Chapter 1
Prelude: A General Overview

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Abstract This chapter provides the reader with a general overview of the various topics discussed in this volume, emphasizing the deep relations existing between them. Following a brief historical account of the emergence of the concept of *"quantization"* both in physics and mathematics, a description of the main concepts and tools appearing in subsequent chapters is presented.

1.1 Introduction

This volume presents various ongoing approaches to the vast topic of quantization, namely to the process of forming a quantum mechanical system starting from a classical one and discusses their numerous fruitful interactions with mathematics.

In its early years, quantum theory was understood in terms of a set of empirical rules that would allow to make sense—to a certain extent—of experimental results. Thus, for instance, in the old quantum theory, an electron would still orbit the nucleus obeying the laws of classical dynamics, but an additional condition, the Bohr–Sommerfeld quantization condition, had to be fulfilled. This reduced the set of allowed orbits to a discrete one, providing a way to explain the quantization of energy levels. The subsequent development of wave mechanics by de Broglie, the

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introduction of Schrödinger's equation and the development of Heisenberg's "matrix mechanics", eventually led to the formulation based on operators in Hilbert space, as presented by Dirac and von Neumann.

In particular, Dirac emphasized that quantum observables—described by operators acting on a Hilbert space—can be obtained by replacing classical observables (i.e., smooth functions on phase space) by self-adjoint operators, in such a way that the Poisson bracket of two classical observables becomes, up to a constant, the commutator of the corresponding quantum observables. Thus, the quantum analogue of the classical Poisson bracket $\{x, p\} = 1$ of classical mechanics is given by the canonical commutation relations $[\hat{x}, \hat{p}] = i\hbar$, with $\hbar$ the Planck constant. So quantization brings in non-commuting operator algebras due to the presence of the parameter $\hbar$.

A first approach to quantization presented in this volume, called deformation quantization, an approach initiated by M. Flato, A. Lichnerowicz, and D. Sternheimer, in viewing the Planck constant $\hbar$ as a small parameter, provides a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables. It is defined in terms of a star product viewed as a formal deformation in the parameter $\hbar$ of the algebraic structure of the space of smooth functions on a Poisson manifold. When symmetries come into play, deformation quantization needs to be merged with group actions, the topic of Chap. 2, by Simone Gutt.

The non-commutativity arising from quantization is the main concern of non-commutative geometry, which has become an autonomous area of research under the impulse of A. Connes. His and Chamseddine's spectral action principle applied to an appropriate non-commutative space yields the standard model action coupled to Einstein and Weyl gravity. Allowing for the presence of symmetries requires working with principal fiber bundles in a non-commutative setup, the topic of Chap. 3, by Christian Kassel. Non-commutativity is central to N. Andruskiewitsch's contribution which presents Nichols algebras that provide a unifying concept for various viewpoints on the quantized enveloping algebra of a simple finite-dimensional Lie algebra $\mathfrak{g}$ at a generic parameter q.

An alternative quantization procedure which claims to encompass gravity was born in the late 1960s and early 1970s under the name of string theory. Indeed, one of the many vibrational states of the string is supposed to correspond to the graviton, a quantum mechanical particle that carries gravitational force. It went through a first golden age in the late 1980s and early 1990s known as the first string theory revolution, and a revival around the concept of duality in the late 1990s and early 2000, known as the second string theory revolution. In Chap. 6, N. Berkovits and H. Gomez present its supersymmetric version which encodes both the bosons and the fermions. Superstrings have drawn the attention of many a mathematician, due to its various fruitful interactions with algebraic geometry, some of which are described here by M. Esole.

The quantization of gauge theories entails many subtleties, in great part due to the presence of gauge invariance. From the point of view of classical dynamics, in gauge theories we are faced with the problem that the theory, initially defined in terms of a Lagrangian density, cannot be described in a Hamiltonian setting without taking into account the presence of constraints. An appropriate treatment of the quantum problem leads to BRST symmetry, as illustrated in the example of the superstring in Berkovits' lectures. The quantization of a field theory can also be performed in a Lagrangian setting, making use of path integrals. In the case of gauge theories, the problems reappear in the form of the Gribov ambiguity [1]. A very general approach devised to properly dealing with the gauge-fixing problem is the Batalin–Vilkovisky formalism, which is the topic of Chaps. 8 and 9.

Reflecting the deep relations between the various topics discussed in the lectures to follow are the many common mathematical or physical concepts and tools they bring into play. Let us name a few transversal concepts to various lectures that can serve as guiding threads for the reader:

- **Group actions** which arise wherever there are symmetries, so in any quantization procedure which claim to take symmetries into account, such as deformation quantization in a G-equivariant setup in Simone Gutt's contribution. In Ch. Kassel's lectures, group actions are generalized to the non-commutative world in the form of comodule algebras over a Hopf algebra. In N. Berkovits' lectures, which uses the BRST formalism, the local symmetries are fixed and ghost and antighost parameters (parameters with inverse statistics) are introduced, thus giving rise to global symmetries and an associated conserved charge, the BRST charge.
- **Hopf algebras**, the dual counterparts of groups, that correspond to structures encoding simultaneously an (unital associative) algebra and a (counital coassociative) coalgebra, with compatibility conditions between these structures together with an antiautomorphism satisfying a certain property. Hopf algebras naturally occur in algebraic topology, in group theory (via the concept of a group ring), quantum groups as can be seen from the lectures by Ch. Kassel where they are used to quantize homogeneous spaces and in the context of Nichols algebras presented by N. Andruskiewitsch, that play a crucial role in the classification program of Hopf algebras. They also have diverse applications ranging from condensed-matter physics and quantum field theory to string theory.
- **Fibrations** that arise wherever quantization meets geometry, here in the form of (i) elliptic fibrations, describing an elliptic curve moving along a variety, the topic of M. Esole's lectures, whose physical background lies in the realm of strings where elliptic curves arise naturally via conformal field theory, (ii) the non-commutative principal fiber bundles discussed in Ch. Kassel's lectures, a non-commutative generalization of ordinary principal fiber bundles that developed with gauge theory, (iii) the Weyl bundle, a bundle used in S. Gutt's lectures, whose fibers are modeled on the Weyl algebra, and on whose flat sections one builds a star product, (iv) as an instance of the more general concept of foliation arising in A. Ashtekar's lectures as globally hyperbolic space-time in the context of quantum field theory on curved space-time.
- **Supersymmetry** which takes different forms depending on the context, e.g., that of a supersymmetric action in N. Berkovits' lectures. Supersymmetry is a key ingredient in string theory; there are various string theories in ten dimensions related by dualities which give rise to challenging questions in mathematics requiring sophisticated tools such as the elliptic fibrations of M. Esole's lectures.

- **Quantization**, a deep and rich concept which is a unifying thread throughout these lectures where it comes up in various disguises, in the form of BRST quantization in Berkovits' lectures, in that of functional quantization used to quantize the strings that serve as one of the motivations for M. Esole's study of elliptic fibrations, as a deformation quantization in S. Gutt's lectures, in the form of unitary representations of the Weyl algebra of an infinite-dimensional symplectic vector space discussed in A. Ashtekar's lectures.
- **Non-commutativity and deformation** inherent in quantization procedures that typically bring—possibly deformed—non-commuting operators into the scene, is reflected in the canonical commutation relations obeyed by the annihilation and creation operators in A. Astekar's lectures and lies behind the operator product expansions in conformal field theory used in N. Berkovits' presentation. In the framework of quantization by deformation discussed in S. Gutt's lectures, Poisson brackets are substituted by $\hbar$-deformed operator brackets, $\hbar$ being the Planck constant. Similarly, in Ch. Kassel's lectures, the coordinate algebra $\mathbb{C}[X, Y]$ of the complex plane is deformed to the q-deformed "coordinate algebra" $\mathbb{C}_q[X, Y]$ of a hypothetical quantum space and symmetry groups such as $SL(2)$ are deformed to quantum groups $SL_q(2)$. Such quantum groups relate to Nichols algebras central to N. Andruskiewitsch's lectures and that appear as the invariant part of Woronowicz's non-commutative differential calculus.

In view of their importance in this volume, the concept of "quantization" and the related concept of non-commutativity deserve further explanations.

The word *quantization* is commonly used to describe a procedure to link the "classical" description of a dynamical system with its "quantum" description. In some cases, such a quantization can be reached exploiting geometric features of the system, but approaches involving rather algebraic or analytical tools are also used when the "quantization rules" can be read of the classical description of the system in algebraic or analytic terms. There is by far no unified approach to quantization, even when only very simple dynamical systems are considered, and in general it is not clear either that such procedure may exist. In any case, the quest for a bridge between the mathematical structures used to describe classical dynamical systems and those used to come up with a quantum description of them gave rise to many deep and interesting ideas in mathematical physics and, in particular, to new mathematical theories.

From the point of view of mathematics, classical dynamics can be achieved using tools borrowed from differential equations, classical analysis, and differential geometry and whenever symmetries are involved, group theory comes into play in more or less sophisticated ways (from special functions and representation theory to the geometry of Lie groups and fiber bundles). Quantum descriptions of dynamics involve functional analysis in an essential way, but they also use non-commutative algebras and shed light on the role of topology for systems sensitive to such type of constraints. In addition, in recent times, new mathematical tools arise from theories inspired by the principles and rules of quantum physics, and by the heuristics of what one expects of a mathematical quantization of the classical structures. Among many others, theories motivated by quantization are non-commutative geometry, quantum groups, and algebraic deformation theory discussed in this volume.

1.2 Poisson Geometry and Classical Dynamics

The basic objects in the commonly used geometric approaches to classical dynamics are smooth manifolds equipped with 2-tensors in terms of which a Lie algebra structure (compatible with differentiation) can be given to the space of functions on the manifold. Alternatively, the starting point can be the operation

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

providing the space of smooth functions (here, we consider real-valued functions on the manifold M, although complex-valued functions can also be considered as observables, see, e.g., Simone Gutt's lectures) with a Lie algebra structure. In other words, the bracket $\{\cdot,\cdot\}$ enjoys the following properties

1. Linearity,
$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\},$$
2. Antisymmetry,
$$\{f, g\} = -\{g, f\},$$
3. Jacobi identity,
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

for all f, g and $h \in C^\infty(M)$, and any scalars α and β, to which we add the compatibility with the usual product of functions, i.e.

4. Leibniz rule,
$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

These four identities define the a *Poisson bracket* on $C^\infty(M)$, and we call M equipped with such a bracket a *Poisson manifold*.

Symplectic manifolds, which are Poisson manifolds for which the Poisson tensor is non-degenerate, are the most popular ground used to model dynamical systems. A symplectic manifold is a pair (M, ω), where ω is a closed and non-degenerate differential 2-form on M (in the context of Poisson geometry, the dual of the Poisson 2-tensor). For example, cotangent bundles are symplectic manifolds particularly well adapted to model phase spaces: If Q is a smooth manifold with local coordinates $(q_1, \ldots, q_n)$, its cotangent bundle T^*Q is a $2n$-dimensional symplectic manifold with local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ whose first n coordinates define the position in the configuration space Q and last n coordinates correspond to their associated generalized momenta. The symplectic structure in this case is canonical, since the cotangent bundle projection onto the configuration space $T^*Q \xrightarrow{\pi} Q$ defines a 1-form θ in terms of which the symplectic form can be written $\omega = d\theta$. This 1-form is called the *symplectic potential*, and in local coordinates it has the form

$$\theta = \sum_{i=1}^{n} p_i dq_i.$$

On a general symplectic manifold (M, ω), given that ω is closed, by the Poincaré lemma such a symplectic potential exists locally (and it is not unique in general). However, Darboux's theorem shows that every symplectic manifold locally has the structure of a cotangent bundle, so that any two symplectic manifolds with the same dimension are locally diffeomorphic since, locally, every symplectic 2-form looks like

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

To illustrate how the symplectic structure can be used to model the classical dynamics of a physical system, let us consider a system whose phase space is the symplectic manifold $(M = T^*Q, \omega)$. A physical observable is, by definition, any real-valued smooth function $f \in C^\infty(M)$; examples are usual physical quantities—energy, momentum, etc. Since the symplectic 2-form ω is non-degenerate, there is a natural linear isomorphism

$$i_\cdot : T_\cdot M \to T^*_\cdot M$$

given by contraction $i(X) = i_X\omega = \omega(X, \cdot)$. This isomorphism can be used to identify tangent and cotangent vectors and in particular, to associate to each smooth function $f \in C^\infty(M)$ a vector field X_f on M by the relation

$$i_{X_f}\omega = -df.$$

Such a vector field X_f is called the *Hamiltonian vector field* associated with f, in terms of which the Poisson bracket operation is given by

$$\{f, g\} = \omega(X_f, X_g).$$

Since the exterior derivative of f can locally be written as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial p_i} dp_i,$$

the Hamiltonian vector field defined by this function is the one given in local coordinates by

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{1.1}$$

In local coordinates, the Poisson bracket of two functions $f, g \in C^\infty(M)$ is the smooth function defined by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}, \tag{1.2}$$

where $2n$ is the dimension of M, which is the usual expression for such an operation used in physics.

Poisson brackets are useful to describe the dynamics of physical systems because, given a Hamiltonian function H for the system, the evolution of classical observables is given by their bracket with the corresponding Hamiltonian [2], i.e.

$$\frac{df}{dt} = \{f, H\}, \tag{1.3}$$

for any smooth function $f \in C^\infty(M)$. Notice that, if $\gamma(t) = (p_i(t), q_i(t))$ is an integral curve of the Hamiltonian vector field (1.1) associated with a function H, the time evolution of the canonical variables on the symplectic manifold is given by

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i = \{p_i, H\} \quad , \quad \frac{\partial H}{\partial p_i} = \dot{q}_i = \{q_i, H\},$$

which are precisely the Hamilton equations in the case in which H is a Hamiltonian for the system. Thus, once a Hamiltonian function is given, dynamics follows directly from the Poisson bracket defined by the 2-form ω in (1.2).

Remark. All the facts illustrated here in the context of symplectic manifolds hold in the more general context of Poisson manifolds, where the expressions before in terms of the 2-form ω must be replaced by their counterparts in terms of the Poisson tensor (see, e.g., Simone Gutt's lectures).

Some years after the birth of quantum mechanics, Paul Dirac realized that the Hamiltonian description of the dynamics, and in particular, the algebraic structure defined by the Poisson bracket $\{\cdot, \cdot\}$ on the algebra of classical observables, is crucial to understand the relationship between classical and quantum dynamics. One of the main features of the quantum description of a physical system is the use of self-adjoint operators acting on Hilbert spaces as quantum observables, highlighting the non-commutative nature of this algebra of observables. Since, with respect to the Poisson bracket operation, classical observables as position and momenta already satisfy commutation relations of the form $\{q_i, p_j\} = \delta_{ij}$, Dirac noticed that to a certain extent, the non-commutativity of quantum observables was already present in the classical setting and, as a consequence, the quantization process should be understood as a morphism between similar algebraic structures in very different contexts. On the one hand the differential-geometric approach of the dynamics in terms of smooth functions on a manifold (as classical observables) a Hamiltonian and a Poisson bracket and, on the other hand, the functional-analytic approach in terms of self-adjoint operators (as quantum observables) acting on a Hilbert space (of "wave functions") with the usual commutator of operators as natural bracket [3]. In this sense, one can think of Poisson manifolds as "maximal noncommutative spaces" between the world of classical physics (commutative algebras of smooth functions on smooth manifolds) and the quantum world of non-commutative algebras, the triple $(C^\infty(M), \{\cdot, \cdot\}, H)$—the algebra of observables plus a distinguished object in terms of which the evolution can be given, see (1.3), often called *dynamical system*— being the starting point of any quantization model.

1.3 Geometric and Deformation Quantization

Quantizing a dynamical system $(C^\infty(M), \{\cdot, \cdot\}, H)$ corresponds to a rule which assigns to the system a representation $f \mapsto \widehat{f}$ of the algebra of classical observables in the algebra of self-adjoint operators $\mathscr{A}$ acting on certain Hilbert space $\mathscr{H}$. How to build the Hilbert space and the representation itself can vary according to the physical system or the mathematical purpose, and in some cases a "complete" quantization cannot be achieved. From the point of view of mathematics, there are two methods which have been successfully studied and given rise to very stimulating ideas beyond their relationship with physics, *geometric quantization* and *deformation quantization*. Both methods strive to fulfill the Dirac quantization conditions [3]:

1. The application $f \mapsto \widehat{f}$ must be linear
2. If f is constant then $\widehat{f}$ must be the multiplication (by the constant f) operator
3. If, for three classical observables, $\{f, g\} = h$ then

$$[\widehat{f}, \widehat{g}] = -i\hbar\widehat{h} \tag{1.4}$$

must be verified by their quantum counterparts.

1.3.1 Geometric Quantization

The goal of geometric quantization is to build both a Hilbert space and a representation of observables from the geometry and the topology of the dynamical system one started from. If ones starts from a symplectic manifold (M, ω), which models the classical phase space for a dynamical system, to quantize *geometrically* such system means finding a map

$$C^\infty(M) \times \Gamma(\mathscr{L}) \to \Gamma(\mathscr{L})$$
$$(f, \psi) \mapsto \hat{f}\psi,$$

where $\Gamma(\mathscr{L})$ denotes the space of sections of a Hermitian line bundle $\mathscr{L} \to M$, modeling "wave functions", satisfying the Dirac quantization conditions. The idea goes back to Kostant and Souriau [4, 5], for whom the "prequantization bundle" $\mathscr{L}$ is a complex line bundle over M, endowed with a connection ∇ with curvature prescribed by the symplectic form, namely equal to $\hbar^{-1}\omega$. Such a bundle exists if and only if the class of $(2\pi\hbar)^{-1}\omega$ in $H^2(M, \mathbb{R})$ is in the image of $H^2(M, \mathbb{Z})$ under the inclusion (see, e.g., [6]) and, if this integrality condition is verified, the Hilbert space of prequantization $\mathscr{H}(M, \mathscr{L})$ is the completion of the space of square integrable sections $s : M \to \mathscr{L}$, denoted by $\Gamma(\mathscr{L})$, with inner product

$$\left(s, s'\right) = \int_M \left\langle s, s'\right\rangle \varepsilon$$

where $\varepsilon = \frac{1}{2\pi\hbar} dp_1 \wedge \cdots \wedge dp_n \wedge dq_1 \wedge \cdots \wedge dq_n$ is the volume element defined by the symplectic form on the manifold M.

Beyond the obvious theoretical importance of this construction, a very relevant feature of this approach is the integrality condition on the symplectic form it involves, namely the topological restriction $[(2\pi\hbar)^{-1}] \in H^2(M, \mathbb{Z})$, which can be used to explain the quantization of certain numbers associated with elementary physical systems (the so-called quantum numbers, see, e.g., [6]). Regarding the representation of observables, in this setting, to each smooth function $f \in C^\infty(M)$, we associate an Hermitian operator according to the Konstant–Souriau representation

$$\hat{f} = f - i\hbar\nabla_{X_f},$$

where X_f denotes the Hamiltonian vector field generated by f. Both the Hilbert space and the representation of observables are determined by the symplectic form; from this point of view, in this quantization, the quantum dynamics is completely determined by the classical dynamics of the system, via a topological condition.

In order to illustrate how this construction works, let us compute a simple example, namely the operators corresponding to position q_i and momentum p_i in the phase space $M = T^*\mathbb{R}^n$ with canonical symplectic form. In this case, the correspondent line bundle associated is $M \times \mathbb{C}$ and the representation corresponding to the observables gives $\hat{f} = f - i\hbar X_f - (p_i dq_i)(X_f)$ so that, since $X_{p_i} = \frac{\partial}{\partial q_i}$ and $X_{q_i} = -\frac{\partial}{\partial p_i}$,

$$\hat{p}_i = p_i - i\hbar\left(\frac{\partial}{\partial q_i}\right) - p_i = -i\hbar\frac{\partial}{\partial q_i}$$

and

$$\hat{q}_i = q_i - i\hbar\left(-\frac{\partial}{\partial p_i}\right) = q_i + i\hbar\frac{\partial}{\partial p_i}.$$

This result disagrees with the usual rules of quantum mechanics (Scrhödinger's version) that read $\hat{p}_i = -i\hbar\frac{\partial}{\partial q_i}$ and $\hat{q}_i = q_i$, and this is why we use the name "prequantization" at this stage for this procedure, which should be promoted to a quantization by means of a polarization (see [6] for details). Both, the geometric prequantization and the polarization procedures, can be carried out on Poisson manifolds, see [7].

1.3.2 Deformation Quantization

The appearance of the Planck constant $\hbar$ in the course of the last few paragraphs is completely incidental, and more related with the wish of recovering the usual commutation rules of quantum mechanics as an output of the quantization process. In contrast, for the deformation theory of quantization, it is the main parameter (actually it is, in this context, everything but a constant), the one in terms of which the algebra of quantum observables will be built as a deformation of the algebra of classic observables $\mathscr{A}_o = (C^\infty(M), \{,\})$.

The algebra $\mathscr{A}_o = C^\infty(M)$ of classical observables is replaced by $\mathscr{A}_\hbar = C^\infty(M)[\![\hbar]\!]$, the algebra of formal power series in $\hbar$ of elements in $\mathscr{A}_o$, whose elements have the form $\mathsf{f} = \sum_{k=0}^{\infty} \hbar^k f_k$. Viewing this formal power series as analogues of symbols in the theory of pseudo-differential operators gives an idea of the composition formula of the corresponding elements in $\mathscr{A}_\hbar$ (Weyl's quantization). A formal deformation quantization of a Poisson manifold M is a couple $(\mathscr{A}_\hbar = C^\infty(M)[\![\hbar]\!], *)$, where

$$* : \mathscr{A}_\hbar \otimes \mathscr{A}_\hbar \to \mathscr{A}_\hbar$$

denotes a star product defined on the algebra of formal power series of elements in $\mathscr{A}_o$ such that, for any $f, g \in \mathscr{A}_o$,

$$f * g = \sum_{l \geq 0} \hbar^k C_k(f, g),$$

where the C_k are defined by bidifferential operators (and define Hochschild 2-cochains on $\mathscr{A}_o$) satisfying $(f * g) * h = f * (g * h)$. Thus, for any $\mathsf{f} = \sum_{k=0}^{\infty} \hbar^k f_k$ and $\mathsf{g} = \sum_{k=0}^{\infty} \hbar^k g_k$ in $\mathscr{A}_\hbar$, with $f_k, g_k \in \mathscr{A}_o$ for all k,

$$\mathsf{f} * \mathsf{g} = \sum_{l+i+j \geq 0} \hbar^l C_l(f_i, g_j), \tag{1.5}$$

where it is assumed that the first two cochains satisfy $C_0(f, g) = fg$ (the usual commutative product of smooth functions) and $C_1(f, g) - C_1(g, f) = \{f, g\}$, so that

$$f * g = fg + O(\hbar)$$

and

$$f * g - g * f = -i\hbar\{f, g\} + O(\hbar^2),$$

a week version of (1.4).

This theory was initiated in the 1970s by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer [8], in the context of symplectic manifolds, and revisited by B. Fedosov in the 1980s [9] in the same context but in a much richer geometric approach. Since then many aspects of the theory have been studied (e.g., classification issues, generalizations to Poisson manifolds and more general differential-geometric/algebraic structures, index theory, etc.) giving rise to very important advances in different areas of mathematical physics. Chapter 2 offers a complete exposition of these and other aspects of deformation quantization by Simone Gutt, a leading expert in the subject who contributed with the theory from an early stage, reaching important developments of the theory such as Kontsevich's formality theorem, the concept of reduction in the formal deformation setting and convergence issues in the deformation quantization programme. Professor Gutt's lectures on deformation quantization, aimed at graduate students in physics or mathematics, are self-contained and contain a very complete list of references to the abundant literature on the subject; we refer the reader to that chapter for more on this interesting point of view on quantization.

1.4 Non-commutative Geometry and Quantum Groups

As mentioned before, the starting point for a description of the dynamics of a classical system is a triple $(\mathscr{A}_o, \{\cdot,\cdot\}, H)$, where $\mathscr{A}_o = C^\infty(M)$ denotes the algebra of classical observables and $\{\cdot,\cdot\}$ the Poisson bracket of smooth functions. Instead of a "constructive" quantization of such dynamical system by a deformation as indicated before, or an explicit construction of the quantum algebra of observables from geometric data, there are methods involving mathematical objects supposed to represent the quantum counterparts of classical dynamical systems without explicit mention to some particular quantization process. From these points of view, the non-commutative algebras involved in the description of the quantum dynamics of a system must, in some appropriate limit, give back the classic algebraic setting of classical dynamics, but they must not necessarily be built from them. Among these theories, we want to mention Alain Connes' *non-commutative differential geometry* and the theory of *quantum groups*.

The basic object in non-commutative geometry is a *spectral triple* $(\mathscr{A}, \mathscr{H}, D)$ (also called unbounded Fredholm module), and it involves an involutive algebra $\mathscr{A}$ represented in a Hilbert space $\mathscr{H}$, together with a self-adjoint operator D with compact resolvent in $\mathscr{H}$ such that $[D, a]$ is bounded for any $a \in \mathscr{A}$ [10]. This triple is the non-commutative generalization of the natural triple $(C^\infty(M), L^2(S, M), D)$ of classical differential spin geometry, where the algebra is the one of smooth functions on a (spin) manifold M, which is commutative with respect to the usual product of functions, the Hilbert space is the one of L^2-spinors (sections of the spin bundle $S \to M$ over M) and D is the classical Dirac operator (the square root of the Laplacian). Thus, the algebra $\mathscr{A}$ models the algebra of functions of a “noncommutative space” which we only see through the spectral properties of the operator D the same way as, for example, the Riemannian metric on M is encoded in the operator spectrum of D in classical global analysis. Conditions can be imposed on a spectral triple to generalize many important features of the usual spectral theory of pseudo-differential operators on manifolds to these non-commutative spaces, obtaining in addition to the usual notions of differential geometry (distances, scalar curvature, etc.) more involved constructions as index theory (see B. Iochum’s lectures in [11], and references therein).

In physics, non-commutative spectral triples have been used to describe elementary particle models over non-commutative space-times, conformal field theories, and dualities among many other uses (see M. Marcolli’s lectures in [11]). Many interesting examples of non-commutative spaces in mathematics come from the theory of *quantum groups*, objects which are deformations of (algebras of functions on) groups, but still have a very similar representation theory. The notion of quantum group comes from the one of Hopf algebras, which are algebraic structures often used to describe deformations of the function algebras on semisimple Lie groups or enveloping algebras of semisimple Lie algebras (see Christian Kassel’s lectures in this volume). These deformations are commonly parametrized by a parameter q which, for some authors, is related to $\hbar$ as $q = \exp(c\hbar)$ for some appropriate scalar c and is used to exhibit explicit deformations of their classical counterparts. For example, the algebra of the quantum group $SU_q(2)$ is the polynomial algebra generated by four elements a, b, c and d satisfying the following relations, for a parameter $0 < q < 1$,

$$\begin{aligned} ba &= qab, \quad & ca &= qac, \\ db &= qbd, \quad & dc &= qcd, \\ bc &= cb; \quad & ad - q^{-1}bc &= da - qbc = 1, \end{aligned}$$

so that the case $q = 1$ would correspond to the classical matrix representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of elements of the Lie group $SU(2)$ in terms of the (commuting) coordinates a, b, c and d.

It is interesting to notice that, if we forget the group-like features of these objects, it has been possible to use the representation theory of many classes of quantum groups to define appropriate *Dirac operators* and, as a consequence, it is possible to realize them as a class of non-commutative spaces in the context of spectral triples. Although the spectral triples $(\mathscr{A}_q, \mathscr{H}_q, D)$ associated with such classes of quantum groups often use the classical Dirac operator on the corresponding classical group, they have interesting properties with potential applications both in mathematics and theoretical physics (see, e.g., [12, 13]).

In classical field theory, principal fiber bundles play a very important role to model gauge symmetries, i.e., internal symmetries of classical systems modeled by the fiber (a Lie group) of a fibration over the space-time manifold; when these classical objects are replaced by their quantum analogues, we obtain different types of objects which appear often in the following pages. First, in Fedosov's approach to deformation quantization, fibrations of non-commutative algebras over symplectic manifolds appear in a natural way (the Weyl bundle) and their geometry is used to build up start products as explained in Simone Gutt's lecture notes. Fibrations in the context of non-commutative geometry play an important role in applications in physics and come in very different flavors which can be used in different situations: classical fibrations on non-commutative spaces (i.e., classical fibers on non-commutative base manifolds), parametrized families of non-commutative spaces or fibrations with quantum fibers on non-commutative spaces. The role of quantum groups in equivariant non-commutative algebraic geometry, in particular the notion of non-commutative principal bundle, or Hopf–Galois extension, will be discussed by Christian Kassel in this volume. Many other examples of fibrations involving Hopf algebras can be studied from the spectral point of view of non-commutative differential geometry; let us just quote the case of non-commutative Hopf fibrations considered by Giovanni Landi and Walter van Suijlekom in [14] and the non-commutative homogeneous spaces studied by Joseph Várilly in [15].

1.5 Quantum Fields

Quantization of a classical field theory brings new features, such as the existence of inequivalent representations of the algebra generated by creation and annihilation operators. This is due to the fact that (by definition) such a theory is a dynamical system with an infinite number of degrees of freedom.

As an example, let us consider the Klein–Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\varphi(x) = 0, \tag{1.6}$$

which is the simplest one compatible with the Poincaré symmetry of Minkowski space $\mathscr{M}$. This equation can be obtained from a Lagrangian density $\mathscr{L}(\varphi, \partial_\mu \varphi)$ as a solution of the corresponding Euler–Lagrange equations, but the dynamics can also be described in terms of a symplectic structure that is naturally associated with the differential equation (1.6). In fact, let V denote the space of real smooth solutions of the Klein–Gordon equation, in a suitable topology. Then, given a choice of a space-like hypersurface Σ, we can define a symplectic form on V,

$$\sigma(\varphi_1, \varphi_2) := \int_\Sigma (\varphi_1 \nabla_\alpha \varphi_2 - \varphi_2 \nabla_\alpha \varphi_1)\, n^\alpha d\mathrm{vol}_\Sigma,$$

which is independent of the choice of Σ. Let $\mathbb{E}^{\pm} : C_0^{\infty}(\mathscr{M}) \to C^{\infty}(\mathscr{M})$ denote the retarded/advanced fundamental solutions of (1.6). Then, for any $f \in C_0^{\infty}(\mathscr{M})$, it is easy to see that $\mathbb{E}f$ is a solution to the field Eq. (1.6), where $\mathbb{E} = \mathbb{E}^{-} - \mathbb{E}^{+}$. From this, we obtain an isomorphism $V \cong C_0^{\infty}(\mathscr{M}) / \ker(\mathbb{E})$. Under this isomorphism, the symplectic form can be written as follows:

$$\sigma([f], [g]) = \int_{\mathscr{M}} f(x)(\mathbb{E}g)(x) d^4x.$$

The quantized field corresponding to this dynamical system can be described in terms of a unital $*$-algebra generated by symbols $\Phi(f)$ (with f in the complexification of $C_0^{\infty}(\mathscr{M})$), that are subject to the following relations:

$$\Phi(\bar{f}) = \Phi(f)^*, \tag{1.7}$$

$$\Phi((\partial_\mu \partial^\mu + m^2) f) = 0, \tag{1.8}$$

$$[\Phi(f), \Phi(g)] = i\sigma([f], [g]). \tag{1.9}$$

Physically, the generators $\Phi(f)$ can be regarded as "smeared" field operators. In terms of the more familiar operator-valued distribution $\hat{\varphi}(x)$ ("the quantum field") we have, at least formally,

$$\Phi(f) = \int \hat{\varphi}(x) f(x) d^4x.$$

Thus, Eq. (1.8) expresses the idea that the quantized field is still a solution of the field equation, whereas (1.9), when written in terms of the quantum field $\hat{\varphi}(x)$, takes the more familiar form of the canonical commutation relations (CCR):

$$[\hat{\varphi}(x), \hat{\varphi}(y)] = i\Delta(x, y). \tag{1.10}$$

Here, $\Delta(x, y)$ denotes the *Pauli–Jordan function*, a distributional solution of (1.6) with causal support [16]. The relation between (1.9) and (1.10) is due to the fact that $\Delta(x, y)$ is also the kernel of the (integral) operator $\mathbb{E}$.

An alternative point of view consists in starting with the symplectic vector space (V, σ) and constructing the corresponding Weyl algebra. The commutation relations obeyed by the generators of the Weyl algebra can be understood as the exponentiated form of the CCR (1.10)

One of the main differences between (standard) quantum mechanics and quantum field theory comes from the Stone–von Neumann theorem, which asserts that, up to unitary equivalence, there is only one irreducible representation of the CCR,

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{i,j}. \tag{1.11}$$

In this case, the symplectic space is $T^*\mathbb{R}^n$, a finite dimensional symplectic vector space. These two assumptions (that the symplectic manifold is a vector space and of finite dimensionality) are essential for the proof of the Stone–von Neumann theorem. Its failure in the case of finite dimensional symplectic manifolds leads to the richness of interplay between topology and symplectic geometry, as discussed previously. In the case of a (free, scalar) quantum field, we are still working with a symplectic vector space, but now of infinite dimensionality.

For the example of the scalar field discussed here, the Hilbert space where the CCR are represented is a *bosonic* Fock space. It can be described in terms of the symmetric tensor algebra of V.

On the other hand, quantization of fermionic fields (such as the one described by the Dirac equation) differs from its bosonic counterpart for commutation relations have to be substituted by *anticommutation* relations due to the spin-statistics connection. The Fock space is then accordingly related to the *exterior algebra* of the space of solutions of the classical equation [17]. As mentioned in Sect. 1.1, quantization of a *gauge theory* entails new difficulties, since the Lagrangian describing such a theory is *singular*, meaning that there are constraints that have to be dealt with in a proper way. Examples of such theories and their quantization are the subject of Chap. 6 (in the context of string theory) and Chaps. 8 and 9 (dealing with different aspects of the Batalin–Vilkovisky formalism).

From the point of view of both mathematics and physics, the appearance of *renormalization* is perhaps one of the most intriguing, as well as interesting, aspects of quantum field theory. Although not discussed in this volume, it is convenient to observe that, at the core of renormalization calculations arising in perturbative quantum field theory, there is a Hopf algebra structure, known as the Connes–Kreimer Hopf algebra [17, 18], which provides an algebraic interpretation of the mechanisms underlying the "forest formula" used by physicists. Another point of view, stemming from the algebraic approach to quantum field theory, uses ideas from deformation quantization to study perturbative renormalization [19]. These two examples provide further illustrations as to how deeply interconnected are the topics discussed in this volume.

References

1. V.N. Gribov, Quantization of non-abelian gauge theories. Nuclear Phys. B **139**(1), 1–19 (1978)
2. V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, 1989)
3. P.A.M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, 1982)
4. B. Kostant, Quantization and unitary representations, in *Lectures in Modern Analysis and Applications III*, vol. 170, Lecture Notes in Mathematics (Springer, 1970), pp. 87–208
5. J.-M. Souriau, *Structure des Systèmes Dynamiques* (Dunod, Paris, 1970)
6. N.M.J. Woodhouse, *Geometric Quantization* (Clarendon Press, 1991)
7. I. Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Birkhauser, 1991)
8. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation Theory and Quantization. I. Deformations of Symplectic Structures* (Annals of Physics, 111, 1978)

9. B. Fedosov, A simple geometrical construction of deformation quantization. J. Differ. Geom. **40**(2), 213–238 (1994)
10. A. Connes, *Noncommutative Geometry* (Academic press, 1995)
11. A. Cardona, C. Neira-Jiménez, H. Ocampo, S. Paycha, A.F. Reyes-Lega, (eds.) *Geometric, Algebraic and Topological Methods for Quantum Field Theory* (World Scientific, 2013)
12. A. Connes, Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$. J. Inst. Math. Jussieu **3**, 203–225 (2004)
13. L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom, J.C. Varilly, The Dirac operator on $SU_q(2)$. Commun. Math. Phys. **259**, 729–759 (2005)
14. G. Landi, W. van Suijlekom, Principal fibrations from noncommutative spheres. Commun. Math. Phys. **260**, 729–759 (2005)
15. A. Cardona, H. Ocampo, S. Paycha (eds.), Geometric and topological methods for quantum field theory, in *Proceedings of the Summer School held in Villa de Leyva, July 9–27, 2001* (World Scientific Publishing Co., Inc., River Edge, 2003)
16. F. Scheck, *Quantum Physics* (Springer, 2007)
17. J.M. Gracia-Bondía, J.C. Várilly, H. Figueroa, *Elements of Noncommutative Geometry* (Springer Science & Business Media, 2013)
18. D. Manchon. Hopf algebras, from basics to applications to renormalization (2004), arXiv:math/0408405
19. R. Brunetti, C. Dappiaggi, K. Fredenhagen, J. Yngvason, *Advances in Algebraic Quantum Field Theory* (Springer, 2015)