From Formal Theory of Knowledge to Non-Fregean Logic

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Abstract The terms: 'diachronic logic' and 'non-Fregean logic' we owe to Roman Suszko. He called 'diachronic logic' an application of classical logic to study of the development of knowledge. But Non-Fregean logic is a logical calculus obtained from the classical logic by adding identity connective and axioms for it. The main goal of the paper is to proof that the non-Fregean logic is a continuation of diachronic logic. The article is divided into following parts: 1. Diachronic logic, 2. Non-Fregan logic, which contain 2.1. Introduction, 2.2. Axiomatic form of non-Fregean Logic, 2.3. Properties of non-Fregean logic, and Bibliography.

Roman Suszko was logician who united in their research the mathematical form of logic with its philosophical content. One of his work starts with the following epigraph: "Abstract mathematics may be a thorough philosophy". This epigraph may be interpreted in various ways, but the most natural is that we can solve certain philosophical problems in mathematical way, by creating at the beginning their formal representation, and next—by solving formal problems related with them. Solving those related formal problem constitutes at the same time looking for an answer to the initial philosophical question. In practice the formulation of a given philosophical problem in a formally strictly and adequate way is a considerable achievement as such. As a matter of fact we restrict ourselves to creating certain pattern or formal model which has a proper philosophical interpretation.

During 1957–1968 Suszko devoted certain number of works to the formal analysis of development of knowledge; this analysis was done with the aid of the models theory for the classical predicate logic calculus.

Suszko called the research on development of knowledge "diachronic logic".

Hence diachronic logic is application of the classical models theory for first-order predicate languages to research development of knowledge.

Suszko used here eloquent terminology offered earlier by Ferdinand Saussure in the monography *Cours Lingistique Gènèrale*, where it was offered to distinguish synchronic

linguistic from a diachronic one. Suszko defined "synchronic logic" as syntax and semantics of formal languages where classical logic is in force, and "diachronic logic"— as application of synchronic logic to research on development of knowledge. He devoted to those issues several articles in 1957–1968 [7, 9].

In 1968–1979 Suszko worked extremely intensely on non-Fregean logic, which he himself invented. This logic was inspired by Wittgenstein's *Tractatus Logico-Philosophicus*. Suszko invented this logic because he thought that the ontology of Tractatus—considered as certain philosophical theory—goes beyond the formal means that are available in the framework of the classical first-order predicate calculus.

In the present work I am trying to show, that research on non-Fregean logic is continuation of earlier research on diachronic logic.

1   Diachronic Logic

A lot of effort was devoted in the 1950s of twentieth century to problems on the border between formal logic and theory of knowledge. In Suszko's opinion the contemporary logic can describe certain aspects of development of knowledge in a precise way and can throw new light on traditional epistemological questions.

Theory of knowledge explores the epistemological opposition $< S, O >$ where S is a subject of knowledge and O is an object of knowledge. Suszko represented the subject of knowledge S by the sequent

(*) (L, Cn, A, T),

where:

  (i) L is a formalized version of language used by the subject of knowledge,
 (ii) Cn is operation of consequence defined on language L,
(iii) A is a set of analytical axioms formulated in language L,
(iv) T is set of sentences of the language L accepted by the subject of knowledge S.

Every sentence of language L represents certain thought, and every predicate of language L represents certain notion at the disposal of subject S. Operation of consequence Cn defined on language L, is the totality of logical thinking rules. Set of analytical axioms A contains logical and extra-logical thinking principles, which are at the disposal of subject. As far as the set of accepted sentences T is concerned we assume that $Cn(T) \neq T$, because subject of knowledge usually does not know all logical consequences accepted by himself and does not know all of his own assumptions. It is assumed also that $T - Cn(A) \neq \emptyset$, i.e. that the subject accepts certain sentences that are not true in an analytical way.

In order to study in a formal way the epistemological opposition $< S, O >$, the reality which is the object of knowledge has also to be defined in a strict way. For a finite subject the world as a the whole reality R is never an object of knowledge; at every moment t subject of knowledge sees only certain fragment of the world, which at the moment t becomes an object of knowledge O_t for subject S. Suszko calls the period during which given subject studies one and the same fragment of the world "epoch in development of

knowledge". If during the given period t certain fragment of knowledge constitutes an object of knowledge, subject attaches to this world-fragment certain language in order to speak about it. Speaking loosely, the object of discursive knowledge O on a given stage of development of knowledge constitutes such a fragment of reality R, that may be caught by the net of notions being at the disposal of the subject, i.e. such a fragment, which serves as a model for language L used by the subject. For the sake of simplicity Suszko assumes that the language used by subject of knowledge is certain first-order predicate language L, and subject of knowledge is represented by certain intended model of language, it means by the structure of the following type:

$$M = < U, d(C_1), d(C_2), \ldots >,$$

where: U is universe of language L i.e. it is a set, from which the nominal variables of the language take their values, and $d(C_k)$ is denotation of the extra-logical constant C_k.

Selected objects from the set U, or certain sets of objects which represent adequate properties of objects and relations and relations between objects or—possibly—functions defined on the set of objects U serve as denotations of extra-logical constants of the given language. Therefore epistemological opposition
$< S, O >$ is represented by Suszko as the following:

(**) $< (L, Cn, A, T), M >,$

where (L, Cn, A, T) represents a subject of knowledge and model M of the language L constitutes a formal representation of object of knowledge. One of the consequences of the assumption, that the subject is equipped with the language L, which is—with the first-order predicate language is the following: we represent the object of knowledge with the intended model of the language.

For a given epistemological opposition we mark the set of true sentences of the language L in model M as $Ver(L, M)$. Let us notice that we have here to do with a double relativisation of the notion of truth: to the language and to the model.

Because the language L is a language defined by the structure of the model M, we can mark this set shorter with the symbol $Ver(M)$.

Research on the formal epistemological opposition (**) is a matter of synchronic logic. In turn research on changes of this opposition in time belongs to diachronic logic.

Development of knowledge consists—according to apt formulation of Suszko—in gaining more and more amount of truths about wider and wider object of knowledge. Within the framework of diachronic logic a development of knowledge is represented by transformation of epistemological oppositions

$$< (L, Cn, A, T), M > / < (L^*, Cn^*, A^*, T^*), M^* >$$

in such a way, that the set of sentences which are simultaneously true and which are accepted by the subject at the next stage of knowledge contains the set of sentences which were true and were accepted by the subject at the previous knowledge stage, i.e.

$$T \cap Ver(M) \subseteq T^* \cap Ver(M^*).$$

Presenting the basic ideas of diachronic logic Suszko used eloquent terminology, which—among others—was a result of modification of Ajdukiewicz's terminology used in works Ajdukiewicz [1, 2].

Let us present briefly those terminology. During every given period of development of knowledge t subject of knowledge sees certain fragment of the really existing reality R, which is called by Suszko "the world-layer in period t" and which is represented in the system of diachronic logic by certain model M_t. The period during which the subject sees one and the same layer of the world is called by Suszko an "epoch in development of knowledge". While initial research on object of knowledge, i.e. of a given world-layer, subject attaches the language which fits for speaking about this fragment of the world. Language L_t tailored to speak about the world-layer in epoch t is called by Suszko "conceptual apparatus in epoch t".

Let $< (L_t, Cn, A_t, T_t), M_t >$ be the epistemological opposition in a given epoch t. Suszko uses further the following terminology: the set of sentences T, accepted by the subject of knowledge is called by him the picture of the world in epoch t and in turn the set $T_t \cap Ver(M_t)$ is a true fragment of world-picture in epoch t, or "real world-knowledge in epoch t". Suszko calls the set $Cn(T_t)$ a "world-perspective in epoch t" or "potential world-knowledge in epoch t" and the set of sentences $Cn(T_t) \cap Ver(M_t)$ constitutes than "true fragment of world-perspective in epoch t".

Suszko distinguishes two main types of knowledge development:

(1) evolutionary, by which the object of knowledge does not change, what means that the subject of knowledge sees the same world-layer and the syntactic structure of the language remains also unchanged.
(2) revolutionary, by which the object of knowledge does change.

The evolutionary process of knowledge consists above all in the situation where the sequence of the sets of sentences accepted by subject of knowledge $T, T^*, T^{**}, \ldots$ contains more and more true sentences about the same object of knowledge. It happens that during the evolutionary development of knowledge the extra-logical principles of thinking change, i.e. set of axioms A becomes set of axioms A^*. Suszko considers following two cases of that kind:

(i) $A \neq A^*$ and $Cn(A) = Cn(A^*)$,
(ii) $A \subseteq A^*$ and $Cn(A) \neq Cn(A^*)$.

In the case (i) the systematization of axioms takes place. In the case (ii) we have to do with reinforcement of axioms. The case (ii) embraces the following sub-case:

$$(L, Cn, A, T)/(L, Cn, A^*, T^*)/(L, Cn, A^{**}, T^{**})$$

which consists in fact that certain sentence α the given language L, is initially not accepted, i.e. $\alpha \notin T$ and the development of knowledge goes in such a way, that at the beginning α becomes a non-analytical theorem, $\alpha \in (T^* - Cn(A^*))$, and at the next stage of knowledge this sentence becomes one of the analytical sentences of given language, i.e. $\alpha \in Cn(A^{**})$; according to Suszko this kind of development of knowledge was noticed by conventionalists. Evolutionary development of knowledge corresponds with what T.S. Kuhn calls in his book *The structure of Scientific Revolutions*, Chicago, 1962 "normal stage of science development".

In turn revolutionary development of knowledge consists in the situation, in which the object of knowledge—represented here by the model of language—change. It means that the change M/M^* takes place. Suszko shows two kinds of such a change:

(2a) The universe of objects U does not change, but new properties of objects and new relations between them are discovered, and—as a result—new model M^* is built, which constitutes an extension of model M

$$M = < U, R_1, R_2, \ldots, R_n > /M^* = < U, R_1, R_2, \ldots, R_n, Q_1, Q_2, \ldots, Q_m >$$

Such an extension of knowledge object causes an extension of the language L to language L^*, which corresponds to the model M^*. In this case

$$L \cap Ver(L^*, M^*) = Ver(L, M).$$

This condition means, that all sentences of language L which are true in model M remain to be true also in model M^*. Additionally there are in model M^* also true sentences, that were impossible to formulate in language L:

$$Ver(L^*, M^*) - Ver(L, M) \neq \emptyset$$

(2b) Universe of model $M = < U, R_1, R_2, \ldots, R_n >$ becomes extended and as a result the object of knowledge becomes $M^* = < U^*, R_1^*, R_2^*, \ldots, R_n^* >$, where $U \subseteq U^*$ and $U \neq U^*$, and relations R_i^* for $i = 1, 2, \ldots, n$ are extensions of—accordingly— relations R_i. New objects become then known. Model M constitutes sub-model of model M^*, it may happen that $Ver(L, M) \neq Ver(L, M^*)$ and certain sentences that are true in model M may not be true in model M^*.

Suszko considered the following as very important: the theory of models can be applied to formulate precisely and to study everything what traditional theory of knowledge had studied just intuitively. Suszko was fascinated by the possibility of precise grasping the traditional issues, although he remained skeptical about the possibility to achieve in such a way any more important results.

Suszko's works on diachronic logic constitute one of the first in the world attempts to apply the theory of models to extra-mathematical questions, and especially—to philosophical problems. His works initiated applications of the theory of models to methodological research carried out in Poland. Those research flourished in Poland especially in 1960s and 1970s of twentieth century.

2   Non-Fregean Logic

2.1   Introduction

During "diachronic logic" period Suszko assumed that subject of knowledge S is equipped with language L, which was the language of the classical predicate calculus.

One of the consequences of that assumption was that object of knowledge M is a model the language L, i.e. it is a structure of the type: $(U, R_1, R_2, \ldots, R_n)$. From philosophical point of view it means that the world is considered as universe of objects which inhere properties and stay in certain relations. Names of the language L refer to objects which are elements of the universe of the model M. One-place predicates refer to properties of objects and many-place predicates refer to relations occurring between objects. The question arises: what sentences of language L in model M refer to?

Answers this question are in papers Suszko [6, 8]. He introduces there the notion of generalized denotation for sentence formulas. And namely: any sentence formula $\alpha(x)$ of language L refers in a model M to all of those objects which satisfy this formula in the model M, i.e. $\{x \in U : \alpha(x)\}$. If the formula $\alpha(x)$ is a sentence, it means, if there are no free variables in the formula, then whether every object of the universe satisfies it or no object in the universe satisfies it. Hence there are in model M only two generalized denotations of sentences and all true sentences have the same generalized denotation and all false sentences have one common generalized denotation. The generalized denotation of true sentence in model M constitutes universe of model U, and the generalized denotation of false sentence is empty set. Because the sum total of generalized denotations of sentences in model M consists of two elements, both sentence variables and quantifiers binding those variables are redundant. What is important from our point of view, Suszko almost from the beginning of his scientific carrier assigned to sentences not only logical values (truth and false) but also semantic correlates, which he called "generalized denotations of sentences".

If set of generalized denotations of sentences equipped in set-theoretical operations corresponding with logical connective, then algebra semantic correlate sentences will be isomorphic with two-elements algebra of logical values.

Because of that isomorphism, Frege could supposed that sentences are names of special objects called "logical values of sentences".

Under the influence of *Tractatus* Suszko modified his view and started to consider situation presented in a sentence as semantic correlate of this sentence. Since Wittgenstein wrote in Tractatus: 4.03 [. . .] *A proposition communicates a situation to us, and so it must be essentially connected with the situation. And the connection is precisely that it is its logical picture.*

Besides Suszko was convinced that logic should not impose any quantitative restraints on the universe of semantic correlates, except the one: there are at least two correlates of sentences, because correlate of any true sentence is different from a correlate of false sentence.

The name "non-Fregean logic" is justified by the fact that in this logic there no theorems asserting how many semantic correlates of sentences there can be.

At the base of non-Fregean logic lies also convictions, that syntactic categories of linguistic expressions should conform to ontological categories of theirs semantic correlates. On behalf of that conformity Suszko postulated—after Wittgenstein—that situations stated by the sentences constitute semantic correlates of these sentences, and sentential variables take their values from the universe of all situations correlated with a given language.

Syntax and semantics of non-Fregean logic displays a logic-philosophical parallelism between language and reality: we have names, functors and quantifiers in language

and—objects, situations and functions in the reality; and functors and quantifiers refer to certain kind of functions.

However one can't conclude from above, that every object in a given universe of our discourse has a name, but one can conclude that every object may be a value of certain name variable. Similarly not every situation or a state of affairs occurring between objects of our discourse's universe may be described with sentences of our language, but if there are in the language sentential variables, then every of those situations may be a value of certain sentential variable. Let us notice that sentential variable differ fundamentally from other kinds of variables, because they are at the same time sentential formulas, and therefore they enter into logical connections with the rest of sentences and sentential formulas of a given language. Because of that the logical consequence influences the interpretation of sentential formulas.

2.2 Axiomatic Form of Non-Fregean Logic

To speak in formal way about the structure of universe of situations and universe of objects, Suszko introduced to literature of logic languages which he called W-languages (in honor of L. Wittgenstein). In the alphabet of these languages, there are:

(1) two kinds variables: sentential variables: $p, q, r, \ldots$, and nominal variables: $x, y, z, \ldots$; (2) truth-functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\leftrightarrow$ (implication), $\leftrightarrow$ (equivalence); (3) predicate-letters: $P_1, P_2, \ldots, P_n$; (4) function-symbols: $F_1, F_2, \ldots, F_m$; (5) symbols identity: identity connective and identity predicate which both symbolized by the sign "$\equiv$"; (6) quantifiers: $\forall, \exists$ binding both kinds of variables.

Each of the quantifiers may bind both sentential variable or nominal one, depending on which variable follows directly after it. Analogous the context uniquely determines whether we have to do with identity connective or identity predicate since the expression $x \equiv p$ is not a formula of the language discussed. A detailed description of the syntax of the W-kind languages has been presented in the papers: Bloom [3], Suszko [10, 12] Operation Cn on L is generated by the Modus Ponens rule and the schemas of logical axioms. To describe the consequence Cn in W-languages the following notations are introduced:

Letters: $v, w, v_1, w_1, v_2, w_2, \ldots$ will be metalanguage variables denoted depending on the context, either sentential variables or the nominal variables. By the letters: $\alpha, \beta, \gamma, \ldots$, will be denote any sentential formulas, by the letters: $\zeta, \xi, \tau, \ldots$, we denote any nominal formulas, and finally: $\phi, \varphi, \psi, \ldots$, denote sentential formulas or nominal ones, depending on the context. Symbols $\alpha[v/\phi]$ denoted result substitution in formula $\alpha(v)$ for free variable v the expression ϕ.

The result of proceeding the formula α by any finite number of universal quantifiers, i.e. $\forall v_1 \forall v_2 \ldots \forall v_n \alpha$, where $n \geq 0$ is called generalization of the formula α. For any set of sentential formulas X, by $Gen(X)$ will be denoted the set of all generalizations of formulas in the set X.

The formulas of the form $\phi \equiv \varphi$, are called equations.

The structural version non-Fregean logic in W-language L is introduced by accepting logical axioms and the only inference rule Modus Ponens.

The logical axioms are those formulas which are generalizations of any formula of the following sorts:

(**A1**) Axiom Schemata for truth-functional connective (they are classical)

(**A2**) Axiom Schemata for quantifiers

> (i) $\forall v\alpha \rightarrow \alpha[v/\phi]$
> (ii) $\alpha \rightarrow \forall v\alpha$ (if v is not free in α)
> (iii) $\forall v(\alpha \rightarrow \beta) \rightarrow (\forall v\alpha \rightarrow \forall v\beta)$
> (iv) $\exists v\alpha \leftrightarrow \neg\forall v\neg\alpha$

(**A3**) Axiom Schemata for identity connective and predicate:

> (**A3.1**) Congruence axioms. All formulas of the form:
>
> > (i) $\varphi \equiv \phi$ (when ϕ, φ vary in at most bound variables)
> >   for every functor Ψ we accept the invariance axiom:
> > (ii) $\varphi_1 \equiv \phi_1 \wedge \varphi_2 \equiv \phi_2 \wedge \ldots \wedge \varphi_n \equiv \phi_n \rightarrow \Psi(\varphi_1, \varphi_2, \ldots, \varphi_n) \equiv \Psi(\phi_1, \phi_2, \ldots, \phi_n)$
> > (iii) $\forall v(\alpha \leftrightarrow \beta) \rightarrow (Qv\alpha \equiv Qv\beta)$, where $Q = \forall, \exists$.
>
> (**A3.2**) Special axiom for identity:
> $\varphi \equiv \phi \rightarrow (\alpha[v/\varphi]) \rightarrow \alpha[v/\phi])$

The set logical axioms **LA** is the sum of three sets: **A1**, **A2**, **A3** i.e. **LA = A1 ∪ A2 ∪ A3**.

A set of all the sentential formulas which are derivable from any set X and from logical axioms in any finite number of steps through the application of MP rule is called theory and is denoted by $Cn(X)$.

A formula α is called logical theorem non-Fregean logic iff $\alpha \in Cn(\emptyset)$.

2.3 Properties of Non-Fregean Logic

If in logical theorems of non-Fregean logic we replace at every place the sign "$\equiv$" with sign "$\leftrightarrow$", then we receive theorems of classical logic. It means that the non-Fregean logic constitutes a generalisation of classical logic and the classical logic constitutes a reinforcement of non-Fregean logic.

The sentence:

(AF) $\forall p\forall q[(p \equiv q) \equiv (p \leftrightarrow q)]$

Suszko called "ontological version of Frege's axiom". In non-Fregean theories, in which (AF) is a theorem the connectives "$\equiv$" and "$\leftrightarrow$" are indistinguishable. The non-Fregean theorie in which (AF) is a theorem are classical theories expressed in non-Fregean language. We can also derive classical logic by addition to its theorems the—seemingly weaker than (AF)—axiom:

$$\forall p\forall q[(p \leftrightarrow q) \rightarrow (p \equiv q)]$$

From the philosophical point of view the most important properties of non-Fregean logic is its logical bivalence and extentionality. The logical bivalence of non-Fregean logic finds expression in the following theorem of that logic:

$$\forall p \forall q \forall r [(p \leftrightarrow q) \vee (q \leftrightarrow r) \vee (p \leftrightarrow r)]$$

In turn the extentionality of this logic finds its expression in the fact, that schemas (2) and:

$$\varphi \equiv \phi \rightarrow (\alpha[v/\varphi] \equiv \alpha[v/\phi])$$
$$\varphi \equiv \phi \rightarrow (\alpha[v/\varphi] \rightarrow \alpha[v/\phi])$$

are schemas of logical theorems. These schemas state that expressions that have the same semantical correlates are mutually interchangeable in all sentential contexts without accordingly changing semantic correlate of those contexts (salva identitate) and without changing logical values of those contexts (salva veritate).

To logical theorems of non-Fregean logic belong theorems:

$$\exists x (x \equiv x), \qquad \exists p \exists q \neg (p \equiv q)$$

which state accordingly that the universe of objects is non-empty and that universe of situations contains at least two elements. To logical theorems of non-Fregean logic though do not belong any conditions, that limit "from above" the number of objects and situations in universe, what means that for every natural number n the following formulas are not logical theorems:

$$(x_1 \equiv x_2) \vee (x_1 \equiv x_3) \vee \ldots \vee (x_{n-1} \equiv x_n)$$
$$(p_1 \equiv p_2) \vee (p_1 \equiv p_3) \vee \ldots \vee (p_{n-1} \equiv p_n)$$

Non-Fregean logic—as every calculus—can be developed without any philosophical presumptions. Nonetheless for this logic the ontology of situations contained in *Tractatus* served as a fundament. This logic presumes the ontology, according to which there exist objects, situations and functions. Suszko introduced attention to the fact that the division all of beings into objects, situations and functions has logical character, i.e. it results the fact, that we describe the world with languages in which we have names, predicates, connectives and quantifiers.

Suszko introduced non-Fregean logic because he was convinced that there exist in the world certain beings and aspects of beings, which can be properly told about with the aid of sentential variables. In other words, the ontology that underlies non-Fregean logic contains the view, that it is not enough to consider the world as the universe of objects only, but we have to consider it also as the universe of possibilities, among which some of them become realized, i.e.—there are facts. In [11] Suszko wrote: "…*perceiving an object x consists of perceiving at least one situation that x is so-an-so*".

It is worth stressing that the importance of non-Fregean logic goes beyond its ontological applications. Since this logic has initiated research on abstract logics. Those logics constitutes new chapter in application of algebra to logic. This problems are discussed among others in monographs Czelakowski [4], Dzik [5].

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