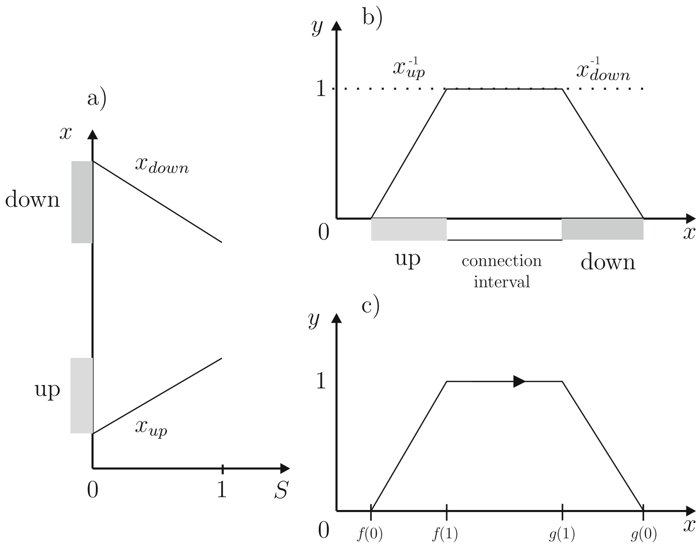


Fig. 3. OFN example, (b) OFN presented in relation to a classic fuzzy number, (c) Arrow denotes the orientation and the order of inverted functions: first UP and then DOWN.

individual defuzzyfication methods were assessed. The main conclusion is that there is no all-purpose defuzzyfication method. Defuzzyfication methods should be oriented to their field of application. For example, maximization methods, which include LOM (Last Of Maxima), FOM (First Of Maxima), are more suitable for inference systems. Research, which has been carried out by authors of the above mentioned study, proved that the distribution and field methods are more suitable for applications where control systems are used. Those methods include COG (Center Of Gravity) and COA (Center Of Area).

Upon development of ordered fuzzy numbers, authors of the paper [7,20,26] proposed criteria for defuzzyfication methods. That give grounds for guidelines enabling creation of suitable models of defuzzyfication operators. The following four conditions should be met for most of the methods.

Definition 7. *Each functional ϕ defined on R with the following properties:*

$$\phi(c) = c \tag{8}$$

$$\phi(A + c) = \phi(A) + c \tag{9}$$

$$\phi(cA) = c\phi(A) \tag{10}$$

$$\phi(A) \geq 0 \;\; if \;\;\; A \geq 0 \tag{11}$$

is called a defuzzyfication functional.

where: ϕ is a representation defined on the set of real numbers, $\phi(c)$ is understood as defuzzyfication of c value on the set of real numbers. In other words the *defuzzyfication using Singelton method should give defuzzyfied number (8). Condition (9) is related to additiveness, and it requires the defuzzyfication value for the sum of components to equal the sum of defuzzyfications for individual components. Condition (10) requires the representation ϕ to be homogeneous (first degree), i.e.: if the argument is multiplied by a factor then the result will also be multiplied by some power of this factor. In this case, that power amounts to one. Condition (11) refers to positive sense of a functional. Detailed interpretation of individual conditions is provided in the study [5, 30].*

3 Defuzzyfication Methods

It is well known that the defuzzyfication process reduces the fuzzy set to an individual defuzzyfied value. Mechanism of that operation consists mainly in the use of an appropriate defuzzyfication methods. Available methods include the following classic solutions:

FOM - first of maxima. This method is **FOM** a method concerning the choice of the smallest element of the set core **A**, where the defuzzyfication value represents the relationship (12).

$$FOM(A) = min\, core(A) \tag{12}$$

LOM - last of maxima. Appropriate choice of the maximum value of element from the set core **A**, is a method **LOM**, the formula of which is presented below:

$$LOM(A) = max\, core(A) \tag{13}$$

MOM - mean of maxima. The formula (14) illustrates the use of **FOM** and **LOM** as methods, the defuzzyfication values of which take into account the minimum and maximum element of the fuzzy set core **A**. The resulting value is the mean value of those two methods.

$$MOM(A) = \frac{min\, core(A) + max\, core(A)}{2} \tag{14}$$

RCOM - random choice of maxima. The method is also called defuzzyfication from a core, because the defuzzyfication value is always included in the core of a fuzzy set. The defuzzyfication value of this method is a random element $x \in core(A)$ calculated as a probability:

$$RCOM(A) = P(x) = \frac{\lambda(x)}{\lambda(core(A))} \tag{15}$$

where λ jest the Lebesgue measure in universe X.

MOS -mean of support. Defuzzyfication method **MOS**, the defuzzyfication value of which is the mean value of **A** number carrier.

$$MOM(A) = \frac{supp(A)}{2} \tag{16}$$

COG - center of gravity. The most widespread method, which is based on determination of the center of gravity of the analyzed system. In the fuzzy number A defuzzyfication process, the **COG** method is expressed as the formula (17).

$$COG(A) = \frac{\int_a^b x\mu_A(x)\mathrm{d}x}{\int_a^b \mu_A(x)\mathrm{d}x} \tag{17}$$

BADD - basic defuzzification distribution. The defuzzyfication method proposed [24] as an extension of COG and MOM methods. We obtain the following defuzzyfication value from the fuzzy set A:

$$BADD(A) = \frac{\int_a^b x\mu_A^\gamma(x)\mathrm{d}x}{\int_a^b \mu_A^\gamma(x)\mathrm{d}x} \tag{18}$$

Depending on parameter $\gamma \in [0, \infty]$, BADD may assume the following instances: when $\gamma = 0$, $BADD(A) = MOS(A)$; when $= 1$, $BADD(A) = COG(A)$; when $\gamma \to \infty$, $BADD(A) = MOM(A)$.

3.1 Defuzzyfication Methods for OFN

Classic defuzzyfication methods presented in the above parts of the paper are reflected in ordered fuzzy numbers. In the analysis of methods shown below, one of their explanations shall include important characteristic elements of OFN presented in Fig. 4.

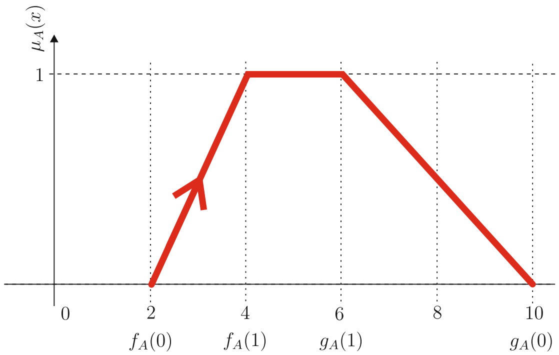


Fig. 4. A OFN number and characteristic elements

In the definition of an ordered fuzzy number expressed by the formula (6), an ordered fuzzy number A can also be defined, according to other approaches to that subject, as an oriented pair of continuous functions:

$$A = (f_A, g_A) \tag{19}$$

where $f_A, g_A : [0,1] \to R$. The function f_A is called the up part UP_A (beginning) of an ordered fuzzy number A, while the function g_A is called the down part $DOWN_A$ (end) of an ordered fuzzy number A.

In the interpretation of OFN defuzzyfication methods, value of f_A function for 0 is f_A, for 1 is $f_A(1)$ and of g_A function it is: for 0 $g_A(0)$, for 1 is $g_A(1)$.

$$\phi_{FOM}(f,g) = f(1) \tag{20}$$

$$\phi_{LOM}(f,g) = g(1) \tag{21}$$

$$\phi_{MOM}(f,g) = \frac{f(1)+g(1)}{2} \tag{22}$$

$$\phi_{ROM}(f,g) = \zeta f(1) + (1-\zeta)g(1), \quad \zeta = [0,1] \tag{23}$$

$$\phi_{COG}(f,g) = \begin{cases} \frac{\int_0^1 \frac{f(s)+g(s)}{2}|f(s)-g(s)|ds}{\int_0^1 |f(s)-g(s)|ds} & , for \ \int_0^1 |f(s)-g(s)|ds \neq 0 \\ \frac{\int_0^1 f(s)ds}{\int_0^1 ds} & , for \ \int_0^1 |f(s)-g(s)|ds = 0 \end{cases} \tag{24}$$

$$\phi_{BADD}(A,\lambda) = \frac{\int_0^1 \frac{f(s)+g(s)}{2}|f(s)-g(s)|\cdot s^{\lambda-1}ds}{\int_0^1 |f(s)-g(s)|\cdot s^{\lambda-1}ds}, for \ \lambda \in [0,1] \tag{25}$$

$$\phi_{GM}(f,g) = \frac{f(1)\cdot g(0) - f(0)\cdot g(1)}{f(1)+g(0)-f(0)-g(1)} \tag{26}$$

The above formulas (20 to 26) are interpretation of classic defuzzyfication methods. In the discussed ordered fuzzy numbers theory [30] and in the earlier studies, the Geometrical Mean method is proposed, created by D. Wilczyńska-Sztyma [38].

4 Definition of Goled Ratio Defuzzyfication Operator

At this point we present a proposal for a new method of defuzzification a fuzzy controller, which is based on the concept of the golden ratio, derived from the Fibonacci series. The origin of the method was the observation of numerous instances of the golden ratio in such diverse fields as biology, architecture, medicine, and painting. A particular area of its occurrence is genetics, where we find the golden ratio in the very structure of the DNA molecule (deoxyribonucleic acid molecules are 21 angstroms wide and 34 angstroms long for each full length of one double helix cycle). This fact makes the ratio in the Fibonacci series in some sense a universal design principle used by man and nature alike.

Fibonacci series is based on the assumption that it starts with two ones, and each consecutive number is the sum of the previous two. The proposal for the Golden Ratio method of defuzzification is based on the proportion of the golden ratio. As a result of dividing each of the numbers by its predecessor, we always obtain quotients oscillating around the value of 1.618 the golden ratio number. The exact value of the limit is the golden number itself:

$$\lim_{n\to 0} \frac{k_{n+1}}{k_n} = 1,618033998875\cdots = \varPhi \tag{27}$$

The possibility of using this formula in the process of defuzzification will be another example of the universality of the method, as it is applied in the new domain of fuzzy logic theory. Calculation of the classical formula of the golden mean assumes that two values of line segments a, and b, are in golden ratio Φ to one another if:

$$\frac{a+b}{a} = \frac{a}{b} = \Phi \tag{28}$$

In this case, one method of finding the value of Φ is to transform the left-hand fraction of Eq. (28) into:

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\Phi},\ where\ \frac{b}{a} = \frac{1}{\Phi} \tag{29}$$

Following subsequent transformations of Eq. (29) we obtain quadratic Eq. (30), for which we will calculate the roots.

$$\Phi^2 - \Phi - 1 = 0 \tag{30}$$

As appropriate, using transformations of the formula in (30) we obtain two square roots (31).

$$\Phi_1 = \frac{1+\sqrt{5}}{2}\ or\ \Phi_2 = \frac{1-\sqrt{5}}{2} \tag{31}$$

In view of the fact that the value of Φ must be positive, in our example we select the positive root, as in Eq. (32).

$$\Phi = \Phi_1 = \frac{1+\sqrt{5}}{2} = 1,618033998875\ldots \tag{32}$$

In sum, the ratio between two objects a, and b is called the golden ratio when the value of $\Phi = 1.61803398875\ldots$

The method of the Golden Ratio (GR) for fuzzy number is the Eq. (33):

Definition 8.

$$\begin{gathered} GR = min(supp(A)) + \frac{|supp(A)|}{\Phi} \\ where\ \Phi = 1,618033998875\ldots \end{gathered} \tag{33}$$

where: GR is the defuzzification operator, $supp(A)$ is support for fuzzy set A in universe X.

4.1 Goled Ratio for OFN

The mathematical formula (33) of the equation as well as graphic interpretation presented in Fig. 5 applies to convex fuzzy numbers. In reference to the number OFN, which has orientation, we should use another equation. Therefore, the interpretation of the proposed method is shown in Fig. 6.

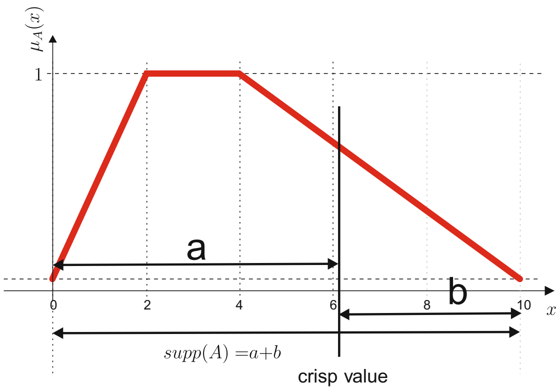
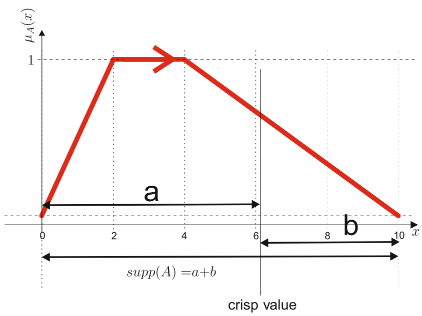
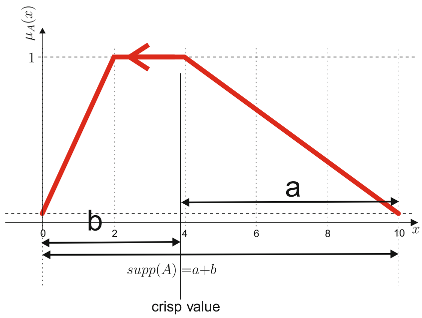


Fig. 5. Golden ration defuzzificaton value



(a) OFN number A=[0,2,4,10]



(b) OFN number B=[10,4,2,0]

Fig. 6. Two OFN of the same shape's, but otherwise ordered

We note that the individual parts of the two values of line segments a, and b take up positions in relation to direction number OFN. In the first case where we have positive number OFN where a larger part of golden ratio GR start from base point $f(0)$. In the second case, which is negative number OFN, we have a base point as $g(0)$. The method of the Golden Ratio (GR) for ordered fuzzy number OFN is the Eq. (34):

Definition 9.

$$GR(A) = \begin{cases} min(supp(A)) + \frac{|supp(A)|}{\Phi} & , if\ order\ (A)\ is\ positive \\ max(supp(A)) - \frac{|supp(A)|}{\Phi} & , if\ order\ (A)\ is\ negative \end{cases} \tag{34}$$

The instrument of the Golden Ratio, as proposed in this paper for fuzzy numbers may serve as another defuzzification method. As a mathematical apparatus that affords wide-ranging possibilities in description and processing of information, it becomes a new solution in constructing models of fuzzy controllers used as tools for inferencing or control.

5 Definition of Mandala Factor Defuzzyfication Operator

Buddhist monks can create amazing pictures of colored sand grains. Those pictures are called mandala. It is difficult to name them paintings because we expect paintings to be rather more lasting effect. Anyone who has ever seen meditating monks creating, grain after grain, a previously designed picture, that remembers such conclusion for a long time. On the one hand, you can observe the beauty of their art while on the other hand the transitory nature (in the literal sense) of the technique they use is evident. The same reverence is seen in Christianity of Eastern Orthodox rite when icons are painted, but fortunately for culture and art, the effects of the work can be watched for a long time. Buddhist mandala is a harmonious combination of a wheel and a square, where the wheel is a symbol of heaven, transcendence, externality and infinity, while the square depicts the inner sphere, i.e. the matters associated with a human and the earth. Both figures are linked by the central point, which is both the start and end of the entire system. The mandala creation process itself, as well as its destruction, is a religious act. The Mandala Factor defuzzyfication operator is inspired by mandala. Let A be a given fuzzy number shown in the figure below. Let it assume a shape of a trapezoid in Fig. 8(a). A trapezoid can in particular case come down to a triangle, but we'll remain at a trapezoid, which will make our analysis more universal. Then one must fill the outline marked by the sides of the number and the OX axis with virtual grains of sand in Fig. 8(a). A number of virtual sand grains will be collected in this way. Then one must construct a rectangle the base of which is equal to the support value of the fuzzy number. The rectangle built in such a manner should be filled with virtual sand grains, starting from the outermost left side in Fig. 8(b). The filling process should be done vertically in columns until all grains are used. A real number obtained as a result of defuzzyfication is the value above which the last filled column was finished (Fig. 7).

Mathematical formalism (35) of the above described Mandala factor visualization is shown below. Calculation of the R value using the Mandala Factor Ψ_M for the rising edge, falling edge and core set function integral. Then the obtained value should be scaled from the center of the coordinate system by adding it to the start of the support value of the fuzzy number. When defuzzyfication is performed in the OFN arithmetic, then in the case of a positive order, one should proceed as described below, while in the case of a negative order, one should deduct the calculated value from the first coordinate of the OFN number corresponding to the outermost right side of the OFN support.

Fig. 7. Mandala creation [http://wellness.gcublogs.org/tag/sand-mandala/]

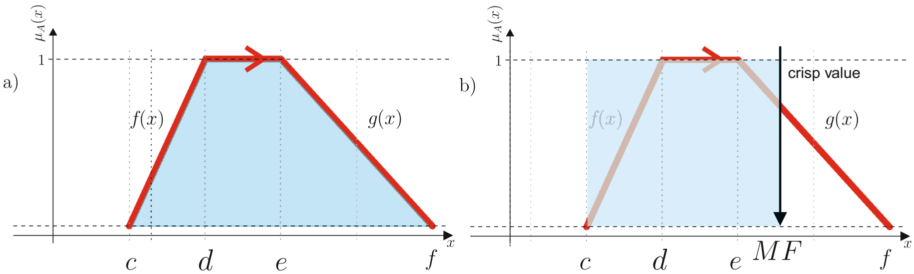


Fig. 8. Mandala factor visualization

Definition 10.

$$MF(A) = \begin{cases} c + r \text{ ,} if\ order\ (A)\ is\ positive \\ c - r \text{ ,} if\ order\ (A)\ is\ negative \end{cases} \tag{35}$$

where:

$$\begin{aligned} r = \frac{1}{d-c}\int_c^d x\,dx - \frac{c}{d-c}\int_c^d dx + \frac{f}{f-e}\int_f^e dx \\ -\frac{1}{f-e}\int_e^f x\,dx + \int_d^e dx \end{aligned} \tag{36}$$

6 Conclusion

The paper presents two new original defuzzyfication methods, i.e. Golden Ratio and Mandala Factor. Each of them is characterized by unique properties worth to be noted. Real number values obtained through the operation of each operator are unique and different from those obtained using known methods. The Golden

Ratio and Mandala factor operation can therefore be applied in well-known and widely used arithmetics of fuzzy numbers such as L-R Dubois and Prade notation [20]. As shown in the calculations presented in the previous paragraph the obtained results distinguish new operators from the classic, commonly known solutions. New operators are also characterized by the feature, which is absent in most of the classic operators. This feature is the sensitivity to order (order sensitive). This feature manifests so that different defuzzyfication values are obtained from one shape of a fuzzy number, depending on the fuzzy number order type (*positive* or *negative*) (Kosiński Ordered fuzzy numbers). This is shown in the previous paragraph. Basic shapes of the Ordered Fuzzy Numbers are visualized in the appendix to this paper. To sum up, it can be concluded that both defuzzyfication methods, i.e. Golen ratio and Mandala Factor, meet all the criteria of defuzzyfication operators, and are adapted to applications in all fuzzy numbers arithmetics, including Kosiński Ordered fuzzy numbers arithmetic.

A Appendix Ordered Fuzzy Numbers Catalog

See Figs. 9, 10, 11, 12, 13, 14, 15, 16 and 17

See Tables 1, 2, 3, 4, 5, 6, 7, 8 and 9

Table 1. The numerical results of defuzzification methods of the fuzzy number $A1$

Method	Value
GR	3.47
MF	3.5
FOM	2
LOM	3

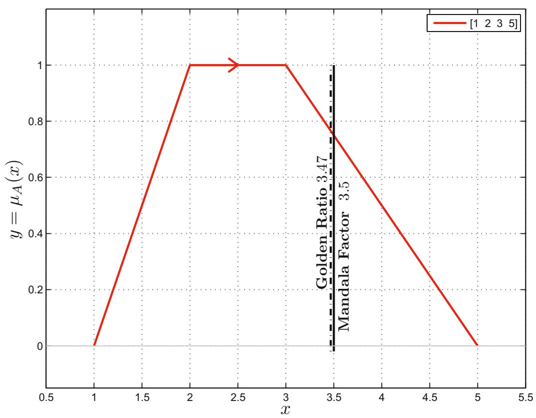


Fig. 9. Example of calculation of the GR and MF defuzzification methods of the OFN number $A1$

Table 2. The numerical results of defuzzification methods of the fuzzy number $A2$

Method	Value
GR	2.53
MF	2.5
FOM	2
LOM	3

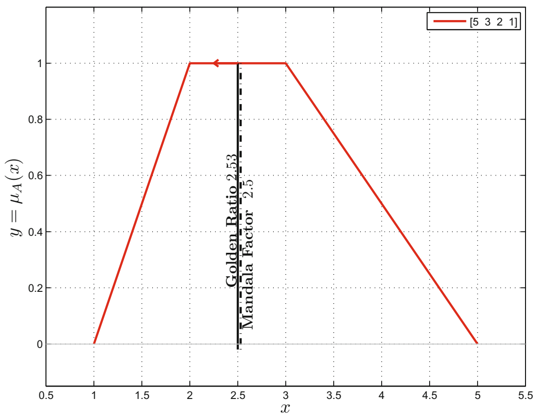


Fig. 10. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A2$

Table 3. The numerical results of defuzzification methods of the fuzzy number $A3$

Method	Value
GR	5
MF	5
FOM	5
LOM	5

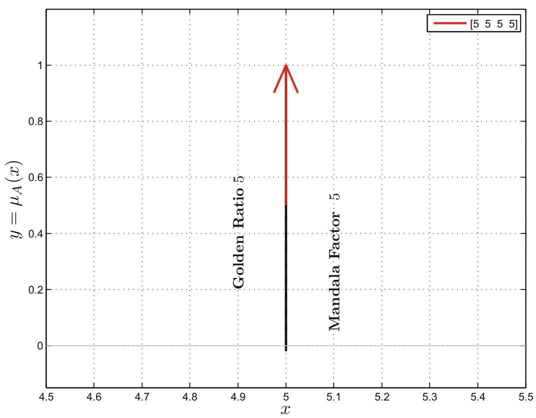


Fig. 11. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A3$

Table 4. The numerical results of defuzzification methods of the fuzzy number $A3$

Method	Value
GR	1.62
MF	2
FOM	2
LOM	3

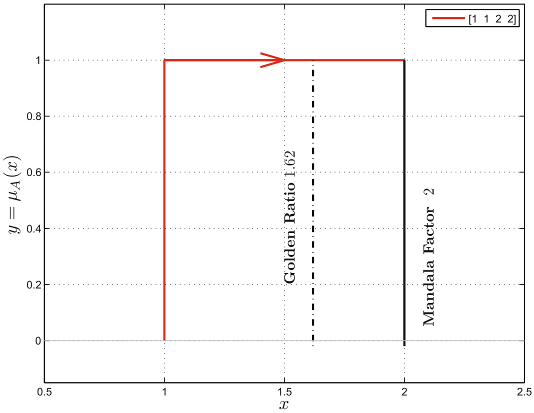


Fig. 12. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A4$

Table 5. The numerical results of defuzzification methods of the fuzzy number $A5$

Method	Value
GR	2.71
MF	2
FOM	2
LOM	2

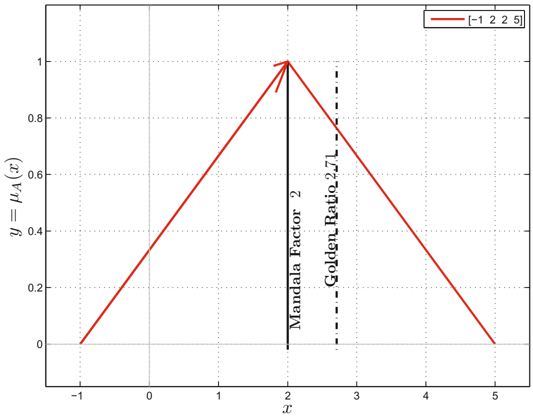


Fig. 13. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A5$

Table 6. The numerical results of defuzzification methods of the fuzzy number $A6$

Method	Value
GR	2.85
MF	3.5
FOM	1
LOM	3

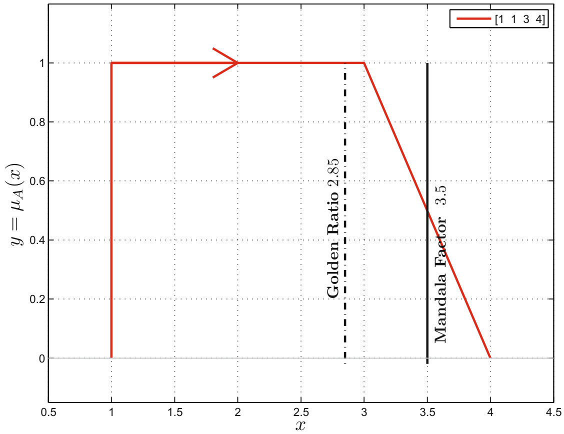


Fig. 14. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A6$

Table 7. The numerical results of defuzzification methods of the fuzzy number $A7$

Method	Value
GR	1.5
MF	2.15
FOM	1
LOM	3

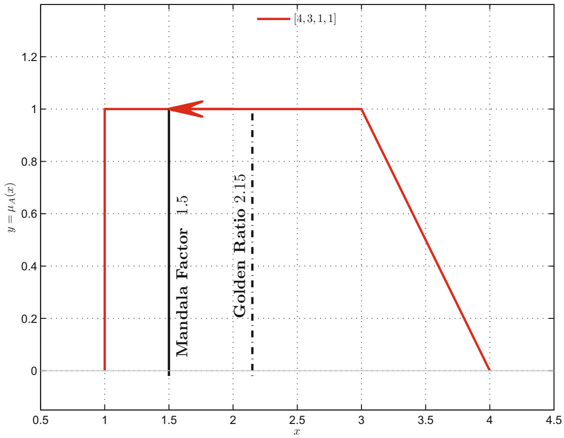


Fig. 15. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A7$

Table 8. The numerical results of defuzzification methods of the fuzzy number $A8$

Method	Value
GR	2.53
MF	3.5
FOM	2
LOM	5

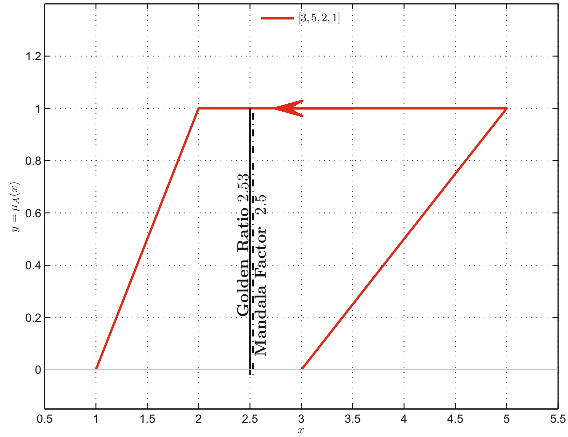


Fig. 16. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A8$

Table 9. The numerical results of defuzzification methods of the fuzzy number $A9$

Method	Value
GR	3.47
MF	3.5
FOM	2
LOM	5

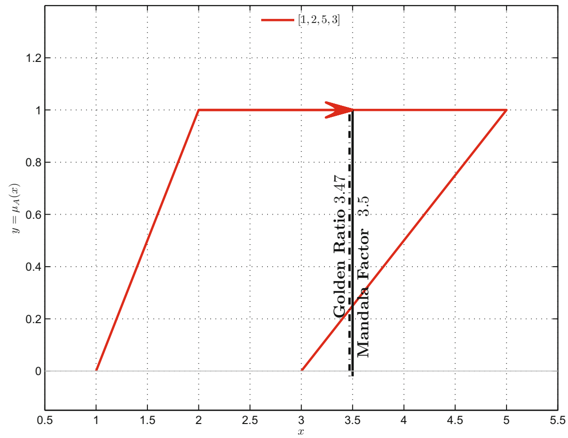


Fig. 17. Example of calculation of the *GR* and *MF* defuzzification methods of the *OFN* number $A9$

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Two Constructions of Ordinal Sums of Fuzzy Implications

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Abstract. This contribution deals with ordinal sums of fuzzy implications. New methods of construction as generalizations of the once previously examined are proposed. The generalizations allow consider summands on intervals of different type: open, closed, or half-open. Sufficient properties of fuzzy implications as summands for obtaining a fuzzy implication as a result are presented.

Keywords: Fuzzy implication · Ordinal sum · Triangular norm

1 Introduction

Fuzzy implications find applications in many fields such as fuzzy control, approximate reasoning, and decision support systems. This is why new families of these connectives are the subject of investigation.

One of the directions of such research is considering an ordinal sum of fuzzy implications on the pattern of the ordinal sum of t-norms. Some interesting results connected to representation of the residual implication corresponding to a fuzzy conjunction (for example continuous or at least left-continuous t-norm) given by an ordinal sum were obtained in [2,5,8]. In [10] Su et al. introduced a concept of ordinal sum of fuzzy implications similar to the construction of the ordinal sum of t-norms. In [3,4] other constructions of ordinal sums of fuzzy implications were described.

In this paper, new possibilities of defining ordinal sums of fuzzy implications are proposed. Sufficient properties for fuzzy implications as summands for obtaining a fuzzy implication are presented. The examined methods are generalizations of the results obtained in [3,4,10]. The generalizations allow consider summands on intervals of different type: open, closed, or half-open.

First, in Sect. 2, we recall definitions and basic results concerning t-norms and fuzzy implications and we recall methods of constructing ordinal sums of fuzzy implications. Then, in Sect. 3, we propose new constructions of ordinal sums of fuzzy implication which generate fuzzy implications. Moreover, we suggest further research directions for the ordinal sums of fuzzy implications.

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2 Preliminaries

Here we recall the notions of a t-norm and a fuzzy implication, as well as some of the constructions of ordinal sums of these fuzzy connectives.

2.1 Fuzzy Connectives

First, we put definition of a t-norm and an important class of t-norms.

Definition 1 ([7], p. 4). *A t-norm is an increasing, commutative and associative operation* $T : [0,1]^2 \to [0,1]$ *with neutral element* 1.

Definition 2 ([7], p. 27). *A t-norm* T *is called Archimedean, if for each* $(x,y) \in (0,1)^2$ *there exists* $n \in \mathbb{N}$ *such that* $x_T^{(n)} < y$.

Example 1 ([7], p. 4, [6], p. 7). Here, we list well-known t-norms, from which T_M is continuous, and T_P, T_L are both continuous and Archimedean.

$$T_M(x,y) = \min(x,y), \qquad T_P(x,y) = xy,$$

$$T_L(x,y) = \max(x+y-1,0), \qquad T_D(x,y) = \begin{cases} x, & \text{if } y=1 \\ y, & \text{if } x=1 \\ 0, & \text{otherwise} \end{cases},$$

$$T_{nM}(x,y) = \begin{cases} 0, & \text{if } x+y \le 1 \\ \min(x,y), & \text{otherwise} \end{cases}.$$

Next, let us recall a representation of continuous t-norms by means of ordinal sums which is based on the ordinal sum of arbitrary t-norms ([7], p. 82).

Theorem 1 ([7], p. 128). *For an operation* $T : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*

(i) T *is a continuous t-norm.*
(ii) T *is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exists a uniquely determined (finite or countably infinite) index set* I, *a family of uniquely determined pairwise disjoint open subintervals* (a_k, b_k) *of* $[0,1]$ *and a family of uniquely determined continuous Archimedean t-norms* $(T_k)_{k\in A}$ *such that (see Fig. 1)*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in [a_k,b_k]^2 \\ \min(x,y) & \text{otherwise} \end{cases}.$$

Now, we focus on fuzzy implications.

Definition 3 ([1], p. 2, [6], p. 21). *A function* $I : [0,1]^2 \to [0,1]$ *is called a fuzzy implication if it satisfies the following conditions:*
(I1) decreasing in its first variable,
(I2) increasing in its second variable,
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

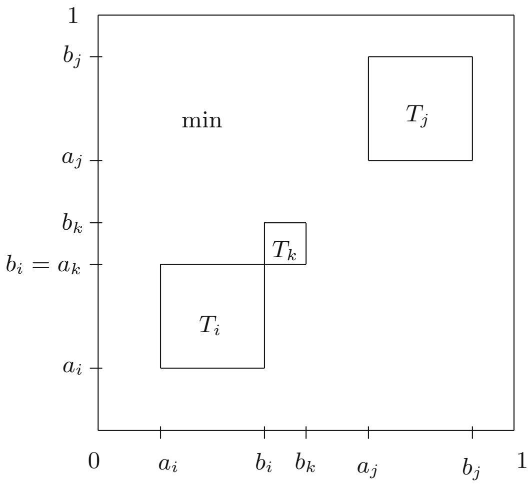

Fig. 1. The structure of an ordinal sum of t-norms

There are many potential properties of fuzzy implications (see, e.g., [1], p. 9). We recall here only one which will be important in the sequel.

Definition 4 ([9]). *We say that a fuzzy implication I fulfils the consequent boundary property* (CB) *if*

$$I(x, y) \geq y, \quad x, y \in [0, 1]. \tag{CB}$$

Example 2 ([1], pp. 4, 5). The following are well-known examples of fuzzy implications. Operations I_0 and I_1 are the least and the greatest fuzzy implication, respectively. Almost all of them, except for I_0 and I_{RS} fulfil property (CB).

$$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1 \\ 0, & \text{otherwise} \end{cases}, \quad I_1(x,y) = \begin{cases} 0, & \text{if } x = 1, y = 0 \\ 1, & \text{otherwise} \end{cases},$$

$$I_{LK}(x,y) = \min(1 - x + y, 1), \qquad I_{GG}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases},$$

$$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}, \qquad I_{RS}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise} \end{cases},$$

$$I_{RC}(x,y) = 1 - x + xy, \qquad I_{YG}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{otherwise} \end{cases},$$

$$I_{DN}(x,y) = \max(1 - x, y), \qquad I_{FD}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1 - x, y), & \text{otherwise} \end{cases},$$

$$I_{WB}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{otherwise} \end{cases}, \qquad I_{DP}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1 - x, & \text{if } y = 0 \\ 1, & \text{otherwise} \end{cases}.$$

2.2 Ordinal Sums of Fuzzy Implications

Now, let us recall a recent approach to the construction of ordinal sum of fuzzy implications. Let start with the method based on the construction of the ordinal sum of t-norms.

Definition 5 ([10]). *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{[a_k, b_k]\}_{k\in A}$ be a family of pairwise disjoint closed subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. The operation $I : [0,1]^2 \to [0,1]$ given by (see Fig. 2)*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x, y \in [a_k, b_k] \\ I_{GD}(x,y), & \text{otherwise} \end{cases} \tag{1}$$

we call an ordinal sum of fuzzy implications $\{I_k\}_{k\in A}$.

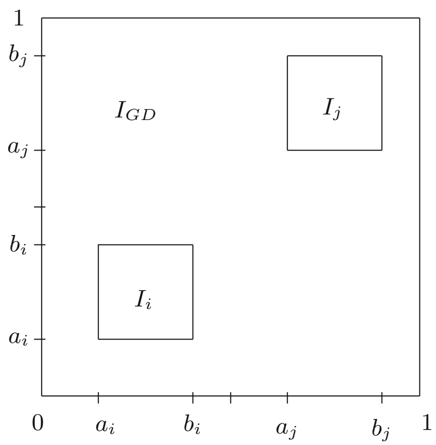

Fig. 2. The structure of an ordinal sum of fuzzy implications given by (1)

The following theorem characterizes the ordinal sum I given by (1) as fuzzy implication.

Theorem 2 ([10]). *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications. The operation I given by (1) is a fuzzy implication if and only if I_k satisfies (CB) whenever $k \in A$ and $b_k < 1$.*

As we can see, not every fuzzy implications can be used in the construction (1) in order to obtain a fuzzy implication. Below we recall a structure in which any fuzzy implications can be used.

Definition 6 ([3]). *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{[a_k, b_k]\}_{k\in A}$ be a family of pairwise disjoint closed subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider an operation $I : [0,1]^2 \to [0,1]$ given by the following formula (see Fig. 3)*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k \left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \textit{if } x,y \in [a_k, b_k] \\ I_{RS}(x,y), & \textit{otherwise} \end{cases}. \tag{2}$$

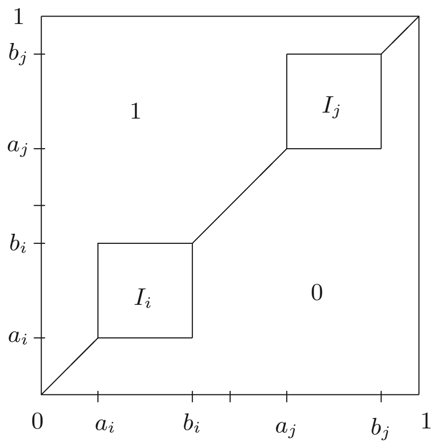

Fig. 3. The structure of an operation given by (2)

Theorem 3 ([3]). *The operation I given by (2) is a fuzzy implication.*

3 Main Results

Here, we propose generalizations of the constructions recalled in Sect. 2.2. The generalizations allow consider summands on intervals of different type: open, closed, or half-open. In this section we will use a symbol $|a_k, b_k|$, which denotes one of the following intervals: (a_k, b_k), $(a_k, b_k]$, $[a_k, b_k)$ or $[a_k, b_k]$. Let start with the generalization of the construction (2).

Definition 7. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k, b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ and $0 \notin |a_k, b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider an operation $I : [0,1]^2 \to [0,1]$ given by the following formula (see Fig. 4)*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k \left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \textit{if } x,y \in |a_k, b_k| \\ I_{RS}(x,y), & \textit{otherwise} \end{cases}. \tag{3}$$

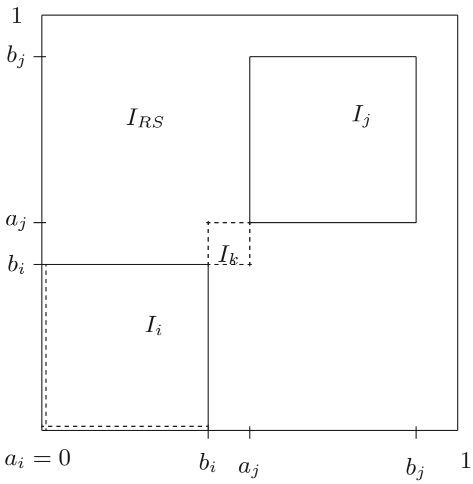

Fig. 4. The structure of an operation given by (3)

For the above defined operation we have the following result.

Theorem 4. *The operation I given by (3) is a fuzzy implication.*

Proof. First, let us consider the condition (I1). Let $x_1, x_2, y \in [0, 1]$, $x_1 < x_2$.

If $y \in |a_k, b_k|$ for some $k \in A$, then we consider the following cases

1. $x_1 \le a_k$, $x_1 \notin |a_k, b_k|$. Then, $I(x_1, y) = I_{RS}(x_1, y) = 1 \ge I(x_2, y)$.
2. $x_1, x_2 \in |a_k, b_k|$. Then, using monotonicity of I_k with respect of the first variable we have

$$I_k\left(\frac{x_1 - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right) \ge I_k\left(\frac{x_2 - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right).$$

This means that $I(x_1, y) \ge I(x_2, y)$.

3. $b_k \le x_2$, $x_2 \notin |a_k, b_k|$. Then $I(x_1, y) \ge 0 = I_{RS}(x_2, y) = I(x_2, y)$.

If $y \notin |a_k, b_k|$ for any $k \in A$, then $I(x_1, y) = I_{RS}(x_1, y) \ge I_{RS}(x_2, y) = I(x_2, y)$, which gives condition (I1).

To prove (I2) let us take $x, y_1, y_2 \in [0, 1]$, $y_1 < y_2$.

If $x \in |a_k, b_k|$ for some $k \in A$, then we obtain the following cases

1. $y_1 \le a_k$, $y_1 \notin |a_k, b_k|$. Then $I(x, y_1) = I_{RS}(x, y_1) = 0 \le I(x, y_2)$.
2. $y_1, y_2 \in |a_k, b_k|$. Then using monotonicity of I_k with respect of the second variable we have

$$I_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y_1 - a_k}{b_k - a_k}\right) \le I_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y_2 - a_k}{b_k - a_k}\right).$$

This means that $I(x, y_1) \le I(x, y_2)$.

3. $b_k \le y_2$, $y_2 \notin |a_k, b_k|$. Then $I(x, y_1) \le 1 = I_{RS}(x, y_2) = I(x, y_2)$.

If $x \notin |a_k, b_k|$ for any $k \in A$, then $I(x, y_1) = I_{RS}(x, y_1) \leq I_{RS}(x, y_2) = I(x, y_2)$. So, we obtain (I2).

Directly from (3) we have $I(0,0) = I_{RS}(0,0) = 1$ and $I(1,0) = I_{RS}(1,0) = 0$. So I fulfils (I3) and (I5). To prove (I4) let us consider two cases. If there exists $k \in A$ such that $1 \in |a_k, b_k|$, then

$$I(1,1) = a_k + (1 - a_k) I_k \left(\frac{1 - a_k}{1 - a_k}, \frac{1 - a_k}{1 - a_k} \right) = a_k + (1 - a_k) I_k (1,1) = 1.$$

Otherwise $I(1,1) = I_{RS}(1,1) = 1$.

So, operation I given by (3) is a fuzzy implication.

Example 3. Let $I : [0,1]^2 \to [0,1]$ such that

$$I(x,y) = \begin{cases} 0.5 I_{DN}(2x, 2y), & \text{if } x, y \in (0, 0.5] \\ 0.5 + 0.1 I_{YG}(10x - 5, 10y - 5), & \text{if } x, y \in (0.5, 0.6) \\ I_{RS}(x,y), & \text{otherwise} \end{cases}.$$

I is a fuzzy implication generated by the ordinal sum (3).

As a corollary we obtain constructions considered in [3,4].

Corollary 1 (cf. [4]). *Let $\{I_k\}_{k \in A}$ be a family of fuzzy implications and $\{(a_k, b_k)\}_{k \in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. The operation $I : [0,1]^2 \to [0,1]$ given by the following formula*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right), & \text{if } x, y \in (a_k, b_k] \\ I_{RS}(x,y), & \text{otherwise} \end{cases} \tag{4}$$

is a fuzzy implication.

Corollary 2 ([4]). *Let $\{I_k\}_{k \in A}$ be a family of fuzzy implications and $\{(a_k, b_k)\}_{k \in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. The operation $I : [0,1]^2 \to [0,1]$ given by the following formula*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right), & \text{if } x, y \in [a_k, b_k) \\ I_{RS}(x,y), & \text{otherwise} \end{cases} \tag{5}$$

is a fuzzy implication.

Now, let us present the generalization of the construction (1).

Definition 8. *Let $\{I_k\}_{k \in A}$ be a family of fuzzy implications and $\{|a_k, b_k|\}_{k \in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ and $0 \notin |a_k, b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider an operation $I : [0,1]^2 \to [0,1]$ given by the following formula (see Fig. 5)*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right), & \text{if } x, y \in |a_k, b_k| \\ I_{GD}(x,y), & \text{otherwise} \end{cases}. \tag{6}$$

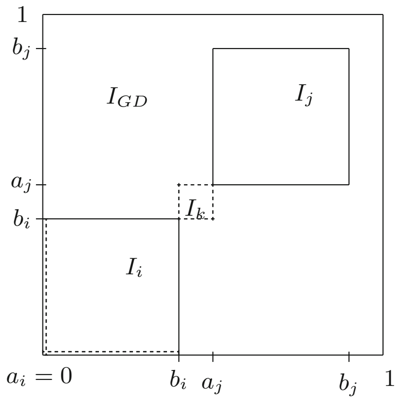

Fig. 5. The structure of an operation given by (6)

For the above defined operation we have the following result.

Theorem 5. *The operation I given by* (6) *is a fuzzy implication if and only if I_k satisfies* (CB) *whenever $k \in A$ and $b_k < 1$.*

Proof. Let I_k satisfies (CB) for $k \in A$ whenever $b_k < 1$. First, let us consider the condition (I1). Let $x_1, x_2, y \in [0, 1]$, $x_1 < x_2$.

If $y \in |a_k, b_k|$ for some $k \in A$, then we consider the following cases
1. $x_1 \leq a_k$, $x_1 \notin |a_k, b_k|$. Then, $I(x_1, y) = I_{GD}(x_1, y) = 1 \geq I(x_2, y)$.
2. $x_1, x_2 \in |a_k, b_k|$. Then, using monotonicity of I_k with respect of the first variable we have

$$I_k\left(\frac{x_1 - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right) \geq I_k\left(\frac{x_2 - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right).$$

This means that $I(x_1, y) \geq I(x_2, y)$.
3. $b_k \leq x_2$, $x_2 \notin |a_k, b_k|$. Then using (CB) for $x_1 \in |a_k, b_k|$ we have

$$I(x_1, y) \geq y = I_{GD}(x_2, y) = I(x_2, y)$$

and for $x_1 \notin |a_k, b_k|$ we have $I(x_1, y) = I_{GD}(x_1, y) \geq I_{GD}(x_2, y) = I(x_2, y)$. If $y \notin |a_k, b_k|$ for any $k \in A$, then $I(x_1, y) = I_{GD}(x_1, y) \geq I_{GD}(x_2, y) = I(x_2, y)$, which gives condition (I1).

To prove (I2) let us take $x, y_1, y_2 \in [0, 1]$, $y_1 < y_2$.

If $x \in |a_k, b_k|$ for some $k \in A$, then we obtain the following cases
1. $b_k \leq y_2$, $y_2 \notin |a_k, b_k|$. Then $I(x, y_1) \leq 1 = I_{GD}(x, y_2) = I(x, y_2)$.
2. $y_1, y_2 \in |a_k, b_k|$. Then using monotonicity of I_k with respect of the second variable we have

$$I_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y_1 - a_k}{b_k - a_k}\right) \leq I_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y_2 - a_k}{b_k - a_k}\right).$$

This means that $I(x,y_1) \le I(x,y_2)$.
3. $y_1 \le a_k$, $y_1 \notin |a_k,b_k|$. Then $I(x,y_1) = I_{GD}(x,y_1) = y_1 \le a_k \le I(x,y_2)$ for $y_2 \in |a_k,b_k|$, and $I(x,y_1) = I_{GD}(x,y_1) \le I_{GD}(x,y_2) = I(x,y_2)$ for $y_2 \notin |a_k,b_k|$.

If $x \notin |a_k,b_k|$ for any $k \in A$, then $I(x,y_1) = I_{GD}(x,y_1) \le I_{GD}(x,y_2) = I(x,y_2)$. So, we obtain (I2).

Directly from (6) we have $I(0,0) = I_{GD}(0,0) = 1$ and $I(1,0) = I_{GD}(1,0) = 0$. Thus, I fulfils (I3) and (I5). To prove (I4) let us consider two cases. If there exists $k \in A$ such that $1 \in |a_k,b_k|$, then

$$I(1,1) = a_k + (1-a_k)I_k\left(\frac{1-a_k}{1-a_k}, \frac{1-a_k}{1-a_k}\right) = a_k + (1-a_k)I_k\,(1,1) = 1.$$

Otherwise $I(1,1) = I_{GD}(1,1) = 1$. So, operation I is a fuzzy implication.

Now, let I given by (6) be a fuzzy implication. In particular I satisfies (I1). If there exists $k \in A$ such that $b_k < 1$ then taking $x,y \in [0,1]$ and $z > b_k$ we have

$$\begin{aligned} I_k(x,y) &= \frac{I(a_k+(b_k-a_k)x, a_k+(b_k-a_k)y) - a_k}{b_k-a_k} \\ &\ge \frac{I(z, a_k+(b_k-a_k)y) - a_k}{b_k-a_k} = \frac{(a_k+(b_k-a_k)y) - a_k}{b_k-a_k} = y \end{aligned}$$

Hence, I_k fulfils condition (CB).

In the above constructions we use two different fuzzy implications I_{RS} and I_{GD} as a complement of component implications I_k. In the first case, where the fuzzy implication I_{RS} is involved, we obtain an ordinal sum which is a fuzzy implication without any additional assumption on component implications I_k. In the second construction, where we use the fuzzy implication I_{GD}, we need an assumption on I_k in order to obtain a fuzzy implication as a result. Let us notice that not every fuzzy implication will be suitable as the complement. For the greatest fuzzy implication I_1 as the complement we do not obtain any other fuzzy implication regardless to properties of summands. It seems worth of studying which fuzzy implications used as a complement of summands allow to generate new useful fuzzy implications taking into account their properties.

Example 4. Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k,b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ and $0 \notin |a_k,b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider an operation $I : [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} a_k + (b_k-a_k)I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x,y \in |a_k,b_k| \\ I_1(x,y), & \text{otherwise} \end{cases}.$$

Let us notice that if there exists $k \in A$ such that $b_k < 1$ then although $b_k < \frac{b_k+1}{2}$ we have $I(b_k,a_k) = a_k < 1 = I\left(\frac{b_k+1}{2}, a_k\right)$. So, (I1) is not fulfilled for any fuzzy implication I_k.

Now, let us assume that $A = \{|a_k, 1|\}$ and consider two cases. If $|a_k, 1| = [a_k, 1]$ then although $\frac{a_k}{2} < a_k$ we have $I(1, \frac{a_k}{2}) = 1 > a_k = I(1, a_k)$, so (I2) is not fulfilled for any fuzzy implication I_k. Otherwise, let us consider $x \in |a_k, 1|$. For any $y < a_k$ we have $I(x, y) = I_1(x, y) = 1$, so to obtain (I2) it is required that $I(x, y) = 1$ for any $y \in |a_k, 1|$. In this case we obtain $I = I_1$.

4 Conclusions

In this paper two methods of constructing ordinal sums of fuzzy implications that generalize the previous considered once were presented. Sufficient properties of summands for obtaining a fuzzy implication as a result were examined. At the end a direction of future research was proposed.

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Novel Tools and Techniques in Modeling, Decision Making and Decision Support: Theory and Applications

A Decision-Making Model in an IVFS Environment Based on Sigma f-Count Cardinality

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Abstract. This article presents a new approach to making decisions when information, possibly incomplete, is provided by many sources. The proposed method is based on IVFS scalar cardinality (sigma f-count). First a general algorithm is introduced, and next an application in supporting medical decisions in ovarian tumor differentiation (based on multiple diagnostic models) is presented and discussed.

Keywords: Cardinality · Interval-Valued Fuzzy Set · Decision making · Incomplete information

1 Introduction

This paper contributes to the study of decision-making based on multiple sources of incomplete information. The possible incompleteness or lack of data is a crucial issue that will be addressed in our considerations. The issue of data uncertainty has recently been studied intensively in many contexts and scientific disciplines, since it has been observed to be a natural and unavoidable feature of many real-life processes. Clearly, missing data can have a significant effect on the conclusions that can be drawn from sets of data. Nevertheless, even when there is no possibility of collecting high-quality data, it is often still possible to make an appropriate decision.

In this paper we present one of the possible approaches to decision-making based on data from multiple, incomplete sources. Assuming we have partial knowledge about the world taken from many sources, we aim to find a single, correct decision to be taken. In the method presented here, information is modeled within the IVFS (Interval-Valued Fuzzy Set) framework, and the decision-making method is based on counting. Moreover, by taking a bipolar perspective, our method makes it possible to express both positive and negative degrees of belief regarding a particular decision. With such an approach, full information is preserved together with its degree of incompleteness.

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The remainder of the paper is organized as follows. Section 2 describes a problem of multiple-source decision-making with incomplete information, and introduces a general method based on the sigma f-count cardinality measure for IVFSs that can be used for some decision-making problems. The method is illustrated with a numerical example. In Sect. 3 we show how the proposed method can be used in the medical decision support system OvaExpert. Section 4 concludes our work and indicates possible paths of further research.

2 Decision-Making Method Based on IVFS Cardinality

2.1 Statement of the Problem

In many decision situations a decision-maker is confronted with the problem that, for various reasons, the information necessary to make a decision is incomplete. In this paper we consider a case where the available information comes from many independent sources (experts, systems). Each source expresses a degree of belief regarding a particular decision. This degree is modeled by a value in the interval [0,1], where 0 denotes a definite vote against a particular decision, 1 fully confirms the decision, and all intermediate values denote some degree of hesitation; in particular 0.5 expresses that there is as much evidence in favor of the decision as there is against. If a source lacks some of the information necessary to specify a precise degree, this degree can be given by an interval. Notice that this interval has an epistemic characterization, since the true degree of belief is contained in this interval and can be specified precisely when a source gathers more information. The aim is to aggregate all of the available information to make an appropriate and high-quality decision, even in the presence of incomplete or missing data.

The natural way for humans to make decisions based on many sources (or many experts) is a strategy of counting. By counting, people determine how many sources or experts vote for and how many vote against a given option, and then they choose the decision for which the greatest number of them have voted. Since the decision in our case is taken on the basis of multiple source decisions represented as intervals, it seems natural to estimate the minimum and maximum possible confidence for a particular decision. Such an approach suggests the use of the cardinality of IVFSs representing the limits of both intervals: supporting and rejecting the decision. It is important to be aware of the fact that, as mentioned in [4], an approach based on voting is justifiable when the sources are independent and identical. In our research we identify sources with experts with different backgrounds and from different research centers, hence we assume their independence and equal importance.

2.2 Scalar Cardinality of IVFS (Sigma f-Counts)

First we introduce some basic concepts from the theory of cardinality of IVFSs. A comprehensive account of this theory can be found in [11]. Another example of an application of fuzzy set cardinalities with weighting functions in decision-making can be found in [3].

A scalar cardinality ($Sigma\, f - Count$) of an IVFS $\widetilde{A}$ can be defined as an interval $sc_f(\widetilde{A}) = [\sigma_f(A_l), \sigma_f(A_u)]$ where

$$\sigma_f(A) = \sum_{x \in supp(A)} f(A(x)) \tag{1}$$

and $f : [0,1] \to [0,1]$ is a weighting function (sometimes called cardinality pattern) fulfilling the conditions $f(0) = 0$, $f(1) = 1$ and $f(x) \leq f(y)$ whenever $x \leq y$. The weighting function plays a crucial role in computing a cardinality. It is worth noting that when $f = id$ the cardinality is equal to the sigma-count defined by Zadeh in [13]:

$$\sigma_f(A) = \sum_{x \in supp(A)} A(x). \tag{2}$$

We define five families of cardinality pattern functions. Each of them entails a different way of treating elements in the counting process.

1. $f_{1,t,p}$, where $t \in [0,1]$ and $p \geq 0$. Use of this type of function resulting in a cardinality called *counting by thresholding* [11] (see Fig. 1)

$$f_{1,t}(x) = \begin{cases} 1, & \text{if } x \geq t, \\ x^p & \text{otherwise} \end{cases} \tag{3}$$

2. $f_{2,t,p}$, where $t \in [0,1]$ and $p \geq 0$. Use of this type of function resulting in a cardinality called *counting by thresholding and joining* [11] (see Fig. 2)

$$f_{2,t}(x) = \begin{cases} x^p, & \text{if } x \geq t, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

3. $f_{3,t_1,t_2,p}$, where $t_1, t_2 \in [0,1]$ and $p \geq 0$. (see Fig. 3)

$$f_{3,t_1,t_2,p} = \begin{cases} 1, & \text{if } x \geq t_2, \\ x^p, & \text{if } x \in (t_1, t_2) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

4. f_{4,t_1,t_2}, where $t_1, t_2 \in [0,1]$. (see Fig. 4)

$$f_{4,t_1,t_2}(x) = \begin{cases} 1, & \text{if } x \geq t_2, \\ 0.5, & \text{if } x \in (t_1, t_2) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

5. $f_{5,t,p}$, where $t > 0$ and $p \in (0,1)$. (see Fig. 5)

$$f_{5,t,p}(x) = \frac{1}{1 + e^{(-t(x-p))}} \tag{7}$$

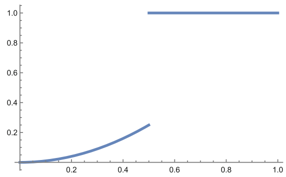


Fig. 1. Weighting function $f_{1,t,p}$ for $t = 0.5$ and $p = 2$

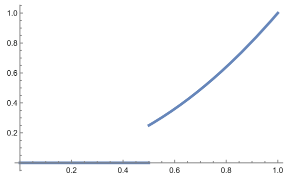


Fig. 2. Weighting function $f_{2,t,p}$ for $t = 0.5$ and $p = 2$

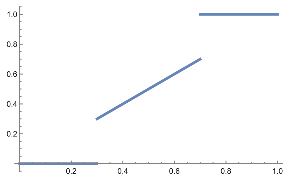


Fig. 3. Weighting function $f_{3,t_1,t_2,p}$ for $t_1 = 0.3$, $t_2 = 0.7$ and $p = 1$

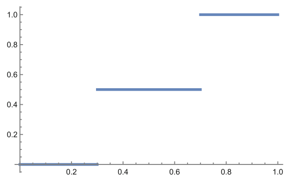


Fig. 4. Weighting function f_{4,t_1,t_2} for $t_1 = 0.3$, $t_2 = 0.7$

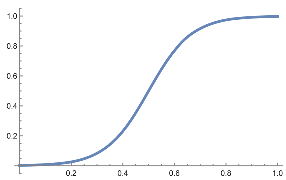


Fig. 5. Weighting function $f_{5,t,p}$ for $t = 0.5$, $p = 12$

2.3 Decision-Making Algorithm

The proposed decision-making model takes as inputs multiple sources of information which may be uncertain. All of them are represented by an IVFS $\widetilde{A} = (A_l, A_u)$ with $A_l \subset A_u$ in the finite universe of n sources of decisions $S_1, S_2, \ldots, S_n$. Fuzzy sets A_l and A_u represent respectively the lower and upper bounds of the degree of belief regarding a decision given by the information sources. These values can be interpreted as follows: a value closer to 1 means a tendency towards a positive decision, while a value closer to 0 means a tendency towards a negative decision.

Figure 6 shows an example of six sources of information for the same problem.

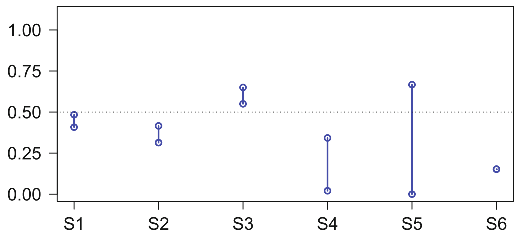


Fig. 6. Example of six sources of decisions as an IVFS

Using a bipolar perspective we define two IVFSs: pro $\widetilde{P}$ (supporting a positive decision) and contra $\widetilde{C}$ (supporting a negative decision). The set $\widetilde{P} = (A_l, A_u)$ models a positive decision, and the set $\widetilde{C} = (A_u^{-1}, A_l^{-1})$ models a negative decision. Next we calculate the cardinality of both sets using sigma f-count, which results in two intervals describing the number of sources tending towards a positive decision $sc_f(\widetilde{P}) = [\sigma_f(A_l), \sigma_f(A_u)]$ and the number of those tending towards a negative decision $sc_f(\widetilde{P}) = [\sigma_f((A_u)^{-1}), \sigma_f((A_l)^{-1})]$. It is worth noting that if we use the weighting function $f_{1,0.5,1}$ (*counting by thresholding*) this approach will be equivalent to a voting strategy (each model will vote 0 or 1). Once the cardinality intervals $sc_f(\widetilde{P})$ and $sc_f(\widetilde{C})$ are obtained, we can make a final decision by comparing them in the following way (see Algorithm 1):

– if the cardinality intervals are overlapping (function *overlap*) or the distance between the ends of the intervals (function *distance*) is smaller than a given number r, the system cannot make a decision. In other words, if the numbers of models voting pro and contra are similar, then it is not possible to make a decision. The parameter r reflects our intuition of how large should be the difference between the numbers of pro and contra votes;
– if the intervals are not overlapping, then compare the centers of the cardinality intervals (function *center*) and select as a decision the option with the greater value. This means that we choose the decision with more models supporting it.

Algorithm 1. Decision Algorithm

if $overlap(sc_f(\widetilde{P}), sc_f(\widetilde{C}))$ **or** $distance(sc_f(\widetilde{P}), sc_f(\widetilde{C})) < r$ **then**
    $Decision \leftarrow NA$
else
    if $center(sc_f(\widetilde{P})) > center(sc_f(\widetilde{C}))$ **then**
        $Decision \leftarrow Positive$
    else
        $Decision \leftarrow Negative$
    end if
end if

The decision algorithm is illustrated by the following example.

Example 1. For the input decisions shown in Fig. 6 we execute the algorithm with parameter $r = 1$ (a decision should be taken by a majority of at least one). When the weighting function $f_{2,0.5,1}$ was used the following sigma f-counts were computed: $sc_f(\widetilde{P}) = [0.55, 1.3]$, $sc_f(\widetilde{C}) = [2.6, 4.1]$. Hence the final decision is positive. For comparison, if the function $f_{3,0.3,0.7,1}$ was used with the same parameters, the cardinality would be $sc_f(\widetilde{P}) = [1.3, 2.6]$, $sc_f(\widetilde{C}) = [3.4, 4.7]$ and hence no decision could be taken.

3 Example of Application to Medical Diagnosis

In this section we present the first experiments using the proposed decision-making method in a real-life application, namely in the medical diagnostic system OvaExpert. OvaExpert is a result of collaboration between scientists from two universities in Poznań, Poland: Adam Mickiewicz University and Poznań University of Medical Sciences. Its development was motivated by one of the most challenging current problems in gynecology – the correct differentiation of adnexal masses. Early identification of malignant ovarian tumors versus benign neoplasms and functional lesions is crucial, because it determines the necessity of surgery, the pre-operative work-up and adequate timing in the operation room [2]. It also has great importance for determining who should perform the surgery – a gynecological oncologist or a general gynecologist. The problem of correct and early diagnosis of this kind of tumor is a difficult task especially for inexperienced gynecologists [2].

OvaExpert is intended to integrate present knowledge about ovarian tumors into a single computer-aided system and to support gynecologists, especially less experienced ones, in stating a correct diagnosis even when some of the data is missing. It has a modular architecture. Some modules implement known prognostic models developed by gynecologists. The most common diagnostic models are based on scoring systems [1,9] and logistic regressions [5,10]. For our research we have selected models from the International Ovarian Tumor Analysis (IOTA) Group – the LR1 and LR2 models, based on logistic regression [7] – as well as RMI [5], Alc. [1], Tim. [10] and SM [9], which have been proposed by different researchers.

In its separate modules, OvaExpert implements novel concepts in medical diagnosis. Among them, an experimental module implements the decision-making method based on cardinality presented in the previous section. A preliminary experiment involved a study group of 375 patients diagnosed and treated for ovarian tumor in the Division of Gynecological Surgery at Poznan University of Medical Sciences. Among them, 232 were diagnosed with a benign tumor and 143 with a malignant tumor. The training set consisted of 200 patients, and the test set of 175 patients. All patients in the training set had a complete set of features. All patients in the test set had up to 50% missing values. The original diagnostic models – LR1, LR2, RMI, Alc., Tim., SM – operate directly on the

Table 1. Cost matrix. The costs were assigned based on expert gynaecologists' opinions.

		Predicted		
		Benign	Malignant	NA
actual	Benign	0	2.5	1
	Malignant	5	0	2

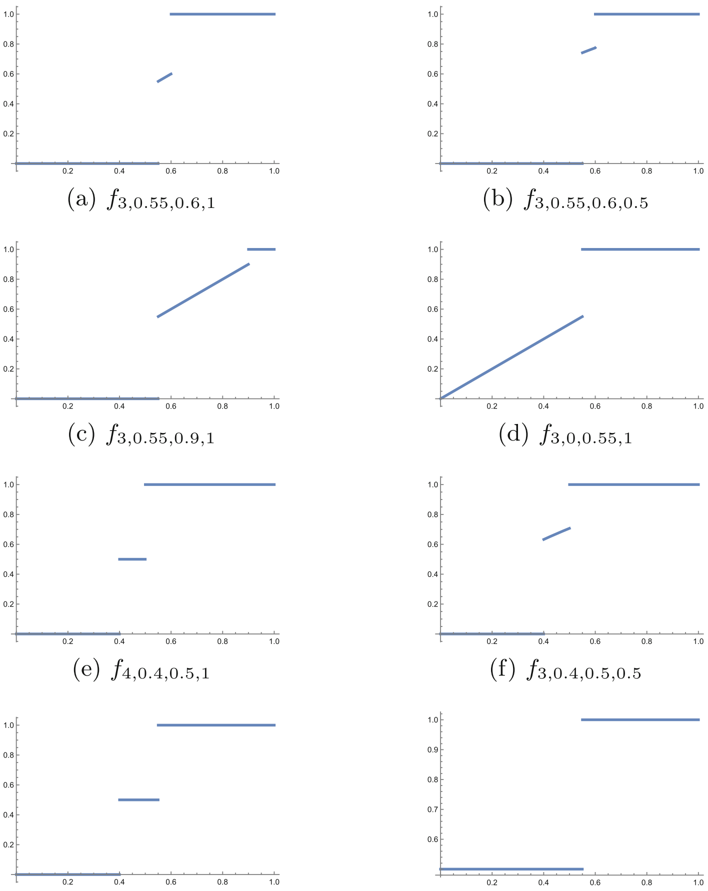
Table 2. Performance measures of the original diagnostic models and new diagnostic modules. Results are obtained on the test set. Abbreviations: Dec. – Decisiveness, Sen. – Sensitivity, Spec. – Specificity, Acc. – Accuracy, OAE – module based on diagnostic models aggregation (cf. [12]), IVFC – module based on interval-valued fuzzy classifier (cf. [8]).

		Total cost	Dec.	Sen.	Spec.	Acc.
Original models	Alc. [1]	189.0	20.6 %	88.2 %	89.5 %	88.9 %
	LR1 [7]	184.0	27.4 %	92.6 %	57.1 %	77.1 %
	LR2 [7]	164.0	33.1 %	94.3 %	65.2 %	82.8 %
	RMI [5]	156.0	56.6 %	75.9 %	87.1 %	83.8 %
	SM [9]	142.0	62.9 %	94.6 %	71.2 %	79.1 %
	Tim. [10]	159.0	47.4 %	66.7 %	97.1 %	91.6 %
OVA modules	OEA [12]	72.0	96.6 %	90.2 %	86.4 %	87.6 %
	IVFC [8]	72.5	100.0 %	90.4 %	84.6 %	86.3 %
Cardinality based models	$f_{3,0.55,0.6,1}$, $r = 1$	67.5	100.0 %	90.4 %	86.2 %	87.4 %
	$f_{3,0.55,0.6,0.5}$, $r = 0$	70.0	100.0 %	90.4 %	85.4 %	86.9 %
	$f_{3,0.55,0.9,1}$, $r = 0$	70.0	100.0 %	90.4 %	85.4 %	86.9 %
	$f_{3,0,0.55,1}$, $r = 0$	70.0	100.0 %	90.4 %	85.4 %	86.9 %
	$f_{4,0.4,0.5,1}$, $r = 0$	70.0	100.0 %	90.4 %	85.4 %	86.9 %
	$f_{3,0.4,0.5,0.5}$, $r = 0.25$	71.5	96.0 %	90.0 %	87.3 %	88.1 %
	$f_{4,0.4,0.55,1}$, $r = 0.25$	71.5	98.3 %	90.2 %	86.0 %	87.2 %
	$f_{4,0,0.55,1}$, $r = 0.25$	71.5	96.0 %	90.0 %	87.3 %	88.1 %

patients' features. The new diagnostic models – based on cardinality with different weighting functions – were evaluated on the intervals returned as results by the original diagnostic models. New diagnostic modules were optimized in the training phase to minimize the total misclassification cost. The cost matrix was based on expert knowledge and is presented in Table 1.

We set the learning and evaluation procedures so that, firstly, in the training phase we simulate possible data incompleteness derived from the patients with the complete set of features and, secondly, we estimate the misclassification error on the patients with real incomplete features. Notably, we extend the training set by a 1000-fold random balanced sampling of 150 patients and random feature obscuration to obtain missing values in the range $[0\%, 50\%]$.

At the optimization stage we choose the best fitting cardinality pattern functions (out of the five defined in Sect. 2, with a wide range of parameter values)



(a) $f_{3,0.55,0.6,1}$ (b) $f_{3,0.55,0.6,0.5}$

(c) $f_{3,0.55,0.9,1}$ (d) $f_{3,0,0.55,1}$

(e) $f_{4,0.4,0.5,1}$ (f) $f_{3,0.4,0.5,0.5}$

(g) $f_{4,0.4,0.55,1}$ (h) $f_{4,0,0.55,1}$, $r = 0.25$

Fig. 7. Cardinality pattern functions used in the selected models

and different values of r (decision interval levels). In Table 2 we present the best eight of them, with performance measures calculated on the test data set. Figure 7 presents plots of the corresponding cardinality pattern functions.

The original diagnostic models vary in their classification characteristics: some of them tend to be more conservative (LR1, LR2, SM) and some more liberal (RMI, Tim.). This can be observed in the considerable differences between values of sensitivity and specificity. One model maintains a balance between these two factors (Alc.). Unfortunately, all six models have low decisiveness (due to the lack of some patient attributes), which results in high total costs for these models. For comparison, Table 2 also presents decision models implemented in OvaExpert on the basis of our previous research (for more information see [8,12]).

It can be noticed that the presented model achieves better results than the previous models in terms of both total cost and specificity. Moreover, for certain choices of f and r it has a high value of decisiveness.

4 Conclusions and Further Work

In the paper we have presented a new method of making decisions by integrating incomplete information from many sources. This method, based on the notion of cardinality, has a solid theoretical foundation, is relatively easy to implement and interpret, and, most importantly, achieves very good effectiveness in real-life applications such as medical diagnostics. We believe that our approach can also be effectively adapted to non-medical problems where data quality is a matter of concern [6]. It could be applied when the information that comes from independent experts is imperfect and it is important to preserve information about this imperfection in the final result. By returning bipolar information – concerning the quantities of positive and negative premises – we are able to evaluate that imperfection and the quality of the information.

We are currently working on developing methods to be used in the process of decision-making in an IVFS environment based on FECount type cardinality. Preliminary results are very promising.

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On Computer-Based Support in Noncooperative Multicriteria Games

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Abstract. The paper deals with noncooperative games in which each player has some number of criteria measuring his payoff. A decision support system is considered as a computer-based tool that allows the players to make an analysis of the conflict situation, taking into account their preferences. The analysis can be done using an interactive, learning procedure utilizing methods of multicriteria optimization. An algorithm supporting analysis of payoffs in the multicriteria game and derivation of the best response strategies satisfying preferences of the players is proposed. The reference point approach with application of the respected achievement function is used in the interactive procedure in which payoffs of players are calculated closely to their preferences. The algorithm utilizes new theoretical results of the theory of noncooperative games. The results presented in the form of theorems include parametric characterization of the multicriteria gains representing preferences of the players and show relations among equilibria in the multicriteria games and the respective classical games.

Keywords: Multicriteria noncooperative games · Noncooperative equilibrium · Multicriteria decision making · Decision support systems

1 Introduction

Decision support problems in the case of conflict situations that can be described as multicriteria noncooperative games are discussed in the paper. The general theory of noncooperative games has already been intensively developed starting from fundamental papers by Nash and after him by Arrow, Debreu, Hurwicz, to mention only the precursors of the theory. In the references only selected papers are mentioned: Nash [14,15], Arrow and Debreu [1], Arrow and Hurwicz [2], Aubin [3]. The last of the references includes a broad bibliography on the subject. The theory has been developed as a mathematical background for analysis of conflict situations under the assumption that each player has an explicitly given one-dimensional utility function measuring his outcome.

In practical problems, it is typical that a player deals with not one but with several criteria which he would like to satisfy. The player has rather in-mind

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preferences on the criteria. The utility function aggregating the criteria is in general not given explicitly. What more, in practice, the decision maker - player can modify his preferences when obtains new information about possible gains and better understands the problem.

The decision support system is considered as a computer-based tool that allows the players to make an analysis of the conflict situation, taking into account their preferences among criteria. The analysis can be done using an interactive, learning procedure utilizing methods of multicriteria optimization. To construct such procedures, a development of the theory of noncooperative games an its generalization for the multicriteria case is required, that is, on the case where different objectives of the players are considered explicitly without the use of any given utility function.

Multicriteria noncooperative games have been formulated by Tzafestas [17] and Szidarovsky et al. [16]. The existence of equilibria in the games has been analyzed by Wang [19], Kruś and Bronisz [11]. Wierzbicki [21] developed concepts and a theory of multicriteria decision analysis in such games. Ideas of solution concepts in the games are developed by Fahem, Radjef [5], Nagy et al. [13], Voorneveld et al. [18].

Discussing the computer-based support we assume that a mathematical model describing the game is given. The model implemented in the system is used to calculate payoffs of players dependently on the strategies assumed. In this paper some theoretical results on n-person noncooperative multicriteria games described in strategic form (normal form) are presented. They relate to the definitions of the noncooperative equilibria and the theorems on the relations of the multicriteria game equilibria to the Nash equilibrium in the respective classical (unicriteria) game. On the basis of the theorems we can simplify the derivation of the multicriteria game equilibria taking into account players preferences. The discussion of decision support problems in the case of the multicriteria noncooperative games is presented. An algorithm supporting multicriteria analysis made by players is proposed. In the papers (Kruś and Bronisz [10], Kruś [6–9]) ideas of computer-based decision support in the case of the multicriteria bargaining problems is developed. These ideas are proposed to be applied in the case of the noncooperative games considered here.

2 Problem Formulation

We assume a given, finite set of the players $N = \{1, \ldots, n\}$.

Each player i has a set of feasible strategies $X(i)$ in a strategy space X^i. The set of feasible multistrategies $X(N)$ is the Cartesian product of the sets $X(i)$, for $i = 1, \ldots, n$, i.e.

$$X(N) = \prod_{i=1}^{n} X(i) \subset X^N = \prod_{i=1}^{n} X^i,$$

where X^N is the space of all multistrategies x.

Each player i has a gain function $g_i : X(N) \to I\!R^k$ associating with any multistrategy x a vector of real numbers representing values of criteria $g_i(x) = (g_{i1}(x), g_{i2}(x), \ldots, g_{ik}(x))$ measuring his gains. The multistrategy set $X(N)$ can be discrete or continuous. For simplicity of notation, without loss of generality, we assume that each player has the same number of criteria. Let $I\!R^{NK} = \prod_{i=1}^{n} I\!R^k$ denote the multigain space. The multigain operator is defined by

$$G : X(N) \to I\!R^{NK},$$

where $G(x) = (g_1(x), g_2(x), \ldots, g_n(x)) \in I\!R^{NK}$.

The operator defines multicriteria gains of all players for the strategies undertaken by all of them. The gains are elements of the multi-gain space which is the Cartesian product of the multicriteria spaces of the gains of particular players.

Definition 1. *An n-person noncooperative multicriteria game* $\{X(N), G\}$ *is described in the strategic form (normal form) by a multigain operator* G *mapping a multistrategy set* $X(N) \in X^N$ *into the multigain space* $I\!R^{NK}$.

In the classical case of the noncooperative game in strategic form, the gain of each player is described by a scalar function. In this paper, we assume that the gain of each player is described by a vector function defining values of the criteria for given decision strategies of all the players. We introduce the domination relation in vector spaces.

Definition 2. *For any space* $I\!R^m$ *and for any* $y, z \in I\!R^m$ *we say that a vector* y ***dominates*** *a vector* z *and write* $y > z$ *if* $y_i > z_i, y \neq z$ *for* $i = 1, 2, \ldots, m$. *We say that a vector* y ***strictly dominates*** *a vector* z *and write* $y \gg z$ *if* $y_i > z_i$ *for* $i = 1, 2, \ldots, m$.

For simplicity of notation, let $\bar{i} = N \backslash \{i\}$. From the point of view of player i, the set of all strategies X^N can be split into the set of strategies of the player i and the strategies of other players $\bar{i}$: $X^N = X^i \times X^{\bar{i}}$, where $X^{\bar{i}} = \prod_{j \neq i} X^j$.

If p^i and $p^{\bar{i}}$ denote the projections from X^N onto X^i and $X^{\bar{i}}$, we set $x = p^i x$ and $x^{\bar{i}} = p^{\bar{i}} x$.

Definition 3. *We say that a multistrategy* $x \in X(N)$ *is a* ***weak noncooperative equilibrium*** *in the n-person multicriteria game* $\{X(N), G\}$ *if for each player* $i \in N$, *there does not exist a multistrategy* $x' \in X(N), p^{\bar{i}} x' = x^{\bar{i}}$ *satisfying* $g_i(x') \gg g_i(x)$.

A multistrategy $x \in X(N)$ *is a* ***noncooperative equilibrium*** *in the n-person multicriteria game* $\{X(N), G\}$ *if for each player* $i \in N$, *there does not exist a multistrategy* $x' \in X(N), p^{\bar{i}} x' = x^{\bar{i}}$ *satisfying* $g_i(x') > g_i(x)$.

Remarks

A multistrategy is a weak equilibrium if no player i can obtain a higher gain for all his criteria (i.e. a gain better according to the strict domination relation), by making an alternative choice under the assumption that the remaining players make no change in their strategies.

A multistrategy is an equilibrium if no player can obtain a higher gain for some of his criteria, not decreasing his other criteria (i.e. a gain better according to the domination relation), by making an alternative choice under the assumption that the remaining players make no change in their strategies.

It is easy to show that if a multistrategy $x \in X(N)$ is a noncooperative equilibrium then it is also a weak noncooperative equilibrium.

In the unicriteria case, i.e. when $k = 1$, these definitions are equivalent and define the Nash equilibrium.

The measure of gain has only ordinal meaning (not cardinal), i.e. for a given criterion, a gain function compares two elements for ordering purposes. We do not also introduce explicitly any "weights of importance" or "priorities" of criteria aggregating them.

Theorem 1. *Suppose that the multistrategy set $X(N)$ is a convex, compact subset, and that for each player i, the gain function g_i is continuous and concave with regard to each coordinate, for all $x \in X(N)$. Then there exists a noncooperative equilibrium.*

The proof is given in (Kruś and Bronisz [11]).

Remarks

The theorem does not say anything about the uniqueness of equilibria. In many cases, there is a set of equilibria.

If we compare our game and a game formulated as a game of $n \times k$ players, i.e. in which each criterion of every player is treated as a "player" in a classical noncooperative game, then the sets of equilibria will be different.

3 Parameter Characterization of Efficient Outcomes of the Multicriteria Game

In multicriteria optimization problems, characterization of the set of in some sense efficient outcomes serves as a mathematical background for the construction of decision support systems enabling the decision maker to scan and analyze the efficient outcomes. Most of the characterizations utilize some substitute scalarizing function. The function typically depends on the objective function but also on additional parameters, for example weighting coefficients (Chankong and Haimes [4]), or levels of objective functions interpreted as reference aspiration levels (Wierzbicki [22,24]).

Using the decision support system, the decision maker can generate some number of efficient outcomes assuming values for the parameters and look for the outcome closest to his preferences. In an analogical way, the scalarizing function could be used in the case of a multicriteria noncooperative game. However, in the last case the problem is much more complicated. Each player has a different vector of objectives. The outcomes are dependent on the strategies of all the players. A question arises: can the scalarizing functions be used for a characterization of efficient outcomes of the game, but also for a characterization of the set of equilibria, or, more precisely, of the set of nondominated equilibria.

Ideas of a selection of game equilibria using the scalarizing function were proposed by Wierzbicki [20]. The scalarizing function can also be considered as a tool aggregating for each player his vector of criteria to unicriteria gain, depending on the selected parameter, and therefore to each multicriteria game we can assign a classical game in which each player has his gain defined by a scalar value.

In the following we assume that each player has in general his own parameter (vector of reference points based on aspiration levels), i.e. we assume that each player i can use a different vector of parameters $w^i = (w_1^i, \ldots, w_i^k)$. Let us consider a set W of such a parameters, $W \subseteq \mathbb{R}^k$. The simplest typical form of scalarization is made using weight coefficients with each player i who is assumed to have his own vector of weights $w^i = \lambda^i$. The scalar gain of each player is calculated as the sum of his weighted criteria. This way of scalarization is not proper in the general case as it does not satisfy the necessity condition formulated in [22]. Not all Pareto optimal points can be derived using this way of scalarization. In the following we consider the scalarization made with the use of reference points and a broader class of scalarizing functions having monotonicity properties (so called strictly and strongly monotone - compare Wierzbicki [22]).

Definition 4. *For any parameter $w \in W$, a scalarizing function $s : \mathbb{R}^k \times W \to \mathbb{R}$ is **strictly monotone** with respect to y if, for any y', $y'' \in \mathbb{R}^k$, y' strictly dominating y'' ($y' \gg y''$) implies $s(y', w) > s(y'', w)$.*

*The function s is **strongly monotone** with respect to y if, for any y', $y'' \in \mathbb{R}^k$, y' dominating y'' ($y' > y''$) implies $s(y', w) > s(y'', w)$.*

Let us consider multicriteria game $\{X(N), G\}$ and a class of associated classical games $\{X(N), G^w\}$ defined for a given player's parameters $w^i \in W$, $i = 1, 2, \ldots, n$, and for a scalarizing function $s(y, w)$ as follows:

$$G^w(x) = (g_1^w(x), \ldots, g_n^w(x)),$$

where $g_i^w(x) = s(g_i(x), w^i)$, with $g_i(x) = (g_{i1}(x), \ldots, g_{ik}(x))$.

In the associated classical game, the gain g_1^w of player i is defined as an aggregation of his multicriteria gains using a scalarizing function.

The scalarizing function depends on a parameter $w_i = (w_1^i, \ldots, w_k^i)$, where i is the number of the player, $i = 1, \ldots, n$. Using the parameter, the player i can express his preferences among his criteria.

The following theorems have been proved.

Theorem 2. *Let $x \in X(N)$ be a Nash equilibrium of a classical game $\{X(N), G^w\}$, i.e. for each player $i \in N$,*

$$g^w(x) = max\{g_i^w(x') : x' \in X(N), p^{\bar{i}}x' = x^{\bar{i}}\}.$$

*If the scalarizing function $s : \mathbb{R}^k \times W \to \mathbb{R}$ is **strongly monotone** with respect to y for any $w \in W$, then x is also a **noncooperative equilibrium of the multicriteria game** $\{X(N), G\}$.*

Theorem 3. *Let $x \in X(N)$ be a Nash equilibrium of a classical game $\{X(N), G^w\}$.*

If the scalarizing function $s : I\!R^k \times W \rightarrow I\!R$ is ***strictly monotone*** *with respect to y for any $w \in W$, then x is also* ***a weak noncooperative equilibrium of the multicriteria game*** *$\{X(N), G\}$.*

The scalarization of gain has only ordinal meaning (not cardinal), i.e. for a given player, the scalarization function compares two strategies for ordering purposes. If player i specifies his parameter w^i properly, the scalarization function should reflect his preferences, and the noncooperative equilibrium should be satisfying.

4 Decision Support in the Case of Multicriteria Noncooperative Games

New methodological problems related to the concepts and construction of decision support systems arise in the case of multicriteria noncooperative games.

By the decision support system we mean a tool that should aid the players in a selection of rational strategies. The system should support players in learning their situation in the game, showing equilibria outcomes, possible conflict escalation, as well as possible cooperation outcomes. The analysis of the game should be made in a multicriteria context i.e. it should allow each player to make a multicriteria analysis according to his preferences.

A simulation of the game is the simplest prototype aiding the players in making such an analysis. In this case the players assume some strategies and the system calculates their outcomes.

The analysis can be made much more effectively when a multicriteria optimization approach, in particular aspiration-led approach [22,23] is used. If we apply the aspiration-led approach in the case of the noncooperative game, each player can analyze the problem assuming his reference points (aspiration levels for his criteria) and assuming the reference points for the counter players. For given reference points assumed by a player i the system generates respective outcome which is Pareto optimal in the set of his attainable outcomes. The system generates the respective Pareto optimal outcome solving an optimization problem with use of so called achievement function.

Let the player i assume a reference point g^{*i} in his space of criteria $I\!R^h$ and assume multistrategies of other players.

The outcome representing the Pareto frontier in the case of the player, $i = 1, 2, \ldots, n$, can be derived solving the optimization problem:

$$\max_{x^i \in X^i} [s(g^i(x^i, x^{\bar{i}}), g^{*i}],$$

subject to given strategies of other players $x^{\bar{i}}$, where: $g^{*i} = (g_1^{*i}, \ldots, g_k^{*i})$ is a reference point assumed by the decision maker i in the space $I\!R^k$, $g^i(x)$ defines the vector of criteria of the i-th decision maker, $i = 1, \ldots, k$ which are dependent on the vector x of decision variables by the model relations, $s(g, g*)$ is an order approximating achievement function.

The following achievement function can be applied:

$$s(g^i(x^i), g^{*i}) = \min_{1\le j\le k, i=1,2\ldots,n}[a_j(g^i_j(x^i) - g^{*i}_j)] + a_{k+1}\sum_{j=1,\ldots,k} a_j(g^i_j(x^i) - g^{*i}_j),$$

where $g^{*i} \in I\!R^k$ is a reference point, a_j, $1 \le j \le k$, are scaling coefficients, and $a_{k+1} > 0$ is a relatively small number.

The following algorithm is proposed supporting analysis made by a given player $i \in N$. It supports multicriteria analysis of the payoffs and calculation of the best response strategies satisfying preferences of the player.

Step 1. Independent analysis made by a given player.
The system invites the player $i = 1, \ldots, n$ to make independently multicriteria analysis of their nondominated payoffs in the multicriteria noncooperative game.
Set the number of round $t = 1$.

Step 2. The player i assumes multistrategies of other players $x^{\bar{i}t}$.

Step 3. Multicriteria analysis of payoffs. Calculation of the best response strategy satisfying preferences of the player.

Step 3.1 The system presents to the player i information about the ideal point I^{it} in the player criteria space $I\!R^k$.
The ideal point is derived as $I^{ti} = (I^{ti}_1, I^{ti}_2, \ldots, I^{ti}_k)$,
where $I^{ti}_j = \max g^i_j(x)$ calculated with respect to x^i subject to $x \in X^N$.

Step 3.2 The player i writes values of the components of his reference point g^{*it}_j, $j = 1, 2, \ldots, k$.

Step 3.3 The system derives the Pareto optimal solution in the criteria space of the player i, maximizing the achievement function and stores the resulting payoff in a data base.

Step 3.4 The player analyzes the generated Pareto optimal payoff. He compares the payoff to other Pareto optimal payoffs stored in the data base, obtained for other reference points. He selects the preferred payoff.

Step 3.5 Has the player i finished multicriteria analysis?
If no - go to Step 3.2, to generate next Pareto optimal payoff.
If yes - system writes in the data base the preferred Pareto optimal payoff indicated by the player $\widehat{g}^{it}$ as well as the optimal response strategies $\widehat{x}^{it}$ for the given strategies $x^{\bar{i}t}$ of other players.

Step 4. The system checks whether the player have finished his analysis for all assumed multistrategies of other players.
If no - set the round number $t = t + 1$ and go to the Step 2 to make analysis for another multistrategy of other players.
If yes - there is in the data base a set of the best response strategies of the player and the respective preferred Pareto optimal outcomes.
End of the procedure.

The system may include a part calculating the equilibrium strategies in the multicriteria game. Let each player $i = 1, \ldots, n$ assume his selected Pareto optimal payoff $\widehat{g}^i$. The presented achievement function can be used as the scalarizing function considered in the Sect. 3 with parameters $w^i = g^i = \widehat{g}^i$. This achievement function is a strongly monotone with respect to the obtained gains. A classical game $\{X(N), G^w\}$ can be constructed for the parameters, and an equilibrium in the game can be derived.

Remarks

According to the Theorem 2 the derived equilibrium of the classical game is also the equilibrium of the considered multicriteria game. That means the equilibrium strategies derived for the classical game define also the equilibrium in the multicriteria game. Let us see that the gains $g^i = \widehat{g}^i$ have been selected according to the preferences of the players. The derived equilibrium in the classical as well as in the multicriteria game expresses the preferences of the player.

In the multicriteria games we deal in general with a set of different equilibria outcomes. It can be said that a multicriteria formulation of decision making problems of players typically leads to nonunique equilibria.

In general, the equilibria can be not Pareto optimal. Therefore, there exist cooperative strategies of the players, such that the players can improve their outcomes in comparison to the case of equilibrium strategies. On the other hand, the nonunique equilibria may lead to a conflict escalation (see Wierzbicki [20]). The escalation can take place when each of the players tries to apply an equilibrium strategy but related to different equilibria points. With regard to the decision support problems, we follow the argument proposed by Wierzbicki [20].

The calculation of equilibrium strategies can lead to some computational problems. Solving the problems we deal with two-level optimization procedures in which a nondifferential objective function is maximized on the second level. Further research in this direction is required.

The decision support system should demonstrate to the players the advantage of possible cooperative strategies. Next we face the problem of how to lead the players into a cooperative outcome, being Pareto optimal in the set of attainable outcomes. In this case another decision support mechanism can be applied - an interactive mediation procedure.

In the mediation we consider the following problem: there is given an disagreement point (it can be assumed as an equilibrium point or as a status quo point) and a set of attainable outcomes in the multicriteria space of all the players. The problem consists in aiding the players in finding a mutually beneficial, unanimously accepted outcome in the set of attainable outcomes. The final outcomes should be selected according to the preferences of each of the players. The mediation procedure can be made according to the rules of the multicriteria bargaining support developed in the papers by Kruś and Bronisz [10], Kruś [6–8]. That approach utilizes the new results of multicriteria bargaining problems and the interactive multicriteria, aspiration-led approach. Initial practical experience of such a support has been obtained when the experimental computer-based system (Kruś et al. [12]) was constructed.

5 Conclusions

In this paper we discuss the problem of decision analysis and support in noncooperative games in a multicriteria context. A theoretical research on the multicriteria games is still required to make a background for construction of decision support systems in the games. In this paper several results in the subject are presented. These include an analysis of equilibria in n-person, multicriteria noncooperative games. In particular, the new theorems describing relations of the equilibria in a multicriteria game to the Nash equilibria in the classical (unicriteria) game are proposed. The classical game is defined by parametrization of the multicriteria game. An algorithm supporting multicriteria analysis of payoffs in the game is proposed. Using the algorithm each player can make the multicriteria analysis and derive the best response strategies satisfying his preferences.

The decision support system in this case is considered as a tool supporting the players: first, in the analysis of the game, and second – aiding selection of a mutually acceptable, cooperative, Pareto optimal outcome. The second case indicates a direction of further research. Application of the aspiration-led approach of multicriteria optimization, multicriteria bargaining, interactive mediation procedures seems to be useful in constructing such systems.

Let the presented interactive algorithm be applied by each player in the noncooperative multicriteria game. According to the algorithm each player makes multicriteria analysis proposing reference points in the space of his criteria and comparing respective Pareto optimal outcomes derived by the computer-based system in a sequence of steps. Each player can select the final reference point closely to his preferences in his criteria space. We may construct the associated classical scalar game using the achievement function and the reference points selected by players. The achievement function applied in the algorithm is strongly monotone with respect to the vector of criteria. According to the Theorem 2 the decision variables for which the noncooperative equilibrium is obtained in this scalar game, define also the noncooperative equilibrium in the multicriteria game. The equilibrium represents preferences of players.

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Multiple Criteria Decision Making and Multiobjective Optimization - A Toolbox

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Abstract. We present an integral approach to solving multiple criteria decision problems in sequences of intelligence, modeling, choice and review phases, often iterated, to identify the most preferred decision variant. The approach taken is human-centric, with the user taking the final decision being a sole and sovereign actor in the decision making process. To ensure generality, no assumption about the Decision Maker preferences or behavior is made. Likewise, no specific assumption about the underlying formal model is made.

The intended goal of the approach is to lower the cognitive barrier related to unsupported use of multicriteria methodologies in day-to-day practice. We present successful application of this approach to a number of practical problems.

Keywords: Multiple objective programming · Multiple criteria decision making

1 Introduction

This paper presents the Authors' experience with bringing laymen people to Multiple Criteria Decision Making (MCDM) and explaining to them its merits and tools. They have been individuals of any sort: operations research, analytics or management science specialists, engineers, politicians, business and industrial practitioners. Last but not least, they have been students of graduate and PhD levels.

We have been constantly confronted with needs to find words and notions to convey principles, strengths and importance of the discipline without explicit referring one to its vast literature. The MCDM literature is rich, but our experience is that for a layman there is a discouragingly high cognitive entering barrier to study it. As MCDM offers no standard approach, and on the contrary, it offers a variety of approaches and methods, this aggravates the problem where to start and with what.

Over years we have attempted, and eventually, we claim it here, we have succeeded, to abstract from the richness of methodological developments in MCDM

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a limited set of notions and a rudimentary methodology which serves the purpose without compromising on formal correctness. We can say that we offer a simple language to communicate the mission and rationale of the discipline to the surrounding world.

In this paper, we engage ourselves in presenting and defending our findings. We also lists examples where those findings have been successfully applied.

2 The General Framework

When talking about MCDM problems, whether to specialists or non-specialists, we always attempt to adopt the general framework by Herbert Simon (Simon 1977). He convincingly argued that any decision process is composed of four phases: *intelligence*, *design*, *choice*, *overview*, with the feedback learning loop from the fourth to the first phase, repeated as many times as the Decision Maker (DM) judges it is worth the effort. The loop is not a static one, each pass of the loop brings the DM into a new state of command over the problem. Hence, the loop is dynamic, and as such it represents a hermeneutic helix (cf. e.g. Wierzbicki, Nakamori 2006, Kaliszewski 2012, Kaliszewski et al. 2016). This means that, in the MCDM parlance, we take an interactive MCDM perspective (Miettinen 1999, Ehrgott 2005, Kaliszewski 2006a) – problems are solved iteratively with the DM in the role of the sovereign of the decision process.

Below we shall concentrate ourselves on the second and the third phase of the process – the design phase and the choice phase. In the design phase, the problem under consideration is shaped into a model. In the choice phase, a problem solution is selected to be scrutinized in the fourth phase as a candidate for the ultimate solution. In the first phase, the problem is formulated and reformulated in general and usually soft terms. The second and the third phase of the decision process are rather formal and therefore cognitively more complex than the other two (however, any phase is as important as another), and therefore can be the "hard nut" for prospective DM, unfamiliar with MCDM methodological and technical issues.

3 The Basic Vocabulary

Below we argue that to convey to a broad audience the meaning, significance, and usefulness of MCDA, it is enough to introduce the notion of *the underlying MCDA model*, the notion of *efficiency*, and just three more notions: *utopia point*, *vector of concessions* and *compromise half line*.

The underlying MCDA model consists of variant set X_0, $X_0 \subseteq \mathcal{R}^n$, and k objective functions $f_l(x)$, $l = 1, ..., k$, $k \geq 2$. We assume that all objective functions are of the "the more, the better" type. The underlying MCDA model has the following form

$$\begin{array}{c} vmax \ f(x) \\ x \in X_0 \ , \end{array} \tag{1}$$

where *vmax* denotes the operator of deriving all efficient variants (as defined below). The derivation of all efficient variants should not be understood literally, but rather as the potential ability to derive any efficient variant, (cf. Miettinen 1999, Ehrgott 2005, Kaliszewski 2006a).

Element $\bar{y}$ of $f(X_0)$ ($f(X_0) \subseteq \mathcal{R}^k$) is *efficient* if $y_l \geq \bar{y}_l$, $l = 1, ..., k$, $y \in f(X_0)$, implies $y = \bar{y}$.

Variant $\bar{x} \in X_0$ is called *efficient* if $f(\bar{x})$ is efficient. Elements of $f(X_0)$ are called *outcomes*.

An alternative way to define efficient variants, present widely in the MCDA literature, is via the notion of the dominance relation.

We denote the set of efficient variants in X_0 by N. Elements of $f(N)$ are called efficient outcomes, because they are efficient in $f(X_0)$ by the definition. Nowadays, $f(N)$ is increasingly often called the *Pareto Front*.

Below we make use of a selected element of $\mathcal{R}^k$, denoted by y^* (*utopia point*), defined as

$$y_l^* = \hat{y}_l + \delta, \ l = 1, ..., k, \tag{2}$$

where $\hat{y}$ (*ideal point*) is calculated as

$$\hat{y}_l = \max_{x \in X_0} \ f_l(x), \ l = 1, ..., k, \tag{3}$$

(or $\hat{y}_i$ is an upper bound, preferably supremum, on $f_l(x)$ over X_0, if for some l the maximum does not exist), and δ is any positive number. By the definition,

$$y_l^* > f_i(x), \ l = 1, ..., k, \text{ for all } \ x \in X_0 \, . \tag{4}$$

Actually, the existence of element y^*, which in practical problems is always guaranteed, is the ONLY assumption on the form of the underlying model (1), under which all considerations presented below are valid.

4 The Language

Element $\hat{y}$ is a natural *anchor point* (Tversky, Kahneman 1974) for the DM to refer (compare) to it any $f(x)$, $x \in X_0$.

Once $\hat{y}$, representing the best values which can be attained by each criterion independently over X_0, is calculated, the DM should be informed whether $\hat{y}$ is attainable, i.e., whether there exists $\hat{x} \in X_0$ such that $f(\hat{x}) = \hat{y}$. If yes, the decision making process terminates with $\hat{x}$ as the most preferred variant. Otherwise, to secure efficient variants x, $x \in X_0$, the DM has *to compromise* on $\hat{y}$ and he/she can do this by selecting a *direction of concessions*

$$\tau, \quad \tau_l > 0, \quad l = 1, ..., k,$$

(Kaliszewski 2006b, Kaliszewski 2012). Vector τ is here the preference carrier; it defines proportions in which the DM agrees to sacrifice unattainable values of criteria represented by $\hat{y}$ in a quest for an efficient outcome (and the corresponding efficient variants). Each direction of concessions τ defines the corresponding *compromise half line* Θ (Kaliszewski 2006b, Kaliszewski 2012), where

$$\Theta = \{y \in \mathcal{R}^k \mid y = \hat{y} - t\tau,\ t \geq 0\}.$$

Vector τ can be defined by the DM explicitly, in the atomistic manner, by indicating its components $\tau_l,\ l = 1, ..., k$, or implicitly, in the holistic manner, by indicating a base point y, i.e., an element from set $\hat{y} - R^k_+$ which defines τ as $\tau = \hat{y} - y$, where R^k_+ is the positive orthant of $\mathcal{R}^k$. The latter is fully analogous to reference point MCDA methods (Wierzbicki 1999, cf. also Miettinen 1999, Ehrgott 2005, Kaliszewski 2006a). Indeed, a pair $(\hat{y}, \text{base point})$ corresponds to an (*aspiration point*, *reservation point*) pair in the reference point method proposed by Wierzbicki (1999), with the only difference that here we fix the aspiration point to $\hat{y}$ to ensure that any efficient outcome can be derived, as explained below.

In what follows, in order to avoid null values of fraction denominators (see formula (6) below), we replace $\hat{y}$ by y^* defined in the previous section, with δ sufficiently small to preserve the interpretation of $\hat{y}$ and y^* in terms of compromises and concessions.

The direction of concessions is thus a simple and intuitive formal construct to represent DM's preferences in a uniform way, irrespective of how they are revealed: in atomistic or holistic manner.

Vectors τ defining compromise half lines are related to weights λ of the *Chebyshev augmented scalarizing function*

$$\max_l \lambda_l(y^*_l - f_l(x)) + \rho e^k(y^* - f(x)), \tag{5}$$

where ρ is a small positive number and $e^k = (1, ..., 1)$ has k elements, by the formula

$$\lambda_i = \frac{1}{\tau_l},\ l = 1, ..., k. \tag{6}$$

(see Kaliszewski 2006a,b, Kaliszewski 2012).

This relationship means that

– if the compromise half line intercepts the set of efficient outcomes at some y, then solving optimization problem

$$\min_{x \in X_0} \max_l \lambda_l(y^*_l - f_l(x)) + \rho e^k(y^* - f(x)), \tag{7}$$

with $\lambda_l,\ l = 1, ..., k$, as in (6), yields efficient x such that $y = f(x)$,
– otherwise, solving optimization problem (7) with $\lambda_l,\ l = 1, ..., k$, as in (6), yields efficient x such that $f(x)$ is the closest (in a sense) element of $f(X_0)$ to y^*,
– in any case the derived variant is efficient.

If so, preferences expressed and represented as directions of concessions can be straightforwardly translated by formula (6) to weights of the Chebyshev augmented scalarizing function (5), enabling the DM to perform preference driven searches for efficient outcomes (and the corresponding efficient variants), as illustrated in the next section.

It is well known that the optimization problem (7) derives only efficient variants. Moreover, the optimization problem (7) has the ability to derive any efficient variant if $\rho \rightarrow 0$. In other words, by using this optimization problem to derive efficient variants, no efficient variant is a priori excluded from consideration (as it can happen in the case of the weighted linear function, cf. Miettinen 1999, Ehrgott 2005, Kaliszewski 2006a).

One may argue that providing the DM with vector τ as a tool to express his/her preferences is a minor contribution to MCDA. In fact, there is little *decision making support* in that. However, this is the price for NO assumptions adopted, neither on the problem model (1) nature (except the assumption about the existence of $\hat{y}$, which in practice is not an issue at all), nor on the DM preference profile or behavior.

The framing outlined above amounts to a form of "what–if" analysis. There are three firm facts about the above MCDA framing:

- it is general – no assumptions are made,
- is practical and versatile – no assumptions are to be verified,
- it is simple – if not explicitly requested, no technicalities need to be presented to the DM, the only "tool" the DM is to deal with are the vectors of concessions.

5 Applications

As there are many competitive methodologies present in MCDM for decades, the framing outlined in the previous section permeates to the community slowly. Below we list a few success stories.

Education

The first Author teaches a course "Computer Aided Multiple Criteria Decision Making" to business management graduate students with no or very little training in basic OR. The course consists of 16 h of lectures, 16 h of individual exercises in a computer laboratory, and an individual project.

In the very tight time schedule it is possible to convey to students the formal basics as presented in Sect. 4. The meaning, significance and working of those notions are clarified with the help of problems solved in the laboratory with *Excel* sheets and optimization add-in *Solver*.

After a short period of hesitation and reluctance, students quickly absorb the basic notions, get a grip on the material and cope successfully with increasing problem complexity. A good command of the whole methodology is reflected in some original projects (business decision cases) which students are to submit to complete the course (cf. e.g. Jastrzebski, Kaliszewski 2011).

The content of the course is closely related to the book by Kaliszewski, Miroforidis, Podkopaev, *Multiple Criteria Decision Making and Multiobjective Optimization - A Toolbox* (Kaliszewski et al. 2016).

Engineering

In Kaliszewski et al. 2016, we report solving two mechanical problems with the aim to attract the engineering community. The first problem solved was a simple, textbook-type problem of round tube beam selection. The solution sought was the most preferred compromise between mass and deflection of a round beam under a given concentrated stress force. The second problem was to find the most preferred compromise between maximal piston velocity and efficiency of a pneumatic high speed machine drive with built-in air reservoir. Both problems were solved for real data and the second problem, highly nonlinear, has been far from trivial. We also solved a three dimensional version of the latter problem where the maximal speed criterion was decomposed into the maximal speed criterion and so called velocity index. It was a mechanical engineer to decide which compromises too search for with the help of vector of concessions. It was also for him/her to decide which of the compromises obtained to regard as the most preferred one.

The decision process was organized as follows. Two teams: the DM and the facilitator (F) were located in different cities and communicated via e-mail. The DM, informed by the F about element y^*, sent vector τ and evaluated the compromise. On the base of his/he evaluations he/she selected another vector of concessions and the procedure repeated till the DM wanted to continue no more. For both problems the DM was able to locate equally good or better compromises than those derived by other approaches well established in mechanical engineering.

Air transport

We applied our approach to the Airport Gate Assignment Problem (AGAP) (Kaliszewski, Miroforidis 2012, Kaliszewski et al. 2013). We formulated a generic problem with two criteria and many simplifications with respect to modeling real airport operations. The model framed only ground side operations. However, it was representative for the core logic of assigning flights (planes) to gates to compromise between total waiting time for gate assignment to unload passengers at gates and total number of passenger unloads on the apron (which involves no waiting extra). To make the problem meaningful, the capacity of the airport was assumed to be to small to serve all flights at gates at specific time windows.

We compared solutions obtained using specialized Evolutionary Multiobjective Optimization algorithm with solutions obtained with CPLEX commercial solver (bi-criteria problems were transformed to linear ones by a simple transformation). The primary goal of that work was to compare the behavior of those two essentially different optimization principles as the sizes of problem instances grew. However, in both cases we were able to carry the DM preferences to optimization engines by vectors of concessions.

Radiotherapy planning

In Kaliszewski, Miroforidis 2015, we report on application of our approach to radiotherapy planning. In the MCDM setting, the radiotherapy planning consists in finding a patient radiation scheme which is the “best” compromise between radiating tumor (to kill the malicious tissue) and sparring the healthy organs.

To deal with nonlinear conditions in the resulting large-scale problems, we used an Evolutionary Multiobjective Optimization approach. This time the DM was a radiotherapy planner whose role was to evaluate a number of radiation plans we provided against the plan (one plan is produced, as a rule) produced by a planning system currently in use in an oncology clinic.

We have solved a number of test problems extracted from anonymized clinical data. The largest problem solved (Head & Neck tumor case) had 3 radiation beams composed of 8064 individual beamlets (problem variables) and 181292 voxels (small body cubes for which individual radiation constraints are set). For that problem, we were able to derive 76 elements of the Pareto Front in 10 min with a desktop, of-the-shelf computer. For radiotherapy plans to have clinical value they have to be prepared with at least 5 radiation beams, therefore the work is still in progress. But the preference passing standard, accepted by planners, will be in all cases in the form of vectors of concessions.

6 Concluding Remarks

Every practical decision problem is conditioned by multiple aspects. Therefore, for each practical decision problem its formal model is the multiobjective optimization problem in which selected aspects become objectives.

In this work, we have presented an exhaustive set of notions and formal constructs which allow solving of any multi aspect problem. This set is also minimal. We have also shown how to express information necessary to guide the process of variant selection in the natural language, without resorting to formal notions. We regard the possibility of translating the natural notions guiding variant selection processes into a formal ones, a step necessary for solving multiobjective optimization problems, as the main contribution of the work.

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Dominance of Binary Operations on Posets

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Abstract. This contribution deals with a dominance property of binary algebraic operations on a partially ordered set. Such dominance is strictly connected with a generalized distributivity of the operations. Consequences of bisymmetry assumption and the existence of neutral element for the operations are presented. Some results known for operations on $[0, 1]$ are generalized to the case of a partially ordered set or a lattice.

Keywords: Partially ordered set · Dominance property · Commuting property · Triangular norms · Uninorms · Aggregation functions

1 Introduction

Diverse types of dominance properties, considered in linear programming, game theory, semigroup theory or graph theory, are based on a direct inequality or inclusion. The dominance property that we consider is a binary relation between algebraic operations. This notion of dominance was introduced by R.M. Tardiff [31] (in the case of triangle functions) and generalized by B. Schweizer and A. Sklar [28] to the class of associative binary operations F, G on a common domain D (and with a common neutral element). The dominance of binary operations is a generalization of the commuting property which was widely studied in [23].

A standard example of the dominance on $D = [0, \infty)$ gives the Minkowski inequality (cf. [8], p. 147)

$$((a+b)^p + (c+d)^p)^{1/p} \leqslant (a^p + c^p)^{1/p} + (b^p + d^p)^{1/p} \text{ for } a, b, c, d \geqslant 0, \ p \geqslant 1, \quad (1)$$

which means that addition dominates operations G_p for $p \geqslant 1$, where $G_p(a, b) = (a^p + b^p)^{1/p}$ on $D = [0, \infty)$. Moreover, G_p dominates addition for $p < 1$, $p \neq 0$. It is worth mentioning that there are also other versions of the dominance property, for example a weak dominance which was considered in [3]. The first examples of dominance in the form used in this paper were described for triangular norms. They have applications for example in construction of product of probabilistic metric spaces (cf. [28]), discussion of products of fuzzy groups (cf. [29]), examination of aggregations preserving transitivity, semi-transitivity or Ferrers property

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of aggregated fuzzy relations (cf. [6,7,10,22]), and considerations on aggregation functions preserving convexity and concavity of functions [5]. H. Sherwood [30] described a family of triangular norms ordered by the dominance. C. Alsina and S. Thomás [2] considered the dominance of triangular norms over quasi–arithmetic means. Conversely, R. Mesiar and S. Saminger [19] dealt with the dominance of weighted averages over triangular norms. In [22] it was introduced a relation of dominance for arbitrary aggregation operators. For example in [21,24] relation of dominance was considered on posets and lattices for aggregation functions and in [4,11,12] on posets and lattices for arbitrary operations.

In our paper, to simplify notations we concentrate only on binary operations, however the results may be transformed into n-argument case. We present algebraic properties of the dominance on a poset, sometimes when needed with wider assumptions on a lattice. We consider consequences of bisymmetry property and existence of a neutral element for the dominance property. We also characterize the reflexivity and describe the antisymmetry property of dominance. Furthermore, we study connections of the dominance with diverse types of distributivity conditions. Some of the presented results are generalizations of the ones from aggregation functions to arbitrary operations.

The structure of the paper is as follows. Firstly, we give basic notions and examples of the dominance (Sect. 2). In Sect. 3 properties of the dominance in the case of bisymmetric operations and operations with a neutral element are presented. Next, in Sect. 4 some results known for operations on $[0,1]$ are generalized to the case of a partially ordered set or a lattice. Then, we consider connections between the dominance and distributivity conditions (Sect. 5).

2 Preliminaries

The minimal requirement in a description of the property of dominance is an order relation. Thus, if not otherwise stated, we assume that $P = (P, \leqslant)$ is a partially ordered set. Firstly, we recall some useful notions.

Definition 1. *An operation $F : P^2 \to P$ is bisymmetric if*

$$F(F(x,y), F(u,v)) = F(F(x,u), F(y,v)) \quad \text{for all } x, y, u, v \in P. \tag{2}$$

Definition 2 (cf. [1], p. 318). *Let $F, G : P^2 \to P$. An operation F is said to be subdistributive with respect to G if both inequalities hold for any $x, y, u \in P$:*

$$F(x, G(y,u)) \leqslant G(F(x,y), F(x,u)), \tag{3}$$

$$F(G(y,u), x) \leqslant G(F(y,x), F(u,x)). \tag{4}$$

An operation F is superdistributive with respect to G if both inequalities hold for any $x, y, u \in P$:

$$F(x, G(y,u)) \geqslant G(F(x,y), F(x,u)), \tag{5}$$

$$F(G(y,u), x) \geqslant G(F(y,x), F(u,x)). \tag{6}$$

Definition 3. *An operation $F : P^2 \to P$ is called:*

- *idempotent if $F(x,x) = x$ for any $x \in P$,*
- *subidempotent if $F(x,x) \leqslant x$ for any $x \in P$,*
- *superidempotent if $F(x,x) \geqslant x$ for any $x \in P$.*

Definition 4 (cf. [28], Definition 12.7.2). *Let $F, G : P^2 \to P$. The operation F dominates the operation G ($F \gg G$) if*

$$F(G(x,y), G(u,v)) \geqslant G(F(x,u), F(y,v)) \quad for \quad x,y,u,v \in P. \tag{7}$$

As a simple example of the dominance we may consider projections. Any projection dominates arbitrary operation $F : P^2 \to P$ and conversely. It might also be noted that the dominance for the case of n-argument aggregation functions (cf. [9]) was considered in [13, Definition 2.5]. Let us notice, that the dominance is a generalization of the commuting property.

Definition 5 ([23]). *Let $F, G : P^2 \to P$. F commutes with G if*

$$F(G(x,y), G(u,v)) = G(F(x,u), F(y,v)) \; for \; x,y,u,v \in P. \tag{8}$$

Commuting operations dominate each other. Any projection commutes with an arbitrary operation $F : P^2 \to P$. Let $x, y \in P$. If $F \equiv x, G \equiv y$ (constant operations), then $F \gg G \Leftrightarrow x \geqslant y$. Here the dominance appears only in the case of comparability. Several examples of the dominance in special cases of ordered sets are presented in [12,14,25]. Some of them are the well-known functional equations and inequalities. Moreover, there exist families of binary operations ordered by the dominance. Let us observe that the dominance relation may be a linear order in certain families of operations.

Some of the operations need to be defined on ordered structures with bounds. Such a structure with minimal assumptions is a bounded poset. However, to obtain more examples of such operations it is more useful to consider a bounded lattice $(L, \vee, \wedge, \mathbf{1}, \mathbf{0})$, where the symbols $\mathbf{1}$, $\mathbf{0}$ denote upper and lower boundary elements, respectively (for short we will write just L).

Definition 6. *Let L be a bounded lattice. An operation $F : L^2 \to L$ which is isotone with respect to both variables is called a conjunction (disjunction) if it fulfils*

$$F(\mathbf{0},\mathbf{0}) = F(\mathbf{0},\mathbf{1}) = F(\mathbf{1},\mathbf{0}) = \mathbf{0}, F(\mathbf{1},\mathbf{1}) = \mathbf{1},$$
$$(F(\mathbf{0},\mathbf{0}) = \mathbf{0}, F(\mathbf{0},\mathbf{1}) = F(\mathbf{1},\mathbf{0}) = F(\mathbf{1},\mathbf{1}) = \mathbf{1}).$$

If a conjunction (disjunction) has a neutral element $\mathbf{1}$ ($\mathbf{0}$), then it is called a triangular seminorm (semiconorm). A triangular seminorm (semiconorm) is called a triangular norm (triangular conorm) if it is associative and commutative.

Some of classical examples of triangular norms (T) and conorms (S) have their extensions on a bounded lattice L in the following form: $T_M = \wedge$, $S_M = \vee$,

$$T_D(x,y) = \begin{cases} x, & y = \mathbf{1} \\ y, & x = \mathbf{1} \\ \mathbf{0}, & otherwise \end{cases}, \quad S_D(x,y) = \begin{cases} x, & y = \mathbf{0} \\ y, & x = \mathbf{0} \\ \mathbf{1}, & otherwise \end{cases}, \quad x, y \in L.$$

Definition 7 (cf. [15,17,20]). *Let L be a bounded lattice. An aggregation operator $A : L^2 \to L$ is an operation which is isotone and has idempotent elements $\mathbf{0}$ and $\mathbf{1}$. An aggregation operator $U : L^2 \to L$ is called a uninorm if it is commutative, associative and has a neutral element $e \in L$. An aggregation operator $V : L^2 \to L$ is called a nullnorm if it is commutative, associative and there is an element $z \in L$ such that $V(x, \mathbf{0}) = x$ for any $x \in [\mathbf{0}, z]$ and $V(x, \mathbf{1}) = x$ for any $x \in [z, \mathbf{1}]$.*

Corollary 1 ([15]). *A nullnorm V has a zero element z.*

Note that conjunctions and disjunctions (triangular seminorms and semiconorms) are aggregation functions. Triangular norms and conorms are uninorms. Let L be a bounded lattice, $z \in L$ and T be a triangular norm on $[\mathbf{0}, z]$, S triangular conorm on $[z, \mathbf{1}]$. Then

$$V(x,y) = \begin{cases} T(x,y) & x,y \in [\mathbf{0},z] \\ S(x,y) & x,y \in [z,\mathbf{1}] \\ z, & otherwise \end{cases}, \quad U(x,y) = \begin{cases} T(x,y) & x,y \in [\mathbf{0},z] \\ y & x \in [\mathbf{0},z],\ y \parallel z \\ x & y \in [\mathbf{0},z],\ x \parallel z \\ x \vee y & otherwise \end{cases},$$

are respectively a nullnorm with zero element z and a uninorm with neutral element z on L.

3 Properties of Dominance Relation

It was mentioned in [22] that T_M dominates arbitrary aggregation function on $L = [0,1]$. This can be generalized for arbitrary infimum in the case when an operation is defined on a lattice $(L, \vee, \wedge)$.

Theorem 1 ([12]). *If an operation $F : L^2 \to L$ is isotone on a lattice $(L, \vee, \wedge)$, then*

$$\wedge \gg F \gg \vee. \tag{9}$$

Apart from supremum and infimum also aggregation functions, including uninorms and nullnorms, are isotone operations.

Corollary 2. *Let L be a bounded lattice. For any aggregation function A (e.g. conjunction or disjunction), uninorm U and nullnorm V we have*

$$\wedge \gg A \gg \vee,\ \wedge \gg U \gg \vee,\ \wedge \gg V \gg \vee. \tag{10}$$

In particular, for any triangular norm T and triangular conorm S we have $T_M \gg T$, $T_M \gg S$, $T \gg S_M$, $S \gg S_M$.

The dominance is a binary relation which is reflexive for example in a family of projections. It follows from the fact that any projection dominates itself. We will look for the greatest family of operations with the reflexivity of dominance.

Theorem 2 ([12], cf. [23]). *Let $F : P^2 \to P$. $F \gg F$ if and only if F is bisymmetric.*

The above criterion means equivalence between selfdominance and bisymmetry and allows us to recognize the domain of the reflexivity of dominance relation. This relation is reflexive, e.g. in the family of weighted means where $P = [0, 1]$. Conversely, non-bisymmetric operations are counter-examples for the reflexivity of dominance (cf. [12]).

Lemma 1. *If $F : P^2 \to P$ is associative and commutative, then it is bisymmetric.*

Let us observe that the assumptions of above lemma are not necessary. Any projection is bisymmetric, but it is not commutative. Similarly, the arithmetic mean (for $P = [0, 1]$) is bisymmetric, but it is not associative. Finally, the bisymmetric weighted means (for $P = [0, 1]$) are non-associative and the majority of them are non-commutative. As a result of Lemma 1 we get

Corollary 3 ([12]). *If $F : P^2 \to P$ is associative and commutative, then $F \gg F$.*

The converse is true with the additional assumption

Corollary 4 ([21]). *Let $F : P^2 \to P$ have a neutral element. Then, $F \gg F$ if and only if F is associative and commutative.*

Corollary 5 ([12]). *The relation of dominance is reflexive in the families of uninorms and nullnorms.*

The above results can be generalized (thanks to the associativity of uninorms and nullnorms) to the case of their n-argument versions (cf. [12,24]).

A dependence between the dominance and order was first observed for triangular norms (cf. [28], p. 209). It is connected with a common neutral element of these operations. In a more general setting we have

Lemma 2. *Let $F, G : P^2 \to P$ have neutral elements $f, g \in P$, respectively. If $F \gg G$, then $f \geqslant g$.*

The converse is not true, since for example triangular norms have the same neutral element, but for example T_D does not dominate T_M.

Corollary 6. *Let $F, G : P^2 \to P$ have neutral elements $f, g \in P$, respectively. If $f < g$ or $f||g$, then F does not dominate G.*

Since triangular conorms have a neutral element $f = \mathbf{0}$, and triangular norms have a neutral element $g = \mathbf{1}$, then as a consequence none of triangular conorms dominates a triangular norm [12]. We can consider similar consequences of the above theorem in the family of all uninorms. As a partial generalization of Corollary 6 we obtain

Theorem 3 (cf. [28], p. 209). *Let $F, G : P^2 \to P$ have a common neutral element $e \in P$. If $F \gg G$, then $F \geqslant G$.*

The converse to the above statement is not true. In the lattice $L = [0,1]$, additively generated strict triangular norms T_f and T_g (a strict triangular norm is a strictly increasing and continuous triangular norm on $(0,1]$, cf. [18], p. 28) are comparable $T_f \geqslant T_g$, where $f(x) = \frac{1}{x} - 1$ on $(0,1]$, $g(x) = \frac{1}{2x}$ on $(0, \frac{1}{2}]$, $g(x) = f(x)$ on $(\frac{1}{2}, 1]$, but there is no dominance $T_f \gg T_g$ (cf. [32], Example 1).

As a consequence of Theorem 3, if two operations with a common neutral element are not comparable, then we have lack of dominance. We have even more general property.

Corollary 7 ([12]). *The dominance relation is antisymmetric in the family of all binary operations with a common neutral element in P. In particular, the dominance is antisymmetric in the family of all uninorms.*

We can consider similar consequences of the above theorem in the family of triangular norms (conorms). However, the converse result is not true (cf. [28], p. 210). The dominance (in general) is not transitive relation. It was shown that the dominance is not transitive for associative operations on a finite domain having a common neutral element (cf. [3]). It was also shown in [27] that the dominance is not transitive in the family of continuous triangular norms. As a result it is also not transitive in the family of uninorms, so the dominance being reflexive and antisymmetric in the family of uninorms, is not a partial order in this family.

4 Dominance in the Class of Conjunctions and Disjunctions

In this section we describe the class of conjunctions which dominate each triangular conorm. Similar considerations for the lattice $L = [0,1]$ where presented in [13,16].

Theorem 4 ([26]). *Let $L = [0,1]$. A triangular seminorm C dominates the class of all triangular conorms if and only if*

$$C(x,y) \in \{0, x, y\} \quad \textit{for any } x, y \in [0,1]. \tag{11}$$

Analogous statement is not true if we take non-linear ordered lattice with two non-comparable elements x, y such that $x \wedge y > \mathbf{0}$. As examples we can take $C = \wedge$. In this case $x \wedge y \notin \{\mathbf{0}, x, y\}$.

If we put in Theorem 4 an arbitrary conjunction instead of a triangular seminorm then such theorem does not hold true. To show it we put the counterexample in $L = [0,1]$. The operation given by formula

$$C(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [\frac{1}{8}, 1]^2 \\ \min(x,y), & \text{otherwise} \end{cases}$$

is a commutative and associative conjunction with the neutral element $\frac{1}{8}$ such that $C(x, y) \in \{x, y\} \subset \{0, x, y\}$ for any $x, y \in [0, 1]$. It does not dominate the t-conorm S_P, where $S_P(x, y) = x+y-xy$, $x, y \in [0, 1]$. Indeed, for $x = 0.3$, $y = 0.4$, $u = 0.6$, $v = 0.2$ we have $C(S_P(x, y), S_P(u, v)) = C(S_P(0.3, 0.4), S_P(0.6, 0.2)) = C(0.58, 0.68) = 0.68$. On the other hand we have

$$S_P(C(x, u), C(y, v)) = S_P(C(0.3, 0.6), C(0.4, 0.2)) = S_P(0.6, 0.4) = 0.76.$$

It means, that C does not dominate the triangular conorm S_P. This is why we add an additional assumption concerning a conjunction.

Theorem 5. *Let L be a bounded lattice. If $C : L^2 \to L$ is a conjunction fulfilling the condition*

$$C(x, y) \in \{\mathbf{0}, x \wedge y\} \quad \textit{for any } x, y \in L, \tag{12}$$

then it dominates every triangular conorm.

Proof. Let C be a conjunction fulfilling (12), S be a triangular conorm and $x, y, u, v \in L$. If $C(x, u) = C(y, v) = \mathbf{0}$ then we get $S(C(x, u), C(y, v)) = S(\mathbf{0}, \mathbf{0}) = \mathbf{0} \leqslant C(S(x, y), S(u, v))$. If $C(x, u) = \mathbf{0}$ and $C(y, v) = y \wedge v > \mathbf{0}$ we have $S(C(x, u), C(y, v)) = S(\mathbf{0}, C(y, v)) = C(y, v) \leqslant C(S(x, y), S(u, v))$. Similarly in the case $C(y, v) = \mathbf{0}$ and $C(x, u) = x \wedge u > \mathbf{0}$. Let $C(x, u) = x \wedge u > \mathbf{0}$ and $C(y, v) = y \wedge v > \mathbf{0}$. At first we observe that $C(S(x, y), S(u, v)) = S(x, y) \wedge S(u, v)$. Since infimum $\wedge$ dominates any triangular conorm and $\wedge \geqslant C$, then we have $C(S(x, y), S(u, v)) = S(x, y) \wedge S(u, v) \geqslant S(x \wedge u, y \wedge v) = S(C(x, u), C(y, v))$. Thus, C dominates every triangular conorm.

Corollary 8. *Every triangular seminorm C fulfilling* (12) *dominates every triangular conorm.*

Let us notice, that on a bounded lattice L triangular norm T_D is a triangular seminorm fulfilling (12), so for any triangular conorm S we have $T_D \gg S$.

The next example shows that there exists a binary operation which fulfils the assumptions of Theorem 5, but it is not a triangular seminorm, so it does not fulfils conditions used in Theorem 4. By Theorem 5 an operation $C : [0, 1]^2 \to [0, 1]$ on $L = [0, 1]$, given by the formula

$$C(x, y) = \begin{cases} \min(x, y), & \text{if } x \in [\frac{1}{2}, 1], y \in [\frac{3}{4}, 1] \\ 0, & \text{otherwise} \end{cases}$$

dominates any triangular conorm.

Analogously to Theorem 5 we obtain similar results for disjunctions which are dominated by any triangular norm.

Theorem 6. *Let L be a bounded lattice. If $D : L^2 \to L$ is a disjunction fulfilling condition*

$$D(x, y) \in \{x \vee y, \mathbf{1}\} \quad \textit{for any } x, y \in L, \tag{13}$$

then it is dominated by every triangular norm.

By the above theorem we obtain $T \gg S_D$, where T is an arbitrary triangular norm.

5 Dominance and Distributivity Inequalities

In our considerations we shall show some of dependencies between the property of relation of dominance and subdistributivity and superdistributivity for operations on a poset or a lattice.

Theorem 7. *Let L be a lattice, $F, G : L^2 \to L$ and G be an isotone operation.*

- *If F is superdistributive with respect to G and $G \geqslant \vee$, then $F \gg G$.*
- *If F is subdistributive with respect to G and $G \leqslant \wedge$, then $G \gg F$.*

Proof. Let $x, y, u, v \in L$. Using assumption on operation G and the superdistributivity conditions for operations F and G we get $F(G(x,y), G(u,v)) \geqslant G(F(G(x,y),u), F(G(x,y),v)) \geqslant G(G(F(x,u), F(y,u)), G(F(x,v), F(y,v))) \geqslant G(F(x,u) \vee F(y,u), F(x,v) \vee F(y,v)) \geqslant G(F(x,u), F(y,v))$. As a result $F \gg G$. In a similar way, using the subdistributivity properties and assumption $G \leqslant \wedge$, we can prove that $G \gg F$.

Theorem 8. *Let $F, G : P^2 \to P$ be isotone operations.*

- *If $F \gg G$, G is subidempotent, then F is superdistributive with respect to G.*
- *If $G \gg F$, G is superidempotent, then F is subdistributive with respect to G.*

Proof. Let $x, y, u \in P$. We consider the left sub- and superdistributivity conditions, i.e. (3) and (5). If $G(x,x) \leqslant x$ and $F \gg G$ then $F(x, G(y,u)) \geqslant F(G(x,x), G(y,u)) \geqslant G(F(x,y), F(x,u))$. Thus F and G fulfil inequality (5). If $G(x,x) \geqslant x$ and $G \gg F$ then $F(x, G(y,u)) \leqslant F(G(x,x),\ G(y,u)) \leqslant G(F(x,y), F(x,u))$, so F and G fulfil (3). Similarly we can prove that F fulfils (4) and (6).

Because the idempotent operation is simultaneously subidempotent and superidempotent, so according to Theorem 8 we have

Corollary 9. *Let $F, G : P^2 \to P$, F be an isotone operation and G be an idempotent operation.*

- *If F dominates G, then F is superdistributive with respect to G.*
- *If G dominates F, then F is subdistributive with respect to G.*

Since triangular seminorms are subidempotent and triangular semiconorms are superidempotent, then from the above theorem we get

Corollary 10. *Let L be a bounded lattice, $F : L^2 \to L$ be isotone.*

- *If F dominates a seminorm G, then F is superdistributive with respect to G.*
- *If a semiconorm G dominates F, then F is subdistributive with respect to G.*

Using the above corollary we may obtain some negative examples for the dominance for some cases of triangular norms, conorms and means (cf. [14]). Moreover by Corollary 10 we get

Corollary 11. *Any triangular norm on a bounded lattice L is superdistributive with respect to T_D. Any triangular conorm on a bounded lattice L is subdistributive with respect to S_D.*

6 Conclusion

In this paper we presented some algebraic properties of a dominance of binary operations considered with minimal assumptions on a poset P or sometimes with wider assumptions on a lattice. The dominance is an interesting inequality appearing in some mathematical problems as for example preservation of some properties of fuzzy relations, which is important from the application point of view in decision making problems.

For the future work there are many open questions regarding the dominance property, for example a characterization of the antisymmetry property of dominance relation (cf. Corollary 7) or a characterization of the transitivity property of dominance relation.

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Game Method for Modelling with Intuitionistic Fuzzy Rules

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Abstract. The mathematical model for predicting the forest dynamics from [16] is extended with intuitionistic fuzzy estimations for the rules for Game Method for Modelling.

Keywords: Game Method for Modelling · Intuitionistic fuzzy rules · Forest dynamics

1 Introduction

The Game Method for Modelling (GMM [1]) was introduced in the middle of 1970s years as a modification and extension of Conway's Game of Life (CGL [12]).

The standard Conway's Game of Life (CGL, see, e.g., [11, 12]) has a "universe", which is an infinite two-dimensional orthogonal grid of square cells, each of which is in one of two possible states, alive or dead, or (as an equivalent definition) in the square there is an asterisk, or not. The first situation corresponds to the case when the cell is alive and the second - to the case when the cell is dead.

In the case of GMM, the asterisks are changed with some symbols from a fixed set S, that represent not only the places of the asterisks in the grid, but also some of their properties or parameters. In this case, the rules are more complex and can include as partial case these from the CGL.

In the present paper, we first describe shortly the GMM and after this, as an illustration, apply it to simulation of some aspects of forest dynamics.

The Intuitionistic Fuzzy Sets (IFSs [2, 3]) are extensions of Zadeh's fuzzy sets [17]. In them, every element x of a fixed universe has a degree of membership ($m(x)$) and a degree of non-membership ($n(x)$), while in fuzzy sets they have only the first degree and the second one is equal to $1 - m(x)$. In the intuitionistic fuzzy case, $m(x) + n(x) \leq 1$, so, here emerges a third degree – a degree of uncertainty (indeterminacy), such that $p(x) = 1 - m(x) - n(x)$. The pair $\langle m(x) - n(x) \rangle$ is called an Intuitionistic Fuzzy Pair (IFP [4]). The IFP $\langle (a, b) \geq (c, d) \rangle$ if and only if $a \geq c$ and $b \leq d$.

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In a series of papers and book [1, 5–7], related to GMM, IFS-estimations of the situations, generated on a fixed (2-dimensional) plane, are defined and studied, but up to now, the rules have only used crisp values.

In the present paper, for a first time, we construct intuitionistic fuzzy rules for GMM functioning and use IFPs as elements of set S.

2 Short Remarks on the Game Method for Modelling

Let us have a set of symbols S and an n-dimensional simplex (in the sense of [14]) comprising of n-dimensional cubes (at $n = 2$, a two-dimensional net of squares).

Let material points (or, for brief, objects) be found in some of the vertices of the simplex and let a set of rules A be given, containing:

1. Rules for the motion of the objects along the vertices of the simplex;
2. Rules for the interactions among the objects.

Let the rules from the i-th type be marked as i-rules, where $i = 1, 2$.

When $S = \{*\}$, we obtain the standard CGL.

We associate with each object its number, n-tuple of coordinates characterizing its location in the simplex, and a symbol from S reflecting the peculiarity of the object (e.g. in physical applications - mass, charge, concentration, etc.). We shall call an *initial configuration* every ordered set of $(n + 2)$-tuples with an initial component being the number of the object; the second, third, etc. until the $(n + 1)$-st – its coordinates; and the $(n + 2)$-nd - its symbol from S. We shall call a *final configuration* the ordered set of $(n + 2)$-tuples having the above form and being a result of a (fixed) initial configuration, modified during a given number of times when the rules from A have been applied.

The single application of a rule from A over a given configuration K will be called an elementary step in the transformation of the model and will be denoted by $A_1(K)$. In this sense, if K is an initial configuration, and L is a final configuration derived from K through multiple application the rules from A, then configurations $K_0, K_i, \ldots, K_m$ will exist, for which

$$\begin{gathered} K_0 = K, \\ K_{i+1} = A_1(K_i) \text{ for } 0 < i < m-1, \\ K_m = L, \end{gathered}$$

(the sign of equality "=" is used in the sense of coincidence in the configurations) and this will be denoted by

$$L = A(K) = A_1(A_1(\ldots A_1(K)\ldots)).$$

Let a rule P be given, which juxtaposes a combination of configurations M to a single configuration $P(M)$, being the mean of the given ones. We shall call this rule a *concentration rule.* The concentration can be made either over the values of the symbols from S for the objects, or over their coordinates, (not over both of them simultaneously).

For example, if k-th element of M ($1 \leq k \leq s$, where s is the number of elements of M) is a rectangle with $p \times q$ squares and if the square staying on (i, j)-th place ($1 \leq i \leq p$, $1 \leq j \leq q$) contains number $d_{i,j}^k \in \{0,1,\ldots, 9\}$, then on the (i, j)-th place of $P(M)$ stays:

- minimal number

$$d_{i,j} = \left[\frac{1}{s}\sum_{k=1}^{s} d_{i,j}^k\right],$$

- maximal number

$$d_{i,j} = \left\lceil\frac{1}{s}\sum_{k=1}^{s} d_{i,j}^k\right\rceil,$$

- average number

$$d_{i,j} = \left[\frac{1}{s}\sum_{k=1}^{s} d_{i,j}^k + \frac{1}{2}\right],$$

where for real number $x = a + \alpha$, where a is a natural number and $\alpha \in [0,1)$: $[x] = a$, and

$$\lceil x \rceil = \begin{cases} a, & \text{if } \alpha = 0 \\ a+1, & \text{if } \alpha > 0 \end{cases}$$

Let B be a criterion derived from physical or mathematical considerations. For two given configurations K_1 and K_2, criterion B answers the question whether these configurations are closed (and we write, e.g., $B(K_1, K_2) = 1$) or not (and we write $B(K_1, K_2) = 0$). For example, for two configurations K_1 and K_2 having the form from the above example,

$$B(K_1, K_2) = \frac{1}{p \cdot q}\sum_{i=1}^{p}\sum_{j=1}^{q}\left|d_{i,j}^1 - d_{i,j}^2\right| < C_1$$

or

$$B(K_1, K_2) = \left(\frac{1}{p \cdot q}\sum_{i=1}^{p}\sum_{j=1}^{q}\left(d_{i,j}^1 - d_{i,j}^2\right)^2\right)^{\frac{1}{2}} < C_2,$$

where C_1 and C_2 are given constants.

For the set of configurations M and the set of rules A we shall define the set of configurations

$$A(M) = \{L|(\exists K \in M)(L = A(K))\}.$$

The rules A will be called statistically correct, if for a great enough (from a statistical point of view) natural number N:

$$(\forall m > N)(\forall M = \{K_1, K_2, \ldots, K_m\})$$
$$(B(A(P(M))(P(\{L_i|L_i = A(K_i), 1 \leq i \leq m\}) = 1). \quad (1)$$

The essence of the method is in the following: the set of rules A, the proximity criterion B and the concentration rule P are fixed in advance. A set of initial configurations M is chosen and the set of the corresponding final configurations is constructed. If the Eq. (1) is valid, we may assume that the rules from the set A are correct in the frames of the model, i.e., they are logically consistent. Otherwise, we replace a part (or all) of them with others. If the rules become correct, then we can add to the set some new rules, or transform some of the existing, and check permanently the correctness of the newly constructed system of rules. Thus, in a stepwise manner, extending and complicating the rules in set A and checking their correctness, we construct the model of the given process. Afterwards, we may check the temporal development (as regards the final system of rules A) of a particular initial configuration.

We initially check the correctness of the modelling rules and only then we proceed to the actual modelling. To a great deal this is due to the fact that we work over discrete objects with rules that are convenient for computer implementation. Thus, a series of checks of the Eq. (1) can be performed only to construct the configuration $A(K)$ for a given configuration K and a set of rules A.

3 Application of GMM for Simulating Some Aspects of Forest Dynamics with Intuitionistic Fuzzy Rules

In the frames of the present research, that is an extension of [9, 10, 16], for brevity, we describe a finite grid with size 36 × 17 to represent a forest area, consisting of different by carbon stocks homogenous forest stands, in which we check the response of tree biomass to a hypothetical recurring disease. We assume that sustainability to that disease decreases with increasing stand biomass, due to aging and/or increasing vulnerability with higher carbon stocks [15]. We assign carbon stocks to our forest stands and general trends between stand biomass carbon and age following [8, 13, 16]. The carbon stocks and corresponding GMM numbers, by which these carbon stocks are represented in the model, are described in Table 1. We also assume that in our grid, i.e. simulated forest complex, there is a river flowing through (marked by R), rocks (marked by S) and open areas without trees, to which zero carbon stocks are assigned (see Fig. 1). We assume that the disease may occur first at reaching the carbon stock of 4000 g/m^2, corresponding to the GMM intuitionistic fuzzy pair $\langle 0.48, 0.48\rangle$.

Table 1. Carbon stocks of boreal black spruce stands, and corresponding GMM numbers, by which these carbon stocks are represented in the model.

GMM corresponding number	Carbon stocks of boreal black spruce stands (g/m^2)
$\langle 0.00, 1.00\rangle$	0
$\langle 0.10, 0.90\rangle$	1000
$\langle 0.20, 0.80\rangle$	2000
$\langle 0.30, 0.70\rangle$	3000
$\langle 0.39, 0.59\rangle$	3500
$\langle 0.48, 0.48\rangle$	4000
$\langle 0.57, 0.37\rangle$	4500
$\langle 0.66, 0.26\rangle$	5000
$\langle 0.75, 0.15\rangle$	5500
$\langle 0.84, 0.06\rangle$	6000

Afterwards, it recurs as the higher carbon stock raises the chances for the disease to reduce part of it. We quantify the expectations that the disease will harm a given stand and reduce its carbon stock by generating a random number $r \in [0, 1]$, which complies with certain criteria, as described below, in order to simulate increasing vulnerability of forest stands of higher biomass.

Thus, we set the rules for the GMM are as follows and if some rule determines that symbol Y should be replaced by symbol Z, we denote this by $Y \to Z$.

The rules for the GMM are as following.

1. $R \to R$
2. $\langle 0.00,\ 1.00\rangle \to \begin{cases} \langle 0.00,\ 1.00\rangle, & \text{if the cell there is not a neighbouring cell containing a digit different than 0} \\ \langle 0.10,\ 0.90\rangle, & \text{otherwise} \end{cases}$
3. $\langle 0.10,\ 0.90\rangle \to \langle 0.20,\ 0.80\rangle$
4. $\langle 0.20,\ 0.80\rangle \to \langle 0.30,\ 0.70\rangle$
5. $\langle 0.30,\ 0.70\rangle \to \langle 0.39,\ 0.59\rangle$
6. $\langle 0.39,\ 0.59\rangle \to \langle 0.48,\ 0.48\rangle$
7. $\langle 0.48,\ 0.48\rangle \to \begin{cases} \langle 0.57,\ 0.37\rangle, & \text{if } r > 0.25 \\ \langle 0.48,\ 0.48\rangle, & \text{if } r \in [0.125, 0.25] \\ \langle 0.39,\ 0.59\rangle, & \text{if } r < 0.125 \end{cases}$
8. $\langle 0.57,\ 0.37\rangle \to \begin{cases} \langle 0.66,\ 0.26\rangle, & \text{if } r > 0.50 \\ \langle 0.57,\ 0.37\rangle, & \text{if } r \in [0.25, 0.50] \\ \langle 0.48,\ 0.48\rangle, & \text{if } r < 0.25 \end{cases}$
9. $\langle 0.66,\ 0.26\rangle \to \begin{cases} \langle 0.75,\ 0.15\rangle, & \text{if } r > 0.75 \\ \langle 0.66,\ 0.26\rangle, & \text{if } r \in [0.40, 0.75] \\ \langle 0.57,\ 0.37\rangle, & \text{if } r < 0.40 \end{cases}$

$$10.\ \langle 0.75,\ 0.15\rangle \rightarrow \begin{cases} \langle 0.84,\ 0.06\rangle, & \text{if } r > 0.90 \\ \langle 0.75,\ 0.15\rangle, & \text{if } r \in [0.50, 0.90] \\ \langle 0.66,\ 0.26\rangle, & \text{if } r < 0.50 \end{cases}$$

$$11.\ \langle 0.84,\ 0.06\rangle \rightarrow \begin{cases} \langle 0.84,\ 0.06\rangle, & \text{if } r > 0.95 \\ \langle 0.75,\ 0.15\rangle, & \text{if } r \leq 0.95 \end{cases}$$

$$12.\ S \rightarrow \begin{cases} \langle 0.10,\ 0.90\rangle, & \begin{array}{l}\text{if the cell there is at least one neighbouring cell} \\ \text{containing a digit higher than } \langle 0.10,\ 0.90\rangle \text{ through 4} \\ \text{sequential steps}\end{array} \\ S, & \text{otherwise} \end{cases}$$

In Figs. 1, 2, 3, 4, 5, 6 and 7, different steps of the forest developments are shown. These figures correspond to the following time-moments of the model functioning, where the duration of one time-moment is 3 years.

Table 2 shows the obtained outcomes from the model. It represents the numbers of the squares with tree biomass corresponding to the GMM intuitionistic fuzzy pair in the grid at the current step of functioning of the model.

Table 3 represents the percentages of the tree biomass corresponding to the GMM intuitionistic fuzzy pair at the current step of functioning of the model.

From the information, obtained by the use of intuitionistic fuzzy rules the following outcomes are obtained:

- Open areas without trees, to which zero carbon stocks with IFPs $\langle 0.00, 1.00\rangle$ are assigned, at Step 0 are 34%. During the next 7 steps they decrease and at Step 8 they becomes to 0%;
- The areas with IFPs estimations of $\langle 0.10, 0.90\rangle$ are 30% at Step 0, and decrease down to 0% at Step 9. Occasionally, during the next steps, they increase up to 2%–3%;
- For the areas with IFPs estimations of $\langle 0.20, 0.80\rangle$, $\langle 0.30, 0.70\rangle$ and $\langle 0.39, 0.59\rangle$, the behavior is approximately the same;
- Since the disease occurs first at reaching the estimation of $\langle 0.48, 0.48\rangle$, and then recurs afterwards, the percentage of the tree biomass changes in a short time interval;
- The areas with estimation of $\langle 0.48, 0.48\rangle$ are 0% at Step 0, but during the last 10 steps they are almost constant, covering around 24%;
- For the areas with estimation of $\langle 0.57, 0.37\rangle$, the behavior is approximately the same, but during the last 10 steps they cover around 36%;
- The areas with rocks are 15% at Step 0 and decrease to 1% at the last step, because the cells in the grid, that represent rocks, have neighbouring cells with digits higher than $\langle 0.10, 0.90\rangle$.

By the model outcome, we can check not only the changes of forest biomass due to induced-by-disease mortality, but also concominant forest colonization of unforested areas (rocks), therefore more aspects of dynamics of the entire forest complex.

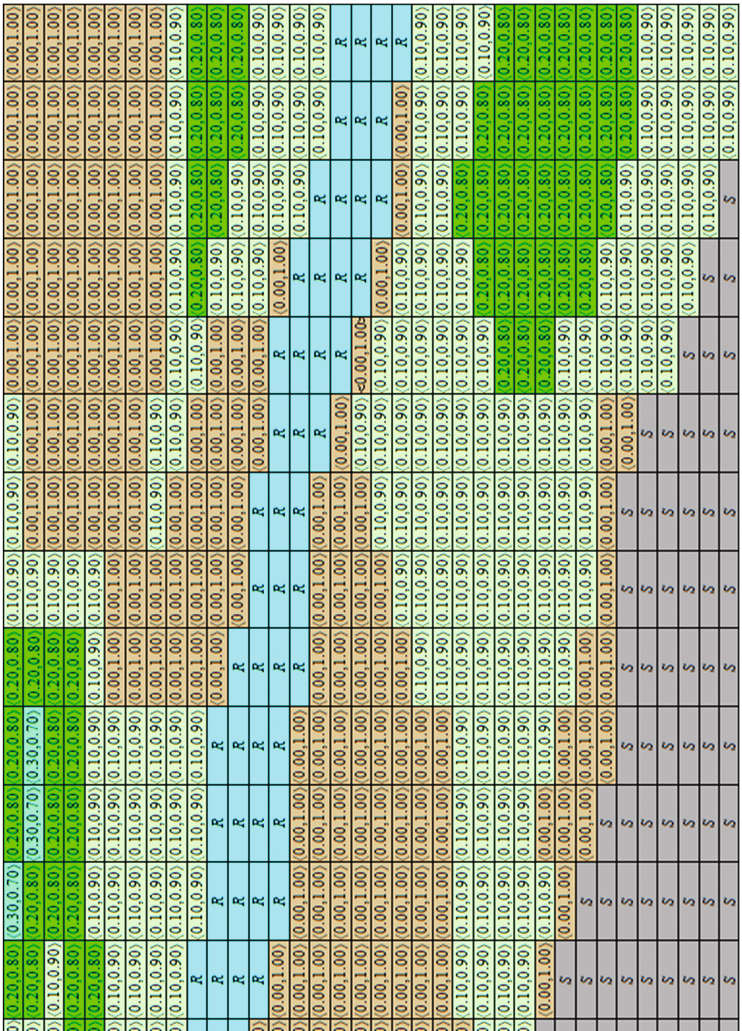

Fig. 1. The initial (zero) time moment.

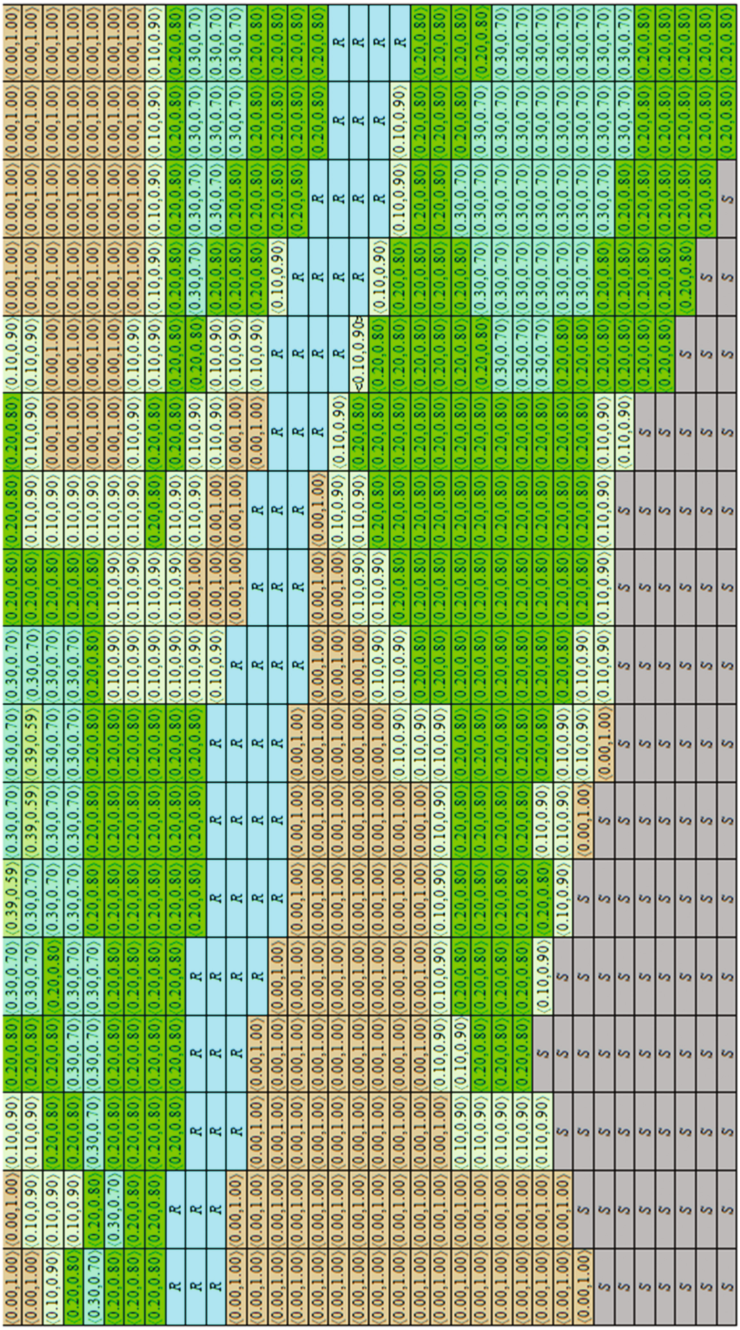

Fig. 2. The first time moment.

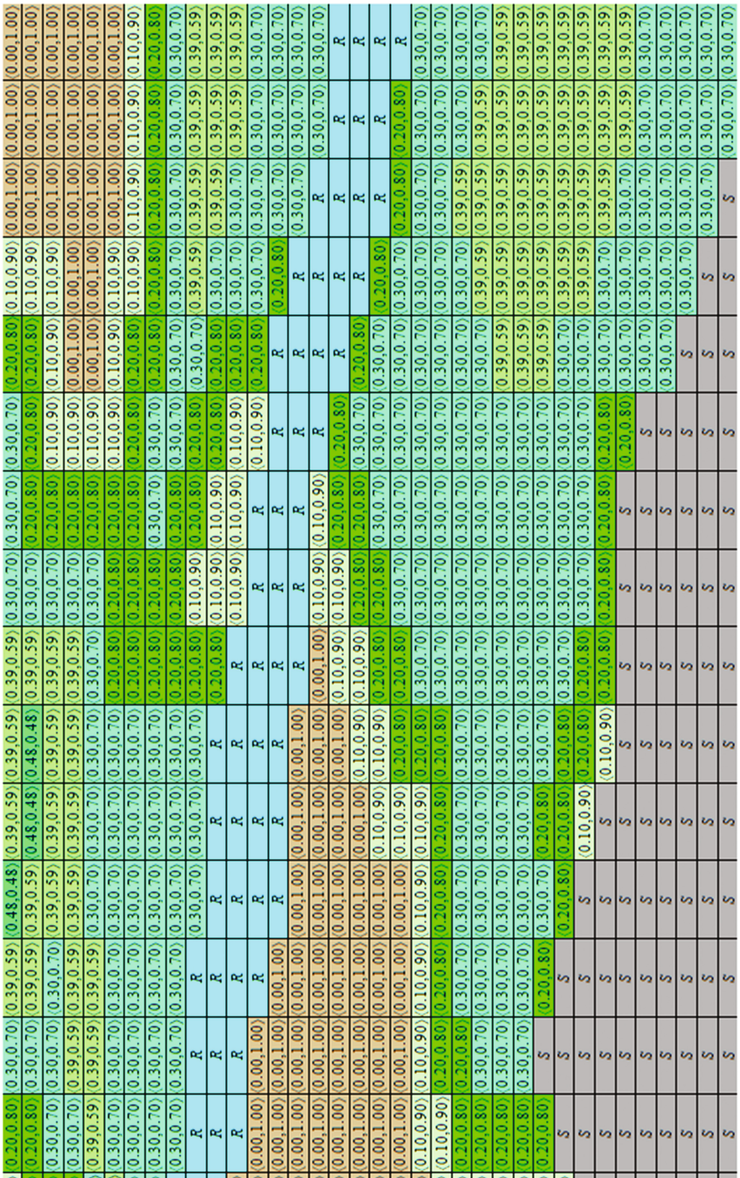

Fig. 3. The second time moment.

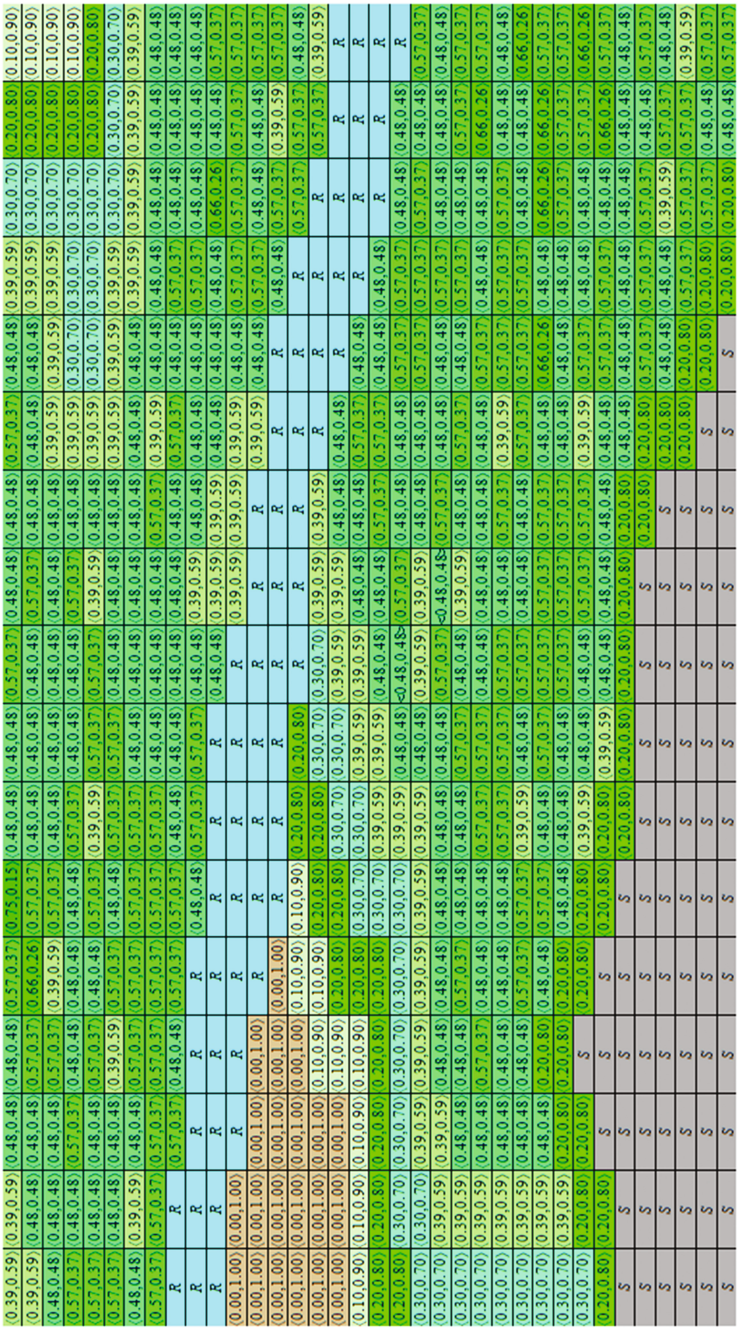

Fig. 4. The fifth time moment.

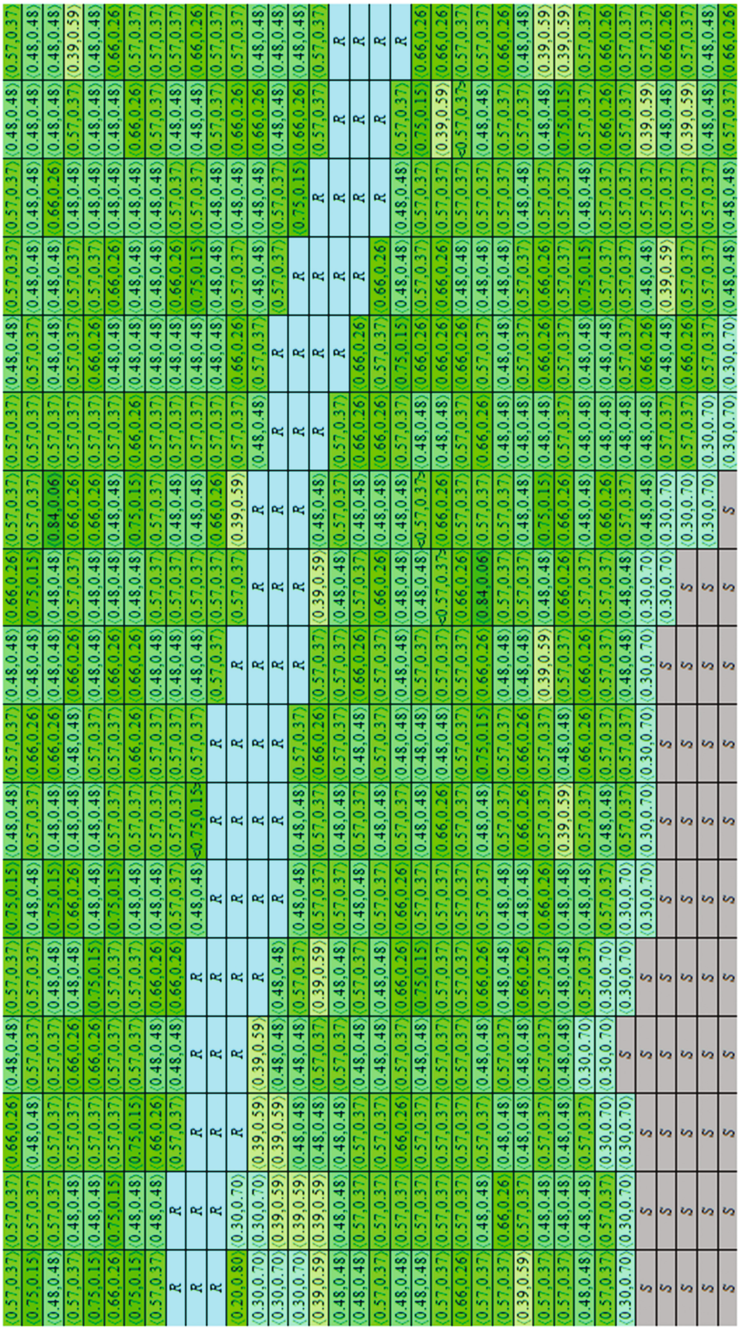

Fig. 5. The tenth time moment.

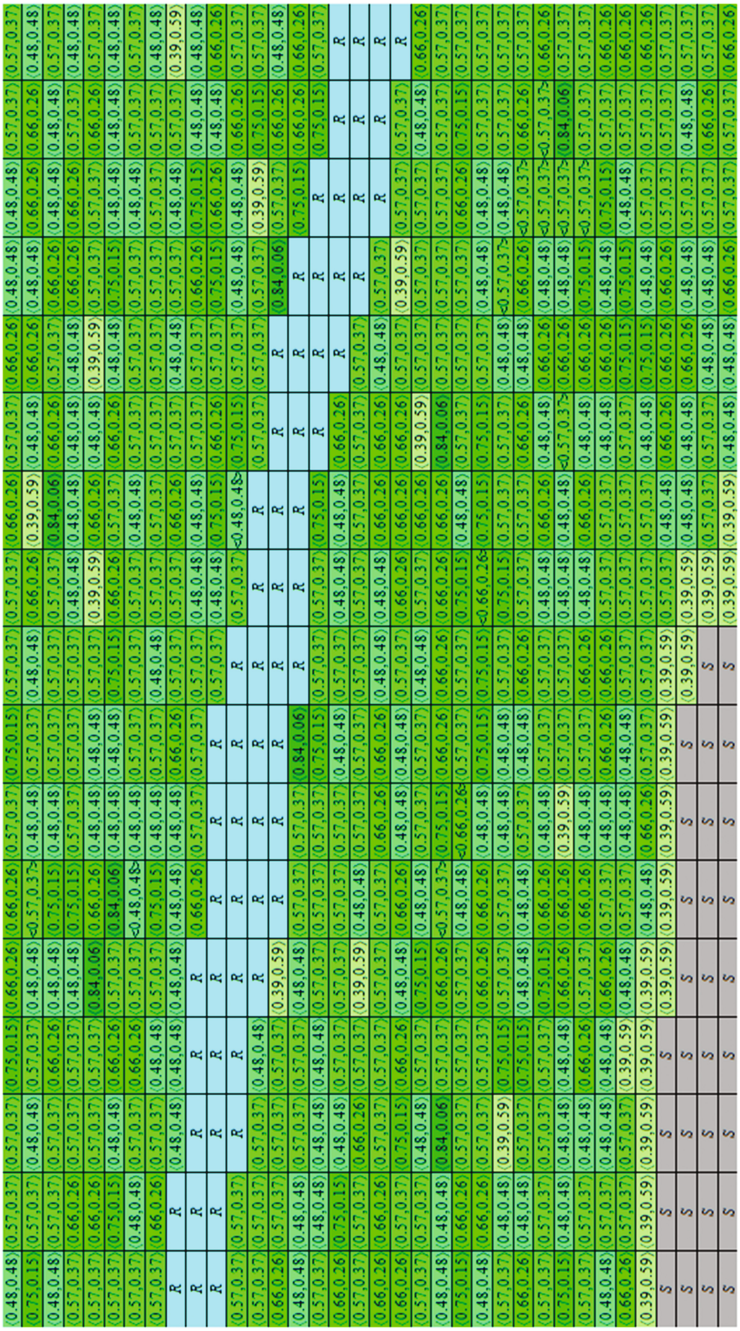

Fig. 6. The fifteenth time moment.

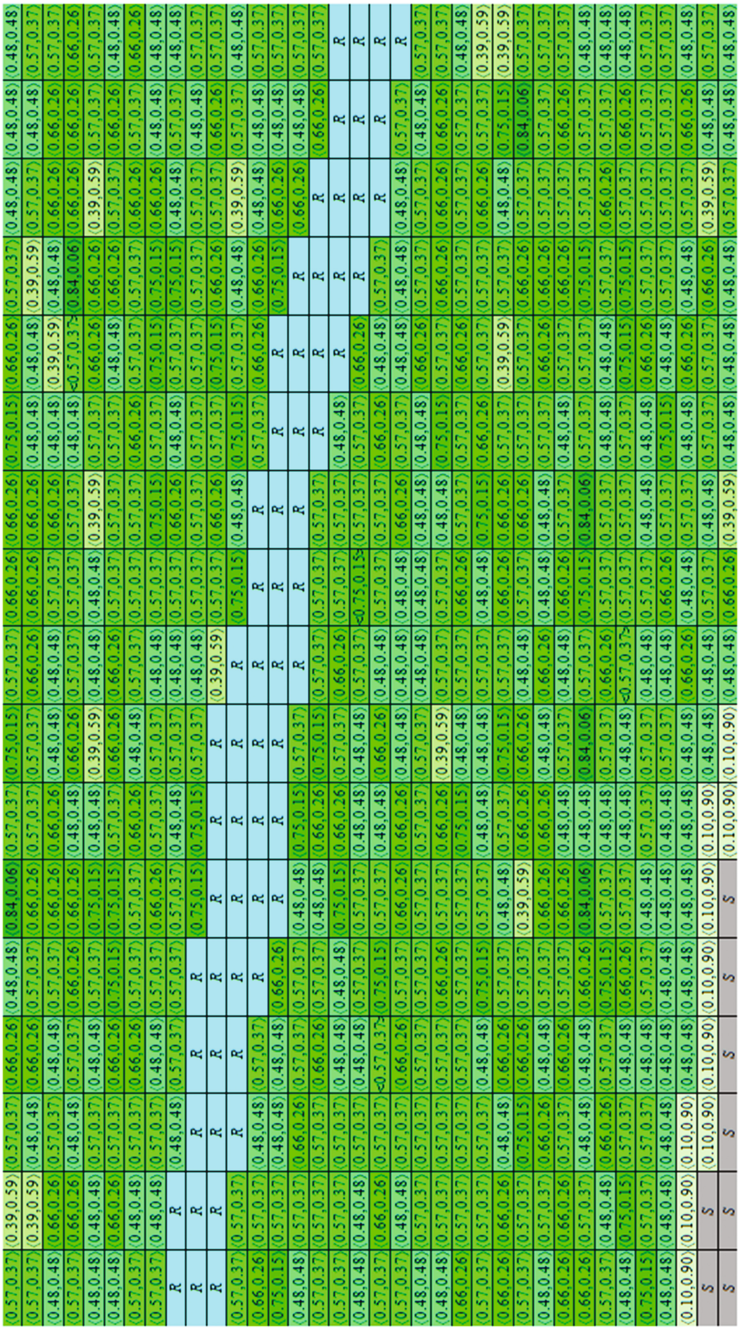

Fig. 7. The twentieth time moment.

Table 2. Numbers of the squares with intuitionistic fuzzy pairs in the grid.

IF pair	Step																				
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\langle 0.00, 1.00\rangle$	211	135	87	53	34	21	13	6	1	0	0	0	0	0	0	0	0	0	0	0	0
$\langle 0.10, 0.90\rangle$	182	76	48	34	45	13	8	7	26	1	0	0	16	0	0	0	12	0	0	0	10
$\langle 0.20, 0.80\rangle$	63	182	76	48	34	45	13	8	7	26	1	0	0	16	0	0	0	12	0	0	0
$\langle 0.30, 0.70\rangle$	3	63	182	76	48	34	45	13	8	7	26	1	0	0	16	0	0	0	12	0	0
$\langle 0.39, 0.59\rangle$	0	3	63	182	82	64	49	61	28	23	20	39	10	9	8	27	9	9	11	23	16
$\langle 0.48, 0.48\rangle$	0	0	3	63	203	178	168	180	176	172	164	146	161	134	148	133	145	152	146	144	145
$\langle 0.57, 0.37\rangle$	0	0	0	3	38	120	159	163	194	192	200	221	228	224	210	218	224	218	234	234	222
$\langle 0.66, 0.26\rangle$	0	0	0	0	1	9	26	43	58	69	72	72	73	112	102	97	103	97	77	83	109
$\langle 0.75, 0.15\rangle$	0	0	0	0	0	1	4	4	7	16	21	20	29	22	33	39	30	33	38	36	36
$\langle 0.84, 0.06\rangle$	0	0	0	0	0	0	0	0	1	0	2	7	5	5	5	8	11	13	16	14	6
Rocs	93	93	93	93	67	67	67	67	46	46	46	46	30	30	30	30	18	18	18	18	8

Table 3. Percentages of the tree biomass corresponding to the intuitionistic fuzzy pair in the grid

IF pair	Step																				
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\langle 0.00, 1.00\rangle$	34%	22%	14%	9%	6%	3%	2%	1%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
$\langle 0.10, 0.90\rangle$	30%	12%	8%	6%	7%	2%	1%	1%	4%	0%	0%	0%	3%	0%	0%	0%	2%	0%	0%	0%	2%
$\langle 0.20, 0.80\rangle$	10%	30%	12%	8%	6%	7%	2%	1%	1%	4%	0%	0%	0%	3%	0%	0%	0%	2%	0%	0%	0%
$\langle 0.30, 0.70\rangle$	0%	10%	30%	12%	8%	6%	7%	2%	1%	1%	4%	0%	0%	0%	3%	0%	0%	0%	2%	0%	0%
$\langle 0.39, 0.59\rangle$	0%	0%	10%	30%	13%	10%	8%	10%	5%	4%	3%	6%	2%	1%	1%	4%	1%	1%	2%	4%	3%
$\langle 0.48, 0.48\rangle$	0%	0%	0%	10%	33%	29%	27%	29%	29%	28%	27%	24%	26%	22%	24%	22%	24%	25%	24%	24%	24%
$\langle 0.57, 0.37\rangle$	0%	0%	0%	0%	6%	20%	26%	27%	32%	31%	33%	36%	37%	37%	34%	36%	37%	36%	38%	38%	36%
$\langle 0.66, 0.26\rangle$	0%	0%	0%	0%	0%	1%	4%	7%	9%	11%	12%	12%	12%	18%	17%	16%	17%	16%	13%	14%	18%
$\langle 0.75, 0.15\rangle$	0%	0%	0%	0%	0%	0%	1%	1%	1%	3%	3%	3%	5%	4%	5%	6%	5%	5%	6%	6%	6%
$\langle 0.84, 0.06\rangle$	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	1%	1%	1%	1%	1%	2%	2%	3%	2%	1%
Rocs	15%	15%	15%	15%	11%	11%	11%	11%	8%	8%	8%	8%	5%	5%	5%	5%	3%	3%	3%	3%	1%

4 Conclusion

In the paper, intuitionistic fuzzy rules for GMM are introduced. An example, describing a forest dynamics is given as an illustration for the possible application of the GMM, which here is implemented with intuitionistic fuzzy rules. This example is an extension of the example from [1, 16], where crisp values were used. The model also extends models from [9, 10], that describe only trees without objects between them, such as the river and rocks in this modelling exercise.

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The Cellular Automata Theory with Fuzzy Numbers in Simulation of Real Fires in Buildings

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Abstract. Many serious real-life problem could be simulated using cellular automata theory. There were lot of fires in public places which kills many people. Proposed simulation method is using cellular automata theory with Fuzzy Numbers and could be used for checking buildings conditions for fire accident. The tests performed on real accident showed that using some extension of Fuzzy Numbers could give a realistic simulation of human evacuation. The authors analyze some real accidents and proved that proposed method appears a very promising solution, especially in the cases of building renovations or temporary unavailability of escape routes.

Keywords: Cellular automata · Fuzzy Numbers · OFN · Fire · Evacuation

1 Introduction

Cellular automata are used by some of the IT branches, including the field of artificial intelligence. They consist of a network of cells, each of which is distinguished by some specific state and a set of rules. The change of the current state of a given cell is the outcome of the above mentioned properties and interrelations with the neighboring cells. The theory of cellular automata was first introduced by an American scientist of Hungarian descent, John von Neumann. He demonstrated, among others, that even simple machines show an ability to reproduce, which was until that time regarded as a fundamental feature of living organisms [44]. For many years cellular automata had been subject to theoretical studies only [15]. With the development of computers and software, optimizing methods based on this approach have been more and more frequently studied and implemented in practice [5,11,13,21]. Due to their versatility, cellular automata

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are applied in many real-life fields, such as biology, physics, mathematics and in various fields of IT, such as cryptography or computer graphics [17,31,32].

1.1 Application of Cellular Automata

Cellular automata have been applied in practice, for example in simulation of the street traffic, where specifically defined cellular automaton controls the traffic. The vehicle flow is managed basically at the specific segment of a given traffic intensity [29]. This applies, for instance, to the traffic intensity control on highways of the Ruhr in Germany [27,39]. The monitoring centers designed exclusively for that purpose collect the data from selected sections of the highways [7,22,41]. The information thus obtained is analyzed and used for preparing short-time simulations of the traffic intensity by means of cellular automata. The project's websites publish the statistical information about the studies performed on the behavior of drivers who were pre-warned about possible traffic problems [28] that might occur over several following hours [22,35]. Another example of cellular automata application are demographic simulations for a given region. The aim of such simulations is to generate a model showing the size of the population at a given area in a form of a map of the forecasted population density. Such simulations can be based on the well-known "Game of Life" [36,43]. By introducing some modification to the algorithm, it is possible to monitor the behavior of the surrounding cells [30,32]. Other examples of cellular automata implementations include image processing, generation of textures, simulation of waves, wind as well as proposed program, which was developed for the purpose of this study [8,19,40]. The aim of the proposed algorithm is to generate the simulations of patterns of human escape from the building on fire with a given number of exits and fire sources with Fuzzy Numbers [45] used to decide what people should do in each step of the algorithm [26,34].

1.2 The Grid of Cellular Automata

A grid or a discrete space, where cellular automata evolution takes place, consists of a set of identical cells. Each of the cells is surrounded by the same number of neighbors and can assume the same number of states. There are three structural factors which significantly influence the grid form and, as a consequence, the behavior of the entire cellular automaton [27,28,42]:

- the size of the space depends on the magnitude of the studied problem, the examples of which are shown in 1D, 2D or 3D grids;
- the provision of regularity, which requires the grid to be filled entirely with identical cells;
- the number of neighbors (dependent on both above-mentioned factors).

2 Forecasting the Fire Hazard

2.1 Fire Accidents in Public Places

Fires are one of the most uncontrollable calamities, especially when they happen indoors. Thus, regardless of the edifice's function - whether it is a residential, business or any other kind of building, its design must comply with fire regulations. The width of corridors, number of emergency exits and the permissible number of people staying inside at the same time has a grave impact on the safety of its users. Simple presence of the doors on floor plan is not sufficient; they have to be open. In many cases the high number of casualties stemmed from the emergency exit doors being locked [12–14]. In the past decades there has been a number of disastrous fires in public places like restaurants and nightclubs. Table 1 presents some examples of such accidents and lists the numbers of victims.

Table 1. Fire accidents in public places

Name	Year	Fatalities	Injuries
Study Club fire	1929	22	50
Karlslust dance hall fire	1947	80–88	150
Stardust fire	1981	48	214
Alcala 20 nightclub fire	1983	82	27
Ozone Disco Club fire	1996	162	95
Gothenburg discothque fire	1998	63	213
Volendam New Years fire	2001	14	241
Canecao Mineiro nightclub fire	2001	7	197
Utopia nightclub fire	2002	25	100
The Station nightclub fire	2003	100	230
Wuwang Club fire	2008	43	88
Santika Club fire	2009	66	222
Lame Horse fire	2009	156	$\leq$160
Kiss nightclub fire	2013	231	168

2.2 The Cocoanut Grove Fire Accident

The Cocoanut Grove was a restaurant built in 1927 and located at 17 Piedmont Street, near Park Square, in downtown Boston, Massachusetts [8,40]. According to Prohibition, it was very popular in 1920's. The building structure was a single-story, with a basement beneath. The basement consists of a bar, kitchen, freezers, and storage areas. The first floor contained a large dining room area and ballroom with a bandstand, along with several bar areas separate from the ballroom. The dining room also had a retractable roof for use during warm weather to allow a view of the moon and stars. The main entrance to the Cocoanut Grove was via a revolving door on the Piedmont Street side of the building. On Saturday, November 28, 1942 there was a very large fire accident. During that evening, a busboy had been ordered to fix a light bulb located at the top of an artificial palm tree in the corner of the basement bar. A moment later decorations started burning. As other furnishings ignited, a fireball of flame and toxic gas raced across the room toward the stairs. The revolving door became jammed due to the crush of panicked patrons [9]. Lots of people stuck in fire. It was later estimated that more than 1000 persons were inside the Grove at the time of the fire. The final death count established by Commissioner Reilly was 490 dead and 166 injured, but the of injured was a count of those treated at a hospital and later released while many patrons were injured but did not seek hospitalization (Fig. 1).

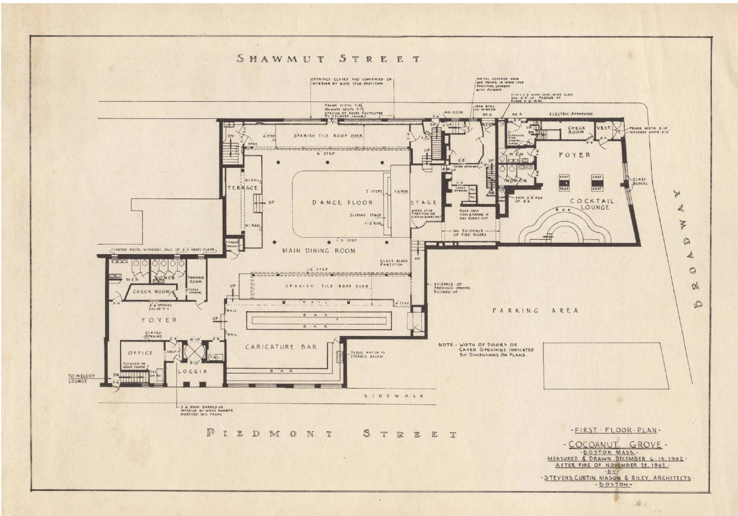

Fig. 1. The Cocoanut Groove scheme [13]

2.3 Simulation Method

The process of implementing Fuzzy Numbers in cellular automata looks as a normal step in practical usage of Fuzzy Logic (Fig. 2). There are lot of possible implementation of Fuzzy Numbers which were described by Zadeh [45], Klir [23], Dubois and Prade [20] and Kosiński [24,26]. The program described in this paper is using 2 dimensional model in which there is neighborhood of cell described by Moore. This give eight possible move from the cell Ni, j. Some example of the move is presented in Fig. 3.

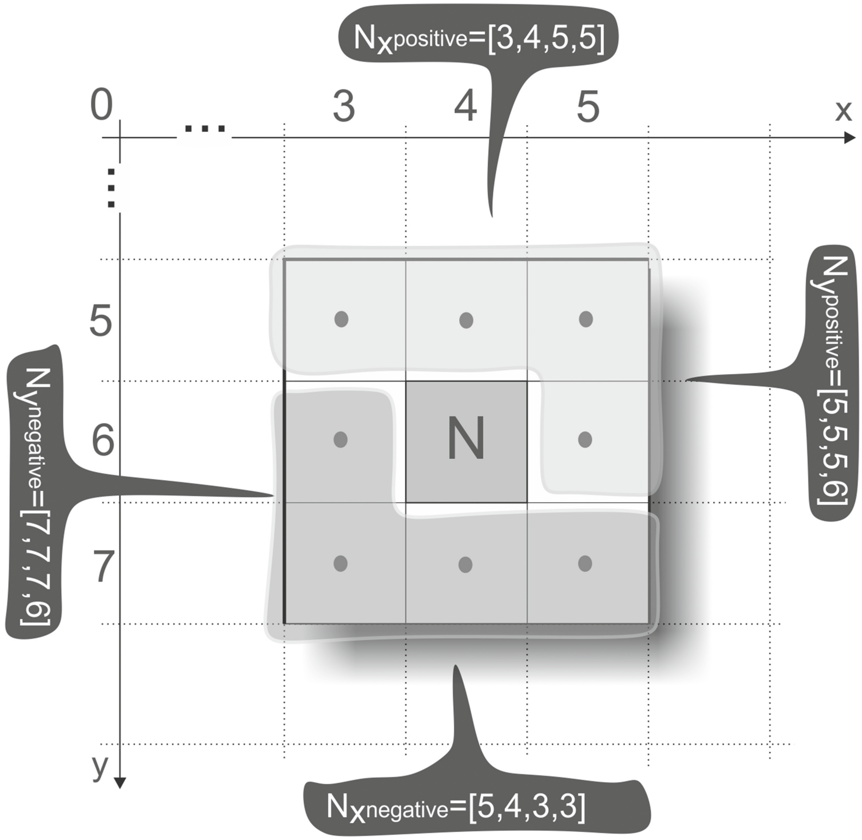

Fig. 2. Example of move in simulation algorithm

In Fig. 3, some of the neighbours half of them are closer to the exit from the building, and the second half of the neighbor cells are in the state "human". For this cell there are two possible move according to the determinants [2,4,33,43]. Both of the sets consist four elements [3,5,16,18]. To describe the cell state authors choose Fuzzy Numbers with some extension [24]. This extension was provided by Kosiński [37,38] and is called Ordered Fuzzy Numbers, and in some work after Kosiński's detah - Kosinski's Fuzzy Numbers [1,6,9–11,25,26].

In this notation each Fuzzy Number A has a trapezoid form defined by four coordinates $[fA(0), fA(1), gA(1), gA(0)]$, which is presented in Fig. 4. The arrow in Fig. 4 defined the direction of the number, which describe the order of the coordinates. Such definition of the fuzzy number let to make an arithmetic operation according to such definition [26]:

- addition $A + B = (f + e, g + h) = C$
  $C \rightarrow [f(0) + e(0), f(1) + e(1), g(1) + h(1), g(0) + h(0)]$

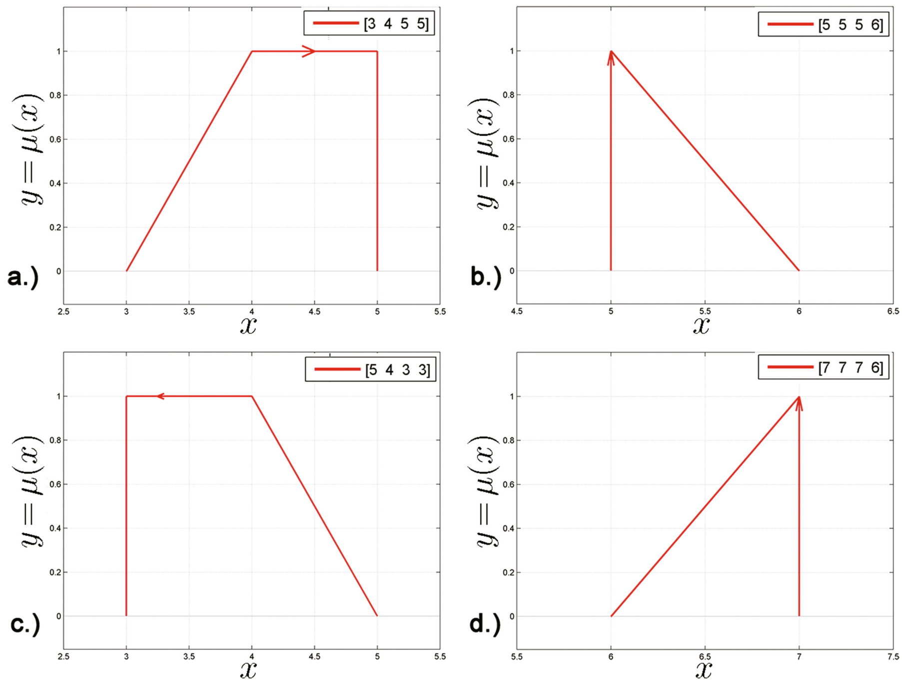

Fig. 3. The OFN vizualization of Nx-positive (a), Ny-positive (b) and Nx-negative (c), Ny-negative (d)

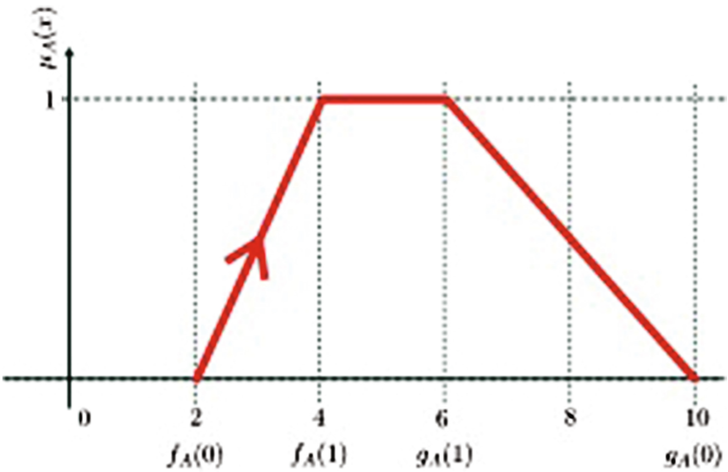

Fig. 4. Fuzzy Number with the extension

– scalar multiplication $C = \lambda A = (\lambda f, \lambda g)$
  $C \rightarrow [\lambda f(0), \lambda f(1), \lambda g(1), \lambda g(0)]$
– subtraction $A - B = (f - e, g - h) = C$
  $C \rightarrow [f(0) - e(0), f(1) - e(1), g(1) - h(1), g(0) - h(0)]$
– multiplication $A * B = (f * e, g * h) = C$
  $C \rightarrow [f(0) * e(0), f(1) * e(1), g(1) * h(1), g(0) * h(0)]$
– division $A/B = (f/e, g/h) = C$
  $C \rightarrow [f(0)/e(0), f(1)/e(1), g(1)/h(1), g(0)/h(0)]$

So, the set of possible move in Moore neighbourhood from the cell $N_{i,j}$ to cell $N'_{(i,j)'}$ is presented in Fig. 5: Implemented algorithm has got a following determinants for the movement:

– follow to the nearest exit,
– follow to the nearest group of people.

The determinants are connected with the order of fuzzy number in Ordered Fuzzy Number notation [24].

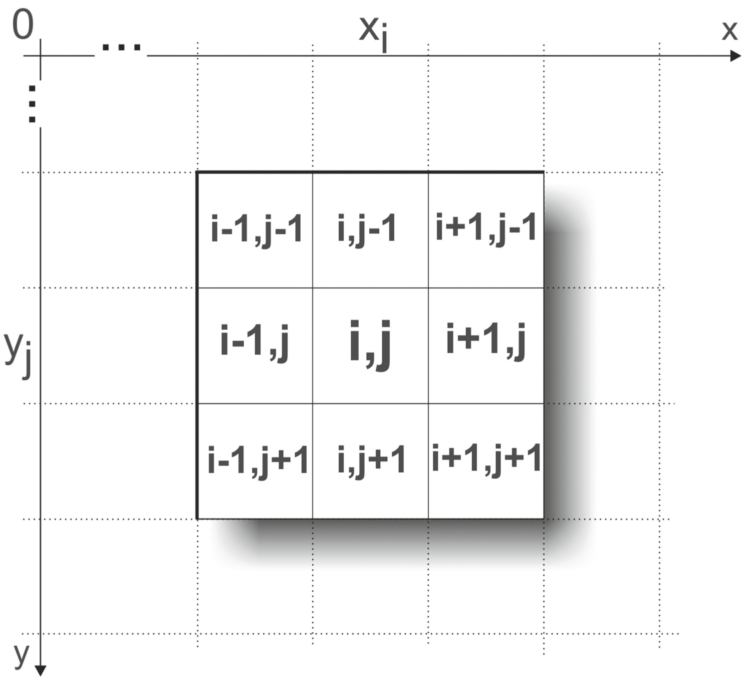

Fig. 5. Possible move

Definition 1. Let there be two pairs of fuzzy numbers (Nx, Ny). Order of the number will be positive for the movement closer to determinants:

$Nx_{positive}[i-1, i, i+1, i+1]$
$Ny_{positive}[j-1, j-1, j-1, j]$

A pair of coordinates of the position indicated further determinant will be negative referral:

$Nx_{negative}[i+1, i, i-1, i-1]$
$Ny_{negative}[j+1, j+1, j+1, j]$

A subset of cells which enable the operation determines the pair of fuzzy numbers satisfying the following rules:

$$IF\ N_{xpositive} * N_{xnegative} \text{ is positive } THEN\ Nx = N_{xpositive}\ ELSE$$
$$Nx = N_{xnegative}\ IF\ N_{ypositive} * N_{ynegative} \text{ is positive THEN } Ny = N_{ypositive}$$
$$\text{ELSE } Ny = N_{ynegative}$$

The set described by fuzzy number $(N_x^{'}, N_y^{'})$ gives the four possible cells for the aim of the move. In next evolution turn a one cell for the move is randomly chosen. Of course, the cells in which motion is impossible to be eliminated from the list. If the movement is not possible for any of the four cells, the state of the cell does not change. This symbolizes the situation in which a person remains motionless.

3 The Experiment with Proposed Method

The authors launched a simulation of the Kiss nightclub scenario in prepared program. They placed "people" inside and set the fire. The building comprised of seven rooms and there was only one exit. The blue points mark people and the red ones - fire. Several tests were performed based on this scheme and the assumed conditions were as follows: The aim of the test was to simulate a fire of the building, basing on certain rules and relations. Setting of the following parameters, selection of versions and inherent rules altogether make up an environment which affect the mortality rate. The variables were:

- the layout of the building floors, including the number and location of doors,
- distribution of a defined number of people inside the building at specified places,
- setting the fire parameters:
- the fire goes out alone if there is if there are no neighbors,
- the fire goes out because of overpopulation if there are more than 3 neighbors,
- new fire is generated when there are at least 3 neighbors, but not more than 4.
- setting of the parameters for people (live cells):
- number of burns resulting in death is by default set to 5,
- location of the fire source on the board;
- specifying the probability of people heading towards the exit (three options): 25%, 50%, 75%;

– specifying whether people move towards the exit in groups (two options): with or without a group effect.

Figure 6 presents Cocoanut Groove schema before the simulation process was started. The red squares represent fire while the blue ones represent people. Figure 7 presents Cocoanut Groove schema after completing the simulation. Table 2 presents the results of the performed simulation. Taking into account the real data concerning the number of fatalities in the Cocoanut Groove fire, the outcome which was closest to the actual death toll was achieved using 50% probability of people going towards the exit and with group effect off. Table 3 compares the results with real numbers. The relative error in all cases did not

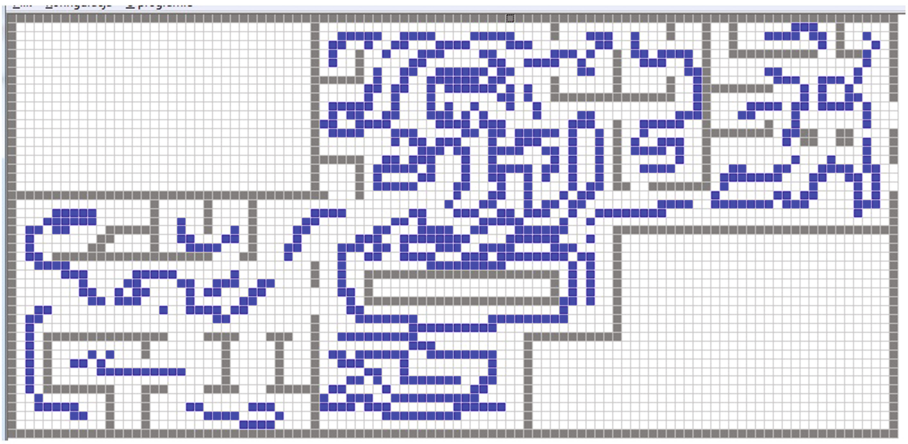

Fig. 6. The Cocoanut Groove schema with people and fire in KFNCA program

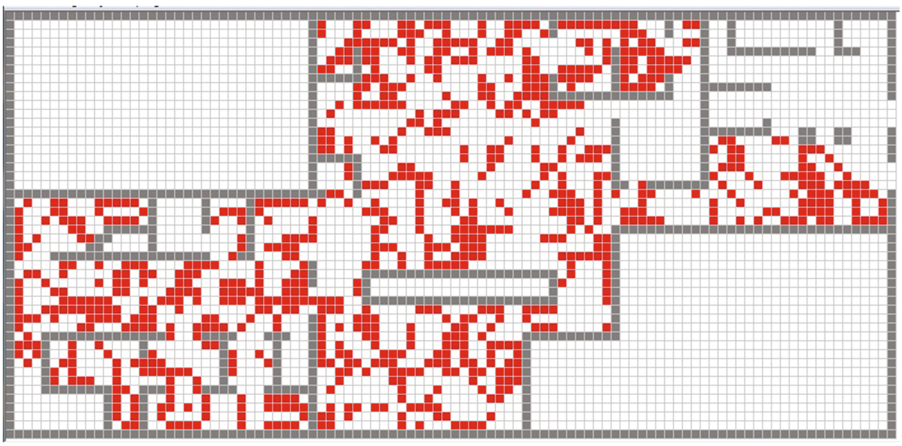

Fig. 7. KFNCA program after simulating fire in the Cocoanut Groove

Table 2. Results of simulation with KFNCA method

Number of people	Group effect					
	No			Yes		
	Probability of people heading towards the exit					
	25%	50%	75%	25%	50%	75%
Died	618	477	355	527	470	442
Trampled	145	178	194	295	228	177
Saved from fire	238	345	451	179	301	381

Table 3. A comparison of the KFNCA method results with actual numbers

Relative error [%]	Group effect					
	No			Yes		
	Probability of people heading towards the exit					
	25%	50%	75%	25%	50%	75%
Died	26	3	28	8	4	10
Trampled	13	7	17	77	37	7
Saved from fire	29	3	35	47	10	14

exceed 7%. The mortality rate depends on the place of the fire outbreak. If the fire blocks any room, then the people staying there are not able to escape and to reach the exit even if they move towards it with 100% probability. The group effect used in the program does not necessarily help in escape of people from the building. It can generate crowd, as people are looking for others to form groups and thus trampling can occur. When a person does not have any direction when he/she could move, he/she is trampled.

4 Conclusions

The comparison of the proposed method with actual case demonstrated that it is extremely difficult to create a simulation of fire escape scenario. The most challenging element is the people's behavior, which may become stochastic and unpredictable. The authors of this study managed to recreate the scenario of the escape of people from a building by means of cellular automata with the fuzzy numbers usage, the implementation of which was the object of this paper. This work proved, that using fuzzy numbers with some specific extension provided by Kosiński could give pretty good results. The results which proved to be closest to the actual numbers were achieved when the value of probability with which people escape was around 50%. The hindrances that affect the decision making process during the evacuation include, among others, limited visibility due to the smoke, resulting from combustion of flammable materials, high temperature and toxic gases. The result achieved in the prepared method can provide valuable information for architects and building constructors. The results obtained from the program confirm the thesis that insouciant or unlawful blocking of escape routes inside buildings may have tragic consequences at each stage of the building operation. The people who are responsible for fire safety and structural safety inspections may apply such tools to justify their decisions that sometimes could seem too strict. To make the simulation even more realistic, it is worth considering the option of automatic change of the parameter related to the probability of a person's moving towards the exit during the simulation. Adding further conditions in order to provide more accurate results is also possible. The future experiments should take this fact into account.

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An Application of Neural Network to Health-Related Quality of Life Process with Intuitionistic Fuzzy Estimation

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Abstract. The Neural networks are a tool that can be used for the modelling of many systems and process behavior. Quality of life has been receiving significant attention in health care systems over the past years. In public health research the 'quality of life' concept is often analyzed in the frame of 'health-related quality of life'. The artificial neural networks can "understand" the information from the health care processes. For the estimations between these two concepts we use intuitionistic fuzzy set.

Keywords: Intuitionistic fuzzy set · Health-related quality of life · Neural networks

1 Introduction

Quality of life (QoL) assessment has been receiving significant attention in health care systems over the past years. QoL concept reflects the strong societal and political interest in optimizing the effectiveness and efficiency of the health services. In public health research the QoL concept is often analyzed in the frame of 'health-related quality of life' (HrQoL). HrQoL has been increasingly acknowledged as a valid and appropriate indicator to measure health needs and outcomes. The terms HrQoL, health, functional status and health status are often used interchangeably [10].

HrQoL-assessment is becoming an indispensable part of the outcome health measures. HrQoL-questionnaires provide an integrated approach to health evaluating physical, mental, spiritual health and the general functioning of the individual. In public health, HrQoL-instruments are essential elements of health surveillance. Further, HrQoL-tools are valid signs about the health service needs and are reliable indicators for the success of any health intervention. Research has proven that self-reported health status often is a more powerful predictor of mortality and morbidity than many objective measures of health [8].

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HrQoL is difficult to be defined being a wholistic concept. However, there is a broad agreement about the fields that HrQoL comprises, which are: functioning and well-being in physical, mental, and social dimensions of health. Functioning involves the individual's ability to operate as well as the performance of usual daily activities of self-care and leisure. Well-being refers to individual's perceptions such as pain and energy and how one feels about life in terms of emotions (negative, neutral or positive) and overall perceptions of quality of life [7].

Herewith is used the data generated during a PhD-project researching HrQoL and its determinants at a small area level. The overall objective of this study was to measure the HrQoL in a community – the residents of Burgas, the fourth largest Bulgarian city, and further to assess the influence of socio-economic, demographic, and behavioural factors on HrQoL. The relationship between HrQoL and social capital (SC) is analyzed through a network-based approach. The key message is that measuring HrQoL could be an alternative or additional approach in understanding health inequalities. The preliminary results have been reported to the 29th Conference of the EuroQol Foundation in Rotterdam, 2012 [13], recently published in the peer-reviewed journal Health Promotion International, [16] and presented during the 8th European Public Health Conference in Milan [14].

The design of this PhD-study implements the mixed-methods approach following the sequential explanatory mixed-methods design. The EQ-5D-3L part is carried out in the first quantitative phase. The cross-sectional study was conducted in 2011 (March–May) with a representative sample of the citizens of Burgas (n = 1050, >18 years old, single exclusion criteria being institutionalised people). The sample was drawn through the method of two-stage random selection - first selecting the living area and then randomly selecting respondents from a particular neighbourhood. The number of the respondents was determined by the official regional statistics for age, gender and number of people living in each neighborhood. If the individual selected at random was unavailable, a substitute was selected at random from the same living quarter, age and gender. Thus the planned sample reached the number of 1050 respondents. The populations in both samples (the population of Burgas from the last census and the EQ-5D-3L sample) are similar in proportions according to the demographic characteristic of the town of Burgas (age and gender). Also the study sample is roughly representing the ethnic structure of the urban Bulgarian population, reported during the 2011 year census. Only pensioners are over-represented which corresponds with our interest towards health-related problems of the elderly in the community.

The information collected from each interview included: data from the answers of the EQ-5D-3-L version, additional details on age, gender, marital status, ethnicity, educational level, regular monthly income per household member (as defined in the last census methodology), occupation, sports, hobbies, smoking behaviour and membership in 2 types of organizations. The EQ-5D-3L instrument was pre-tested to decide on the number of the additional questions (aiming at an interview length of about 15 min) and to improve the wording.

2 The Instrument: EQ-5D-3L Questionnaire

EQ-5D-3L questionnaire for measuring HrQoL is used for the first time in Bulgaria at a community level and the Bulgarian version was provided by the EuroQol Group, especially for the project. The EQ-5D-3L is a standardized generic self-administered questionnaire which defines health in five dimensions including mobility, self-care, usual activities (work, study, housework, family, or leisure), pain or discomfort, and anxiety or depression. Each dimension has three levels ranging from "no problem", then "some problem", and to "extreme problem". Furthermore, EQ-5D-3L consists of two components. The first part is a questionnaire, a descriptive element with the five dimensions. Respondents rating their health status are asked to select the level of dimension which describes at best their "health today". On the second part respondents record each their self-rating of health on a 20-cm Visual Analogue Scale (VAS–"Thermometer" type) anchored by "100- denoting as best imaginable health state" and "0 - denoting as worst imaginable health state." The validity and reliability of the EQ-5D-3L questionnaire have been tested extensively [3, 4, 12]. Since 2013 the EQ-5D-3L instrument has been used in 20 national surveys and 7 regional population surveys [9].

3 Artificial Neural Networks

The artificial neural networks [2, 6] are one of the tools that can be used for object recognition and identification. The neural network is a structure that can adaptively perceive changes in the environment and learn from it. Figure 1 shows in abbreviated notation of a classic two-layered neural network.

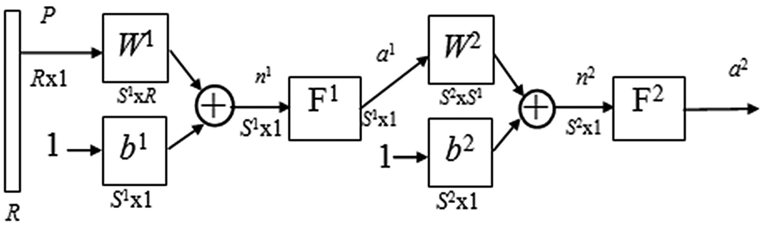

Fig. 1. Abbreviated notation of a two layer multi-layer perceptron

In the two-layered neural networks, one layer's exits become entries for the next one. The equations describing this operation are:

$$a^2 = f^2\left(w^2 f^1\left(w^1 p + b^1\right) + b^2\right), \tag{1}$$

where:

- a^m is the exit of the m-th layer of the neural network for m = 1, 2;
- w^{m} is a matrix of the weight coefficients of the each of the entries of the m-th layer;
- b is neuron's entry bias;
- f^1 is the transfer function of the 1-st layer;
- f^2 is the transfer function of the 2-nd layer.

The neuron in the first layer receives outside entries p. The neurons' exits from the last layer determine the neural network's exits a.

Since it belongs to the learning with teacher methods, to the algorithm are submitted training set (an entry value and an achieving aim – on the network's exit)

$$\{p_1, t_1\},\ \{p_2, t_2\}, \ldots,\ \{p_Q, t_Q\}, \tag{2}$$

$Q \in (1, \ldots, n)$, n – numbers of learning couple, where p_Q is the entry value (on the network entry), and t_Q is the exit's value corresponding to the aim. Every network's entry is preliminary established and constant, and the exit have to corresponding to the aim.

In order the neural network to be trained without overfitting, the training sequence is divided into three parts. In the first part, the neural network is trained (Training set) and is usually about 70% of all data. The second part is for validation (Validation set) of the training of the neural network. In this phase of training a mean square error is accumulated, as this part of the data is not used for training. By increasing the error in this phase above a certain threshold, the training is terminated [17]. The rest of the data used to test the behavior of the neural network. The three parts of the data (total of 100%) are used throughout the whole training process.

4 Intuitionistic Fuzzy Sets

Intuitionistic fuzzy sets (IFS, [1]) are sets whose elements have degrees of belonging and not belonging. They are defined by Krassimir Atanassov (1983) [1] as an extension of fuzzy sets of Lotfi A. Zadeh. In the classical theory, element belongs or does not belong to the summary. Zadeh defines membership in the interval [0, 1]. The theory of intuitionistic fuzzy sets extends above concepts by comparing belonging and not belonging real numbers in the interval [0, 1] and the sum of these numbers must also belongs to the interval [0, 1].

Let the universe is E. Let A be a subset of E. Let us construct the set

$$A* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

We will call $A*$ an IFS.

The functions $\mu_A(x)$: $E \to [0, 1]$ and $\nu_A(x)$: $E \to [0, 1]$ set degree of membership (membership) and non-membership *(non-membership).*

It is defined the function $\pi_A(x)$: $E \to [0, 1]$ through $\pi(x) = 1 - \mu(x) - \nu(x)$, corresponding to the degree of uncertainty *(uncertainty).*

In this paper we introduce the artificial neural network for modeling the data from the healthcare process. The intuitionistic fuzzy estimation is also proposed (Fig. 2).

5 Discussion

For the preparing we use MATLAB and neural network structure 20:25:1 (20 inputs, 25 neurons in hidden layer and one output (Fig. 3). For the inputs data we use quantitative EQ-5D-data taken from the presented above PhD-project [12]. For the output we use EQ_VAS.

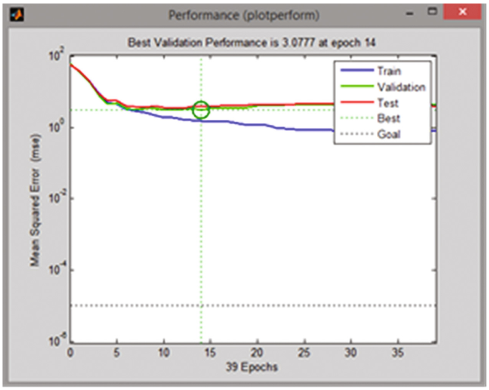

Fig. 2. The learning process

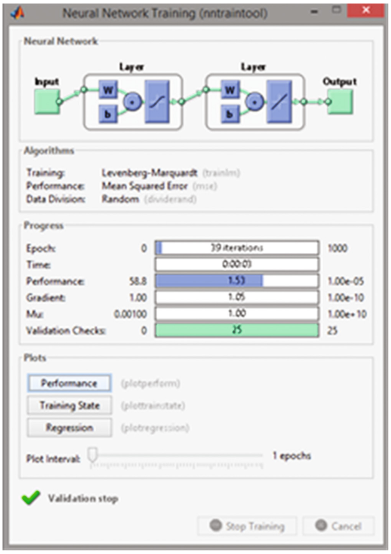

Fig. 3. The neural network structure

For the test we use 8 test vectors.

The output *a* of the neural network are shown on the table below:

N:	Test vectors	Otput
1	[24 2 4 1 1 3 1 2 4 1 2 1 1 2 2 2 2 2 2 2]	8.4085
2	[21 2 9 2 1 2 1 2 4 4 1 3 2 2 2 2 2 2 2 2]	7.9439
3	[34 2 1 3 1 3 1 2 3 1 8 1 2 2 2 2 2 2 2 2]	8.5997
4	[47 2 3 3 1 3 1 2 3 1 2 3 1 2 2 2 2 2 2 2]	7.9411
5	[49 2 5 2 1 1 1 2 4 1 8 3 1 2 2 2 2 2 2 1]	6.8644
6	[65 1 3 3 1 2 1 2 5 1 8 3 2 2 2 2 2 2 2 2]	4.6312
7	[77 1 3 4 1 1 1 2 5 2 2 3 2 2 2 2 2 2 2 2]	2.5528
8	[79 1 5 1 1 1 1 2 5 2 3 1 2 2 2 2 2 1 2 2]	6.2531

In this paper we also introduce intuitionistic fuzzy assessment of the compare of the data of the neural network output a_{nn} and investigation values *val*.

The values of the measurement belong to degree of affiliations μ if $a_{nn} - val > \varepsilon$, where the ε is the threshold value.

If $a_{nn} - val < \varepsilon$ the assessment, belongs to degree of the non-affiliations (ν).

If $(a_{nn} - val) \in [-\varepsilon \div +\varepsilon]$ the assessment, belongs to degree of the uncertainty (π).

The obtained information, are represented by ordered pairs <μ, ν> of real numbers from the set [0, 1] × [0, 1].

The degree of uncertainty also represents as a π = 1−μ−ν.

At the beginning is done statistics of the 1050 values that we used for learning the neural network. Initially when still no information has been obtained, all estimations are given initial values of <0, 0>. When $k \geq 0$, the current (k + 1)-st estimation is calculated on the basis of the previous estimations according to the recurrence relation

$$\mu_{k+1},\ \mathrm{v}_{k+1} = \left\langle \frac{\mu_k k + m}{k+1}, \frac{v_k k + n}{k+1} \right\rangle,$$

where $\langle \mu_k, v_k \rangle$ is the previous estimation, and <μ, ν> is the estimation of the latest measurement, for m, $n \in [0, 1]$ and $m + n \leq 1$.

6 Conclusion

The authors investigate the possibility to analyze HrQoL combining different techniques like Intuitionistic fuzzy set and neural networks. This is the second article which presents the application of Intuitionistic fuzzy estimations [15].

Herewith is designed a neural network, which can predict some of the parameters which characterize HrQoL.

Intuitionistic fuzzy estimations are calculated in order to assess the quality of the designed neural network. This estimates how the real data corresponds to the predicted values. Since, during the assessment process is not always possible to get a precise qualitative estimate; three stages of assessment are introduced: affiliations, non-affiliations and uncertainty.

The use of intuitionistic fuzzy set in order to estimate between the two concepts: artificial neural networks and HrQoL is an innovative step towards the worldwide interest in measuring perceived health status and finding health inequalities. Such analysis could support the research processes of gaining a comprehensive understanding of HrQoL and designing relevant public health policies.

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Application of OFN Notation in the Fuzzy Observation of WIG20 Index Trend for the Period 2008–2016

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Abstract. The article concerns the issue of seeking patterns in trends. In the study a method to detect patterns in trends recorded in a linguistic has been proposed. Linguistic variables take their values as a result of the calculations in Ordered Fuzzy Numbers notation. Thus, in a first stage fuzzyfication of the source data occurs. Transposition of the parameters was applied to daily quotations (min, max, the opening value, the closing value and the direction of change) which were interpreted as a single OFN number. This is the first usage of this notation to describe the stock index which allows to describe five different parameters in a single number. Then the data are converted into linguistic form. The level of trend sequence similarity is determined by following set parameters: the Frame size of the pattern, expressed as a percentage similarity of trend sequence to a frame set at the outset, threshold indicating how many trend fragments is consistent with the frame and the frequency of the pattern occurrences. Patterns detected in this way in the nature of things are characterized by various support, and coefficients of similarity in both the whole pattern and the individual elements. For the purposes of this study, we developed a dedicated computer program performing patterns search. As research material the main index of the Warsaw Stock Exchange, i.e. WIG20 from the years 2008–2016 was used as a data set. This preliminary study is beginning to develop Rule base forecasting methods, and in this direction further experiments will be carried out.

Keywords: WIG20 · OFN · Trend · Rule-based forecasting · Pattern recognition · Financial engineering

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1 Introduction

Rule-based Forecasting (RBF) integrates statistical data and domain expertise to create a more precise forecasting methods. One could say that RBF is a kind of an expert system, which utilizes the properties of time series and selected extrapolation techniques [26–28]. Current implementations of the expert system contain approx [5,12,13] 100 rules that combine four basic forecasts methods of extrapolation, i.e. random walk, linear regression, Holt's exponential smoothing and Brown's exponential smoothing [4,23,37]. These rules were based on the study of literature, interviews and questionnaires as well as domain expertise of five independent experts. Calibration of the basic version of the rules was conducted with the usage of 90 time series while the validation with the next 36 series [38,39]. In this sense, the RBF is a system of knowledge exploration [33,41], which successfully combines statistical techniques with the domain knowledge. The authors report that both the results of independent studies and recent interest in M-3 competition showed that RBF is consistently more accurate than the leading standards, such as the method of random walk or combining equal weights [25]. Rule-based Forecasting is intensively developing field of knowledge, which is gaining growing number of followers [9,10]. Full bibliography is widely available and its evocation, even in part, is not the purpose of this work. However,

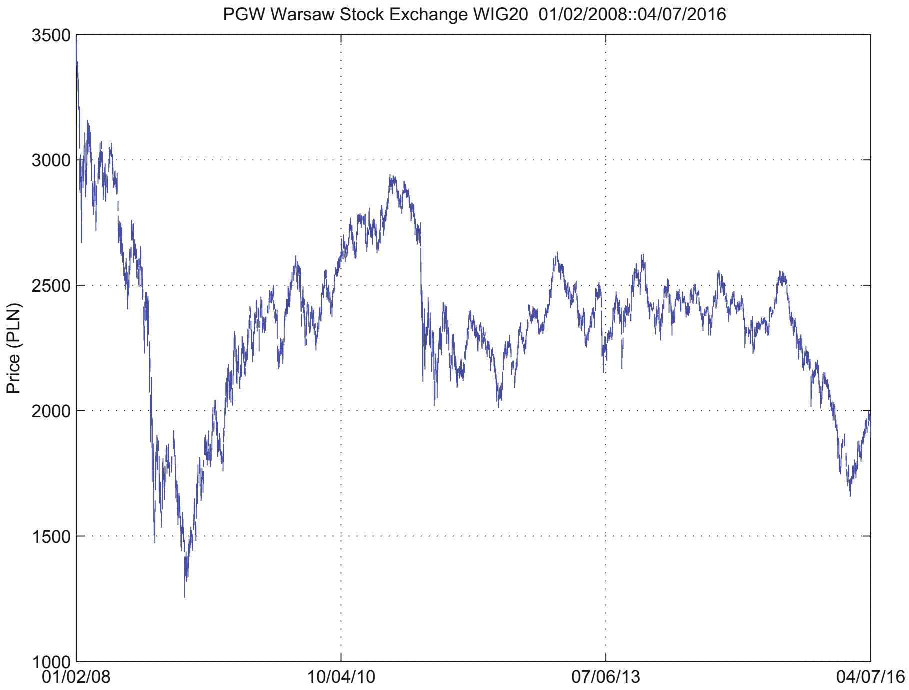

Fig. 1. WIG20 from the years 2008 to 2016

it is worth to mention at least these three authors, i.e. J.S. Armstrong, M. Adya and F. Collopy [1–3], whose shared and individual works drive research trends of the group centered around the International Journal of Forecasting.

This paper focuses on the formulation of the method of detecting patterns in a time series of real data. With the assumption, that the series of data were initially fuzzyfied [40]. The detected patterns of literals sequences, that describe how fuzzy the recurring trends in the test sequence data are, will be used in later studies to formulate short-term trend prediction rules [24,29,37]. The input is considered a collection of linguistic data (created with the help of fuzzy logic) describing the course of the trend of the stock exchange index [34,35]. As research material a data set of the main index of the Warsaw Stock Exchange, i.e. WIG from the years 2008–2016 was used. The Fig. 1 shows the data as a pictorial graph.

The source data are shown in the following table. They contain standard information of the session, i.e. the opening value, closing value, maximum and minimum values or the percentage change as compared to the value of the previous day [32,36]. These data were then converted into linguistic values. In Table 1

Table 1. Selected WIG20 index data. The dataset covers the time period from 1 February 2016 to 29 February 2016

Index	Date	Opening value	Maximum value	Minimum value	Closing value	Change
A	02/01/2016	1781.67	1781.67	1760.71	1777.54	−0.15
B	02/02/2016	1776.26	1779.17	1730.19	1731.64	−2.58
C	02/03/2016	1734.44	1750.02	1726.09	1739.08	0.43
D	02/04/2016	1748.87	1788.65	1748.87	1788.65	2.85
E	02/05/2016	1789.26	1805.25	1782.58	1790.87	0.12
F	02/08/2016	1790.32	1792.73	1759.20	1769.71	−1.18
G	02/09/2016	1766.02	1774.76	1746.44	1767.45	−0.13
H	02/10/2016	1772.40	1792.10	1760.99	1760.99	−0.37
I	02/11/2016	1759.51	1766.82	1730.66	1758.93	−0.12
J	02/12/2016	1758.10	1782.03	1757.28	1765.53	0.38
K	02/15/2016	1780.23	1798.25	1779.97	1787.98	1.27
L	02/16/2016	1788.14	1791.30	1765.47	1783.93	−0.23
M	02/17/2016	1786.14	1847.95	1786.14	1847.95	3.59
N	02/18/2016	1847.00	1878.14	1841.81	1853.76	0.31
O	02/19/2016	1848.09	1852.98	1835.19	1841.06	−0.69
P	02/22/2016	1849.60	1881.08	1849.60	1859.73	1.01
R	02/23/2016	1847.90	1862.13	1838.35	1844.96	−0.79
S	02/24/2016	1844.44	1849.03	1814.33	1818.37	−1.44
T	02/25/2016	1826.34	1844.88	1822.54	1838.17	1.09
U	02/26/2016	1845.08	1863.49	1841.55	1844.57	0.35
W	02/29/2016	1836.93	1837.34	1811.86	1824.08	−1.11

the actual values and the corresponding linguistic data are indicated. The latter have been added to the original dataset.

2 Fuzzy Observation of WIG20 Index with the Usage of OFN Notation

Fragment of the stock exchange index WIG20 data covering one month of the current year is presented in Table 1. The data is presented in the standard format for this type of time-series. Individual rows with daily figures can be identified by the letters of the alphabet.

The Fig. 2 presents a well known graphical charts interpretation of time series representing the indexes. As you can see the graph shows all the five previously introduced attributes of each trading day. These attributes are shown in Table 1.

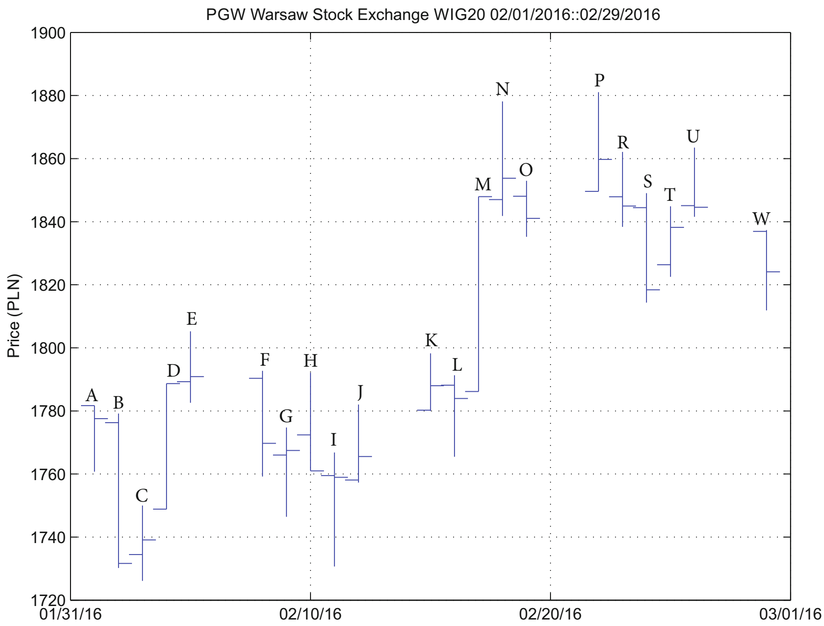

Fig. 2. The WIG20 index values for the period from 1 February 2016 to 29 February 2016

As one can see in Fig. 2 graphic interpretations of lines A and B carry information about daily decline in index value. While the visualizations of records C and D show a substantial increase. It is also worth paying attention to the charts of lines E and U, which despite fluctuations in the value of the index during the day essentially do not reveal any changes between the opening and the closing value [8,17,25]. Since this type of visualization doesn't seem intuitive to everyone we can introduce at this point a new one based directly on the logic of Ordered Fuzzy Numbers [30,31]. Table 2 depicts characteristic points of the fuzzy number in OFN notation [14,15,20]. Descriptions of the attributes are the same as the names of daily notations from Table 1.

Table 2. Characteristic points

OFN number	$f_A(0)$	$f_A(1)$	$g_A(1)$	$g_A(0)$	Orientation of OFN number
WIG20 index	Minimum value	Opening value	Closing value	Maximum value	Change

Table 3. Example of a positively ordered OFN number as an interpretation of the WIG20 index

OFN number	$f_A(0)$	$f_A(1)$	$g_A(1)$	$g_A(0)$	Positive orientation of OFN number
WIG20 index	Minimum value	Opening value	Closing value	Maximum value	Change (positive value)

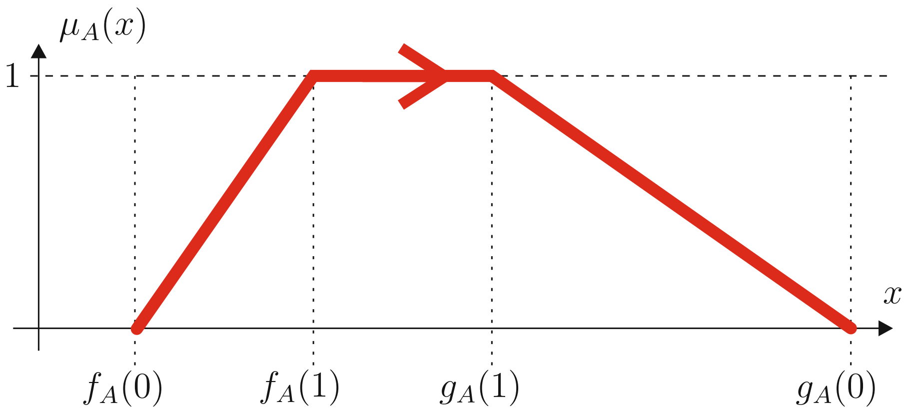

Fig. 3. Graphic interpretation of a *positively* ordered OFN number with characteristic points in relation to the WIG20 index

Figure 4 illustrates an example of a daily record of index value. The day closing brought an uplift in index value. The translation of attributes for this case is shown in Table 3. As a result we get Fig. 3 which presents ordered fuzzy number in OFN notation. This number is oriented (ordered) positively, which is symbolized by an arrow (characteristic for this notation) directed towards increasing values of daily quotes (Table 4).

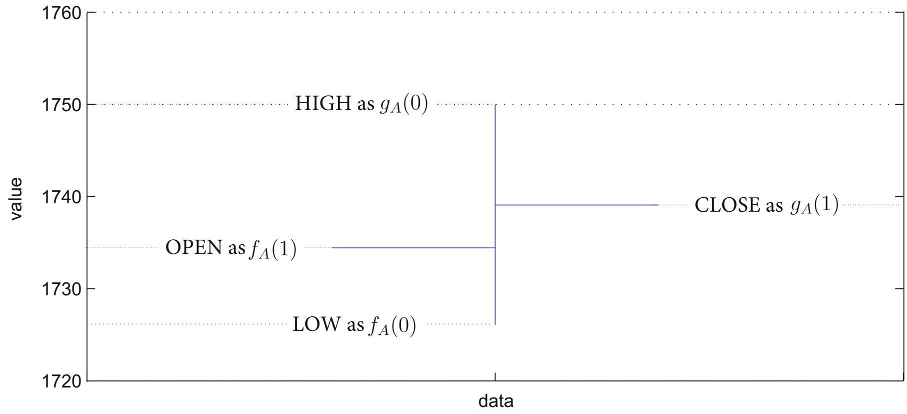

Fig. 4. Graphical interpretation of the *positive* value of the change tag in the high-low chart for the WIG20 index

Table 4. Example of a negatively ordered OFN number as an interpretation of the WIG20 index

OFN number	$f_A(0)$	$f_A(1)$	$g_A(1)$	$g_A(0)$	Negative orientation of OFN number
WIG20 index	Maximum value	Closing value	Opening value	Minimum value	Change (negative value)

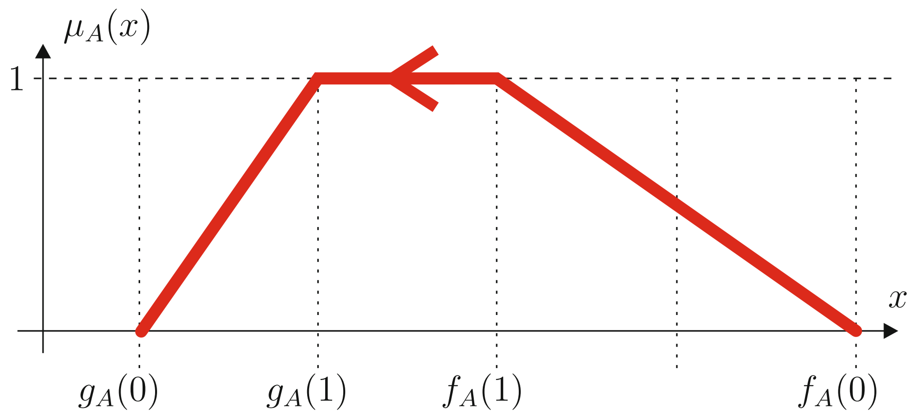

Fig. 5. Graphic interpretation of a *negatively* ordered OFN number with characteristic points in relation to the WIG20 index

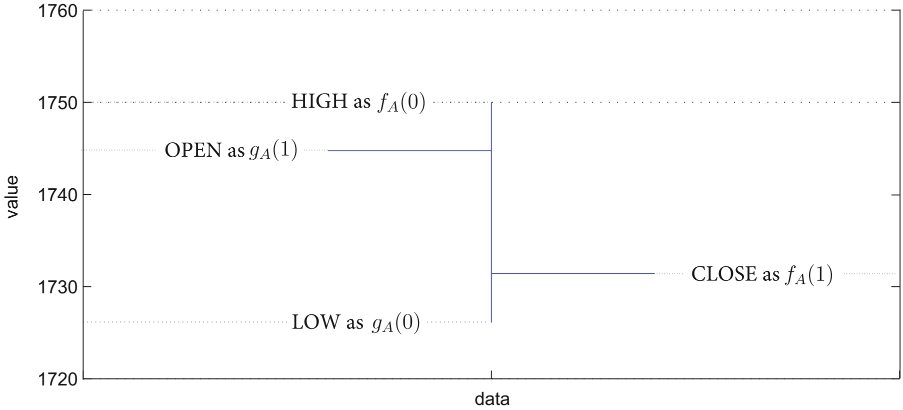

Fig. 6. Graphical interpretation of the *negative* value of the change tag in the high-low chart for the WIG20 index

Figure 5 shows the ordered fuzzy number stretched on the same values as the previous one but carrying different information on the direction of daily index value changes. Figure 6 High-Low chart shows the decline in the value of the daily rate. This is illustrated in Fig. 5 negative orientation of OFN number stretched on the same points as Fig. 3.

3 Formalization of OFN Fuzzy Observation

WIG20 index value data are designated as $\mathbb{R}_1, \ldots, \mathbb{R}_m$ corresponding to daily records of the index. The set R is a base for fuzzy observation of changes in the four independent and one subsidiary parameter of the index. This observation is conducted in an OFN notation. For each day the number $\mathbb{R}_i \in \{\mathbb{R}_1 \ldots \mathbb{R}_m\}$ is created to contain the four values recorded respectively in time. t_i is designated as the day of measurement, while the times t_{OPEN}, t_{MIN}, t_{MAX} and t_{CLOSE} represent the moment of measurement respective for the daily index values of: opening, minimum, maximum and close value.

Definition 1. *Fuzzy observation of index WIG20 at the time t_i is a set of*

$$\mathbb{R}/t_i \in \{\mathbb{R}^{(0)}/t_{OPEN},\ \mathbb{R}^{(1)}/t_{MIN},\ \mathbb{R}^{(1)}/t_{MAX},\ \mathbb{R}^{(0)}/t_{CLOSE}\} \tag{1}$$

where

$$t_{CLOSE} > \{t_{MIN}, t_{MAX}\} > t_{OPEN}$$

$$f_{\mathbb{R}}(0) < f_{\mathbb{R}}(1) < g_{\mathbb{R}}(1) < g_{\mathbb{R}}(0)$$

Measurement time of traffic intensity $t \in \{t_{OPEN}, t_{MIN}, t_{MAX}, t_{CLOSE}\}$ is identical with the notion of the Order in the OFN. This means that the order of measurements is determined and significant. The orientation of the OFN number depicted on the Fig. 3 as the arrow specifies the daily trend of index changes. The default orientation of $\mathbb{R}$ number is positive as shown in Fig. 2.

Lemma 1.

$$\mathbb{R}_{positive} = \begin{cases} \mathbb{R}_{CLOSE} \leq \mathbb{R}_{OPEN} \\ \mathbb{R}_{OPEN}, \mathbb{R}_{MIN}, \mathbb{R}_{MAX}, \mathbb{R}_{CLOSE} \\ f_{\mathbb{R}}(0), f_{\mathbb{R}}(1), g_{\mathbb{R}}(1), g_{\mathbb{R}}(0) \end{cases} \tag{2}$$

and in the opposite case

$$\mathbb{R}_{negative} = \begin{cases} \mathbb{R}_{CLOSE} > \mathbb{R}_{OPEN} \\ \mathbb{R}_{CLOSE}, \mathbb{R}_{MAX}, \mathbb{R}_{MIN}, \mathbb{R}_{OPEN} \\ f_{\mathbb{R}}(0), f_{\mathbb{R}}(1), g_{\mathbb{R}}(1), g_{\mathbb{R}}(0) \end{cases} \tag{3}$$

WIG20 is the most prestigious index on the Warsaw Stock Exchange and is based on quotations of the 20 largest Polish stock companies. It is a type of price index [42,43]. WIG20 Component companies must meet a number of requirements to enter the index including the value of trading in shares and market capitalization [44,45]. The composition of the companies making up the WIG20 is verified every quarter. Additionally the index cannot consist of more than 5 companies coming from the same sector. The index cannot be make up of investment funds. Stock market analysts argue that the updating of the composition of WIG20 companies allows for better representation of market trends. Thus, the WIG20 index is an aggregate of twenty companies, which directly create the index trend. Fuzzy observation of the subaggregate Sm of the index n companies quotations change is expressed with aggregating formula.

Definition 2. *Fuzzy observation of index WIG20 at the time t_i is a set of*

$$s_m = \sum_{i=1}^{n} \left\{ \begin{matrix} \mathbb{R}_{positive} & \mathbb{R}_{negative} \\ \mathbb{R} \cdot w_i & | \quad -\mathbb{R} \cdot w_i \end{matrix} \right\} \tag{4}$$

where

$$n \leq 20, w_i \in \{w_1, \ldots w_n\}$$

is a vector of the individual companies impact, default $w_i = 1$.

The percentage of individual companies is determined as follows

$$P_j = \frac{\mathbb{R}_j * w_j}{\sum_{i=1}^{m} \mathbb{R}_i * w_i} * 100\% \tag{5}$$

where $j \in [1, m]$.

Definition 3. *As a predictor of a change of trend could be a case where a sub-aggregate Sm of the WIG20 aggregate representing n companies (e.g. from one sector), indicate by directing OFN different trend than that seen at the moment in orientation of the index. This can can be expresses as the fulfilment of the following rule:*

$$\begin{aligned} &\textbf{IF } WIG20 \textit{ is positive } \textbf{AND } S_m \textit{ is negative} \\ &\qquad\qquad \textbf{THEN } \textit{Possible change is true} \end{aligned} \tag{6}$$

4 Conclusion

Decisions regarding trading activity are one of the most complicated issues that financial engineering is facing. This is because the common understanding of the processes taking place in the financial markets is far from perfect. These belong to a highly dynamic phenomena exhibiting characteristics of non-linear chaotic behaviour. As one of the most important issues in technical analysis appears detection and proper use of recognizable trends. Therefore financial security requires building a market strategy, which will be profitable during longer periods of strong uptrend, while in times of economic fluctuations will minimize losses. Such strategies are usually derived from expert knowledge stored in the intuitive rules, which can be quite naturally modeled using fuzzy logic [11,16,18,20]. There are a number of indicators and analytical methods created for the technical analysis that can be used to identify the trend [21,22]. Many are insufficiently intuitive and others can even discourage the investors by their overcomplicated form [5,13,14]. The method proposed in the article which describe the trend by ordered fuzzy numbers is relatively simple and intuitive. It refers to the known method of visualization but brings added value to the observations [19,21]. It allows using one number in OFN notation to provide five at-tributes (values) that describe the index trading day. What's more, the manner of description of ordered fuzzy numbers used in the article also allows the early detection of a possible change in a trend, which gives a chance for a short-term prediction. These are not the first study related to the use of OFN notation in financial engineering. It is worth recalling some previous publications related to the issue of how to describe the rate of return on investment. The authors intend to continue research in order to develop a more multilateral models for predicting changes in the trend of the time series [6–8].

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Advanced IT/ICT Applications

Large-Scale Periodic Routing Problems for Supporting Planning of Mobile Personnel Tasks

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Abstract. Implementation concepts of a decision support system for large-scale periodic time-dependent vehicle routing and scheduling problems with complex constraints supporting planning and management of mobile personnel tasks (sales representatives and others) are discussed. Complex nonuniform constraints with respect to frequency, time windows, working time, etc. are taken into account with additional fast adaptive procedures for operational rescheduling of plans in presence of various disturbances. Five individual solution quality indicators with respect to a single personnel person are considered. This paper deals with modeling issues corresponding to the problem and general solution concepts.

1 Introduction

Decision support system for planning and management of mobile personnel tasks has to deal with complex optimization problems. The SATIS software, SATIS GPS (2016), supports various fleet management tasks as well as management of mobile employees who work in the field: sales representatives, service technicians, suppliers, drivers work of a sales team and allows to stimulate sales activity. SATIS Mobile module provides applications for mobile terminals, smartphones and tablets (iOS, Android) connected to the management and analytics system for a manager, dispatcher or forwarding agents. The latter functions include planning of visits and meetings of field employees, assignment of tasks to be performed and preparation and settlement of sales plans. Daily access to data is ensured by embedded indicators and reports such as, for example, planning of work of commercial representatives, monitoring of execution of a plan or report on visits. Gathered data is used to measure effectiveness of employees, it allows to define objectives and motivates to improve sales results. SATIS Optimisation module gives a possibility to set the most advantageous route so that a driver may cover a distance in the appropriate time. Optimization of processes reduces fuel consumption and travel time, due to which mobile employees may work more effectively, maintaining or reducing costs. For efficient support of

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planning and management of mobile personnel tasks there have been developed procedures to deal with large-scale periodic time-dependent vehicle routing and scheduling problems with complex nonuniform constraints with respect to frequency, time windows, working time, etc. Such a problem might be considered as some kind of the Vehicle Routing Problem, Toth and Vigo (2002). Actually, a generalization of the Periodic Traveling Salesman Problem (TDTSP), Francis and Smilowitz (2010), and the Time-Dependent Traveling Salesman Problem (TDTSP), Angelelli et al. (2009), Angelelli and Speranza (2002), Campbell and Wilson (2014), Belhaiza et al. (2014), Favaretto et al. (2007), Savelsbergh (1992), or Time-Dependent Orienteering Problem (TDOP), Souffriau et al. (2013), Tricoire et al. (2010), with several real features. Specifically, the following features we take into account: each point (customer) can define multiple time windows during which it is available and it can be serviced; the travel time between the points varies, due to the traffic, and actually it depends on the traffic time zone, in which the transit actually occurs; starting and ending depots are treated as points so that they also have time windows. Moreover, additional fast adaptive procedures for operational rescheduling of plans in presence of various disturbances are needed and those procedures should work on mobile terminals of the personnel.

Several solution quality indicators with respect to a single personnel person must be considered. The objectives are structured in a two-level lexicographic optimization where on the top level is maximized the number of visits completed according to all the restrictions. At the bottom level quantified schedule quality criteria are aggregated: minimized travel cost, minimized labor cost (including overtime costs), minimized excess deviation from the weekly working time norm, minimized lower and upper deviations from the reference visit frequency (gap between subsequent visits).

2 Model

The major optimization problem may be formalized with a mixed integer linear programming model. While the set of points represents all the customers, the major scheduling objects are visits. There is considered large enough time horizon to allow multiple visits at the same point being scheduled. The sets and indices are referred to as follows:

$d \in \{1, \ldots, \|D\|\}$	days within scheduling horizon
$w \in W$	weeks within scheduling horizon
D^w	days of week w
$p \in P$	points
$i \in I$	visits
I_p	set of visits in point p
$p(i)$	point served in visit i
Θ_i^d	set of time windows of point $p(i)$ on day d
Λ_{ij}^d	set of traffic time zones for path $(p(i), p(j))$ on day d

Further there are given several parameters:

$r_i = [d_i^{\min}, d_i^{\max}]$	possible range of days for visit i
$\tau_i^{dk} = [a_i^{dk}, b_i^{dk}]$	k-th time window of point $p(i)$ on day d; $k \in \Theta_i^d$
$\sigma_{ij}^{dn} = [c_{ij}^{dn}, d_{ij}^{dn}]$	n-th traffic time zone for path $(p(i), p(j))$ on day d
s_i	length of visit i
w_{ij}^{dn}	travel time between visits i and j on day d within traffic time zone n
C_{ij}^{dn}	travel cost between visits i and j on day d within traffic time zone n
T_b^d	maximum basic working hours on day d
T_m^d	maximum total working hours on day d
W_b^d	unit cost for basic working hours on day d
W_o^d	unit cost for overtime hours on day d
T_b^w	maximum basic working hours in week w
T_m^w	maximum total working hours in week w
W_o^w	unit cost for weekly overtime hours in week w
a_p	required gap between subsequent visits in point p
$P_p^+(\delta)$	piecewise linear non decreasing penalty function of excess δ over the required gap between subsequent visits in point p
$P_p^-(\delta)$	piecewise linear non decreasing penalty function of shortage δ to the required gap between subsequent visits in point p

The model is built with the following decision and state variables:

y_i^{dk}	binary variable $= 1$ only if visit i is performed on day d within time window k of point $p(i)$
x_{ij}^d	binary variable $= 1$ only if after visit i on day d visit j is performed (it might be on the subsequent day)
z_i^d	binary variable $= 1$ only if visit i is the first visit on day d
β_i^d	leaving time after completing visit i on day d
v_{ij}^{dn}	binary variable $= 1$ only if travel between visits i and j on day d is performed within traffic time zone n
r^d	total travel time (working time) on day d
r^w	total travel time (working time) in week w
u_p^d	binary variable $= 1$ only if point p is visited on day d
δ_p^{d+}	excess over the required gap between subsequent visits in point p on day d (0 if there is no earlier visit)
δ_p^{d-}	shortage to the required gap between subsequent visits in point p on day d (0 if there is no earlier visit)
$\tilde{u}_p^{d,d'}$	auxiliary binary variable for all p and $d' < d$;

The model relations and restrictions are represented by the following constraints. Each visit is performed at most once and within specified time periods:

$$\sum_{d \in D} \sum_{k \in \Theta_i^d} y_i^{dk} \leq 1 \qquad \forall i \in I \tag{1}$$

$$y_i^{dk} = 0 \qquad \forall i \in I,\ k \in \Theta_i^d,\ d \notin r_i \tag{2}$$

No more than one visit daily in a given point:

$$u_p^d = \sum_{i \in I_p} \sum_{k \in \Theta_i^d} y_i^{dk} \qquad \forall p, d \tag{3}$$

Path continuity (first point visited on day $d+1$ is the last one visited on day d):

$$z_i^d + \sum_{j \neq i} x_{ji}^d = \sum_{j \neq i} x_{ij}^d + z_i^{d+1} \qquad \forall i \in I, d \in D \tag{4}$$

Relations between y and x:

$$\sum_{k \in \Theta_i^d} y_i^{dk} = \sum_{j \neq i} x_{ij}^d \qquad \forall i \in I, d \in D \tag{5}$$

Time constraints – departure after concluding visit:

$$\beta_i^d + \sum_{n \in \Lambda_{ij}^d} w_{ij}^{dn} v_{ij}^{dn} + s_j + M x_{ij}^d \leq \beta_j^d + M \qquad \forall i, j \in I, d \in D \tag{6}$$

Time windows for point $p(i)$:

$$(a_i^{dk} + s_i) y_i^{dk} \leq \beta_i^d \leq b_i^{dk} + M(1 - y_i^{dk}) \qquad \forall i \in I, d \in D, k \in \Theta_i^d \tag{7}$$

Traffic time zones:

$$c_{ij}^{dn} v_{ij}^{dn} \leq \beta_i^d \leq d_{ij}^{dn} + M(1 - v_{ij}^{dn}) \qquad \forall i, j \in I, d \in D, n \in \Lambda_{ij}^d \tag{8}$$

$$\sum_{n \in \Lambda_{ij}^d} v_{ij}^{dn} = x_{ij}^d \qquad \forall i, j \in I,\ d \in D \tag{9}$$

Daily travel time:

$$\beta_i^d + \sum_{n \in \Lambda_{ij}^d} w_{ij}^{dn} v_{ij}^{dn} - (\beta_j^d - s_j) \leq r^d \qquad \forall i, j \in I : i \neq j, d \in D \tag{10}$$

$$r^d \leq T_m^d \qquad \forall d \in D \tag{11}$$

Weekly travel time:

$$r^w = \sum_{d \in D^w} r^d \qquad \forall w \in W \tag{12}$$

$$r^w \leq T_m^w \qquad \forall w \in W \tag{13}$$

Deviations from the required gap between subsequent visits must be nonnegative:

$$\delta_p^{d+}, \delta_p^{d-} \geq 0 \qquad \forall p, d \tag{14}$$

Shortages to the required gap between subsequent visits must satisfy

$$\delta_p^{d-} \geq \max\{a - du_p^d + d'u_p^{d'} \ : \ d' < d,\ u_p^d = 1,\ u_p^{d'} = 1\} \qquad \forall p, d$$

Hence:

$$\delta_p^{d-} \geq a - du_p^d + d'u_p^{d'} - M(1 - u_p^d) - M(1 - u_p^{d'}) \qquad \forall p; d' < d \tag{15}$$

Excesses over the required gap between subsequent visits must satisfy

$$\delta_p^{d+} \geq \min\{du_p^d - d'u_p^{d'} - a \ : \ d' < d,\ u_p^d = 1,\ u_p^{d'} = 1\} \qquad \forall p, d$$

Hence, more complicated constraints with auxiliary variables $\tilde{u}_p^{d,d'}$:

$$\delta_p^{d+} \geq du_p^d - d'u_p^{d'} - a - M(1 - u_p^d) - M\tilde{u}_p^{d,d'} \qquad \forall p; d' < d \tag{16}$$

$$\sum_{d'<d} \tilde{u}_p^{d,d'} \leq d - 1 - \frac{1}{M} \sum_{d'<d} u_p^d \qquad \forall p; d' < d \tag{17}$$

$$\tilde{u}^{d,d'} \geq 1 - u_p^{d'} \qquad \forall p; d' < d \tag{18}$$

Objective represents two level lexicographic optimization:

$$\operatorname{lexmax} \Bigg\{ \sum_{i \in I} \sum_{d \in D} \sum_{k \in \Theta_i^d} y_i^{dk}, \tag{19a}$$

$$- \Big[\sum_{d \in D} \sum_{i,j \in I} \sum_{n \in \Lambda_{ij}^d} c_{ij}^{dn} v_{ij}^{dn} \tag{19b}$$

$$+ \sum_{d \in D} W_b d r^d + \sum_{d \in D} (W_o^d - W_b^d)(r^d - T_b^d) \tag{19c}$$

$$+ \sum_{w \in W} W_o^w (r^w - {T_b}^w) \tag{19d}$$

$$+ \sum_{p \in P} \sum_{d \in D} (P_p^+(\delta_p^{d+}) + P_p^-(\delta_p^{d-})) \Big] \Bigg\} \tag{19e}$$

where the number of visits performed with respect to the constraints (19a) is maximized on the first level. On the second level, there are minimized: travel costs (19b), total work costs taking into account overtime (19c), penalty for exceeding weekly limits on working hours (19d) and total of penalties for deviations from the required gap to subsequent visits after in point p on day d representing a measure of quality of service (19e).

3 Approximate Solution

Since the formulated model is an extremely difficult problem to solve due to the number of binary variables, for medium and larger number of customers it is more efficient to tackle it with heuristic approaches. Business requirements enforce that in short time (operational) perspective every point is visited by the same personnel person. In longer horizon, due to optimization needs or personnel fluctuation, there is possible a change of the personnel allocation. Therefore, while solving the problem one can separate the phase of personnel allocation to points and simultaneous allocation of visits to specific days with respect to required frequency and personnel limitations. Further one has the more-less classical Time-Dependent Traveling Salesman Problem with Multiple Time Windows (TDTSPMTW) with the complex objective function replacing the travel duration. Approximate solution of the TDTSPMTW depends on the use some metaheuristic to examine various sequences of points (master problem) while comparing the route durations (objective function values) computed as a subproblem optimizing the minimum route duration for a given sequence of points. Finally, the whole problem can be decomposed into three problems:

1. Allocation of points (customers) to personnel.
2. Finding optimal sequence of points for one-day schedule.
3. Computing the minimum route duration for a given sequence of points.

3.1 Personnel Allocation

Allocation of points to personnel is related to long-term planning horizon and thereby an exact solution based on optimization of all the routes and schedules would be a problem of immortal size. Therefore, an approximate approach is used based on several iterations of a simplified scheduling algorithm covering the entire planning horizon, all the points and the entire personnel. It is crucial for the approximation method to encompass as many input data, constraints and relations as possible. The asymmetric distance matrix, points' time windows, usual pattern of visits, labor law – all these data were taken into account in our allocation algorithm.

The objective of the allocation includes the forecasted average system cost in long-term horizon.

The basic analysis shows the geographic location of points (travel times and costs between points) and frequency of visits at given points have major influence on the solution. There are many location clustering algorithms available, but none that take into account frequency or sequence of visits. That is why we propose the unique algorithm to deal with the complexity of the above problem.

In the algorithm we use simplified greedy scheduling for the long-term horizon to statistically rate usefulness of the particular assignment of person to the point. The evaluation is iteratively repeated and depending on the rating the personnel set meant for visiting given point is modified (tightened).

Data Preparation. As it is not possible to have exact information on the future visits for very long horizon, we utilize frequency information for the given point to generate visits randomly, range of days r_i for each visit i in particular. Next, all the visits are ordered according to the latest possible visit day $d_i^{\max}$. For each point p the initial personnel set Q_0^p is determined, basing on personnel skills and/or points and personnel preferences, etc.

Simplified Scheduling. Algorithm implements simple greedy construction heuristic. Taking first visit i from the ordered list, routes for each person from set Q_0^p are examined searching for a place to insert an additional visit. Only days in range r_i are considered. The visit is inserted at a place where the cost increase is the smallest. Keep in mind that all the usual penalties (over-hours) are involved when the maximum daily or weekly route duration is exceeded. The algorithm is computationally effective thus allowing construction of routes for very long periods.

The Allocation Algorithm. The algorithm can be formalized as follows.

1. Initialization, input data preparation including initial set Q_p^0, $k := 0$.
2. Simplified scheduling for long time horizon. Only person $q \in Q_p^k$ can be scheduled for a visit to point p.
3. The resulting schedule is analyzed to determine a person in the set Q_p^k with smallest number of visits to point p. If $|Q_p^k| > 1$, the person is removed from the set.
4. If $|Q_p^k| = 1 \; \forall p$, then STOP. Otherwise go to 2.

The maximum number of iterations equals the number of personnel available.

The initial tests have shown that using the above allocation algorithm one can get up to 28% better final schedules, as compared to the random allocation of personnel to points. The tests were performed on two limited geographical areas – the whole Warsaw capital city and one of its districts.

3.2 Algorithms for Computing the Schedule

For computing the one-day schedule, i.e. for solving the TDTSPMTW master problem, we have chosen three different metaheuristics: Simulated Annealing (SA), List-Based Threshold Accepting algorithm (LBTA) and Variable Neighborhood Search (VNS).

Simulated annealing, first introduced by Kirkpatrick et al. (1983), was adapted to solving combinatorial problems by Černý (1985). Applications of the SA algorithm to various optimization problems have been studied. The effectiveness of the algorithm was also inspected by Hurkała and Hurkała (2012). The optimization process of the SA algorithm can be described in the following steps. Before the algorithm can start, an initial solution is required. Then, repeatedly, a candidate solution is randomly chosen from the neighborhood of the current solution. If the candidate solution has the same or better value than the current one, it is accepted and it replaces the current solution. A worse solution than the current one still has a chance to be accepted with, so called, acceptance probability. This probability is a function of difference between objective values of both solutions and depends on a control parameter taken from the thermodynamics, called temperature. The temperature is decreased after a number of iterations (search intensification), and the process continues as described above. The optimization is stopped either after a maximum number of iterations or when a minimum temperature is reached. The best solution found during the annealing process is considered final. The threshold accepting metaheuristic belongs to the randomized search class of algorithms. The search trajectory crosses the solution space by moving from one solution to a random neighbor of that solution, and so on. Unlike the greedy local search methods which consist of choosing a better solution from the neighborhood of the current solution until such can be found (hill climbing), the threshold accepting allows choosing a worse candidate solution based on a threshold value. In general concept of the threshold accepting algorithm a set of decreasing threshold values is constructed prior to the computations. The rate at which the values decrease controls the trade-off between diversification (associated with large threshold values) and intensification (small threshold values) of the search. However, it is immensely difficult to predict how the algorithm will behave when a certain decrease rate is applied for a given problem without running the actual computation. These reflections led to the list-based threshold accepting branch of threshold accepting metaheuristics. In the list-based threshold accepting approach, see Lee et al. (2002, 2004), Tarantilis et al. (2003), Hurkała and Śliwiński (2014), instead of a predefined set of values, the list is dynamically created during the presolve phase of the algorithm. Both phases of the algorithm are very similar. The optimization algorithm starts from an initial solution, creates new solution from the neighborhood of the current one using one of move functions (perturbation operator) chosen at random from a predefined set of functions, and compares both solutions. If the candidate solution is better, it becomes the current one. Otherwise, a threshold value is calculated as a relative change between these two solutions. The threshold value is compared against the largest value at the list. If the new threshold value is larger, then the new solution is discarded. Otherwise, the new threshold value replaces the value from the list, and the candidate solution is accepted to the next iteration. The best solution found during the optimization process is considered final. Within the initial phase the threshold values are accumulated within the list. The list allows to control a gradual switching form the diversification to intensification in the search process. As a consequence, the algorithm is converging to a local or possibly even a global optimum.

Variable neighborhood search is a metaheuristic introduced by Mladenović and Hansen (1997). This global optimization method is based on an idea of systematically changing the neighborhood in the descent to local minima and in the escape from the valleys which contain them. It has already been successfully used in various Vehicle Routing Problems, see Hemmelmayr et al. (2009). The VNS algorithm consists of two building blocks: variable neighborhood descent (VND) and reduced variable neighborhood search (RVNS). The optimization process of VND can be explained as follows. Starting form an initial solution, within a given neighborhood a candidate solution is repeatedly generated and it replaces the current one if it is better. After a specified number of iterations (neighborhood size) the neighborhood is changed to the first one if a better than initial solution has been found. Furthermore, the best solution found so far replaces the initial solution. Otherwise, if the search resulted in no better solutions that the initial one, the current neighborhood is changed to the next one. Either way, the whole operation is repeated again until the search gets stuck in a local optimum. The best solution found during this process is returned. The RVNS is a stochastic algorithm that executes the VND with different initial solutions. If the VND returns a solution that is better than the current one, it gets replaced, and the algorithm starts over from the first neighborhood. Otherwise, the neighborhood is changed to the next one. The whole process can be repeated until the stopping criterion (e.g. specified number of evaluations, time limit) is met.

The most problem-specific mechanism of SA, LBTA and VNS algorithms, that always needs a different approach and implementation, is the procedure of generating a candidate solution from the neighborhood of the current one, which is called a perturbation scheme, transition operation/operator or a move function. Although there are many ways to accomplish this task for the traveling salesman problem, we have chosen the following three operators: interchanging two adjacent customers, interchanging two random customers, moving a single, random customer to a randomly chosen position.

3.3 Minimum Route Duration Algorithm

The problem of finding the minimum route duration for a given sequence of visits (points) in the TDTSPMTW can be defined as follows, see Hurkala (2015). The visit i defines time interval $[\alpha_i^d, \beta_i^d]$, where $\alpha_i^d = \beta_i^d - s_i$ is the time of arrival at point $p(i)$, while β_i^d is the departure time of the visit. Since the customers can already have multiple time windows, we can assume that they do not span across multiple traffic time zones (i.e. they have been partitioned if necessary). That means, there is defined a function $\gamma : \Theta_i^d \rightarrow \Lambda_{ij}^d$ that maps the time windows into time zones. Having the travel time unambiguously defined for every time window one can normalize the model, similarly to Jong et al. (1996), by merging the service time and the travel time into one parameter denoting the visit duration. At the same time, the end of every time window is postponed by the travel time associated with it. In order to put the minimum route duration algorithm to work, some time windows have to be dismissed and some cropped. This is done in the preprocessing phase. The aim of the preprocessing is to ensure that the route starting at the beginning of the first point time window and the route ending at the end of the last point time window are both feasible (in some cases it can be one and the same route). If no feasible route can be found during the preprocessing, then the problem has no solution. Similar approach has been used by Tricoire et al. (2010).

The minimum route duration algorithm consists of iteratively reviewing schedules of which the one with the shortest duration is chosen. The procedure of constructing each schedule is divided into two phases. The first phase consist of computing a feasible sub-schedule that starts from a given point, and ends with the last one. Repeatedly, among the time windows that have enough room to accommodate the visit after the given departure time, the earliest visit departure time is searched for. The second phase constructs the so-called dominant solution, cf. Jong et al. (1996). To put it simply, a schedule with starting time α^1 dominates schedule with starting time α^2, if $\alpha^1 > \alpha^2$ and at the same time ending time $\beta^1 = \beta^2$, cf. Tricoire et al. (2010). In the procedure, instead of fixed visit departure time, the arrival time is used. Nevertheless, the principle is the same - repeatedly search for the latest possible starting time of a visit among the time windows, that are suitable, i.e. when the currently considered visit does not overlap the initial one.

The main procedure of the algorithm enumerates schedules, constructing them in a particular fashion. For each virtual time window, first, a feasible sub-schedule is constructed with the initial departure time set to the end of the current time window. Second, a dominant sub-schedule is constructed with the initial arrival time set to the latest possible start of the visit in the considered window. The route duration is than computed as the difference between the departure time from last customer and arrival time at the first customer. The best solution found during the process is stored and returned. Savelsbergh (1992) has introduced the concept of forward time slack to postpone the beginning of service at a given customer. It can be proven, that the optimal schedule can be postponed until one of the visits ends with the time window. Hence, by iteratively reviewing schedules one by one so that every possible time window with visit at the end of it is taken into consideration, an optimal schedule is found by our algorithm. The entire procedure has $O(|V^d|)$ time complexity, where $V^d = \cup_{i \in I^d} \Theta_i^d$ is the set of all time windows within the set I^d of points allocated to the route.

3.4 Numerical Experiments

The numerical experiments were performed on a number of randomly generated problem instances while using the real-life maps and varying traffic data of two limited geographical areas – the whole Warsaw capital city and one of its districts. The number of customers (points to be visited) was ranging from 15 to 50. Larger routes were not widely tested since they were usually infeasible due to the time constraints. The algorithms were implemented in C++. All the computations were executed on the Intel Core i7 3.4 GHz microprocessor.

To better compare relative performance of the three algorithms, the only stopping criterion for single run was reaching exactly 10000 schedule evaluations – the same for all computations and problem sizes. This way we could compare the speed as well as the convergence per iteration.

The algorithms produced similar results in terms of both the solution quality and the computation time. For some instances one algorithm produces better results, while for some other it is the other way around. Generally, the VNS tends to find a little bit better solutions: 2.49% on average better than LBTA, and 2.97% than SA. LBTA and SA perform very similar. The former is on average better by only 0.49% than the latter.

The computation time for a given problem instance was almost the same for every algorithm due to the (identical) stopping criterion. It was ranging from half a second for smaller routes of about 20 points to less than 10 s for up to 50 points. Hence, the procedures for computing the one-day schedule and recompute it due to changing traffic data or other disturbances, can be implemented and effectively used on mobile terminals.

4 Conclusions

Task planning and management for mobile personnel requires solving large-scale periodic time-dependent vehicle routing and scheduling problems with complex constraints and goals. The main difficulty there lies in the problem size, real, atypical constraints and requirements and the limited computation times. Our three level approach allows for efficient tactical and operational optimization of the problem. The algorithm embedded in the SATIS fleet management software creates ready-to-use sophisticated, highly optimized decision support system, leading to an substantial increase of personnel productivity and total cost savings.

Our work was to design, implement and test the above problem decomposition, and all the particular optimization procedures. Among others we have compared three metaheuristic algorithms that compute the one-day schedule in Time-Dependent Traveling Salesman Problem with Multiple Time Windows problem with route duration criterion, which can be successfully utilized in time-oriented TSP, VRP, OP, and other mixed routing and scheduling problems. To our knowledge, this is the first attempt of solving this time-dependent problem with multiple time windows.

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Network Dimensioning with Minimum Unfairness Cost for the Efficiency

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Abstract. Network dimensioning is a specific kind of optimization problems. In general the main goal in this task is to ensure a connection between given pairs of nodes (source-target) with possible high efficiency. When each pair (demand) brings different revenue, the problem of blocking less attractive demands appears. Usually this situation is caused by not including any fairness criterion into optimization and thus optimizing only the total (revenue) efficiency of the system. Another complication is the fact of inverse proportionality of these criteria. In this paper an optimization model has been examined which takes into account a fairness criterion and minimizes the loss of system efficiency. It may also be understood as optimizing the ratio of fairness degree to the mean of the traffic flow in the network. For implementation of the model the CPLEX package was used. As input data the example of backbone Polish network structure was chosen. To evaluate the approach, basic statistics which help in describing the equity of distribution such as standard deviation, kurtosis and Gini coefficient are used.

1 Introduction

Telecommunications networks are facing increasing demand for many services. Therefore, the problem to determine how much traffic of every demand (traffic stream) should be admitted into the network and how the admitted traffic should be routed through the network, so as to satisfy the requirements of high network utilization and to guarantee fairness to the users, is one of the most challenging problems referred to telecommunications networks design. The problem, usually called network dimensioning is related to planning the deployment of network resources (bandwidth, link capacity etc.). Dimensioning of telecommunications networks requires the allocation of the flows (loads, bandwidth) to given traffic demands for the source-destination pairs of nodes. Unit flow allocated to the given demand is associated with a revenue that may vary for different demands. Besides fundamental assumptions which model the basic traffic flow relations in network in case of given decision-maker preferences. To describe a demand one can define it as a path. The path between two nodes in the net where first is the source and second is the target. It is possible to define path in two ways. One way is to allow it to bifurcation and the second is to prohibit this. There is two models to choose between this assumptions. One is called node-link and the

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second the link-path approach. In addition the path could be described as request to ensure a connection between source-target pair on possible best quality. There is used the measure of reserved bandwidth on the link to estimate a quality of the connection. In this paper the quality measure would be also called as traffic load allocated on given demand. It is also assumed that for each unit of load there is a unit of revenue (profit) which has an impact on main objective function of optimization problem. Each link in the network has also capacity limit which could not be exceed while dimensioning process. The simplest objective functions commonly used for system optimization is the total outcome (the total throughput in typical network problems) $T(\mathbf{y}) = \sum_{i \in I} y_i$, equivalently as the mean (average) outcome $\mu(y) = \frac{T(y)}{m}$. The mean (total) outcome maximization is primarily concerned with the overall efficiency. Indeed, in our network dimensioning problem it represents the value of revenue gained from allocated load unit when the vector of unit profits is known. It may generate solutions where some demands are discriminated. Such a situation sometimes even leading to starvation of some paths problem. A straightforward network dimensioning could be thought of as a search for such a network flows that will maximize the aggregated network throughput while staying within a budget constraint for the costs of link bandwidth. However, maximizing aggregated throughputs can result in extremely unfair solutions allowing even for starvation of flows for certain services. On the other hand, while looking at the problem from the perspective of a network user, the network flows between different nodes should be treated as fairly as possible. The so-called Max-Min Fairness (MMF) is widely considered as such ideal fairness criteria. Indeed, the lexicographic max-min optimization used in the MMF approach generalizes equal sharing at a single link bandwidth to any network while maintaining the Pareto optimality. Certainly, allocating the bandwidth to optimize the worst performance, may cause a large worsening of the overall throughput and of the network and thus the total revenue. Therefore, network management must consider two conflicting goals: increasing revenue (throughput) and providing fairness. Fairness criterion often appears in many cases of allocation tasks. For example it can be a problem of allocation public resources. Each person wants to has a highest possible access to goods such as healthcare, safety, public transport and many others public resources financed from taxes. There have made a lot of work in theory of justice in this area. Fairness criterion is also important and must not be omitted in government decisions. These approaches should refer to network optimization process. It is easy to imagine that while dimensioning a network where exist any demands returning different unit profits, optimization problem basing only on the efficiency criterion, would return a solution with some paths blocked. This is the situation when in the network are links belonging to more than two different paths with different unit revenues. In this case solution maximizing the total profit would allocate the whole possible load on the more profitable path. Situation when at least one demand is such discriminated example is not acceptable in terms of fairness. Other concepts to gain a fair and more efficient solution are methods such a Proportional-Fairness (PF), Ordered Weighted Averages (OWA), Reference Point Method (RPM) or the others. Generally used to evaluate the fairness of the resource allocation are inequality measures. In particular, the Gini coefficient which defines the percentage of inequality of the obtained values. Index equal to 0 gives an information about full equitability of the distribution and is growing to 1 while distribution's inequality is increasing.

In the paper the model of fair network optimization based on the minimization the ratio of loss of fairness for possible high increase of total system efficiency is analyzed. Total efficiency measure describes the mean of all values of revenues gained from allocated load on demands. As a fairness measure some index of equity of allocated traffic stream is used.

2 Measures of Equality and Fairness

In system analysis fairness is usually qualified with so-called fairness measures (or inequality measures), which are functions ϱ that maps $\mathbf{y}$ into (nonnegative) real numbers. Various measures have been proposed throughout the years, e.g., in and references therein. Typical inequality measures are deviation type dispersion characteristics. They are *translation invariant* in the sense that $\varrho(\mathbf{y}+\mathbf{a}e) = \varrho(\mathbf{y})$ for any real number a (where $\mathbf{e}$ vector of units (1, …, 1)), thus being not affected by any shift of the outcome scale. Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of perfectly equal outcomes while taking positive values for unequal ones, thus to be minimized for fairness. Although some fairness measures, like Jain's index requires maximization.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *maximum absolute difference* or the *mean absolute difference*

$$\Gamma(\mathbf{y}) = \frac{1}{2m^2}\sum\nolimits_{i\in I}\sum\nolimits_{j\in I}\left|y_i - y_j\right|. \tag{1}$$

Another group of measures us related to deviations from the mean outcome, like the *maximum absolute deviation* or the *mean absolute deviation*

$$\delta(\mathbf{y}) = \frac{1}{m}\sum\nolimits_{i\in I}|y_i - \mu(\mathbf{y})|. \tag{2}$$

The *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\sigma^2(\mathbf{y}) = \frac{1}{m}\sum\nolimits_{i\in I}(y_i - \mu(\mathbf{y}))^2 = \frac{1}{2m^2}\sum\nolimits_{i\in I}\sum\nolimits_{j\in I}\left(y_i - y_j\right)^2. \tag{3}$$

Deviational measures may be focused on the downside semideviations as related to worsening of outcome while ignoring upper semideviations related to improvement of outcome. One may define the *mean (downside) semideviation*

$$\bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum\nolimits_{i\in I}(\mu(\mathbf{y}) - y_i)_+ \tag{4}$$

and the standard (downside) semideviation

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\frac{1}{m}\sum\nolimits_{i\in I}(\mu(\mathbf{y}) - y_i)_+^2} \tag{5}$$

where $(.)_+$ denotes the nonnegative part of a number.

Similarly, the *maximum (downside) semideviation* is given as

$$\Delta(\mathbf{y}) = \max_{i\in I}(\mu(\mathbf{y}) - y_i) = \mu(\mathbf{y}) - M(\mathbf{y}) \tag{6}$$

where $M(y) = \min_{i\in I} y_i$ represent the minimum outcome. Its generalization is the worst conditional *k/m*-semideviation defined as the average of k largest semideviations:

$$\Delta_{k/m}(\mathbf{y}) = \frac{1}{k}\sum_{i=1}^{k}(\mu(\mathbf{y}) - \Theta_i(\mathbf{y})) \tag{7}$$

where it can be generalized to $\Delta_{\beta(\mathbf{y})}$ for any real $0 < \beta \le 1$ as

$$\Delta_\beta(\mathbf{y}) = \frac{1}{\beta}\int_0^\beta (\mu(\mathbf{y}) - F_y^{(-1)}(\alpha))d\alpha \tag{8}$$

where the quantile function:

$$\mathrm{F}_{\mathrm{y}}^{(-1)}(\beta) = \inf\{\eta : F_y(\eta) \ge \beta\} \;\; for \;\; 0 < \beta \le 1$$

is the inverse of the cumulative distribution function (CDF):

$$\mathrm{F}_{\mathrm{y}}(d) = \sum_{i\in I}\frac{1}{m}\delta_i(d) \;\; where \;\; \delta_i(d) = \begin{cases} 1 & if\, y_i \le d \\ 0 & otherwise \end{cases} \tag{9}$$

which for any real (outcome) value d provides the measure of outcomes less or equal to d.

It follows from the duality relations that the worst conditional β-semideviations can be found as the optimal solution to the following problem:

$$\min\left\{\sum_{i\in I}^{m}\frac{1}{m}y_i - y + \frac{1}{\beta}\sum_{i\in I}^{m}\frac{1}{m}(y - y_i)_+ : \mathbf{y} \in Q\right\}, \tag{10}$$

For the special case $\beta = k/m$ one gets the model:

$$\min\left\{\sum_{i\in I}^{m}\frac{1}{m}y_i - y + \frac{1}{k}\sum_{i\in I}^{m}\frac{1}{m}(y-y_i)_+ : \mathbf{y}\in Q\right\}, \tag{11}$$

In economics there are usually used relative inequality measures normalized by mean outcome, so-called indices. The most commonly accepted is the Gini index (Gini coefficient) which is the relative mean difference. Considered in networking the Jain's index computes a normalized square mean. One can easily notice that direct minimization of individual outcomes resulting in equal but very low outcomes. The same applies to the Jain's index maximization. Moreover, this contradiction cannot completely be resolved with the standard bicriteria mean-equity model which takes into account both the efficiency with optimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$. However, for several inequality measures, the inequality-reward ratio optimization

$$\min\left\{\frac{\varrho(y)}{\mu(y)-\tau} : y\in Q\right\} \tag{12}$$

guarantees fairness of the solution provided that $\mu(\mathbf{y}) > \tau$ and $\varrho(\mathbf{y}) > 0$. This applies, in particular, to the worst conditional semideviation. For practical implementation the tolerance ε is introduced into formula:

$$\min\left\{\frac{\varrho(y)+\varepsilon}{\mu(y)-\tau} : \mu(y)\geq\tau+\varepsilon, y\in Q\right\}. \tag{13}$$

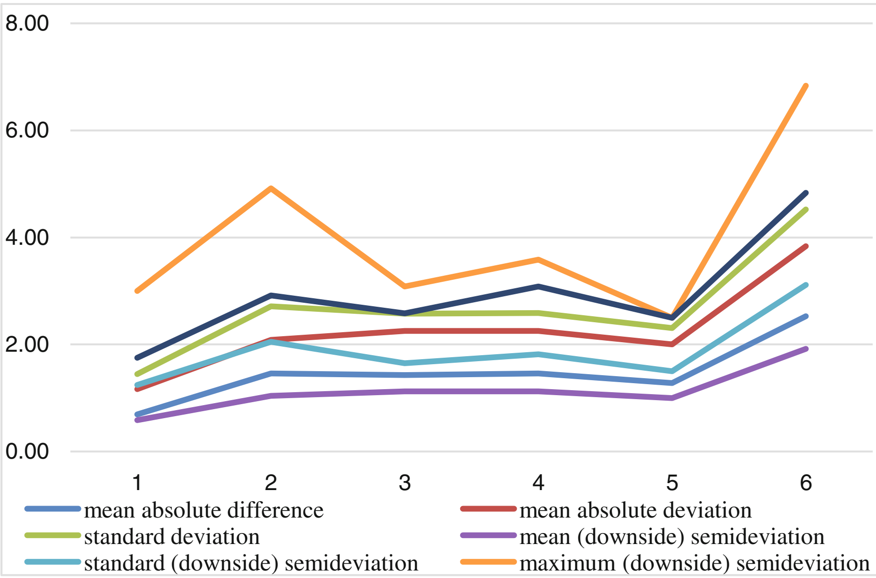

Fig. 1. Results of calculated equity measures for hospital allocation example

Consider the problem of allocation the hospital in sample area and six control points which should be the closest as possible to it. In the table below are presented several possible locations of the hospital with the distances from the control points (Fig. 1 and Table 1).

Table 1. Results of calculation of chosen equity measures for spatial allocation example.

Allocation	1	2	3	4	5	6
Distance from control point 1 [m]	6	4.5	3.5	10.5	9	0
Distance from control point 2 [m]	6	4	6	11	10	5
Distance from control point 3 [m]	2	0	2.5	8	5.5	4
Distance from control point 4 [m]	6	8	10	5	6.5	13
Distance from control point 5 [m]	5.5	8	7.5	4	4	12
Distance from control point 6 [m]	4.5	5	4	7	4	7
Mean absolute difference	0.69	1.46	1.43	1.46	1.28	2.53
Mean absolute deviation	1.17	2.08	2.25	2.25	2.00	3.83
Standard deviation	1.44	2.71	2.57	2.59	2.31	4.52
Mean (downside) semideviation	0.58	1.04	1.13	1.13	1.00	1.92
Standard (downside) semideviation	1.24	2.05	1.65	1.82	1.50	3.11
Maximum (downside) semideviation	3.00	4.92	3.08	3.58	2.50	6.83
k largest semideviations (50%)	1.00	2.08	2.25	2.25	2.00	3.83
k largest semideviations (33%)	1.75	2.92	2.58	3.08	2.50	4.83

3 Mathematical Optimization Model

In the network dimensioning problem, a decision-maker is interested in obtaining the possible highest volume of load on each of given demands and in maintaining the fairness. The graph of network is undirected. It means that traffic flow could move in two sides on each link. Let us denote the set of links as $l \in \{1,..,L\}$ and append a capacity limit c_l for each of them (14).

$$\sum_{d \in D} x_{ld} \leq c_l, \forall l \in L \tag{14}$$

The problem is to allocate a bandwidth reservation on given demands $d \in \{1,..,D\}$. It could also be called as paths which have one source and target belonging to set of nodes $n \in \{1,..,N\}$. Each unit of load provides a revenue which value was determined by p_d.

$$\sum\nolimits_{l \in L} a_{nl} \cdot x_{ld} - \sum\nolimits_{l \in L} b_{nl} \cdot x_{ld} = \begin{cases} h_d \; if \; n = s_d \\ 0 \; if \; n \neq s_d, t_d \\ -h_d \; if \; n = t_d \end{cases}, \forall n \in N \tag{15}$$

Equation (15) defines traffic flow constraint. Other parameters have been defined as follows. First of all there are binary values of node-link incidence matrixes a_{nl} and b_{nl}.

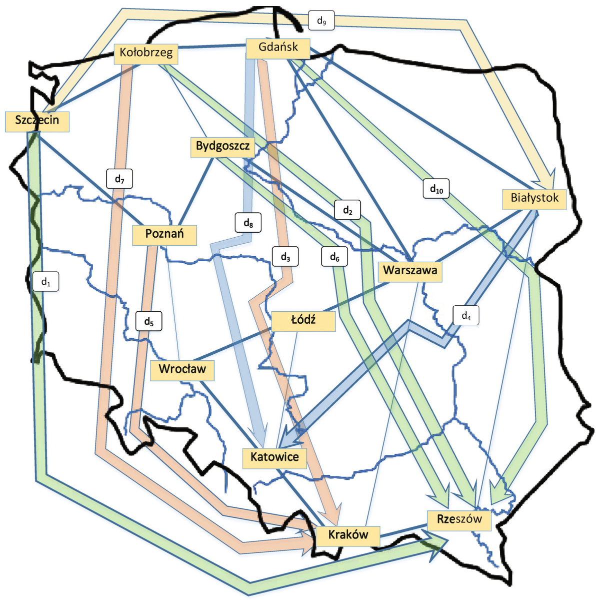

Fig. 2. An illustration of considered network with given demands

Parameter a_{nl} takes value of 1 when *l-th* link comes out from *n-th* node and 0 otherwise. Similarly, parameter b_{nl} which takes value of 1 when *l-th* link comes into *n-th* node and 0 otherwise. The equation concerns three cases of node types. Each *n-th* node could be a source node (s_d), target node (t_d) and connection node when $n \neq s_d, t_d$ in *d-th* demand.

In the paper an example of Polish backbone network is considered. Characteristic parameters and structure of the network are presented in Fig. 2. Dimensioning in the simplest form of maximization of the total revenue on bandwidth allocated to demands has an objective function:

$$\max \sum_{d \in d} h_d \cdot p_d, h_d \geq 0 \tag{16}$$

Model (16) guarantees the maximum value of total revenue from the given network. This simple implementation, unfortunately, usually returns non-acceptable solutions with respect to fairness. There may appear blocked demands by assigning the value of 0 to the least profitable paths. Such a starvation is caused by using strict rational preference model in the optimization process. Method which includes fairness criterion has to use anonymous and equitable model of preference. It could be done by extend the problem formulation by additional constraints and change the objective function.

In the paper there is proposed formulation that uses such constraints but also optimizes the ratio of fairness and total efficiency of the system. Referring to the ratio formulation (13) with the worst conditional β-semideviations as the inequality measure applied to outcomes $y_d = h_d \cdot p_d$, the new network dimensioning optimization model is built as follows.

$$\min \frac{z_o + \in}{z - \tau} \tag{17}$$

$$z - u + \frac{\sum_{d \in D} k_d}{\beta \cdot \bar{D}} = z_o, k_d \geq 0 \tag{18}$$

$$z = \frac{1}{\bar{D}} \sum_{d \in D} h_d \cdot p_d, h_d \geq 0, \forall d \in D \tag{19}$$

$$k_d + h_d \cdot p_d \geq u, \forall d \in D \tag{20}$$

$$z \geq \tau + \in, \in = \tau \cdot \beta \tag{21}$$

$$\sum_{l \in L} a_{nl} \cdot x_{ld} - \sum_{l \in L} b_{nl} \cdot x_{ld} = \begin{cases} h_d \text{ } if \text{ } n = s_d \\ 0 \text{ } if \text{ } n \neq s_d, t_d, \forall n \in N \\ -h_d \text{ } if \text{ } n = t_d \end{cases} \tag{22}$$

$$\sum_{d \in D} x_{ld} \leq c_l, \forall l \in L \tag{23}$$

Except of fundamental constraints of node-link formulation problem (22 and 23), there are included necessary assumptions to maintain the anonymous and special treat of percentage of the least attractive demands in case of efficiency. Main constraint that has an influence for the index of fairness is given in Eq. (18). Here appears the extra unrestricted variable u. The criterion of the total system efficiency was append us mean of total bandwidth that has been allocated. The ratio model mentioned in introduction has also $\in$ parameter which means the additional improvement of the solution with respect to fairness criterion. Considering the objective function of the model (17), it is easy to see the non-linearity of equation. To solve this problem and make it linear, formulation can be simplify by making the substitution: $v = \frac{z}{z-\tau}, v_0 = \frac{1}{z-\tau}, \widetilde{h_d} = \frac{h_d}{z-\tau}, \widetilde{k_d} = \frac{k_d}{z-\tau}, \tilde{u} = \frac{u}{z-\tau}, \widetilde{x_{ld}} = \frac{x_{ld}}{z-\tau}$. The simplified model takes the following form:

$$\min v - \tilde{u} + \frac{\sum_{d \in D} \widetilde{k_d}}{\beta \cdot \bar{D}} + \in \cdot v_0 \tag{24}$$

$$v - v_0 \tau = 1 \tag{25}$$

$$v = \frac{1}{\bar{D}} \sum_{d \in D} \widetilde{h_d} \cdot p_d \tag{26}$$

$$\widetilde{k_d} + \widetilde{h_d} \cdot p_d \geq \tilde{u}, \forall d \in D, \widetilde{k_d} \geq 0 \tag{27}$$

$$1 \geq \in \cdot v_0 \tag{28}$$

$$\sum_{l \in L} a_{nl} \cdot \widetilde{x_{ld}} - \sum_{l \in L} b_{nl} \cdot \widetilde{x_{ld}} = \begin{cases} \widetilde{h_d} \; if \; n = s_d \\ 0 \; if \; n \neq s_d, t_d \\ -\widetilde{h_d} \; if \; n = t_d \end{cases} \tag{29}$$

$$\sum_{d \in D} \widetilde{x_{ld}} \leq c_l \cdot v_0, \forall l \in L \tag{30}$$

The τ parameter means the given value of total efficiency function of maximum fair solution. In the most of cases it is the Max-Min solution. The goal of optimization method is to increase the efficiency of the system while reducing the fairness as much as possible. This procedure takes place for given parameter β which is related to degree of decision-maker preferences between two main criteria. There is necessary to append additional constraint (25) to guarantee the possibility of finding solution. β parameter in this model is related to fairness index and it is the one of the most important control parameter in this formulation. It accepts the values from 0 to 1. More precisely, for value of 0 it should be expected the solution similar to Max-Min approach and for value of 1, the solution close to simple maximization of total revenue which does not take into account the fairness criterion.

All other equations are different from the previous model only in a terms of just-mentioned substitutions. Modified model meets the requirements of linear programming and can be implemented in a simple way in a general purpose in linear programming package, like CPLEX, GLPK, GUROBI, etc. Any variables in the model takes continuous values.

4 Results

As optimization package the CPLEX environment has been chosen. Experiments have been conducted on several sets of input data, the revenue vectors. In the paper one of them is analysed in details. First of all on the set of data the Max-Min solution has been obtained. This is necessary to get the value and substitute it as τ. Described model was used to obtain the results of several cases of input data and control parameter β where $\beta \in (0; 1 >$. It refers to the degree of unfairness of the solution. In other words this is the percentage of the most discriminated paths or demands obtaining the smallest revenues. Parameter β close to 0 should allow to obtain the most possible fair solution. It can be also compare to the solution obtained from concept of Max-Min regularized by the sum of revenues.

The method first maximizes the least profitable demand and then maximizes each of the rest as much as possible. For many cases this solution will not satisfy the decision-maker. Moreover, sometimes it can be even dominated at all. When β takes the higher values the solution is losing on equitability and gaining on efficiency. Solution control parameter equal to 1 refers to simple maximization of the system efficiency with no regard to fairness criterion. In Table 2 are presented the results for ten values of the control parameter β and one vector of revenues P. In the lower part of

Table 2. Results in case of growing parameter β, and statistics of values distribution

d	p_d	*Max-Min*	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1$
1	110	3917	3917	2500	2328	2328	2889	2889	2889	1639	1639	0
2	70	3917	3917	2500	2328	0	0	0	0	0	0	0
3	130	3917	3917	2500	2328	2328	2889	2889	2889	5837	5837	3900
4	110	3917	9808	8250	8439	8439	8678	8678	11856	11000	11000	13440
5	70	3917	3917	2500	2328	1400	1400	1400	1400	0	0	0
6	60	3917	3917	2500	2328	2328	1733	1733	0	0	0	0
7	200	3917	3917	29989	32462	41767	39192	39192	39192	47020	47020	56000
8	100	3917	3917	2500	2328	2328	2889	2889	2889	0	0	0
9	50	3917	3917	2500	2500	2500	2889	2889	2889	1500	1500	0
10	100	3918	3917	2500	2328	2328	2889	2889	2889	3510	3510	5000
mean (h)		3917	4506	5824	5970	6575	6545	6545	6689	7051	7051	7834
kurtosis (K)		10.00	10.00	8.81	8.88	8.21	8.04	8.04	6.25	5.89	5.89	3.84
std. dev. (σ)		0	1767	8236	9015	11911	11089	11089	11282	13734	13734	16562
Gini (G(x))		0.00	0.12	0.49	0.53	0.65	0.63	0.63	0.66	0.77	0.77	0.81

the table are included the basic statistics. Considering the table it can be deduced that in described example for $\beta = 0.4$ the solution is not fair. System gains on efficiency when the most profitable demand ($d = 7$) takes more of load, taking the capacity from the links used by the others and starving them. Mean gives an information about total efficiency of the system and is strictly related to them.

Other statistics are kurtosis and standard deviation. Kurtosis is the measure made on forth moment of probability distribution. This fact affects it to be very sensitive for values which are clearly stand out from the mean. Kurtosis was calculated for non-zero values of outcome vector. In judgment of the equitability of the solution also helps a lot the calculation of standard deviation measure. In difference to kurtosis, standard deviation is increasing together with the mean. In Fig. 3 are presented the changes of statistics and gives an information about degree and nature of the proportionality between them. Referring to Table 2 the last fair solution is for $\beta = 0.3$. Each further value of β greater than 0.3 includes at least one starved demand in the outcome vector. On the plot of the Gini coefficient and the standard deviation the highest grow can be see for $\beta = 0.2$. On the kurtosis and mean value plots have not existed noticeable changes between those values of β. Those facts should lead the decision-maker to choose the solution obtained with $\beta = 0.2$ as the control parameter, since it provides the highest degree of efficiency and fairness criteria.

Results have also been presented in the graph in Fig. 4. Blue color is related to demand seven ($d = 7$) which is the most profitable. It can be seen that paths bringing the lowest revenue are $d = 6$, $d = 5$, $d = 9$, $d = 2$. In summary, the proposed method allows to obtain a solution maintaining the efficiency and fairness criterion with the higher possible ratio of them. Method could be adopted to the other problems which are similar to allocation problems. Described formulation has only one main control parameter which affects the ease of use. For various preferences, the decision-maker can also

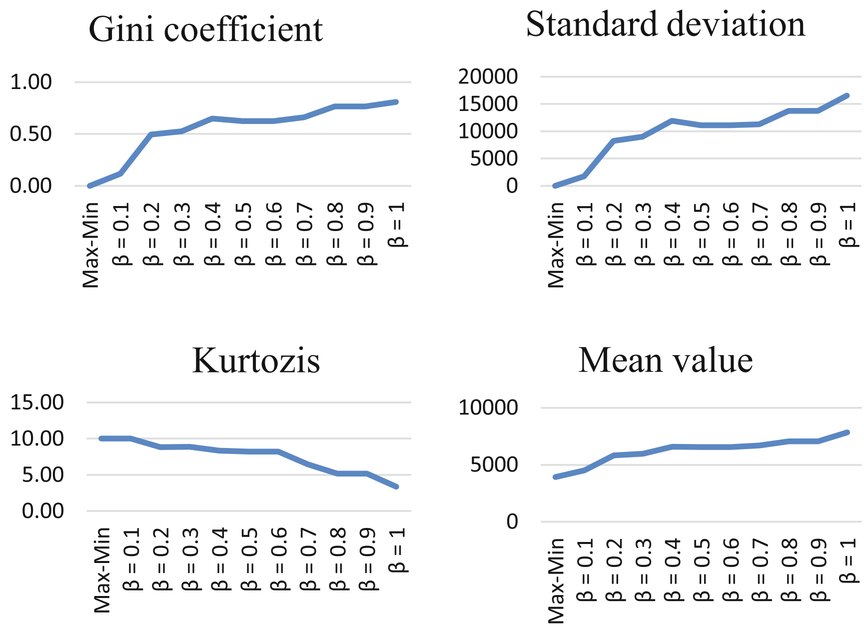

Fig. 3. Plots of calculated statistics for each further solution

choose the solution from whole spectrum of more or less efficient solutions. More precisely, in Fig. 4 are presented combined values of allocated bandwidth on each demand. Each further bar of the plot is related to different value of β. For calculations were chosen ten values of the β parameter between 0 and 1 at intervals of 0.1. As it has been mentioned in the previous section, the highes the value of β is, the more unequal solution is. In the extreme cases, when the β parameter is equal to 0, it is derived solution of direct revenue maximization and for $\beta = 0$ is returned solution of simple MaxMin optimization. Each bar consists of several color parts which are related to suitable demand. For the most equitable solution each color should has size like the others.

Described problem is one of possible uses of ratio-model. Deployed approach of fairness optimization is possible to implement in a lot of different domains such as:

- spatial distribution
- allocation of scarce resources
- air and train traffic
- decision-making in organization management
- machine designs
- various constructions
- architecture
- medicine
- transport
- others

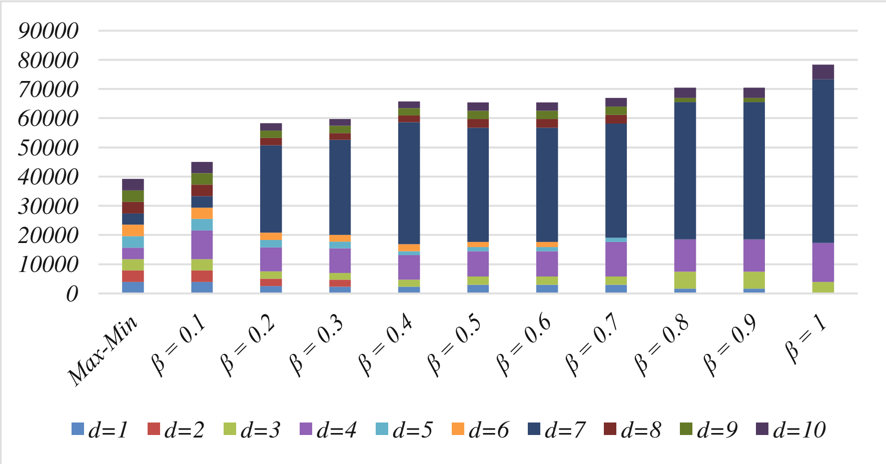

Fig. 4. Cumulated plot of total revenues obtained from load allocated on demands

In each of these areas it can appear a problem of fairness optimization. Spatial distribution of public benefit institutions is the problem where fairness criterion is the most important. For example, in limited resource allocations it may be important to distribute fairly held good. Airlines have problem in giving permissions for airspace access while landing or starting of airplanes. The problem appears more complicated when some airlines want to be treated in special way by paying extra fees for the use of airport services. The similar appears in train traffic management, where on given railroad is allowed a limited amount of trains. Another example is associated with organizations, where manager has to make fairness decisions for each department or employer. While designing many of devices, machines, vehicles, buildings, etc. appears a problem of consumption of limited financial resources. Then project manager is allowed to use some of optimization methods to allocate it between safety, design or other attribute in a fair way. In some degree of fairness there has to treat patients in hospitals but it is well known that sometimes such decisions are very worthwhile from the finance point of view. Many of optimization problems exist in transport branch of economy. In logistics, in each kind of transport, constantly appear new problems to solve, also with the use of fairness optimization technics.

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Spatial Data Analysis in Archaeology: Computer-Aided Selection of Priority Location for Archaeological Survey

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Abstract. The article describes a method to help experts selecting the most probable locations of settlements inhabited by the people in Crete under the Venetian rule (1204–1669). The method ranks the possible location in the designated area based on a set of criteria, but also considering the natural obstacles, e.g. terrain. This multi-criteria task is solved using a modification of the Bellman-Ford algorithm for each criteria and then combining them using weighted sum. The explanation is supported by a small test case.

1 Introduction

The possibility of mathematical modelling in modern archaeology depends on the assumption that human behaviour can be described by patterns. Thanks to this assumption, we have the possibility to observe preferences as to location of settlements inhabited by the investigated community ([3] p. 269).

Archaeological survey is a long time project which involves archaeologists walking around the investigated area and looking for signs on the ground surface which can suggest existence of an archaeological site. It is a costly and time-consuming enterprise and it requires a team composed of several archaeologist. Also a versification survey is needed, even if the area has been already investigated once. At the foundation of the methodology of survey lays the necessity to conduct preliminary survey of whole investigated area ([17] pp. 68–75). In order to reduce the costs of conducting a survey, often only a statistical sample of the area is tested. However, the use of a statistical sample can strongly affect the data collected during a survey ([17] pp. 71–73).

The main idea behind the presented project is to create a model that allows to objectively choose the most promising places to conduct an archaeological survey. Such a model would lead to a more efficient use of resources and time, while still

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allowing to investigate the whole area, without the necessity to arbitrary select some parts of the area over others.

The multi-criteria approach has been chosen for the above problem since it adequately reflects its complexity. Our goal is to apply the weighted objectives method to optimize the choice of locations that are most likely to hold an archaeological site. We bring a multidimensional problem to a one dimensional problem, and using the algorithms available we will optimize the goal function. The algorithm used is a modification of Bellman-Ford algorithm [1] for graphs. It was adapted to work with gridded data and it is recalculated multiple times for different target cells to achieve the shortest paths. The criteria are based on a general knowledge of the archaeological site location.

In Sect. 2 a short state of the art is presented. In Sect. 3 the problem is formulated. The next Sect. 4 describes the data sets that were used for testing the algorithms. Section 5 present a algorithm itself and its workflow. In Sect. 6 the results of an experiment that was conducted on the data presented in Sect. 4 are discussed. The paper ends with short conclusions and discussion about possible future works.

2 State of the Art

So far numerous scientists worked with spatial data analysis in archaeology. The main problems discussed in literature are:

1. creating a model of probability of the appearance of an archaeological site [3,7,12,16];
2. a model of the economic activity areas [4,6,13];
3. reconstruction of connection grids [8,14];
4. spatial data analysis of the environment [2,10].

Most of the models published were created for micro-regions and included many previously known sites, and do not give the possibility to be applied in a wider geographical region or with a small amount of archaeological data.

The problem of finding the locations on gridded data can be expressed as finding the nodes on graphs. The transition from grid to graph can be easily made by interpreting the cell in graph as a node and the neighbourhood operation as an edge of the graph. There is a number of algorithm of finding the way in the graph, e.g. Breadth First Search [15], Dijkstra's Algorithm [5], A* [11]. But that is not sufficient in the situation when the beginning point of the route has to be found. For that purpose the Bellma-Ford algorithm [1] or Floyd algorithm [9] are better suited.

3 Problem Statement

Archaeology as a science is using mostly inductive reasoning – deriving theory from a set of facts, while the formal science is looking for rules and theorems that are proved or disproved. That is a source of different approach to the discussed problem.

1. From **archaeological** point of view the problem is defined as identification of potential location of archaeological sites. It is based on prior knowledge about the considered society and its preferences.
2. From **formal** perspective the investigated problem is the problem of identification of potential archaeological sites based on chosen criteria and geographical data. The formal perspective requires a model and well defined criteria. The importance of each criteria for the investigated case has to be set a priori.

The aim of our research is to create a model that would allow to objectively choose the most promising places for the given criteria. The goal is not to find one optimal solution, but to rank all possible locations based on considered criteria. It is unlikely that there is a single solution that is a good choice for all of the chosen criteria.

In order to provide maximum clarity in the reasoning presented, below are explained some terms that are used throughout the paper:

1. **Criteria** – by criteria, we understand factors, in our case of a geographical or anthropological nature, that affect the choice of site location by past societies, e.g. access to resources such as water and arable land;
2. **Cost** – by cost we understand the profitability of a location meeting one or several criteria. The cost is measured by taking in to account factors such as distance from the point of origin or land relief and angle of inclination of hills. It is a measure that describes how profitable each location is. It is calculated for every cell of the map.

The criteria are divided in two main groups: geographical and anthropogenic. The division in two groups is due to significant differences between the two groups. The first group, the geographical criteria, uses mostly data from maps and satellite photographs. The second group, the anthropogenic criteria, are based on archaeological and historical data, and reflect the cultural preferences of the investigated community, and have to be provided by a qualified investigator.

The model developed for the investigated problem considers gridded data, the map is represented by a $n \times m$ grid V. v_{ij} is a value of each cell of a grid. Let $T \subset V$ be a target set. It is assumed that $T = T_1 \cup T_2$, where T_1 represents a water sources and T_2 represents a arable lands. In Figs. 1 and 2 the target data is presented, they are respectively, water and arable land areas.

The aim is to compute the cost of travel from all points in V to target sets T_1 and T_2. The distances are non-negative, denoted by

$$d_{T_i}(v,t) \quad \text{for } v \in V,\ t \in T_i,\ i = 1, 2.$$

The cost is calculated based on the given cost matrices. To present the method two matrices were chosen: the distance and landforms. The distance represents the distance that has to be travelled in each cell of the grid. It is assumed that the distance in each cell is equal. The important aspect of the algorithms is that overall distance is proportional to the amount of cells on the way to the target. The grid used for distance measurements is presented in Fig. 3. The distance cost matrix is denoted as $\{c_1(v_{ij})\}_{i,j=1}^{50}$.

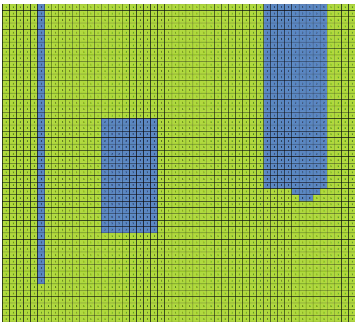

Fig. 1. The matrix representing water sources

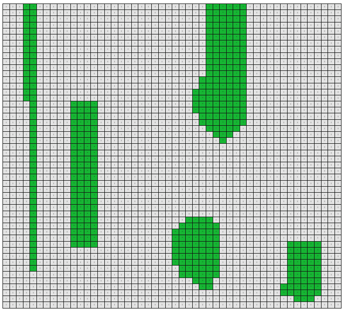

Fig. 2. The matrix representing arable land areas

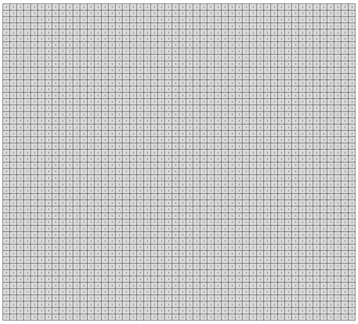

Fig. 3. The matrix representing the distance cost

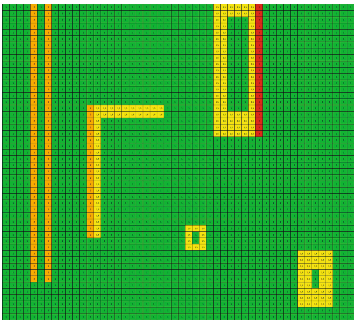

Fig. 4. The matrix representing land relief cost

The landform is representing the difficulty of travelling – the bigger the value the more steep is the terrain and will need additional time and effort to reach the target. It can be argumented that going 'up' is more difficult to going 'down', but in this case the traveler has to go to the target and come back to the settlement. For the considered example the terrain is fairly simple, as presented in Fig. 4. The land relief cost matrix is denoted as $\{c_2(v_{ij})\}_{i,j=1}^{50}$.

Merged cost matrix is defined by

$$c(v_{ij}) := c_1(v_{ij}) + c_2(v_{ij}) \quad i, j = 1, \ldots, 50$$

The grids can be converted to graphs, where each possible transition between neighbouring cells is expressed as a vertex in graph. In this approach a path is defined as a sequence $(v, i_1), (i_1, i_2), \ldots, (i_{n-1}, t)$ from cell $v \in V$ to cell $t \in T_i, i = 1, 2$. The aim is find the Pareto-optimal set of paths from point v to point t of T_l, $l = 1, 2$. For optimal path $(v, i_1), (i_1, i_2), \ldots, (i_{n-1}, t)$, algorithm calculate summing cost value

$$F_l(v) := c_l(i_1) + \cdots + c_l(i_{n-1}) \quad l = 1, 2.$$

Next step is to combine these results by weighted sum method i.e.

$$F_w(v) := w_1 F_1(v) + w_2 F(w_2), \text{ where } w_1 + w_2 = 1.$$

4 Data Sets

For a case study the island of Crete in the Venetian period was chosen. As it was stated before, the criteria for locating settlements where divided in two main groups:

1. The geographical criteria are distance of the settlement from a water sources and from arable fields or the form of the terrain. This group includes all criteria related to the environmental conditions, affecting the population and its choice of location activity.
2. The anthropogenic criteria could be the distance from the nearest settlement, location on a trade route (described in terms "yes" or "no") or defensible location (described in terms "yes" or "no"). This group includes all criteria that are due to a human factor.

As previously stated, for the experiment to verify our approach the island of Crete in the Venetian period was chosen. We chose a specific region the south-west coast, it is presented in Fig. 5, as it was previously thoroughly investigated using a classical method of archaeological survey. This allows us to compare our results to the results obtained in a traditional way. For the purpose of the experiment a GIS database was created with numerical terrain models. On those models a map of elevations was generated. The hydrological and arable land model was build using vectors on the bases of orthogonal photographs of investigated area. The whole investigated area was divided into roughly squared areas, about 500 by 500 m each.

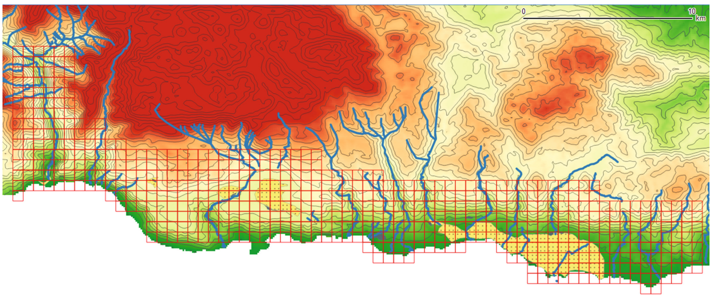

Fig. 5. Map of the investigated area

5 Algorithm and Implementation

The computation of the areas that have the shortest distance to all of the required resources is an interesting problem as there might be no optimal solution and the calculations have to reflect the human limitation of distance and difficulty of the road. The path that is going via river beds and steep hills is much less interesting than a longer path that goes via relatively flat empty area.

The algorithmic problem in this case is not finding the path – there is no beginning or ending point chosen. The aim of the algorithm is to find potential starting point (the location of settlement), considering distance not to the single goal point but to the areas with some properties. That is the reason why the classical path finding algorithms were not directly used. The algorithm implemented is a simple version of Bellman-Ford algorithm [1]. It is similar to Dijkstra [5], but searches the distance from single source vertex to all of the other vertices in a weighted digraph.

The algorithm works as presented in the pseudocode in Algorithm 5.1. The target grids are loaded to the targets set and the cost grids are loaded to the cost set. Then the distances are calculated using *calculateDistance* function for each target grid separately. Then the merging of distances calculates the weighted sum using the given vector of weights.

Algorithm 5.1. Algorithm calculating the weighted distance from each of the target.

```
Data: T_i, i = 1,...n – target grids, C_i, i = 1,...n – cost grids, W – vector of
        weights regarding the importance of each target grid;
Result: The weighted cost grid of distances to the set of target grids;
loading all target grids T_i to the T – set of target grids;
loading all cost grids C_i to the C – set of cost grids;
forall the T_i in T do
  | DistanceGrids set add the grid calculated by calculateDistance(T_i, C);
end
mergeDistance(DistanceGrids, W);
save distances ;
```

The core of the algorithm is a *calculateDistance* function that takes as arguments: the target grid for which the distances should be calculated, the set of cost grids that define the constraints for the distance. The function returns the grid of costs of reaching the target for each cell in the grid. The general outline of the algorithm is presented in Algorithm 5.2. As in [1] the set of cells that are caller frontier is the set of cell to be calculated next. The construction of the frontier starts from the cell that belongs to the target, which cost is 0. Then its neighbouring cells are added to the frontier. Neighbourhood consists of 3, 5 or 8 cells depending on the location of the cell in the grid, 3 neighbours has a cell that is in the corner of the grid, 5 has cell on the border and 8 in all other cases. The cell for which all the neighbours were calculated is removed from the frontier and marked as visited. Then the algorithm considers the next cell from the frontier, calculates its neighbours, and so on, till the frontier gets empty. The neighbours that were in visited set are not calculated again nor included in the frontier unless the new cost is lower than the previously calculated cost for that neighbour. Such condition allows algorithm to come back to already visited cells and assign them lower cost when the shorter/cheaper path is found. Such situation happens when there are multiple target points and multiple paths.

Algorithm 5.2. The pseudocode of calculateDistance method. It calculates the cost for one target grid.

```
Data: T – target grid for which the cost is calculated, C – set of cost grids;
Result: The cost grid containing the costs of reaching the target from each
        point of the grid;
Initialization of Frontier set;
Initialization of Visited set;
forall the t_ij in T_l, l = 1, 2 do
    add t_ij to Frontier;
    add t_ij to Visited;
    while Frontier is not empty do
        current = take Cell from Frontier;
        add current to Visited;
        neighbours = get neighbours for current;
        forall the neighbour in neighbours do
            if neighbour not in Visited OR cost of neighbour > cost of getting
            from current then
                if neighbour in T_l, l = 1, 2 then
                    cost is 0;
                end
                else
                    calculate cost of arriving to neighbour;
                end
                add neighbour to Frontier;
                add neighbour to Visited;
            end
        end
        remove current from Frontier;
    end
end
```

6 Results

In Figs. 6, 7, 8, 9 and 10 the grids of weighted sums of costs with different values of weights are presented. Figure 6 presents the situation when the agral areas are

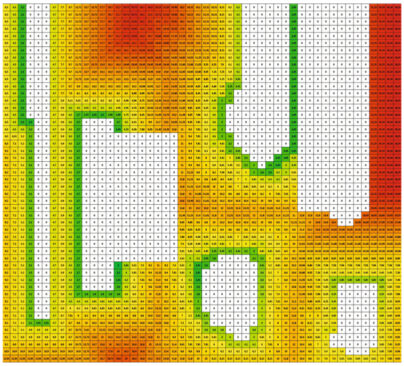

Fig. 6. $w_1 = 0.1$, $w_2 = 0.9$

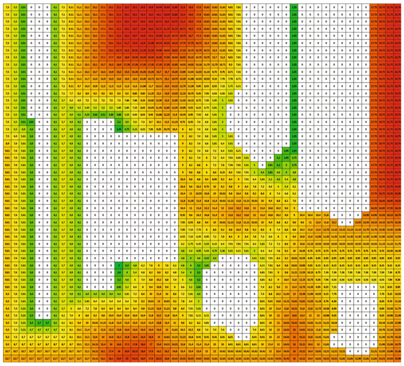

Fig. 7. $w_1 = 0.3$, $w_2 = 0.7$

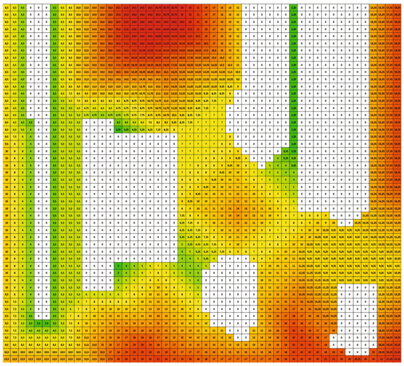

Fig. 8. $w_1 = 0.5$, $w_2 = 0.5$

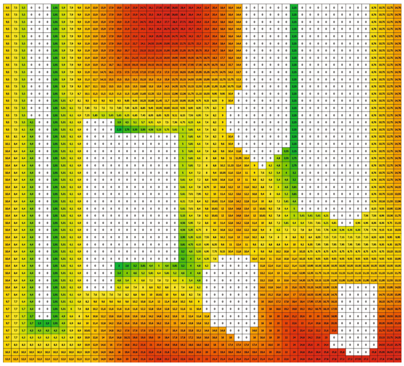

Fig. 9. $w_1 = 0.7$, $w_2 = 0.3$

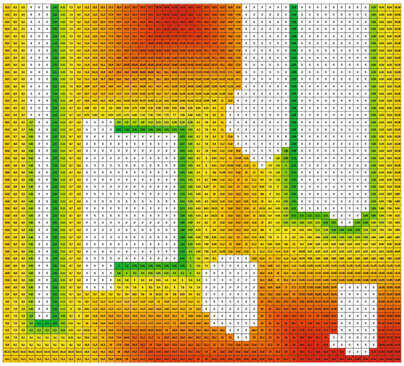

Fig. 10. $w_1 = 0.9$, $w_2 = 0.1$

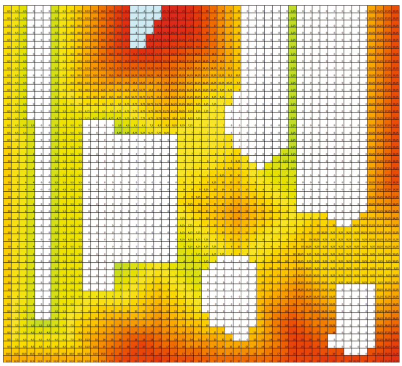

Fig. 11. $w_1 = 0.5$, $w_2 = 0.5$, with distance limited to 10 cells

considered more important and are given weight $w_1 = 0.9$, the water was given weight $w_2 = 0.1$. In Fig. 10 the opposite situation can be seen, where $w_1 = 0.1$ and $w_2 = 0.9$. The colours in the figures represent the distance, green means the areas closest to the target, while red colour represent large distance. As can be seen on this set of figures the weighted sum represents well the influence of each target on the merged cost grid.

The presented results have a very real interpretation, the potential archeological sites should be found where the targets are reachable withing walking distance. There might be additional limitation added, which is a limitation on the distance a person can go to reach the water or agral area. Figure 11 presents the potential areas with exclusion of those that are more than 10 cells from the targets (which might be interpreted as limitation of around 5 km). In the given simplified example there is only one of such area in the top of the figure.

7 Conclusions and Future Work

The algorithmic methods of finding cost and distance on the maps are known for a very long time. But these methods are not commonly used in archaeology, where the human judgement and investigator's knowledge has priority.

The values of weights cannot be chosen without thorough knowledge of the meaning of the targets and constraints. The values of the weights have to be assigned by a specialist. To this aim no automatic method can be applied, as the attribution of weights is based on many factors. The number of the set elements in sets of targets T can be extended, for this example we assumed just two elements of the target set ($l = 2$). We can use our algorithm for the set $T = T_1 \cup \cdots \cup T_k$, then the grid V_w will be defined by the function

$$F(v) = w_1 F_1(v) + \cdots + w_k F_k(v), \quad \text{where } w_1 + \cdots + w_k = 1.$$

The presented results show that even a simple method can give promising results. The focus now will be moved to defining more criteria and costs, first, to determine which are really valid and second to test them on different examples (different areas).

When considering future research, the next goal is to optimise the algorithm and extend our main function with more criteria. Furthermore, the model should be adopted to new data and the estimation of optimal weights should be calculated. For given weights we can find weakly $\varepsilon-$ solutions (i.e. $F(x) \leq F(\bar{x}) - \varepsilon$, see [18] for definition) and by changing ε we can analyse our data, predict promising places.

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New Proposal of Fuzzy Observation of DDoS Attack

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Abstract. This article presents the potential use of implementation fuzzy observance for discovering and protecting network from suffering of Distributed Denial of Service attacks. DDoS attack are able to block web servers and could be started from any place in the network. In this article some real experimental results are presented. Prepared network and DDoS attack tool was used for collecting IP packets during attack, then some extension to fuzzy logic was implemented and used for discovering attack. As a results, the authors present a problem and tool which implemented in IP network could deal with DDoS attack using fuzzy logic.

Keywords: DDoS · Security · Fuzzy logic · Fuzzy observation

1 Introduction

Today networks are mostly Wide Area Network. Huge companies poses lot of offices in different location which are connected with one network, often it is Internet with some special Virtual Private Network created between them. Distributed Denial of Services attack on such systems to block them are used to often. Blocking resources are generating financial lost for a companies. That is why this problem is very common and should be solved somehow. Observation of the environment by humans is usually unmeasurable. This also applies to observers in the network monitoring center, who communicate with each other reporting normal traffic or increased activity. The observer can describe it such linguistic terms like more, lots of or very much. Such linguistic description of reality is characteristic to powerful and dynamically developing discipline of artificial intelligence like fuzzy logic. The author of Fuzzy logic is an American professor of the Columbia University in New York City and of Berkeley University in California Lotfi A. Zadeh, who published the paper entitled Fuzzy sets in the journal Information and Control in 1965 [8]. He defined the term of a fuzzy set there, thanks to which imprecise data could be described using values from the interval (0,1). The number assigned to them represents their degree of membership in this set. It is worth mentioning that in his theory L. Zadeh used the article on 3-valued logic published 45 years

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before by a Pole Jan ukasiewicz [23,34,35,37]. That is why many scientists in the world regard this Pole as the father of fuzzy logic. Next decades saw rapid development of fuzzy logic. Another milestones of the history of that discipline should necessarily mention L-R representation of fuzzy numbers proposed by D. Dubois and H. Prade [23,31], which enjoys great successes today. Coming back to the original analogy, one can see some trend, i.e. general increase during rising tide or decrease during low tide, regardless of momentary fluctuations of the water surface level. This resembles a number of macro and microeconomic mechanisms where trends and time series can be observed. The most obvious example of that seem to be bull and bear market on stock exchanges, which indicate to the general trend, while shares of individual companies may temporarily fall or rise. The aim is to capture the environmental context of changes in economy or another limited part of reality. Changes in an object described using fuzzy logic seem to be thoroughly studied in many papers. But it is not necessarily the case as regards linking those changes with trend. Perhaps this might be the opportunity to apply generalization of fuzzy logic which are, in the opinion of authors of that concept, W. Kosiski [36,47] and his team, Ordered Fuzzy Numbers (OFN) [37]. In this paper authors present applications of such notation for the purpose of denoting trend of changes [39,42] in the server activity. Potential DDoS attacks and standard mechanisms of their detection often cause users' irritation. This feeling has been experienced by many CitiBank clients who tried to perform nonstandard operation with a card. The same has happened to many network users, the administrators of which tightened the security policy so much that made the network virtually unavailable to users. The authors hope that by implementing presented solution to description of phenomena occurring in the network they shall make the protections more "humane" to users [9,26,33]. The paper is organized as follows: In Sect. 2 Distributed Denial of Service attack used in this experiment is presented. Section 3 consists the proposed algorithm of using fuzzy numbers in KFN notations for detecting the DDoS attack. Finally, we conclude the paper in Sect. 4.

2 Distributed Denial of Service Attack Description, Recognition and Simulation

DDoS attack description and recognition

DDoS attacks are widely described in the literature [13,43]. These attacks can be performed on various system resources: TCP/IP sockets [1,2,44] or Domain Name System (DNS) servers. Regardless of the method, the main principle is to simulate so many correct user connections that their number exceeds the actual system performance and drives it to abnormal operation. Papers [2,32,43] describe methods for dealing with DDoS attacks by their global detection and the necessity of cooperation between network providers. The transmission of the attackers packets is done through the providers network and if it cannot be blocked, it leads to data link saturation. Such saturation results in lack of connection to the server [48] and could block some valuable portals like e-learning platforms [5,46]. Even when some extra intrusion detection systems with usage of general purpose computing on graphics processing units [45], it is not enough.

Lot of DDoS analysis paper's are concerning Local Area Network protected with one firewall [11,12]. Databases used by Intrusion Prevention Systems and Intrusion Detection Systems consist data collected during simple attack from one place of the network [11]. Wide Area Network are much bigger and detecting DDoS attack source is more complicated. There is a requirement for cooperation of routers, firewalls and Internet providers to detect attack source. There are some papers which describe how to do it [29,30]. Other solution are using fuzzy logic for detecting DDoS attacks [3,4]. This solution require lot of work from experts. These experts have to describe the attack and generate the rules for the system. This rules are used for attack recognition.

DDoS simulation
To made bigger research paper's authors have prepared special IP network inspired the situation when Polish government tries to sign ACTA papers. In this situation lot of users have tried to block government web pages. It could be made using simple tools for DDoS attack which could be find on Internet [7], for example DDOSIM - Layer 7 DDoS Simulator [41]. DDOSIM was chosen because it provide source code. When attackers collect appropriate amounts of hosts, they can start the attack. It is simple, because it is enough to just run the DDOSIM. So, network with mesh topology showed on Fig. 1 was prepared. There were DDOSIM software running on host User 1 to 5. This station was running on real machine (Windows 7 64-bit system and was equipped with Intel i3 processor) as a virtual machine (virtual VMWare environment, and were equipped with 512 MB of memory and 1 processor, with Debian operating system). Web Server was working on server equipped in two Intel Xeon processors with Windows 2008 Server operating system and Internet Information Service as a HTTP server was started. Routers R1 - R6 were Cisco 2600 series routes. DDOSIM software was establishing TCP connection to Web server in this steps:

- User sends TCP SYN packet on port 80,
- Web server was answering with TCP SYN ACK and reserved resources,
- User sends TCP ACK packet,
- User sends HTTP/GET packet.

Process of attack was made in such steps:

- Web server worked on IP 192.168.10.12,
- 5 user host on IP 192.168.x.4, where x is from 1, 2, 3, 4, 5,
- packets were sniffed using Wireshark on 6 places: User 1 to 5 machines and Web server,
- User 1 host started attack on Web server after 1 min network running continously,
- Users machine were sending 1000 HTTP GET messages to Web servers every 30 s,
- when all User's hosts started the attack then all of them was attacking together for 5 min,
- when the attack stopped packets were sniffed another 5 min.

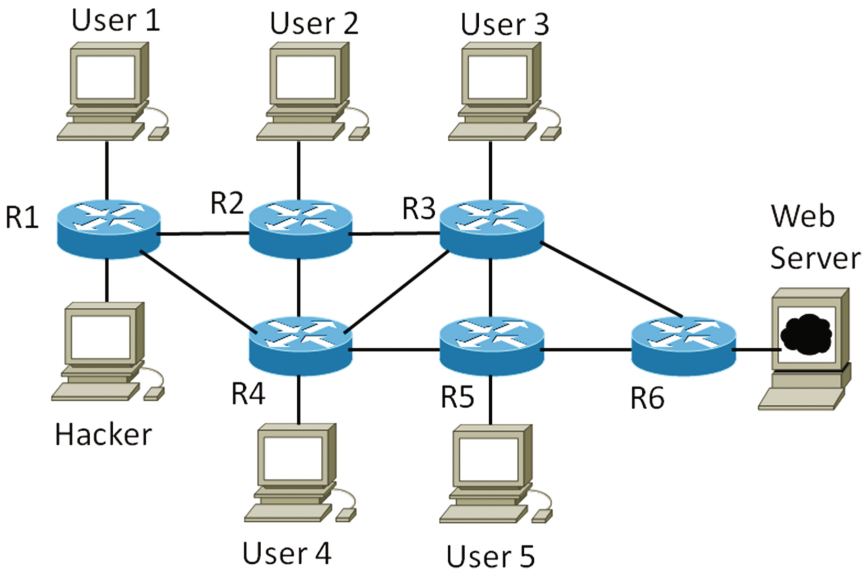

Fig. 1. Used network

Database from this simulation with all connection traffic could be downloaded from data resource page [40]. There were OSPF running between routers as in ordinary network routing protocols are used [15,16,34]. There was not any Quality of Services method implemented and firewall or IDS/IPS implemented in the network while it should be [49].

3 Fuzzy DDoS Detection

Detection procedure
During the network observation system administrator could check that some users are connected to server. He also possess knowledge about main time of the day in which users work with the servers. Next thing which could be checked is how many TCP SYN connection are coming to the server. The last one which could be checked are router statistics of packet transmission. The problem which has to be solved is when DDoS attack started. Administrator knows, that the connection number start growing, but it could not decide only on that fact. The better solution is to measure the packet count during network work on router statistics (Fig. 2). In proposed algorithm administrator should measure specific packet count four times:

$$t_i, t_{(i-1)}, t_{(i-2)}, t_{(i-3)} \tag{1}$$

where t_i is a current timeslot All four measure together gives an fuzzy number in KFN notation where

- $f_A(0)$ respond to $t_{(i-3)}$,
- $f_A(1)$ respond to $t_{(i-2)}$,
- $g_A(1)$ respond to $t_{(i-1)}$,
- $g_A(0)$ respond to t_i.

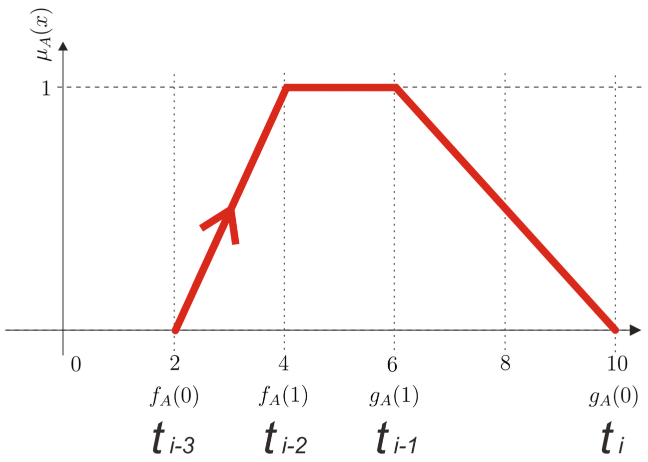

Fig. 2. Fuzzy number in KFN notation

Definition 1. Fuzzy observance of R router in time t_i is a set

$$R/t_i = \{f_R(0)/t_{i-3}, f_R(1)/t_{i-2}, g_R(1)/t_{i-1}, g_R(0))/t_i\} \tag{2}$$

where

$$t_i > t_{i-1} > t_{i-2} > t_{3-1}|t_i - t_{i-1}| = |t_{i-1} - t_{i-2}| = |t_{i-2} - t_{i-3}| = t_n,$$
$$\text{timeslot of the measurement}$$
$$f_R(0) \leq f_R(1) \leq g_R(1) \leq g_R(0)$$

Lemma 1.

$$R_{positive} = \begin{cases} f_R(0) < f_R(1) < g_R(1) \\ or \\ f_R(1) < g_R(1) < g_R(0) \end{cases} \tag{3}$$

In other situation $R_{negative}$.

According to this definition we get:

- possitive order of KFN when the packet count increase,
- negative order of KFN when the packet count decrease.

When the packet statistics on routers are made and appropriate counters gives a results for preparing fuzzy number, there could be a fuzzy observance of the group of routers defined (Fig. 3).

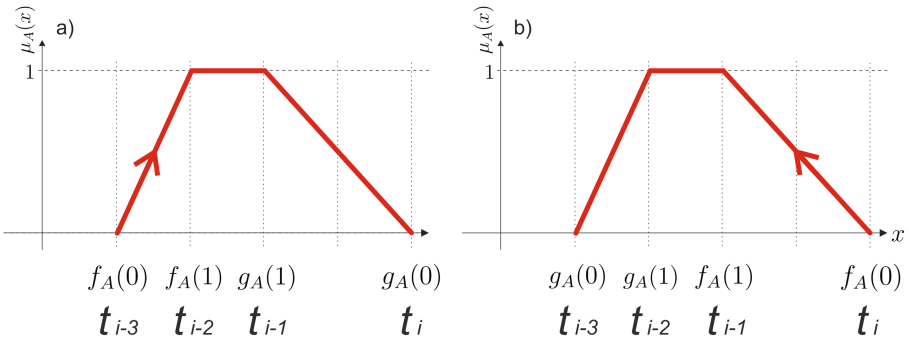

Fig. 3. Order interpretation in KFN notation

Definition 2. Fuzzy observance of the group of routers describe following formula:

$$S_m = \sum_{i=1}^{n} \left\{ \begin{array}{l} R_{positive}|R_{negative} \\ R_i * w_i| - R_i * w_i \end{array} \right\}. \tag{4}$$

where $w_i \in \{w_i, \ldots, w_n\}$ desribe an impact on all router.

Definition 3. An attack on router is recognized in such conditions:

If Ri is pasitive AND Ri is negative THEN Attack = *true*

where *Ri* is an orded from history statistical results according to that router on time of the day of the observance.

Definition 4. An attack on group of routers is recognized in such conditions:

If Sm is positive AND Sm' negative THEN Attack = *true*

where Sm is an orded from history statistical results according to that group of rout-ers on time of the day of the observance.

Experiment results
According to described test DDoS attack, packet statistics on router was measured. The result are presented in Table 1. The timeslots during the test was 30 s. The numbers in Table 1 was normalized by dividing by 1000. Then the fuzzy numbers was prepared and the sum was calculated. The results are presened in Table 2. According to the sum results, the trend was described and sutation about the attack could be recognized. According the fact, that till the time when four measurement result are achieved, the KFN number could be prepared with some approximation. This give a result of possible DDoS attack with the appropriate probability:

1. 50% when we have got two measurement results,
2. 75% when we have got three measurement results.

Method comparision. The presented method could be compared with the method proposed in the literature. This comparision is presened in Table 3.

Table 1. Packets count on routers during test

Timeslots:														
Router	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	t11	t12	t13
R1	0	1	1	1	1	1	1	1	1	1	0	0	0	0
R2	0	0,5	1,5	1,5	1,5	1,5	1,5	1,5	1,5	1,5	0	0	0	0
R3	0	0,5	1	2	2,5	2,5	2,5	2,5	2,5	2,5	0	0	0	0
R4	0	0,5	1	1	2	2	2	2	2	2	0	0	0	0
R5	0	0,5	1	1	1,5	2,5	2,5	2,5	2,5	2,5	0	0	0	0
R6	0	1	2	3	4	5	5	5	5	5	0	0	0	0

Table 2. The sum calculation results

Time	Sum	Trend	Situation
t0	[0,0,0,0]	-	No attack
t1	[0,0,2.5,2.5]	Positive	Attack in 50%
t2	[0,2.5,6,6]	Positive	Attack in 75%
t3	[0,4,7.5,9.5]	Positive	Attack
t4	[4,7.5,9.5,12.5]	Positive	Attack
t5	[7.5,9.5,12.5,14.5]	Positive	Attack
t6	[9.5,12.5,14.5,14.5]	Positive	Attack
t7	[12.5,14.5,14.5,14.5]	Positive	Attack
t8	[14.5,14.5,14.5,14.5]	Positive	Attack
t9	[14.5,14.5,14.5,14.5]	Positive	Attack
t10	[14.5,14.5,14.5,0]	Negative	No attack
t11	[14.5,14.5,0,0]	Negative	No attack
t12	[14.5,0,0,0]	Negative	No attack
t13	[0,0,0,0]	Negative	No attack

Table 3. Method comparison

Functionality	Method in literature	Proposed method
Expert work	Require expert to define the rules	Do not require expert to define the rules
Result of observance DDoS attack	Results made by comparing list of rules	Made by calculations of equation
Possibility to implement in real environment	Require lot of computional power	Require only computing simple mathematical equation

As it could be notice, the proposed algorithm require only to compute some simple mathematical equation, while proposed method require lot of work from expert in securing IP network. The result are achieved very quick and in easy way.

4 Conclusions

In this article, a new concept of detecting DDoS attacks was presented using fuzzy numbers. Determining the server activity trend changes by specification of the direction of changes using KFN was an original solution, all the more that it was executed on the data set of about 2 GB. Expert rules classifying query sequences as classic ones or those which may be treated as anomalies turned out to be more helpful.

It would also seem to be very interesting to associate OFN numbers with trend of changes taking place for studied part of the reality. We are sure that new applications of this property of KFN, shown here in the example of the fuzzy observation of the DDoS attack, will be introduced in the future. Hence it seems that introduction of KFN provides new possibilities for designers of highly dynamic systems. With this approach it is possible to define trend of changes, which offers new possibilities for the development of fuzzy control and it charts new ways of research in the fuzzy logic discipline. When it is broadened by the theory of ordered fuzzy numbers, it seems to enable more efficient use of imprecise operations. By simple algorithmization of ordered fuzzy numbers, it is possible to use them in a new control model. It also inspires scientists to search for new solutions. Authors did not use defuzzyfication operators in this paper, which as such are interesting subject of many researches. They shall also contribute to development of the comparative calculator created here.

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Ant Colony Optimization Application to GPS Surveying Problems: InterCriteria Analysis

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Abstract. Ant Colony Optimization (ACO) has been used successfully to solve hard combinatorial optimization problems. This metaheuristics method is inspired by the foraging behavior of ant colonies, which manage to establish the shortest routes between their colonies to feeding sources and back. In this paper, ACO algorithms are developed to provide near-optimal solutions for Global Positioning System surveying problem (GSP). In designing Global Positioning System (GPS) surveying network, a given set of earth points must be observed consecutively (schedule). The cost of the schedule is the sum of the time needed to go from one point to another. The problem is to search for the best order in which this observation is executed, minimizing the cost of the schedule. We apply InterCriteria Analysis (ICrA) on the achieved results. Based on ICrA we examine some relations between considered GSPs and ACO algorithm performance.

Keywords: InterCriteria Analysis · Ant Colony Optimization · GPS surveying

1 Introduction

InterCriteria Analysis (ICrA) is a recently developed approach [2] aiming to go beyond the nature of the criteria involved in a process of evaluation of multiple objects against multiple criteria, and, on this basis, to discover any existing correlations between the criteria themselves. Given in details in [2], ICrA has been developed further in [11,38].

Up to now ICrA has been applied to several problem fields, namely:

- for the purposes of temporal, threshold and trends analyses of an economic case-study of European Union member states' competitiveness [8–10];
- Genetic Algorithms (GA) performance for a parameter identification problem [1,22,28,29,31,32];

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- to establish the relations and dependencies of parameters referred to various metaheuristic algorithms, namely hybrid schemes using GA and Ant Colony Optimization (ACO) [30];
- for evaluation of pollution indicators of rivers [23,24];
- to universities ranking [12];
- in radar detection threshold analysis [17];
- for Neural Network preprocessing procedure [36];
- etc.

In this paper, ICrA is applied to analysis of an ACO algorithm used to provide near-optimal solutions for Global Positioning System surveying problem (GSP).

Satellite navigation systems have an impact in geoscience, in particular on surveying work in quick and effective determining positions and changes in positions networks. The most widely known space systems are: the American NAVSTAR global positioning system, the Russian GLObal Navigation Satellite System (GLONASS), and the forthcoming European satellite navigation system (GALILEO).

GPS satellites continuously transmit radio signals to the Earth while orbiting it. A receiver, with unknown position on Earth, has to detect and convert the signals received from all of the satellites into useful measurements. These measurements would allow a user to compute a three-dimensional coordinate position: location of the receiver.

Solving such problems – GSPs – to optimality requires a very high computational time. Therefore, metaheuristic methods are used to provide near-optimal solutions for large networks within acceptable amount of computational effort [13,14,18–21,33,34]. In this paper, we implement Max-Min Ant System (MMAS) [37].

2 Background

GPS has a strong impact on the art and practice of most forms of positioning and navigation. However, GPS has already had a tremendous impact on surveying, initially as a technology for geodetic surveys [25]. The GPS navigation supports the safe passage of a vessel, aircraft or vehicle from the departure, while underway to its point of arrival; while GPS surveying is mostly associated with the traditional functions of establishing geodetic control, supporting engineering constructions, cadastral surveys and map making. Any GPS observation is proven to have biases, hence, in order to survey an appropriate combination of measurement and processing strategies must be used to minimize their effect on the positioning results. Differencing data collected simultaneously from two or more GPS receivers to several GPS satellites allows to eliminate or significantly reduce most of the biases. All position results are therefore expressed relative to datum stations. GPS relative technology can, in practice, be employed for a wide range of activities and it is competitive against conventional terrestrial technologies of surveying.

Some main uses of GPS surveying will be mentioned:

– Geodetic Surveys: GPS has already replaced other methods for establishing, maintaining and densifying geodetic networks. Over distances of a few hundred kilometers, multi stations method gives relative position accuracies of a few decimeters. Unlike it, GPS can give accuracies of 1ppm even over distances as short as a few kilometers. Furthermore, GPS is much more faster [25].
– Scientific Survey: The measurement of crystal deformation is central to our understanding of earthquake processes, plate motion, ruffing, mountain building mechanism and the near-surface behavior of volcanoes. In this case are measured changes in position, displacement or strain with time. Hence one seeks to repeat the measurements under as nearly an identical set of circumstances and as high an accuracy as possible (few centimeters) [25].

The GPS positioning technology is superior to the conventional and theodolite procedures, and it can be used by users with no previous experience in satellite surveying.

3 Problem Description

The GPS network can be defined as a set of stations $(a_1, a_2, \ldots a_n)$, which are co-ordinated by placing receivers $(X_1, X_2, \ldots)$ on them to determine sessions $(a_1a_2, a_1a_3, a_2a_3, \ldots)$ among them. The problem is to search for the best order in which these sessions can be organized to give the best schedule. Thus, the schedule can be defined as a sequence of sessions to be consecutively observed.

The solution is represented by a linear graph with weighted edges. The nodes represent the stations and the edges represent the moving cost.

The objective function of the problem is the cost of the solution which is the sum of the costs (time) to move from one point to another:

$$C(V) = \sum C(a_i, a_j),$$

where a_ia_j is a session in solution V.

For example, if the number of points (stations) is 4, a possible solution is

$$V = (a_1, a_3, a_2, a_4)$$

and it can be represented by a linear graph

$$a_1 \to a_3 \to a_2 \to a_4.$$

The moving costs are as follows:

$$C(a_1, a_3), C(a_3, a_2), C(a_2, a_4).$$

Thus, the cost of the solution is

$$C(V) = C(a_1, a_3) + C(a_3, a_2) + C(a_2, a_4).$$

In practice, this determines how each GPS receiver should be moved between stations to be surveyed in an efficient manner taking into account some important factors such as time, cost, etc. The problem is to search for the best order, with respect to the time, in which these sessions can be observed to give the cheapest schedule or to minimize $C(V)$. The initial data is a cost matrix, which represents the cost of moving a receiver from one point to another. The cost could be evaluated purely upon the time or purely upon the distance; for more details see Dare [14].

4 Ant Colony Optimization for GPS Surveying Problem

Real ants foraging for food lay down quantities of pheromone (chemical cues) marking the path that they follow. An isolated ant moves essentially at random but an ant encountering a previously laid pheromone will detect it and decide to follow it with high probability and thereby reinforce it with a further quantity of pheromone. The repetition of the above mechanism represents the auto-catalytic behavior of real ant colony where the more the ants follow a trail, the more attractive that trail becomes.

The ACO algorithm uses a colony of artificial ants that behave as cooperative agents in a mathematics space were they are allowed to search and reinforce pathways (solutions) in order to find the optimal ones [15]. The problem is represented by a graph and the ants walk on the graph to construct solutions. The solution is represented by a path in the graph. After initialization of the pheromone trails, ants construct feasible solutions, starting from random nodes, then the pheromone trails are updated. At each step, ants compute a set of feasible moves and select the best one (according to some probabilistic rules) to carry out the rest of the tour. The structure of ACO algorithm is shown in Fig. 1. The transition probability p_{ij}, to chose the node j when the current node is i, is based on the heuristic information η_{ij} and pheromone trail level τ_{ij} of the move, where $i, j = 1, \ldots, n$.

$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{k \in Unused} \tau_{ik}^{\alpha}\eta_{ik}^{\beta}}. \tag{1}$$

The higher value of the pheromone and the heuristic information, the more profitable is to select this move and resume the search. In the beginning, the initial pheromone level is set to a small positive constant value τ_0 and then ants update this value after completing the construction stage. ACO algorithms adopt different criteria to update the pheromone level.

In our implementation, we use MMAS [15,37], which is ones of the best ant approaches. In MMAS, the main point is using fixed upper bound τ_{max} and lower bound τ_{min} of the pheromone trails. Thus, the accumulation of a large amount of pheromone by a part of the possible movements and repetition of same solutions is partially prevented. The main features of MMAS are:

Ant Colony Optimization

```
Initialize number of ants;
Initialize the ACO parameters;
while not end-condition do
    for k=0 to number of ants
        ant k starts from random node;
        while solution is not constructed do
            ant k selects the node with higher probability;
        end while
    end for
    Local search procedure;
    Update-pheromone-trails;
end while
```

Fig. 1. Pseudocode for ACO

- Strong exploration of the space in search for the best found solution. This can be achieved by only allowing one single ant to add pheromone after each iteration, the best one.
- Wide exploration of the best solution. After the first iteration the pheromone trails are reinitialized to τ_{max}. In the next iteration, only the movements that belong to the best solution receive a pheromone, while other pheromone values are only evaporated.

The aim of using only one solution is to make solution elements, which frequently occur in the best found solutions, get large reinforcement. Pheromone trail update rule is given by:

$$\tau_{ij} \leftarrow \rho\tau_{ij} + \Delta\tau_{ij}, \tag{2}$$

$$\Delta\tau_{ij} = \begin{cases} 1/C(V_{best}) \text{ if } & (i,j) \in \text{best solution}, \\ \\ 0 & \text{otherwise}, \end{cases}$$

where V_{best} is the iteration best solution and $i, j = 1, \ldots, n$. To avoid stagnation of the search, the range of possible pheromone values on each movement is limited to an interval $[\tau_{min}, \tau_{max}]$. τ_{max} is an asymptotic maximum of τ_{ij} and $\tau_{max} = 1/(1-\rho)C(V^*)$, while $\tau_{min} = 0.087\tau_{max}$, where V^* is the optimal solution, but it is unknown, therefore, we use V_{best} instead of V^*.

When all ants have completed their solutions, the pheromone level is updated by applying the global update rule. Only the pheromone corresponding to the best found solution is increased in a way, similar to the MMAS. The global update rule is intended to provide a greater amount of pheromone on the paths of the best solution. It is a kind of intensification of the search around the best found solution.

The heuristic information we use equals to $1/C(V)$.

5 Experimental Results

One of the main things in implementation of ACO algorithms is the graph of the problem. In our implementation, the nodes of the graph correspond to the stations and the arcs correspond to the cost of the sessions. The ants deposit the pheromone on the arcs.

To achieve good results we need to carefully choose the ACO parameter settings and to determine structural elements of the algorithms. On every iteration, ants start constructing their solution from a random node of the graph, therefore the number of the ants can be much less than the number of nodes (stations). Thus, we decrease the needed computing resources like time and memory. Experimentally we found that 10 ants are enough. Using more ants, we increase the computational time without improving achieved results, when the number of iterations is the same. Tabu list is associated with each ant in order to prevent it from visiting a node more than once.

The control parameters govern the work of the algorithms and are mainly concerned with pheromone information and transition probability. The parameter α is the pheromone intensity parameter and β is the heuristic control parameter. When the value of α is high, the pheromone information is more important when the ant chooses next node to move to. When the β is high the heuristic information is more important. After some tests, we found that best values for transition probability control parameters for GPS surveying problem are $\alpha = 1$ and $\beta = 2$.

Parameter ρ is used in pheromone update rules to diversify the search by regulating the influence of the old pheromone. $\rho \in (0, 1)$, when ρ is close to 0 only the new added pheromone is important in the next iteration. When ρ is close to 1, the influence of old pheromone (experience from previous iterations) is great. We found that our algorithms achieve the best results when $\rho = 0.2$.

The initial pheromone value τ_0 has not influenced the algorithm performance. What is important is to have τ_0 less than $\Delta\tau_{ij}$. Therefore, we choose the initial pheromone value to be $\tau_0 = 0.005$.

The number of iterations is equal to the number of the stations.

In this section, we analyze the experimental results obtained using MMAS algorithm described in the previous section. For a test problem, we use real data from Malta and Seychelles GPS networks. The Malta GPS network is composed of 38 sessions and the Seychelles GPS network is composed of 71 sessions. We use 6 larger test problems from: http://www.informatik.uni-heidelberg.de/groups/comopt/software/TSLIB95/ATSP.html. The test problems range from 100 to 443 sessions.

The results are obtained by performing 30 independent runs, for every experiment.

In Table 1, we show the obtained costs for every test problem. Let us compare our results with the results in [35]. For tests, they use only small examples of Malta and Seychelles and for improving the performance of their ACO algorithm they combine it with a local search procedure. They report only the best

Table 1. GSP results over 30 runs

Tests	Sessions	MMAS	MMAS-best
Malta (GSP_1)	38	899.5	895
Seychelles (GSP_2)	71	922	865
rro124 (GSP_3)	100	40910.6	39096
ftv170 (GSP_4)	170	3341	3115
rgb323 (GSP_5)	323	1665.9	1615
rgb358 (GSP_6)	358	1692.6	1648
rgb403 (GSP_7)	403	3428.5	3382
rgb443 (GSP_8)	443	3765.8	3701

results they found, as follows: 895 for Malta and 865 for Seychelles by 200 iterations. Thus, we can conclude that our algorithm is better, because we achieve similar best cost for the same test problems, but without local search and with less number of iterations. Moreover, we observe a larger difference between the algorithms in larger test problems.

6 InterCriteria Analysis

Following [2,6], an Intuitionistic Fuzzy Pair (IFP) [3] with the degrees of "agreement" and "disagreement" between two criteria applied on different objects is obtained. An IFP is an ordered pair of real non-negative numbers $\langle a, b\rangle$ such that: $a + b \leq 1$.

Let an Index Matrix (IM) (see [4,5]) whose index sets consist of the names of the criteria (for rows) and objects (for columns) be given. The elements of this IM are further supposed to be real numbers. An IM with index sets consisting of the names of the criteria (for rows and for columns) with elements IFPs corresponding to the "agreement" and "disagreement" of the respective criteria will be obtained. Two things are further assumed: (i) all criteria provide an evaluation for all objects and all these evaluations are available; (ii) all the evaluations of a given criteria can be compared amongst themselves.

The set of all objects being evaluated is denoted by O, and the set of values assigned by a given criterion C to the objects by $C(O)$, i.e.

$$O \stackrel{\text{def}}{=} \{O_1, O_2, \ldots, O_n\},\ C(O) \stackrel{\text{def}}{=} \{C(O_1), C(O_2), \ldots, C(O_n)\}.$$

Let $C^*(O) \stackrel{\text{def}}{=} \{\langle x, y\rangle |\ x \neq y\ \&\ \langle x, y\rangle \in C(O) \times C(O)\}$.

In order to compare two criteria, the vector of all internal comparisons of each criteria which fulfill exactly one of three relations R, $\overline{R}$ and $\tilde{R}$ must be constructed. It is required that for a fixed criterion C and any ordered pair $\langle x, y\rangle \in C^*(O)$ it is true:

$$\langle x, y\rangle \in R \Leftrightarrow \langle y, x\rangle \in \overline{R}, \tag{3}$$

$$\langle x, y\rangle \in \tilde{R} \Leftrightarrow \langle x, y\rangle \notin (R \cup \overline{R}), \tag{4}$$

$$R \cup \overline{R} \cup \tilde{R} = C^*(O). \tag{5}$$

From the above it is seen that only a subset of $C(O) \times C(O)$ has to be considered for the effective calculation of the vector of internal comparisons (denoted further by $V(C)$), since from (3), (4) and (5) it follows that if the relation between x and y is known, the relation between y and x is known as well. Thus, only lexicographically ordered pairs $\langle x, y\rangle$ are considered. Let, for brevity, $C_{i,j} = \langle C(O_i), C(O_j)\rangle$. Then, for a fixed criterion C the following vector is constructed:

$$V(C) = \{C_{1,2}, C_{1,3}, \dots, C_{1,n}, C_{2,3}, C_{2,4}, \dots, C_{2,n}, C_{3,4}, \dots, C_{3,n}, \dots, C_{n-1,n}\}.$$

It can be easily seen that it has exactly $\frac{n(n-1)}{2}$ elements. Further, to simplify our considerations, the vector $V(C)$ is replaced with $\hat{V}(C)$, where for each $1 \le k \le \frac{n(n-1)}{2}$ for the k-th component it is true:

$$\hat{V}_k(C) = \begin{cases} 1 \text{ iff } V_k(C) \in R, \\ -1 \text{ iff } V_k(C) \in \overline{R}, \\ 0 \text{ otherwise.} \end{cases}$$

Then, when comparing two criteria, the degree of "agreement" between the two is the number of matching components (divided by the length of the vector for normalization purposes). The degree of "disagreement" is the number of components of opposing signs in the two vectors (again normalized by the length).

The above described algorithm for calculating the degrees of "agreement" (μ) and degrees of "disagreement" (ν) between two criteria C and C' is realized in Matlab environment according to [26,31].

7 InterCriteria Analysis Results

In Table 2, the results of every 30 runs for the eight GSPs are presented. The average results of the first 5 runs (C_1), first 10 runs (C_2), first 15 runs (C_3), etc., and finally of all 30 runs (C_6) are calculated and presented in Table 3. ICrA has been applied on the results in Table 3.

Table 2. Results of 30 runs

run	GSP_1	GSP_2	GSP_3	GSP_4	GSP_5	GSP_6	GSP_7	GSP_8
1	905	897	41773	3333	1670	1711	3434	3786
2	900	914	40486	3115	1636	1648	3412	3791
3	900	934	41005	3279	1663	1668	3454	3731
4	895	910	41311	3230	1652	1657	3416	3754
5	895	927	42107	3267	1660	1684	3384	3729
6	895	908	40089	3422	1641	1697	3397	3723
7	900	963	41384	3299	1665	1699	3486	3723
8	895	915	40894	3448	1679	1702	3454	3779
9	900	865	40471	3314	1681	1688	3396	3769
10	895	923	41004	3326	1661	1681	3474	3772
11	900	911	40776	3450	1692	1695	3447	3776
12	895	918	41299	3408	1615	1708	3484	3784
13	900	959	40541	3265	1691	1689	3435	3720
14	900	930	41046	3391	1661	1715	3382	3794
15	900	963	40693	3358	1692	1719	3442	3750
16	895	934	42219	3405	1679	1664	3364	3733
17	900	918	40947	3290	1646	1690	3422	3717
18	900	914	40602	3398	1654	1673	3395	3803
19	895	930	40827	3630	1677	1684	3431	3827
20	905	963	39244	3263	1684	1703	3428	3749
21	920	916	41137	3341	1694	1726	3464	3701
22	900	914	41773	3370	1659	1693	3430	3778
23	900	916	39477	3291	1646	1675	3422	3754
24	900	963	39096	3338	1678	1735	3446	3758
25	895	913	40604	3409	1698	1652	3437	3819
26	900	964	41147	3302	1642	1699	3393	3791
27	900	865	40984	3286	1641	1692	3407	3785
28	905	897	41311	3333	1657	1705	3472	3803
29	895	928	41747	3335	1674	1705	3405	3796
30	900	890	41324	3362	1689	1723	3444	3779

Table 3. IM for ICrA

	C_1	C_2	C_3	C_4	C_5	C_6
GSP_1	899.00	898.00	898.33	898.50	899.40	899.50
GSP_2	916.40	915.60	922.47	924.80	924.72	922.07
GSP_3	41336.40	41052.40	40991.93	40935.90	40832.20	40910.60
GSP_4	3244.80	3303.30	3327.00	3344.55	3345.60	3341.93
GSP_5	1656.20	1660.80	1663.93	1664.95	1666.96	1665.90
GSP_6	1673.60	1683.50	1690.73	1688.75	1690.24	1692.67
GSP_7	3420.00	3430.70	3433.13	3426.85	3429.44	3428.57
GSP_8	3758.20	3755.70	3758.73	3760.50	3760.80	3765.80

The resulting degrees of "agreement" ($\mu_{C,C'}$) are as follows:

$$\mathrm{IM}_1 = \begin{array}{c|cccccccc} & GSP_1 & GSP_2 & GSP_3 & GSP_4 & GSP_5 & GSP_6 & GSP_7 & GSP_8 \\ \hline GSP_1 & \mathbf{1} & 0.60 & 0.27 & 0.67 & 0.73 & 0.67 & 0.33 & 0.87 \\ GSP_2 & 0.60 & \mathbf{1} & 0.27 & 0.80 & 0.73 & 0.53 & 0.47 & 0.73 \\ GSP_3 & 0.27 & 0.27 & \mathbf{1} & 0.07 & 0 & 0.20 & 0.40 & 0.13 \\ GSP_4 & 0.67 & 0.80 & 0.07 & \mathbf{1} & 0.93 & 0.73 & 0.53 & 0.80 \\ GSP_5 & 0.73 & 0.73 & 0 & 0.93 & \mathbf{1} & 0.80 & 0.60 & 0.87 \\ GSP_6 & 0.67 & 0.53 & 0.20 & 0.73 & 0.80 & \mathbf{1} & 0.67 & 0.80 \\ GSP_7 & 0.33 & 0.47 & 0.40 & 0.53 & 0.60 & 0.67 & \mathbf{1} & 0.47 \\ GSP_8 & 0.87 & 0.73 & 0.13 & 0.80 & 0.87 & 0.80 & 0.47 & \mathbf{1} \end{array}$$

The resulting degrees of "disagreement" ($\nu_{C,C'}$) are as follows:

$$\mathrm{IM}_2 = \begin{array}{c|cccccccc} & GSP_1 & GSP_2 & GSP_3 & GSP_4 & GSP_5 & GSP_6 & GSP_7 & GSP_8 \\ \hline GSP_1 & \mathbf{0} & 0.40 & 0.73 & 0.33 & 0.27 & 0.33 & 0.67 & 0.13 \\ GSP_2 & 0.40 & \mathbf{0} & 0.73 & 0.20 & 0.27 & 0.47 & 0.53 & 0.27 \\ GSP_3 & 0.73 & 0.73 & \mathbf{0} & 0.93 & 1 & 0.80 & 0.60 & 0.87 \\ GSP_4 & 0.33 & 0.20 & 0.93 & \mathbf{0} & 0.07 & 0.27 & 0.47 & 0.20 \\ GSP_5 & 0.27 & 0.27 & 1 & 0.07 & \mathbf{0} & 0.20 & 0.40 & 0.13 \\ GSP_6 & 0.33 & 0.47 & 0.80 & 0.27 & 0.20 & \mathbf{0} & 0.33 & 0.20 \\ GSP_7 & 0.67 & 0.53 & 0.60 & 0.47 & 0.40 & 0.33 & \mathbf{0} & 0.53 \\ GSP_8 & 0.13 & 0.27 & 0.87 & 0.20 & 0.13 & 0.20 & 0.53 & \mathbf{0} \end{array}$$

Regarding ICrA, we observed that the ACO algorithm performs in a similar way for the GSP_2, GSP_4, GSP_5 and GSP_8. They are GPS networks with different numbers of sessions, but may have a similar structure, therefore, the value of "agreement" is high. For other networks, we can conclude that they have very different structure.

The GSP that are in positive consonance are shown in Fig. 2. For these GSP, ACO obtains close and similar results, i.e., the algorithm performance is identical when solving the given tasks. In Fig. 3 are shown these couples of GSP, for which

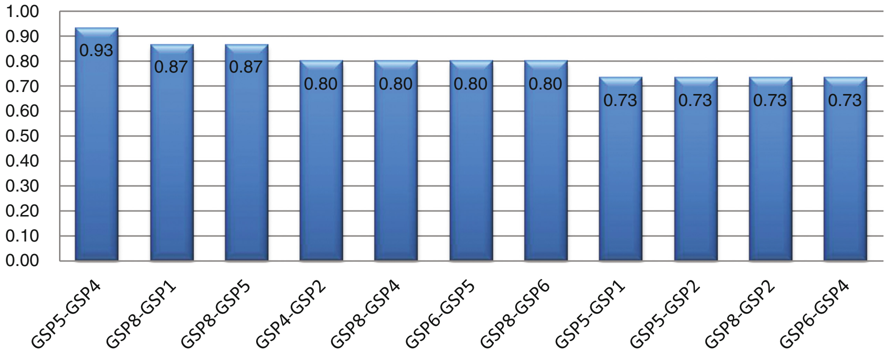

Fig. 2. GSPs relation in positive consonance scale

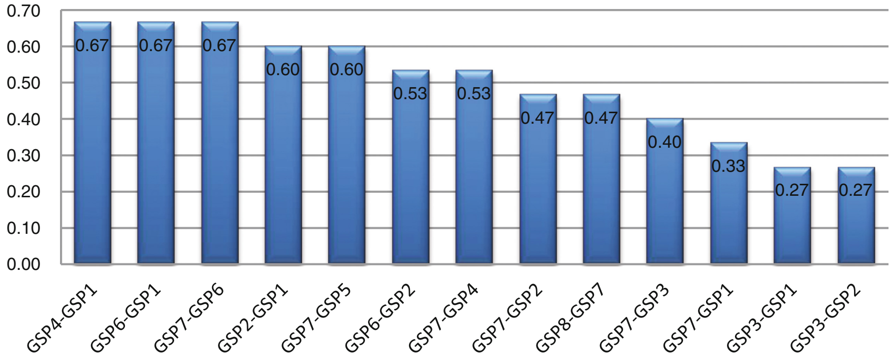

Fig. 3. GSPs relation in dissonance scale

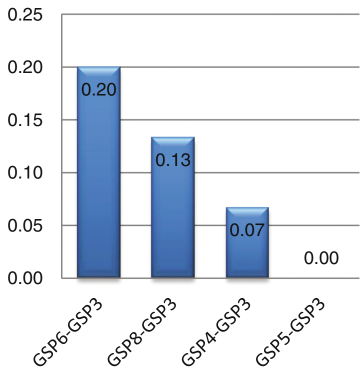

Fig. 4. GSPs relation in negative consonance scale

ACO does not exhibit identical performance, i.e., these GSP are in dissonance. Finally, in Fig. 4 are presented couples of GSP, which are in negative consonance. The definitions of the positive/negative consonance and dissonance are according to [7].

8 Conclusions

The GPS surveying problem and the InterCriteria Analysis are addressed in this paper. Instances containing from 38 to 443 sessions have been solved using MMAS algorithm. A comparison of the performance of the both ACO algorithms applied to various GPS networks is reported. The obtained results are encouraging and the ability of the developed techniques to generate fast high-quality solutions for observing GPS networks can be observed. The problem is important because it arises in wireless communications like GPS and mobile phone networks, and can improve the services in the networks. Thus, the problem has an economical importance.

ICrA has been applied to the obtained results. By InterCriteria Analysis we were able to find some relations and dependences between the considered 8 GSPs and MMAX algorithm performance.

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Ontology Usage for Database Conversion in Practical Solution for Military Systems - Case Study

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Abstract. Nowadays IT systems are used for quick share lot of data. These data are collected in databases which are working in a special rules. These rules describe how data are stored the structure of database and how it can be added or removed from them. The process of changing structures in databases is very complicated. Polish company TELDAT develop mechanism called ADTA (Automated Data Transformation and Aggregation) to achieve interoperability between systems which are using different data exchange protocol versions and different database structures. This mechanism is using semantic transformation. In this paper an ADTA mechanism is presented and some practical results of test made on international exercises are also described.

Keywords: Semantic algorithm · Database conversion

1 Introduction

Nowadays IT systems are used for quick share lot of data. These data are collected in databases which are working in a special rules [1,12]. These rules describe how data are stored the structure of database and how it can be added or removed from them. Military systems are many time the most innovative ones. This fact is a reason while there often a requirement to change their databases structures. The process of changing structures in databases is very complicated. The countries who are a part of North Atlantic Treaty Organization required the same standardized databases for military systems. To achieve this goal, there are a Multilateral Interoperability Programme [27] established. The engineers who worked in this program are developing the protocols for exchanging data. During the last years there are a several protocols normalized: MIP 2.0, 3.0 and 3.1. But during the work there are not thinking about possibility to exchange information between systems which are used different version of MIP protocols. This is because these protocols are developed for working with different structures of databases. Polish company TELDAT develop mechanism

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called ADTA (Automated Data Transformation and Aggregation) to achieve interoperability between systems which are using different MIP protocol version and different database structures [28]. This mechanism is using semantic transformation [24,30]. Without this transformation there will be no possibility to connect lot of NATO systems. In this paper an ADTA mechanism is presented according to this semantic algorithm [1,8]. Some practical results of test made on international exercises are also described. Finally some conclusion are made.

2 ADTA Transformation Mechanism

ADTA *Mechanism Decription*
ADTA mechanism is a unique Gateway which translate the MIP protocol between each version: 2.0, 3.0, and 3.1, which let to translate data between different database model. The protocol 2.0 is using C2IEDEM database model while protocols 3.0 and 3.1 are using the newer database model JC3IEDEM. This mechanism was implemented in Network Centric Data Communication Platform JASMINE by polish engineers from company TELDAT. This platform uses JC3IEDEM as a native standard of database. JC3IEDEM is a huge complex database format which describe the situation on military battlefield. This format was created using NATO requirements and many countries who participate in Multilateral Interoperability Program. During the time this model has evolved. The consequence of this evolution is a lot of command support systems which are using different version. These systems are unable to communicate. The engineers developed universal language which describe the changes in the model. This let to generate the program for translating message which comes from different model in automatic way. The final effect is a ADTA mechanism which let to translate the old model of database - C2IEDEM and the new one JC3IEDEM (Fig. 1).

Semantic Algorithm of Data Transformation
The ADTA transformation mechanism is primary based on the formal definition of JC3IEDM and C2IEDM ontologies. These ontologies are represented as the series of entities interconnected with relationships. As a result we can treat each ontology as a directed graph with nodes being Entities and edges being relationships [11–13].

The Entity is defined by its unique name and the series of attributes describing its current state. Every relationship also has its own unique name and set of specified attributes. [9,14,23] The most important relationship attribute is the Relationship Type descriptor which can have one of following values: Identifying, Non-identifying, Subtype. The example combination of entity and relationship may be expressed as a logical sentence: <Parent entity> <verb phase> <Child entity>, eg. ABSOLUTE-POINT is a CARTESIAN-POINT, ACTION-TASK references OBJECT-ITEM-ASSOCIATION.

The crucial part of defining transformation between two meta models it to define a semantic id for each of schema element. This way it is now possible to select a subset of similar entities in C2IEDM and JC3IEDM models. In most

Fig. 1. ADTA description

cases semantic identifier will be equal to entity name but there are some minor changes in JC3IEDM that requires manual involvement into this process. When the process of entity to semantic id mapping is complete for entities, relationships and attributes we have now a basic identity transformation that could handle some basic scenarios (assuming that any missing elements will be skipped) but in most cases it is not enough [7,18,20]. Each of the model has also set of business rules that needs to be fulfilled and as a result an additional mechanism was developed. The heart of ADTA is a special DSL (Domain Specific Language) that specifies what kind of operations should be performed in order to transform Entity between model representation.

The basic ADTA transformations are as following:

– Dictionary lookup - if the old domain value does not fit into new domain attribute accepted values a new candidate should be selected according to provided value mapping. In most cases it will be NOS (Not otherwise specified) but for some Entities a nearly the same values exists in target model,
– Math expression - if there is need for degradation of information an additional computation is needed. For example C2IEDM model contains limited set of location and as a result some 3d location should be simplified and transformed into 2d representation. In such cases DSL provides and mathematic formula for performing transformation
– Entity creation - some concepts such as Plans And Orders are not present in C2IEDM and as a result additional steps need to be taken. In this particular case ADTA tries to mimic Plans And Ordered by creating other Entities and associating them with existing objects
– Model query - in order to perform transformation additional information from source or target model needs to be gathered. In order to perform

that another DSL has been introduced to get particular information. For example to obtain all friendly polish units from model state we should provide following query: /org: all()/ctx:status() -eq FRORG/ctx:status() -eq FRNO/oi:affiliations().geo().affiliation -eq POL/ which means to search in contexts of every organizations for all object items containing geopolitical affiliation of Poland/

Example rule expressed in ADTA with use of presented language may take following form:

if ENTITY[type] is OBJECT-ITEM:

TRANSFORM-ATTR LOCATION− > name func:truncate(12)

if (ENTITY− > LOCATION is-present) and (ENTITY− > LOCATION [type] is SPHERE-VOLUME):

TRANSFORM-CREATE ENTITY− > LOCATION as FAN-AREA func:geometry:convert 3d_sphere_volume_into_fan_area()

As we can see such rule can be easily transformed into natural language and be showed to any domain expert. In this case we can say that: "In order to transform any Object Item location its name should be truncated to 12 characters and if it has a Spherical Volume location it should be transformed into its 2d representation - a fan area".

Translation Example

The example of data transformation between different MIP model is an information about object located on battlefield. This example is presented on Figs. 2 and 3.

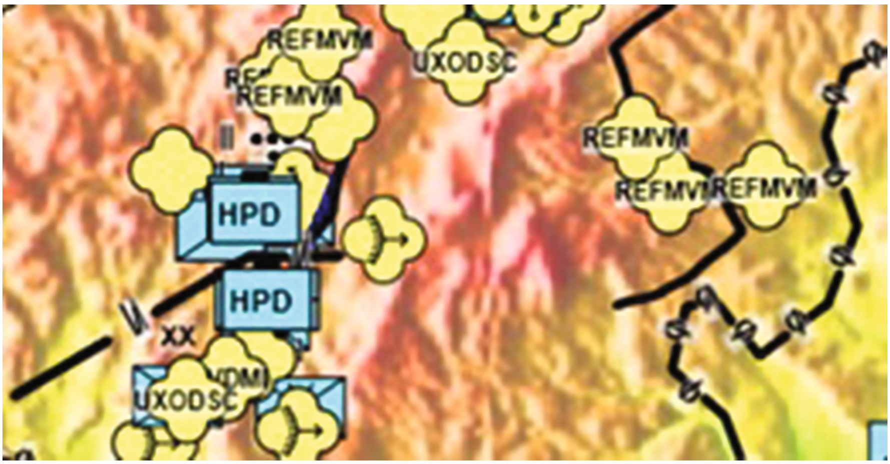

Fig. 2. The example battlefield in MIP Baseline 3

Some of presented tactical symbols have not got direct reflection in second model. In such situation the generic symbols are used and some geometrical approximation.

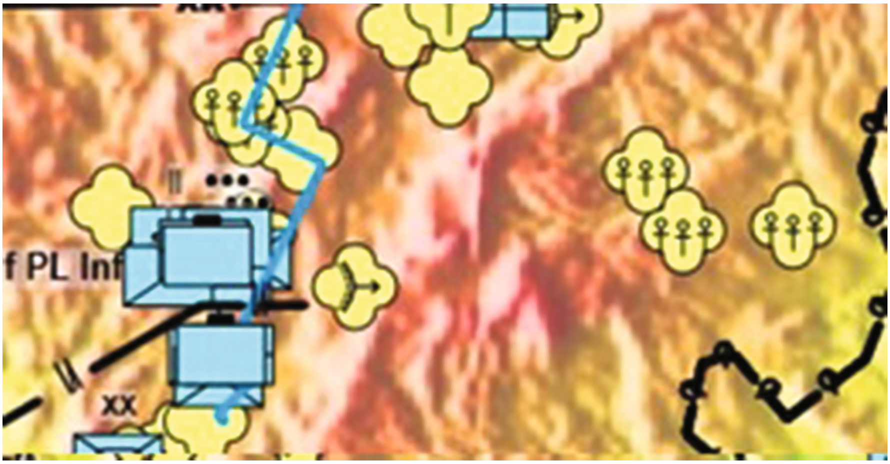

Fig. 3. The example battlefield in MIP Baseline 2

Test Results

Nowadays an ADTA mechanism was widely tested during international military exercises: CWIX (Coalition Warrior Interoperability eXploration, eXperimentation, eXamination, eXercise) [29] and Combined Endeavor [26]. It was used as a main element in the schema of the scenario which was tested. This mechanism let to use different NATO countries systems and build one operational view of battlefield [15–17, 22]. The schema of the connection on CWIX 2011 exercises is presented on Fig. 4. There was two world of MIP Baseline protocols: 2.0 and 3.1. Unfortunately the CWIX exercises are made on NATO secret network. There is no possibility to provide more test results and present database view according to security policy. During such exercises there is possibility to exchange expertise, collect experience and test solutions.

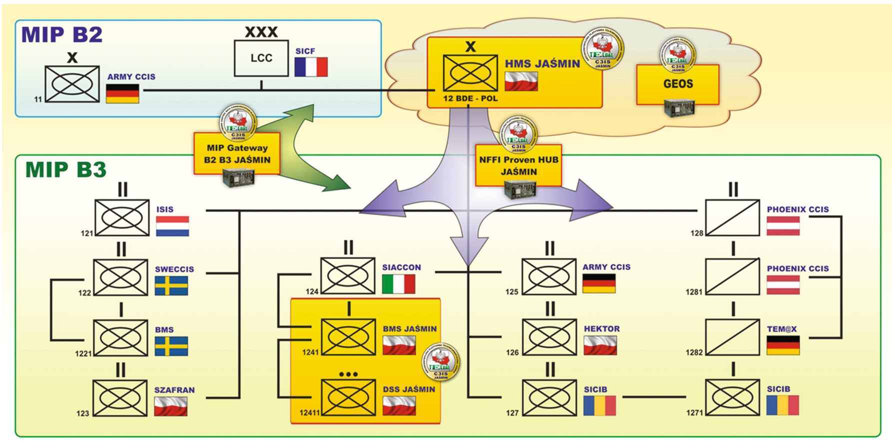

Fig. 4. The schema of the CWIX 2011

ADTA mechanism is one of the main innovative component of ICT JASMINE System. Of course military solutions require appropriate network protection using the best possible solutions [2,4,14]. In JASMINE system there are also implemented some mechanism for network protection [7] like Information Exchange Gateway [3,18,25].

3 Conclusions

In this paper authors tries to describe the practical solution of using semantic transformation for different database model. This proves that using such mechanism could be effective [10,19,21]. The MIP programme was not provided for ensuring transformation for different database model but practical usage of the science could provide good results. Presented solutions is an innovative on military market developed by polish engineers and gives them lot of respect on international stage. The TELDAT engineers are known as innovative ones using some other research results like artificial intelligence [5,6,23].

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Effective Search of Proteins in the DNA Code

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Abstract. The publication discusses the important issues that relate to the processing of large amounts of data associated with the genome. The statistics of gene content of the human genome is demonstrated. Article also presents the most important information about how the proteins are stored in the DNA code and the process of the RNA transcription, splicing and protein translation. This is followed by description of practical algorithm for searching protein and pattern matching in the DNA code given complete genome and protein data.

Keywords: Proteins · DNA code

1 Formulation of the Problem

The aim of this paper is to demonstrate how to look for a pattern consisting of protein sequence in a DNA code. It is assumed that we look for exact occurrences of given pattern in a long text (genome). We actually know what we are looking for, but the large size and organization of genomes makes it problematic. In this paper the statistics of gene content of the human genome is presented, RNA Transcription, RNA splicing, protein translation and other aspects that can affect the solution. It will be demonstrated that organizing data into efficient data structure such as suffix tree, would be a crucial part of solving the problem. The issue of searching protein data in DNA is related closely to the problem of exact and approximate pattern matching in a long text. There are approaches in bioinformatics [8,11], Big Data, genetics [9,17,19,38,39], databases architecture [5,6,34,42], data analysis [6,21], computational solutions [32], software systems [7,41,42], rough sets [2,13,15,23,33], which can provide variety of solutions to interpreting and process human genome big data. In the mentioned papers there has been demonstrated different varieties of problems related to DNA data. Following chapters present the most important information about genes in the human genome, how the proteins are stored in the DNA code, the process of the transcription between DNA code and mRNA as well as Protein Translation [1,22,30,36,37]. This is followed by proposal of algorithm and short conclusions.

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2 Gene Content of the Human Genome

DNA carries inheritance information from generation to generation. It can be compared to an instruction book with detailed information on how to form and maintain a living being. Thanks to the available free genome data [27,28] one can make many types of operation on strings. For instance:

– DNA sequence analysis (For a DNA sequence annotation, searching coding sequences, regulatory and repetitive, markers, etc.) cataloging and processing of biological information contained in databases
– Genome sequence analysis (comparing the genomes search genes responsible for genetic diseases)
– Analysis of the evolutionary relationships between sets of sequences (phylogenetics)
– Analysis of gene expression (microarrays - to do with pictures)
– Protein sequence analysis (comparing the sequences, search, domains and themes, anticipating functions and locations within the cell
– Cataloging functions of genes/proteins, determining metabolic pathways

Diversity of genome information (data sequence, structural, literature, etc.) is still rising and there is increase in the amount of available raw genomes [26,30,31]. This makes finding any further valuable information difficult without appropriate algorithms and tools such as Blast [9,14,16]. The size of the human genome, expressed in base pairs is $3,079 \times 10^9$. This is divided into 22 chromosomes, plus the X chromosome (one in males, two in females) and, in males only, one Y chromosome. The genome also includes the mtDNA - mitochondrial DNA [22,29]. Human genome contains about 20–25 thousand protein-coding genes. Protein-coding genes make up only 1.5% of human genome. The rest of the genome encodes no protein, but manufactured on the basis of the DNA molecule of RNA, or non-coding DNA, which contains, among other regulatory sequences, repetitive or moving elements such as transposons and retrotransposons. Recent biochemical studies revealed that the mere RNA is able to catalyze a series of chemical reactions in the cell. Additionally observed phenomenon of blocking the expression of certain genes by their complementary copies elsewhere in the genome [18–20].

According to the Table 1 protein-coding genes are not distributed evenly across the chromosomes, ranging from a few dozen to more than 2000, with an especially high gene density within chromosomes 19, 11, 1 and 2. Moreover every chromosome contains various gene-rich and gene-poor regions. The reason for these gene density nonrandom patterns is not well known [24]. The deoxyribonucleic acid (DNA) has complementary pair of guanine (G), cytosine (C) and adenine (A) with thymine (T). The ribonucleic acid (RNA) in place of thymine (T) has uracil (U). The following DNA sequences demonstrate pair double-stranded patterns. The top strand is written from the $5'$ end to the $3'$ end and the corresponding bottom strand is written $3'$ to $5'$. Nucleic acids can only be synthesized by polymerase in the $5 - to - 3$ direction. A base-paired DNA sequence:

Table 1. Gene content of the human genome based on [22]

Chromosome	Base pairs	Variations	Proteins
1	249,250,621	4,401,091	2,012
2	243,199,373	4,607,702	1,203
3	198,022,430	3,894,345	1,04
4	191,154,276	3,673,892	718
5	180,915,260	3,436,667	849
6	171,115,067	3,360,890	1,002
7	159,138,663	3,045,992	866
8	146,364,022	2,890,692	659
9	141,213,431	2,581,827	785
10	135,534,747	2,609,802	745
11	135,006,516	2,607,254	1,258
12	133,851,895	2,482,194	1,003
13	115,169,878	1,814,242	318
14	107,349,540	1,712,799	601
15	102,531,392	1,577,346	562
16	90,354,753	1,747,136	805
17	81,195,210	1,491,841	1,158
18	78,077,248	1,448,602	268
19	59,128,983	1,171,356	1,399
20	63,025,520	1,206,753	533
21	48,129,895	787,784	225
22	51,304,566	745,778	431
X	155,270,560	2,174,952	815
Y	59,373,566	286,812	45
mtDNA	16,569	929	13

AGTCGATTGAGCTCTAGCG
TCAGCTAACTCGAGATCGC

The corresponding RNA sequence, in which uracil is substituted for thymine where uracil takes its place in the RNA strand:

AGUCGAUUGAGCUCUAGCG
UCAGCUAACUCGAGAUCGC

Later in the article we will use directly thymine (T) for the search of protein in DNA algorithm, skipping conversion of the text to uracil (U) and then to thymine (T) will save the run time.

3 RNA Transcription and Splicing

Taking a book as an example, in order not to risk destroying a DNA, we need to copy the pages (genes) we are interested in and return the book (DNA) intact to the library. Our pages (genes) are copied into RNA which is next translated (decrypted) into proteins in the cytoplasm. For security reasons RNA is used as an intermediate to produce protein. The cytoplasm could harm DNA. Hence we can think of RNA as a working version of the information to produce proteins, while the DNA is a safe place for long storage needed to sustain life [3,10,12,17]. RNA polymerase travels along the DNA template and polymerizes ribonucleotides into an RNA copy of the gene. Polymerase moves at a constant speed of approximately 30 base pairs per second. Even in the case of very long genes polymerase effectively keeps DNA (for instance giant 2.4 million dystrophin). At the end of the gene the RNA polymerase detaches DNA and the process is finished. In some prokaryotic and mammalian genes the polymerase may detach the DNA template before it finishes the task. This functionality allow a cell to control the number of complete RNA copies [1]. The RNA copy of a protein encoding gene must be modified before it will be transported from the nucleus to cytoplasm and translated into protein. The primary transcription product of a gene is therefore called a precursor of mRNA, pre-mRNA. Firstly both ends 5′ and 3′ of the pre-mRNA are modified in processes called capping and Poly(A)-tail. In second step splicing is conducted. Splicing is the process of removing the intron (non-coding sequences) and combining the exons (coding sequences) of eukaryotic mRNA precursor. This process occurs during post-transcriptional processing in order to mature mRNA and prepare it for the translation. Then mRNA encodes the continuous polypeptide chain (from start to stop codon). Splicing is catalyzed by a complex of proteins and RNA called the spliceosome. Processing of pre-mRNA in many cases takes place during transcription. Introns may be as short as 30 bp or as long as 70–80 thousand

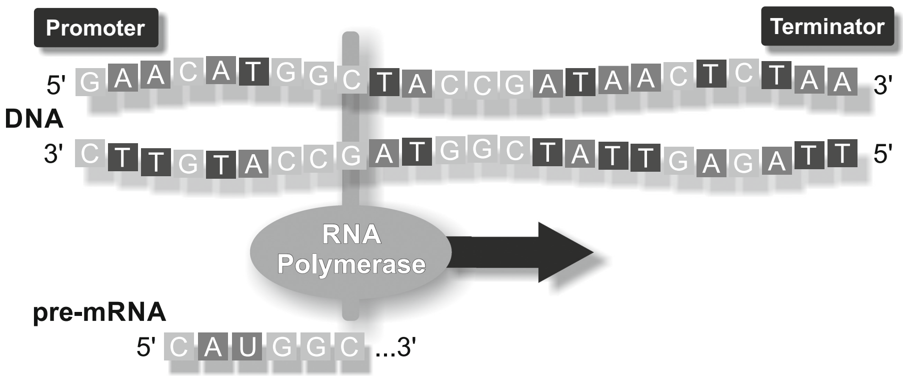

Fig. 1. RNA transcription

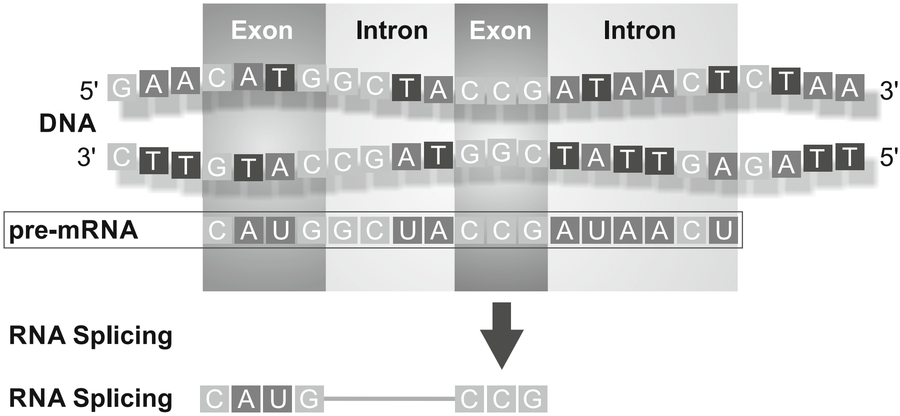

Fig. 2. Splicing alternative splicing may lead to creation of different mRNA.

base pairs. It should be marked that the pre-mRNA is associated with hnRNP proteins and not naked at any time (these are not shown in the illustrations) It is known short sequence motifs GU at the 5′ end and a sequence AG at the 3′ end in the pre-mRNA are critical for splicing. When the intron is liberated gets rapidly degraded. The splicing factors are believed to be reused [1] (Figs. 1 and 2).

4 Protein Translation

Proteins perform a vast array of functions within living organisms, including catalyzing metabolic reactions, DNA replication, responding to stimuli, and transporting molecules from one location to another. Translation is a very complicated process and requires a coordinated interaction over a hundred different types of macromolecules: ribosomes, tRNA, mRNA, and many accessory proteins. Translation is process of synthesis of the polypeptide chain of the protein from mRNA. As a result, there is a final translation of genetic information originally contained in the genetic code of DNA to a specific protein structure, depending on the ranking of amino acids in the polypeptide chain (Fig. 3).

The ribosome provides the main machinery for the translation process. The major role of the ribosome is to catalyse coupling of amino acids into protein according to the mature mRNA sequence are by Additional important molecule is tRNA (transfer RNA) which bring the amino acids to the ribosome. Translation may be divided into three distinct steps. In first - initiation of transaltion ribosome is bound to the specific start site on the mRNA while the initiator tRNA is bound to the ribosome. The second stage, elongation, consists of joining amino acids to the polypeptide chain according to the sequence. This is repeated over and over again until the stop codon is reached. Then the ready-made protein

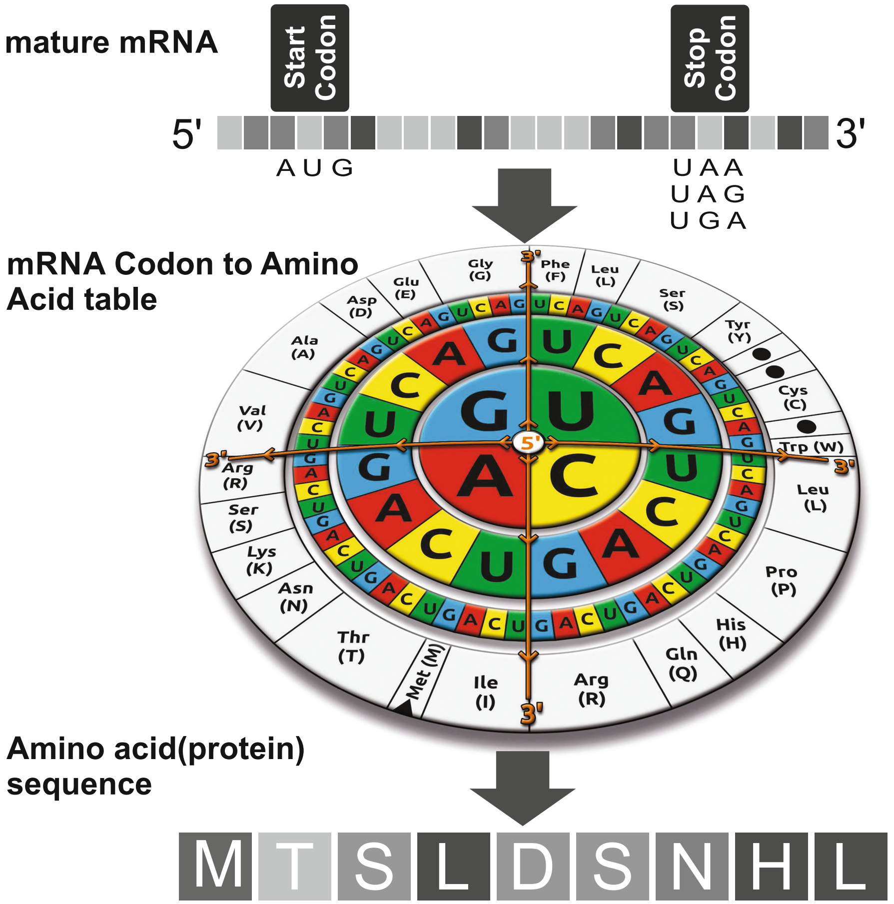

Fig. 3. Protein translation

is released from the ribosome. Protein is built up from 20 amino acids whereas the mRNA is composed of four different nucleotides. In order to specify 20 different amino acids from the four nucleotides, the nucleotide sequence is interpreted in codons, groups of three nucleotides. Each of the codons specify one amino acid. This is referred to as the genetic code [35] (see Table 2 below).

Reverse Transcription is the process of rewriting the single-stranded RNA (ssRNA) by the enzyme reverse transcriptase (RT) into double stranded DNA. The process of reverse transcription is used by some RNA viruses, such as HIV to integrate their genetic material into the host cell genome and replicate [6]. Reverse transcription is also used in the process of playback of telomeres by telomerase and is accompanied by a movement of retrotransposons in the host genome. Perhaps in the future it will be possible to repair or improve DNA with reverse transcription.

Table 2. Genetic code, mRNA to amino acid table, based on [35]

1st base	2nd base								3rd base
	T		C		A		G		
T	TTT	Phe/F Phenylalanine	TCT	Ser/S Serine	TAT	Tyr/Y Tyrosine	TGT	Cys/C Cysteine	T
	TTC		TCC		TAC		TGC		C
	TTA	Leu/L Leucine	TCA		TAA	Stop Ochre	TGA	Stop Opal	A
	TTG		TCG		TAG	Stop Amber	TGG	Trp/W TryptophanG	G
C	CTT		CCT	Pro/P Proline	CAT	His/H Histidine	CGT	Arg/R Arginine	T
	CTC		CCC		CAC		CGC		C
	CTA		CCA		CAA	Gln/Q Glutamine	CGA		A
	CTG		CCG		CAG		CGG		G
A	ATT	Ile/I Isoleucine	ACT	Thr/T Threonine	AAT	Asn/N Asparagine	AGT	Ser/S Serine	T
	ATC		ACC		AAC		AGC		C
	ATA		ACA		AAA	Lys/K Lysine	AGA	Arg/R Arginine	A
	ATG	Met/M Methionine	ACG		AAG		AGG		G
G	GTT	Val/V Valine	GCT	Ala/A Alanine	GAT	Asp/D Aspartic acid	GGT	Gly/G Glycine	T
	GTC		GCC		GAC		GGC		C
	GTA		GCA		GAA	Glu/E Glutamic acid	GGA		A
	GTG		GCG		GAG		GGG		G

This is the standard genetic code, in amino acid⟶ codon form. By default it is the DNA code; for the RNA code (using Uracil rather than Thymine), add template parameter "T = U". The characteristics of the genetic code: The genetic code is ternary. Each amino acid is encoded by three juxtaposed nucleotides in

Table 3. Reverse transcription table. Amino acid to codon form.

Amino acid	Codons	Amino acid	Codons
Ala/A	GCT, GCC, GCA, GCG	Leu/L	TTA, TTG, CTT, CTC, CTA, CTG
Arg/R	CGT, CGC, CGA, CGG, AGA, AGG	Lys/K	AAA, AAG
Asn/N	AAT, AAC	Met/M	ATG
Asp/D	GAT, GAC	Phe/F	TTT, TTC
Cys/C	TGT, TGC	Pro/P	CCT, CCC, CCA, CCG
Gln/Q	CAA, CAG	Ser/S	TCT, TCC, TCA, TCG, AGT, AGC
Glu/E	GAA, GAG	Thr/T	ACT, ACC, ACA, ACG
Gly/G	GGT, GGC, GGA, GGG	Trp/W	TGG
His/H	CAT, CAC	Tyr/Y	TAT, TAC
Ile/I	ATT, ATC, ATA	Val/V	GTT, GTC, GTA, GTG
START	ATG	STOP	TAA, TGA, TAG

the DNA molecule. Each three are Codon. Codon - 3 consecutive nucleotides in the DNA chain. The genetic code is without commas - between codons there are no additional elements and they are read one by one. The genetic code is not overlaping i.e. the codons in no way overlap. The genetic code is degenerate - one amino acid may be encoded by several codons. The genetic code is unambiguous - the codon builds always one, the same type of amino acid. The genetic code is universal. This means that in all organisms genetic code is the same (Table 3).

5 Algorithm for Exact Pattern Matching

To search a database of sequences for a known sequence is a well-known problem in bioinformatics. Given a string s = s1...sn and a longer string ls = ls1...lsm, The problem is to find any occurrences of s in ls. For instance, if ls = TACCAGCCA and s = CCA, then the solution is position 3 and 7. With a brute force algorithm we need to verify at every position of string ls whether whole s starts there. For worst-case scenario, the time can be estimated as O(nm). Best case scenario is O(m). It's worth to remind that for human genome m would be 3 billion (base pairs), not to mention 6 reading frames. To deal with that problem, authors propose algorithm which use original data structure developer by Peter Weiner called a suffix tree [40]. It solves the pattern matching problem in linear time for any text or pattern [2,4,11]. With the use of the suffix trees one is able to search a text in a way that for any pattern of length n, one can determine whether or not it occurs in the text. No matter how long the text is, it is using only the O(n) time. Of course, we need time create the suffix trees (Step 1) which takes O(m) time. Once the trees are created for a given human genome data, they can be reuse (Steps 2–5) any number of times for protein search. The algorithm to find s in ls includes:

STEP 1: create the suffix tree for string ls
STEP 2: pattern s need to be threaded through the tree
STEP 3: **if** threading is finished print every matching leaf in the tree
STEP 4: **else** print that there are no occurences of s in ls
STEP 5: STOP Above figure shows searching for the pattern TAC within suffix tree of TACGTATGTACC. Grey leafs provide positions in the tree where text s occurs in ls, in case of TAC it starts at position 1 and 9. Data for the real cases search should be prepared with transcription tables. Below part of the solution in c++ code using the tables: One can use additional insights and properties of the genome that can be used in the solution construction as described in [39]

- The smallest exon size in human genes is 10 base pairs (but there is exceptionally short exon 4 bp in the troponin I gene TNNI1),
- average exon size is 288 bp,
- largest exon size is 18,2 kilobases,
- smallest intron size is about 30 bp,

– largest intron size is 1.1 MB,
– A small minority of human protein-coding genes lack introns and are generally small,
– As genes get larger, exon size remains fairly constant but intron sizes canbecome very large.
– Internal exons tend to be fairly uniform in size, but the terminal exon or some exons near the 3′ end can be many kilobases long;
– number of exons in a gene ranges from 1 (no introns) to 363 (titin gene).

Thus one have to divide the protein into exons and search for instances of texts at least with a length of 10 characters (the smallest exon). The distance between exons is at least 30 characters. Searches should be performed separately for each chromosome. Algorithm should start with the chromosomes with the highest number of protein occurrences 19, 11, 1, 2 etc. (Fig. 4).

Algorithm 1.

```
int fChrToPrn(string & s1, char & c1)
// references{
  switch(s1[0]){
    case 'A':
       switch(s1[1]){
           case 'A':
    switch(s1[2]){
        case 'A': c1='K'; return 1;
        // Lysine
        case 'T': c1='N'; return 1;
        // Asparagine
        case 'G': c1='K'; return 1;
        // Lysine
            case 'C': c1='N'; return 1;
        // Asparagine
    }
          case 'T':
    switch(s1[2]){
        case 'A': c1='I'; return 1;
        // Isoleucine
        case 'T': c1='I'; return 1;
        // Isoleucine
        case 'G': c1='M'; return 1;
        // Methionine
        case 'C': c1='I'; return 1;
        // Isoleucine
.....
```

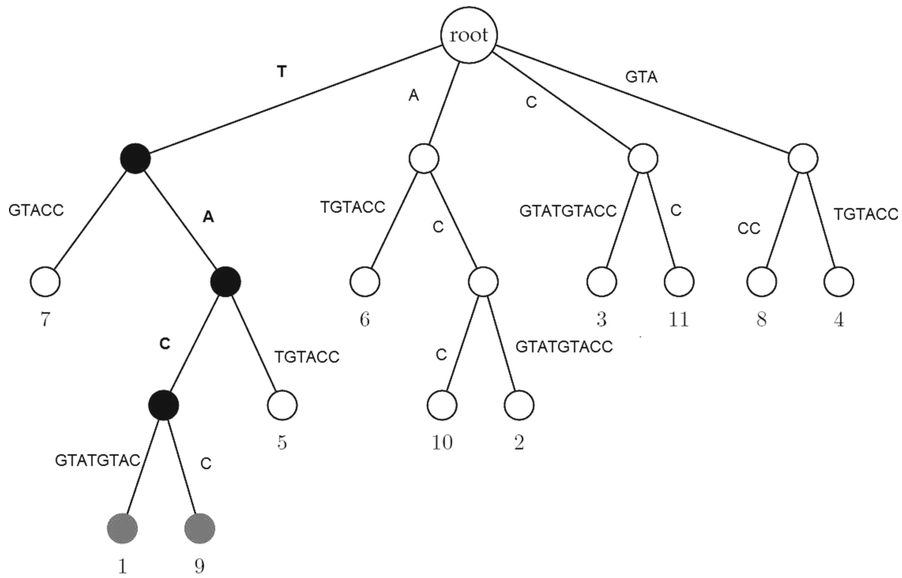

Fig. 4. Suffix tree for the string TACGTATGTACC, based on [7]

6 Conclusions

The article studies the process of protein production based on the DNA sequence as well as issues related to the processing of large amounts of data associated with the genome. Process of mRNA transcription, splicing, and protein translation and other aspects that can affect the proposed solution were investigated. An algorithm and solution based on the suffix trees has been introduced. It has been concluded that the algorithm can be efficiently used for searching of the required patterns. The algorithm is simple to implement and allows for additional improvements. Additional insights such as exon size and properties of the genome may help to accelerate the algorithm. In-depth investigation could be performed to measured it with the further requirements described in the literature [25]. Similar solutions are an essential part of modern bioinformatics tools including popular services such as BLAST [6]. Therefore a further development of the algorithm should be conducted. The adjustment should allow it for the effective searching for approximate occurences of given sequences. This would make it practical for many contemporary issues related to DNA.

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Generalized Reciprocity Property for Interval-Valued Fuzzy Setting in Some Aspect of Social Network

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Abstract. In this paper we study interval-valued fuzzy relations. For this setting we introduce the fuzzy negation based reciprocity property. We examine what is the connection of this property with weak transitivity and some equivalence relation for interval-valued fuzzy relations. We also study preservation of such reciprocity by some operators. Moreover, we present an algorithm to find the best alternative in decision making problem with the use of new reciprocity concept.

Keywords: Interval-valued fuzzy reciprocal relations · Fuzzy negations · Transitivity properties · Operators for interval-valued fuzzy relations

1 Introduction

Social Network Analysis (SNA) methodology studies the relationships between social entities like members of a group, corporations or nations and it is a useful methodology to examine structural and locational properties such as: centrality, prestige and structural balance [21]. In this article, we focus on one type of social networks in which the users explicitly express their opinion on other users. Furthermore, to represent the uncertainty or fuzziness of relationship between group experts, this article develops an interval-valued fuzzy SNA based on decision making.

Interval-Valued Fuzzy Relations (IVFRs) (Sambuc, Zadeh 1975), which are extensions of fuzzy relations (Zadeh 1965), are applied also in databases, pattern recognition, neural networks, fuzzy modelling, economy, medicine or multicriteria decision making [14,16]. In recent applications to image processing [2] or classification [19] it has been proven that, under some circumstances, the use of IVFSs together with the total order defined by Xu and Yager [22] provide results that are better than their fuzzy counterparts.

Preference relations appear for example in choice and utility theories. The concept of a preference relation has been studied by many authors, both in crisp or fuzzy environments [8] (see also [11]). Detailed discussion on transitivity of

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reciprocal relations (for fuzzy setting) can be found in [9,10,20]. Transitivity is an important property of relations, since it may guarantee consistency of choices of decision makers. Diverse properties of IVFRs (also for the case of interval-valued fuzzy reciprocal relations IVFRRs) have been studied by a range of authors [2,12,15,23].

Our main goal is to introduce fuzzy negation based reciprocity property, which means that instead of using classical negation in definition of reciprocity, we apply a fuzzy negation. We consider the problem of preservation of such reciprocity property of interval-valued fuzzy relations by some operators and connection of the new reciprocity with weak transitivity and some equivalence relation. Presented considerations have possible applications in decision making problems.

This work is composed of the following parts. Firstly, some concepts and results useful in further considerations are recalled (Sect. 2). Next, weak transitivity property is analyzed in the context of introduced here reciprocity property (Sect. 3). Later, preservation of the reciprocity by some operators is discussed (Sect. 4). Moreover, (Sect. 5) there is considered an equivalence relation for the interval-valued fuzzy relation and its connection with the reciprocity property. Finally, (Sect. 6) it is presented an algorithm to find the best alternative in decision making problem with new reciprocity concept.

2 Preliminaries

We recall the notion of a fuzzy negation and some concepts for interval-valued fuzzy relations.

Definition 1 **([13]).** *A fuzzy negation function is a decreasing function* $N : [0,1] \to [0,1]$ *verifying the boundary conditions* $N(0) = 1$ *and* $N(1) = 0$. *Strictly decreasing and continuous negation functions are known as strict negations, whereas involutive negation functions (i.e. those verifying* $N(N(x)) = x$ *for all* $x \in [0,1]$*) are known as strong negations (and constitute a subclass of strict negations).*

Typical examples of fuzzy negations are (cf. [17]):

- $N(x) = 1 - x$, which is a strong negation and is called the classical or standard negation;
- $N(x) = 1 - x^2$, which is strict but not strong;
- $N_S^{\lambda}(x) = \frac{1-x}{1+\lambda x}$, the Sugeno family of fuzzy (strong) negations, where $\lambda \in (-1, \infty)$, and for $\lambda = 0$ we get the classical fuzzy negation.

Let X, Y, Z be non-empty sets and $L^I = \{[x_1, x_2] : x_1, x_2 \in [0,1], x_1 \leqslant x_2\}$. Note that L^I endowed with the partial order $[x_1, x_2] \leqslant_{L^I} [y_1, y_2]$, if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$, is a complete bounded lattice with the top element given by $\mathbf{1} = [1,1]$ and the bottom element given by $\mathbf{0} = [0,0]$. In this lattice, the supremum and infimum of any two elements are defined by

$[x_1, x_2] \vee [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)]$,
$[x_1, x_2] \wedge [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)]$, respectively.

Definition 2 (cf. [18,24]**).** *An IVFR R between universes X, Y is a mapping $R : X \times Y \to L^I$ such that*

$$R(x, y) = [\underline{R}(x, y), \overline{R}(x, y)] \text{ for all pairs } (x, y) \in X \times Y.$$

The class of all IVFRs between universes X, Y is denoted by $\mathcal{IVFR}(X \times Y)$, or $\mathcal{IVFR}(X)$ for $X = Y$.

$(\mathcal{IVFR}(X \times Y), \vee, \wedge)$ with $\leqslant_{L^I}$ is a complete and distributive lattice.

Definition 3. *Let* $\text{card}(X) = n$*. An Interval-Valued Fuzzy Reciprocal Relation (IVFRR) R on the set X is a matrix $R = (r_{ij})_{n \times n}$ with $r_{ij} = [\underline{R}(i, j), \overline{R}(i, j)]$, for all $i, j \in \{1, \ldots, n\}$, where $r_{ij} \in L^I$*

$$r_{ii} = [0.5, 0.5], \quad r_{ji} = N_{IV}(r_{ij}) = [N(\overline{R}(i, j)), N(\underline{R}(i, j))] \text{ for } i \neq j,$$

where N is a fuzzy negation.

This is the fuzzy negation based reciprocity property considered, for application reasons, on a finite set. This notion is a generalization of the reciprocity property considered for example in [23], where N was a classical fuzzy negation. However, the assumption $r_{ji} = 1 - r_{ij}$ for $i, j \in \{1, \ldots, n\}$, is rather strong and frequently violated by decision makers in real-life situations. This is why we use a fuzzy negation instead of the classical fuzzy negation $N(x) = 1 - x$. Firstly, we give some results for the basic operations on relations, which are here generalized to fuzzy negation based versions (cf. [6]). Let $N : [0, 1]^2 \to [0, 1]$ be a fuzzy negation. For $R \in \mathcal{IVFR}(X)$ we define:

- the converse relation, $R^{-1}(x, y) = [\underline{R}(y, x), \overline{R}(y, x)], x, y \in X$;
- the complement, $R'(x, y) = [N(\overline{R}(x, y)), N(\underline{R}(x, y))], x, y \in X$;
- the dual relation, $R^d(x, y) = [N(\overline{R}(y, x)), N(\underline{R}(y, x))], x, y \in X$.

Note that, as a result of the previous, $R^d = (R')^{-1} = (R^{-1})'$.

Proposition 1. *Let $N : [0, 1]^2 \to [0, 1]$ be a strong fuzzy negation. If R is an IVFRR, then R' and R^{-1} are IVFRRs as well. Moreover, $R^{-1} = R'$ and $R^d = R$.*

Proof. Let R be IVFRR, $i, j \in \{1, ..., n\}$, with $S = R^{-1}$, $S = [s_{ij}]$, N be a fuzzy negation (we do not need assumption of a strong negation). Then

$$s_{ji} = [N(\overline{S}(i, j)), N(\underline{S}(i, j))] = [N(\overline{R}(j, i)), N(\underline{R}(j, i))] = r_{ij}.$$

As a result $S = R^{-1}$ is an IVFRR. Let $T = R'$, $T = [t_{ij}]$, $N : [0, 1]^2 \to [0, 1]$ be a strong fuzzy negation. Then

$$t_{ji} = [N(\overline{T}(i, j)), N(\underline{T}(i, j))] = [\underline{R}(i, j), \overline{R}(i, j)] = r_{ij}.$$

This means that $T = R'$ is an IVFRR and, by analyzing both parts of the proof, we see that $R^{-1} = R'$, so we get $R^d = R$.

3 Weak Transitivity and Reciprocal Relations

We recall the notion of transitivity and weak transitivity. $R \in \mathcal{IVFR}(X)$ is called transitive if $R^2 \leqslant R$ (cf. [3]). Transitivity for $X = \{x_1, \ldots, x_n\}$ implies that the value $R(i,j)$, representing the preference between the alternatives x_i and x_j, where $R(i,j) \geqslant \max_{1\leqslant k\leqslant n}(\min(R(i,k), R(k,j))$, should be greater than or equal to maximum of the minimum partial values between the alternatives x_i, x_k and x_k, x_j. Now we consider more general kind of transitivity.

Definition 4 (cf. [23]). *$R \in \mathcal{IVFR}(X)$ is said to be weakly transitive, if for all $x, y, z \in X$*

$$R(x,z) \geqslant [0.5, 0.5], R(z,y) \geqslant [0.5, 0.5] \Rightarrow R(x,y) \geqslant [0.5, 0.5]. \tag{1}$$

In the context of preference relations, for $X = \{x_1, \ldots, x_n\}$, weak transitivity captures the fact that, if the alternative x_i is preferred to x_k and x_k is preferred to x_j, then x_i should be preferred to x_j.

Theorem 1. *Let $R \in \mathcal{IVFR}(X)$. If the following conditions hold:*

$$\forall_{x,z\in X}\ \underline{R}^2(x,z) \geqslant 0.5 \Rightarrow \underline{R}(x,z) \geqslant 0.5, \tag{2}$$

$$\forall_{x,z\in X}\ \overline{R}^2(x,z) \geqslant 0.5 \Rightarrow \overline{R}(x,z) \geqslant 0.5, \tag{3}$$

then R is weakly transitive.

Proof. We use definition of classical max-min composition of fuzzy relations $\overline{R}$, $\overline{R}$ and definition of weak transitivity.

The converse statement to Theorem 1 is not true. Let $\operatorname{card}(X) = 3$ and $R \in \mathcal{IVFR}(X)$,

$$R = \begin{bmatrix} [0.5, 0.5] & [0.5, 0.6] & [0.3, 0.6] \\ [0.4, 0.7] & [0.5, 0.5] & [0.3, 0.4] \\ [0.7, 0.7] & [0.6, 0.7] & [0.5, 0.5] \end{bmatrix}.$$

R is weakly transitive, relation $\underline{R}$ fulfils condition (2) but relation $\overline{R}$ does not fulfil (3).

Definition 5. *Let $\operatorname{card}(X) = n$. $R \in \mathcal{IVFR}(X)$ is said to be a relation with strictly dominating upper (lower) triangle, if*

$$\forall_{1\leqslant i,j\leqslant n,\ i<j\ \ (i>j)} R(i,j) > [0.5, 0.5]. \tag{4}$$

Strictly dominating upper (lower) triangle property in the set of alternatives $\{x_1, x_2, ...x_n\}$ means that $x_1 \succ x_2 \succ ... \succ x_n$ $(x_1 \prec x_2 \prec ... \prec x_n)$.

Theorem 2. *Let N be a fuzzy negation and $N(0.5) \leqslant 0.5$. If R is IVFRR with strictly dominating upper (lower) triangle, then it is weakly transitive.*

Proof. Let R be IVFRR with strictly dominating upper triangle. If $i = j$, then $R(i, j) = [0.5, 0.5]$. Thus implication (1) is true. If $i \neq j$, then we consider the following cases:

1. For $i > j$ we have by (4) and the reciprocity property $R(i, j) < [0.5, 0.5]$, because for $i < j$ we have $R(i, j) > [0.5, 0.5] \Leftrightarrow N_{IV}(R(i, j)) < [N(0.5), N(0.5)]$, what implies $R(j, i) < [0.5, 0.5]$ and we examine the following cases:
   - if $i \geqslant k > j$, then $R(k, j) < [0.5, 0.5]$;
   - if $k > i > j$, then $R(k, j) < [0.5, 0.5]$;
   - if $i > j \geqslant k$, then $R(i, k) < [0.5, 0.5]$.

   In all these cases we obtain the false antecedent and consequence, so (1) is true.
2. For $i < j$ we have $R(i, j) > [0.5, 0.5]$ so implication (1) is true. The proof for strictly dominating lower triangle property is similar.

Examples of fuzzy negations fulfilling condition $N(0.5) \leqslant 0.5$ are the Sugeno negations for $\lambda \geqslant 0$. Let $\text{card}(X) = 3$. The converse to Theorem 2 is not true. The following IVFRR (for the classical fuzzy negation)

$$R = \begin{bmatrix} [0.5, 0.5] & [0.5, 0.6] & [0.3, 0.3] \\ [0.4, 0.5] & [0.5, 0.5] & [0.3, 0.4] \\ [0.7, 0.7] & [0.6, 0.7] & [0.5, 0.5] \end{bmatrix}$$

is weakly transitive (see Theorem 1), but it is not a relation with strictly dominating lower (upper) triangle.

4 Operators on Interval-Valued Fuzzy Relations

There exist several operators designed to act on intervals, a notable case being Atanassov's operators, which follow the ideas in [1], and were studied in [5]. Let $[a, b] \in L^I$, $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leqslant 1$. The Atanassov's operator is defined as $F_{\alpha,\beta} : L^I \to L^I$ with

$$F_{\alpha,\beta}([a, b]) = [a + \alpha(b - a), b - \beta(b - a)].$$

Moreover, we can also define, for $\alpha \leqslant \beta$,

$$P_{\alpha,\beta}([a, b]) = [\max(\alpha, a), \max(\beta, b)], \quad Q_{\alpha,\beta}([a, b]) = [\min(\alpha, a), \min(\beta, b)].$$

Directly by the above definition we observe that for $\alpha, \beta \in [0, 1]$ and every $x, y \in X$ we have $\underline{R}(x, y) \leqslant \underline{F_{\alpha,\beta}(R)}(x, y) \leqslant \overline{F_{\alpha,\beta}(R)}(x, y) \leqslant \overline{R}(x, y)$, where $\alpha + \beta \leqslant 1$. The above property means that the operator $F_{\alpha,\beta}(R)$ applied to R makes the amplitude of the interval of R smaller, so in the interpretation in preference modelling the uncertainty is smaller. For $\alpha \leqslant \beta$ we have $R \leqslant P_{\alpha,\beta}(R)$ and $Q_{\alpha,\beta}(R) \leqslant R$.

Now we recall the concept of a partial inclusion of a relation in which the function sgn : $\mathbb{R} \to \mathbb{R}$ occurs, where

$$\mathrm{sgn}(t) = \begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t = 0 \\ -1, & \text{for } t < 0 \end{cases}. \tag{5}$$

Definition 6 (cf. [4]). *A relation $R \in \mathcal{IVFR}(X)$ is said to be partially included if*

$$\mathrm{sgn}(\underline{R}(x,z) - \underline{R}(z,y)) = \mathrm{sgn}(\overline{R}(x,z) - \overline{R}(z,y)), \quad x,y,z \in X. \tag{6}$$

The concept of partial inclusion was originally introduced in [5] as a mean to study the preservation of transitivity in IVFRs under the action of some specific operators.

Proposition 2 ([3]). *Let $R \in \mathcal{IVFR}(X)$, $\alpha, \beta \in [0,1]$.*

- *If R is partially included and transitive, then $F_{\alpha,\beta}(R)$ is transitive for $\alpha+\beta \leqslant 1$.*
- *If R is transitive, then $P_{\alpha,\beta}(R)$ is transitive for $\alpha \leqslant \beta$.*
- *If R is transitive, then $Q_{\alpha,\beta}(R)$ is transitive for $\alpha \leqslant \beta$.*

Proposition 3 ([3]). *Let R be IVFRR, N be the classical fuzzy negation and $\alpha, \beta \in [0,1]$, with $\alpha + \beta \leqslant 1$, such that at least one of the preferences is given by an interval of positive length. $F_{\alpha,\beta}(R)$ is an IVFRR if and only if $\alpha = \beta$.*

Proposition 4. *Let R be IVFRR, N be a fuzzy negation and $\alpha, \beta \in [0,1]$, with $\alpha \leqslant \beta$, such that at least one of the preferences is given by an interval of positive length. Thus:*

- *$P_{\alpha,\beta}(R)$ is an IVFRR if and only if $\alpha \leqslant \underline{R}(i,j) \leqslant N(\beta)$ and $\alpha \leqslant 0.5$ for all $i,j \in \{1,...,n\}$;*
- *$Q_{\alpha,\beta}(R)$ is an IVFRR if and only if $N(\beta) \leqslant \underline{R}(i,j) \leqslant \alpha$ for all $i,j \in \{1,...,n\}$.*

Proof. Now we will examine the operator $P_{\alpha,\beta}$. For $\alpha \leqslant \underline{R}(i,j) \leqslant N(\beta)$ we have $\max(\alpha, \underline{R}(i,j)) = \underline{R}(i,j) = N(\overline{R}(j,i)) = \min(N(\beta), N(\overline{R}(j,i))) = N(\max(\beta, \overline{R}(j,i)))$, what proves that $P_{\alpha,\beta}(R)$ is an IVFRR.

If $P_{\alpha,\beta}(R)$ and R are IVFRRs, then $P_{\alpha,\beta}(r_{ii}) = [\max(\alpha, \underline{R}(i,i)), \max(\beta, \overline{R}(i,i))] = [\max(\alpha, 0.5), \max(\beta, 0.5)] = [0.5, 0.5]$. As a result, $\alpha \leqslant 0.5 = \underline{R}(i,i) \leqslant N(\beta)$ and $\beta \leqslant 0.5 = \overline{R}(i,i)$.

For $i \neq j$ we obtain $\max(\alpha, \underline{R}(i,j)) = N(\max(\beta, \overline{R}(j,i))) = \min(N(\beta), N(\overline{R}(j,i))) = \min(N(\beta), \underline{R}(i,j))$. This condition is true only for $\alpha \leqslant \underline{R}(i,j) \leqslant N(\beta)$, so these inequalities are also true. The case of $Q_{\alpha,\beta}(R)$ may be proven in a similar way.

By Proposition 2, and by imposing $R_{ij} + R_{ji} = [1,1]$, i.e. $\underline{R}(i,j) + \underline{R}(j,i) = 1$ and $\overline{R}(i,j) + \overline{R}(j,i) = 1$ (which means that an IVFRR reduces to a fuzzy relation), we obtain

Proposition 5 **([3]).** *Let $\alpha, \beta \in [0,1]$, N be the classical fuzzy negation and $\alpha+\beta \leqslant 1$. If R is an IVFRR satisfying $R_{ij}+R_{ji}=[1,1]$ for all $i,j \in \{1,\ldots,n\}$ and the transitivity property, then $F_{\alpha,\beta}(R)$, $(F_{\alpha,\alpha}(R))$ is transitive (is transitive IVFRR).*

Directly by Propositions 2 and 4 we have

Proposition 6. *Let N be a fuzzy negation and $\alpha, \beta \in [0,1]$, $\alpha \leqslant \beta$.*

- *If R is an IVFRR satisfying the transitivity property, $\alpha \leqslant \underline{R}(i,j) \leqslant N(\beta)$ and $\alpha \leqslant 0.5$, then $P_{\alpha,\beta}(R)$ is also an IVFRR satisfying the transitivity property.*
- *If R is an IVFRR satisfying the transitivity property and $N(\beta) \leqslant \underline{R}(i,j) \leqslant \alpha$, then $Q_{\alpha,\beta}(R)$ is also an IVFRR satisfying the transitivity property.*

5 Equivalence and Reciprocal Relations

The considered relation $\sim$ is an equivalence relation in the family $\mathcal{IVFR}(X)$. This fact enables the classification of interval-valued fuzzy information and the selection of subordinations of that information.

Definition 7 **([3]).** *Let $R=[\underline{R},\overline{R}]$, $S=[\underline{S},\overline{S}] \in \mathcal{IVFR}(X)$. We say that relations R and S are equivalent ($R \sim S$), if for all $x,y,u,v \in X$*

$$\underline{R}(x,y) \leqslant \underline{R}(u,v) \;\Leftrightarrow\; \underline{S}(x,y) \leqslant \underline{S}(u,v)$$

and

$$\overline{R}(x,y) \leqslant \overline{R}(u,v) \;\Leftrightarrow\; \overline{S}(x,y) \leqslant \overline{S}(u,v).$$

Some results on the operations supremum and infimum may be applied in verifying the equivalence between two given IVFRs.

Proposition 7 **([3]).** *Let $R=[\underline{R},\overline{R}]$, $S=[\underline{S},\overline{S}] \in \mathcal{IVFR}(X)$. If $R \sim S$, then for every non-empty subset P of $X \times X$ and each $x,y,z,t \in P$, the following conditions are fulfilled*

$$\begin{cases} \underline{R}(x,y) = \sup\limits_{(u,v)\in P} \underline{R}(u,v) \;\Leftrightarrow \underline{S}(x,y) = \sup\limits_{(u,v)\in P} \underline{S}(u,v) \\ \overline{R}(z,t) = \sup\limits_{(u,v)\in P} \overline{R}(u,v) \;\Leftrightarrow \overline{S}(z,t) = \sup\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}, \tag{7}$$

$$\begin{cases} \underline{R}(x,y) = \inf\limits_{(u,v)\in P} \underline{R}(u,v) \;\Leftrightarrow \underline{S}(x,y) = \inf\limits_{(u,v)\in P} \underline{S}(u,v) \\ \overline{R}(z,t) = \inf\limits_{(u,v)\in P} \overline{R}(u,v) \;\Leftrightarrow \overline{S}(z,t) = \inf\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}, \tag{8}$$

$$\begin{cases} \underline{R}(x,y) = \sup\limits_{(u,v)\in P} \underline{R}(u,v) \;\Leftrightarrow \underline{S}(x,y) = \sup\limits_{(u,v)\in P} \underline{S}(u,v) \\ \overline{R}(z,t) = \inf\limits_{(u,v)\in P} \overline{R}(u,v) \;\Leftrightarrow \overline{S}(z,t) = \inf\limits_{(u,v)\in P} \overline{S}(u,v) \end{cases}, \tag{9}$$

$$\begin{cases} \underline{R}(x,y) = \inf_{(u,v)\in P} \underline{R}(u,v) \ \Leftrightarrow \underline{S}(x,y) = \inf_{(u,v)\in P} \underline{S}(u,v) \\ \\ \overline{R}(z,t) = \sup_{(u,v)\in P} \overline{R}(u,v) \ \Leftrightarrow \overline{S}(z,t) = \sup_{(u,v)\in P} \overline{S}(u,v) \end{cases} . \tag{10}$$

We note that the converse statement to Proposition 7 is true, and it is enough to assume that only one of the conditions in Eqs. (7)–(10) is fulfilled.

Proposition 8 ([3]). *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. If for every finite, non-empty subset P of $X \times X$ and each $x, y, z, t \in P$ one of the conditions in Eqs. (7)–(10) holds, then $R \sim S$.*

Equivalent relations have connection with the transitivity property. We can obtain for IVFRs the following property

Proposition 9 ([3]). *Let $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}] \in \mathcal{IVFR}(X)$. If $R \sim S$, then R is transitive if and only if S is transitive.*

For IVFRR we can weaken assumptions from Proposition 9. By exploring equivalence one can generate classes of IVFRRs, so that all of its members express the same preferences over the different alternatives.

Proposition 10. *If $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}]$ are IVFRRs and for an arbitrary non-empty set $P \subset X \times X$ and $(i, j) \in P$, we have:*

$$\underline{R}(i,j) = \max_{(v,w)\in P} \underline{R}(v,w) \Leftrightarrow \underline{S}(i,j) = \max_{(v,w)\in P} \underline{S}(v,w) \tag{11}$$

or

$$\underline{R}(i,j) = \min_{(v,w)\in P} \underline{R}(v,w) \Leftrightarrow \underline{S}(i,j) = \min_{(v,w)\in P} \underline{S}(v,w), \tag{12}$$

then R is transitive if and only if S is transitive.

Proof. For an IVFRR (meaning fuzzy negation based reciprocity), satisfying (11) we obtain adequate condition (with minimum) for $\overline{R}, \overline{S}$. Similarly, for an IVFRR, satisfying the condition (12) we obtain adequate condition (with maximum) for $\overline{R}, \overline{S}$. Then by Propositions 7 and 8 we obtain $R \sim S$. Moreover, we observe that if $R = [\underline{R}, \overline{R}]$, $S = [\underline{S}, \overline{S}]$ are IVFRRs and $\underline{R} \sim \underline{S}$, then $\overline{R} \sim \overline{S}$. As a result, by Proposition 9 we have transitivity property both for R and S.

6 Choice of Alternatives by a Linear Order

Our above results allow to perform the following applications. We consider an interval-valued fuzzy relation on $X = \{x_1, \ldots, x_n\}$ (set of users) which represents the opinion of each user over another one. The following algorithm gives an alternative (a user) who has the worst/best relationships in the considered group X.

Algorithm
Inputs: $X = \{x_1, \ldots, x_n\}$ set of alternatives; N a fuzzy negation; $R \in \mathcal{IVFR}(X)$: interval-valued fuzzy reciprocal and transitive relation; linear order $\preceq_{\alpha,\beta}$ generated by $F_{\alpha,1-\alpha}$ and $F_{\beta,1-\beta}$, where the relation $\preceq_{\alpha,\beta}$ on L^I is given by $[a,b] \preceq_{\alpha,\beta} [c,d] \Leftrightarrow F_{\alpha,1-\alpha}(a,b) < F_{\alpha,1-\alpha}(c,d)$
or $(F_{\alpha,1-\alpha}(a,b) = F_{\alpha,1-\alpha}(c,d)$ and $F_{\beta,1-\beta}(a,b) \leqslant F_{\beta,1-\beta}(c,d))$.
This is an admissible order on L^I (linear order refining $\leqslant_{L^I}$ on L^I, cf. [7]).
Output: Solution alternative: $x_{selection}$.
(Step 1) Build the nondominance interval-valued fuzzy set:
$\mathrm{ND}_{IV} = \{(x_j, \mathrm{ND}_{IV}(x_j) = [\bigvee_{i=1}^{n}(\underline{r}_{ij}), \bigvee_{i=1}^{n}(\overline{r}_{ij})])|x_j \in X\}$;
(Step 2) Apply N_{IV}, generated by a fuzzy negation, to the set ND_{IV}:
$N_{IV}(\mathrm{ND}_{IV})(x_j) = [N(\bigvee_{i=1}^{n}(\overline{r}_{ij})), N(\bigvee_{i=1}^{n}(\underline{r}_{ij}))]$;
(Step 3) Order the alternatives in a non-increasing way using $\preceq_{\alpha,\beta}$.

Analysis of the position in the network - which individuals occupy the best positions in the structure of the network? Who will first get valuable information? Are leaders occupying central positions in the network, or perhaps are located on the outskirts? Social network analysis allows us to answer these questions, providing among other things, reliable basis for an efficient allocation of tasks. In summary, social network analysis provides a new quality in the analysis of phenomena and processes in the organization.

7 Conclusion

We introduced the new type of reciprocity property for interval-valued fuzzy setting. This property is based on a fuzzy negation concept and has a potential in applications. We focused on one type of social networks in which the users explicitly express their opinion over other users. We used an interval-valued fuzzy SNA based on decision making to represent the uncertainty or fuzziness of relationships in group of experts.

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Generalized Nets: Theory and Applications

Generalized Net of a Centralized Embedded System

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Abstract. In embedded controlling computer systems, multiple servicing subsystems can process multiple tasks. The problem arises of the tasks optimal centralized distribution among the multiple subsystems. In the present paper, this problem is addressed by designing a generalized net model to organize the behaviour of these servicing devices and the performance of the various tasks assigned to them in parallel.

Keywords: Embedded controlling systems · Generalized net modelling · Servicing devices

1 Introduction

Embedded controlling computer systems are basic components of all mobile devices (robots, drones, etc.). On their own turn, they consist of multiple different subsystems, responsible for the implementation of various tasks. Some of these subsystems (that we will also refer to as servicing devices) are strictly specific (e.g., device for defining the azimuth/direction of movement, altitude, acceleration, etc.). Others, like the memory and the central processor unit, are functionally universal ones, i.e., capable of handling a variety of versatile tasks (e.g. calculations, data storage, etc.) [3, 4]. In the universal subsystems, multiple tasks are received for processing and the problem arises of their optimal distribution among the multiple subsystems. This problem is solvable by designing a mathematical model, which organizes the behaviour of these servicing devices and the performance of the various tasks assigned to them. For the sake of this modelling, it is necessary to know in details their capabilities, such as performance parameters, memory, etc.

In this work, we will discuss a model, developed using the apparatus of the generalized nets that abstracts the process of centralized distribution of a set of tasks over a set of servicing devices. Generalized nets have been used in a similar application in [5, 6].

Let n servicing devices D_1, D_2, D_3, …, D_n designed for m types of tasks T_1, T_2, T_3, …, T_m are given. The task T_i sends two requests to the first device, one request for

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Table 1. Table of the tasks

	D_1	D_2	D_3	...	D_n
T_1	w_{11}	w_{12}	w_{13}	$\ldots$	w_{1n}
T_2	w_{21}	w_{22}	w_{23}	$\ldots$	w_{2n}
T_3	w_{31}	w_{32}	w_{33}	$\ldots$	w_{3n}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
T_m	w_{m1}	w_{m2}	w_{m3}	$\ldots$	w_{mn}

the second and third devices and no requests to device D_4. The table will look as follows (Table 1):

The state of the servicing system of type:

D_1	D_2	D_3	...	D_n

at certain moment will be represented by the vector $(a_1,\ a_2,\ a_3,\ \cdots,\ a_n), a_i \geq 0$, $i = 1, 2, \ldots, n$ consisting of the number of requests to the servicing system.

2 Generalized Net Model

Generalized nets (GNs) are a discrete tool for universal description of adaptable, flexible, structured and reusable models of complex systems with many different and interacting components, not necessarily of the homogeneous structure and origin, usually involved in parallel, simultaneous activities. Generalized nets represent a significant extension and generalization of the concept of Petri nets, as well as of other Petri nets extensions and modifications. The concept of Generalized Nets (GNs) is described in [1, 2].

The GN-model described here (Fig. 1) contains (n + 1) transitions and (4n + 8) places. The places are related to the two types of the tokens:

- α-tokens represent the task and activities with them,
- β-tokens represent the devices and connected with them queries, functions, activities.

For brevity, we shall use the notation α- and β-tokens instead of α_i- and β_j-tokens, where i, j are the numbers of the respective tokens.

Initially the α- and β-tokens remain, respectively, in places L_{T2} and L_{D2} with initial characteristics:

x_0^{α} = "list of the tasks T_1, T_2, T_3, …, T_m and their current characteristics",

x_0^{β} = "list of the devices D_1, D_2, D_3,…, D_n and their current status".

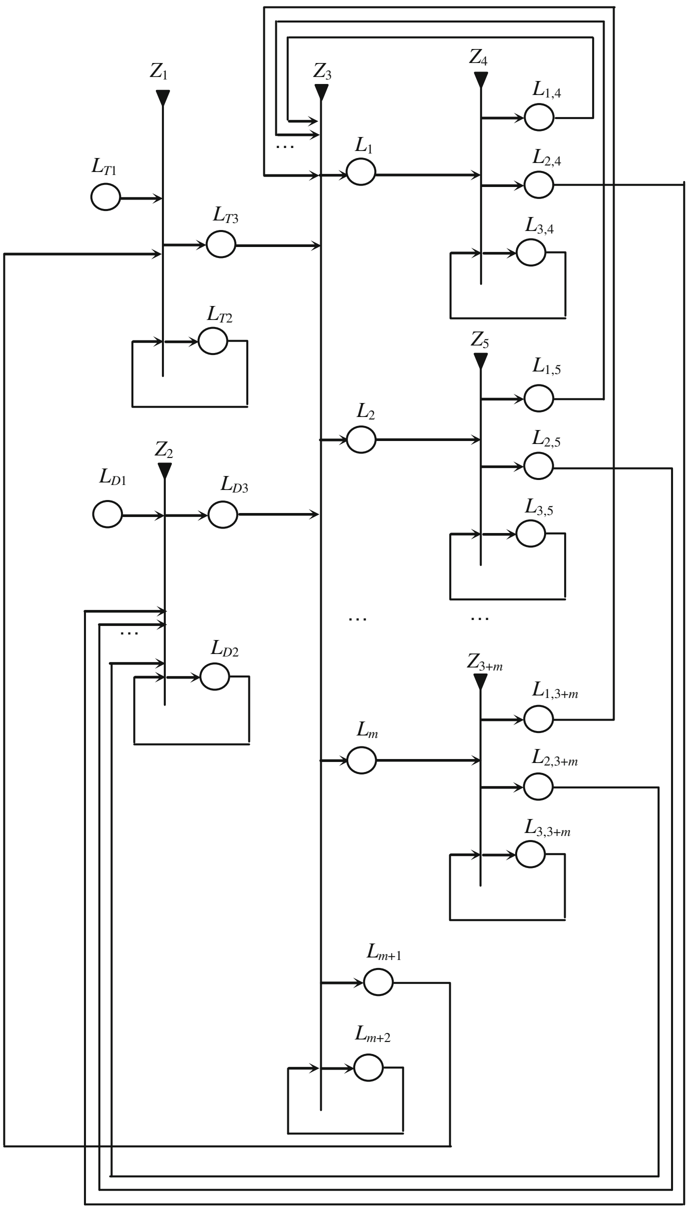

Fig. 1. GN model of the discrete states and discrete time

Let x_{cu}^{α} and x_{cu}^{β} be the current characteristics of the α- and β-tokens, respectively. The forms of the transitions are the following. The first transition Z_1 has the following form:

$$Z_1 = \langle \{L_{T1}, L_{T2}, L_{m+1}\}, \{L_{T2}, L_{T3}\}, \begin{array}{c|cc} & L_{T2} & L_{T3} \\ \hline L_{T1} & true & false \\ L_{T2} & W_{T2,T2} & W_{T2,T3} \\ L_{m+1} & true & false \end{array} \rangle,$$

where:

- $W_{T2,T2}$ = "There is at least one task",
- $W_{T2,T3}$ = "The task is chosen".

The α-tokens do not obtain new characteristic in place L_{T2} and they obtain the characteristic x_{cu}^{α} = "task T_i, list of devices that have to service the task" in place L_{T3}.

The second transition Z_2 has the following form:

$$Z_2 = \langle \{L_{D1}, L_{D2}, L_{2,4}, L_{2,5}, \ldots, L_{2,3+m}\}, \{L_{D2}, L_{D3}\}, \begin{array}{c|cc} & L_{D2} & L_{D3} \\ \hline L_{D1} & true & false \\ L_{D2} & true & W_{D2,D3} \\ L_{2,4} & true & false \\ L_{2,5} & true & false \\ \ldots & \ldots & \ldots \\ L_{2,3+m} & true & false \end{array} \rangle,$$

where $W_{D2,D3}$ = "The list of devices is chosen".

The β-tokens do not obtain new characteristic in place L_{D2} and they obtain the characteristic x_{cu}^{β} = "list of devices that have to service the chosen task" in place L_{D3}.

The third transition Z_3 has the following form:

$$Z_3 = \langle \{L_{T3}, L_{D3}, L_{1,4}, L_{1,5}, \ldots, L_{1,3+m}, L_{m+2}\}, \{L_1, L_2, \ldots, L_m, L_{m+1}, L_{m+2}\},$$

$$\begin{array}{c|cccccc} & L_1 & L_2 & \ldots & L_m & L_{m+1} & L_{m+2} \\ \hline L_{T3} & false & false & \ldots & false & false & true \\ L_{D3} & false & false & \ldots & false & false & true \\ L_{1,4} & false & false & \ldots & false & true & false \\ L_{1,5} & false & false & \ldots & false & true & false \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ L_{1,3+m} & false & false & \ldots & false & true & false \\ L_{m+2} & W_{n+2,1} & W_{n+2,2} & \ldots & W_{m+2,1} & false & W_{m+2,m+2} \end{array} \rangle,$$

where:

- $W_{m+2,j}$ = "The queries for device j from task T_i are defined", for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$,
- $W_{m+2,m+2}$ = "There is a task that must be serviced".

The α-tokens and β-tokens that enter place L_{m+2} do not obtain new characteristic.

The α-tokens and β-tokens that enter places $L_{1,4}$, $L_{1,5}$, …, $L_{1,3+m}$ obtain the characteristics "task T_i, queries for device j, parameters".

The α-tokens that enter place L_{m+2} do not obtain new characteristic.

For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, we have all the respective transitions described likewise:

$$Z_{j+3} = \langle \{L_j, L_{3,j+3}\}, \{L_{1,j+3}, L_{2,j+3}, L_{3,j+3}\},$$

$$\begin{array}{c|ccc} & L_{1,j+3} & L_{2,j+3} & L_{3,j+3} \\ \hline L_j & false & false & true \\ L_{3,j+3} & W_{1,j+3} & W_{2,j+3} & W_{3,j+3} \end{array} \rangle,$$

where:

- $W_{1,\,j+3} = W_{2,\,j+3}$ = "There is an executed query for task T_i",
- $W_{3,\,j+3}$ = "There is a query that must be serviced from device D_j".

The α-tokens and β-tokens that enter place $L_{3,j+3}$ form place L_j do not obtain new characteristic.

The α-tokens and β-tokens that enter places $L_{1,4}$, $L_{1,5}$, …, $L_{1,3+m}$ obtain the characteristics "task T_i, queries for device j, parameters".

The α-tokens that enter place $L_{1,j+3}$ obtain characteristic "task T_i, executed queries for device j, current status".

The β-tokens that enter place $L_{2,j+3}$ obtain characteristic "device j, current status".

3 Conclusion

The GN model above shows a possible working approach to modelling the problem of centralized assigning multiple tasks to multiple servicing systems.

The model can be further expanded by adding specialized interfaces to the servicing components of the embedded system. Every embedded device is portable which provides the opportunity of certain mobility of the user.

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Generalized Net of MapReduce Computational Model

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Abstract. The topic in the current research work is one of the Big data frameworks known as MapReduce paradigm. This is a programming model for parallel processing of large volumes of data in distributed environment. MapReduce is applied in the clusters of commodity machines. The workflow of the MapReduce computational model is constructed using the possibilities of Generalized nets. The Generaliezed net model which is presented allows us to observe and monitor the Mapreduce framework in detail.

Keywords: Big Data · Cloud computing · Generalized net · Distributed systems · Map-Reduce · Parallel computing · Distributed algorithms

1 Overview of Big Data

In the recent years the term Big Data becomes very popular. Big data is a combination of old and new technologies that helps companies gain actionable insight. Big data provides tools for storing, managing and manipulating the vast amounts of data at the right speed and at the right time. The data may came from internal and external sources from multiple locations in the form of multiple formats and can be generated by multiple applications. The data sources can be various - transactions, social media, sensors, digital images, videos, audios and clickstreams. These recent development of different technologies for collecting data from different devices increases the amount of data [19]. Big Data is often described in terms of the “*V’s of Big Data*”: *Volume*, *Variety*, *Velocity*, *Veracity* (*Variability*), *Valence* and *Value* (Fig. 1). The three main characteristics describing Big Data are *Volume*, *Velocity* and *Variety* [1, 11, 13, 17].

- *Volume* (*size*) refers to the vast amount of data generated in every second. This can be shared data sets or events collecting over time. *Volume* defines the data which are being generated from various sources in huge volumes and are in the very huge form (i.e., of Terabyte, Petabyte, Exabyte, Zettabyte). The challenges with working

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with volumes of Big Data are cost, scalability and performance related to their storage, access and processing.

- *Variety* (*complexity*) refers to the different forms of data. There are mainly three types of data: structured data, unstructured data and semi-structured data. The structured data is in the form of text or in numeric and it is stored in external databases. Unstructured data is in the form of PDF files, video files, audio files, images, likes, tweets, comments. Semi-structured data is in the form of XML files, JSON files, emails, JavaScript files, sever log files, sensor data.
- *Velocity* (*speed*) refers to the speed of creating, storing and processing the data. *Velocity* provides fast processing of huge volume of different types of data, generated from various sources (called analysis of streaming data). Big data processing can be classified into two types of methods. One is the batch based stored data processing and the other is the real-time data-stream processing.

These main characteristics describe the big amounts of data in different formats and varying quality which must be processed quickly. In the recent years several additional characteristics appears - *Veracity*, *Valence* and *Value*.

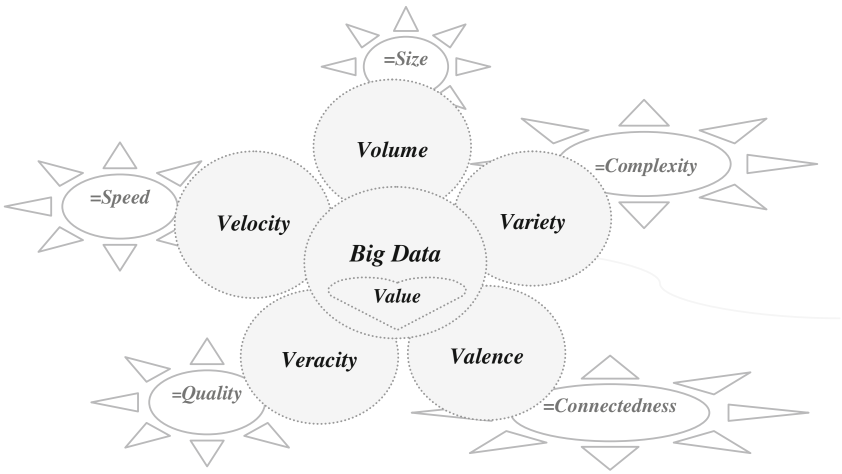

Fig. 1. "V's of Big Data"

- *Veracity* (*variability*) refers to the level of quality, accuracy and uncertainty of data. The quality of the data can vary greatly. Accuracy of analysis depends on the veracity of the source data. It refers to necessary and sufficient data to test many different hypotheses.
- *Valence* (*connectedness*) is a measure of connectivity. The term originates from the theory of chemistry. The data could be directly or indirectly connected. Data items are connected when there are related to each other (e.g. two Facebook users are connected because there are friends). Then the *Valence* is a fraction of data items that are connected out of total number of possible connections. *Valence* increases

over time (e.g. network graphs). Possible challenge for valence of Big Data can be the analysis the behavior of users in social networks.

- *Value* is the “heart” of big data. The Value refers to discovering actionable insights. Value is the importance or usefulness of the target data. Value is the result should be generated after all of the procedures. Data science can help us to determine the target interactions.

In the literature exists several explanations of the Big data process. One of them is visualized in Fig. 2 [15]: The presented workflow is consisted of two sub-processes: Data Management and Analytics. Data Management include three stages: Acquisition and Recording; Extraction, Cleaning and Annotation; Integration, Aggregation and Representation. Analytics consists of two steps: Modeling and Analysis; Interpretation. Big data can be analyzed with the software tools used as part of advanced analytics as predictive analytics, data mining, text analytics and statistical analysis. Newer class of technologies that includes Hadoop and related tools such as YARN, MapReduce, Spark, Hive and Pig as well as NoSQL databases can be used also [1, 11, 13, 17].

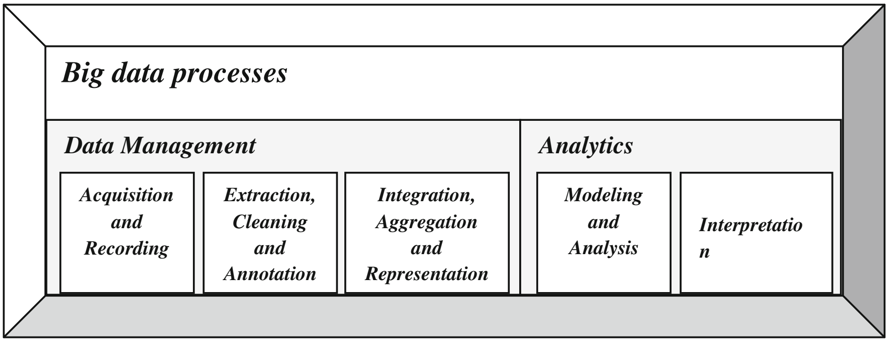

Fig. 2. Big Data process

Big data systems are constructed using some typical components as distributed databases (HBase, Hive, Cassandra), distributed processing frameworks (Apache Hadoop), distributed processing systems (MapReduce) and distributed file systems (HDFS, GFS). Cloud computing, Internet of things (IoT), data centers and Hadoop are several fundamental technologies that are closely related to Big Data [9, 10, 23].

Mapreduce programming model is the topic of the following part of the paper. This is a powerful Big Data framework. MapReduce is not a completely new paradigm because it is used in functional programming many years ago (Lisp, Haskell, Prolog, R). The Apache Hadoop platform starts the new era of MapReduce. Google File System and open-source Hadoop platform are presented in [8, 18]. Hadoop is a combination of HDFS (Hadoop Distributed File System) and MapReduce framework. Several additional tools are provided to facilitate the work with the platform [12]. Nowadays exists several extensions of MapReduce included in different platforms with different system design (distribution, shared resources, communication) to improve the speed. Six forms of MapReduce paradigm are discussed in the litterature: Map Only

(Pleasingly parallel algorithms, e.g. transform pdf document in png), Classic MapReduce (it can be used for distributed search, recommender systems), Iterative MapReduce or Map-Collective (Iterative Mapreduce extends MapReduce programming model and support iterative algorithms for Data Mining and Data Analysis - e.g. PageRank, Clustering), Point to Point or Map-Communication (different type of communications), Map Streaming (online classifiers) and Shared memory Map Communicates.

In the current research work the authors are focused on the classical MapReduce framework. Typically it works with HDFS which is organized as master-slave. Information for HDFS can be found in the literature and the authors do not describe it in the paper [8, 13, 17]. Hadoop is used in some of the largest sites in Internet: Amazon Web Services (Elastic MapReduce), EBay (search optimization), Facebook (reporting and machine learning), LinkedIn (proposed people), Twitter (process tweets, log files), Yahoo.

2 Generalized Net of MapReduce Computational Model

The concept of Generalized nets (GNs) is introduced in [2, 3]. In the recent years many data mining concept and algorithms were discussed, analyzed and explored using Generalized nets. Little part of the models are given in [4–7, 14, 16, 20, 21, 24]. In the current research work the authors are focused on the MapReduce paradigm as framework of Big Data. The Generalized Net model of MapReduce computational model is presented in Fig. 3. Obviously the newest technologies allow us to cross the border of the standard methods for storing, transforming and analyzing the data sets and extend them to biggest platforms for distributed data management and their parallel processing.

Let us describe briefly the tasks in the MapReduce paradigm. Firstly it is necessary to determine the type of the input files. It is typical for input files to be very large. The default format is *"text"*. This treats each line of each input file as a separate record. The next step of the process divides the input files in n chunks (64 MB by default = the same size as blocks in HDFS). The information in these input splits (chunks) is transformed in the type of <key, value> pairs. The parallel processing begins. Map functions are applied over chunks. Each input split is processed by map function (mapper). The output <key, value> pairs have three scenarios: they are output of map-only task and go to record the result; they need to be grouped by combiner function (optionally) and they need to be send to the partition function. The combiner step is optional. It combines the output of map function locally. Obviously the combiner has the function as "mini-reducer" and it optimize the Mapreduce paradigm. The <key, value> pairs received after map function has the name intermediate <key, value> pairs and the step between the mapper and reducer are known as intermediate step. Intermediate <key-value> pairs must be grouped by key through a distributed sort and shuffled. The partition function receives the keys of intermediate pairs and the number of reducers. For each key in the partition assigned to a Reducer. Hash function is used by default. In the next step the reducer function is applied over grouped <key, value> pairs. The number of reducers (m) must be smaller from the number of mappers (n),

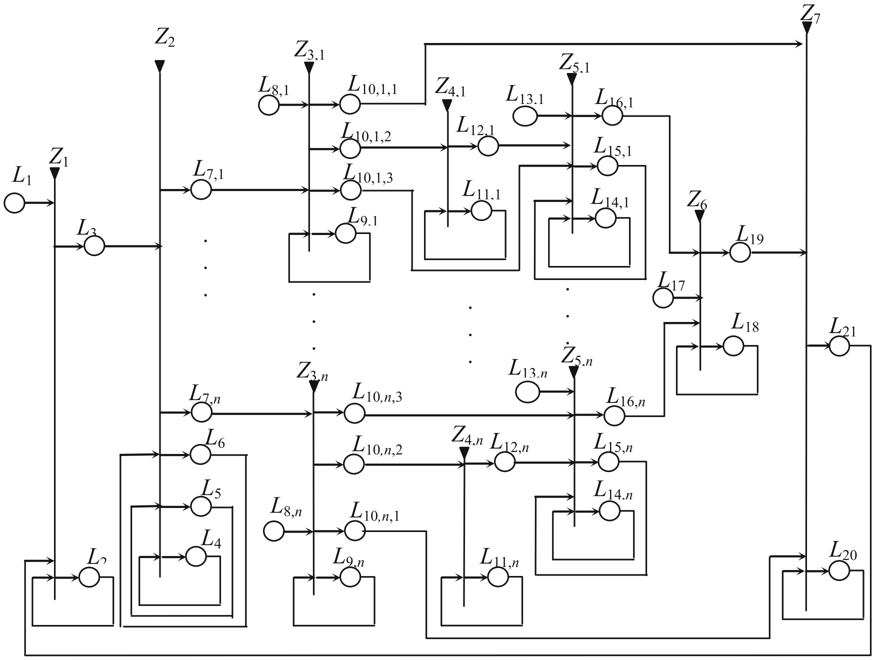

Fig. 3. Generalized net of MapReduce computational model

i.e. $n > m$. The output pairs are always written on HDFS. MapReduce framework supports key algorithms to filter, search, sort and aggregate the data.

Let us resume that the inputs and outputs of MapReduce are usually stored in a distributed file system but the intermediate data is usually recorded on local disk. The reducers accessed it remotely. Reducers start to work when all the mappers are finished. According this explanation the following Generalized net represents the steps of standard MapReduce paradigm. The GN model of MapReduce computational model contains the following set of transitions (Fig. 3):

- Z_1 - "Input files (typically stored in HDFS - Hadoop Distributed File System)"
- Z_2 - "Determine the Input format of the file, Split it in chunks (input splits) and Transform it into <key1, value1> pairs"
- $\{Z_{3,1} \ldots Z_{3,n}\}$ - "Applying Map function for each chuck of data"
- $\{Z_{4,1} \ldots Z_{4,n}\}$ - "Applying optional combine function over output of Map function"
- $\{Z_{5,1} \ldots Z_{5,n}\}$ - "Partition & Shuffle and Sort the data (local machines)"
- Z_6 - "Applying reduce function over data in chunks"
- Z_7 - "Recording the output format of data in files"

Initially in the place L_2 there is one α-token. It will be in his own place during all the time of GN functioning. It has the following characteristic: "*files stored in distributed file system*". Also initially in the place L_{18} there is one token with characteristic "*current status of reduce functions and number of reducers*".

Token enters the net via place L_1 with initial characteristics: "*input files*".

The transition Z_1 has the form:

$$Z_1 = \langle \{L_1, L_{21,} L_2\}, \{L_2, L_3\}, R_1, \vee(L_1, L_{21,} L_2)\rangle,$$

where

$$R_1 = \begin{array}{c|cc} & L_2 & L_3 \\ \hline L_1 & true & false \\ L_{21} & true & false \\ L_2 & W_{2,2} & W_{2,3} \end{array}$$

and:

- $W_{2,3}$ = "input file for MapReduce processing is selected";
- $W_{2,2} = \neg W_{2,3}$;

The tokens, entering in place L_2 from places L_1 and L_{21} don't obtain new characteristics. The token from place L_2 enters in place L_3 with characteristic:

$$''selected\, file\, for\, MapReduce\, processing''.$$

The transition Z_2 has the form:

$$Z_2 = \langle \{L_3, L_4, L_5, L_6\}, \{L_{7,1}, \ldots, L_{7,n}, L_4, L_5, L_6\}, R_2, \vee(L_3, L_4, L_5, L_6)\rangle,$$

where

$$R_2 = \begin{array}{c|cccccc} & L_{7,1} & \ldots & L_{7,n} & L_4 & L_5 & L_6 \\ \hline L_3 & false & \ldots & false & true & false & false \\ L_4 & false & \ldots & false & W_{4,4} & W_{4,5} & false \\ L_5 & false & \ldots & false & false & W_{5,5} & W_{5,6} \\ L_6 & W_{6,7,1} & \ldots & W_{6,7,n} & false & false & W_{6,6} \end{array}$$

and:

- $W_{4,5}$ = "input format of file for MapReduce processing is determined";
- $W_{4,4} = \neg W_{4,5}$;
- $W_{5,6}$ = "input file for MapReduce processing is divided in input splits (chunks)";
- $W_{5,5} = \neg W_{5,6}$;
- $W_{6,7,1}$ = "the data in input split 1 is transformed into <*key, value*> pairs";
- $W_{6,7,n}$ = "the data in input split n is transformed into <*key, value*> pairs";
- $W_{6,6} = \neg(W_{6,7,1} \wedge W_{6,7,n})$;

The token, entering in place L_4 from place L_3 doesn't obtain new characteristic. When the value of $W_{4,5}$ is "true", the token from place L_4 enters in place L_5 with characteristic:

$$''determined\ input\ format\ of\ file''.$$

At the second activation of the transition the token from place L_5 enters in place L_6 with characteristic (when $W_{5,6}$ is "true"):

$$''input\ splits\ (chunks)''.$$

At the third activation of the transition the token from place L_6 splits into new tokens that enter in places $L_{7,1}$,..., $L_{7,n}$ with characteristics:

$$''transformed\ data\ in\ input\ spit\ 1\ in\ <key,\ value>\ pairs'',$$
$$''transformed\ data\ in\ input\ spit\ n\ in\ <key,\ value>\ pairs''.$$

Everywhere below $i = 1, 2, \ldots, n$.

Via place $L_{8,i}$ n in number tokens enter the net with initial characteristic: *"map function and number of mappers"*.

The transition $Z_{3,i}$ has the form:

$$Z_{3,i} = \langle \{L_{8,i}, L_{7,i}, L_{9,i}\},\ \{L_{10,i,1}, L_{10,i,2}, L_{10,i,3}, L_{9,i}\}, R_{3,i}, \vee(\wedge(L_{8,i}, L_{7,i}), L_{9,i})\rangle,$$

where

$$R_{3,i} = \begin{array}{c|cccc} & L_{10,i,1} & L_{10,i,2} & L_{10,i,3} & L_{9,i} \\ \hline L_{8,i} & false & false & false & true \\ L_{7,i} & false & false & false & true \\ L_{9,i} & W_{9,i,1} & W_{9,i,2} & W_{9,i,3} & W_{9,i} \end{array}$$

and:

- $W_{9,i,1}$ = "*map* function is applied over data for Map-only applications";
- $W_{9,i,2}$ = "*map* function is applied over data for following *combine* function";
- $W_{9,i,3}$ = "*map* function is applied over data for following *partition* function";
- $W_{9,i} = \neg(W_{9,i,1} \wedge W_{9,i,2} \wedge W_{9,i,3})$;

The tokens, entering in place $L_{9,i}$ from places $L_{7,i}$ and places $L_{8,i}$ don't obtain new characteristics. The token in place $L_{9,i}$ spits into tree new tokens, that enter in places $L_{10,i,1}$, $L_{10,i,2}$ and $L_{10,i,3}$ with characteristics:

$$''applied\ map\ function\ for\ Map-only\ applications'',$$
$$''applied\ map\ function\ for\ following\ combine\ function'',$$
$$''applied\ map\ function\ for\ following\ partition\ function''.$$

The transition $Z_{4,i}$ has the form:

$$Z_{4,i} = \langle \{L_{10,i,2}, L_{11,i}\},\ \{L_{12,i}, L_{11,i}\}, R_{4,i}, \vee(L_{10,i,2}, L_{11,i})\rangle,$$

where

$$R_{4,i} = \begin{array}{c|cc} & L_{12,i} & L_{11,i} \\ \hline L_{10,i,2} & false & true \\ L_{11,i} & W_{11,12,i} & W_{11,11,i} \end{array}$$

and:

- $W_{11,12,i}$ = "*combine* function is applied over <key, value> pairs";
- $W_{11,11,i} = \neg\ W_{11,12,i}$.

The token, entering in place $L_{11,i}$ from place $L_{10,i,2}$ doesn't obtain new characteristic. The token that enters in places $L_{12,i}$ from place $L_{11,i}$ with characteristic:

$$''applied\ combine\ function''.$$

The new token enters the net via place $L_{13,i}$ with initial characteristic: *"partition function (hash function by default)"*.

The transition $Z_{5,i}$ has the form:

$$Z_{5,i} = \langle \{L_{12,i}, L_{13,i}, L_{10,i,3}, L_{14,i}, L_{15,i}\},\ \{L_{16,i}, L_{15,i}, L_{14,i}\}, R_{5,i}, \vee(L_{12,i}, L_{13,i}, L_{10,i,3}, L_{14,i}, L_{15,i})\rangle,$$

where

$$R_{5,i} = \begin{array}{c|ccc} & L_{16,i} & L_{15,i} & L_{14,i} \\ \hline L_{13,i} & false & true & false \\ L_{12,i} & false & false & true \\ L_{10,i,3} & false & false & true \\ L_{15,i} & W_{15,16,i} & W_{15,15,i} & false \\ L_{14,i} & false & W_{14,15,i} & W_{14,14,i} \end{array}$$

and:

- $W_{15,16,i}$ = "*received data are shuffled and partitioned for reducers on local machines* ";
- $W_{15,15,i} = \neg\ W_{15,16,i}$;
- $W_{14,15,i}$ = "*received <key, value> pairs are sorted*";
- $W_{14,14,i} = \neg W_{14,15,i}$;

The tokens, entering in place $L_{14,i}$ from places $L_{12,i}$ and $L_{10,i,3}$, and in place $L_{15,1}$ from place $L_{13,i}$ don't obtain new characteristics. The token in place $L_{14,i}$ enters in places $L_{15,1}$ (when predicate $W_{14,15,i}$ is "true") with characteristic:

$$''sorted < key,\ value > \ pairs''.$$

At the second activation of the transition the token from place $L_{15,i}$ enters in place $L_{16,i}$ (when predicate $W_{15,16,i}$ is "true") with characteristic:

$$''shuffled\ and\ partitioned < key,\ value > \ pairs\ by\ key\ for\ reducers''.$$

A new m in number token enter the net via place L_{17} and has initial characteristic: *"reduce function and number of reducers"*, for $j = 1, 2,\ldots,m$.

Everywhere below $j = 1,2,\ldots,m$.

The transition Z_6 has the form:

$$Z_6 = \langle \{L_{17}, L_{16,1}, \ldots, L_{16,n}, L_{18}\},\ \{L_{18}, L_{19}\}, R_6, \vee(L_{17}, L_{16,1}, \ldots, L_{16,n}, L_{18})\rangle,$$

where

$$R_6 = \begin{array}{c|cc} & L_{18} & L_{19} \\ \hline L_{17} & true & false \\ L_{16,1} & true & false \\ \ldots & \ldots & \ldots \\ L_{16,n} & true & false \\ L_{18} & W_{18,18} & W_{18,19} \end{array}$$

and:

- $W_{18,19}$ = "*intermediate <key, value> pairs are reduced*";
- $W_{18,18} = \neg W_{18,19}$.

The tokens, entering in place L_{18} from places L_{17}, $L_{16,i}$ don't obtain new characteristics. The token from place L_{18} that enters in places L_{19} (when predicate $W_{18,19}$ is "true") with characteristic:

$$''reduced < key,\ value > \ pairs''.$$

The transition Z_7 has the form (for $i = 1, 2,\ldots n$):

$$Z_7 = \langle \{L_{10,1,1},\ \ldots, L_{10,n,1}, L_{19}, L_{20}\},\ \{L_{20}, L_{21}\}, R_7,$$
$$\vee(L_{10,1,1},\ \ldots, L_{10,n,1}, L_{19}, L_{20})\rangle,$$

where

$$R_7 = \begin{array}{c|cc} & L_{20} & L_{21} \\ \hline L_{10,1,1} & true & false \\ \dots & \dots & \dots \\ L_{10,n,1} & true & false \\ L_{19} & true & false \\ L_{20} & W_{20,20} & W_{20,21} \end{array}$$

and:

- $W_{20,21}$ = "*output file is written*";
- $W_{20,20} = \neg W_{20,21}$.

The tokens, entering in place L_{20} don't obtain new characteristics. The token that enters in places L_{21} obtains characteristic:

$$''written\ output\ file''.$$

3 Conclusion

In the current paper is presented a Generalized net of Mapreduce computational model. The constructed model represents in detail the computation process of Mapreduce framework. Generalized net can be used to explore and observe the steps of the process. These model can be extended to be valid for the extensions of MapReduce paradigm. The authors consider to divide these improvements according the system design which is used for execute the proposed extensions.

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Generalized Net Model of Optimization of the Self-Organizing Map Learning Algorithm

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Abstract. This paper describes an optimization of the algorithm of self-organizing map neural network. The proposed algorithm takes place during the learning trial. We take into consideration the number of the epochs so their number needs to be decreased. In order to do that, for each epoch the distance from each cluster unit to all training vectors is measured. If the total distance is the same as the distance estimated from the previous epoch, it is assumed that the network is trained and the learning trial stops. The process of optimization is described with the generalized net.

Keywords: Neural networks · Generalized net · Self-Organizing map

1 Introduction

The goal of Self-Organizing Map Neural Network [2, 4] is to transform incoming data of arbitrary dimension into a one or two-dimensional discrete map and to perform the transformation adaptively in a topologically ordered fashion [5, 6]. The structure of a self-organizing map is presented in Fig. 1.

Each neuron (denoted as black circle in the figure) is connected to all other neurons in the input layer. The input signal enters all neurons and the neuron closest to the input pattern is the winner in the competition phase [3, 7].

The algorithm responsible for the formation of the self-organizing map proceeds by initializing the synaptic weights in the network [5, 8]. Once the network is properly initialized, there are three essential processes involved in the formation of the self-organizing map:

1. Competition: For each input pattern the neurons compute their respective values.
2. Cooperation: The winning neuron determines the spatial locations of the topological neighbourhood of excited neurons.
3. Synaptic adaptation: This last mechanism enables the excited neurons to increase their individual values of the discriminant function in relation to the input pattern through suitable adjustment applied to their synaptic weights.

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Advances in Intelligent Systems and Computing 559, DOI 10.1007/978-3-319-65545-1_29

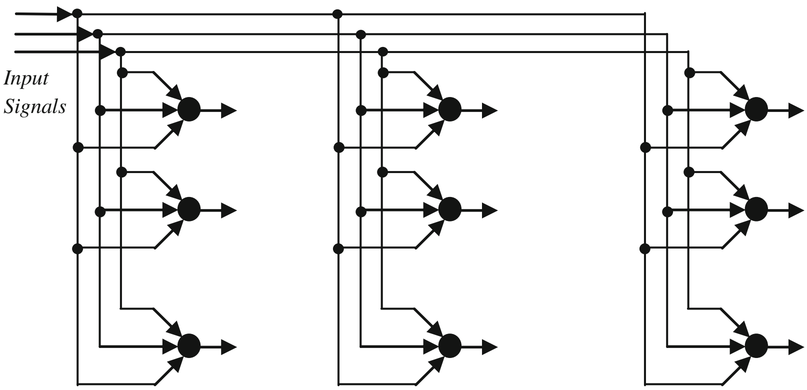

Fig. 1. Structure of self-organizing map neurons in a lattice

According to [5], the summary of SOM algorithm is as follows:

1. Initialization: Choose random values for the initial weight vectors $W_j(0)$. The only restriction here is for W_j to be different for $j = 1, 2, \ldots, l$, where l is the number of neurons in the lattice.
2. Sampling: Draw a sample x from the input space with a certain probability; the vector x represents the activation pattern that is applied to the lattice. The dimension of x is equal to m, where m denotes the dimension of the input (data) space.
3. Similarity matching: Find the best matching (winning) neuron $i(x)$ at the time step n by using the minimum-distance Euclidean criterion:

$$i(x) = \arg\min_{\mathrm{j}} ||x(n) - w(j)||, j = 1, 2, \ldots, l \tag{1}$$

4. Updating: Adjust the synaptic weight vectors of all neurons by using the update formula:

$$w_j(n+1) = w_j(n) + \eta(n)h_{j,i(x)}(n)\big(x(n) - w_j(n)\big) \tag{2}$$

where

- η is the learning rate parameter:

$$\eta(n) = \eta_0 \exp\left(-\frac{n}{\tau_2}\right) \tag{3}$$

τ_2 = number of epochs; η_0 = initial learning rate, typical value is 0.1; n = current time (i.e., current iteration from the training vector set $n = 0, 1, 2, \ldots$).

- $h_{j,i(x)}$ is the topological neighbourhood:

$$h_{j,i(x)} = \exp\left(-\frac{d_{j,i}^2}{2\sigma^2(n)}\right) \tag{4}$$

$d_{j,i}^2$ is the lateral distance between the winning neuron i and excited neuron j.

$$d_{j,i}^2 = ||r_j - r_i||^2 \tag{5}$$

r_j is the discrete position of the current excited neuron; r_i is the discrete position of winning neuron i.

σ is the width of topological neighbourhood function it decreases in time n.

$$\sigma(n) = \sigma_0 \exp\left(-\frac{n}{\tau_1}\right) \tag{6}$$

where σ_0 is the value at the initiation of the SOM algorithm; τ_1 is the time constant:

$$\tau_1 = \frac{epochs}{\log(\sigma_0)} \tag{7}$$

Equation (2) is applied to all the neurons in the lattice that lie inside the topological neighbourhood of the winning neuron i. Equation (2) has the effect of moving the synaptic weight vector W_j of the winning neuron i toward the input vector $\boldsymbol{x}$.

5. Continue: Continue with Step 2 until no noticeable changes in the feature map are observed.

2 Generalized Net Model

In this section, the optimization of the self-organizing map neural network is represented with generalized net (GN, see [1]). Initially the following tokens enter the GN:

In place L_1 – "α token with characteristic training vector set $x_0^{\alpha} = 1, 2, 3, \ldots, m$".

In place L_2 – "β token with characteristic number of epochs for training the SOM $x_0^{\beta} = 1, 2, 3, \ldots, n$".

In place L_6 – "x_0^{χ} token with characteristic parameters for updating the weights".

In place L_{13} – "δ token with characteristic number of neurons $x_0^{\delta} = 1, 2, 3, \ldots, l$".

The GN (see Fig. 2) is introduced by the following set of transitions

$$A = \{Z_1, Z_2, Z_3, Z_4, Z_5\},$$

where

- Z_1 = "Extracting a vector from the training set";
- Z_2 = "Defining the current winning neuron";
- Z_3 = "Calculating the parameters for training the SOM";
- Z_4 = "Updating the weights";
- Z_5 = "Verification of the clusters".

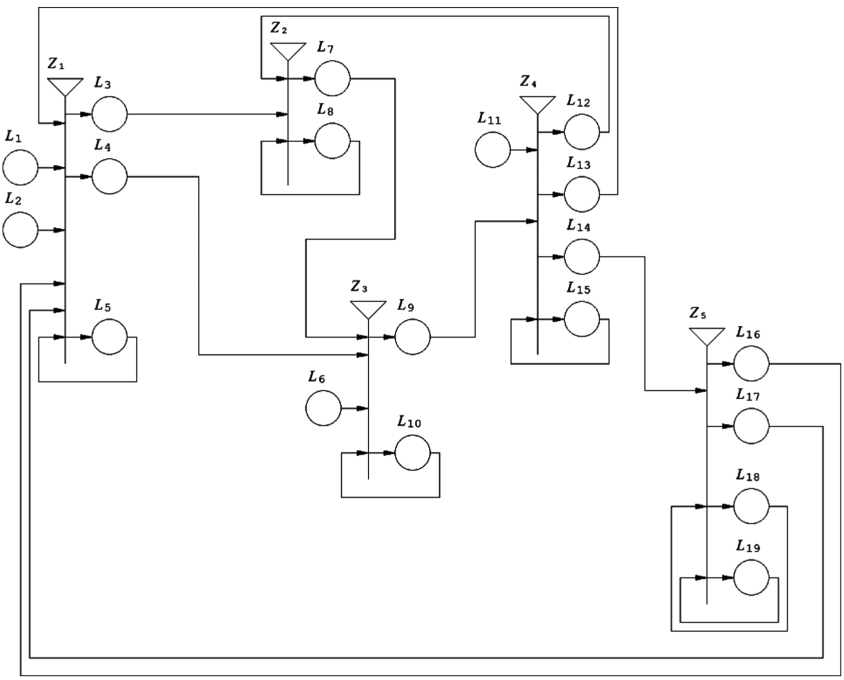

Fig. 2. GN model of the optimization of SOM

The GN consists of five transitions, identified by the following descriptions. The first transition has the form:

$$Z_1 = \langle \{L_1, L_2, L_5, L_{13}, L_{16}, L_{17}\}, \{L_3, L_4, L_5\}, R_1, \\ \vee (L_1, L_2, L_{17} \wedge (L_5, L_{13}, L_{16})) \rangle,$$

where

$$R_1 = \begin{array}{c|ccc} & L_5 & L_4 & L_3 \\ \hline L_1 & false & false & true \\ L_2 & false & false & true \\ L_5 & W_{5,3} & W_{4,5} & true \\ L_{13} & false & false & true \\ L_{16} & false & false & true \\ L_{17} & false & false & true \end{array},$$

where $W_{5,3}$ = $W_{5,4}$ = “Extracted vector from the training set”.

Tokens α and β from places L_1 and L_2 that enter place L_5 do not obtain new characteristic.

Token ζ' from place L_{13} unites with token α in place L_5 and obtains the characteristic $x_{cu}^{\alpha'} = \alpha_{cu}(i+1)$, where i is the current input vector.

Token λ'' from place L_{16} unites with token β in place L_5 and obtain characteristic $x_{cu}^{\beta'} = \beta_{cu}(n-1)$, where n is the current epoch.

Token λ''' from place L_{17} unites with token β in place L_5 and obtain characteristic $x_{cu}^{\beta''} =$ "Stopped learning".

Token α' from place L_5 splits in two tokens that enter place L_3 and L_4. In place L_3 does not obtain new characteristic, while in place L_4 obtains characteristic $x_{cu}^{\alpha''} = \langle m(x_1, x_2, \ldots, x_i, \ldots, x_k) \rangle$, where m is the number of the training set.

Tokens from place L_5 that enter place L_5 does not obtain new characteristic.

The second transition has the form:

$$Z_2 = \langle \{L_3, L_8, L_{12}\}, \{L_7, L_8\}, R_2, \vee(L_8 \wedge (L_3, L_{12})) \rangle,$$

$$R_2 = \begin{array}{c|cc} & L_7 & L_8 \\ \hline L_3 & false & true \\ L_8 & W_{7,8} & true \\ L_{12} & false & true \end{array},$$

where $W_{8,7}$ = "The cluster unit is determined".

Token α' from place L_3 that enters place L_8 does not obtain new characteristic.

Token ζ that enters place L_8 does not obtain new characteristic.

Tokens α' and ζ unites in place L_8 and obtains the characteristic $x_{cu}^{\varepsilon} = \langle i(x), j \rangle$, where $i(x) = \arg\min_{j} ||x(n) - w(j)||$ is the current cluster winner, and j (where $j = 1, 2, \ldots, l$) is the distance between the winning neuron i and the excited neuron j.

Token ε from place L_8 that enters place L_7 does not obtain new characteristic. Likewise, token ε from place L_8 that enters place L_8 does not obtain new characteristic.

The third transition has the form:

$$Z_3 = \langle \{L_4, L_6, L_7, L_{10}\}, \{L_9, L_{10}\}, R_3, \vee(L_6 \wedge (L_4, L_7)) \rangle,$$

$$R_3 = \begin{array}{c|cc} & L_9 & L_{10} \\ \hline L_4 & false & true \\ L_6 & false & true \\ L_7 & false & true \\ L_{10} & W_{10,9} & true \end{array},$$

where $W_{10,9}$ = "The parameters for weight updating are calculated".

Token α'' from place L_4 that enters place L_{10} does not obtain new characteristic.

Token ε from place L_7 that enters place L_{10} obtain characteristic $x_{cu}^{\varepsilon'} = \langle \eta(n), h_{j,I\,(x)}$, $\mathrm{pr}_1 x_{cu}^{\varepsilon}, \mathrm{pr}_2 x_{cu}^{\varepsilon}, \mathrm{pr}_1 x_{cu}^{\alpha''} \rangle$, where

- $\eta(n) = \eta_0 \exp\left(-\frac{n}{\tau_2}\right)$ is the calculated learning rate parameter.
- $h_{j,i(x)} = \exp\left(-\frac{d_{j,i}^2}{2\sigma^2(n)}\right)$ is the topological neighborhood.

Token ε′ from place L_{10} that enters place L_9 does not obtain new characteristic.

The fourth transition has the form:

$$Z_4 = \langle \{L_9, L_{11}, L_{15}\}, \{L_{12}, L_{13}, L_{14}, L_{15}\}, R_4, \vee(L_9, L_{11}, L_{15}) \rangle,$$

$$R_4 = \begin{array}{c|cccc} & L_{12} & L_{13} & L_{14} & L_{15} \\ \hline L_9 & false & false & false & true \\ L_{11} & false & false & false & true \\ L_{15} & W_{15,12} & W_{15,13} & W_{15,14} & true \end{array},$$

where

- $W_{15,12}$ = "Weights are trained";
- $W_{15,13}$ = "Request for next input vector";
- $W_{15,14}$ = "Current epoch finished".

Token δ from place L_{11} unites with token ε′ from place L_9 in place L_{15} and obtain characteristic $x_{cu}^{\varphi} = \langle w_{j1,j2,\ldots,jl}(n+1) \rangle$, where

$$w_j(n+1) = w_j(n) + \eta(n) h_{j,i(x)}(n) \big(x(n) - w_j(n)\big)$$

where $j = 1, 2, \ldots, l$ and l is the total number of neurons in SOM.

Token ζ from place L_{15} that enters place L_{12} does not obtain new characteristic. Token ζ that enters place L_{13} obtains characteristic $x_{cu}^{\varphi'} = \langle x_{i+1} \rangle$. Token ζ that enters place L_{14} from place L_{15} obtains characteristic

$$x_{cu}^{\varphi''} = \left\langle w_{j1,j2,\ldots,jl}(n+1),\ \mathrm{pr}_1 x_{\mathrm{cu}}^{\alpha''} \right\rangle.$$

Token ζ that loops from place L_{15} to place L_{15} does not obtain new characteristic.

The fifth transition has the form:

$$Z_5 = \langle \{L_{14}, L_{18}, L_{19}\}, \{L_{16}, L_{17}, L_{18}, L_{19}\}, R_5, \vee(L_9, L_{11}, L_{15}) \rangle,$$

$$R_5 = \begin{array}{c|cccc} & L_{16} & L_{17} & L_{18} & L_{19} \\ \hline L_{14} & false & false & false & true \\ L_{18} & W_{18,16} & W_{18,17} & true & false \\ L_{19} & false & false & W_{19,18} & true \end{array},$$

where

- $W_{18,16} = W_{18,17}$ = “The SOM is verified”;
- $W_{19,18}$ = “The distance is measured”.

Token ζ'' from place L_{14} that enters place L_{19} obtains the characteristic

$$x_{cu}^{\lambda} = \left\langle ||pr_1 x_{cu}^{\varphi''} - pr_2 x_{cu}^{\varphi''}|| \right\rangle.$$

Token λ from place L_{19} that enters place L_{18} does not obtain new characteristic.
Token λ from place L_{18} that enters place L_{18} obtains characteristic

$$x_{cu}^{\lambda'} = \langle \text{comp}(\lambda - 1, \lambda) \rangle.$$

where comp is the comparison between the current and the previous distances.
Token λ' from place L_{18} that enters place L_{16} obtains characteristic

$$x_{cu}^{\lambda''} = \left\langle pr_1 x_{cu}^{\lambda'} > pr_2 x_{cu}^{\lambda'} \right\rangle.$$

Likewise, token λ' from place L_{18} that enters place L_{17} obtains characteristic

$$x_{cu}^{\lambda''} = \left\langle pr_1 x_{cu}^{\lambda'} \leq pr_2 x_{cu}^{\lambda'} \right\rangle.$$

3 Results and Discussion

This paper explains the optimization of the SOM algorithm. The aim of the research is to break the learning phase when the network is properly trained and no more epochs are necessary. In order to achieve that goal, the learning algorithm explained in the introduction section is optimized as follows:

First, the input vector **x** is taken from the input training data set. Then the closest cluster unit updates its weights and the weights of its neighbor neurons.

When all vectors go through Steps 1 and 2 (i.e., the current epoch has finished), then the process of leaning stops in order to verify the network.

The same data set for learning is used for verifying the network. Each vector is extracted and then enters into the learned network in order to verify the SOM.

In the testing procedure the distance is measured between the current cluster unit and the input vector **x**; the same procedure is applied over all network cluster units and input vectors.

The distance results are saved for further measurements.

In the next epoch as the distances are measured too, they are compared to the previous ones.

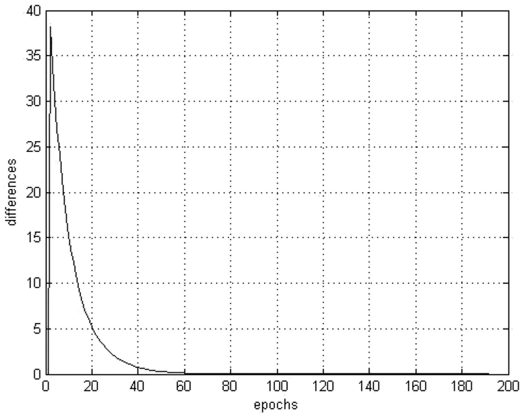

Fig. 3. Differences at each epoch

If the distances are the same, the learning phase stops, if not - the learning phase continues until they become equal.

In Fig. 3 a graphic is plotted, describing the differences at each epoch. Here the training set consists of 3 vectors and the number of cluster units is 2, the number of epochs is 300 and the learning stops at epoch 192.

The epoch depicted in the figure denotes the current epochs. The differences in the figure denotes the differences between the current and the previous epoch.

4 Conclusion

The optimization of the Self-Organizing Map neural network algorithm is described here. First, the algorithm is represented step by step and is explained with formulas. The optimization is based on number of epochs. When the necessarily condition is reached, the learning phase stops. In order to determine the stopping condition, the distances between the training set and the clusters' weights were measured. The algorithm is tested by the training set consisting of three vectors as the number of clusters is two. The total number of epochs is 300 and the algorithm stops at epoch 192. It has been seen that the research can successfully be used in order to extend the learning algorithm and consequently the time for learning decreases. The generalized net model employed here proves as a useful tool for describing processes that are parallel in time and it has been used here to describe the process of optimization of the Self-Organizing Map learning algorithm.

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Generalized Net Model of Synchronous Binary Counter

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Abstract. This paper is a continuation of the previous two papers [7, 8], representing generalized net model of one of the major binary counter types – a synchronous binary counter. We can perform the measurements, if we have a set of several logic circuits that can be used to obtain identical output data. These logic circuits must be composed of different logic elements. By using several measurement points and different schematics, we can infer the best solution for the considered type of task.

Keywords: Generalized nets · Synchronous binary counter

1 Introduction

Formally, every transition (Fig. 1) is described by seven components:

$$Z = \left\langle L^{'}, L^{''}, t_1, t_2, r, M, \square \right\rangle$$

where:

- $L^{'}$ and $L^{''}$ are finite, non-empty sets of the transition's input and output positions, respectively (for the transition in Fig. 1 these positions are $L^{'} = \{l_1^{'}, l_2^{'}, \ldots, l_m^{'}\}$, $L^{''} = \{l_1^{''}, l_2^{''}, \ldots, l_n^{''}\}$);
- t_1 is the current time-moment of the transition's firing;
- t_2 is the current value of the duration of the transition's active state;
- r is the transition's *condition* that determine which tokens will transfer from the input to the output of the transition. The parameter r can be represented as an index matrix [5]:
- M is an IM of the capacities of transition's arcs:
- $\square$ is called transition type.

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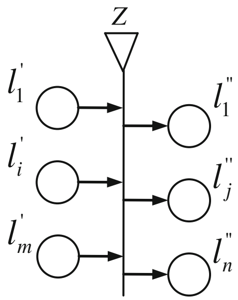

Fig. 1. GN in general form, featuring one transition Z with m inputs and n outputs

A comprehensive description of the theory of Generalized Nets (GN) is provided in [4, 6]. Useful information concerning the research application of GNs can be found in [3, 4, 6].

In the previous two papers [7, 8], we have presented a GN model of basic logic gates "AND" and "OR"; some basic combinational logic circuits as n-bit binary to decimal decoder, n-bit digital comparator, n-bit digital adder; some sequential logic circuits as D-type flip-flop and n-bit binary counter.

The GN model of the n-bit binary counter was represented with the single transition Z_C, comprising two input and n outputs (Fig. 2).

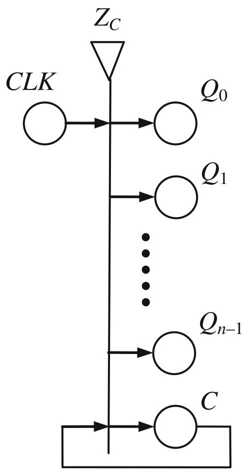

Fig. 2. GN model of n-bit binary counter

The transition Z_C is represented by the following expression:

$$Z_C = \langle \{CLK, C\}, \{Q_0, Q_1, \ldots, Q_{n-1}, C\}, R_C, \wedge(CLK, C) \rangle$$

The matrix R_C, has the following form:

$$R_C = \begin{array}{c|ccccc} & Q_0 & Q_1 & \ldots & Q_{n-1} & C \\ \hline CLK & false & false & \ldots & false & true \\ C & W_{CQ} & W_{CQ} & \ldots & W_{CQ} & W_{CQ} \end{array},.$$

where W_{CQ} = "There is a clock edge". The token from place C splits in n + 1 tokens, where the token that enters place C obtains a new characteristic $x_t^\alpha = x_{t-1}^\alpha + 1$.

The token that enters place Q_i obtains characteristic: "i-th bit of x_t^α", where i = 0, …, n–1.

Here we present a more detailed GN model of a particular type of binary counter – the synchronous binary counter.

2 Generalized Net Model of Synchronous Binary Counter

Depending on the way the clock signal is propagated through the elements (flip-flops) of the counter, there are two major types of digital counters [1, 2, 9–11]: asynchronous and synchronous counters. As can be seen (Fig. 3), the clock input of every subsequent flip-flop (except the first one) is connected to the output of the previous one. Thus, the clock signal is distributed successively to each of the flip-flops, causing some delay between triggering of the flip-flops.

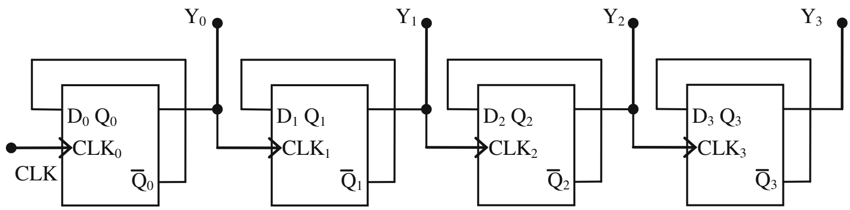

Fig. 3. The 4-bit asynchronous binary counter schematic diagram

The clock signal in the synchronous counter is distributed to all of the flip-flops in parallel. This in turn requires the use of additional logic circuits (Fig. 4). In short, the principle of operation of the synchronous counter is as follows: the clock signal triggers the particular flip-flop, when all of the Q-outputs of the previous flip-flops are "1".

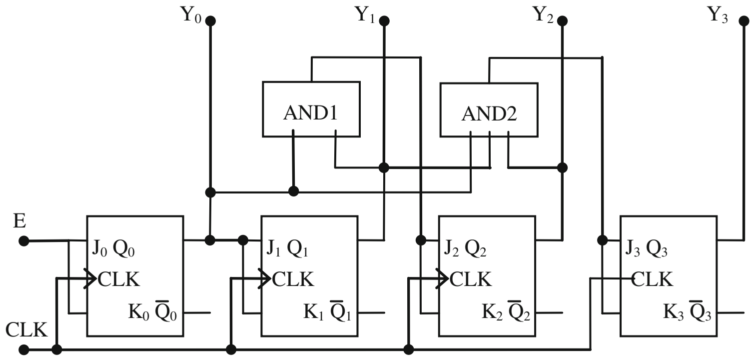

Fig. 4. The 4-bit synchronous binary counter schematic diagram

The GN model of the 4-bit synchronous binary counter consists of six transitions:

$$A = \{Z_{JK0}, Z_{JK1}, Z_{JK2}, Z_{JK3}, Z_{AND1}, Z_{AND2}\}$$

Initially, the following tokens are present in the model:

- in place E – α token with characteristic: $x_0^{\alpha} = false$;
- in place CLK – β token with characteristic $x_0^{\beta} = false$;
- in place Y_0 – γ_0 token with characteristic $x_0^{\gamma 0} = false$;
- in place Y_1 – γ_1 token with characteristic $x_0^{\gamma 1} = false$;
- in place Y_2 – γ_2 token with characteristic $x_0^{\gamma 2} = false$;
- in place Y_3 – γ_3 token with characteristic $x_0^{\gamma 3} = false$;

The transition Z_{JK0} is represented by the following expression:

$$Z_{JK0} = \langle \{E, CLK\}, \{Y_0, JK_1, C_0, AND_{01}, AND_{02}, CLK\}, R_{JK0}, \wedge(E, CLK) \rangle$$

The matrix R_{JK0} has the following form:

$$R_{JK0} = \begin{array}{c|cccccc} & Y_0 & JK_1 & C_0 & AND_{01} & AND_{02} & CLK \\ \hline E & false & false & false & false & false & true \\ CLK & W_{JK0} & W_{JK0} & true & W_{JK0} & W_{JK0} & true \end{array},$$

where $W_{JK0} = (x_{cu}^{\beta} \wedge \neg x_{pre}^{\beta}) \wedge x_{cu}^{\alpha}$, where x_{cu}^{β} and x_{pre}^{β} are the current and the previous characteristics of the β token, respectively, and x_{cu}^{α} is the current characteristic of the α token.

The token from place CLK splits in five tokens. The token that enters place C_0 does not change its characteristic. The remaining four tokens enter places Y_0, JK_1, AND_{01}, AND_{02}, respectively, and obtain the same characteristic $x_{cu}^{\gamma 0} = \neg x_{pre}^{\gamma 0}$, where $x_{pre}^{\gamma 0}$ is the previous characteristic of the γ_0 token (Fig. 5).

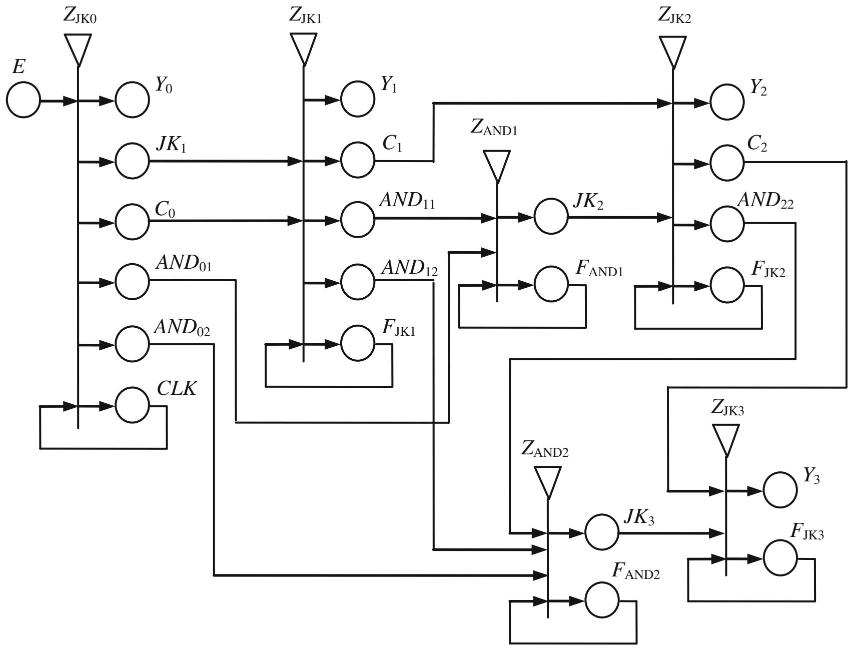

Fig. 5. GN model of a 4-bit synchronous binary counter

The transition Z_{JK1} is represented by the following expression:

$$Z_{JK1} = \langle \{JK_1, C_0, F_{JK1}\}, \{Y_1, C_1, AND_{11}, AND_{12}, F_{JK1}\}, R_{JK1}, \wedge(JK_1, C_0, F_{JK1})\rangle.$$

The matrix R_{JK1} has the following form:

$$R_{JK1} = \begin{array}{c|ccccc} & Y_1 & C_1 & AND_{11} & AND_{12} & F_{JK1} \\ \hline JK_1 & false & false & false & false & true \\ C_0 & false & true & false & false & true \\ F_{JK1} & W_{JK1} & W_{JK1} & W_{JK1} & W_{JK1} & true \end{array},$$

where $W_{JK1} = (x_{cu}^{\beta} \wedge \neg x_{pre}^{\beta}) \wedge x_{cu}^{JK1}$.

The token from place F_{JK1} splits in four tokens. The token that enters place C1 does not change its characteristic. The remaining three tokens enter places Y_1, AND_{11}, AND_{12} respectively, and obtain same characteristic $x_{cu}^{\gamma 1} = \neg x_{pre}^{\gamma 1}$, where $x_{pre}^{\gamma 1}$ is the previous characteristic of the γ_1 token.

The transition Z_{JK2} is represented by the following expression:

$$Z_{JK2} = \langle \{JK_2, C_1, F_{JK2}\}, \{Y_2, C_2, AND_{22}, F_{JK2}\}, R_{JK2}, \wedge(JK_2, C_1, F_{JK2})\rangle$$

The matrix R_{JK2} has the following form:

$$R_{JK2} = \begin{array}{c|cccc} & Y_2 & C_2 & AND_{22} & F_{JK2} \\ \hline JK_2 & false & false & false & true \\ C_1 & false & true & false & true \\ F_{JK2} & W_{JK2} & W_{JK2} & W_{JK2} & true \end{array},$$

where $W_{JK2} = (x_{cu}^{\beta} \wedge \neg x_{pre}^{\beta}) \wedge x_{cu}^{JK2}$.

The token from place $\mathrm{F_{JK2}}$ splits in three tokens. The token that enters place C_2 does not change its characteristic. The remaining two tokens enter places Y_2, $\mathrm{AND_{22}}$ respectively, and obtain same characteristic $x_{cu}^{\gamma 2} = \neg x_{pre}^{\gamma 2}$, where $x_{pre}^{\gamma 2}$ is the previous characteristic of the γ_2 token.

The transition Z_{JK3} is represented by the following expression:

$$Z_{JK3} = \langle \{JK_3, C_2, F_{JK3}\}, \{Y_3, F_{JK3}\}, R_{JK3}, \wedge(JK_3, C_2, F_{JK3}) \rangle$$

The matrix R_{JK3} has the following form:

$$R_{JK3} = \begin{array}{c|cc} & Y_3 & F_{JK3} \\ \hline JK_3 & false & true \\ C_2 & false & true \\ F_{JK3} & W_{JK3} & true \end{array},$$

where $W_{JK3} = (x_{cu}^{\beta} \wedge \neg x_{pre}^{\beta}) \wedge x_{cu}^{JK3}$.

The token from place F_{JK3} enters place Y_3 and obtains the characteristic $x_{cu}^{\gamma 3} = \neg x_{pre}^{\gamma 3}$, where $x_{pre}^{\gamma 3}$ is the previous characteristic of the γ_3 token.

The transition Z_{AND1} is represented by the following expression:

$$Z_{AND1} = \langle \{AND_{01}, AND_{11}, F_{AND1}\}, \{JK_2, F_{AND1}\}, R_{AND1}, \wedge(AND_{01}, AND_{11}, F_{AND1}) \rangle$$

The matrix R_{AND1} has the following form:

$$R_{AND1} = \begin{array}{c|cc} & JK_2 & F_{AND1} \\ \hline AND_{01} & false & true \\ AND_{11} & false & true \\ F_{AND1} & W_{AND1} & true \end{array},$$

where W_{AND1} = “The logic function yields output”.

The token from place F_{AND1} enters place JK_2 and obtains characteristic

$$x_{cu}^{AND1} = x_{cu}^{AND01} \wedge x_{cu}^{AND11}.$$

The transition Z_{AND2} is represented by the following expression:

$$Z_{AND2} = \langle \{AND_{02}, AND_{12}, AND_{22}, F_{AND2}\}, \{JK_3, F_{AND2}\}, R_{AND2}, \\ \wedge\,(AND_{02}, AND_{12}, AND_{22}, F_{AND2})\rangle$$

The matrix R_{AND2} has the following form:

$$R_{AND2} = \begin{array}{c|cc} & JK_3 & F_{AND2} \\ \hline AND_{02} & false & true \\ AND_{12} & false & true \\ AND_{22} & false & true \\ F_{AND2} & W_{AND2} & true \end{array},$$

where W_{AND2} = "The logic function yields output".
The token from place F_{AND2} enters place JK_3 and obtains characteristic

$$x_{cu}^{AND2} = x_{cu}^{AND02} \wedge x_{cu}^{AND12} \wedge x_{cu}^{AND22}.$$

3 Discussion

The literature review in the area of modelling of logic schemes has shown that while there are different tools for modelling, there is none that renders account of time delay of the scheme's reaction. Rendering account of these time delays is important in order to perform synchronization, and hence yield a correct value at the output of the logical scheme. While the discussed sample here in Fig. 4 contains only four triggers in transitions Z_{JK0} to Z_{JK3}, the larger the logic scheme, the more important the question of the parallelization and synchronization of the time delays.

The authors consider that the above model of a synchronous binary counter demonstrates that generalized nets are an appropriate tool for modelling of parallelism featuring schemes with logic outputs.

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Generalized Net Model of Adolescent Idiopathic Scoliosis Diagnosing

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Abstract. Of the many postural disorders commonly found in the population, scoliosis is the most complex and the most difficult to diagnose and treat. Adolescent idiopathic scoliosis affects 2% to 4% of adolescent population. Although most of those patients will not develop clinical symptoms, scoliosis can progress to rib deformity and respiratory compromise, and can cause significant cosmetic problems and emotional distress for some patients. Accordingly, early detection is the key factor to prevent the curve magnitude progress, as well as to ensure an appropriate and successful treatment. Proper diagnosis is extremely important for designing a coordinated exercise programs and reliable monitoring progress during treatment. The purpose of the present study is to present a successful example of Generalized Nets application in orthopedics and to propose a novel approach to timely detection of adolescent idiopathic scoliosis and its categorization.

Keywords: Scoliosis · Adolescent idiopathic scoliosis · Generalized Nets

1 Introduction

Scoliosis is one of the most common deformities of the spine. Currently, scoliosis is defined as spinal deformity consisting of lateral deviation of the normal vertical line of the spine in the coronal plane of greater than 10° as measured using the Cobb angle method from a standing upright radiograph. Although the lateral curvature is the main component, it can also be associated with rotation of the spine and different plane curvatures. These additional curvatures and rotation make scoliosis a complex three-dimensional deformity. As the curve progresses, the vertebrae and spinous processes in the area of the major curve rotate toward the concavity of the curve. For a given degree of curvature, the rotation is always more pronounced at the lumbar level than the thoracic level. Rotation causes typical deformations of the vertebrae and the ribs. It may develop as a single primary curve (resembling the letter C) or as two curves (a primary curve along with a compensating secondary curve that forms an S-shape). Scoliosis may occur only in the upper back (the thoracic area) or lower back (lumbar), but most commonly it develops in the area between the thoracic and lumbar area (thoracolumbar area). The causes of scoliosis vary and are classified broadly as

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congenital, neuromuscular, syndrome-related, idiopathic and spinal curvature due to secondary reasons. The exact pathophysiologic mechanism for scoliosis is unknown and this disorder can arise from a number of underlying conditions. The most common form is idiopathic, which means that the condition has no identifiable causes, although scientists have identified that it can result from genetic and epigenetic factors. Approximately 30% of idiopathic patients have some family history of scoliosis, which would indicate a genetic connection [17]. If both parents have idiopathic scoliosis, their children are 50 times more likely to require scoliosis treatment compared with the general population [9]. Arbitrarily, idiopathic scoliosis is divided into three groups by the age of presentation: infantile (0 to 3 years), juvenile (3 to 10 years), and adolescent (greater than 10 years). Adolescent idiopathic scoliosis (AIS) is the single most common form of spinal deformity seen in orthopedic practice, affecting approximately 2% to 4% of adolescents [9, 12, 16]. Other studies indicate an overall prevalence of 0.47–5.2% [8]. AIS developing at the age of 11–18 years, accounts for approximately 90% of cases of idiopathic scoliosis in adolescents. According to the Scoliosis Research Society (SRS), the prevalence of AIS in adolescents from 10 to 16 years of age is 2% to 3%. According to Stirling et al. [21], the point prevalence of idiopathic scoliosis in patients aged 6–14 years is 0.5% and the prevalence of scoliosis was highest (1.2%) in patients aged 12–14 years. The prevalence of idiopathic scoliosis in adolescents at the end of their growth period has been variously reported as being between 1.5–3% of the population [4, 13]. AIS is significantly more common in female children with an average female to male prevalence ratio of 2.1:1 to 2.36:1 based on data from recent large-scale studies [20, 22]. Approximately 85% to 90% of adolescent idiopathic scoliosis cases involve a right thoracic curve (the spinal curve is convex to the right) [10, 11]. A left thoracic curve (convex to the left) is more likely to be associated with additional pathology, including spinal cord tumors, neuromuscular disorders, Arnold-Chiari malformations, or occult syrinx. The size of the curve tends to increase over the entire lifetime, but the degree of progression over a lifetime and the time-at-risk varies with many factors. Curve progression is unpredictable, though a subset of children with AIS may exhibit rapid progression. Several studies have investigated the natural history and natural course of curve progression in AIS. All report the strongest predictive factors in the development of idiopathic scoliosis are age, magnitude of curve, and gender [5, 24, 26]. The more skeletally and sexually immature the patient is, the greater is the probability of curve progression [1, 25]. Other studies suggest that the initial Cobb angle magnitude is the most important predictor of long-term curve progression and behavior past skeletal maturity [23].

In all its forms, early diagnosis of AIS is a primary goal. The purpose of the present study is to give an example how the apparatus of generalized nets might be successfully applied in orthopedics and as such – to be proposed a novel mathematical approach for diagnosing the adolescent idiopathic scoliosis. Generalized nets (GNs; see [2, 3]) are an apparatus for modeling of parallel and concurrent processes, developed as an extension of the concept of Petri nets and some of their modifications. In general the GNs may or may not have some of the components in their definition. GNs which do not have some of the components form special classes called reduced GNs [2]. The presented reduced GN-model has similar features with previous models for medical diagnosing [14, 18, 19], but this is the first one GN-model highlighting the diagnostic algorithm for AIS and thus representing an application of GNs in orthopedics.

2 Diagnostic Approaches of AIS

For the completeness of the present study it is important some short remarks on the diagnostic approaches of AIS to be given. Clinical evaluation of AIS is based on a detailed history, physical examination and a plan for image testing. The focus of investigation should center on several main topics: exclusion of other causes of spinal deformity, pubertal status and estimation of remaining growth potential, determination of the degree of the spinal curve and the curve pattern. In order to make a proper diagnosis and rule out other possible conditions, the patient's past medical history is taken. Many examination techniques are used to evaluate patients suspected to have AIS. Traditionally, the Adam forward bend test with level plane and a scoliometer evaluation [4] of the patient is used to guide clinical decision making. A positive result is spinal asymmetry, unleveled shoulders, scapula asymmetry, unleveled hips and one of > 5° at any paraspinal prominence measured with the scoliometer. Generally, an angle of trunk rotation less than 5° is insignificant and may not require follow-up. A measurement of 10° or greater requires further investigation with X-rays and evaluation for Cobb angle measurement. The Cobb angle is used as a standard measurement to determine the degree of the spinal curve and track the progression of AIS. The method remains "gold standard" diagnostic tool of choice due to being well understood within the orthopedic community, as well as facilities capable of acquiring full spinal radiographs are readily available in most clinics. Cobb [6] suggested that the angle of curvature be measured by drawing lines parallel to the upper border of the upper vertebral body and the lower border of the lowest vertebra of the structural curve, then erecting perpendiculars from these lines to cross each other, the angle between these perpendiculars being the "angle of curvature". A Cobb angle greater than 10° establishes a diagnosis of scoliosis. Scoliosis is considered mild at 10–25°, moderate at 25–50° and severe at greater than 50°. Since scoliosis curves grow larger during rapid growth, the best predictors of curve progression are growth potential and growth velocity, both of which are functions of skeletal maturity. The potential for growth is evaluated taking into consideration the chronologic age, skeletal age, menarchal stage, and stage of development of the iliac crest apophysis (Risser stage). The Risser grading system [15] is often used to determine a child's skeletal maturity on the pelvis, which correlates with how much spine growth is left. The Risser grading system rates a child's skeletal maturity on a scale of 0 to 5. Higher Risser grades indicate greater skeletal ossification, hence less potential for growth and curve progression. Grade 0 describes an X-ray on which no ossification center is seen in the apophysis, grade 1 is 25% ossification, grade 2 is 50% ossification, grade 3 is 75% ossification, grade 4 is 100% ossification and grade 5 represents complete ossification and fusion of the iliac apophysis. The Cobb angle and Risser grading system can be combined to predict the likelihood of curve progression. Taking into consideration the curve pattern, magnitude and flexibility of the scoliosis deformity, King et al. [7] described their classification system in 1983, commonly known as the King classification system for adolescent idiopathic scoliosis. King et al. defined five curve types:

- **Type 1:** An S-shape deformity, in which both curves are structural and cross the center sacral vertical line (CSVL), with the lumbar curve being larger than the thoracic one.
- **Type 2:** An S-shape deformity, in which both curves are structural and cross the CSVL, with the thoracic curve being larger or equal to the lumbar one.
- **Type 3:** Major thoracic curve in which only the thoracic curve is structural and crosses the CSVL.
- **Type 4:** Long C-shape thoracic curve in which the fifth lumbar vertebra is centered over the sacrum and the forth lumbar vertebra is tilted into the thoracic curve.
- **Type 5:** Double thoracic curve.

In the next two sections of the presented paper we will describe two GN-models. The first one represents the diagnostic algorithm for patient with possible diagnosis of AIS. The next one is devoted to categorisation and further evaluation of estabilished diagnosis of AIS.

3 GN-Model of AIS Diagnosing

Here is represented a reduced GN-model which represents the diagnosing plan for AIS. The proposed model is shown in Fig. 1.

The GN-model has 8 transitions and 27 places with the following meanings:

- Transition Z_1 represents the personal data of the patient.
- Transition Z_2 represents the past medical history of the patient and physical examination techniques.
- Transition Z_3 represents the results from the physical and neurological examination.
- Transition Z_4 represents the set of imaging tests.
- Transition Z_5 represents the results from the X-ray.
- Transition Z_6 represents the results from magnetic resonance imaging (MRI), EMG and nerve conducting testing.
- Transition Z_7 represents the final diagnosis.
- Transition Z_8 represents the possible differential diagnosis.

The GN-model contains 7 types of tokens: α, β, μ, η, ε, γ, and φ. Some of the model transitions contain the so called “special place” where a token stays and collects information about the specific parts of the diagnosing process which it represents as follows:

- In place l_3, token β stays permanently and collects the overall information obtained from the diagnostics steps in the personal record (personal data).
- In place l_7, token μ stays permanently and collects information about the medical history of the patient.

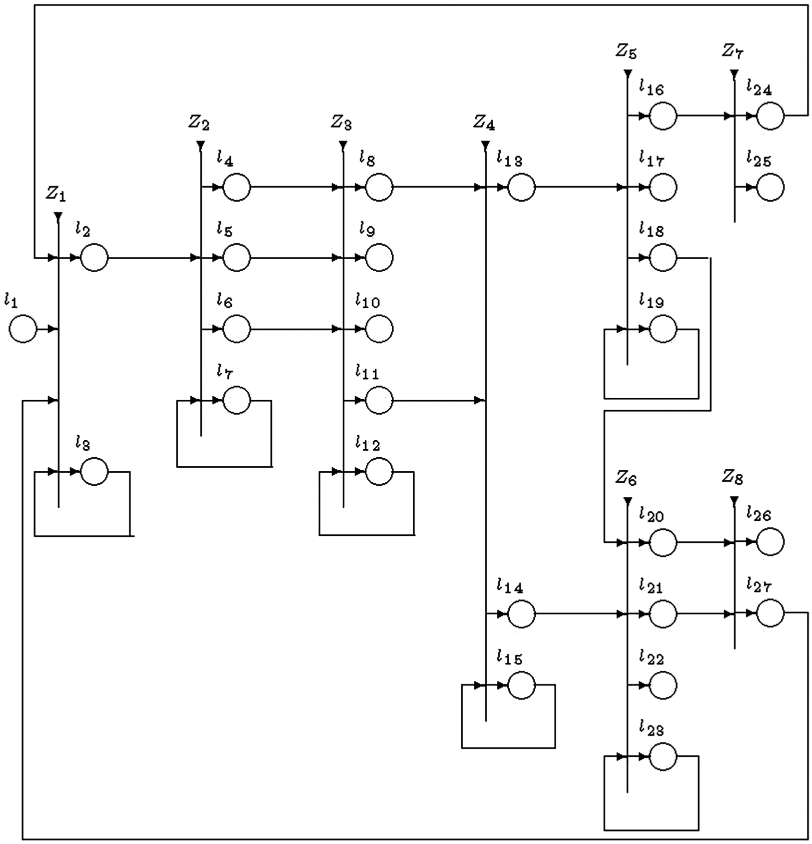

Fig. 1. GN-model of adolescent idiopathic scoliosis diagnosing

- In place l_{12}, token η stays permanently and collects information about the current functional status of the patient obtained from the physical and neurological examination.
- In place l_{15}, token ε stays permanently and collects information about the possible imaging techniques for diagnosing.
- In place l_{19}, token γ stays permanently and collects information about the results from the X-ray.
- In place l_{23}, token φ stays permanently and collects information about the results from MRI, EMG and nerve conducting tests.

During the GM-model functioning, the α-tokens will unite with the tokens from the rest types (β, μ, η, ε, γ and φ). After that, some of these tokens can split in order to generate new α-tokens obtaining corresponding characteristics. When there are some α-tokens (α_1, α_2 and eventually, α_3), on the next time-moment, all they will unite with a token from another type.

The token α enters the net in place l_1 with an initial characteristic:

"*patient between 10–18 years of age, who is suspected to have AIS*".

The transition Z_1 of the GN-model has the following form:

$$Z_1 = \langle \{l_1, l_3, l_{24}, l_{27}\},\ \{l_2, l_3\}, r_1 \rangle,$$

where:

$$r_1 = \begin{array}{c|cc} & l_2 & l_3 \\ \hline l_1 & \mathit{false} & \mathit{true} \\ l_3 & W_{3,2} & \mathit{true} \\ l_{24} & \mathit{false} & \mathit{true} \\ l_{27} & \mathit{false} & \mathit{true} \end{array}$$

and,

$W_{3,2}$ = "*past medical history and physical examination are necessary*".

The tokens from the four input places of transition Z_1 enter place l_3 and unite with token β with the above mentioned characteristic. On the next time-moment, token β splits to two tokens – the same token β and token α. When the predicate $W_{3,2}$ is true, token α_1 enters place l_2 and there it obtains a characteristic:

"*results of the medical history and physical examination*".

The transition Z_2 has the following form:

$$Z_2 = \langle \{l_2, l_7\},\ \{l_4, l_5, l_6, l_7\}, r_2 \rangle,$$

where:

$$r_2 = \begin{array}{c|cccc} & l_4 & l_5 & l_6 & l_7 \\ \hline l_2 & \mathit{false} & \mathit{false} & \mathit{false} & \mathit{true} \\ l_7 & W_{7,4} & W_{7,5} & W_{7,6} & \mathit{true} \end{array}$$

and,

$W_{7,4}$ = "*there are evidences of recent growth spurts and postural asymmetry during daily activities*";

$W_{7,5}$ = "$\neg\ W_{7,4}$";

$W_{7,6}$ = "*there is severe back pain and alterations in sensation or motor control*".

The tokens from all input places of transition Z_2 enter place l_7 and unite with token μ with the characteristic, as mentioned above. On the next time-moment, token μ splits to two or three tokens – the same token μ that stays permanently in the place l_7 and tokens α_1 and α_2. When the predicate $W_{7,4}$ is true, token α_1 enters place l_4 and there it obtains a characteristic:

"*perform Adam forward bend test with scoliometer and examination on standing position*".

When the predicate $W_{7,5}$ is true, token α_1 enters place l_5 and there it obtains a characteristic:

"*consider leg-length discrepancy and perform leg-length measurement and examination on sitting position*".

When the predicate $W_{7,6}$ is true, token α_2 enters place l_6 and there it obtains a characteristic:

"*consider underlying intraspinal pathology and perform a neurological examination*".

The transition Z_3 has the following form:

$$Z_3 = \langle \{l_4, l_5, l_6, l_{12}\}, \{l_8, l_9, l_{10}, l_{11}, l_{12}\}, r_3 \rangle,$$

where:

$$r_3 = \begin{array}{c|ccccc} & l_8 & l_9 & l_{10} & l_{11} & l_{12} \\ \hline l_4 & false & false & false & false & true \\ l_5 & false & false & false & false & true \\ l_6 & false & false & false & false & true \\ l_{12} & W_{12,8} & W_{12,9} & W_{12,10} & W_{12,11} & true \end{array}$$

and,

$W_{12,8}$ = "*the result from the Adam forward bend test with scoliometer is positive and shows presence of asymmetric paraspinal prominence with scoliometer reading > 5°*";

$W_{12,9}$ = "*the result from the Adam forward bend test with scoliometer is negative*";

$W_{12,10}$ = "*there is a correction of the spinal deformity on sitting position*";

$W_{12,11}$ = "*neurological examination shows: motor weakness, spasticity, sensory and motor deficits*".

The tokens from all input places of transition Z_3 enter place l_{12} and unite with token η with the characteristic, as mentioned above. On the next time-moment, token η splits to three tokens – the same token η that stays permanently in the place l_{12} and tokens α_1 and α_2. When the predicate $W_{12,8}$ is true, token α_1 enters place l_8 and there it obtains a characteristic:

"*perform X-ray*".

When the predicate $W_{12,9}$ is true, token α_1 enters place l_9 and there it obtains a characteristic:

"*follow-up in 6 to 12 months according to grown potential*".

When the predicate $W_{12,10}$ is true, token α_2 enters place l_{10} and there it obtains a characteristic:

"*inform patient and give leaflet on good posture,*
shoe lifts and regular follow-up examinations".

When the predicate $W_{12,11}$ is true, token α_2 enters place l_{11} and there it obtains a characteristic:

"*perform MRI, EMG and nerve conducting testing*".

The transition Z_4 of the GN-model has the following form:

$$Z_4 = \langle\{l_8, l_{11}, l_{15}\},\ \{l_{13}, l_{14}, l_{15}\}, r_4\rangle,$$

where:

$$r_4 = \begin{array}{c|ccc} & l_{13} & l_{14} & l_{15} \\ \hline l_8 & false & false & true \\ l_{11} & false & false & true \\ l_{15} & W_{15,13} & W_{15,14} & true \end{array}$$

and,

$W_{15,13}$ = "*the patient was suitable for plain film radiographs and X-rays of the entire spine were taken*";

$W_{15,14}$ = "*the patient was with atypical presentation, suitable for MRI, EMG and nerve conducting tests and past trough the tests*".

The tokens from all input places of transition Z_4 enter place l_{15} and unite with token ε with the characteristic, as mentioned above. On the next time-moment, token ε splits to three tokens – the same token ε and tokens α_1 and α_2 that enter respectively places l_{13} and l_{14}. When the predicate $W_{15,13}$ is true, token α_1 obtains a characteristic in place l_{13}:

"*results from the X-ray*"

When the predicate $W_{15,14}$ is true, token α_2 obtains a characteristic in place l_{14}:

"*results from the MRI, EMG and nerve conducting tests*".

The transition Z_5 has the following form:

$$Z_5 = \langle\{l_{13}, l_{19}\},\ \{l_{16}, l_{17}, l_{18}, l_{19}\}, r_5\rangle,$$

where:

$$r_5 = \begin{array}{c|cccc} & l_{16} & l_{17} & l_{18} & l_{19} \\ \hline l_{13} & false & false & false & true \\ l_{19} & W_{19,16} & W_{19,17} & W_{19,18} & true \end{array}$$

and,

$W_{19,16}$ = "*the result from the X-ray is: there is a visible coronal plane spinal curvature and hypo- or hyperkyphosis of the thoracic spine*";

$W_{19,17}$ = "$\neg W_{19,16}$";

$W_{19,18}$ = "*the result from the X-ray is: there is atypical curve pattern (left thoracic curve, short angular curve, apical kyphosis)*".

The tokens from all input places of transition Z_5 enter place l_{19} and unite with token γ with the characteristic, as mentioned above. On the next time-moment, token γ splits to two or three tokens – the same token γ that stays permanently in the place l_{19} and tokens α_1 and α_2. When the predicate $W_{19,16}$ is true, token α_1 enters place l_{16} and there it obtains a characteristic:

"*consider AIS*".

When the predicate $W_{19,17}$ is true, token α_1 enters place l_{17} and there it obtains a characteristic:

"*exclude AIS and consider functional scoliosis*".

When the predicate $W_{19,18}$ is true, token α_2 enters place l_{18} and there it obtains a characteristic:

"*consider MRI*".

The transition Z_6 has the following form:

$$Z_6 = \langle \{l_{14}, l_{18}, l_{23}\}, \{l_{20}, l_{21}, l_{22}, l_{23}\}, r_6, \rangle$$

where:

$$r_6 = \begin{array}{c|cccc} & l_{20} & l_{21} & l_{22} & l_{23} \\ \hline l_{14} & false & false & false & true \\ l_{18} & false & false & false & true \\ l_{23} & W_{23,20} & W_{23,21} & W_{23,22} & true \end{array}$$

and,

$W_{23,20}$ = "*the MRI demonstrates tethered spinal cord and/or cystic cavitation of the spinal cord*";

$W_{23,21}$ = "*EMG and nerve conducting tests demonstrate evidence of upper motor neuron lesions*";

$W_{23,22}$ = "*the results from MRI and EMG testing are normal*".

The tokens from all input places of transition Z_6 enter place l_{23} and unite with token φ with the characteristic, as mentioned above. On the next time-moment, token φ splits to three or four tokens – the same token φ that stays permanently in the place l_{23} and tokens α_1, α_2 and α_3. When predicate $W_{23,20}$ is true, token α_1 enters place l_{20} and there it obtains a characteristic:

"*consider Syringomyelia or Arnold-Chiari malformation*".

When predicate $W_{23,21}$ is true, token α_2 enters place l_{21} and there it obtains a characteristic:

"*consider neuromuscular scoliosis*".

When predicate $W_{23,22}$ is true, token α_3 enters place l_{22} and there it obtains a characteristic:

"*follow-up observation is necessary*".

The transition Z_7 has the following form:

$$Z_7 = \langle \{l_{16}\},\ \{l_{24}, l_{25}\}, r_7 \rangle,$$

where:

$$r_7 = \begin{array}{c|cc} & l_{24} & l_{25} \\ \hline l_{16} & true & true \end{array}$$

The token α obtains the characteristics "*the diagnosis of the patient is AIS*" in place l_{25} and token β_1 with the same characteristic for the personal record of the patient, in place l_{24}. The token returns to place l_3 to extend the personal record of the current patient.

The transition Z_8 of the GN-model has the following form:

$$Z_8 = \langle \{l_{20}, l_{21}\},\ \{l_{26}, l_{27}\}, r_8 \rangle,$$

where:

$$r_8 = \begin{array}{c|cc} & l_{26} & l_{27} \\ \hline l_{20} & true & false \\ l_{21} & false & true \end{array}$$

Tokens α_1 and α_2 split to two tokens: token $\alpha_{1,1}$ that enters place l_{26} and token $\alpha_{1,2}$ that enters places l_{27} and token $\alpha_{2,1}$ that enters place l_{26} and token $\alpha_{2,2}$ that enters places l_{27}. Both tokens $\alpha_{1,1}$ and $\alpha_{2,1}$ unite in place l_{26} in token α that obtains a characteristic: "*possible differential diagnoses are Syringomyelia or Arnold-Chiari malformation or neuromuscular scoliosis*". Both tokens $\alpha_{1,2}$ and $\alpha_{2,2}$ unite in place l_{27} in token β_1 that obtains the same characteristic and it returns to place l_3 to extend the personal record of the current patient.

4 GN-Model of AIS Classification and the Curve Progression Probability

Described in Sect. 3 GN-model represents the diagnostic algorithm for patient with possible diagnosis of AIS. After the final diagnosis is made it is extremely important to define the curve pattern and to estimate the risk for curve progression which will help to categories idiopathic scoliosis for easier communication, prognosticate the disease and guide the treatment strategy. Accordingly, here we will construct a reduced GN-model of AIS classification and the curve progression probability. The proposed GN-model (Fig. 2) is based on Cobb angle measurements, King classification system and the Risser grading system discussed in Sect. 2.

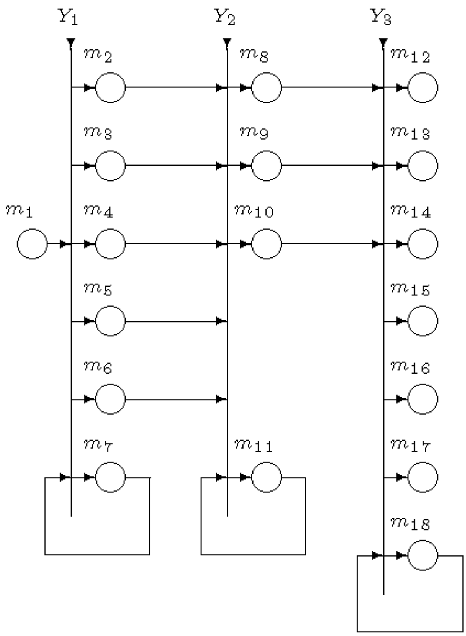

Fig. 2. GN-model of AIS classification and the curve progression probability

The GN-model presented in Fig. 2 has 3 transitions and 18 places with the following meaning

- Transition Y_1 represents the King classification system.
- Transition Y_2 represents the Cobb angle measurements.
- Transition Y_3 represents the Risser grading system.

The GN-model contains four types of tokens: α, β, μ and η. The three transitions of the GN-model have three "special places", where a token stays and collects information about the specific parts of the diagnosing process which it represents as follows:

- In place m_7, token β stays permanently and collects information about the curve patterns according to the King classification system.
- In place m_{11}, token μ stays permanently and collects information about the results from the Cobb angle measurements.
- In place m_{18}, token η stays permanently and collects information about the risk for curve progression based on the Risser grading system.

Token α enters the net with an initial characteristic "*patient with confirmed AIS*" in place m_1.

The transition Y_1 of the GN-model has the following form:

$$Y_1 = \langle \{m_1, m_7\},\ \{m_2, m_3, m_4, m_5, m_6, m_7\}, r_1 \rangle,$$

where:

$$r_1 = \begin{array}{c|cccccc} & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 \\ \hline m_1 & false & false & false & false & false & true \\ m_7 & W_{7,2} & W_{7,3} & W_{7,4} & W_{7,5} & W_{7,6} & true \end{array}$$

and,

$W_{7,2}$ = "*there is an S-shape deformity, in which both curves are structural and cross the CSVL, with the lumbar curve being larger than the thoracic one*";

$W_{7,3}$ = "*there is an S-shape deformity, in which both curves are structural and cross the CSVL, with the thoracic curve being larger or equal to the lumbar one*";

$W_{7,4}$ = "*there is a major thoracic curve in which only the thoracic curve is structural and crosses the CSVL*;

$W_{7,5}$ = "*there is a long C-shape thoracic curve in which the fifth lumbar vertebra is centered over the sacrum and the forth lumbar vertebra is tilted into the thoracic curve*;

$W_{7,6}$ = "*there is a double thoracic curve*".

The tokens from the two input places of transition Y_1 enter place m_7 and unite with token β that obtains the above mentioned characteristics. On the next time-moment, token β splits to two tokens – the same token β and token α. When the predicate $W_{7,2}$ is true, token α enters place m_2 and there it obtains a characteristics:

"*AIS is of King's Type 1*".

When the predicate $W_{7,3}$ is true, token α enters place m_3 and there it obtains a characteristics:

"*AIS is of King's Type 2*".

When the predicate $W_{7,4}$ is true, token α enters place m_4 and there it obtains a characteristics:

"*AIS is of King's Type 3*".

When the predicate $W_{7,5}$ is true, token α enters place m_5 and there it obtains a characteristics:

"*AIS is of King's Type 4*".

When the predicate $W_{7,6}$ is true, token α enters place m_5 and there it obtains a characteristics:

"*AIS is of King's Type 5*".

It is important to note that in each time-moment only one of the five predicates is valid. So, there is only one token α in the net at any moment.

The transition Y_2 of the GN-model has the following form:

$$Y_2 = \langle \{m_2, m_3, m_4, m_5, m_6, m_{11}\},\ \{m_8, m_9, m_{10}, m_{11}\}, r_2 \rangle,$$

where:

$$r_2 = \begin{array}{c|cccc} & m_8 & m_9 & m_{10} & m_{11} \\ \hline m_2 & false & false & false & true \\ m_3 & false & false & false & true \\ m_4 & false & false & false & true \\ m_5 & false & false & false & true \\ m_6 & false & false & false & true \\ m_{11} & W_{11,8} & W_{11,9} & W_{11,10} & true \end{array}$$

and,

$W_{11,8}$ = "*AIS of King's Type 1, 2, 3, 4, or 5 is with the Cobb measurement of any spinal curve above 10°*";

$W_{11,9}$ = "*AIS of King's Type 1, 2, 3, 4, or 5 is with the Cobb measurement of any spinal curve above 25°*";

$W_{11,10}$ = "*AIS of King's Type 1, 2, 3, 4, or 5 is with the Cobb measurement of any spinal curve above 50°*".

The tokens from all input places of transition Y_2 enter place m_{11} and unite with token μ that obtains the above mentioned characteristics. On the next time-moment, token μ splits to two tokens – the same token μ and token α. When predicate $W_{12,8}$ is true, token α enters place m_8 and there it obtains a characteristics:

"*AIS is of King's Type 1, 2, 3, 4 or 5 with 10 to 25° curve*".

When predicate $W_{12,9}$ is true, token α enters place m_9 and there it obtains a characteristics:

"*AIS is of King's Type 1, 2, 3, 4 or 5 with 25 to 50° curve*".

When predicate $W_{12,10}$ is true, token α enters place m_{10} and there it obtains a characteristics:

"*AIS is of King's Type 1, 2, 3, 4 or 5 with 50 to 100/or above degrees curve*".

The transition Y_3 has the following form:

$$Y_3 = \left\langle \{m_8, m_9, m_{10,} m_{18}\},\ \{m_{12}, m_{13}, m_{14}, m_{15}, m_{16}, m_{17}, m_{18}\}, r_3 \right\rangle,$$

where:

$$r_3 = \begin{array}{c|ccccccc} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} & m_{17} & m_{18} \\ \hline m_8 & false & false & false & false & false & false & true \\ m_9 & false & false & false & false & false & false & true \\ m_{10} & false & false & false & false & false & false & true \\ m_{18} & W_{18,12} & W_{18,13} & W_{18,14} & W_{18,15} & W_{18,16} & W_{18,17} & true \end{array}$$

and,

$W_{18,12}$ = "*AIS is of King's Type 1, 2, 3, 4 or 5 with 10 to 25° curve and a Risser grade from 0 to 1*";

$W_{18,13}$ = "*AIS is of King's Type 1, 2, 3, 4 or 5 with 10 to 25° curve and a Risser grade from 2 to 4*";

$W_{18,14}$ = "*AIS is of King's Type 1, 2, 3, 4 or 5 with 25 to 50° curve and a Risser grade from 0 to 1*";

$W_{18,15}$ = "*AIS is of King's Type 1, 2, 3, 4 or 5 with 25 to 50° curve and a Risser grade from 2 to 4*";

$W_{18,16}$ = "*AIS is of King's Type 1, 2, 3, 4 or 5 with 50 to 100/or above degrees curve and a Risser grade from 0 to 1*";

$W_{18,17}$ = "*AIS is of King's Type 1, 2, 3, 4 or 5 with 50 to 100/or above degrees curve and a Risser grade from 2 to 4*".

The tokens from all input places of transition Y_3 enter place m_{18} and unite with token η that obtains the characteristic, as mentioned above. On the next time-moment, token η splits to two tokens – the same token η that stays permanently in the place m_{18} and token α. When predicate $W_{18,12}$ is true, token α enters place m_{12} and there it obtains a characteristic:

"*there is a moderate risk of curve progression*".

When predicate $W_{18,13}$ is true, token α enters place m_{13} and there it obtains a characteristic:

"*there is a low risk of curve progression*".

When predicate $W_{18,14}$ is true, token α enters place m_{14} and there it obtains a characteristic:

"*there is a high risk of curve progression*".

When predicate $W_{18,15}$ is true, token α enters place m_{15} and there it obtains a characteristic:

"*there is a very high risk of curve progression*".

When predicate $W_{18,16}$ is true, token α enters place m_{16} and there it obtains a characteristic:

"*there is a high risk of curve progression*".

When predicate $W_{18,17}$ is true, token α enters place m_{17} and there it obtains a characteristic:

"*there is a very high risk of curve progression*".

It is important to note that in each time-moment only one of the six predicates is valid. So, there is only one token α in the net at any moment.

5 Conclusions

AIS is a serious and important postural disorder commonly found in the population. Early detection of AIS is the key factor for preventing progression of the curve magnitude, appropriate and successful treatment. Proper diagnosis is extremely important for designing a coordinated exercise programs and reliably monitoring progress during treatment.

In this paper two GN-models are presented: (i) the first one represents the diagnostic algorithm for patient with possible diagnosis of AIS, (ii) the second one is devoted to categorisation and further evaluation of estabilished diagnosis of AIS. The so described GN-models may provide a framework that can be used by primary care practitioners to guide diagnostic processes for patient suspected to have AIS, enabling more accurate and efficient identification of that condition and would assist in optimizing patient outcomes and more effective treatment.

The presented in this paper GN-model of diagnostic algorithm for patient with possible diagnosis of AIS is a part of a series of studies for diagnosing through GN-modeling assistance and can be improved in multiple ways to yield improvements in results. This model can be complicated and detailed, which will significantly improve the accuracy of the primary diagnosis and the reliability of the proposed algorithm. The obtained results could be used for assisting the process of decision making in the diagnostic processes for other diseases as well.

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