Chapter 7
Introduction to Elliptic Fibrations

Mboyo Esole

Abstract The modern study of elliptic fibrations started in the early 1960s with seminal works by Kodaira and by Néron. Elliptic fibrations play a central role in the classification of algebraic surfaces, in many aspects of arithmetic geometry, theoretical physics, and string geometry. In these notes, we introduce the reader to basic geometric properties of elliptic fibrations over the complex numbers. We start with an introduction to the geometry of elliptic curves defined over the complex numbers. We then discuss Weierstrass models, Kodaira's classification of singular fibers of elliptic surfaces, Tate's algorithm, and Miranda's regularization of elliptic threefolds.

7.1 Introduction

The theory of elliptic curves is an elegant and vast subject in mathematics that can be traced back to ancient Greece and beyond. An elliptic curve is a non-singular projective curve of genus one, with a choice of a rational point. The chosen rational point plays the role of the neutral element of the Mordell–Weil group of the elliptic curve. An elliptic fibration is the relative case of an elliptic curve. Intuitively, an elliptic fibration is the variety swapped by an elliptic curve moving over a base variety. The study of elliptic fibrations started in 1962–1963 with Kodaira's work on compact complex analytic surfaces [12] followed in 1964 by Néron's paper on minimal models of Abelian varieties [19].

Elliptic curves are a pillar of number theory; they are instrumental in cryptography and geometric modeling. Elliptic curves have also invaded many branches of theoretical physics through their modular properties. Elliptic fibrations are at the heart of F-theory, the theory that describes (among other things) the non-perturbative regime

M. Esole (✉)
Department of Mathematics, Northeastern University, 360 Huntington Ave., Boston, MA 02115, USA
e-mail: j.esole@northeastern.edu

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of type IIB string theory. Elliptic fibrations also provide geometric constructions of certain superconformal field theories including some that do not have a Lagrangian description.

In these notes, we will focus on the basic properties of elliptic fibrations over the complex numbers. We do not have space for complete proofs, but we will give appropriate references. In Sect. 7.2, we review the theory of elliptic curves over the complex numbers. In Sect. 7.3, we study the theory of elliptic fibrations. In particular, we start in Sect. 7.3.1 by reviewing the Riemann–Roch argument to derive the Weierstrass model of an elliptic curve. In Sects. 7.3.2 and 7.3.3, we explain in detail how the Riemann–Roch argument is combined with an appropriate base change theorem to obtain Weierstrass models for an elliptic fibrations. In Sect. 7.4, we introduce the Kodaira–Néron classification of singular fibers of a minimal elliptic surface and discuss Tate's algorithm. In Sect. 7.5, we study Miranda's regularization of elliptic threefolds and the notion of collisions of singularities.

There are many important questions that we will not address. As an apology, we give the following reading list for elliptic fibrations and Weierstrass models:

- The classical reference for Weierstrass models is the original paper of Deligne (in French) known as the "Formulaire" [4]. Deligne beautifully explains how to derive a Weierstrass model for an elliptic fibration with a ration section. It also introduces Tate's notation widely used today. The construction of Weierstrass models is also discussed in detail by Mumford and Suominen in [16, Chap. 3] and Nakayama [17, 18].
- The original paper of Kodaira on elliptic surfaces [12], Néron [19], and Tate [28] contain significant details not usually covered in reviews.
- In [21], Schütt and Shioda give a short introduction to the theory of elliptic surfaces.
- Chapter 3 of the book of Mumford and Suominen on the theory of moduli [16] has a self-contained section on elliptic curves and elliptic fibrations where the authors carefully derive the existence of a Weierstrass model for an elliptic fibration with a rational section.
- For more advanced topics, we refer to Liu's book on arithmetic geometry [13].
- Miranda's lecture notes on elliptic surfaces [15] are another classic review for the study of elliptic surfaces over an algebraically closed field.
- Nakayama analyzes the global and local structure of elliptic fibrations [17, 18]. He takes the interesting point of view of the variation of Hodge structure to describe elliptic fibrations. He shows that a polarized variation of Hodge structures of rank two, weight one over a base B is equivalent to a Weierstrass model.
- In [14], Miranda studies the problem of finding regular models for Weierstrass models over a smooth surface. He discusses the phenomena of collisions of Kodaira fibers and classifies the singular fibers that appear over codimension two points after the specific regularization that he considers. These are some of the first examples of non-Kodaira singular fibers.
- In his Ph.D. thesis [26], Szydlo generalizes the regularization of Miranda to the case of elliptic n-folds under the same assumptions as Miranda. He also considers the arithmetic case, when the field is not of characteristic zero and provides a

generalization of Tate's algorithm to the case of a complete discrete valuation ring with non-perfect residue field [27].

- Dolgachev and Gross have computed the Ogg–Shfarevich Theory of elliptic three-folds using Miranda's models [7].
- Conrad has an elegant unpublished paper on minimal models for elliptic curves with a strong EGA flavor in which he promises to "free the theory of elliptic curves from the curse of Weierstrass equations" [2]. However, before doing this, he presents a systematic derivation of the Weierstrass equation over $Spec(R)$.
- In Chapter IX of [1], Beauville gives a short introduction to the theory of elliptic surfaces from the point of view of the Kodaira dimension. Cossec and Dolgachev study genus-one fibration in Chap. 5 of [3].

7.2 Elliptic Curves over $\mathbb{C}$

In this section, we collect basic facts about elliptic curves over the complex numbers. This topic is elegantly covered in numerous books. For the proofs, we refer to Chap. 1 (Sects. 1–6) of [23], Chap. 3 of [11], and Chap. VII of [22].

We denote by $\mathbb{C}$ the field of complex numbers and by $\mathbb{Z}$ the ring of integers.

7.2.1 *Modular Group and Complex Tori*

Modulo similitude transformations, an elliptic curve over the complex numbers is equivalent to a complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, that is, the quotient of the complex plane by the double-lattice $\mathbb{Z}+\tau\mathbb{Z}$ generated by 1 and the complex number τ (the *period*). The Abelian group structure on the elliptic curve is then induced from the addition in $\mathbb{C}$. Geometrically, the period τ characterizes the shape of the complex torus. By convention, τ is restricted to be in the upper-half plane:

$$\mathscr{H} = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}. \tag{7.1}$$

More generally, for a complex torus $\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ with periods (ω_1, ω_2), after a rescaling by ω_1^{-1}, we get $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\tau = \frac{\omega_2}{\omega_1}$. We can permute ω_1 and ω_2 if necessary to ensure that $\operatorname{Im}(\tau) > 0$ (Fig. 7.1).

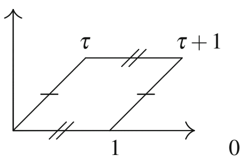


Fig. 7.1 Torus seen as the quotient $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$

Theorem 7.1 *Two 2-tori are equivalent modulo similitudes if and only if their periods are related by a modular transformation:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{7.2}$$

Proof See Lemmas 1.1 and 1.2 in Chap. 1. Sect. 1 of [23].

In particular, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, $Im(\tau)$ (the imaginary part of τ) transforms as

$$\mathrm{Im}(g \cdot \tau) = \frac{\mathrm{Im}\tau}{|c\tau + d|^2}. \tag{7.3}$$

We denote by I_2 the 2×2 identity matrix. Since the matrix $(-\mathrm{I}_2)$ acts trivially on τ, to have a faithful action, we consider the *modular group* to be the quotient

$$\Gamma(1) := \mathrm{SL}(2, \mathbb{Z})/\{\pm \mathrm{I}_2\}. \tag{7.4}$$

We use the same symbol for a matrix in $\mathrm{SL}(2, \mathbb{Z})$ and its projection to $\Gamma(1)$.

Theorem 7.2 *The group $SL(2, \mathbb{Z})$ is generated by the following two elements:*

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{7.5}$$

They act as

$$S \cdot \tau = \frac{-1}{\tau}, \quad T \cdot \tau = \tau + 1. \tag{7.6}$$

S and T satisfy the following relations in $SL(2, \mathbb{Z})$:

$$S^2 = (ST)^3 = -\mathrm{I}_2. \tag{7.7}$$

Proof See Remark 1.3 on p. 10 of [23].

When S and T are considered as elements of $\Gamma(1)$, we have $S^2 = (ST)^3 = \mathrm{Id}$ so that $\Gamma(1)$ can be considered as the free group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$:

$$\Gamma(1) \cong \langle a, b : a^2 = b^3 = 1 \rangle. \tag{7.8}$$

7.2.2 The Weierstrass Equation

The Weierstrass $\wp$-function provides a natural description of a complex torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ as a cubic curve in $\mathbb{P}^2$ in Weierstrass form. It is defined as follows:

$$\wp(z,\tau)=\frac{1}{z^2}+\sum_{\substack{w\in\Lambda_\tau\\ w\neq 0}}\Big(\frac{1}{(z-w)^2}-\frac{1}{w^2}\Big), \tag{7.9}$$

where $\Lambda_\tau=\mathbb{Z}+\tau\mathbb{Z}$. The Weierstrass $\wp$-function is a meromorphic function with double poles at the lattice points $w\in\Lambda_\tau$, and doubly periodic:

$$\wp(z+1,\tau)=\wp(z,\tau),\quad \wp(z+\tau,\tau)=\wp(z,\tau). \tag{7.10}$$

The Weierstrass $\wp$-function has a pole of order 2 at the origin, while its derivative $\wp'$ (with respect to z) has a pole of order 3. Together, they satisfy the Weierstrass equation

$$(\wp')^2=4\wp^3-g_2\wp-g_3,\quad \text{where } g_2(\tau):=60G_4(\tau) \text{ and } \quad g_3(\tau):=140G_6(\tau). \tag{7.11}$$

For a given lattice Λ_τ, the Eisenstein series G_{2k} of weight $2k$ are by definition

$$G_{2k}(\tau)=\sum_{\substack{w\in\Lambda_\tau\\ w\neq 0}} w^{-2k} \tag{7.12}$$

Theorem 7.3 *The map*

$$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\to\mathbb{P}^2 : z\mapsto\left[\wp:\frac{1}{2}\wp':1\right], \tag{7.13}$$

provides an analytic isomorphism between the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the following cubic in $\mathbb{P}^2$:

$$E : zy^2=x^3+fxz^2+gz^3, \tag{7.14}$$

with $f=-g_2/4$, $g=-g_3/4$, $y=\wp'/2$, $x=\wp$. For a regular curve $E : y^2=x^3+fx+g$, there is a unique lattice Λ_τ (up to modular transformation on τ) such that E and $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ are analytic isomorphic as complex Lie groups through the previous map.

Proof Corollary 4.3 on p. 35 of [23].

7.2.3 Moduli Space of Smooth Elliptic Curves

To classify smooth elliptic curves up to isomorphisms, we introduce the *Klein j-invariant* (also called the modular invariant).The j-invariant is a rational function of G_4^3/G_6^2, which ensures that it is a modular invariant. Any modular invariant is a rational function of G_4^3/G_6^2 or equivalently a rational function of the j-invariant. The j-invariant maps bijectively the moduli space of complex tori modulo similitudes (and therefore the moduli space of smooth elliptic curves) to the complex plane $\mathbb{C}$. Two elliptic curves over $\mathbb{C}$ are isomorphic if and only if they have the same j-invariant.

Definition 7.1 The j-invariant of a Weierstrass equation $y^2 = x^3 + fx + g$ is defined as follows:

$$j(\tau) = 1728 \frac{4f^3}{4f^3 + 27g^2} = 1728 - \frac{27g^2}{4f^3 + 27g^2}. \tag{7.15}$$

The coefficient 1728 is chosen to ensure that the j-invariant has residue 1 at infinity.

Theorem 7.4 *The j-invariant can be expressed (as a function of τ) by a Laurent series in $q = \exp(2\pi i \tau)$ of the form:*

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n>0} c_n q^n, \quad c_n \in \mathbb{N}. \tag{7.16}$$

Since $\text{Im}\tau > 0, q = \exp(2\pi i\tau)$ is in the unit disk $|q| < 1$. The modular group admits as a *fundamental domain* the closure of the open region:

$$R_\Gamma = \{\tau \in \mathscr{H} : |\tau + \overline{\tau}| < 1 \text{ and } |\tau| > 1\}, \tag{7.17}$$

with a $\mathbb{Z}/2\mathbb{Z}$ identification on the boundary given by $\tau \cong -\overline{\tau}$. When we have to make a choice between two points on the boundary, we will take the one with negative real part. We recall some additional properties of the j-invariant:

$$j(\mathrm{i}) = 1728, \quad j(\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}) = 0, \quad j(-\overline{\tau}) = \overline{j(\tau)}, \quad \lim_{Im(\tau)\to+\infty} |j(\tau)| = \infty. \tag{7.18}$$

Geometrically, the moduli space of complex tori modulo similitude is the orbifold

$$Y(1) := \mathscr{H}/\Gamma(1). \tag{7.19}$$

If is useful to also include tori admitting an infinite value for the j-invariant. This corresponds to allowing an infinite value for the imaginary part of τ. By the action of the modular group, we should then also include all the rational points of the real line. This defines the *extended upper-half plane*

$$\mathscr{H}^* := \mathscr{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathscr{H} \cup \mathbb{Q} \cup \{\infty\} \quad \text{and } X(1) := \mathscr{H}^*/\Gamma(1). \tag{7.20}$$

$X(1)$ is called the *modular curve*. The points of $X(1)\backslash Y(1)$ are called the *cups*. They are the orbit of $\tau = i\infty$ under the action of $\Gamma(1)$. The name cusp can be confusing as $\tau = i\infty$ actually corresponds to a nodal elliptic curve, but the name cusp in this context refers to the singularities of $X(1)$ and not to a singular elliptic curve.

The following classical theorem is proven, for example, in Sect. 4.1 of Chap. 1 of [23].

Theorem 7.5 *The j-invariant is an isomorphism between $X(1)$ and the Riemann sphere $\mathbb{P}^1$:*

$$j : X(1) \to \mathbb{P}^1 : \tau \mapsto [U : V] = [1728 \cdot 4f^3 : 4f^3 + 27g^2], \tag{7.21}$$

where $[U : V]$ denotes the projective coordinates of $\mathbb{P}^1$. The value of the j-invariant at $\tau = i\infty$ is the point at infinity $[1 : 0]$.

One can define an appropriate topology and complex structure on the modular curve $X(1)$. This is explained in Chap. 1 of [23]. Every meromorphic function on $X(1)$ is then a rational function of j. For this reason, the j-invariant is also called the *modular invariant*.

Theorem 7.6 *There is an elliptic curve with a given j-invariant for any $j_0 \in \mathbb{C}$:*

$$y^2z = x^3 + gz^3, \quad g \neq 0, j_0 = 0, \tag{7.22}$$

$$y^2z = x^3 + fxz^2, \quad f \neq 0, j_0 = 1728, \tag{7.23}$$

$$y^2z = x^3 - \frac{27j_0\lambda^2}{4(j_0 - 1728)}xz^2 - \frac{27\lambda^3 j_0}{4(j_0 - 1728)}z^3, \quad \lambda \neq 0, j_0 \neq 0, 1728. \tag{7.24}$$

Proof Direct computation using the definition of the j-invariant.

Remark 7.1 A Weierstrass equation with a nodal singularity is given, for example, by the following equation:

$$zy^2 = x^3 - 3xz^2 + 2z^3. \tag{7.25}$$

The presence of a node can be seen by factorizing the r.h.s. to get

$$zy^2 = (x - z)^2(x + 2z). \tag{7.26}$$

Such a curve has an infinite j-invariant since f and g are nonzero, while the discriminant vanishes.

Theorem 7.7 (Automorphism of an elliptic curve) *The group of automorphisms of an elliptic curve E_j with invariant j is $\mathbb{Z}/2\mathbb{Z}$ for $j \neq 0, 1728$. It is $\mathbb{Z}/4\mathbb{Z}$ for $j = 1728$ and $\mathbb{Z}/6\mathbb{Z}$ for $j = 0$.*

Proof See [11, Chap. 3 Sect. 4].

Remark 7.2 (*Ramifications and Automorphisms of elliptic curves*) Since the j-invariant can be expressed as $j = 1728(4f^3)/(4f^3 + 27g^2)$, it has a ramification of degree 3 at $f = 0$ for which $j = 0$. As we can also write $j - 1728 = -27g^2/(4f^3 + 27g^2)$, there is also a ramification of degree 2 at $g = 0$ for which

$j = 1728$. An elliptic curve with invariant $j = 0$ is given by $E : y^2 = x^3 + g$ with $g \neq 0$. An elliptic curve with invariant $j = 1728$ is $E : y^2 = x^3 + fx$ with $f \neq 0$. The $\mathbb{Z}/2\mathbb{Z}$ automorphism of an elliptic curve with invariant $j \neq 0, 1728$ is given by $(x, y) \to (x, -y)$. It is the inverse of the group law. For $j = 0$, it is induced by $(x, y) \to (\omega x, -y)$ where ω is a choice of a cubic root of the unit ($\omega^3 = 1$). For $j = 1728$, it is generated by $(x, y) \to (-x, \mathrm{i}y)$ where $\mathrm{i}^2 = -1$.

Remark 7.3 (*Cusps and jump phenomena*) Consider a regular elliptic curve in Weierstrass form $E : y^2 = x^3 + fx + g$ defined over a field k. For any nonvanishing $\lambda \in k$, we can define the curve $E_\lambda : y^2 = x^3 + f\lambda^4 x + g\lambda^6$. For $\lambda \neq 0$, E_λ is isomorphic to E after the redefinition $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$. However, at $\lambda = 0$ we always have the cusp $E_0 : y^2 = x^3$. It follows that an elliptic curve with an arbitrary j-invariant can jump to a cusp. For this reason, cusps are excluded in the moduli space of elliptic curves. When considering only smooth curves, the j-invariant maps the space of elliptic curve modulo isomorphism onto $\mathbb{C}$. This space can be compactified by allowing curves of arithmetic genus one with a nodal singularity.

7.3 Elliptic Fibrations

We work over an algebraically closed field k of characteristic zero. The reader is welcome to think of the base field k as the field of complex numbers $\mathbb{C}$. Most of the results do not require the field to be algebraically closed nor of characteristic zero. But we still assume it out of convenience. We denote by $\mathbb{Z}$ the ring of relative integers. By a variety we mean a reduced and irreducible algebraic scheme [10]. Given a variety X, we denote by $\mathscr{O}_X$ the sheaf of regular functions of X. Given a Cartier divisor D in a normal variety X, we denote by $\mathscr{O}_X(-D)$ the normal bundle of D in X. The sheaf $\mathscr{O}_X(nD)$ ($n \in \mathbb{Z}$) is the sheaf of rational functions with a pole of degree n over the divisor D. The dual sheaf of $\mathscr{O}_X(nD)$ is denoted $\mathscr{O}_X(-nD)$. In particular, D is the vanishing locus of a section of $\mathscr{O}_X(D)$.

Definition 7.2 (*Genus-one fibration*) A genus-one fibration is a surjective proper morphism $\varphi : Y \to B$ between algebraic varieties such that the generic fiber is a regular projective curve of genus one. The variety B is called the *base of the fibration*.

Definition 7.3 (*Discriminant locus*) The locus of singular fibers of the fibration $\pi : Y \to B$ is called the *discriminant locus* of π and is denoted Δ.

To avoid trivialities, we assume that a genus-one fibration has a non-trivial discriminant locus (there is at least one singular fiber).

Definition 7.4 (*Rational section*) A *rational section* of a fibration $\varphi : Y \to B$ is a rational map $\sigma : B \to Y$ such that $\varphi \circ \sigma$ is the identity away from a Zariski closed set of B.

Definition 7.5 (*Elliptic fibration*) An elliptic fibration is a genus-one fibration endowed with a *rational section*.

7.3.1 Weierstrass Models for Elliptic Curves

Before discussing elliptic fibrations, we first review the classical argument to get a Weierstrass equation for a regular curve of arithmetic genus 1 with a choice of a rational point S. We follow Mumford and Suominen [16].

Let Y be a non-singular projective curve of genus one over k. Denote by S the divisor associated with a fixed base k-rational point O of Y. The Riemann–Roch theorem asserts that $\dim_k H^0(Y, \mathscr{O}_Y(nS)) = n$ for $n > 0$. Hence, the vector space $H^0(Y, \mathscr{O}_Y(2S))$ has dimension two. Since the only rational functions with at most a pole of degree one on an elliptic curve are the constants, there exists a rational function x with a double pole at O such that $\{1, x\}$ is a basis of $H^0(Y, \mathscr{O}_Y(2S))$. In the same way, since $\dim_k H^0(Y, \mathscr{O}_Y(3S)) = 3$, there is a rational function $y \in H^0(Y, \mathscr{O}_Y(3S))$ with a triple pole at O. Using the basis $\{1, x, y\}$ of $H^0(Y, \mathscr{O}_Y(3S))$, we can prove the following lemma:

Lemma 7.1 *The set* $\{1, x, x^2, \ldots, x^m, y, xy, yx^2, \ldots, yx^{m-2}\}$ *is a basis of* $H^0(Y, \mathscr{O}_Y(nS))$ *for* $n = 2m$. *We get a basis for* $H^0(Y, \mathscr{O}_Y(nS))$ *for* $n = 2m + 1$ *by adding the monomial* yx^{m-1}.

Proof By Riemann–Roch, $H^0(Y, \mathscr{O}_Y(nS))$ has dimension n for $n > 0$. The basis presented in the lemma contains n elements that are linearly independent since each function has a pole at the origin with a different order.

For a curve of genus g, any divisor of degree $2g + 1$ or bigger is a very ample divisor. For a curve of genus 1, any divisor of degree 3 is very ample. It follows that the divisor $3S$ provides a closed embedding of the elliptic curve into $\mathbb{P}^2$. All is left is to give the equation of that curve. Since $H^0(Y, \mathscr{O}_Y(3S))$ is generated by $\{1, x, y\}$, there is a unique embedding $Y \to \mathbb{P}^2$ such that $\mathscr{O}_Y(3S)$ is the pullback of the tautological line bundle $\mathscr{O}_{\mathbb{P}^2}(1)$ and $(x, y, 1)$ are the affine coordinates. The punch line of the proof of the existence of an isomorphic cubic curve in Weierstrass form for an elliptic curve is the following. Since $y^2 \in H^0(Y, \mathscr{O}_Y(6S))$, there are constants $a_0, a_1, a_3, a_4, a_6 \in k$ such that

$$y^2 + a_1 xy + a_3 y = a_0 x^3 + a_2 x^2 + a_4 x + a_6. \tag{7.27}$$

Finally, we have to show that a_0 cannot be zero. If $a_0 = 0$, $\{y^2, xy, y, x^2, x, 1\}$ would be linearly dependent. But this is not possible since there is no terms to cancel out the pole (of order 6) of y^2. We can then redefine $(x, y, a_1, a_2, a_3, a_4, a_6) \to (a_0 x, a_0^2 y, a_0 a_1, a_0^2 a_3, a_2, a_0^3 a_4, a_0^4 a_6)$ and eliminate the overall factor of a_0^4 to get the Weierstrass equation in Tate form:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \tag{7.28}$$

7.3.2 Preparation for the Relative Case

Given an elliptic fibration $\varphi : Y \longrightarrow B$ with a section $\sigma : B \rightarrow Y$, we construct a Weierstrass model birational to Y. We assume the following conditions:

1. φ is a flat projective morphism between (quasi)-projective varieties.
2. Y is normal, and the base B is smooth.
3. The section σ is a morphism.
4. All fibers are irreducible projective curves.

As varieties, Y and B are in particular Noetherian schemes. Hence, the projectivity of φ implies that φ is also a proper morphism by [10, Chap. II, Theorem 4.9]. Since we work over an algebraically closed field, φ is flat if and only if φ is equidimensional (every fiber has the same dimension). Hence, the assumption (4) implies that φ is also a flat morphism.

Since $\varphi : Y \rightarrow B$ is a proper morphism, φ is in particular separated and the section σ defines a closed immersion of B in Y (an isomorphism from B onto a closed subscheme of Y) by [5, Corollary 5.4.6]. Let $\mathscr{I}$ denote the ideal sheaf of that subscheme; its support is a Cartier divisor S of Y. We denote by $\mathscr{N}_{S/Y}$ the normal sheaf of S in Y.

Using the Riemann–Roch Theorem, we can write a Weierstrass equation for each smooth fiber Y_p as in the previous section by studying the cohomology of the fiber. The challenge is now to understand how the cohomology along the fiber varies as a function of the fiber. This is a question of cohomology and base change, an important topic in algebraic geometry covered, for example, in Chap. 3 of [10].

In algebraic geometry, a family of schemes is simply a morphism $f : X \rightarrow Y$ and the members of the family are the fibers $X_y = X \times_Y \operatorname{Spec} k(y)$, where $k(y)$ is the residue field at the point $y \in Y$. To study the cohomology of family of schemes, the higher direct image functors $R^i f_*$ are introduced. They describe the "relative cohomology of X over Y".

Definition 7.6 Let X be any topological space, we denote by $\mathscr{U}(X)$ the category of sheaves of Abelian groups on X. Given a continuous function $f : X \rightarrow Y$ between topological spaces, for any integer $i \geq 0$, we define $R^i f_* : \mathcal{U}(X) \rightarrow \mathscr{U}(Y)$ as the right derived functors of the direct image function f_*.

The following theorem gives a local description of $R^i f_*(\mathscr{F})$:

Theorem 7.8 (Chapter III. Proposition 8.1 of [10]) *For each $i \geq 0$ and each $\mathscr{F} \in \mathscr{U}(X)$, $R^i f_*(\mathscr{F})$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)})$ on Y.*

Given a fibration $f : X \rightarrow Y$, one would like to find some relation between the fiber cohomology groups $H^i(X_y, \mathscr{F}_y)$ and the globally defined sheaves $R^i f_*(\mathscr{F})$. The Proper Base Change Theorem is discussed by Mumford in Sect. 5 of Chap. 2 of "Abelian Varieties."

In the case of an elliptic fibration $\varphi : Y \to B$, the crucial step is the introduction of the fundamental line bundle $\mathscr{L}$ (over the base of the elliptic fibration) which provides a splitting of $\varphi_* \mathscr{O}_Y(nS)$ where S is a Cartier divisor defined by the section. When this is established, one can just follow the usual Riemann–Roch argument and define an embedding of the fibration in a projective bundle $\mathbb{P}^2 \to B$ using the fact that $\varphi_* \mathscr{O}_Y(3S)$ is very ample relatively to the base B. The projective bundle will be defined by the projectivation of $\varphi_* \mathscr{O}_Y(3S)$.

Lemma 7.2 (See [16] Chap. 3 Lemma 2)

1. *The ideal sheaf $\mathscr{I}$ corresponding to the subscheme S defined by the section is an invertible sheaf.*
2. *The sheaf of functions $\mathscr{O}_Y(nS)$ with n-fold poles along S is isomorphic to $\mathscr{I}^{\otimes(-n)}$ for any integer $n > 0$.*

Working fiber by fiber, we get the following lemma summarizing the (cohomological) properties of the pushforward of $\mathscr{O}_Y(nS)$:

Lemma 7.3 *For an elliptic fibration $\varphi : Y \to B$ with a section $\sigma : B \to Y$ defining a closed subscheme S of Y, $R^1\varphi_*(nS)$ and $\varphi_* \mathscr{O}_Y(nS)$ are both locally free for all n and we have:*

1. $\varphi_* \mathscr{O}_Y = \mathscr{O}_B$
2. $\varphi_* \mathscr{O}_Y(nS)$ *is locally free of rank n for all $n > 0$.*
3. $R^1\varphi_* \mathscr{O}_Y(nS) = 0, \quad$ *for all $n > 0$, and locally free for of rank one for $n = 0$.*
4. $R^1\varphi_* \mathscr{O}_Y \cong \varphi_* \mathscr{N}_{S/Y}$ *is an invertible sheaf.*
5. $R^i\varphi_* \mathscr{O}_Y(nS) = 0, \quad$ *for all $i > 1$, and all integers n.*

Proof See Mumford–Suominen [16, Chap. 3], Deligne [4], or Miranda [15, Lecture II Sect. 3].

The line bundle $R^1\varphi_* \mathscr{O}_Y$ is a fundamental invariant of the elliptic fibration $\varphi : Y \longrightarrow B$. This motivates the following definition [15].

Definition 7.7 (*Fundamental line bundle of an elliptic fibration*) The fundamental line bundle of an elliptic fibration $\varphi : Y \to B$ is the invertible sheaf $\mathscr{L}$ defined as:

$$\mathscr{L} := \left(R^1\varphi_* \mathscr{O}_Y\right)^{-1}. \tag{7.29}$$

Remark 7.4 The fundamental line bundle $\mathscr{L}$ is often defined as $(\varphi_* \mathscr{N}_{S/Y})^{-1}$. By the previous Lemma, the two definitions agree since $\varphi_* \mathscr{N}_{S/Y} \simeq R^1\varphi_* \mathscr{O}_Y$. It also follows from the Lemma that the sheaf $\mathscr{N}_{S/Y}$ does not depend on the section S.

For an elliptic fibration with a section, the fundamental line bundle provides a splitting of $\varphi_* \mathscr{O}_Y(nS)$ for $n > 1$:

Theorem 7.9 *For $n > 1$, we have*

$$\varphi_* \mathscr{O}_Y(nS) \cong \mathscr{O}_B \oplus \mathscr{L}^{-2} \oplus \mathscr{L}^{-3} \oplus \cdots \oplus \mathscr{L}^{-n}. \tag{7.30}$$

Proof See Lemma II.4.3 of [15].

Equipped with Theorem 7.9, we can now apply the familiar Riemann–Roch argument to derive the Weierstrass equation in the relative case.

7.3.3 Weierstrass Models for Elliptic Fibrations

Lemma 7.4 (Deligne [4]) *Given an invertible section μ of $\mathscr{L}$, there exists locally for Zariski topology a basis $\{1, x, y\}$ of $\varphi_*\mathscr{O}_Y(3S)$ such that:*

1. 1 *is a generator of* $\mathscr{O}_B$.
2. $\{1, x\}$ *is a basis of* $\varphi_*\mathscr{O}_Y(2S)$*, and the image of* x *along* $\mathscr{L}^{-2}$ *is* μ^{-2}.
3. y *belongs to* $\varphi_*\mathscr{O}_Y(3S)$ *and the image of* y *along* $\mathscr{L}^{-3}$ *is* μ^{-3}.
4. $\{1, x, x^2, \dots, x^n, y, yx, \dots, yx^{n-2}\}$ *is a basis for* $\varphi_*\mathscr{O}_Y(mS)$ *if* $m = 2n$ *with* $n > 1$.
5. $\{1, x, x^2, \dots, x^n, y, yx, \dots, yx^{n-2}, yx^{n-1}\}$ *is a basis for* $\varphi_*\mathscr{O}_Y(mS)$ *if* $m = 2n + 1$ *with* $n > 1$.

Given a different choice μ' *of an invertible section of* $\mathscr{L}$*, there exists well-defined* u, r, s, t *such that the new basis* $(1, x', y')$ *is related to the previous one as follows:*

$$\begin{cases} x' = u^2 x + r \\ y' = u^3 y + s u^2 x + t \\ \mu' = u\mu \end{cases} \tag{7.31}$$

These transformations (7.31) will be called *admissible transformations of a Weierstrass model*. For $\varphi_*\mathscr{O}_Y(6S)$, we have the basis $\{1, x, x^2, x^3, y, yx\}$ composed of six generators, but the space of monomials generated by $\{1, x, y\}$ in $\varphi_*\mathscr{O}_Y(6S)$ is seven-dimensional. The missing monomial is y^2, and its image along $\mathscr{L}^{-\otimes 6}$ is μ^6, which matches the image of x^3. It follows that $y^2 - x^3$ can be uniquely written as a linear combination of generators of $\varphi_*\mathscr{O}_Y(5S)$. This gives the Weierstrass equation in Tate form:

$$y^2 + a_1 xy + a_3 = x^3 + a_2 x^2 + a_4 x + a_6. \tag{7.32}$$

For each index i, the coefficient a_i is a section of $\mathscr{L}^{\otimes i}$. The line bundle $\mathscr{O}_Y(3S)$ is very ample relatively to the base B. The basis $(1, x, y)$ can be seen as affine coordinates of a $\mathbb{P}^2$ in which each fiber is embedded. We have an immersion of the elliptic fibration Y into a $\mathbb{P}^2$ projective bundle over the base B:

$$Y \to \mathbb{P}(\mathscr{E}) := \text{Proj}(\text{Sym}\, \mathscr{E}^*), \quad \text{where } \mathscr{E} := \varphi_*\mathscr{O}_Y(3S), \tag{7.33}$$

and $(x, y, 1)$ are the affine coordinates as they generate $\varphi_*\mathscr{O}_Y(3S)$. When the characteristic is different from 2 and 3, the Weierstrass equation can be reduced to the shorter form: $y^2 = x^3 + fx + g$ where f and g are, respectively, sections of $\mathscr{L}^{\otimes 4}$ and $\mathscr{L}^{\otimes 6}$. We quickly review our conventions for projective bundles.

Remark 7.5 (*Conventions for projective bundles*)

- We use the classical convention for the projectivization $\pi : \mathbb{P}(\mathscr{E}) \to B$ of a locally free sheaf $\mathscr{E}$ over B: The fibers of $\mathbb{P}(\mathscr{E})$ are the lines of $\mathscr{E}$ passing through the origin and not the hyperplanes. In our conventions $\mathbb{P}(\mathscr{E}) := \text{Proj}(\text{Sym } \mathscr{E}^*)$. In other words, what we call $\mathbb{P}(\mathscr{E})$ corresponds to $\mathbb{P}(\mathscr{E}^*)$ in the convention of EGA II.4.1.1 or Hartshorne.
- We denote the tautological line bundle of the projective bundle $\mathbb{P}(\mathscr{E})$ by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1)$. Its dual is the canonical line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. By an abuse of notation, we will write $\mathscr{O}(-1)$ and $\mathscr{O}(1)$, respectively, for $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1)$ and $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. We also write $\mathscr{O}(-n)$ (for $n > 0$) for the nth tensor product of $\mathscr{O}(-1)$. Its dual is $\mathscr{O}(n)$, the nth tensor product of $\mathscr{O}(1)$. In particular, in our notation $\pi_*\big(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)\big) = \mathscr{E}^*$.
- Given a locally free sheaf $\mathscr{E} = \mathscr{O}_B \oplus \mathscr{L}^{\otimes a} \oplus \mathscr{L}^{\otimes b}$, there are natural embeddings $\mathscr{O}_B \hookrightarrow \mathscr{E}$, $\mathscr{L}^{\otimes a} \hookrightarrow \mathscr{E}$, and $\mathscr{L}^{\otimes b} \hookrightarrow \mathscr{E}$. We use these embeddings to define projective coordinates $[z : x : y]$ for $\mathbb{P}(\mathscr{E})$:

$$\begin{cases} z \text{ is a section of } \mathscr{O}(1) \\ x \text{ is a section of } \mathscr{O}(1) \otimes \pi^*\mathscr{L}^{\otimes a} \\ y \text{ is a section of } \mathscr{O}(1) \otimes \pi^*\mathscr{L}^{\otimes b} \end{cases}$$

We can now introduce the definition of a Weierstrass model.

Definition 7.8 (*Weierstrass models*) Given a base B endowed with a line bundle $\mathscr{L}$, the Weierstrass model $\mathscr{W}_B(\mathscr{L}|f, g)$ defines an elliptic fibration $Y \to B$ where Y is the zero-scheme of a section of $\mathscr{O}(3) \otimes \pi^*\mathscr{L}^{\otimes 6}$ in $\mathbb{P}[\mathscr{O}_B \oplus \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 3}]$ cuts by the Weierstrass normal equation:

$$y^2 z = x^3 + fxz^2 + gz^3. \tag{7.34}$$

In the previous equation, $[z : x : y]$ are projective coordinates of the projective bundle as explained earlier. The coefficient f is a section of $\mathscr{L}^{\otimes 4}$ and g a section of $\mathscr{L}^{\otimes 6}$. It is assumed that the discriminant $\Delta := -16(4f^3 + 27g^2)$ is not identically zero and defines a Cartier divisor in the base B.

Definition 7.9 (*Canonical section of a Weierstrass model*) A Weierstrass model admits a section given by $x = z = 0$ which is always in the smooth locus of the elliptic fibration. It is called the *canonical section*.

Definition 7.10 (*Discriminant locus*) The discriminant locus of the Weierstrass model $\mathscr{W}_B(\mathscr{L}|f, g)$ is given by the zero-scheme of the following section of $\mathscr{L}^{\otimes 12}$:

$$\Delta = -16(4f^3 + 27g^2), \quad \Delta \in H^0(B, \mathscr{L}^{\otimes 12}). \tag{7.35}$$

Remark 7.6 The factor of (-16) is there to match the definition of the discriminant for a Weierstrass model in Tate form as it is given in the formulaire of Deligne and Tate. It also matches the definition $\Delta(\tau) = g_2^3 - 27g_3^2$ of the cusp form associated to the Weierstrass equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

Theorem 7.10 (Equivalence of Weierstrass models) *Two Weierstrass models* $\mathscr{W}_B(\mathscr{L}_1|f_1, g_1)$ *and* $\mathscr{W}_B(\mathscr{L}_2|f_2, g_2)$ *over the same base* B *are equivalent if and only if there is a nowhere vanishing* $u \in H^0(B, \mathscr{L}_2 \otimes \mathscr{L}_1^{-1})$ *such that* $f_2 = u^4 f_1$ *and* $g_2 = u^6 g_1$.

We have proven the following.

Theorem 7.11 ([4, 15, 16]) *Let* $\varphi : Y \to B$ *be a smooth elliptic fibration admitting a section* $\sigma : B \to Y$*. Then, there exists a triplet* $(\mathscr{L}, f, g)$ *and an isomorphism* $\mu : Y \to \mathscr{W}_B(\mathscr{L}|f, g)$ *over* B *such that* $\mu \circ \sigma$ *is the canonical section and* $\mathscr{L}^{-1} \simeq R^1\varphi_*\mathscr{O}_Y$*. Moreover, the discriminant* Δ *is invertible over the locus of regular fibers.*

One can have a similar result in the presence of mild singularities.

Theorem 7.12 (Nakayama) *An elliptic fibration* $\varphi : X \to B$ *with a section* $\sigma : B \to X$ *is birationally equivalent to a Weierstrass model* $\mathscr{W}_B(\mathscr{L}|f, g)$ *with canonical singularities and such that* $\mathscr{L}$ *is the fundamental line bundle associated to the elliptic fibration.*

As a direct consequence of the adjunction formula, we have the following theorem.

Theorem 7.13 *The canonical bundle of a smooth Weierstrass model* $\mathscr{W}_B(\mathscr{L}|f, g)$ *for a smooth elliptic fibration* $\varphi : Y \to B$ *is*

$$\omega_Y \cong \varphi^*(\omega_B \otimes \mathscr{L}). \tag{7.36}$$

Lemma 7.5 (Elliptically fibered Calabi–Yau) *A Weierstrass model* $Y = \mathscr{W}_B(\mathscr{L}|f, g)$ *has a trivial canonical divisor if and only the dual of its fundamental line bundle is the canonical line bundle of the base. That is*

$$K_Y = 0 \iff \mathscr{L}^{-1} = \omega_B.$$

7.3.4 The j-Invariant

Given a Weierstrass model $\mathscr{W}_B(\mathscr{L}|f, g)$, for any nonvanishing section μ of $\mathscr{O}_B$, we can rescale $(f, g) \mapsto (u^4 f, u^6 g)$ and get an equivalent Weierstrass model with the same fundamental line bundle $\mathscr{L}$. It follows that there is a unique invariant f^3/g^2 that we can write. However, it is more convenient to use $f^3/(4f^3 + 27g^2)$ since $4f^3 + 27g^2$ is nonvanishing over regular fibers.

Definition 7.11 (*j-invariant*) To a Weierstrass model $\mathscr{W}_B(\mathscr{L}|f, g)$, we associate the j-invariant:

$$j(f, g) := 1728\frac{4f^3}{4f^3 + 27g^2} \in H^0(B, \mathscr{O}_B). \tag{7.37}$$

Remark 7.7 Kodaira uses the normalization $j = 4f^3/(4f^3 + 27g^2)$. The one we use here with the extra factor of $1728 = 12^3$ is the normalization used by number theorists. It matches the conventions of Deligne and Tate.

Remark 7.8 (*The j-map is not injective*) Let K be a field containing a nonzero element λ which has no square root in K. Two elliptic curves with the same j-invariant are isomorphic in a quadratic or cubic extension of the field. The elliptic curves $E_1 : y^2 = x^3 + fx + g$ and $E_2 : y^2 = x^3 + \lambda^2 fx + \lambda^3 g$ are not isomorphic over K even though they have the same j-invariant. They become isomorphic over any field extension K' of K containing a square root of λ.

7.3.5 Deligne's Formulaire

In this section, we would like to collect important definitions and formulas for an elliptic curve in Weierstrass form:

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3. \tag{7.38}$$

We follow Tate's notation [4, 29]. Geometrically, the marked point of the Weierstrass form of an elliptic curve is its intersection point with the line at infinity $z = 0$, namely the point $[x : y : z] = [0 : 1 : 0]$, which is a point of inflection and the only point at infinity of the curve. The curve is called a Weierstrass normal form since (in characteristic different from 2 and 3) after the change of variables:

$$\wp = x + \frac{1}{12}(a_1^2 + 4a_2), \quad \wp' = 2y + a_1x + a_3, \tag{7.39}$$

it reduces to the traditional cubic equation satisfied by the Weierstrass $\wp$-function and its derivative:

$$E : (\wp')^2 = 4\wp^3 - g_2\wp - g_3. \tag{7.40}$$

The *Néron differential* associated to the elliptic curve is the the differential invariant under translations in the group law and defined as follows:

$$\omega = \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4 - a_1y} = \Big(= \frac{d\wp(z)}{\wp'(z)} = dz\Big). \tag{7.41}$$

A curve given by a Weierstrass equation is singular if and only if its discriminant Δ is zero. If we denote by $\bar{k}$ the algebraic closure of k, two smooth elliptic curves are isomorphic over $\bar{k}$ if and only if they have the same j-invariant. We recall the formulaire of Deligne and Tate which is useful to express the discriminant Δ, the j-invariant and to reduce the Weierstrass equation into simpler forms:

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6, \tag{7.42}$$
$$b_8 = b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \tag{7.43}$$
$$c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6, \tag{7.44}$$
$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \tag{7.45}$$
$$j = \frac{c_4^3}{\Delta} \tag{7.46}$$

These quantities are related by the following relations:

$$4b_8 = b_2b_6 - b_4^2 \quad \text{and} \quad 1728\Delta = c_4^3 - c_6^2. \tag{7.47}$$

The variables b_2, b_4, b_6 are used to express the Weierstrass equation after completing the square in y by a redefinition

$$y \mapsto y - \frac{1}{2}(a_1x + a_3z), \tag{7.48}$$

which gives

$$zy^2 = x^3 + \frac{1}{4}b_2x^2z + \frac{1}{2}b_4xz^2 + \frac{1}{4}b_6z^3. \tag{7.49}$$

The variables c_2, c_4, and c_6 are then obtained after eliminating the term in x^2 by the redefinition

$$x \mapsto x - \frac{1}{12}b_2z, \tag{7.50}$$

which finally gives the short form of the Weierstrass equation:

$$zy^2 = x^3 - \tfrac{1}{48}c_4xz^2 - \tfrac{1}{864}c_6z^3. \tag{7.51}$$

We will use the following normalization of the short Weierstrass equation (obtained by introducing $f = -\frac{1}{48}c_4$ and $g = -\frac{1}{864}c_6$):

$$E : zy^2 = x^3 + fxz^2 + gz^3, \quad \Delta = -16(4f^3 + 27g^2), \quad j = 1728\frac{4f^3}{4f^3 + 27g^2}. \tag{7.52}$$

A Weierstrass equation is unique up to the following admissible coordinate transformation - (with $r, s, t, u \in k$ and $u \neq 0$):

$$x = u^2 x' + r, \quad y = u^3 y' + s u^2 x' + t, \tag{7.53}$$

under which we have

$$\begin{aligned}
u a_1' &= a_1 + 2s, \\
u^2 a_2' &= a_2 - s a_1 + 3r - s^2, \\
u^3 a_3' &= a_3 + r a_1 + 2t, \\
u^4 a_4' &= a_4 - s a_3 + 2 r a_2 - (t + rs) a_1 + 3r^2 - 2st, \\
u^6 a_6' &= a_6 + r a_4 + r^2 a_2 + r^3 - t a_3 - r t a_1 - t^2,
\end{aligned} \tag{7.54a}$$

$$\begin{aligned}
u^2 b_2' &= b_2 + 12r, \\
u^4 b_4' &= b_4 + r b_2 + 6r^2, \\
u^6 b_6' &= b_6 + 2 r b_4 + r^2 b_2 + 4r^3, \\
u^8 b_8' &= b_8 + 3 r b_6 + 3 r^2 b_4 + r^3 b_2 + 3r^4,
\end{aligned} \tag{7.54b}$$

$$u^4 c_4' = c_4, \quad u^6 c_6' = c_6, \tag{7.54c}$$

$$u^{12} \Delta' = \Delta, \quad u\omega' = \omega, \qquad j' = j. \tag{7.54d}$$

7.4 Kodaira–Néron Classification of Singular Fibers

For an elliptic fibration $\varphi : Y \to B$, a smooth fiber is isomorphic to a torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ where τ lives in the upper-half plane $\mathscr{H}$. Two elliptic curves with period τ and τ' are isomorphic if and only if they are related by a modular transformation:

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{7.55}$$

The elliptic fibration admits a discriminant locus over which the fibers are singular. Let B^* be the locus of points p of B such that the fiber Y_p over p is a smooth curve. By considering the ambiguity, we have a period mapping function $\tau : U \to \mathscr{H}$ from the universal covering space U of B^* into the upper-half plane $\mathscr{H}$ and a monodromy representation

$$\mu : \pi_1(B^*) \to \mathrm{SL}(2, \mathbb{Z}), \tag{7.56}$$

such that for $\gamma \in \pi_1(B^*)$ and $p \in U$

$$\tau(\gamma p) = \frac{a_\gamma \tau + b_\gamma}{c_\gamma \tau + d_\gamma}, \quad \mu(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{7.57}$$

Table 7.1 Quasi-unipotent matrices in SL(2, ℤ)

I_a ($a \in \mathbb{Z}$)	II	III	IV
$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = T^{\otimes a}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = -(ST)^2$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -S$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = -ST$
I^*_b ($b \in \mathbb{Z}$)	II*	III*	IV*
$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} = -T^{-b}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = ST$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (ST)^2$

7.4.1 Monodromy

For a proper map $\varphi : Y \to B$ between smooth projective varieties, the monodromy around a point of the discriminant locus with at most normal crossing singularity is a quasi-unipotent matrix by Borel's lemma [20]. We recall the definition of quasi-unipotent and give a classification for SL(2, ℤ) following Kodaira.

Definition 7.12 A matrix M is said to be *quasi-unipotent* if all its eigenvalues are roots of the unit. That is, there are integers $n, k \geq 1$ such that $(M^k - \mathrm{Id})^n = 0$.

In the case of SL(2, ℤ), quasi-unipotent matrices up to conjugation form eight different classes:

Lemma 7.6 (Kodaira [12]) *A quasi-unipotent matrix in SL*$(2, \mathbb{Z})$ *is conjugated exactly to one of the matrices in Table 7.1.*

These eight conjugation classes provide a classification of the type of singular fibers over a general point of a component of the discriminant locus assuming that the singularity at that point is at most a normal crossing singularity.

7.4.2 Fiber Type

The local ring of a subvariety S of X is denoted $\mathscr{O}_{X,S}$, its maximal ideal is $\mathscr{M}_{X,S}$, and the quotient field is the residue field $\kappa(V) = \mathscr{O}_{X,S}/\mathscr{M}_{X,S}$. The local ring $\mathscr{O}_{X,S}$ is the stalk of the structure sheaf of X at the generic point η_S of S, and $\kappa(S)$ is the function field of S. If S is a divisor, $\mathscr{O}_{X,S}$ is a one-dimensional local domain. In case X is non-singular along S, $\mathscr{O}_{X,S}$ is a discrete valuation ring and the order of vanishing is given by the usual valuation.

Definition 7.13 (*Fiber over a point*) Let $\varphi : Y \longrightarrow B$ be a morphism of schemes. For any $p \in B$, the fiber over p is denoted Y_p and defined using a fibral product[1] as

[1]Given three sets (A_1, A_2, and S) and two maps $\varphi_1 : A_1 \to B$ and $\varphi_2 : A_2 \to B$, we define the fibral product $A_1 \times_S A_2$ as the subset of $A_1 \times A_2$ composed of couples (a_1, a_2) such that $\varphi_1(a_1) = \varphi_2(a_2)$.

Table 7.2 Allowed collisions of a Miranda model

$j=\infty$	$j=0$	$j=1728$
$I_{M_1}+I_{M_2}$	$\mathrm{II}+\mathrm{IV}$	$\mathrm{III}+\mathrm{I}_0^*$
$I_{M_1}+I^*_{M_2}$	$\mathrm{II}+I_0^*$	
	$\mathrm{II}+\mathrm{IV}^*$	
	$\mathrm{IV}+I_0^*$	

$$Y_p = Y \times_B \text{Spec}\,\kappa(\text{p}).$$

The first projection $Y_p \longrightarrow Y$ induces an homeomorphism from Y_p onto $f^{-1}(p)$ [13, Sect. 3.1 Proposition 1.16]. The second projection gives Y_p the structure of a scheme over the residue field $\kappa(p)$.

If p is not a closed point,[2] the residue field $\kappa(p)$ is not necessarily algebraically closed. Certain components of Y_p could be $\kappa(p)$-irreducible (i.e., irreducible when defined over $\kappa(p)$), while they become reducible after an appropriate field extension. An irreducible scheme over a field k is said to be *geometrically irreducible* when it stays irreducible after any field extension. The most refined description of the fiber Y_p is always the one corresponding to the algebraic closure $\overline{\kappa(p)}$ of $\kappa(p)$. This motivates the following definition (Table 7.2).

Definition 7.14 The geometric fiber over p is the fiber $Y_p \times_{\kappa(p)} \overline{\kappa(p)}$, the fiber Y_p after the base change induced by the field extension $\kappa(p) \to \overline{\kappa(p)}$ to the algebraic closure of $\kappa(p)$.

By construction, a geometric fiber is always composed of geometrically irreducible components.

Definition 7.15 We say that the type of a fiber Y_p is *geometric* if it does not change after a field extension.

For an elliptic n-fold, the Kodaira fibers are also the *geometric generic fibers* of the irreducible components of the reduced discriminant locus.

Definition 7.16 (*Algebraic cycle*) An algebraic cycle of a Noetherian scheme X is a finite formal sum $\sum_i n_i V_i$ of subvarieties V_i with integer coefficients n_i. If all the subvarieties V_i have the same dimension d, the cycle is called a d-cycle. The free group generated by subvarieties of dimension d is denoted $Z_d(X)$. The group of all cycles, denoted $Z(X) = \bigoplus_d Z_d(X)$, is the free group generated by subvarieties of X.

Definition 7.17 (*Degree of a zero-cycle* [9, Chap. 1, Definition 1.4, p. 13]) Let X be a complete scheme. The *degree* of a zero-cycle $\sum n_i p_i$ of X is $\deg(\sum_i n_i p_i) = \sum_i n_i[\kappa(p_i) : k]$, where $[\kappa(p_i) : k]$ is the degree of the field extension $\kappa(p_i) \to k$.

Let Θ be an algebraic one-cycle with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$. We denote by $\Theta_i \cdot \Theta_j$ the zero-cycle defined by the intersection of Θ_i and Θ_j for

[2]For example, if p is the generic point of a subvariety of B.

$i \neq j$. A *n-point* of an algebraic one-cycle Θ is a point in $\bigcup_i \Theta_i$, which belongs to exactly n distinct irreducible components Θ_i. An algebraic one-cycle Θ is said to be a *tree* if it does not have n-points for $n > 2$. Two curves intersect transversally if their intersection consists of isolated reduced closed points.

Following Kodaira [12], we introduce the following definition:

Definition 7.18 (*Fiber type*) By the *type* of an algebraic one-cycle $\Theta \in Z_1(X)$ with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$, we mean the isomorphism class of each irreducible curve Θ_i, together with the topological structure of the reduced polyhedron $\sum \Theta_i$ (that is, the collection of zero-cycles $\Theta_i \cdot \Theta_j$ $(i \neq j)$), and the homology class of $\Theta = \sum_i m_i \Theta_i$ in the Chow group $A_1(X)$.

Example 7.1 For instance, $\Theta_1 \cdot \Theta_2 = 2p_1 + 3p_2$ indicates that the two curves Θ_1 and Θ_2 meet at two points p_1 and p_2 with respective intersection multiplicity 2 and 3.

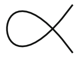
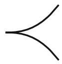
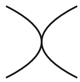
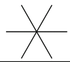
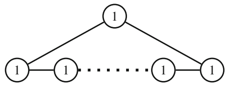
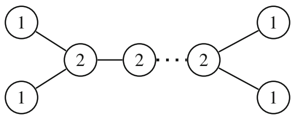
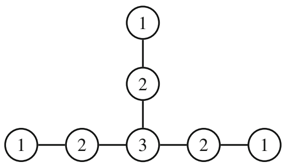
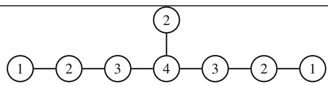
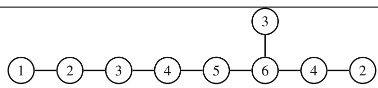
Definition 7.19 (*Dual graph*) To an algebraic one-cycle Θ with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$, we associate a weighted graph (called the *dual graph* of Θ) such that:

- The vertices are the irreducible components of the fiber.
- The weight of a vertex corresponding to the irreducible component Θ_i is its multiplicity m_i. When the multiplicity is one, it can be omitted.
- The vertices corresponding to the irreducible components Θ_i and Θ_j $(i \neq j)$ are connected by $\hat{\Theta}_{i,j} = \deg(\Theta_i \cdot \Theta_j)$ edges.

Definition 7.20 (*Kodaira symbols, See* [12]) Kodaira has introduced the following symbols characterizing the type of one-cycles appearing in the study of minimal elliptic surfaces. See Table 7.3 for a visualization of these fibers.

1. Type I_0: a smooth curve of genus 1.
2. Type I_1: an irreducible nodal rational curve.
3. Type II: an irreducible cuspidal rational curve.
4. Type I_2: $\Theta = \Theta_1 + \Theta_2$ and $\Theta_1 \cdot \Theta_2 = p_1 + p_2$: two smooth rational curves intersecting transversally at two distinct points p_1 and p_2. The dual graph of I_2 is $\tilde{A}_1$.
5. Type III: $\Theta = \Theta_1 + \Theta_2$ and $\Theta_1 \cdot \Theta_2 = 2p$: two smooth rational curves intersecting at a double point. Its dual graph is $\tilde{A}_1$.
6. Type IV: $\Theta = \Theta_1 + \Theta_2 + \Theta_3$ and $\Theta_1 \cdot \Theta_2 = \Theta_1 \cdot \Theta_3 = \Theta_2 \cdot \Theta_3 = p$: a 3-star composed of smooth rational curves. Its dual graph is $\tilde{A}_2$.
7. Type I_n $(n \geq 3)$: $\Theta = \Theta_0 + \cdots \Theta_n$ with $\Theta_i \cdot \Theta_{i+1} = p_i$ $i = 0, \ldots, n-1$ and $\Theta_n \cdot \Theta_0 = p_n$. Its dual graph is the affine Dynkin diagram $\tilde{A}_{n-1}$.
8. Type I_n^* $(n \geq 0)$: $\Theta = \Theta_0 + \Theta_1 + 2\Theta_2 + \cdots + 2\Theta_{n+2} + \Theta_{n+3} + \Theta_{n+4}$, with $\Theta_i \cdot \Theta_{i+1} = p_i$ $(i = 1, \ldots, n+2)$, $\Theta_0 \cdot \Theta_2 = p_0$, $\Theta_{n+4} \cdot \Theta_{n+2} = p_{n+4}$. The dual graph is the affine Dynkin diagram $\tilde{D}_{4+n}$.
9. Type IV^*: $\Theta = \Theta_0 + \Theta_1 + 2\Theta_2 + 2\Theta_3 + 3\Theta_4 + 2\Theta_5 + \Theta_6$ with $\Theta_i \cdot \Theta_{i+1} = p_i$ $(i = 3, \ldots, 6)$, $\Theta_1 \cdot \Theta_3 = p_1$, $\Theta_0 \cdot \Theta_2 = p_0$, $\Theta_2 \cdot \Theta_4 = p_2$. The dual graph is the affine Dynkin diagram $\tilde{E}_6$.

Table 7.3 Kodaira–Néron classification of geometric fibers over codimension-one points of the base of an elliptic fibration [12, 19]. The type of the fiber is given by its Kodaira symbol. In the second, third, and fourth column, $\nu(A)$ is the valuation of A. The j-invariant of the I_0^* is never ∞ and can take any finite value

Type	$\nu(c_4)$	$\nu(c_6)$	$\nu(\Delta)$	j	Monodromy	Fiber	Dual Graph
I_0	≥ 0	≥ 0	0	$\mathbb{C}$	I_2	Smooth Elliptic Curve	-
I_1	0	0	1	∞	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	 (curve of arithmetic genus 1 with a nodal singularity)	$\widetilde{\mathrm{A}}_0$
II	≥ 1	1	2	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	 (curve of arithmetic genus 1 with a cuspidal singularity)	$\widetilde{\mathrm{A}}_0$
III	1	≥ 2	3	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	 Two rational curves intersecting at a double point	$\widetilde{\mathrm{A}}_1$
IV	≥ 2	2	4	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$		$\widetilde{\mathrm{A}}_2$
I_n	0	0	$n>1$	∞	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	 n nodes	$\widetilde{\mathrm{A}}_{n-1}$
I_n^*	2	≥ 3	$n+6$	∞	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	 $n+5$ nodes	$\widetilde{\mathrm{D}}_{n+4}$
	≥ 2	3	$n+6$				
IV^*	≥ 3	4	8	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$		$\widetilde{\mathrm{E}}_6$
III^*	3	≥ 5	9	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$		$\widetilde{\mathrm{E}}_7$
II^*	≥ 4	5	10	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$		$\widetilde{\mathrm{E}}_8$

10. Type III*: $\Theta = \Theta_0 + 2\Theta_1 + 2\Theta_2 + 3\Theta_3 + 4\Theta_4 + 3\Theta_5 + 2\Theta_6 + \Theta_7$ with $\Theta_i \cdot \Theta_{i+1} = p_i$ $(i = 3, \ldots, 6)$, $\Theta_1 \cdot \Theta_3 = p_1$, $\Theta_0 \cdot \Theta_1 = p_0$, $\Theta_2 \cdot \Theta_4 = p_2$. The dual graph is the affine Dynkin diagram $\widetilde{E}_7$.
11. Type II*: $\Theta = 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 6\Theta_4 + 5\Theta_5 + 4\Theta_6 + 3\Theta_7 + 2\Theta_8 + \Theta_0$, with $\Theta_i \cdot \Theta_{i+1} = p_i$ $(i = 3, \ldots, 7)$, $\Theta_1 \cdot \Theta_3 = p_1$, $\Theta_8 \cdot \Theta_0 = p_8$, and $\Theta_2 \cdot \Theta_4 = p_2$. The dual graph is the affine Dynkin diagram $\widetilde{E}_8$.

While the dual graph of a Kodaira fiber is an affine Dynkin diagram of type $\widetilde{A}_k$, $\widetilde{D}_{4+k}$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$, the dual graph of the generic (arithmetic) fiber itself can also be a twisted Dynkin diagram of type $\widetilde{B}^t_{3+k}$, $\widetilde{C}^t_{2+k}$, $\widetilde{G}^t_2$, or $\widetilde{F}^t_4$. This is reviewed in Table 7.5. These dual graphs are not geometric in the sense that after an appropriate base change, they become $\widetilde{D}_{4+n}$, $\widetilde{A}_{2+2k}$ or $\widetilde{A}_{1+2k}$, and $\widetilde{E}_6$, respectively. The Kodaira fibers of the following type never need a field extension: I_1, II, III, III*, and II*.

The remaining Kodaira fibers (IV, $\mathrm{I}_{n>1}$, I^*_n, and IV*) can come from fibers Y_p whose types are not geometric and require at least a field extension of degree 2 to describe a fiber with a geometric type. When the fiber Y_p has a geometric type, the type of the fiber is said to be *split*. Otherwise, the type of Y_p is said to be non-split. When that is the case we mark the fiber with an "ns" superscript: $\mathrm{IV}^{\mathrm{ns}}$, $\mathrm{I}^{\mathrm{ns}}_n$, $\mathrm{I}^{*\mathrm{ns}}_n$, $(n \geq 2)$ and $\mathrm{IV}^{*\mathrm{ns}}$. When a field extension is not needed, the fibers are marked with an "s" superscript ("split"): IV^{s}, $\mathrm{I}^{\mathrm{s}}_n$, $\mathrm{I}^{*\mathrm{s}}_n$, $(n \geq 2)$ and $\mathrm{IV}^{*\mathrm{s}}$. The fiber of type I^*_0 can be split, semi-split, or non-split if the Kodaira types require no field extension, at field extension of degree 2, or a field extension of degree 3. The corresponding dual graphs are, respectively, $\widetilde{G}^t_2$, $\widetilde{B}^t_3$, and $\widetilde{D}_4$.

7.4.3 Tate's Algorithm

Let R be a complete discrete valuation ring with valuation v, uniformizing parameter s, and perfect residue field $\kappa = R/(s)$. We are interested in the case where κ has characteristic zero. We recall that a discrete valuation ring has only three ideals, the zero ideal, the ring itself, and the principal ideal sR. It follows that the scheme $\mathrm{Spec}(R)$ has only two points[3]: the generic point (defined by the zero ideal) and the closed point (defined by the principal ideal sR).

Let E/R be an elliptic curve over R with Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in R.$$

The generic fiber is a regular elliptic curve. After a resolution of singularities, we have a regular model $\mathscr{E}$ over R and the *special fiber* is the fiber over the closed point of Spec R.

Tate's algorithm determines the type of the geometric special fiber over the closed point of $\mathrm{Spec}(R)$ by manipulating the valuations of the coefficients and the discriminant and the arithmetic properties of some auxiliary polynomials. The type of the

[3] As usual we take the convention in which the ring itself is not a prime ideal.

geometric fiber is given by its Kodaira's symbol. The special fiber becomes geometric after a quadratic or a cubic field extension κ'/κ. Keeping track of the field extension used gives a classification of the special fiber as a κ-scheme as discussed, for example, in [13, Sect. 10.2]. The information on the required field extension needed to have geometrically irreducible components is already carefully encoded in Tate's original algorithm, as it is needed to compute the local index.

Tate's algorithm consists of the following eleven steps (see [28], [23, Sect. IV.9], [6]).

Step 1. $v(\Delta) = 0 \implies \mathrm{I}_0$.

Step 2. If $v(\Delta) \geq 1$, change coordinates so that $v(a_3) \geq 1$, $v(a_4) \geq 1$, and $v(a_6) \geq 1$.
If $v(b_2) = 0$, the type is $\mathrm{I}_{v(\Delta)}$. To have a fiber with geometric irreducible components, it is enough to work in the splitting field κ' of the following polynomial of $\kappa[T]$:

$$T^2 + a_1 T - a_2.$$

The discriminant of this quadric is b_2. If b_2 is a square in κ, then $\kappa' = \kappa$, otherwise $\kappa' \neq \kappa$:
(a) $\kappa' = \kappa \implies \mathrm{I}_n^{\mathrm{s}}$ (b) $\kappa' \neq \kappa \implies \mathrm{I}_n^{\mathrm{ns}}$

Step 3. $v(b_2) \geq 1$, $v(a_3) \geq 1$, $v(a_4) \geq 1$, and $v(a_6) = 1 \implies$ II.

Step 4. $v(b_2) \geq 1$, $v(a_3) \geq 1$, $v(a_4) = 1$, and $v(a_6) \geq 2 \implies$ III.

Step 5. $v(b_2) \geq 1$, $v(a_3) \geq 1$, $v(a_4) \geq 2$, $v(a_6) \geq 2$, and $v(b_6) = 2 \implies$ IV.
The fiber has geometric irreducible components over the splitting field κ' of the polynomial

$$T^2 + a_{3,1} T - a_{6,2}$$

Its discriminant is $b_{6,2}$. If $b_{6,2}$ is a square in κ, then $\kappa' = \kappa$ otherwise $\kappa' \neq \kappa$.
(a) $\kappa' = \kappa \implies \mathrm{IV}^{\mathrm{s}}$ (b) $\kappa' \neq \kappa \implies \mathrm{IV}^{\mathrm{ns}}$

Step 6. $v(b_2) \geq 1$, $v(a_3) \geq 1$, $v(a_4) \geq 2$, $v(a_6) \geq 3$, $v(b_6) \geq 3$, $v(b_8) \geq 3$. Then, make a change of coordinates such that $v(a_1) \geq 1$, $v(a_2) \geq 1$, $v(a_3) \geq 2$, $v(a_4) \geq 2$, and $v(a_6) \geq 3$. Let

$$P(T) = T^3 + a_{2,1} T^2 + a_{4,2} T + a_{6,3}$$

If $P(T)$ is a separable polynomial in κ, that is, if $P(T)$ has three distinct roots in a field extension of κ, then the type is I_0^*. The geometric fiber is defined over the splitting field κ' of $P(T)$ in κ. The type of the special fiber before to go to the splitting field depends on the degree of the field extension $\kappa' \to \kappa$:

- $[\kappa' : \kappa] = 3$ or $6 \implies \mathrm{I}_0^{*\mathrm{ns}}$ with dual graph $\tilde{G}_2^t$.
- $[\kappa' : \kappa] = 2 \implies \mathrm{I}_0^{*\mathrm{ss}}$ with dual graph $\tilde{B}_3^t$.
- $[\kappa' : \kappa] = 1 \implies \mathrm{I}_0^{*\mathrm{s}}$ with dual graph $\tilde{D}_4$.

where "ns", "ss", and "s" stand, respectively, for "non-split", "semi-split", and "split". In the notation of Liu, these fibers are, respectively, $\mathrm{I}^*_{0,3}$, $\mathrm{I}^*_{0,2}$, and I^*_0.

Step 7. If $P(T)$ has a double root, then the type is I^*_n.
Make a change of coordinates such that the double root is at the origin. Then $v(a_1) \geq 1, \quad v(a_2) = 1, v(a_3) \geq 2, \quad v(a_4) \geq 3, \quad v(a_6) \geq 4$, and $v(\Delta) = n + 6 \ (n \geq 1)$.

Step 8. If $P(T)$ has a triple root, change coordinates such that the triple root is zero. Then, $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 4$.
Let

$$Q(T) = T^2 + a_{3,2}T - a_{6,4}$$

If Q has two distinct roots ($v(b_6) = 4$ or equivalently $v(\Delta) = 8$), the type is IV*. The split type depends on the rationality of the roots. If $b_{6,4}$ is a perfect square modulo s, the fiber is $\mathrm{IV}^{*\mathrm{s}}$, otherwise the fiber is $\mathrm{IV}^{*\mathrm{ns}}$.
The split form can be enforced with $v(a_6) \geq 5$ and hence $v(a_3) = 2$ to ensure that $v(b_6) = 4$.

Step 9. If Q has a double root, we change coordinates so that the double root is at the origin. Then:
$v(a_1) \geq 1, \ v(a_2) \geq 2, \ v(a_3) \geq 3, \ v(a_4) = 3, \ v(a_6) \geq 5 \implies$ type III*.

Step 10. $v(a_1) \geq 1, \ v(a_2) \geq 2, \ v(a_3) \geq 3, \ v(a_4) \geq 4, \ v(a_6) = 5 \implies$ type II*.

Step 11. Else $v(a_i) \geq i$ and the equation is not minimal. Divide all the a_i by s^i and start again with the new equation.

7.5 Miranda Models

The theory of elliptic surfaces has been treated by Kodaira. The geometry of the singular fibers is specially elegant. Singular fibers appear over isolated points on the base where their positions are given by the zeros scheme of the reduced discriminant. The complete list of singular fibers encompassed two infinite series (I_n and I_n^*) and six exceptional cases (II, III, IV, IV*, III*, II*). They can also be classified by their monodromies, and they can be attributed a well-defined value for the j-invariant. Namely $j = 0$ for the two infinite series (I_n, I_n^* with $n > 0$), arbitrary for I_0^*, $j = 0$ for II, IV, IV*, II* and $j = 1728$ for III and III*. If a Weierstrass model is given, the singular fibers can also be classified purely algebraically by Tate's algorithm.

If the base of the fibration is higher dimensional, we can still use Kodaira results and Tate's algorithm over codimension-one loci in the base. But there is a new challenge in determining the structure of singular fibers over higher codimensional loci in the base, for example, at the collisions of several components of the discriminant locus.

Miranda has given an explicit algorithm for finding a resolution of an elliptic threefold given by a (singular) Weierstrass models. Blow up the base until the reduced discriminant locus has simple normal crossings. Continue further so that only one of a small list of possible collisions between component of the discriminant locus occurs. Namely the following seven possibilities:

This list of collisions is obtained by requiring three conditions:

1. The reduced discriminant has simple normal crossing.
2. Only fibers with the same j-invariant are allowed to collide. This ensures that after the resolution, the j-invariant is a morphism.
3. Collisions that do not admit a small resolution are excluded. This ensures that the resolution gives a flat fibration.

The only places where one leaves the category of schemes in Miranda's resolution is in using a small resolution of an ordinary double point in resolving the collision $I_{M_1} + I_{M_2}$ when M_1 and M_2 are both odd. One has to contract a ruled surface to a $\mathbb{P}^1$ to ensure that the fibers are unidimensional. However, if one blows up such a collision point, one obtains over the exceptional curve a fiber of type $I_{M_1+M_2}$. Since $M_1 + M_2$ would be even, we can avoid collisions $I_{M_1} + I_{M_2}$ with M_1 and M_2 odd. Miranda's construction is purely local. But he also shows that it is well defined globally.

Following Dolgachev and Gross [7], we define a Miranda elliptic fibrations as follows:

Definition 7.21 (*Miranda elliptic fibrations*) A Miranda elliptic fibration is an elliptic fibration $\varphi : Y \to B$ such that (1) Y and B are regular and φ is flat and admits a section. (2) The discriminant locus has simple normal crossing. (3) All collisions are of the following seven types $I_{M_1} + I_{M_2}$, $I_{M_1} + I^*_{M_2}$, II+IV, II+I_0^*, II+IV*, IV+I_0^* or III+I_0^*.

7.5.1 Fibers at the Collisions of a Miranda Model

In Miranda models, in addition to the usual Kodaira fibers, there are new fibers that appear in higher codimensions. For an elliptic threefold, we have fibers in codimension-2 that could be one of the five exceptional types that are essentially sequences chains of 3, 4 or 5 rational curves with multiplicities (see Table 7.4). There is also the fibers I_n and I_n^* that can appear and a new infinite family called I_n^{*+} which admits as a dual graph the Dynkin diagram of D_{n+5} (we recall that a fiber I_n^* has a dual graph $\tilde{D}_{n+4}$. It consists of two rational curves of multiplicity one connected to a chain of $n+2$ rational curves of multiplicity 2. One can think of a I_n^{*+} fiber as a I_n^* fiber in the limit in which one of the two pairs of curves of multiplicity one is identified. Since there is a section, it is necessary the pair that does not intersect the section.

Table 7.4 Colliding singularities in an elliptic threefold as constructed by Miranda. The non-Kodaira fiber I_n^{*+} has the shape of a diagram of type D_{n+4}. The last column shows the fiber that would be obtained for an elliptic with base a smooth curve passing through the point of collision. The last column is what would be predicted by "applying" Tate algorithm in higher codimension

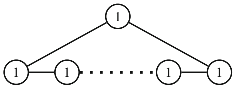
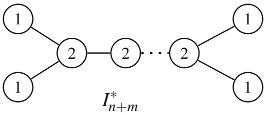
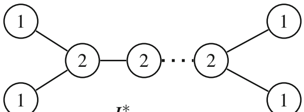
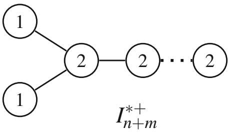
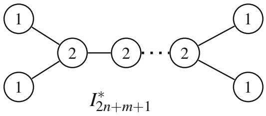
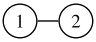
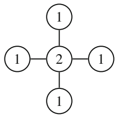
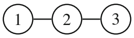
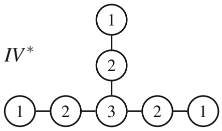
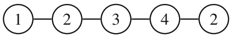
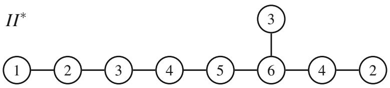
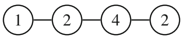
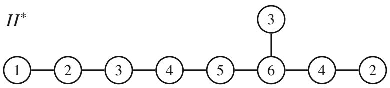
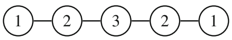
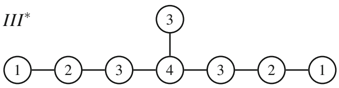
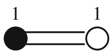
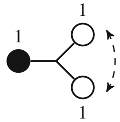
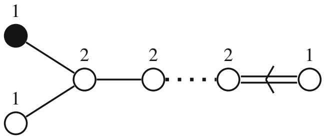
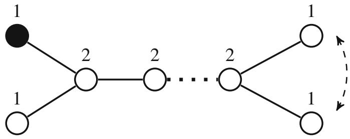
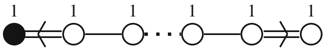
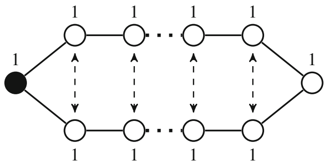
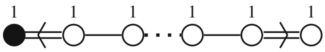
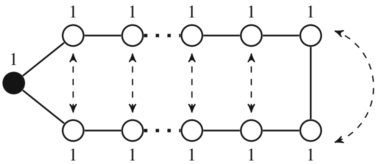
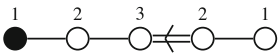
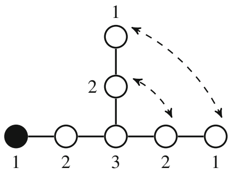
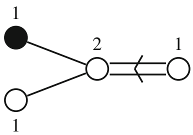
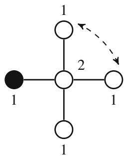
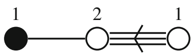
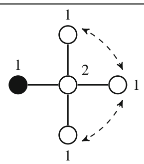
j-inv	Collision	Dual graph	if the base was a smooth curve through the collision point
∞	$I_{M_1}+I_{M_2}$	 $\mathrm{I}_{M_1+M_2}$ nodes	same
∞	$I_{2n}+I_m^*$	 I_{n+m}^*	 I_{2n+m}^*
∞	$I_{2n+1}+I_m^*$	 I_{n+m}^{*+} ($n+m+4$ components)	 I_{2n+m+1}^*
0	$II+IV$		I_0^* 
0	$II+I_0^*$		IV^* 
0	$II+IV^*$		II^* 
0	$IV+I_0^*$		II^* 
1728	$III+I_0^*$		III^* 

Table 7.5 Dual graphs for elliptic fibrations. The fiber type follows the notation of Liu [13, Sect. 10.2]. A fiber type is called T_d if the corresponding geometric fiber has Kodaira type T and a field extension of at least degree d is necessary to make all the components of the fiber geometrically irreducible. This indicates some nodes are not geometrically irreducible and split into d geometrically irreducible curves after a field extension of degree d

Fiber Type	Dual graph	Dual graph of the geometric fibre
IV_2 $\widetilde{A}_1$		
$I^*_{\ell-3,2}$ $\widetilde{B}^t_\ell$ $(\ell \geq 3)$		
$I_{2\ell+2,1}$ $\widetilde{C}^t_{\ell+1}$ $(\ell \geq 1)$		
$I_{2\ell+3,2}$ $\widetilde{C}^t_{\ell+1}$ $(\ell \geq 1)$		
IV^*_2 $\widetilde{F}^t_4$		
I^*_2 $\widetilde{B}^t_3$		
$I^*_{0,3}$ $\widetilde{G}^t_2$		

7.5.2 Szydlo's Generalization of Miranda Models

Assuming the same conditions as Miranda, Szydlo has analyzed the general case of collisions in higher codimensions [26]. He assumes that the base scheme of the fibration is Noetherian, n-dimensional, regular, integral, and separated. He also allows mixed characteristic.

Szydlo does not assume that the residue field is perfect, it follows that an irreducible polynomial can have roots with multiplicity so that the roots only exist in non-separable extension of the residue field. The translation needed in Tate's algorithm translates the singular point of a Weierstrass model to the origin and the multiple root of certain quadratic polynomial to the origin (Table 7.5).

Interestingly, starting from codimension-three, the only collisions possible are those with $J = \infty$ (type I_n and I_n^*) with the following restrictions: There are at most one fiber of type I_n^* and at most one fiber of type I_{2m+1}, and the number of fiber of type I_{2n} is bounded by the codimension of the collision. Taking this into account, we have the following four types of collisions:

$$\begin{aligned} J = \infty : I_{2n_1} + \cdots I_{2n_k} &\longrightarrow I_{2n}, \quad n = n_1 + \cdots + n_k \\ I_{2n_1} + \cdots + I_{2n_k} + I_{2r+1} &\longrightarrow I_{2n+2r+1}, \\ I_{2n_1} + \cdots + I_{2n_k} + I_m^* &\longrightarrow I_{n+m+1}^*, \\ I_{2n_1} + \cdots + I_{2n_k} + I_{2r+1} + I_m^* &\longrightarrow I_{n+r+m+1}^{*+}. \end{aligned} \tag{7.58}$$

The resolution of the singularities at the collisions depends on some discrete choices. In particular, the order in which the blowups are performed is crucial for the final result. For example, Miranda and Szydlo don't have the same results for the collision $IV + I_0^*$ and the justification can be traced back to different conventions on how to order the blowups:

(Miranda)	(Szydlo)
$IV + I_0^*$: (1)–(2)–(4)–(2)	$I_0^* + IV$: (1)–(2)–(3)–(2)

(7.59)

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