Nonlinear Evolution Equation for Propagation of Waves in an Artery with an Aneurysm: An Exact Solution Obtained by the Modified Method of Simplest Equation

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Abstract We study propagation of traveling waves in a blood filled elastic artery with an axially symmetric dilatation (an idealized aneurysm) in long-wave approximation.The processes in the injured artery are modelled by equations for the motion of the wall of the artery and by equation for the motion of the fluid (the blood). For the case when balance of nonlinearity, dispersion and dissipation in such a medium holds the model equations are reduced to a version of the Korteweg-deVries-Burgers equation with variable coefficients. Exact travelling-wave solution of this equation is obtained by the modified method of simplest equation where the differential equation of Riccati is used as a simplest equation. Effects of the dilatation geometry on the travelling-wave profile are studied.

1 Introduction

The theoretical investigation of pulse wave propagation in human arteries has a long history. Over the past decade the scientific efforts have been concentrated on theoretical investigations of nonlinear wave propagation through the blood in arteries with a variable radius. Clearing how local imperfections appeared in an artery can disturb the blood flow can help in predicting the nature and main features of various cardiovascular diseases, such as stenoses and aneurysms. In order to study propagation of nonlinear waves in a stenosed artery, Tay and co-authors treated the artery as a homo-

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geneous, isotropic and thin-walled elastic tube with an axially symmetric stenosis. The blood was modeled as an incompressible inviscid fluid [1], Newtonian fluid with constant viscosity [2], and Newtonian fluid with variable viscosity [3]. Using a specific perturbation method, in a long-wave approximation the authors obtained the forced Korteweg-de Vries (KdV) equation with variable coefficients [1], forced perturbed KdV equation with variable coefficients [2], and forced Korteweg-de Vries-Burgers (KdVB) equation with variable coefficients as evolution equations [3]. The same theoretical frame was used in [4, 5] to examine nonlinear wave propagation in an artery with a variable radius. Considering the artery as a long inhomogeneous prestretched thin elastic tube with an imperfection (presented at large by an unspecified function $f(z)$), and the blood as an incompressible inviscid fluid the authors obtained again the forced KdV equation with variable coefficients. Apart from solitary propagation waves in such a system, in [5], possibility of periodic waves was discussed at appropriate initial conditions. In this text we shall focus on consideration of the blood flow through an artery with a local dilatation (an aneurysm). The aneurysm is a localized, blood-filled balloon-like bulge in the wall of a blood vessel [6]. In many cases, its rupture causes massive bleeding with associated high mortality. Motivated by investigations in [1–5], the main goal of this paper is to investigate effects of the aneurismal geometry and the blood characteristics on the propagation of nonlinear waves through an injured artery. For that purpose, we use a reductive perturbation method to obtain the nonlinear evolution equation. Exact solution of this equation is obtained by using the modified method of simplest equation. Recently, this method has been widely used to obtain general and particular solutions of economic, biological and physical models, represented by partial differential equations. The paper is organized as follows. A brief description about the derivation of equations governing the blood flow trough a dilated artery is presented in Sect. 2. In Sect. 3 we derive a basic evolution equation in long-wave approximation. A traveling wave solution of this equation is obtained in Sect. 4. Numerical simulations of the solution are presented in Sect. 5. The main conclusions based on the obtained results are summarized in Sect. 6 of the paper.

2 Mathematical Formulation of the Basic Model

It is well-known that the pulsate motion of blood causes wave propagation in arteries. In order to model the interaction of the blood with its container we shall consider two types equations which represent (i) the motion of the arterial wall and (ii) the motion of the blood. To model such a medium we shall treat the artery as a thin-walled incompressible prestretched hyperelastic tube with a localized axially symmetric dilatation. We shall assume the blood to be an incompressible viscous fluid. A brief formulation of the above-mentioned equations follows in the next two subsections.

2.1 Equation of the Wall

It is well-known, that for a healthy human, the systolic pressure is about 120 mm Hg and the diastolic pressure is 80 mm Hg. Thus, the arteries are initially subjected to a mean pressure, which is about 100 mm Hg. Moreover, the elastic arteries are initially prestretched in an axial direction. This feature minimizes its axial deformations during the pressure cycle. Experimental studies show that the longitudinal motion of arteries is very small [7], and it is due mainly to strong vascular tethering and partly to the predominantly circumferential orientation of the elastin and collagen fibers. Taking into account these observations, and following the methodology applied in [1–4], we consider the artery as a circularly cylindrical tube with radius R_0. We assume that such a tube is subjected to an initial axial stretch λ_z and a uniform (mean) inner pressure $P_0^*(Z)$ which cause relatively high circumferential and axial initial stresses. On the other hand, the pressure deviation in the course of periodic motion of heart is about ±20 mm Hg. Then the dynamical deformation due to this pressure deviation can be assumed to be smaller than the initial deformation. Therefore, the theory of small deformations superimposed on initial static deformation can be used in studying the wave propagation in such a complex medium. Under the action of such a variable pressure the position vector of a generic point on the tube can be described by

$$\mathbf{r_0} = [r_0 + f^*(z^*)]\mathbf{e_r} + z^*\mathbf{e_z}, \; z^* = \lambda_z Z^* \tag{1}$$

where $\mathbf{e_r}$ and $\mathbf{e_z}$ are the unit basic vectors in the cylindrical polar coordinates, $\mathbf{r_0}$ is the deformed radius at the origin of the coordinate system, Z^* is axial coordinate before the deformation, z^* is the axial coordinate after static deformation and $f^*(z^*)$ is a function describing the dilatation geometry. We shall specify the concrete form of $f^*(z^*)$ later. Upon the initial static deformation, we shall superimpose only a dynamical radial displacement $u^*(z^*, t^*)$, neglecting the contribution of axial displacement because of the experimental observations, given above. Then, the position vector $\mathbf{r}$ of a generic point on the tube is

$$\mathbf{r} = [r_0 + f^*(z^*) + u^*]\mathbf{e_r} + z^*\mathbf{e_z} \tag{2}$$

The arc-lengths along meridional and circumferential curves respectively, are:

$$ds_z = [1 + (f^{*\prime} + \frac{\partial u^*}{\partial z^*})^2]^{1/2} dz^*, \; ds_\theta = [r_0 + f^* + u^*]d\theta \tag{3}$$

In this way, the stretch ratios in the longitudinal and circumferential directions in final configuration are

$$\lambda_1 = \lambda_z \Lambda, \; \lambda_2 = \frac{1}{R_0}(r_0 + f^* + u^*) \tag{4}$$

where

$$\Lambda = [1 + (f^{*\prime} + \frac{\partial u^*}{\partial z^*})^2]^{1/2} \quad (5)$$

The notation '$\prime$' denotes the differentiation of f^* with respect to z^*. Then, the unit tangent vector **t** along the deformed meridional curve and the unit exterior normal vector **n** to the deformed tube are

$$\mathbf{t} = \frac{(f^{*\prime} + \frac{\partial u^*}{\partial z^*})\mathbf{e_r} + \mathbf{e_z}}{\Lambda}, \quad \mathbf{n} = \frac{\mathbf{e_r} - (f^{*\prime} + \frac{\partial u^*}{\partial z^*})\mathbf{e_z}}{\Lambda} \quad (6)$$

According to the assumption made about material incompressibility the following restriction holds:

$$h^* = \frac{H}{\lambda_1 \lambda_2} \quad (7)$$

where H and h^* are the wall thicknesses before and after deformation, respectively, and λ_1 and λ_2 are the current stretch ratios in longitudinal and circumferential directions, respectively. For hyperelastic materials, the tensions in longitudinal and circumferential directions have the form:

$$T_1 = \frac{\mu^* H}{\lambda_2}\frac{\partial \Pi}{\partial \lambda_1}, \; T_2 = \frac{\mu^* H}{\lambda_1}\frac{\partial \Pi}{\partial \lambda_2} \quad (8)$$

where $\mu^* \Pi$ is the strain energy density function of wall material as μ^* is the material shear modulus. Although the elastic properties of an injured wall section differ from those of the healthy part, here, we assume that the wall is homogeneous, i.e. μ^* is a constant through the axis z. A detailed analysis of the forces acting on an element of the artery including a free-body diagram can be found in [8, 9]. Finally, according to the second Newton's law, the equation of radial motion of a small tube element placed between the planes $z^* = const$, $z^* + dz^* = const$, $\theta = const$ and $\theta + d\theta = const$ obtains the form:

$$-\frac{\mu^*}{\lambda_z}\frac{\partial \Pi}{\partial \lambda_2} + \mu^* R_0 \frac{\partial}{\partial z^*}\left\{\frac{(f^{*\prime} + \partial u^*/\partial z^*)}{\Lambda}\frac{\partial \Pi}{\partial \lambda_1}\right\} + \frac{P^*}{H}(r_0 + f^* + u^*)\Lambda = \rho_0 \frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}} \quad (9)$$

where t^* is the time parameter, P^* is the inner blood pressure and ρ_0 is the mass density of the tube material.

2.2 Equation of the Fluid

Experimental studies over many years demonstrated that blood behaves as an incompressible non-Newtonian fluid because it consists of a suspension of cell formed elements in a liquid well-known as blood plasma. However, in the larger arteries (with a vessel radius larger than 1 mm) it is plausible to assume that the blood has an approximately constant viscosity, because the vessel diameters are essentially larger than the individual cell diameters. Thus, in such vessels the non-Newtonian behavior becomes insignificant and the blood can be considered as a Newtonian fluid. Here, for our convenience we assume a 'hydraulic approximation' and apply an averaging procedure with respect to the cross-sectional area to the Navier-Stokes equations. Then, we obtain

$$\frac{\partial A^*}{\partial t^*} + \frac{\partial}{\partial z^*}(A^*\omega^*) = 0 \tag{10}$$

$$\frac{\partial \omega^*}{\partial t^*} + \omega^* \frac{\partial \omega^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial P^*}{\partial z^*} = \frac{\mu_f}{\rho_f}\frac{\partial^2 \omega^*}{\partial z^{*2}} + \frac{2\mu_f}{r_f^2 \rho_f}(r\frac{\partial V_z^*}{\partial r}) \mid_{r=r_f} \tag{11}$$

where A^* denotes the inner cross-sectional area, i.e., $A^* = \pi r_f^2$ as $r_f = r_0^* + f^* + u^*$ is the final radius of the tube after deformation, ω^* is the averaged axial fluid velocity, V_z^* is the velocity component in the axial direction, ρ_f is the fluid density and μ_f is the dynamical viscosity of the fluid. The substitution of A^* in Eq. (10) leads to

$$2\frac{\partial u^*}{\partial t^*} + 2\omega^*[f^{*\prime} + \frac{\partial u^*}{\partial z^*}] + [r_0 + f^*(z^*) + u^*]\frac{\partial \omega^*}{\partial z^*} = 0 \tag{12}$$

We introduce the following non-dimensional quantities

$$t^* = (\frac{R_0}{c_0})t,\ \ z^* = R_0 z,\ \ u^* = R_0 u,\ \ f^* = R_0 f,\ \ \omega^* = c_0\omega,\ \ \mu_f = c_0 R_0 \rho_f \nu,$$
$$P^* = \rho_f c_0^2 p,\ \ r_0 = R_0\lambda_\theta,\ \ c_0^2 = \frac{\mu^* H}{\rho_f R_0},\ \ m = \frac{\rho_0 H}{\rho_f R_0}, V_z^* = c_0 V_z,\ \ r = R_0 x \tag{13}$$

where c_0 is the Moens-Korteweg velocity, ν is the kinematic viscosity of the fluid and λ_θ is the initial stretch ratio in a circumferential direction. We put (13) in Eqs. (12), (11) and (9), respectively. Thus the final model takes the form:

$$2\frac{\partial u}{\partial t} + 2\omega[f' + \frac{\partial u}{\partial z}] + [\lambda_\theta + f(z) + u]\frac{\partial \omega}{\partial z} = 0 \tag{14}$$

$$\frac{\partial \omega}{\partial t} + \omega\frac{\partial \omega}{\partial z} + \frac{\partial p}{\partial z} = \nu\frac{\partial^2 \omega}{\partial z^2} + \frac{2\nu}{(\lambda_\theta + f + u)^2}(\frac{\partial V_z}{\partial x}) \mid_{x=\lambda_\theta+f+u} \tag{15}$$

$$p = \frac{m}{\lambda_z(\lambda_\theta + f(z) + u)}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda_z(\lambda_\theta + f(z) + u)}\frac{\partial \Pi}{\partial \lambda_2}$$
$$-\frac{1}{(\lambda_\theta + f(z) + u)}\frac{\partial}{\partial z}(\frac{f' + \partial u/\partial z}{\Lambda})\frac{\partial \Pi}{\partial \lambda_1} + \nu\frac{(f' + \partial u/\partial z)\omega}{\lambda_\theta + f + u} \tag{16}$$

3 Derivation of the Evolution Equation in a Long-wave Approximation

In this section we shall use the long-wave approximation to study the propagation of waves in a fluid-solid structure system, presented by Eqs. (14)–(16). In the long-wave limit, it is assumed that the variation of radius along the axial coordinate is small compared with the wave length. As this condition is valid for large arteries, the reductive perturbation method [10] can be applied to study the asymptotic behaviour of dispersive waves in the medium. According to this method an appropriate scale transformation with a perturbation expansion of the dependent variables is introduced. The choice of coordinate transformation (known also as stretching) depends on the dispersion relationship. The dispersion relationship for such systems is derived, e.g., in [8, 9]. According to this relationship the following stretched coordinates are introduced

$$\xi = \epsilon^{1/2}(z - ct), \quad \tau = \varepsilon^{3/2} z \tag{17}$$

where ε appears in the dispersion relationship. It is a small parameter ($\varepsilon = r/l \prec 1$, where l is the characteristic wavelength) measuring the weakness of dispersion. In Eq. (17) c is the phase velocity of the harmonic wave propagation in the medium in the long-wave limit. Then, $z = \varepsilon^{-3/2}\tau$, and $f(\varepsilon^{-3/2}\tau) = \chi(\xi, \tau)$. Thus, the variables u, ω and p are functions of the variables (ξ, τ) and the small parameter ε. Taking into account the effect of dilatation, we assume f to be of order of 5/2, i.e.

$$\chi(\xi, \tau) = \varepsilon h(\tau) \tag{18}$$

In addition, taking into account the effect of viscosity, the order of viscosity is assumed to be $O(1/2)$, i.e.

$$\nu = \varepsilon^{1/2}\overline{\nu} \tag{19}$$

The last assumption ensures balance of nonlinearity, dispersion and dissipation in the system. We introduce also the following perturbation expansions of the variables u, ω and p in term of ε

$$\begin{aligned} u &= \varepsilon u_1 + \varepsilon^2 u_2 + \ \dots, \ \omega = \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \ \dots, \\ V_z &= \varepsilon V_{z1} + \varepsilon^2 V_{z2} + \ \dots, \ p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ \dots, \end{aligned} \tag{20}$$

where $u_1 \dots p_2$ are some unknown functions of the stretched coordinate (ξ, τ). To close the system (14)–(16) p must be presented as a function of u. Therefore we expand the other quantities in Eq. (16) in asymptotic series as follows:

$$\begin{aligned} &\lambda_1 \cong \lambda_z,\ \lambda_2 = \lambda_\theta + \varepsilon(u_1 + h) + \varepsilon^2(u_2 + (u_1 + h)^2) + \ \dots, \\ &\frac{1}{\lambda_\theta \lambda_z}\frac{\partial \Pi}{\partial \lambda_1} = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial \Pi}{\partial \lambda_z} = \gamma_0 \\ &\frac{1}{\lambda_\theta \lambda_z}\frac{\partial \Pi}{\partial \lambda_2} = \beta_0 + \beta_1(u_1 + h)\varepsilon + (\beta_1 u_2 + \beta_2(u_1 + h)^2)\varepsilon^2 + \ \dots \end{aligned} \tag{21}$$

where

$$\beta_0 = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial \Pi}{\partial \lambda_\theta},\ \ \beta_1 = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial^2 \Pi}{\partial \lambda_\theta^2},\ \ \beta_2 = \frac{1}{2\lambda_\theta \lambda_z}\frac{\partial^3 \Pi}{\partial \lambda_\theta^3} \tag{22}$$

Substituting (17)–(21) into Eqs. (14)–(16), we obtain the following differential sets:

$O\,(\varepsilon)$ equations

$$-2c\frac{\partial u_1}{\partial \xi} + \lambda_\theta \frac{\partial \omega_1}{\partial \xi} = 0,\ -c\frac{\partial \omega_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} = 0,\ p_1 = \gamma_1(u_1 + h) \tag{23}$$

$O\,(\varepsilon^2)$ equations

$$\begin{aligned} -2c\frac{\partial u_2}{\partial \xi} + 2\omega_1\frac{\partial u_1}{\partial \xi} + \lambda_\theta\frac{\partial \omega_2}{\partial \xi} + [u_1 + h]\frac{\partial \omega_1}{\partial \xi} + \lambda_\theta\frac{\partial \omega_1}{\partial \tau} &= 0 \\ -c\frac{\partial \omega_2}{\partial \xi} + \omega_1\frac{\partial \omega_1}{\partial \xi} + \frac{\partial p_2}{\partial \xi} + \frac{\partial p_1}{\partial \tau} - \overline{\nu}\frac{\partial^2 \omega_1}{\partial \xi^2} &= 0 \\ p_2 = (\frac{mc^2}{\lambda_\theta \lambda z} - \gamma_0)\frac{\partial^2 u_1}{\partial \xi^2} + \gamma_1 u_2 + \gamma_2(u_1 + h)^2 & \end{aligned} \tag{24}$$

From the solution of Eqs. (23), we obtain

$$u_1 = U(\xi, \tau),\ \ \omega_1 = \frac{2c}{\lambda_\theta}U,\ \ p_1 = \frac{2c^2}{\lambda_\theta}U + \gamma_1 h \tag{25}$$

where $U(\xi, \tau)$ is an unknown function whose governing equation will be obtained later. The averaged axial velocity ω_1 in Eq. (25) is determined also by a function depending on τ. However if we consider the process in infinity content this function can be removed. Comparing p_1 in Eqs. (23) and (25) leads to the following relationship $\gamma_1 = \frac{2c^2}{\lambda_\theta}$. We introduce (25) in Eqs. (24), and obtain

$$-2c\frac{\partial u_2}{\partial \xi} + \frac{4c}{\lambda_\theta}U\frac{\partial U}{\partial \xi} + \lambda_\theta\frac{\partial \omega_2}{\partial \xi} + 2c\frac{\partial U}{\partial \tau} + \frac{2c}{\lambda_\theta}(U + h)\frac{\partial U}{\partial \xi} = 0 \tag{26}$$

$$-c\frac{\partial \omega_2}{\partial \xi}+\frac{4c^2}{\lambda_\theta^2}U\frac{\partial U}{\partial \xi}+\frac{2c^2}{\lambda_\theta}\frac{\partial U}{\partial \tau}+\gamma_1 h'+\frac{\partial p_2}{\partial \xi}-\frac{4c^2}{\lambda_\theta^2}\overline{\nu}\frac{\partial^2 U}{\partial \xi^2}=0 \tag{27}$$

$$p_2=(\frac{mc^2}{\lambda_\theta \lambda z}-\gamma_0)\frac{\partial^2 U}{\partial \xi^2}+\gamma_1 u_2+\gamma_2 U^2+\gamma_2 h(\tau)U+\gamma_2 h(\tau)^2 \tag{28}$$

Replacing Eq. (28) into Eq. (27), and eliminating ω_2 between Eqs. (26) and (27), the final evolution equation takes the form:

$$\frac{\partial U}{\partial \tau}+\mu_1 U\frac{\partial U}{\partial \xi}-\mu_2\frac{\partial^2 U}{\partial \xi^2}+\mu_3\frac{\partial^3 U}{\partial \xi^3}+\mu_4(\tau)\frac{\partial U}{\partial \xi}+\mu(\tau)=0 \tag{29}$$

where

$$\mu_1=\frac{5}{2\lambda_\theta}+\frac{\gamma_2}{\gamma_1},\quad \mu_2=\frac{\overline{\nu}}{\lambda_\theta},\quad \mu_3=\frac{m}{4\lambda_z}-\frac{\gamma_0}{2\gamma_1}, \tag{30}$$
$$\mu_4(\tau)=h(\tau)(\frac{1}{2\lambda_\theta}+\frac{\gamma_2}{\gamma_1}),\quad \mu(\tau)=\frac{1}{2}h'(\tau)$$

and

$$\gamma_1=\beta_1-\frac{\beta_0}{\lambda_\theta},\ \gamma_2=\beta_2-\frac{\beta_1}{\lambda_\theta} \tag{31}$$

Finally we have to objectify the idealized aneurysm shape. For an idealized abdominal aortic aneurysm (AAA), $h(\tau)=\delta exp(\frac{-\tau^2}{2L^2})$, where δ is the aneurysm height, i.e. $\delta=r_{max}-r_0$, and L is the aneurysm length [11]. In order to normalize these geometric quantities, we non-dimensionalize δ by the inlet radius (diameter). Then, the non-dimensional coefficient can be presented by $\delta'=DI-1$, where $DI=2r_{max}/2r_0=D_{max}/D_0$ is a geometric measure of AAA, which is known as a diameter index or a dilatation index [12]. In the same manner, the aneurysm length L is normalized by the maximum aneurysm diameter (D_{max}), i.e. $l'=L/D_{max}=1/SI$, where SI is a ratio, which is known as a sacular index of AAA [12]. For AAAs, D_{max} varies from 3 to 8.5 cm, and L varies from 5 to 10–12 cm.

4 Analytical Solution for the Nonlinear Evolution Equation: Application of the Modified Method of Simplest Equation

In this section we shall derive a travelling wave solution for the variable coefficients evolution equation, presented by Eq. (29). We shall make change of the function and the variables in the the evolution equation with variable coefficients as follows:

Let us introduce a new dependent variable such as $U(\xi,\tau) = V(\xi,\tau) - \int \mu(\tau)d\tau$. Then Eq. (29) reduces to:

$$\frac{\partial V}{\partial \tau} + \mu_1 V\frac{\partial V}{\partial \xi} - \mu_2\frac{\partial^2 V}{\partial \xi^2} + \mu_3\frac{\partial^3 V}{\partial \xi^3} + [\mu_4(\tau) - \mu_1 \int \mu(\tau)d\tau]\frac{\partial V}{\partial \xi} = 0. \quad (32)$$

Now, we introduce the coordinate transformation

$$\tau' = \tau,\ \xi' = \xi - \int [\mu_4(\tau) - \mu_1 \int \mu(\tau)d\tau]d\tau$$

Then, Eq. (29) is reduced to the generalized KdVB equation:

$$\frac{\partial V}{\partial \tau'} + \mu_1 V\frac{\partial V}{\partial \xi'} - \mu_2\frac{\partial^2 V}{\partial \xi'^2} + \mu_3\frac{\partial^3 V}{\partial \xi'^3} = 0. \quad (33)$$

Next, we shall find an analytical solution of Eq. (33) applying the modified method of simplest equation [13–16]. The short description of the modified method of simplest equation is as follows. First of all by means of an appropriate ansatz (for an example the traveling-wave ansatz) the solved of nonlinear partial differential equation for the unknown function η is reduced to a nonlinear ordinary differential equation that includes η and its derivatives with respect to the traveling wave coordinate ζ

$$\Phi\left(\eta, \eta_\zeta, \eta_{\zeta\zeta}, \dots\right) = 0 \quad (34)$$

Then the finite-series solution

$$\eta(\zeta) = \sum_{\mu=-\kappa}^{\kappa_1} a_\mu [g(\zeta)]^\mu \quad (35)$$

is substituted in (34). a_μ are coefficients and $g(\zeta)$ is solution of simpler ordinary differential equation called simplest equation. Let the result of this substitution be a polynomial of $g(\zeta)$. Equation (35) is a solution of Eq. (34) if all coefficients of the obtained polynomial of $g(\zeta)$ are equal to 0. This condition leads to a system of nonlinear algebraic equations. Each nontrivial solution of the last system leads to a solution of the studied nonlinear partial differential equation. In addition, in order to obtain the solution of Eq. (34) by the above method we have to ensure that each coefficient of the obtained polynomial of $g(\zeta)$ contains at least two terms. To do this within the scope of the modified method of the simplest equation we have to balance the highest powers of $g(\zeta)$ that are obtained from the different terms of the solved equation of kind (34). As a result of this we obtain an additional equation between some of the parameters of the equation and the solution. This equation is called a balance equation.

We introduce transformation of a traveling-wave type, i.e. $\zeta = \xi' - v^*\tau'$, where v^* is the velocity of the traveling wave. We substitute the last expression in Eq. (33)

and obtain:

$$-v^* \frac{dV}{d\zeta} + \mu_1 V \frac{dV}{d\zeta} - \mu_2 \frac{d^2 V}{d\zeta^2} + \mu_3 \frac{d^3 V}{d\zeta^3} = 0. \tag{36}$$

Now we search for solution of Eq. (36) of kind $V = V(\zeta) = \sum_{r=0}^{q} a_r g^r$, where $g_\zeta = \sum_{j=0}^{m} b_j g^j$. Here a_r and b_j are parameters, and $g(\zeta)$ is a solution of some ordinary differential equation, referred to as the simplest equation. The balance equation is $q = 2m - 2$. We assume that $m = 2$, i.e. the equation of Riccati will play the role of simplest equation. Then

$$V = a_0 + a_1 g + a_2 g^2, \;\; \frac{dg}{d\zeta} = b_0 + b_1 g + b_2 g^2 \tag{37}$$

The differential equation of Riccati can be written as

$$\left(\frac{dg}{d\zeta}\right)^2 = c_0 + c_1 g + c_2 g^2 + c_3 g^3 + c_4 g^4 \tag{38}$$

where

$$c_0 = b_0^2; \; c_1 = 2b_0 b_1; \; c_2 = 2b_0 b_2 + b_1^2; \; c_3 = 2b_1 b_2; \; c_4 = b_2^2 \tag{39}$$

and its solutions are given in [14]. The relationships among the coefficients of the solution and the coefficients of the model are derived by solving a system of five algebraic equations, and they are

$$a_0 = -\frac{1}{25} \frac{-3\mu_2^2 - 30\mu_2\mu_3 b_1 + 75\mu_3^2 b_1^2 + 25 v \mu_3}{\mu_1 \mu_3};$$

$$a_1 = -\frac{12}{5} \frac{b_2(5\mu_3 b_1 - \mu_2)}{\mu_1}; \quad a_2 = -12 \frac{\mu_3 b_2^2}{\mu_1}; \quad b_0 = \frac{1}{100} \frac{25\mu_3^2 b_1^2 - \mu_2^2}{b_2 \mu_3^2} \tag{40}$$

Here b_1, b_2 are free parameters. Then substituting (40) in the first equation of (37) the solution of the evolution equation with constant coefficients (Eq. (33)) is

$$V(\zeta) = -\frac{1}{25} \frac{-3\mu_2^2 - 30\mu_2\mu_3 b_1 + 75\mu_3^2 b_1^2 + 25 v \mu_3}{\mu_1 \mu_3} - \tag{41}$$
$$-\frac{12}{5} \frac{b_2(5\mu_3 b_1 - \mu_2)}{\mu_1} g(\zeta) - 12 \frac{\mu_3 b_2^2}{\mu_1} g(\zeta)^2$$

where

$$g(\zeta) = -\frac{b_1}{2b_2} - \frac{\Delta}{2b_2}\tanh\left(\frac{\Delta(\zeta+\zeta_0)}{2}\right) + \tag{42}$$
$$+\frac{\exp\left(\frac{\Delta(\zeta+\zeta_0)}{2}\right)}{2\cosh\left(\frac{\Delta(\zeta+\zeta_0)}{2}\right)\frac{b_2}{\Delta} + 2C^*\exp\left(\frac{\Delta(\zeta+\zeta_0)}{2}\right)\cosh\left(\frac{\Delta(\zeta+\zeta_0)}{2}\right)}$$

In Eq. (42) $\Delta = \sqrt{b_1^2 - 4b_0b_2} > 0$, and ξ_0 and C^* are constants of integration. The solution of the evolution equation with variable coefficients (Eq. (29)) is

$$U(\xi,\tau) = V(\zeta) - \int \mu(\tau)d\tau \tag{43}$$

where

$$\zeta = \xi - v^*\tau - \int [-\mu_1 \int \mu(\tau)d\tau + \mu_4(\tau)]d\tau \tag{44}$$

5 Numerical Findings and Discussions

It is obvious that the wave profile of the radial displacement U (Eq. (43)) depends on the material properties of the arterial wall, on the initial deformations and on the arterial geometry. In order to see their effect on the wave profile of U we need the values of coefficients $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2, \mu_1, \mu_2, \mu_3, \mu_4(\tau)$ and $\mu(\tau)$. For that purpose, the constitutive relation for tube material must be specified. Here, unlike [1–5], we assume that the arterial wall is an incompressible, anisotropic and hyperelastic material. The mechanical behaviour of such a material can be defined by the strain energy function of Fung for arteries [17]:

$$\Pi = C(e^Q - 1), \quad Q = C_1E_{QQ}^2 + C_2E_{ZZ}^2 + 2C_3E_{QQ}E_{ZZ} \tag{45}$$

where E_{QQ} and E_{ZZ} are the Green-Lagrange strains in the circumferential and axial directions, respectively, and C, C_1, C_2, C_3 are material constants. Taking into account that $E_{QQ} = 1/2(\lambda_\theta^2 - 1)$ and $E_{ZZ} = 1/2(\lambda_z^2 - 1)$, we substitute (45) in (22), (30) and (31), and obtain:

$$\begin{aligned}
\beta_0 &= \frac{1}{\lambda_z}\left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1)\right)F(\lambda_\theta\lambda_z)\\
\beta_1 &= \frac{1}{\lambda_z\lambda_\theta}\left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1)\right)\left(1 + \lambda_\theta^2\left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1)\right)\right)F(\lambda_\theta\lambda_z)\\
\beta_2 &= \frac{1}{2\lambda_z}\left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1)\right)^2\left(3 + \lambda_\theta^2\left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1)\right)\right)F(\lambda_\theta\lambda_z)\\
\gamma_0 &= \frac{1}{\lambda_\theta}\left(\frac{C_2}{2} + C_3(\lambda_\theta^2 - 1)\right)F(\lambda_\theta\lambda_z),\ \gamma_1 = \frac{1}{\lambda_z}\left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1)\right)^2F(\lambda_\theta\lambda_z),
\end{aligned} \tag{46}$$

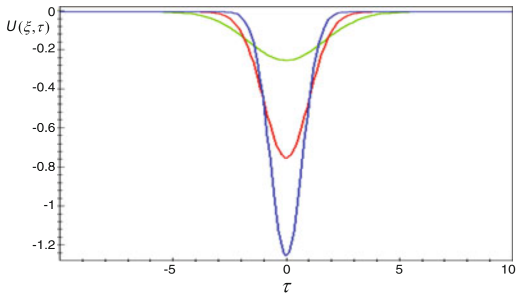

Fig. 1 Variations of the radial displacement for different values of δ' and l': for $\delta' = 0.5, l' = 1.66$ ($D_{max} = 3$ cm) (the *green line* in the figure); for $\delta' = 1.5, l' = 1$ ($D_{max} = 5$ cm) (the *red line* in the figure); for $\delta' = 2.5, l' = 0.7$ ($D_{max} = 7$ cm) (the *blue line* in the figure) ($L = 5$ cm)

$$\gamma_2 = \frac{1}{\lambda_z}(\frac{C_1}{2} + C_3(\lambda_z^2 - 1))(\frac{\lambda_\theta^2}{2}(\frac{C_1}{2} + C_3(\lambda_z^2 - 1))^2 + \frac{5}{2}(\frac{C_1}{2} + C_3(\lambda_z^2 - 1))$$
$$-\frac{1}{\lambda_\theta^2})F(\lambda_\theta \lambda_z)$$

where

$$F(\lambda_\theta \lambda_z) = C \exp(\frac{C_1}{4}(\lambda_\theta^2 - 1) + \frac{C_2}{4}(\lambda_z^2 - 1) + \frac{C_3}{2}(\lambda_\theta^2 - 1)(\lambda_z^2 - 1)) \qquad (47)$$

The numerical values of material coefficients in (45) are as follows: $C = 2.5$ kPa, $C_1 = 14.5, C_2 = 7, C_3 = 0.1$. They were derived in [18] from experimental data of human aortic wall segments applying a specific inverse technique. Assuming the initial deformation $\lambda_z = 1.5, \lambda_\theta = 1.2$, we obtain the following values for the coefficients: $\beta_0 = 554.97$, $\beta_1 = 5374$, $\beta_2 = 27872.89$, $\gamma_0 = 333.36$, $\gamma_1 = 4911.52$, $\gamma_2 = 23394.55$. Then, the numerical values of the coefficients in Eq. (29) are:

$$\mu_1 = 6.85;\ \mu_2 = 2.73.10^{-5} \mathrm{m}^2/\mathrm{s};\ \mu_3 = -0.017; \qquad (48)$$
$$\mu_4(\tau) = 5.36\delta' \exp(-\tau^2/2l'^2), ;\ \mu(\tau) = -\delta'\tau \exp(-\tau^2/2l'^2)/2l'^2.$$

We take into account that $\nu = 3.28.10^{-6}\ \mathrm{m}^2/\mathrm{s}$ when calculating μ_2. Using these numerical values, the travelling-wave solution of Eq. (29) for $\xi = 1$ is plotted in Fig. (1). In all simulations $v^* = 1, m = 0.1$ and $b_1 = 1, b_2 = 1$, which are defined by the symmetry condition at $\tau = 0$ and $\tau = \pm\infty$. In more detail Fig. 1 demonstrates the effect of aneurysm geometrical characteristics such as the maximal aneurysm diameter and in particular the aneurysmal length (*DI* and *SI* indexes of AAA defined in the end of Sect. 3) on the wave profile of wall displacement. Taking into account that the healthy aortic diameter is about 2 cm, various wave profiles of *U* are obtained for various values of the maximal aneurysm diameter D_{max} (in particular δ' or *DI*). In all these cases, a constant aneurysm length *L* is assumed, but l' (in particular *SI*) also varies, because D_{max} involves in this ratio. As it is seen from Fig. (1) wave elastic drop, followed by a prompt wave elastic jump is observed in presence of arterial dilatation. The graph also demonstrates that the wave amplitude increases but wave length decreases when the maximal aneurysm diameter increases. (in particular when *DI* and *SI* of AAA increase). The increasing wave amplitude of the wall displacement can lead to aneurysm rupture. Thus the obtained results are conformable with observations in the medical practice.

6 Conclusions

Modelling the injured artery as a thin-walled prestetched, anisotropic and hyperelastic tube with a local imperfection (an aneurysm), and the blood as a Newtonian fluid we have derived an evolution equation for propagation of nonlinear waves in this complex medium. Numerical values of the model parameters are determined for specific mechanical characteristics of the arterial wall and specific aneurismal geometry. We have obtained a traveling wave analytical solution of the model evolution equation. The numerical simulations of this solution demonstrate that solitary waves are observed when a local arterial dilatation appears.

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