New Research Directions in Modern Actuarial Sciences

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Abstract The aim of the paper is to outline the new trends in modern actuarial sciences in order to help the researchers to find new domains of activity and university professors teaching future actuaries to prepare special courses. The paper begins by description of actuarial profession and a brief historical sketch. After recalling the main achievements of the first two periods in actuarial sciences, we describe the new research directions of the third and fourth periods characterized by interplay of insurance and finance, unification of reliability and cost approaches, as well as, consideration of complex systems. Sophisticated mathematical tools are used for analysis and optimization of insurance systems including dividend payment, reinsurance, and investment. Discrete-time models turned out to be more realistic in some situations for investigation of insurance problems.

Keywords Risk · Dividends · Reinsurance · Investment · Ruin · Bankruptcy

1 Introduction

Web site CareerCast.com has ranked actuary the fourth-best job of 2014 taking into account environment, income, hiring outlook, and stress. Data from the U.S. Department of Labor and the Bureau of Labor Statistics, as well as other government agencies, trade associations, and private survey firms were used to evaluate the 200 jobs included in its annual Jobs Rated report. The top three jobs, according to the report, are mathematician, tenured university professor, and statistician.

Math skills are key in landing some of the best jobs in the nation, according to CareerCast's 2015 Jobs Rated report, with four of the nation's ten best jobs focusing on mathematics. An actuary—who uses mathematics, statistics, and financial theory to assess the risk that an event will occur—came in at No. 1 on the list, just ahead of mathematician (No. 3), statistician (No. 4), and data scientist (No. 6).

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Thus, it is natural to ask the following questions. What is an actuary? What is actuarial science? One of the answers is given below.

Actuarial science is the discipline that applies mathematical and statistical methods to assess risk in insurance, finance, and other industries and professions. Actuaries are professionals who are qualified in this field through intense education and experience. In many countries, actuaries must demonstrate their competence by passing a series of thorough professional examinations.

Actuarial profession was formally established in 1848, with the formation of Institute of Actuaries, London. The Faculty of Actuaries, Edinburgh, was organized in 1856, and in 2010 it merged with Institute of Actuaries. The International Actuarial Association (IAA) is a worldwide association of local professional actuarial associations. It was established in 1895 as an association of individuals under the name the "Comité Permanent des Congrés d'Actuaires", renamed IAA in 1968 and restructured at the 26th International Congress of Actuaries, held in Birmingham on 7–12 June 1998. Nowadays IAA includes 69 Full Member Associations, representing 98% of qualified actuaries worldwide, and 28 Associate Member Associations. It has seven sections.

ASTIN, the section for **A**ctuarial **ST**udies **I**n **N**on-life insurance, was created in 1957 as the first section of the IAA. ASTIN's main objective is to promote actuarial research, particularly in non-life insurance. ASTIN is continually working to further develop the mathematical foundation of non-life insurance and reinsurance.

Another section of the IAA, created in 1986, was **AFIR**, which stands for **A**ctuarial Approach for **FI**nancial **R**isks. Its objective was defined as promotion of actuarial research in financial risks and problems, see, e.g., [407]. Effective from 2011, the section mandate was extended to formally include **E**nterprise **R**isk **M**anagement (ERM), hence, the section was named **AFIR/ERM**. The purpose of this change was to expand the discussion beyond market risk issues and provide a strong home for international discussion and research on ERM topics. It is a reflection of the expanding and developing role of ERM in actuarial practice and the IAA efforts to provide support for this growing area of actuarial practice. It is a natural extension and many ERM papers and topics have been presented at past AFIR colloquia, see, e.g., [135] and references therein.

In November 2009, a group of actuarial professional bodies took the unprecedented step of agreeing to collaborate to develop and administer a new qualification in enterprise risk management (ERM)—the Chartered Enterprise Risk Actuary (CERA)—a ground breaking achievement, and the birth of the Global CERA Treaty. The first nine actuaries received this certificate in July 2010.

We do not consider in this paper the scientific activity of such important IAA Sections as the Health Section (IAAHS) created in 2003, the Pensions, Benefits and Social Security Section (PBSS) also started in 2003, and Life Section (IAALS) created in 2005, although these branches of research deserve a special consideration, see, e.g., [398].

The Russian Actuarial Society was organized on September 14, 1994, the first President was Professor A.N. Shiryaev, see [381]. The Russian Guild of Actuaries was founded in 2002 on the basis of the Society of actuaries, established in 1994.

On November 4, 2008, the Russian Guild of Actuaries became the full member of the IAA and was acknowledged as an integral part of international actuarial community.

Actuarial education at the Moscow State University was initiated by Professor B.V. Gnedenko in 1993, see [91]. It is necessary to mention the achievements of Russian actuarial science before 1917. The most well-known person is S.E. Savitsch (1864–1936) who was a vice-president of the first four International Congresses of Actuaries and a member of Organizing Committee of the 8th Congress which was planned to take place in St. Petersburg in 1915 (however canceled due to war). He was a permanent member of the Insurance Committee at the Ministry of Internal Affairs, which carried out insurance supervision in the Russian Empire. For the most part, he was interested in life insurance, health and pensions (see, e.g., his book [367]). However, there also exists his paper [368] dealing with premiums in fire insurance. The worldwide known specialists in probability theory, V.Ya. Bunyakovski [97] and A.A. Markov ([297], chap. VIII) have also contributed to the development of life insurance.

This paper is organized as follows. Historical background is provided in Sect. 2. In particular, we sketch the main steps in the history of actuarial sciences and describe what is actuary of the fourth kind. General description of applied probability models is given in Sect. 3. This description clearly demonstrates the similarity of models arising in different research domains. It is also useful for models classification. New research directions in modern actuarial sciences are presented in Sect. 4 (for continuous-time models) and in Sect. 5 (for discrete-time models). Three examples are treated in Sect. 6. Conclusion is given in Sect. 7.

2 Historical Background

The keyword in all definitions of actuarial sciences is risk. According to the Concise Oxford English Dictionary, “risk is a hazard, a chance of bad consequences, loss or exposure to mischance” (see also [330]).

There exists the following classification of risks. First of all, risk can be pure or speculative. Pure risk entails loss only, whereas speculative one can provide gain, as well as loss. The most known sources of the latter risks are gambling and stock exchange. In its turn, pure risk is subdivided into physical and moral. Both are typical for insurance. Insurance is a means of protection from financial loss. It is a form of risk management primarily used to hedge against the risk of a contingent, uncertain loss. Not all pure risks can be insured. To be insurable a risk must be random, not depend entirely on the will of insured and materialize in the future. Randomness can be of two types. Either the event under consideration (insured’s death) will happen with certainty sometimes, however the occurrence time is random, or the event (e.g., theft of auto) may not occur at all. In the latter case, the occurrence time is also a random variable however its distribution is improper. That is one of the reasons for different approaches in life insurance and non-life insurance.

Risk is present whenever the outcome is uncertain, whether favorable or unfavorable. Traditional actuarial mathematics work best on hazard risks, as they are generally independent and discontinuous. Actuaries and other risk professionals have done a remarkably good job assessing and evaluating hazard risks. Organizations rarely become insolvent due to failure to manage hazard risks.

Financial risks are those that affect assets, including interest rates, inflation, equity values, and foreign exchange rate. These risks are correlated, continuous, and require an understanding of stochastic calculus to be measured appropriately. Unlike hazard risks, financial risks provide the possibility of a gain, not just a loss. The techniques for managing financial risks—financial derivatives such as forwards, futures, options, and swaps—are relatively new. Misuse of these techniques and the resulting financial debacles they caused have actually led to the need for enterprise risk management (ERM), see [133].

According to the Casualty Actuarial Society (CAS), enterprise risk management is defined as: "The process by which organizations in all industries assess, control, exploit, finance and monitor risks from all sources for the purpose of increasing the organization's short and long term value to its stakeholders."

In other words, ERM is the systematic evaluation of all the significant risks facing an organization and how they affect the organization in an aggregate way. Hence, categorizing risks as hazard, financial, operational, or strategic is most useful. Operational risks represent the failure of people, processes, or systems. Strategic risk reflects the business decisions of an organization or the impact of competition or regulation. Examples of strategic risk for insurance are the benefits produced for those first to use credit scoring (see, e.g., [359]) as a rating variable, and the market share losses of those companies that were slow to adopt this approach. ERM originally focused on loss prevention, controlling negative surprises, and reducing downside risk. Now ERM deals with the entire range of potential outcomes, not just downside risk. Accepting risks where it has a comparative advantage, and transferring or avoiding risks where it does not, a company is adding value by efficient risk treatment.

Methods for transferring or distributing risk were practiced by Chinese and Babylonian traders as long ago as the 3rd and 2nd millennia BC, respectively. Chinese merchants traveling treacherous river rapids would redistribute their wares across many vessels to limit the loss due to any single vessel's capsizing. The Babylonians developed a system which was recorded in the famous Code of Hammurabi, c. 1750 BC, and practiced by early Mediterranean sailing merchants. If a merchant received a loan to fund his shipment, he would pay the lender an additional sum in exchange for the lender's guarantee to cancel the loan should the shipment be stolen or lost at sea. Further history of insurance development is described, e.g., in [381].

Why actuarial science emerged significantly later (in the 17th century) one can read in the interesting book by P.L. Bernstein [64] written in 1996 about the risk history.

According to classification given in 1987 by H. Bühlmann [82], there were three periods in actuarial sciences. However less than two decades later, in 2005, P. Embrechts declared the beginning of the fourth period, namely, appearance of actuaries of the fourth kind. S. D'Arcy, in his Presidential address [133] to CAS

(Casualty Actuarial Society), has told that such a term is applied to actuaries working in ERM (enterprise risk management) and explained how to become an actuary of the fourth kind.

Actuaries of the first kind are life actuaries. According to Bühlmann, the primary methods of life actuaries involve deterministic calculations. Actuaries of the second kind, the casualty actuaries, develop probabilistic methods for dealing with risky situations. The actuaries of the third kind deal with the investment side of insurance and incorporate stochastic processes into actuarial calculations. Nowadays, almost all aspects of insurance product development and pricing involve a combination of investment and insurance characteristics. This change requires all actuaries to become actuaries of the third kind. How to reach this goal one can read in [134]. The actuaries of the third kind, who were the object of Bühlmann's editorial, were the investment actuaries applying stochastic processes, contingent claims, and derivatives to assets and liabilities. This specialty developed in the 1980s as financial risk became more important and tools to manage financial risk were created. In order to become the actuary of the fourth kind, one has to learn to deal not only with hazard and financial risks but with operational and strategic risks as well, see [130].

According to [133], in ERM, as in traditional risk management, the first step is *risk identification*. Focus on the most significant risks an organization faces. Deal with those first, then in future iterations expand the focus to the next level of risk elements, as advised one of ERM pioneers, J. Lam (see [248]).

Step two in ERM, as in traditional risk management, is to *quantify the risks*. Actuaries are well skilled in this area, at least for hazard risks, but ERM also requires the quantification of the correlations among different risks. Two risks can be generally uncorrelated, but, if an extreme event were to occur, then they could be highly correlated. Techniques for evaluating these forms of correlations, filters, tail dependency, copulas, and other numerical techniques must be incorporated. Much needs to be done to be able to quantify operational and strategic risk to the standards common in hazard and financial risk (see, e.g., [130, 131]).

Step three of the risk management process involves *evaluating the different methods for handling risk*. Risks can be assumed, transferred, or reduced. A variety of methods exist for transferring (subcontracting, insurance, or securitization) or reducing risk (loss control, contract, or reinsurance).

Step four is to *select the best method for handling the risk*, which in most cases will involve a combination of different techniques. Moreover, the organization wants to make consistent choices about all the risks it faces, how much risk it will accept, and what return it would require for accepting a particular level of risk.

Step five is to *monitor and adjust the risk management approach* selected. It is an iterative process that entails identifying additional significant risks, quantifying those risks, and improving the quantification of previously identified risks based on additional information and improved mathematical techniques. It also entails reevaluating the different approaches to handle risk, implementing an improved strategy, and then, once more, monitoring the result.

In addition to references in [133], the following papers and books may be helpful: [51, 105, 107, 135, 136, 140, 151, 164, 194, 196, 228, 233, 302, 317, 323, 365, 384, 392, 396, 419].

Thus, the first period in actuarial science development was deterministic, see, e.g., [398]. It is characterized by E. Haley's mortality tables which appeared in 1693 and D. Bernoulli's utility functions introduced in 1738. Although some researchers claim that idea of mortality tables belongs to Roman juror Ulpian, the first life tables appeared in the seventeeth century. They were issued by John Graunt in 1662 (some historians attribute them to William Petty, who introduced the new subject named "political arithmetic") and Johan de Witt, 1671. However, E. Halley "was the first individual to describe the principles of actuarial mathematics on scientifically accurate lines" (see [200]).

The second (stochastic) period is marked by the application of probability theory and stochastic processes to solving the actuarial problems. The main achievement of this period is the collective risk theory, in particular, a well-known Cramér–Lundberg model, see, e.g. [299]. It is worth mentioning that one of widely used in practice stochastic processes with independent increments, namely, Poisson process was introduced for the first time in the dissertation of F. Lundberg in 1903, see, e.g. [129]. The other process with independent increments called Brownian motion or Wiener process appeared as a model for stock exchange performance in dissertation of L. Bachelier "Theory of speculation" in 1900. The results of Lundberg were explained and further developed by H. Cramér in 1930s. It is said that the reason for Swedish insurance companies successes was the attention paid to actuarial sciences. Thus, in 1929, a special chair of Statistics and Actuarial Sciences was created at Stockholm university for H. Cramér.

The science has gone through revolutionary changes during the last 40 years due to the proliferation of high-speed computers and the union of stochastic actuarial models with modern financial theory. Thus, one can call the third period financial. It was very short, not more than three decades. The fourth (modern) period has brought, in addition to achievements of previous periods, development of enterprise risk management.

Hence, the modern period is characterized by strong interaction of insurance and finance, investigation of complex systems and employment of sophisticated mathematical tools. The aim of the paper is to outline the new research directions which emerged during the last two decades. Further on, we are going to focus on non-life (general) insurance, mentioning in passing that life insurance is thoroughly treated in the book [234] by M. Koller, see also [165]. The models used in health insurance can be found in [347] by E. Pitacco. Those interested in the famous Wilkie's investment model and its generalizations are referred to [420] and original papers [362, 363, 406, 408, 409].

It is important to underline that the books [127, 158, 195, 217, 232, 237, 241, 326, 329, 341, 348, 352, 361, 383, 413, 421] demonstrate the similarity between the models arising in insurance, finance, and other research fields. Thus, methods used in one research field may turn out fruitful in others. The books [71, 73, 83, 126, 141, 142, 145, 208, 221, 231, 300, 304, 349, 370, 375, 380, 420] also can be useful, along with traditional textbooks such as [56, 75, 81, 138, 173, 371], for professors planning the special courses for actuarial students.

Although the bibliography of this paper contains 446 papers and books, almost all of them published in this century, the list is far from being complete. Further references can be found in the mentioned books and reviews [8, 28, 40, 202, 272, 379]. The last review was published during the preparation of this paper, so the material was rearranged in order not to repeat [379]. Thus, the taxation problems (see, e.g., the loss-carry-forward tax model for Lundberg risk process in [7], or [245] where general tax processes are investigated for Lévy insurance model, as well as [305] dealing with a compound Poisson process under absolute ruin) and statistical estimation (see, e.g., [436] where nonparametric estimation for ruin probability in Lévy risk model is treated) are only mentioned, the interested reader is referred to [379].

3 General Description of Applied Probability Models

Not only insurance, but other applied probability research domains such as inventory and dams, finance, queueing theory, reliability, and some others can be considered as special cases of decision-making under uncertainty (or risk management) aimed at the systems performance optimization, thus eliminating or minimizing risk, see, e.g., [85, 301]. "The capacity to manage risk, and with it appetite to take risk and make forward-looking choices, are key elements of the energy that drives the economic system forward"—one reads in [64]. The ability of businesses to survive and thrive often requires unconventional thinking and calculated risk taking. The key is to make the right decisions—even under the most risky, uncertain, and turbulent conditions, see, e.g., [168].

For correct decision-making, one needs an appropriate mathematical model. For several centuries, mathematics has been the language of the exact sciences. Only in the twentieth century has mathematics become predominant in other fields, particularly economics and finance. Obviously, it is possible to construct a lot of models describing the same real-life event or process more or less precisely. Furthermore, the same mathematical model can arise in different research domains.

Constructing an insurance company model one has to take into account its twofold nature. Originally all insurance societies were designed for risk sharing. Hence, their primary task is policyholders indemnification. Nowadays, for the most part, they are joint-stock companies. Thus, the secondary but very important task is dividend payments to shareholders.

It is well known that insurance company performance generates two cash flows. Namely, the inflow consists of premiums paid by insureds and outflow is determined by claim process. Premiums are paid by all policyholders (insureds) however reimbursement is obtained only by those who suffered from risk realization. Clearly, the insurance company models are of input–output type. They can be described by the following six-tuple $(T, Z, Y, U, \Psi, \mathscr{L})$.

Table 1 Interpretation of model parameters for different research domains

Research field	Input	Output	System state
Insurance	Premium	Indemnity	Surplus
Finance	Money inflow	Money outflow	Capital
Inventory	Supply	Demand	Inventory level
Storage	Water inflow	Water outflow	Water level in a dam
Reliability	New & repaired	Broken elements	Working elements
Queueing	Customers arrival	Served customers	Queue length
Population growth	Birth and immigration	Death and emigration	Population size

Here, T is the planning horizon, $Z = \{Z(t), t \in [0, T]\}$ and $Y = \{Y(t), t \in [0, T]\}$ being input and output processes, respectively. The next element $U = \{U(t), t \in [0, T]\}$ is a control, whereas Ψ represents the system configuration and operation mode. Hence, $X = \Psi(Z, Y, U)$ is the system state, so, $X = \{X(t), t \in [0, T]\}$. All the above-mentioned processes may be multidimensional, moreover, their dimensions may differ. Finally, $\mathscr{L}_T(U) = \mathscr{L}(Z, Y, U, X, T)$ is an objective function (target, valuation criterion, risk measure) evaluating the system performance quality.

Definition 1 A control $U_T^* = \{U^*(t), t \in [0, T]\}$ is called *optimal* if

$$\mathscr{L}_T(U_T^*) = \inf_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T), \quad (\text{or} \quad \mathscr{L}_T(U_T^*) = \sup_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T)), \tag{1}$$

where $\mathscr{U}_T$ is a class of all feasible controls. Furthermore, $U^* = \{U_T^*, T \geq 0\}$ is called an *optimal policy* (or strategy).

If the extremum in (1) cannot be attained, one has to use either the ε-optimal or asymptotic optimal policies.

Giving another interpretation to input and output processes, one can pass (see, e.g., [90]) from one research field to another as shown in Table 1.

3.1 Models Classification

Now, we turn to models classification according to parameters of their general description.

1. The planning horizon can be finite ($T < \infty$) or infinite ($T = \infty$). Furthermore, one can consider continuous or discrete time. In the first case, the system is observed at any time $t \in [0, T]$, in the second one, its behavior is known in a finite or countable set of points belonging to the planning horizon.

2. Input and output processes can be deterministic or stochastic. In the latter case, their distributions may be known completely, partly (unknown parameters), or

unknown at all. Thus, a system will be deterministic or stochastic if the same true for both (input and output) processes. It is called mixed if one process is deterministic while the other one is stochastic.

3. According to the set of feasible controls, the system can be static (control is applied only one time) or dynamic (control is applied many times or continuously). Moreover, one can control input or/and output processes, as well as the system initial state, configuration, and operation mode. Hence, dimensions of underlying processes (input, output, control, and system state) can differ and change in course of system functioning.

4. The last but not least element of systems description is an objective function (risk measure). At first, in many research fields, an objective function was not considered at all. Hence, there was no control and optimization. One can mention here queueing and dam theory. Nowadays, in all applied probability research fields, one is interested in the choice of optimal control providing extremum of some prescribed objective function. Multi-objective optimization (see, [318]) can also be studied.

The most widely used approaches in choosing the objective function are reliability and cost ones. It is clear that reliability approach has arisen in reliability theory. The researchers were always interested in survival time of the system under consideration, in other words, the time until the system failure, as well as, in survival probability. The reliability approach was also used in insurance. Since company solvency is very important for its existence, for a long time the primary task of actuarial sciences was investigation of ruin time and ruin probability.

On the other hand, the cost approach was applied from the beginning in finance and inventory theory. The expected (discounted) costs were typical for inventory models optimization. Mean variance principle was used for portfolio optimization and capital allocation since 1952 when the seminal paper [298] was published. Insurance application of this principle is presented, e.g., in [53, 210], whereas optimal portfolio choice for a loss averse insurer is treated in [192] (see also references therein). Other well-known financial risk measures, such as VaR (Value at Risk) or CVaR (Conditional Value at Risk), were widely used in insurance as well. Coherent risk measures (see, e.g., [22, 162]), deviation measures and expectation bounded risk measures (see, e.g., [360]) became very popular during the last two decades. Now, reliability and cost approaches (along with their various combinations) are used in any applied probability domain.

3.2 New Trends in Actuarial Sciences

Further on, the following characteristics of modern period of actuarial sciences are treated.

- Interplay of actuarial and finance methods, in particular, unification of reliability and cost approaches.

- Investigation of complex systems including dividend payment, reinsurance, investment, and bank loans, as well as taxes. Hence, the necessity of dealing with more intricate models and processes, application of sophisticated mathematical tools.
- Consideration of discrete-time models which turned out to be more appropriate for the description of some aspects of insurance company performance.

Historically, most insurance-related problems deal with jump processes due to the nature of insurance claims which occur at discrete-time points, whereas many classical models in financial mathematics rely on continuous processes to reflect fluctuations in the constantly changing financial markets. Although the two disciplines of applied probability have evolved rather independently, there is a common trend in recent years to incorporate stochastic models with both continuous and jump components, see, e.g., [385].

For example, on the ruin theory side, in addition to the random jumps which account for insurance claims, diffusion components have gained increasing popularity to describe investment returns in sophisticated risk models.

4 New Results for Continuous-Time Insurance Models

It was already mentioned that functioning of insurance company generates two cash flows. Namely, input $Z(t)$ describes the premiums acquired up to time t, whereas output $Y(t)$ represents the payments of company to policyholders in order to satisfy their claims. In other words, $Y(t)$ is the aggregate claim amount up to time t.

Thus, insurance company capital (surplus or reserve) at time t is given by

$$X(t) = x + Z(t) - Y(t), \tag{2}$$

where x is the initial capital.

Continuous-time models were used during the last century and still are very popular. The famous *Cramér–Lundberg model*, which appeared in 1903 (see [129]), has a mixed type. Its input is deterministic $Z(t) = ct$, $c > 0$ is the premium rate, whereas the output $Y(t)$ is a stochastic process

$$Y(t) = \sum_{i=1}^{N(t)} Y_i. \tag{3}$$

Here, the claim number $N(t)$ is a Poisson process with parameter λ, Y_i being the amount of the ith claim. The sequence $\{Y_i\}$ of i.i.d. r.v.'s and $N(t)$ are supposed independent. Thus, $Y(t)$ is a compound Poisson process with intensity λ. It is interesting to mention that $X(t)$ given by (2) and (3) is a particular case of spectral negative Lévy process.

4.1 Decision Problems and Objective Functions

The problems of interest for any insurance company are choice of underwriting procedure, premium principles (see, e.g., [114, 255, 274, 277, 288, 351, 440, 441]) and reserves (see, e.g., [1, 29, 84, 211, 303, 418]) to ensure the company solvency. Moreover, very important decisions are dividend payments, reinsurance, and investment. Hence, very popular research topics are

- calculation of ruin probabilities,
- estimation of ruin severity (Gerber–Shiu function),
- investigation of the rate of capital growth,

as well as, thorough study of models incorporating

- dividends, investment, reinsurance, tax.

4.1.1 Ruin Probability

From the beginning, the *ruin probability* attracted attention of actuaries occupied with company solvency. There exists a vast bibliography pertaining to this problem, see, e.g., [24, 25, 186] and references therein.

Denote by $\tau = \inf\{t > 0 : X(t) < 0\}$ the ruin time of the company. Then, finite-time ruin probability (ruin in interval [0, T]) is defined as follows:

$$\psi(x, T) = P(\tau \le T | X(0) = x) = P(\inf_{0<t\le T} X(t) < 0),$$

whereas the probability of ultimate ruin is given by

$$\psi(x) = P(\tau < \infty | X(0) = x) = \lim_{T\to\infty} \psi(x, T).$$

Much of the literature on ruin theory is concentrated on classical risk model, in which the insurer starts with an initial surplus, and collects premiums continuously at a constant rate, while the aggregate claims process follows a compound Poisson process.

In 1957, Sparre Andersen (see, [19]) let claims occur according to a more general renewal process and derived an integral equation for the corresponding ruin probability. Since then, random walks and queuing theory have provided a more general framework, which has led to explicit results in the case where the interclaim times or the claim severities have distributions related to the Erlang or phase-type distributions.

Some *other generalizations of the basic model* will be outlined in the next subsection. Now, we only mention that ruin probability was investigated under various assumptions. Thus, the explicit formulas for ruin probability with dependence between risks, arising due to mixing over simple model parameters, were established in [13]. Archimedean dependence structure can be considered as a particular case of such procedure. Other classes of processes for which explicit expressions for ruin probability exist can be found in [25]. Estimates of ruin probabilities for Cramér–Lundberg model with stochastic premiums are established in [20]. Review of fluid methods in ruin theory is given in [40]. Generalization of De Vylder approximation for ruin probability is provided in [41].

The author of [80] deals with obtaining the optimal investment policy in a risky asset minimizing the ruin probability. The related objective of minimizing the expected discounted penalty paid at ruin is treated as well. Minimization of ruin probability by choosing the optimal investment is also the object of [57]. The authors consider an insurance company whose surplus is represented by the classical Cramér–Lundberg process. The company can invest its surplus in a risk-free asset and in a risky asset, governed by the Black–Scholes equation. There is a constraint that the insurance company can only invest in the risky asset at a limited leveraging level. The minimal ruin probability as a function of the initial surplus is characterized by a classical solution to the corresponding Hamilton–Jacobi–Bellman (HJB) equation. It is shown that the optimal investment policy significantly differs from those established in [203] for unrestricted case or in [35] for the case of no shortselling and no borrowing. Minimization of the ruin probability by investment and reinsurance is considered in [369].

Ruin probabilities with dependent rate interests are treated [99], whereas in [100] stochastic rates of interest and in [102] Markov Chain interests are assumed. The bounds for ruin probabilities in multivariate risk model are obtained in [103]. The ruin for the Erlang(n) risk process is tackled in [266]. Ruin probabilities for two classes of risk processes are studied in [269]. Ruin theoretical and financial applications of the first passage time for compound Poisson process perturbed by diffusion are given in [251]. Lundberg type bounds are obtained in [372] by investigation of renewal equations.

An important question in insurance is how to evaluate the probabilities of (non-) ruin of a company over any given horizon of finite length. The paper [261] aims to present some (not all) useful methods that have been proposed for computing, or approximating, these probabilities in the case of discrete claim severities. The starting model is the classical compound Poisson risk model with constant premium and independent and identically distributed claim severities. Two generalized versions of the model are then examined. The former incorporates a nonconstant premium function and a nonstationary claim process. The latter takes into account a possible interdependence between the successive claim severities. Special attention is paid to a recursive computational method that enables us to tackle, in a simple and unified way, the different models under consideration. The approach, still relatively little known, relies on the use of remarkable families of polynomials which are of Appell or generalized Appell (Sheffer) types.

Asymptotic behavior and estimates of ruin probabilities are given, e.g., in [26, 63, 123, 159, 187, 235]. Two papers, [321, 322], are devoted to investigation of ruin probabilities under capital injections. The paper [205] establishes the asymptotics of ruin probabilities for controlled risk processes in the small claims case.

A thorough survey of the ruin problem in risk models with investment income (until 2008) is presented in [333] (see also [331]). In addition to a general presentation of the problem, topics covered are a presentation of the relevant integro-differential equations, exact and numerical solutions, asymptotic results, bounds on the ruin probability and also the possibility of minimizing the ruin probability by investment and possibly reinsurance control. The main emphasis is on continuous-time models, but discrete-time models are also covered.

4.1.2 Gerber–Shiu Function

- *The ruin probability is a popular but not always a good risk measure*. To treat solvency problems, it is important to know the ruin time distribution and severity of ruin.

Already in 1988, Dufresne and Gerber (see [154]) in the classical compound Poisson model of the collective risk theory considered U, the surplus before the claim that causes ruin, and V, the deficit at the time of ruin. Let $f(x; u, v)$ be their joint density (x initial surplus) which is a defective probability density (since U and V are only defined, if ruin takes place). For an arbitrary claim amount distribution, they established that $f(0; u, v) = ap(u + v)$, where $p(z)$ is the probability density function of a claim amount and a is the ratio of the Poisson parameter and the rate of premium income. After that, the distribution of the surplus prior to ruin and that of the claim causing ruin were studied in [143, 144], respectively.

During 1997–1998, Gerber and Shiu (see [176, 177]) introduced the expected discounted penalty function (EDPF) taking into account the surpluses immediately before and at ruin. Since then, many researchers studied the following function

$$m(x) = \mathsf{E}(e^{-\delta\tau} w(X(\tau^-), |X(\tau)|) I(\tau < \infty) | X(0) = x),$$

where δ is the force of interest, $I(A)$ is indicator of event A and $w(x_1, x_2)$ is a nonnegative penalty function defined on $[0, \infty) \times [0, \infty)$.

- So one can see the unification of reliability and cost approaches. (The ruin probability is obtained for $\delta = 0$, $w(x_1, x_2) \equiv 1$.)

The joint analysis of these random variables, which had been traditionally studied separately, allowed to offer an elegant characterization of the ruin event in terms of a renewal equation.

The function $m(x)$ (called frequently EDPF) is useful whenever one wishes to place a value on cash flows triggered by the first passage of a process across a given barrier. Applications of the EDPF are natural not only in the context of solvency where it can be used to determine the initial capital required by a company to avoid insolvency with a minimum level of confidence, but in option pricing or dividends optimization as well. This is the case for credit risky securities, whose cash flows depend on a firm's assets falling below its liabilities, or for American options, whose exercise is triggered by the underlying security's market value crossing an exercise boundary.

By the end of the last century, Gerber and Landry (see [175]) and Gerber and Shiu (see [178]), for example, used the EDPF to price perpetual American options and reset guarantees.

The deficit at ruin and surplus before ruin were studied in [43] for a correlated risk process. Moments of the surplus before ruin and the deficit at ruin are obtained in [109] for Erlang-2 risk model. Approximations for moments of deficit at ruin for the case of exponential and sub-exponential claims are given in [110]. The distribution of the deficit at ruin when claims are phase-type is provided in [153]. The maximum surplus before ruin in an Erlang(n) risk process is treated in [265]. The moments of the time of ruin, the surplus before ruin, and the deficit at ruin are tackled in [281].

The ruin probability and the Gerber–Shiu function in a compound renewal (Sparre Andersen) risk process with interclaim times that have a K_n distribution (i.e., the Laplace transform of their density function is a ratio of two polynomials of degree at most $n \in N$) was studied in [267]. The Laplace transform of the expected discounted penalty function at ruin is derived. This leads to a generalization of the defective renewal equations given in [179, 410]. The explicit results are established for rationally distributed claim severities. The case of Erlang interclaim times has been studied in [179, 266].

By now, EDPF is usually called Gerber–Shiu function according to the names of its inventors. It was investigated in many papers under various assumptions about the underlying risk model. The almost universal approach of analysis is the derivation of some (defective) renewal equations, coming from a set of integro-differential equations which are obtained via Itô's formula or the infinitesimal generator of the risk reserve process. There exists already a special book [242] devoted to Gerber–Shiu risk theory.

Gerber–Shiu function is studied in [247] for the following generalization of Cramér–Lundberg model. The claim sizes are allowed to take positive as well as negative values. Depending on the sign of these amounts, they are interpreted either as claims made by insureds or as income from deceased annuitants, respectively. The classical risk model with a two-step premium rate is treated in [437]. Gerber–Shiu analysis in a perturbed risk model with dependence between claim sizes and interclaim times is implemented in [435]. A Sparre Andersen risk process perturbed by diffusion is dealt with in [268]. The Gerber–Shiu discounted penalty functions for a risk model with two classes of claims is investigated in [438].

In [116], a generalization of the usual penalty function is proposed, and a defective renewal equation is derived for the Gerber–Shiu discounted penalty function in the classical risk model. This is used to derive the trivariate distribution of the deficit at ruin, the surplus prior to ruin, and the surplus immediately following the second last claim before ruin. The marginal distribution of the last interclaim time before ruin is derived and studied, and its joint distribution with the claim causing ruin is derived. In [117], the results of previous paper are extended on the Sparre Andersen models allowing for possible dependence between claim sizes and interclaim times. The penalty function is assumed to depend on some or all of the surplus immediately prior to ruin, the deficit at ruin, the minimum surplus before ruin, and the surplus immediately after the second last claim before ruin. Defective joint and marginal distributions involving these quantities are derived. A discussion of Lundberg's fundamental equation and the generalized adjustment coefficient is given, and the connection to a defective renewal equation is considered.

The analysis of the Gerber–Shiu discounted penalty function for risk processes with Markovian arrivals is performed in [2]. The paper [32] concerns an optimal dividend distribution problem for an insurance company whose risk process evolves as a spectrally negative Lévy process (in the absence of dividend payments). The management of the company is assumed to control timing and size of dividend payments. The objective is to maximize the sum of the expected cumulative discounted dividend payments received until the moment of ruin and a penalty payment at the moment of ruin, which is an increasing function of the size of the shortfall at ruin. Compound geometric residual lifetime distributions and the deficit at ruin are studied in [411], whereas in [412] the author treats the discounted penalty function in the renewal risk model with general interclaim times.

The penalty delivered by the classical EDPF has local nature, in the sense that it only characterizes the surplus in a neighborhood of the ruin time. So, one can explore the possibility of introducing path-dependent variables in the EDPF such as the last minimum of the surplus before ruin (see [68]).

A generalized Gerber–Shiu measure for Markov additive risk processes with phase-type claims and capital injections is studied in [79]. It is supposed that the arrivals (either claims or capital injections) occur according to a Markovian point process. Both claim and capital injection sizes are phase-type distributed and the model allows for possible correlations between these and the interclaim times. The premium income is modeled by a Markov-modulated Brownian motion which may depend on the underlying phases of the point arrival process. For this risk reserve model, the authors derive a generalized Gerber–Shiu measure that is the joint distribution of the time to ruin, the surplus immediately before ruin, the deficit at ruin, the minimal risk reserve before ruin, and the time until this minimum is attained. The investigation is based on the results concerning the joint distribution of the space-time positions of overshoots and undershoots derived in [78] for Markov additive processes with phase-type jumps.

An explicit characterization of a generalized version of the Gerber–Shiu function in terms of scale functions is provided in [67] for spectrally negative Lévy insurance risk processes. The joint analysis of discounted aggregate claim costs until ruin is carried out in the recent thesis [282], the other ruin-related quantities are also examined.

- There arose the new research directions in actuarial sciences specific for modern period. They include, along with dividend payments, reinsurance, and investment problems.
- Thus, the treatment of complex models and consideration of new classes of processes, such as Markov-modulated processes, martingales, diffusion, Lévy processes or generalized renewal ones is needed.

- Several types of objective functions and various methods are used to implement the stochastic models optimization.

In order to understand the papers treating the modern actuarial problems, it is necessary to possess solid knowledge in the field of probability theory and stochastic processes. Let us mention Lévy processes (see, e.g., [21, 65, 241, 366]), point processes [215], Brownian motion and stochastic calculus [156, 227, 230], convergence of probability measures [69], limit theorems for stochastic processes [216] which are widely used nowadays by researchers. One has to be also acquainted with stochastic control and dynamic programming (see, e.g., [37, 327, 343, 370]), Markov decision processes [52] and controlled Markov processes [163]. Among the others, one finds in [40] fluid flow matrix analytic methods, in [238] Volterra integro-differential equations, in [309] renewal processes. It is necessary to be able to deal not only with ordinary differential equations (ODE), see, e.g., [197], but with SDE (stochastic differential equations), see, e.g. [328]. Very important area is risk management (see, e.g., [208, 300]). As previously, we stress that it is impossible to mention all the needed mathematical tools and sources to study them.

4.1.3 Dividends

- Now we turn to the decision problems arising in actuarial sciences.

We briefly recall that a *dividend* is a distribution of a portion of a company's earnings, decided by the board of directors, to a class of its shareholders. Dividends can be issued as cash payments, as shares of stock, or other property.

The study of dividends in insurance was proposed by B. de Finetti in 1957, see [139]. He argued that under net profit condition the company surplus could become infinitely large as time grows that is not realistic. So, it is necessary to decide when and how much to pay, in other words, to choose a dividend strategy.

There exist a lot of possible dividends strategies. The simplest one is a *barrier strategy* with barrier level b. Such a strategy means that there is no dividends payment if $X(t) < b$, whereas the payment intensity equals c (the premium rate), if $X(t) = b$.

Let $V(x, b) = \mathsf{E}\left[\int_0^\tau e^{-\delta t}\, dD(t)\right]$ be the expected discounted dividends until ruin time τ under barrier strategy with parameter b, whereas x denotes the initial company surplus, $0 \le x \le b$. Then, according to [181], $V(x, b)$, as a function of x, satisfies the following equation

$$cV'(x, b) - (\lambda + \delta)V(x, b) + \lambda \int_0^x V(y, b)p(x - y)dy = 0, \quad 0 < x < b, \tag{4}$$

with the boundary condition $V'(b, b) = 1$.

In [11], exact solutions for dividend strategies of threshold and linear barrier type in a Sparre Andersen model are established. Barrier strategies are studied in, e.g., [4, 9, 287, 291, 378, 429] under various assumptions.

The main drawback of a barrier strategy is that sooner or later the company surplus becomes negative bringing the ruin (or bankruptcy).

In a *threshold strategy*, no dividends are paid when the risk reserve is below a certain threshold, while above this threshold dividends are paid at a rate that is less than the rate of premium income, see, e.g., [38, 39, 44, 45, 121, 172, 279, 280, 284, 285, 311, 319, 400]. Such a strategy leads to probability of ruin less than 1.

It is necessary to mention multi-threshold (see, e.g., [6]) and band strategies (see, e.g., [36] or [370]) as well.

In insurance risk theory, dividend and aggregate claim amount are of great research interest as they represent the insurance company's payments to its shareholders and policyholders, respectively. Since the analyses of these two quantities are performed separately in the literature, the Gerber–Shiu expected discounted penalty function was generalized in [120] by further incorporating the moments of the aggregate discounted claims until ruin and the discounted dividends until ruin. While in [120], the authors considered the compound Poisson model with a dividend barrier in which ruin occurs almost surely, the paper [115] looks at this generalized Gerber–Shiu function under a threshold dividend strategy where the insurer has a positive survival probability. Because the Gerber–Shiu function is only defined for sample paths leading to ruin, the joint moments of the aggregate discounted claims and the discounted dividends without ruin occurring are also studied. Some explicit formulas are derived when the individual claim distribution follows a combination of exponentials. Numerical illustrations involving the correlation between aggregate discounted claims and discounted dividends are given.

Optimal dividend payments under a time of ruin constraint in case of exponential claims are considered by the authors of [199]. In [201], optimal dividend payment is studied under ruin constraint in three cases: de Finetti model in which time and space are discrete, continuous-time Brownian motion with drift model and Cramér–Lundberg model with exponential claims. Value function at each time point is supposed to depend on two variables (current surplus and current ruin probability). Dynamic equations are derived on the base of assumption that ruin probability does not exceed a given small α. They can be solved numerically in the discrete model and might be used to identify the optimal strategy in the other cases.

Dividend problems are also discussed in [8, 10, 15, 16, 28, 31, 32, 34, 36, 54, 55, 87, 89, 108, 124, 149, 160, 180, 181, 219, 220, 225, 239, 264, 270, 271, 275, 279, 287, 291, 292, 358, 376, 401, 431, 434, 439].

4.1.4 Investment

- Another notion we are going to use is *investment*.

To invest is to allocate money (or sometimes another resource, such as time) in the expectation of some benefit in the future. In finance, the expected future benefit from investment is a return. The return may consist of capital gain and/or investment income, including dividends, interest, rental income, etc.

Investment generally results in acquiring an asset, also called an investment. If the asset is available at a price worth investing, it is normally expected either to generate income, or to appreciate in value, so that it can be sold at a higher price (or both). It is worth mentioning that the Code of Hammurabi provided a legal framework for investment. Various aspects of investment role in company performance optimization are studied in many papers. We mention below only some recent results.

The optimal dividend problem for an insurance company whose uncontrolled reserve process evolves as a classical Cramér–Lundberg process is considered in [36]. The firm has the option of investing part of the surplus in a Black–Scholes financial market. The objective is to find a strategy consisting of both investment and dividend payment policies which maximizes the cumulative expected discounted dividend payouts until the time of bankruptcy. It is shown that the optimal value function is the smallest viscosity solution of the associated second-order integro-differential Hamilton–Jacobi–Bellman equation. The regularity of the optimal value function is studied. It is proved that the optimal dividend payment strategy has a band structure. A method is found to construct a candidate solution and obtain a verification result to check optimality. Finally, an example is given where the optimal dividend strategy is not barrier and the optimal value function is not twice continuously differentiable.

A combination of investment and reinsurance is treated in [66] under assumption of diffusion approximation. The aim is minimization of the absolute ruin risk (this notion will be discussed later). The paper [223] addresses the situation where the reserve of an insurance business is currently invested in an asset that may yield negative interest. Upper and lower bounds for the probability of ruin are obtained in the case where the cash flow of premiums less claims and the logarithm of the asset price are both Lévy processes. These bounds are in general power functions of the initial reserve. Thus, it is shown that risky investments may impair the insurer's solvency just as severely as do large claims. One can also find in this paper references on previous results concerning ruin problem and investment.

The paper [276] focuses on the optimal investment problem for an insurer and a reinsurer. The insurer's and reinsurer's surplus processes are both approximated by a Brownian motion with drift and the insurer can purchase proportional reinsurance from the reinsurer. In addition, both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. First, the optimization problem of minimizing the ruin probability for the insurer is studied. Then according to the optimal reinsurance proportion chosen by the insurer, two optimal investment problems for the reinsurer are investigated, namely, the problem of maximizing the exponential utility and the problem of minimizing the ruin probability. By solving the corresponding Hamilton–Jacobi–Bellman (HJB) equations, optimal strategies for both the insurer and the reinsurer are derived explicitly. Furthermore, it is established that the reinsurer's optimal strategies in these two cases are equivalent for some special parameters. Finally, numerical simulations are presented to illustrate the effects of model parameters on the optimal strategies.

In [203], the ruin probability of the risk process, modeled as a compound Poisson process, is minimized by the choice of a suitable investment strategy in a risky asset (market index) that follows a geometric Brownian motion. The optimal strategy is computed using the Bellman equation. The existence of a smooth solution and a verification theorem are proved. The explicit solutions in some cases with exponential claim size distribution, as well as numerical results in a case with Pareto claim size, are given. For this last case, the optimal amount invested will not be bounded.

Optimal investment and proportional reinsurance in the Sparre Andersen model are treated in [278]. Optimal investment and risk control for an insurer under inside information are considered in [337]. Optimal investment, consumption, and proportional reinsurance under model uncertainty are studied in [338]. An extension of Paulsen–Gjessing's risk model with stochastic return on investments is dealt with in [430]. Expected utility maximization for insurer by optimal investment and risk control is provided in [445], see also [446].

Insurance models with stochastic return on investments are also considered in [57–62, 166, 169–171, 188, 189, 222, 332, 334, 340].

4.1.5 Reinsurance

- Now we have to answer what is *reinsurance*.

Reinsurance is the practice of insurers transferring portions of risk portfolios to other parties by some form of agreement in order to reduce the likelihood of having to pay a large obligation resulting from an insurance claim. The intent of reinsurance is for an insurance company to reduce the risks associated with underwritten policies by spreading risks across alternative institutions. It is well known as *insurance for insurers*. Legal rights of the policyholders (insureds) are in no way affected by reinsurance, and the insurer remains liable to the insureds for insurance policy benefits and claims.

The most popular approach is to minimize some measure of the first insurer's risk after reinsurance, although the interests of reinsurer are sometimes also taken into account. Thus, in [104] a "reciprocal reinsurance" was treated to consider the objectives of both companies, while in [213], portfolio selection problem for an insurer as well as a reinsurer aiming at maximizing the probability of survival is tackled. The authors of [48] propose a risk sharing approach in order to diversify the risk as much as possible, so as to make the "global market risk" (or systemic risk, in this paper) as close as possible to the total sum of partial risks. In other words, the paper deals with "reciprocal reinsurance contracts" involving n companies.

An optimal reinsurance strategy combining a proportional and an excess of loss reinsurance is obtained in [185] for a collective risk theory model with two classes of dependent risks. The aim is to maximize the expected utility of the terminal wealth. Using the control technique, the Hamilton–Jacobi–Bellman equation is written and, in the special case of the only excess of loss reinsurance, the optimal strategy and the corresponding value function are given in a closed form. In [315], more general case is studied, namely, optimal reinsurance in the model with several risks within one insurance policy.

A two-dimensional risk model with proportional reinsurance is treated in [47]. A review concerning optimal reinsurance up to year 2009 can be found in [106], whereas optimal reinsurance under ruin probability constraint is surveyed in [224]. In [49], the authors deal with optimization of reinsurance taking into account not only risk but uncertainty (or ambiguity) of statistical data possessed by insurer and reinsurer. The levels of uncertainty of insurer and reinsurer do not have to be identical. Furthermore, the decision variable is not the retained (or ceded) risk, but its sensitivity (mathematical derivative) with respect to the total claims. Thus, if one imposes strictly positive lower bounds for this variable, the reinsurer moral hazard is totally eliminated. Necessary and sufficient optimality conditions are given. The optimal reinsurance problem is shown to be equivalent to other linear programming problem (the double-dual problem), despite the fact that risk and uncertainty (and many pricing principles) cannot be represented by linear expressions. This fact explains why the nonlinear optimal reinsurance problem may be solved by a bang-bang reinsurance. Optimal investment, consumption, and proportional reinsurance under model uncertainty is treated in [338].

Optimal control of capital injections by reinsurance in a diffusion approximation is investigated in [157]. A correlated aggregate claims model with common Poisson shocks, which allows the dependence in n ($n \geq 2$) classes of business across m ($m \geq 1$) different types of stochastic events is presented in [209]. The dependence structure between different claim numbers is connected with the thinning procedure. Under combination of quota-share and excess of loss reinsurance arrangements, the properties of the proposed risk model are examined. An upper bound for the ruin probability determined by the adjustment coefficient is established through martingale approach. Optimal risk control and dividend policies under excess of loss reinsurance are considered in [313].

Optimal reinsurance under distortion risk measures is treated in [440, 441]. In the first paper, the authors impose a premium constraint, in the second one, expected value premium principle is applied for reinsurer. The paper [443] investigates optimal reinsurance strategies for an insurer with multiple lines of business under the criterion of minimizing its total capital requirement calculated based on the multivariate lower orthant Value at Risk. The reinsurance is purchased by the insurer for each line of business separately. The premium principles used to compute the reinsurance premiums are allowed to differ from one line of business to another, but they all satisfy three mild conditions: distribution invariance, risk loading and preserving the convex order, which are satisfied by many popular premium principles. It is shown that an optimal strategy for the insurer is to buy a two-layer reinsurance policy for each line of business, and it reduces to be a one-layer reinsurance contract for premium principles satisfying some additional mild conditions, which are met by the expected value principle, standard deviation principle, and Wang's principle among many others.

The risk models incorporating reinsurance can be also found in [34, 66, 72, 77, 86, 89, 96, 125, 206, 207, 225, 278, 295, 336, 390, 391].

4.1.6 Solvency

The *solvency problems* (see, e.g. [339, 365]), a company bankruptcy or liquidation gave rise to the introduction of new notions of ruin.

- *Absolute ruin*

Since its practical importance, the absolute ruin problem has attracted growing attention in risk theory. When the surplus is below zero or the insurer is on deficit, the insurer could borrow money at a debit interest rate to pay claims. Meanwhile, the insurer will repay the debts from the premium income. The negative surplus may return to a positive level. However, when the negative surplus is below a certain critical level, the surplus is no longer able to become positive. Absolute ruin occurs at this moment. One of the first papers in this direction is [174].

One of the latest is [167] where the dividend payments in a compound Poisson model with a constant debit interest r are considered. That is to say, the insurer can borrow an amount of money equal to the deficit at a debit interest force r when the surplus is negative. Meanwhile, the insurance company will repay the debts continuously from its premium income (acquired at rate c). Denoting the surplus of the insurer at time t with the debit interest r by $X(t)$, one easily gets the following equation satisfied

$$dX(t) = \begin{cases} cdt - dY(t), & X(t) \geq 0, \\ (c + rX(t))dt - dY(t), & -c/r \leq X(t) < 0, \end{cases} \tag{5}$$

where $Y(t)$ is given by (3). It is also assumed that dividends are paid to shareholders according to a barrier strategy with parameter $b > 0$. Under the barrier strategy, the premium incomes are paid out as dividends when the surplus reaches b, that is, when the value of the surplus hits b, dividends are paid continuously at rate c and the surplus remains at level b until the next claim occurs. Denote the aggregate dividends paid in the time interval $[0, t]$ by $D(t)$. So the modified surplus $X_b(t) = X(t) - D(t)$. The time of absolute ruin is defined as $T_b = \inf\{t > 0 : X_b(t) \leq -c/r\}$. Then $D_{x,b} = \int_0^{T_b} e^{-\delta t}\, dD(t)$ is the present value of all dividends payable to shareholders, till absolute ruin time T_b, calculated at a constant force of interest $\delta > 0$, whereas x is the initial surplus of insurer.

The authors investigate the moment generating function of $D_{x,b}$, that is, $M(x, y, b) = \mathsf{E} \exp(yD_{x,b})$. They put $M(x, y, b) = M_1(x, y, b)$ for $0 \leq x \leq b$ and $M(x, y, b) = M_2(x, y, b)$ for $-c/r \leq x < 0$. Then, assuming the functions to be smooth in x and y and using the strong Markov property of the surplus process, they establish the following integro-differential equations. For $0 < x < b$

$$c(\partial/\partial x)M_1(x,y,b) = \delta y(\partial/\partial y)M_1(x,y,b) + \lambda M_1(x,y,b)$$
$$-\lambda\left[\int_0^x M_1(x-u,y,b)\,dF(u) + \int_x^{x+c/r} M_2(x-u,y,b)\,dF(u) + \bar{F}(x+c/r)\right]$$

and for $-c/r < x < 0$

$$(rx+c)(\partial/\partial x)M_2(x,y,b) = \delta y(\partial/\partial y)M_2(x,y,b) + \lambda M_2(x,y,b)$$
$$-\lambda\left[\int_0^{x+c/r} M_2(x-u,y,b)\,dF(u) + \bar{F}(x+c/r)\right].$$

Here, $\bar{F}(t) = 1 - F(t)$ and F is the distribution function of claim size. Additionally, $M(x,y,b)$ satisfies the following conditions

$$(\partial/\partial x)M_1(x,y,b) = yM_1(x,y,b), \quad x = b, \quad M_2(-c/r,y,b) = 1,$$

right and left limits of $M(x,y,b)$, as $x \to 0$, coincide.

This result allows to establish the equations for the moments of $D_{x,b}$ and calculate the explicit form of moments and $M(x,y,b)$ for the case of exponential claim distribution. Thus, it is possible to find the optimal dividend barrier for exponential claims.

Minimization of the risk of absolute ruin under a diffusion approximation model with reinsurance and investment is considered in [66]. On the contrary, in [101] it is assumed that the surplus of an insurer follows a compound Poisson surplus process. The expected discounted penalty function at absolute ruin is studied. Moreover, it is shown that when the initial surplus goes to infinity, the absolute ruin probability and the classical ruin probability are asymptotically equal for heavy-tailed claims, while the ratio of the absolute ruin probability to the classical ruin probability goes to a positive constant that is less than one for light-tailed claims. Explicit expressions for the function in exponential claims case are also given. Absolute ruin probability in a Markov risk model is treated in [286].

An Ornstein–Uhlenbeck type risk model is considered in [290]. The time value of absolute ruin in the compound Poisson process with tax is studied in [305]. First, a system of integro-differential equations satisfied by the expected discounted penalty function is derived. Second, closed-form expressions for the expected discounted total sum of tax payments until absolute ruin and the Laplace–Stieltjes transform (LST) of the total duration of negative surplus are obtained. Third, for exponential individual claims, closed-form expressions for the absolute ruin probability, the LST of the time to absolute ruin, the distribution function of the deficit at absolute ruin, and the expected accumulated discounted tax are given. Fourth, for general individual claim distributions, when the initial surplus goes to infinity, it is shown that the ratio of the absolute ruin probability with tax to that without tax goes to a positive constant which is greater than one. Finally, the asymptotic behavior of the absolute ruin probability is investigated for a modified risk model where the interest rate on a positive surplus is involved.

In [312], the absolute ruin in a Sparre Andersen risk model with constant interest is considered, whereas in [401, 402], the absolute ruin problems are treated for the classical risk model.

In the paper [296], it is assumed that the surplus process of an insurance entity is represented by a pure diffusion. The company can invest its surplus into a Black–Scholes risky asset and a risk-free asset. The following investment restrictions are imposed. Only a limited amount is allowed in the risky asset and no short-selling is allowed. When the surplus level becomes negative, the company can borrow to continue financing. The ultimate objective is to seek an optimal investment strategy that minimizes the probability of absolute ruin, i.e., the probability that the lim inf of the surplus process is $-\infty$. The corresponding Hamilton–Jacobi–Bellman (HJB) equation is analyzed and a verification theorem is proved. Applying the HJB method authors obtain explicit expressions for the S-shaped minimal absolute ruin function and its associated optimal investment strategy. In the second part of the paper, the optimization problem with both investment and proportional reinsurance control is studied. There, the minimal absolute ruin function and the feedback optimal investment-reinsurance control are found explicitly as well.

Absolute ruin probability for a multi-type-insurance risk model is treated in [422].

- *Parisian ruin*

In the last few years, the idea of Parisian ruin has attracted a lot of attention. The idea comes from Parisian options (see, e.g., [111]), the prices of which depend on the excursions of the underlying asset prices above or below a barrier. An example is a Parisian down-and-out option, the owner of which loses the option if the underlying asset price S reaches the level l and remains constantly below this level for a time interval longer than d.

In Parisian type ruin models, the insurance company is not immediately liquidated when it defaults: a grace period is granted before liquidation. More precisely, Parisian ruin occurs if the time spent below a predetermined critical level (red zone) is longer than the implementation delay, also called the clock. Originally, two types of Parisian ruin have been considered, one with deterministic delays (see, e.g., [132, 293]) and another one with stochastic delays ([253, 257]). These two types of Parisian ruin start a new clock each time the surplus enters the red zone, either deterministic or stochastic. A third definition of Parisian ruin, called cumulative Parisian ruin, has been proposed very recently in [191]; in that case, the race is between a single deterministic clock and the sum of the excursions below the critical level.

In the paper [289], the time of Parisian ruin with a deterministic delay is considered for a refracted Lévy insurance risk process.

In [293], for a spectrally negative Lévy process, a compact formula is given for the Parisian ruin probability, which is defined by the probability that the process exhibits an excursion below zero, with a length that exceeds a certain fixed period r. The formula involves only the scale function of the spectrally negative Lévy process and the distribution of the process at time r.

Another relevant paper is [257]. Here the authors study, for a spectrally negative Lévy process of bounded variation, a somewhat different type of Parisian stopping time, in which, loosely speaking, the deterministic, fixed delay r is replaced by an independent exponential random variable with a fixed parameter $p > 0$. To be a little bit more precise, each time the process starts a new excursion below zero, a new independent exponential random variable with parameter p is considered, and the stopping time of interest, let us denote it by $k_{exp}(p)$, is defined as the first time when the length of the excursion is bigger than the value of the accompanying exponential random variable. Although in insurance the stopping time $k_{exp}(p)$ is arguably less interesting than k_r (corresponding to a fixed delay r), working with exponentially distributed delays allowed the authors to obtain relatively simple expressions, for example, the Laplace transform of $k_{exp}(p)$ in terms of the so-called (q-)scale functions of X. In order to avoid a misunderstanding, we emphasize that, in the definition of $k_{exp}(p)$, by [257], there is not a single underlying exponential random variable, but a whole sequence (each attached to a separate excursion below zero); therefore $P_x(k_{exp}(p) \in dz)$ does not equal $\int_0^\infty pe^{-pr} P_x(k_r \in dz)\,dr$.

In the paper [137], a single barrier strategy is applied to optimize dividend payments in the situation where there is a time lag $d > 0$ between decision and implementation. Using a classical surplus process with exponentially distributed jumps, the optimal barrier b^* maximizing the expected present value of dividends is established.

Parisian-type ruin is treated in [357] for an insurance ruin model with an adaptive premium rate, referred to as restructuring/refraction, in which classical ruin and bankruptcy are distinguished. In this model, the premium rate is increased as soon as the wealth process falls into the red zone and is brought back to its regular level when the wealth process recovers. The analysis is focused mainly on the time a refracted Lévy risk process spends in the red zone (analogous to the duration of the negative surplus). Building on results from [243], the distribution of various functionals related to occupation times of refracted spectrally negative Lévy processes is obtained. For example, these results are used to compute both the probability of bankruptcy and the probability of Parisian ruin in this model with restructuring.

Other Parisian problems are treated in [414].

- *Omega model*

In classical risk theory, a company goes out of business as soon as ruin occurs, that is, when the surplus is negative for the first time. In the Omega model, there is a distinction between ruin (negative surplus) and bankruptcy (going out of business). It is assumed that even with a negative surplus, the company can do business as usual and continue until bankruptcy occurs. The probability for bankruptcy is quantified by a bankruptcy rate function $\omega(x)$, where x is the value of the negative surplus. The symbol for this function leads to the name Omega model. The idea of distinguishing ruin from bankruptcy comes from the impression that some companies and certain industries seem to be able to continue doing business even when they are technically ruined. This may especially be true for companies that are owned by governments or other companies. Such a model was introduced in [14]. Assuming that dividends can only be paid with a certain probability at each point of time, the authors derive closed-form formulas for the expected discounted dividends until bankruptcy under a barrier strategy. Subsequently, the optimal barrier is determined, and several explicit identities for the optimal value are found. The surplus process of the company is modeled by a Wiener process (Brownian motion). A similar model was also treated in [182] where the probability of bankruptcy and the expectation of a discounted penalty at the time of bankruptcy are determined. Explicit results are derived under assumption that the surplus process is described by the Brownian motion.

In [403], the Omega model with underlying Ornstein–Uhlenbeck type surplus process for an insurance company is considered. Explicit expressions for the expected discounted penalty function at bankruptcy with a constant bankruptcy rate and linear bankruptcy rate are derived. Based on random observations of the surplus process, the differentiability for the expected discounted penalty function at bankruptcy, especially at zero, is examined. Finally, the Laplace transforms for occupation times are given.

- *Drawdown analysis*

Another important research direction associated with solvency problems is drawdown analysis. The concept of drawdown is being used increasingly in risk analysis, as it provides surplus-related information similar to ruin-related quantities. For the insurer's surplus $\{X_t, t \geq 0\}$, the drawdown (or reflected) process Y_t is defined as the difference between its running maximum $M_t = \sup_{0\leq s\leq t} X_s$ at time t and X_t.

A new drawdown-based regime-switching (DBRS) Lévy insurance model in which the underlying drawdown process is used to describe an insurer's level of financial distress over time, and to trigger regime-switching transitions is proposed in [259]. Explicit formulas are derived for a generalized two-sided exit problem. Conditions under which the survival probability is not trivially zero (which corresponds to the positive security loading conditions of the proposed model) are stated. The regime-dependent occupation time until ruin is later studied. As a special case of the general DBRS model, a regime-switching premium model is given further consideration. Connections with other existing risk models (such as the loss-carry-forward tax model of [7]) are established.

Some drawdown-related quantities in the context of the renewal insurance risk process with general interarrival times and phase-type distributed jump sizes are treated in [249]. Some recent results on the two-sided exit problem for the spectrally negative Markov additive process (see, e.g., [214]) and a fluid flow analogy between certain queues and risk processes (see, e.g., [4]) are used to solve the two-sided exit problem of the renewal insurance risk process. The two-sided exit quantities are later shown to be central to the analysis of such drawdown quantities as the drawdown time, the drawdown size, the running maximum (minimum) at the drawdown time, the last running maximum time prior to drawdown, the number of jumps before drawdown and the number of excursions from running maximum before drawdown. Finally, another application of this methodology is proposed for the study of the expected discounted dividend payments until ruin.

4.2 Generalization of the Classical Cramér–Lundberg Model

Generalization of the model has gone in the following directions.

- *Another type of counting process* for representation of claim number was introduced (instead of Poisson one).

A well known *Sparre Andersen model* appeared in 1957, see [19], and was studied in many papers afterwards. This model has also the mixed type, since the premium is supposed to be acquired continuously at a constant rate. Thus, the company surplus is described by (2) with $Y(t)$ given by (3) where $N(t)$ is assumed to be a renewal process. That means the intervals between the claims are nonnegative independent identically distributed random variables however their distribution is arbitrary (not exponential), see, e.g. [11, 146–148, 179, 254, 278, 312, 356, 388, 416].

Polya–Aeppli counting processes are treated in [306]. Generalized renewal process can be also considered as claim number, see, e.g., [88]. Two classes of claims were studied in [47, 184, 438], for multivariate case see, e.g., [103].

- *Dependence conditions*

In previous models, the counting process (number of events) and claim severities were supposed to be independent. Recently, this restriction was taken away. Various types of dependence exist between claim amounts and interarrival times.

In [33], a one-dimensional surplus process is considered with a certain Sparre Andersen type dependence structure under general interclaim times distribution and correlated phase-type claim sizes. The Laplace transform of the time to ruin is obtained as the solution of a fixed-point problem, under both the zero-delayed and the delayed cases. An efficient algorithm for solving the fixed-point problem is derived together with bounds that illustrate the quality of the approximation. A two-dimensional risk model is analyzed under a bailout-type strategy with both fixed and variable costs and a dependence structure of the proposed type.

In [46], the authors consider an extension of the Sparre Andersen insurance risk model by relaxing one of its independence assumptions. The newly proposed dependence structure is introduced through the assumption that the joint distribution of the interclaim time and the subsequent claim size is bivariate phase-type (see, e.g. [27, 240]). Relying on the existing connection between risk processes and fluid flows (see, e.g., [3, 4, 42, 44, 354]), an analytically tractable fluid flow is constructed. That leads to the analysis of various ruin-related quantities in the aforementioned risk model. Using matrix analytic methods, an explicit expression for the Gerber–Shiu discounted penalty function is obtained when the penalty function depends on the deficit at ruin only. It is investigated how some ruin-related quantities involving the surplus immediately prior to ruin can also be analyzed via the fluid flow methodology.

The discounted penalty function in a Markov-dependent risk model is considered in [5], whereas a correlated aggregate claims model with Poisson and Erlang risk processes is studied in [432]. Optimal dynamic proportional and excess of loss reinsurance under dependent risks are obtained in [185].

Other examples can be found in [13, 43, 74, 117, 128, 193, 198, 209, 252, 256, 258, 260, 274, 342, 350, 374, 389, 424, 432], as well as [119, 268, 283, 377, 423].

Several types of claims to treat heterogeneous insurance portfolios are considered in [438]. The authors obtain integro-differential equations for the Gerber–Shiu discounted penalty function, generalized Lundberg equation and Laplace transforms for the Gerber–Shiu discounted penalty function under assumption that the surplus process $X(t) = x + ct - Y(t)$, $t \geq 0$, is of the Cramér–Lundberg type where the aggregate claim process $Y(t)$ is generated by two classes of insurance risks, i.e.,

$$Y(t) = Y_1(t) + Y_2(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0,$$

and $N_1(t)$ is a Poisson process and $N_2(t)$ is Erlang(n).

- *The Markovian claim arrivals, Markov additive processes (MAP), and Markov-modulated risk processes*

Beginning with [23, 355], researchers start consideration of risk processes in the Markovian environment.

Potential measures for spectrally negative Markov additive processes with applications in ruin theory are studied in [161]. Markovian arrivals were treated in [2, 112], where a unified analysis of claim costs up to ruin is given. In [118], a generalization of the risk model with Markovian claim arrivals is introduced. Moments of the discounted dividends in a threshold-type Markovian risk process are obtained in [38], whereas a multi-threshold Markovian risk model is analyzed in [45]. Analysis of a threshold dividend strategy for a MAP risk model is implemented in [44], while generalized penalty function with the maximum surplus prior to ruin in a MAP risk model is studied in [113], see also [252], where occupation times in the MAP risk model are treated.

For a Markov-modulated risk model, probability of ruin is obtained in [294], moments of the dividend payments and related problems are treated in [270], and decompositions of the discounted penalty functions and dividends-penalty identity are established in [271]. Bounds for the ruin probability in a Markovian modulated risk model are obtained in [417], while expected discounted penalty function is treated in [378], under additional assumption of constant barrier and stochastic income.

- *Spectrally negative Lévy processes* are considered in [31, 67, 132, 244, 253, 291, 293, 387, 429] and many other papers.

- *Perturbed and diffusion processes*

Ruin theory models incorporating a diffusion term aim to reflect small fluctuations in the insurance companies' surplus. Such fluctuations might be due to the uncertainty in the premium income or in the economic environment as a whole. Extensive research in this area has been carried out during the past 25 years.

We mention just a few papers. Thus, one of the first papers in this direction is [155] devoted to risk theory for the compound Poisson process perturbed by diffusion. In [400], dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion are considered. In [395], a generalized defective renewal equation for the surplus process perturbed by diffusion is studied, whereas in [264] the research focuses on the distribution of the dividend payments. The paper [251] treats the first passage times for compound Poisson processes with diffusion and provides actuarial and financial applications. The threshold dividend strategy is dealt with in [121] for a generalized jump-diffusion risk model. Gerber–Shiu function is investigated in [122] for a classical risk process perturbed by diffusion, while a linear barrier dividend strategy is a subject of [287]. The perturbed compound Poisson risk model with constant interest and a threshold dividend strategy is treated in [172]. The Gerber–Shiu function in a Sparre Andersen risk model perturbed by diffusion is studied in [268], whereas in [283] a generalized discounted penalty function is considered. A multi-threshold compound Poisson process perturbed by diffusion is investigated in [311]. Gerber–Shiu analysis in a perturbed risk model with dependence between claim sizes and interclaim times is provided in [435]. Absolute ruin minimization under a diffusion approximation model is carried out in [296]. The optimal dividend strategy in a regime-switching diffusion model is established in [405]. In contrast to classical case, it is assumed there that the dividends can be only paid at arrival times of a Poisson process. By solving an auxiliary optimization problem, it is shown that optimal is a modulated barrier strategy. The value function can be obtained by iteration or by solving a system of differential equations.

- *Stochastic premiums*

To reflect the cash flows of the insurance company more realistically, some papers assumed that the insurer earns random premium income. In the simplest case, the company surplus at time t is given by (2) where $Z(t)$ and $Y(t)$ are independent compound Poisson processes (with different intensities and jumps distributions). An interesting example is presented in the book [237] for modeling the speculative activity of money exchange point and optimization of its profit by using such a process.

In [308], the authors consider a generalization of the classical risk model when the premium intensity depends on the current surplus of an insurance company. All surplus is invested in the risky asset, the price of which follows a geometric Brownian motion. An exponential bound is established for the infinite-horizon ruin probability.

Models with stochastic premiums or income were also studied in [20, 76, 183, 195, 246, 288, 378, 393, 394, 444] and many others.

- *Dual processes*

In a model dual to classical Cramér–Lundberg one, see, e.g. [12], the surplus (without dividends) is described by the following equation

$$X(t) = x - ct + Y(t), \tag{6}$$

where c is now the rate of expenses, assumed to be deterministic and fixed. The process $Y(t)$ is compound Poisson. Such a model is natural for companies that have occasional gains whose amount and frequency can be modeled by the process $\{Y(t)\}$. For companies such as pharmaceutical or petroleum companies, the jump should be interpreted as the net present value of future gains from an invention or discovery. Other examples are commission-based businesses, such as real estate agent offices or brokerage firms that sell mutual funds of insurance products with a front-end load. Last but not least, a model of the form (6) might be appropriate for an annuity or pension fund. In this context, the probability of ruin has been calculated, see, e.g., [371]. The dividend problem for such a model is treated in [12]. A key tool is the method of Laplace transforms. A more general case where surplus is a skip-free downwards Lévy process is considered as well. The optimal strategy is of barrier type, the optimal barrier b^* is obtained. It is also shown that if the initial surplus is b^*, the expectation of the discounted dividends until ruin is the present value of a perpetuity with the payment rate being the drift of the surplus process.

A short proof of the optimality of barrier strategies for all spectrally positive Lévy processes of bounded or unbounded variation is given in [54]. Moreover, the optimal barrier is characterized using a functional inverse of the scale functions. A variant of the dividend payment problem in which the shareholders are expected to give capital injection in order to avoid ruin is also considered. The form of the value function for this problem is very similar to the problem in which the horizon is the time of ruin. The optimal dividend problem for a spectrally positive Lévy process is also considered in [428].

Optimal dividends in the dual model under transaction costs are treated in [55].

The time value of Parisian ruin in (dual) renewal risk processes with exponential jumps is considered in [415]. Other dual models are also considered in [108, 284, 319, 320].

- *Interest rates*

In recent years, the classical risk process has been extended to more practical and real situations. Thus, it is very important to deal with the risks that rise from monetary inflation in the insurance and finance market, and also to consider the operation uncertainties in administration of financial capital.

An optimal control problem is considered in [204] under assumption that a risky asset is used for investment, and this investment is financed by initial wealth as well as by a state dependent income. The objective function is accumulated discounted expected utility of wealth. Solution of this problem enables the authors to deal with the problem of optimal investment for an insurer with an insurance business modeled by a compound Poisson or a compound Cox process, under the presence of constant as well as (finite state space Markov) stochastic interest rate.

The aim of the paper [351] is to build recursive and integral equations for ruin probabilities of generalized risk processes under rates of interest with homogenous Markov chain claims and homogenous Markov chain premiums, while the interest rates follow a first-order autoregressive process. Generalized Lundberg inequalities for ruin probabilities of this process are derived by using recursive technique. Interest bearing surplus model with liquid reserves is considered in [373].

Asymptotic finite-time ruin probability for a two-dimensional renewal risk model with constant interest force and dependent sub-exponential claims is studied in [424]. The absolute ruin problems taking into account debit and credit interest rates are investigated, e.g., in [401, 402, 442] under some additional assumptions. A model with interest is studied in [310]. A multi-threshold compound Poisson surplus process is introduced there as follows. When the initial surplus is between any two consecutive thresholds, the insurer has the option to choose the respective premium rate and interest rate. Also, the model allows for borrowing the current amount of deficit whenever the surplus falls below zero. Explicit expressions for the Gerber–Shiu function are obtained if claim sizes are exponentially and phase-type(2) distributed.

5 Discrete-Time Models

A review [272] on discrete-time insurance models appeared in 2009. The authors underline that although most theoretical risk models use the concept of time continuity, the practical reality is discrete. Thus, dividend payment is usually based on results of financial year, whereas reinsurance treaties are discussed by the end of a year. It is important that recursive formulas for discrete-time models can be obtained without assuming a claim severity distribution and are readily programmable. The models, techniques used, and results for discrete-time risk models are of independent scientific interest. Moreover, results for discrete-time risk models can give, in addition, a simpler understanding of their continuous-time analog. For example, these results can serve as approximations or bounds for the corresponding results in continuous-time models. The expected discounted penalty functions and their special cases in the compound binomial model and its extensions are reviewed. In particular, the discrete-time Sparre Andersen models with K_m interclaim times and general interclaim times are treated, as well as other extensions to the compound binomial model including time-correlated claims and general premium rates, the compound Markov binomial risk model, and the compound binomial model defined in a Markovian environment.

Two papers [344, 345], not included in [272], deal with finite-time and ultimate ruin probability, respectively, for the following discrete-time model. It is supposed that the cumulative loss process has independent and stationary increments, the increments per unit of time take nonnegative integer values and their distribution $\{a_k\}_{k\ge 0}$ has a finite mean $\bar{a}$. The premium receipt process $\{c_k\}_{k\ge 0}$ is deterministic, nonnegative, and nonuniform. In addition, it is assumed that there exists a constant $c > \bar{a}$ such that the deviation $\sum_{k=0}^{t}(c_k - c)$ is bounded as the time t varies. In particular, $P(\tau = \infty)$, where τ is the ruin time, is obtained as $\lim_{t\to\infty} P(\tau > t)$, first, if $c = d^{-1}$ for some positive integer d, then general case if $a_0 > 0.5$.

A class of compound renewal (Sparre Andersen) risk processes with claim waiting times having a discrete K_m distribution is studied in [262, 263]. The classical compound binomial risk model is a special case when $m = 1$. A recursive formula is derived in the former paper for the expected discounted penalty (Gerber–Shiu) function, which can be used to analyze many quantities associated with the time of ruin. In the latter paper, an explicit formula for the Gerber–Shiu function is given in terms of a compound geometric distribution function. The finite-time ruin probability under the compound binomial model is treated in [273].

Discrete-time multi-risks insurance model is considered in [346]. The model describes the evolution in discrete time of an insurance portfolio covering several interdependent risks. The main problem under study is the determination of the probabilities of ruin over a finite horizon, for one or more risks. An underlying polynomial structure in the expression of these probabilities is exhibited. This result is then used to provide a simple recursive method for their numerical evaluation.

The discounted factorial moments of the deficit in discrete-time renewal risk model are treated in [50]. The discrete stationary renewal risk model and the Gerber–Shiu discounted penalty function were considered in [335].

We would also like to mention some papers considering other aspects of discrete-time models. Thus, two discrete-time risk models under rates of interest are dealt with in [98]. Stochastic inequalities for the ruin probabilities are derived by martingales and renewal recursive techniques.

In [149], the authors discuss a situation in which a surplus process is modified by the introduction of a constant dividend barrier. They extend some known results relating to the distribution of the present value of dividend payments until ruin in the classical risk model by allowing the process to continue after ruin. Moreover, they show how a discrete-time risk model can be used to provide approximations when analytic results are unavailable. Discrete-time financial surplus models for insurance companies are proposed in [218]. A generalization of the expected discounted penalty function in a discrete-time insurance risk model is introduced in [250].

Survival probabilities for compound binomial risk model with discrete phase-type claims are dealt with in [397]. Asymptotic ruin probabilities for a discrete-time risk model with dependent insurance and financial risks are obtained in [427], the ruin probability in a dependent discrete-time risk model with insurance and financial risks is studied in [426], whereas asymptotic results are established for a discrete-time risk model with Gamma-like insurance risks in [425]. Discrete-time insurance risk models with dependence structure are treated in the thesis [404]. A thorough analysis of the generalized Gerber–Shiu function in discrete-time dependent Sparre Andersen model is presented in the quite recent thesis [350].

Randomized observation periods were considered for compound Poisson risk model in [16] in connection with dividend payments. The authors study a modification of the horizontal dividend barrier strategy by introducing random observation times at which dividends can be paid and ruin can be observed. This model contains both the continuous-time and the discrete-time risk model as a limit and represents a certain type of bridge between them which still enables the explicit calculation of moments of total discounted dividend payments until ruin. In [17] for Erlang(n) distributed inter-observation times, explicit expressions for the discounted penalty function at ruin are derived. The resulting model contains both the usual continuous-time and the discrete-time risk model as limiting cases, and can be used as an effective approximation scheme for the latter. Optimal dividend payout in random discrete time is treated in [15].

In [434], a Markov additive insurance risk process under a randomized dividend strategy in the spirit of [16] is considered. Decisions on whether to pay dividends are only made at a sequence of dividend decision time points whose intervals are Erlang(n) distributed. At a dividend decision time, if the surplus level is larger than a predetermined dividend barrier, then the excess is paid as a dividend as long as ruin has not occurred. In contrast to [16], it is assumed that the event of ruin is monitored continuously (as in [30, 433]), i.e., the surplus process is stopped immediately once it drops below zero. The quantities of interest include the Gerber–Shiu expected discounted penalty function and the expected present value of dividends paid until ruin. Solutions are derived with the use of Markov renewal equations. Numerical examples are given, and the optimal dividend barrier is identified in some cases.

In [229], the authors focus on the development of a recursive computational procedure to calculate the finite-time ruin probabilities and expected total discounted dividends paid prior to ruin associated with a model which generalizes the single threshold-based risk model introduced in [152]. Namely, a discrete-time dependent Sparre Andersen risk model with multiple threshold levels is considered in an effort to characterize an insurer's minimal capital requirement, dividend paying scenarios, and external financial activities related to both investment and loan undertakings.

Computational aspects are also treated in [18]. A Sparre Andersen insurance risk model in discrete time was analyzed there as a doubly infinite Markov chain to establish a computational procedure for calculating the joint probability distribution of the time of ruin, the surplus immediately prior to ruin, and the deficit at ruin. Discounted factorial moments of the deficit in discrete-time renewal risk model are studied in [50].

Cost approach for solving discrete-time actuarial problems was introduced in [90], see also [89, 93–96].

The paper [70] deals with the discrete-time risk model with nonidentically distributed claims. The recursive formula of finite-time ruin probability is obtained, which enables one to evaluate the probability of ruin with desired accuracy. Rational valued claims and nonconstant premium payments are considered.

In [226], a discrete-time model of insurance company is considered. It is supposed that the company applies a dividend barrier strategy. The limit distribution for the time of ruin normalized by its expected value is found. It is assumed that shareholders cover the deficit at the time of ruin. The barrier strategies maximizing shareholders' dividends and profit accumulated until ruin are investigated. In case the additional capital is injected right after the ruin to enable infinite performance of the company, existence of optimal strategies is proved both for expected discounted dividends and net profit.

A discrete-time model for the cash flow of an insurance portfolio/business in which the net losses are random variables, while the return rates are fuzzy numbers was studied in [399]. The shape of these fuzzy numbers is assumed trapezoidal, Gaussian or lognormal, the last one having a more flexible shape than the previous ones. For the resulting fuzzy model, the fuzzy present value of its wealth is evaluated. The authors propose an approximation for the chance of ruin and a ranking criterion which could be used to compare different risk management strategies. A discrete-time insurance model with reinvested surplus and a fuzzy number interest rate is investigated in [307].

The discrete-time risk model with nonidentically distributed claims is studied in [70]. The recursive formula of finite-time ruin probability is obtained, which enables one to evaluate the probability of ruin with desired accuracy. Rational valued claims and nonconstant premium payments are considered. Some numerical examples of finite-time ruin probability calculation are presented. Ruin probability in the three-seasonal discrete-time risk model is obtained in [190]. It is also interesting to mention a discrete-time pricing model for individual insurance contracts studied in [325].

6 Examples

Below, we give three simple examples to demonstrate the problems and methods we did not discuss earlier and present some results of the author. At first, we deal with dividends optimization by reinsurance treaty with liability constraint, published in [87]. Then the stability of the periodic review model of insurance company performance with capital injections and reinsurance, introduced in [96], is studied. The full version of this results will be submitted for publication elsewhere. Finally, some limit theorems for generalized renewal processes introduced in [88] are provided.

6.1 Limited Liability of Reinsurer and Dividends

Below, we give some results proved in [87] concerning the dividend payments under barrier strategy and excess of loss reinsurance with limited liability of reinsurer in the framework of Cramér–Lundberg model.

Denote by d the retention level and by l the reinsurer's liability. Let Y be the initial claim size of direct insurer. Then, his payment under the above mentioned treaty is $Y_l = \min(d, Y) + \max(Y - l - d, 0)$, whereas the reinsurer's payment is equal to $Y_l' = \min(\max(Y - d, 0), l)$. We assume that $X(0) = x \le b$, hence, the insurer's surplus $X(t)$ never exceeds the dividend barrier b.

Let us suppose that direct insurer and reinsurer use for premiums calculation the expected value principle with loads θ and θ_1 respectively (and $\theta_1 > \theta > 0$). Then the insurer's premium net of reinsurance c_l has the form

$$c_l = \lambda(1+\theta)p_1 - \lambda(1+\theta_1)\int_d^{d+l}(1 - F(y))\,dy$$

where λ is the intensity of the Poisson process describing claim arrivals, $F(y)$ is the claim distribution function with density $p(y)$ and the expected claim value $p_1 = \int_0^\infty yp(y)\,dy$.

Theorem 1 *The integro-differential equation for expected total discounted dividends until ruin, under reinsurance treaty, $V(x,b,d,l)$ can be written for $0 < x < d$ as follows*

$$\tilde{c}_l V'(x,b,d,l) - (1+\alpha)V(x,b,d,l) + \int_0^x V(y,b,d,l)p(x-y)\,dy = 0$$

and for $d \le x < b$

$$\tilde{c}_l V'(x,b,d,l) - (1+\alpha)V(x,b,d,l) + \int_{x-d}^{x} V(y,b,d,l)p(x-y)\,dy$$
$$+V(x-d,b,d,l)(F(d+l)-F(d)) + \int_0^{x-d} V(y,b,d,l)p(l+x-y)\,dy = 0$$

with $\tilde{c}_l = c_l\lambda^{-1}$, $\alpha = \delta\lambda^{-1}$ and boundary condition $V'(b,b,d,l) = 1$.

Turning to exponential claim distribution with parameter β, one obtains the following results.

Theorem 2 *For $0 < x < d$, the function $V(x,b,d,l)$ satisfies the second-order differential equation*

$$\tilde{c}_l V''(x,b,d,l) + (\beta\tilde{c}_l - (1+\alpha))V'(x,b,d,l) - \alpha\beta V(x,b,d,l) = 0,$$

whereas for $d \le x < b$ one has

$$\tilde{c}_l V''(x,b,d,l) + (\beta\tilde{c}_l - (1+\alpha))V'(x,b,d,l) - \alpha\beta V(x,b,d,l)$$
$$= -e^{-\beta d}F(l)V'(x-d,b,d,l).$$

Here, $\tilde{c}_l = \beta^{-1}\left((1+\theta) + (1+\theta_1)e^{-\beta d}(e^{-\beta l}-1)\right)$.

Theorem 3 *For the exponential claim distribution, the optimal dividend barrier, under excess of loss reinsurance treaty with limited liability of reinsurer and assumption $0 < x \le b < d$, is given by*

$$b_l^* = b^*(r_l, s_l) = \frac{1}{r_l - s_l} \ln \frac{s_l^2(s_l+\beta)}{r_l^2(r_l+\beta)}\ .$$

Here, $r_l > 0, s_l < 0$ *are the roots of the characteristic equation*

$$\tilde{c}_l \xi^2 + (\beta \tilde{c}_l - (1+\alpha))\, \xi - \alpha\beta = 0.$$

Assume the claims to be uniformly distributed on the interval $[0, h]$. It is reasonable to suppose that $d + l < h$.

Theorem 4 *For* $0 < x < d$, *the function* $V(x, b, d, l)$ *satisfies the second-order differential equation*

$$\tilde{c}_l V''(x, b, d, l) - (1+\alpha)V'(x, b, d, l) + \frac{1}{h} V(x, b, d, l) = 0, \tag{7}$$

whereas for $d \le x < h - l$ *one has*

$$\tilde{c}_l V''(x, b, d, l) - (1+\alpha)V'(x, b, d, l) + \frac{1}{h} V(x, b, d, l) + \frac{l}{h} V'(x-d, b, d, l) = 0 \tag{8}$$

and for $h - l \le x < b$

$$\begin{aligned} \tilde{c}_l V''(x, b, d, l) - (1+\alpha)V'(x, b, d, l) + \frac{1}{h} V(x, b, d, l) \\ + \frac{l}{h} V'(x-d, b, d, l) - \frac{1}{h} V(x-(h-l), b, d, l) = 0. \end{aligned} \tag{9}$$

Here, $\tilde{c}_l = (1+\theta)\frac{h}{2} - l(1+\theta_1)(1 - \frac{2d+l}{2h})$.

Theorem 5 *If the claim distribution is uniform on interval* $[0, h]$ *and the roots of characteristic equation corresponding to differential equation (7) are real then the optimal dividend barrier b under assumption* $0 < x \le b < d$ *is equal to initial capital of insurance company* x.

To calculate $V(x, b, d, l)$ for $d \le x < b$ it is possible to use the following algorithm

1. Find expression of $V(x, b, d, l)$ on interval $(0, d)$.
2. Let $h - l \in (nd, (n+1)d]$ for $n = 1, 2, \ldots$. The form of the function on half-interval $[kd, (k+1)d)$ for $1 \le k \le n-1$ can be obtained using its form on half-interval $[(k-1)d, kd)$ and Eq. (8), the same is true for the last half-interval $[nd, h-l)$.
3. For $x \in [h-l, (n+1)d)$ according to (9) the function $V(x, b, d, l)$ depends on $V'(x-d, b, d, l)$ and $V(x-(h-l), b, d, l)$. The same is true for $x \ge (n+1)d$. Similarly, for $h - l \le x < b$, we use the expression of the function on two previous half-intervals.

Thus, for the exponential and uniform claim distributions, we have considered the barrier dividend strategy and obtained the form of optimal barrier level b_l^* for the model with limited reinsurer's liability l in the excess of loss reinsurance treaty having retention d.

Some results pertaining to the case of variable barriers which are changed after each claim arrival are obtained in [316], whereas a generalization of Lundberg inequality for the case of a joint-stock insurance company one can find in [314].

6.2 Discrete-Time Model with Reinsurance and Capital Injections

A periodic review insurance model is considered under the following assumptions. In order to avoid ruin, the insurer maintains the company surplus above a chosen level a by capital injections at the end of each period. One-period insurance claims form a sequence $\{\xi_n\}_{n\geq 1}$ of independent identically distributed nonnegative random variables with a known distribution function and finite mean. The company concludes at the end of each period the stop-loss reinsurance treaty. If the retention level is denoted by $z > 0$ then $c(z)$ is the insurer premium (net of reinsurance). It is necessary to choose the sequence of retention levels minimizing the total discounted injections during n periods.

Let x be the initial surplus of insurance company. One-period minimal capital injections are defined as follows

$$h_1(x) := \inf_{z>0} \mathsf{E} J(x,z), \quad \text{where} \quad J(x,z) = (\min(\xi, z) - (x-a) - c(z))^+ .$$

For the n-step model, $n \geq 1$, the company surplus $X(n)$ at time n is given by the relation

$$X(n) = \max(X(n-1) + c(z) - \min(\xi, z), a), \quad X(0) = x.$$

It was proved in [96] that the minimal expected discounted costs injected in company during n years satisfy the following Bellman equation

$$h_n(x) = \inf_{z>0}(\mathsf{E} J(x,z) + \alpha \mathsf{E} h_{n-1}(\max(x + c(z) - \min(\xi, z), a))), \quad h_0(x) = 0, \tag{10}$$

where $0 < \alpha < 1$ is the discount factor.

Under assumption that premiums of insurer and reinsurer are calculated according to mean value principle, the optimal reinsurance strategy was established. It turned out that its character depends on the relationship between the safety loading of insurer and reinsurer.

An important problem is investigation of the system asymptotic behavior and its stability with respect to parameters fluctuation and perturbation of underlying processes. It was established in [96] that $h_n(x) \to h(x)$ as $n \to \infty$ uniformly in x.

The analysis of the model sensitivity to cost parameters fluctuations is carried out, in the same way as in [95], using the results of [92, 324, 364, 386].

To study the perturbations of the processes one has to use the probability metrics, see, e.g., [353].

Definition 2 For random variables X and Y defined on some probability space $(\Omega, \mathscr{F}, P)$ and possessing finite expectations, it is possible to define their distance on the base of Kantorovich metric in the following way

$$\kappa(X,Y)=\int_{-\infty}^{+\infty}|F(t)-G(t)|dt,$$

where F and G are the distribution functions of X and Y respectively.

This metric coincides (see, e.g. [150] or [382]) with Wasserstein L_1 metric defined as $d_1(F,G)=\inf \mathsf{E}|X-Y|$ where infimum is taken over all jointly distributed X and Y having marginal distribution functions (d.f.'s) F and G. It is supposed that both d.f.'s belong to $\mathscr{B}_1$ consisting of all F such that $\int_{-\infty}^{+\infty}|x|\,dF(x)<\infty$.

Lemma 1 *The following statements are valid.*

1. Let $F^{-1}(t)=\inf\{x : F(x)\ge t\}$, then $d_1(F,G)=\int_0^1|F^{-1}(t)-G^{-1}(t)|\,dt$.

2. $(\mathscr{B}_1, d_1)$ is a complete metric space.

3. For a sequence $\{F_n\}_{n\ge 1}$ from $\mathscr{B}_1$ one has $d_1(F_n,F)\to 0$ if and only if $F_n \stackrel{d}{\to} F_n$ and $\int_{-\infty}^{+\infty}|x|\,dF_n(x)\to\int_{-\infty}^{+\infty}|x|\,dF(x)$, as $n\to\infty$. Here $\stackrel{d}{\to}$ denotes, as usual, convergence in distribution.

The proof can be found in [150].

Lemma 2 *Let X, Y be nonnegative random variables possessing finite expected values and $\kappa(X,Y)\le\rho$. Assume also that $g : R^+\to R^+$ is a nondecreasing Lipschitz function. Then $\kappa(g(X),g(Y))\le C\rho$ where C is the Lipschitz constant.*

The next result enables us to estimate the difference between infimums of two functions.

Lemma 3 *Let functions $f_1(z)$, $f_2(z)$ be such that $|f_1(z)-f_2(z)|<\delta$ for some $\delta>0$ and any $z>0$. Then $|\inf_{z>0}f_1(z)-\inf_{z>0}f_2(z)|<\delta$.*

Note that we are going to add the label X to all functions depending on ξ if $\xi\sim law(X)$.

Putting $\Delta_1:=\sup_{u>a}|h_{1_X}(u)-h_{1_Y}(u)|$, we prove the following result.

Theorem 6 *Let X, Y be nonnegative random variables possessing finite expectations, moreover $\kappa(X,Y)\le\rho$. Then*

$$\Delta_1\le(1+l+m)\rho$$

where l and m are the safety loading coefficients of insurer and reinsurer premiums, respectively. Both premiums are calculated according to expected value principle and $1 < l < m$.

For the multistep case, we get the following results.

Lemma 4 *Function $h_n(u)$ defined by (10) is non-increasing in u.*

Lemma 5 *For each $n \geq 0$ and any $u \geq a$, the following inequality is valid*

$$|h_n(u + \Delta u) - h_n(u)| \leq C_n \Delta u,$$

where $C_n = (1 - \alpha^n)(1 - \alpha)^{-1}$.

To establish the model stability, we put $\Delta_n = \sup_{u>a} |h_{n_X}(u) - h_{n_Y}(u)|$ and formulate the following result.

Theorem 7 *Let X, Y be nonnegative random variables having finite means and $\kappa(X, Y) \leq \rho$. Then*

$$\Delta_n \leq \left(\sum_{i=0}^{n-1} \alpha^i C_{n-i} \right) (1 + l + m)\rho,$$

here $0 < \alpha < 1$ is the discount factor, $1 < l < m$ are the safety loadings of insurer and reinsurer and C_k, $k \leq n$, were defined in Lemma 5.

Furthermore, in practice neither the exact values of parameters nor the processes distributions are known. Thus, it is important to study the systems behavior under incomplete information. The estimates of distribution parameters are easily obtained on the base of previous observations.

If there is no a priori information, it may be useful to employ the empirical processes, see, e.g., [382].

For each fixed $t \in R$, the difference $H_n(\omega, t) =: F_n(\omega, t) - G_n(\omega, t)$ of two empirical distribution functions is a real-valued function of the random vector $(X_1, Y_1, \ldots, X_n, Y_n)$ defined on a probability space $(\Omega, \mathscr{F}, P)$, namely,

$$H_n(\omega, t) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \leq t\} - \frac{1}{n} \sum_{i=1}^{n} I\{Y_i \leq t\} = \frac{1}{n} \sum_{i=1}^{n} \zeta_i(t),$$

where $\zeta_i(t) = I\{X_i \leq t\} - I\{Y_i \leq t\},\ i = \overline{1, n}$.

According to properties of convergence in distribution, we get immediately the following result

Lemma 6 *For any $t \in R$, as $n \to \infty$,*

$$\sqrt{n}\, |F_n(\omega, t) - G_n(\omega, t) - (F(t) - G(t))| \xrightarrow{d} \sqrt{F(t) + G(t) - (F^2(t) + G^2(t))}|N(0, 1)|.$$

We have also obtained a functional limit theorem. The established results are used to construct the empirical asymptotically optimal policies for the discrete-time model. The following three-step algorithm is proposed

1. Find the optimal control for known parameters and distributions.
2. Obtain stationary asymptotically optimal policy.
3. Calculate empirical asymptotically optimal policy using previous observations.

6.3 Generalized Renewal Processes

It is well known that ordinary renewal processes are widely used in various applications of probability theory not only in insurance, see, e.g., [309]. However, they are appropriate for the study of systems with time-homogeneous evolution.

In order to take into account the initial phase of a system functioning or its seasonal variations several generalizations of renewal processes are introduced, see, e.g., [88]. We focus here on delayed periodic processes and investigate their asymptotic behavior, in particular, state the strong law of large numbers and functional limit theorem. Some results concerning the reward-renewal processes are also provided.

Definition 3 Let $\{T_n\}_{n\geq 1}$ be a sequence of independent nonnegative random variables, F_j, $j = 1, \ldots, l$, being the distribution function of variable T_{ql+j} for some fixed integer $l \geq 1$, $q = 0, 1, \ldots$. Let $\{X_i\}_{i=0,\ldots,k-1}$ be another sequence of nonnegative independent r.v.'s with distribution functions G_i, respectively. The sequences $\{T_n\}$ and $\{X_i\}$ are also supposed to be independent.

The delayed periodical renewal process is formed in the following way: $S_n = X_0 + \cdots + X_n$, $0 \leq n \leq k-1$, whereas $S_n = S_{k-1} + T_1 + \cdots + T_{n-k+1}$ for $n \geq k$. The partial sums S_n are called the renewals (or renewal epochs) and the summands X_j and T_i are the intervals between the renewals.

It is reasonable to call l the *process period* and k the *length of delay*, thus we have, say (k, l)-process. Taking $l = 1$, $k = 1$ and $X_0 = 0$, we obtain the ordinary renewal process. We can also consider the following types of generalized renewal processes

- generalized delayed process corresponds to $l = 1$, $k > 1$,
- putting $k = 1$, $X_0 = 0$ and leaving $l > 1$ we obtain a periodic renewal process;
- a special case of the periodic process with $l = 2$ is a well-known alternating process.

The asymptotic behavior of ordinary renewal process is thoroughly studied. Central limit theorem (CLT), strong law of large numbers (SLLN) and functional limit theorem (FCLT) are proved for them.

We have proved the same theorems for our generalized processes. In order to do this, we established that the delay length does not have any influence on the asymptotic behavior of a renewal process.

Lemma 7 *Let $\{T_n\}_{n\geq 1}$ and $\{X_i\}_{i\geq 0}$ be two independent sequences of independent r.v.'s. Put $S_j = \sum_{i=1}^{j} T_i$, the delayed sequence $\{\Sigma_n\}_{n\geq 0}$ is given by $\Sigma_n = X_0 + \cdots + X_n$, $0 \leq n \leq k-1$, while $\Sigma_n = \Sigma_{k-1} + S_{n-k+1}$ for $n \geq k$.*

If there exists an almost sure (a.s.) convergence $n^{-1}S_n \to \mu$, then for any fixed k there exists the same limit for the delayed sequence:

$$n^{-1}\Sigma_n \to \mu \quad a.s. \quad as \quad n \to \infty.$$

Lemma 8 *Let $\{S_n\}$ and $\{\Sigma_n\}$ be the sequences defined in Lemma 7. Assume that all random variables have finite mathematical expectations. If there exists a number $\sigma > 0$ such that*

$$\frac{S_n - \mathsf{E}S_n}{\sigma\sqrt{n}} \xrightarrow{d} \xi, \quad as \quad n \to \infty,$$

where ξ has a standard Gaussian distribution, then for any fixed k the same statement is true for Σ_n, that is,

$$\frac{\Sigma_n - \mathsf{E}\Sigma_n}{\sigma\sqrt{n}} \xrightarrow{d} \xi \quad as \quad n \to \infty.$$

Symbol $\xrightarrow{d}$ denotes weak convergence of random variables.

Lemma 9 *If there exists a finite number μ such that $n^{-1}S_n \to \mu$ a.s., then there is an a.s. convergence*

$$t^{-1}N_t \to \mu^{-1} \quad as \quad t \to \infty.$$

Lemma 10 *If there exist numbers μ and σ such that*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} \xi \quad as \quad n \to \infty,$$

where ξ is a random variable having a standard Gaussian distribution, then

$$\frac{N_t - t\mu^{-1}}{\sigma\sqrt{t\mu^{-3}}} \xrightarrow{d} \xi \quad as \quad t \to \infty.$$

Theorem 8 (SLLN) *Let S_n be a delayed periodical renewal process. Suppose that all the summands T_{ql+i} have finite mathematical expectation $\mu_i < \infty$, $i = 1, \ldots, l$. Then a.s.*

$$\frac{N_t}{t} \to \frac{l}{\mu} \quad as \quad t \to \infty.$$

Here, the counting process N_t is defined as earlier, $N_t = \min\{n \geq 0 : S_n > t\}$ and $\mu = \mu_1 + \cdots + \mu_l$.

Theorem 9 (CLT) *Suppose that r.v.'s T_{lq+i} have finite mathematical expectations μ_i and variances $0 < \sigma_i^2 < \infty$ respectively, $i = 1, \ldots, l$, and r.v. X_j has finite* *mathematical expectation* ν_j, $j \geq 1$. *Then, as* $t \to \infty$*, we have*

$$\frac{N_t - tl\mu^{-1}}{\sigma l\sqrt{t\mu^{-3}}} \xrightarrow{d} \xi$$

where $\mu = \mu_1 + \cdots + \mu_l$, $\sigma^2 = \sigma_1^2 + \cdots + \sigma_l^2$ *and r.v.* ξ *has the standard Gaussian distribution.*

Next, we state the functional limit theorem for the generalized renewal process $\{S_n\}$ treated in Theorem 9.

Theorem 10 (FCLT) *Put* $\mu = \mu_1 + \cdots + \mu_l$, $\sigma^2 = \sigma_1^2 + \cdots + \sigma_l^2$ *and*

$$Z_n(t, \omega) = \frac{N_{nt}(\omega) - ntl\mu^{-1}}{\sigma l\sqrt{n\mu^{-3}}},$$

Then, $Z_n \xrightarrow{D} W$ *as* $n \to \infty$.

It is interesting to deal with controlled processes introduced in [236].

Definition 4 X_t is a controlled version of N_t if it is formed by the sequence of $S_n' = \sum_{i=0}^n T_i'$ where $T_i' = T_i/\nu(i)$, $0 < \nu(i) < \infty$. In other words, the ith inter-renewal time is scaled by a (deterministic) function of the number of previous times. The function ν is called the speed of the process.

Note that for a constant speed $\nu(i) = c$ one gets $X_t = N_{tc}$.

For controlled versions of renewal processes, one can consider the so-called fluid (deterministic) and diffusion approximations. More precisely, consider a twice continuously differentiable function $c : (0, \infty) \to (0, \infty)$, and define the nth approximation X^n to N as the controlled renewal process with the speed $\nu^n(i) = nc(i/n)$.

Thus, X^n is a point process with points generated by $T_j^n = T_j/nc(j/n)$. We assume T_i to have finite mean and variance denoted by μ and σ^2, respectively.

Theorem 11 (Fluid approximation) *Consider the ODE* $x_t' = \mu^{-1}c(x_t)$, $t \geq 0$, *with* $x_0 = 0$ *and assume that* c *is such that* x_t *remains finite for all* $t > 0$. *Let* $x_t^n = n^{-1}X_t^n$. *Then,* x^n *converges to the solution* x *of the ODE, as* $n \to \infty$*, in the sense that for any* $\varepsilon > 0$ *and any* $T > 0$,

$$\lim_{n\to\infty} P(\sup_{0\leq t\leq T} |x_t^n - x_t| > \varepsilon) = 0.$$

Theorem 12 (Diffusion approximation) *Consider the process* $\xi_t^n = \sqrt{n}(x_t^n - x_t)$. *Let* $D[0, \infty)$ *denote the space of càdlàg functions endowed with the Skorokhod topology. Then* ξ^n *converges weakly, as* $n \to \infty$*, to the solution of the following SDE*

$$d\xi_t = \mu^{-1}c'(x_t)\xi_t dt + \sqrt{\mu^{-3}\sigma^2 c(x_t)}dW_t, \quad t \geq 0,$$

$\xi_0 = 0$. *Here* W_t *is a Wiener process and* x_t *is the solution of ODE.*

At last, we turn to *reward-renewal* processes.

Definition 5 Let $(T_i, Y_i)_{i\geq 0}$ be a bivariate renewal sequence (vectors are i.i.d. for $i > 0$ and $T_i \geq 0$). Then, $Y_t = \sum_{i=0}^{N_t} Y_i$ is called a reward-renewal process.

Theorem 13 *If there exist* $\mathsf{E}T_i = \mu$ *and* $\mathsf{E}Y_i = \delta$, $i \geq 1$, *then almost surely*

$$\frac{Y_t}{t} \to \frac{\delta}{\mu}, \quad as \quad t \to \infty.$$

Theorem 14 *If* $\{T_n\}_{n\geq 1}$ *and* $\{Y_n\}_{n\geq 1}$ *are periodic renewal sequences with periods* l_1 *and* l_2 *respectively and there exist* $\mathsf{E}T_i = \mu_i$, $\mathsf{E}Y_i = \delta_i$, *then almost surely*

$$\lim_{t\to\infty} \frac{Y_t}{t} = \frac{l_1}{l_2} \frac{\sum_{i=1}^{l_2} \delta_i}{\sum_{i=1}^{l_1} \mu_i}.$$

Note that $\lim_{t\to\infty} t^{-1} Y_t$ represents the long-run costs and widely used as objective function in various applications.

It is possible to consider purely stochastic model (difference of two reward-renewal processes) generalizing the model introduced in [247].

$$X(t) = x + Z(t) - Y(t)$$

where $Z(t) = \sum_{i=1}^{N_1(t)} Z_i$, $N_1(t)$ is generated by l_3 periodic process and $\{Z_i\}$ form a l_4 periodic process, the corresponding means being μ_i' and δ_i'. Then

$$\lim_{t\to\infty} \frac{X(t)}{t} = \frac{l_1}{l_2} \frac{\sum_{i=1}^{l_2} \delta_i}{\sum_{i=1}^{l_1} \mu_i} - \frac{l_3}{l_4} \frac{\sum_{i=1}^{l_4} \delta_i'}{\sum_{i=1}^{l_3} \mu_i'}. \tag{11}$$

The positivity of rhs in (11) is analog of classical net profit condition. Its fulfillment enables us to state that ultimate ruin probability is less than 1.

Diffusion approximation for insurance models was proposed for the first time by D.L. Iglehart in 1969, see [212]. It can be useful for estimation of ruin probabilities.

Denote by $W_{a,\sigma^2}(t)$ the Wiener process with the mean at and variance $\sigma^2 t$. This random process is stochastically equivalent to $at + \sigma W(t)$, where $W(t)$ is a standard Wiener process.

The process with stochastic premiums can be approximated (see, e.g., [444]) by $x + W_{a,\sigma^2}(t)$ where $\mathsf{E}R(t) = at$ and $\mathsf{Var}R(t) = \sigma^2 t$ for $R(t) = Z(t) - Y(t)$. So, parameters a and σ^2 can be easily calculated.

Hence, ultimate ruin probability is approximated as follows:

$$\psi(x) \approx P(\inf_{t>0} W_{a,\sigma^2}(t) < -x) = \exp\{-2xa/\sigma^2\}$$

and ruin probability on finite interval

$$\psi(x,T) \approx P(\inf_{0<t\le T} W_{a,\sigma^2}(t) < -x)$$

$$= 1 - \Phi\left(\frac{aT + x}{\sigma\sqrt{T}}\right) + \exp\{-2xa/\sigma^2\}\Phi\left(\frac{aT - x}{\sigma\sqrt{T}}\right).$$

We have obtained the diffusion approximation and FLCT for the difference of two periodic renewal-reward processes to be published elsewhere.

7 Conclusion

Actuarial science is a fast growing research domain, so it turned out impossible even to include all recent publications. In this review, a classification of existing so far models is given, emphasizing the role of the new ones. Since some of the models possess several characteristics such as implementation of investment, reinsurance, capital injections, and so on, they can be mentioned not only in one group. Summing up, it is necessary to stress that three new notions of ruin (absolute, Parisian and Omega) were introduced for treating the solvency and bankruptcy problems. Many generalizations of Gerber–Shiu function, which unified reliability and cost approach, allow to investigate more precisely the company surplus behavior in order to control it avoiding bankruptcy. On the other hand, various extensions of classical Cramér–Lundberg and Sparre Andersen models aim at better description of reality, although they demand more profound knowledge of mathematics. So, hopefully, the review will be useful for the researcher in applied probability and professor teaching future actuaries, as well as, students themselves.

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References

1. Abdallah, A., Boucher, J.P., Cossette, H.: Modeling dependence between loss triangles with hierarchical Archimedean copulas. ASTIN Bull. **45**, 577–599 (2015)
2. Ahn, S., Badescu, A.L.: On the analysis of the Gerber-Shiu discounted penalty function for risk processes with Markovian arrivals. Insur.: Math. Econ. **41**(2), 234–249 (2007)
3. Ahn, S., Ramaswami, V.: Efficient algorithms for transient analysis of stochastic fluid flow models. J. Appl. Probab. **42**, 531–549 (2005)
4. Ahn, S., Badescu, A.L., Ramaswami, V.: Time dependent analysis of finite buffer fluid flows and risk models with a dividend barrier. Queueing Syst. **55**, 207–222 (2007)
5. Albrecher, H., Boxma, O.J.: On the discounted penalty function in a Markov-dependent risk model. Insur.: Math. Econ. **37**, 650–672 (2005)
6. Albrecher, H., Hartinger, J.: A risk model with multi-layer dividend strategy. N. Am. Actuar. J. **11**, 43–64 (2007)

7. Albrecher, H., Hipp, C.: Lundberg's risk process with tax. Bl. DGVFM **28**(1), 13–28 (2007)
8. Albrecher, H., Thonhauser, S.: Optimal dividend strategies for a risk process under force of interest. Insur.: Math. Econ. **43**, 134–149 (2008)
9. Albrecher, H., Thonhauser, S.: Optimality results for dividend problems in insurance. Rev. R. Acad. Cien. Serie A Mat. **103**(2), 295–320 (2009)
10. Albrecher, H., Hartinger, J., Tichy, R.: On the distribution of dividend payments and the discounted penalty function in a risk model with linear dividend barrier. Scand. Actuar. J. **2005**(2), 103–126 (2005)
11. Albrecher, H., Hartinger, J., Thonhauser, S.: On exact solutions for dividend strategies of threshold and linear barrier type in a Sparre Andersen model. ASTIN Bull. **37**(2), 203–233 (2007)
12. Albrecher, H., Badescu, A.L., Landriault, D.: On the dual risk model with taxation. Insur.: Math. Econ. **42**(3), 1086–1094 (2008)
13. Albrecher, H., Constantinescu, C., Loisel, S.: Explicit ruin formulas for models with dependence among risks. Insur.: Math. Econ. **48**(2), 265–270 (2011)
14. Albrecher, H., Gerber, H.U., Shiu, E.S.W.: The optimal dividend barrier in the Gamma-Omega model. Eur. Actuar. J. **1**, 43–55 (2011)
15. Albrecher, H., Bäuerle, N., Thonhauser, S.: Optimal dividend pay-out in random discrete time. Stat. Risk Model. **28**, 251–276 (2011)
16. Albrecher, H., Cheung, E.C.K., Thonhauser, S.: Randomized observation periods for the compound Poisson risk model: dividends. ASTIN Bull. **41**(2), 645–672 (2011)
17. Albrecher, H., Cheung, E.C.K., Thonhauser, S.: Randomized observation periods for the compound Poisson risk model: the discounted penalty function. Scand. Actuar. J. **2013**(6), 224–252 (2013)
18. Alfa, A.S., Drekic, S.: Algorithmic analysis of the Sparre Andersen model in discrete time. ASTIN Bull. **37**, 293–317 (2007)
19. Andersen, S.: On the collective theory of risk in case of contagion between claims. Bull. Inst. Math. Appl. **12**, 275–279 (1957)
20. Andrusiv, A., Zinchenko, N.: Estimates for ruin probabilities and invariance principle for Cramér-Lundberg model with stochastic premiums. Prykladna Statist. Actuarna ta Financ. Matematyka. **2**, 5–17 (2004)
21. Applebaum, D.: Lévy Process and Stochastic Calculus. Cambridge University Press, Cambridge (2004)
22. Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Coherent measures of risk. Math. Financ. **9**, 203–228 (1999)
23. Asmussen, S.: Risk theory in a Markovian environment. Scand. Actuar. J. **1989**(2), 69–100 (1989)
24. Asmussen, S.: Ruin Probabilities. World Scientific, Singapore (2000)
25. Asmussen, S., Albrecher, H.: Ruin Probabilities, 2nd edn. World Scientific, New Jersey (2010)
26. Asmussen, S., Kluppelberg, C.: Large deviation results for subexponential tails, with applications to insurance risk. Stoch. Process. Appl. **64**, 105–125 (1996)
27. Assaf, D., Langberg, N.A., Savits, T.H., Shaked, M.: Multivariate phase-type distributions. Oper. Res. **32**, 688–702 (1984)
28. Avanzi, B.: Strategies for dividend distribution: a review. N. Am. Actuar. J. **13**(2), 217–251 (2009)
29. Avanzi, B., Cheung, E.C.K., Wong, B., Woo, J.-K.: On a periodic dividend barrier strategy in the dual model with continuous monitoring of solvency. Insur.: Math. Econ. **52**(1), 98–113 (2013)
30. Avanzi, B., Taylor, G., Vu, P.A., Wong, B.: Stochastic loss reserving with dependence: a flexible multivariate Tweedie approach (30 March 2016). UNSW Business School Research Paper No. 2016ACTL01. Available at SSRN. https://dx.doi.org/10.2139/ssrn.2753540
31. Avram, F., Palmowski, Z., Pistorius, M.R.: On the optimal dividend problem for a spectrally negative Lévy process. Ann. Appl. Probab. **17**, 156–180 (2007)

32. Avram, F., Palmowski, Z., Pistorius, M.R.: On Gerber-Shiu functions and optimal dividend distribution for a Lévy risk process in the presence of a penalty function. Ann. Appl. Probab. **25**(4), 1868–1935 (2015)
33. Avram, F., Badescu, A.L., Pistorius, M.R., Rabehasaina, L.: On a class of dependent Sparre Andersen risk models and a bailout application. Insur.: Math. Econ. **71**, 27–39 (2016)
34. Azcue, P., Muler, N.: Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model. Math. Financ. **15**, 261–308 (2005)
35. Azcue, P., Muler, N.: Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constraints. Insur.: Math. Econ. **44**(1), 26–34 (2009)
36. Azcue, P., Muler, N.: Optimal investment policy and dividend payment strategy in an insurance company. Ann. Appl. Probab. **20**, 1253–1302 (2010)
37. Azcue, P., Muler, N.: Stochastic Optimization in Insurance. A Dynamic Programming Approach. Springer, Berlin (2014)
38. Badescu, A.L., Landriault, D.: Moments of the discounted dividends in a threshold-type Markovian risk process. Braz. J. Probab. Stat. **21**, 13–25 (2007)
39. Badescu, A.L., Landriault, D.: Recursive calculation of the dividend moments in a multi-threshold risk model. N. Am. Actuar. J. **12**(1), 74–88 (2008)
40. Badescu, A.L., Landriault, D.: Applications of fluid flow matrix analytic methods in ruin theory - a review. Rev. R. Acad. Cien. Serie A Mat. **103**(2), 353–372 (2009)
41. Badescu, A.L., Stanford, D.A.: A generalization of the De Vylder approximation for the probability of ruin. Econ. Comput. Econ. Cybern. Stud. Res. **40**(3–4), 245–265 (2006)
42. Badescu, A., Breuer, L., da Silva Soares, A., Latouche, G., Remiche, M.-A., Stanford, D.: Risk processes analyzed as fluid queues. Scand. Actuar. J. **2005**(2), 127–141 (2005)
43. Badescu, A., Breuer, L., Drekic, S., Latouche, G., Stanford, D.: The surplus prior to ruin and the deficit at ruin for a correlated risk process. Scand. Actuar. J. **2005**, 433–445 (2005)
44. Badescu, A.L., Drekic, S., Landriault, D.: Analysis of a threshold dividend strategy for a MAP risk model. Scand. Actuar. J. **2007**(4), 227–247 (2007)
45. Badescu, A.L., Drekic, S., Landriault, D.: On the analysis of a multi-threshold Markovian risk model. Scand. Actuar. J. **2007**(4), 248–260 (2007)
46. Badescu, A.L., Cheung, E.C.K., Landriault, D.: Dependent risk models with bivariate phase-type distributions. J. Appl. Probab. **46**(1), 113–131 (2009)
47. Badescu, A.L., Cheung, E., Rabehasaina, L.: A two dimensional risk model with proportional reinsurance. J. Appl. Probab. **48**(3), 749–765 (2011)
48. Balbás, A., Balbás, B., Balbás, R.: Optimal reinsurance: a risk sharing approach. Risks **1**, 45–56 (2013)
49. Balbás, A., Balbás, B., Balbás, R., Heras, A.: Optimal reinsurance under risk and uncertainty. Insur.: Math. Econ. **60**(1), 61–74 (2015)
50. Bao, Zh., Liu, He.: On the discounted factorial moments of the deficit in discrete time renewal risk model. Int. J. Pure Appl. Math. **79**(2), 329–341 (2012)
51. Barton, T.L., Shenkir, W.G., Walker, P.L.: Making Enterprise Risk Management Pay Off: How Leading Companies Implement Risk Management. Prentice Hall, Upper Saddle River (2002)
52. Bäuerle, N., Rieder, U.: Markov Decision Processes with Applications to Finance. Springer, Berlin (2011)
53. Bäuerle, N.: Benchmark and mean-variance problems for insurers. Math. Methods Oper. Res. **62**(1), 159–165 (2005)
54. Bayraktar, E., Kyprianou, A.E., Yamazaki, K.: On optimal dividends in the dual model. ASTIN Bull. **43**(3), 359–372 (2013)
55. Bayraktar, E., Kyprianou, A.E., Yamazaki, K.: Optimal dividends in the dual model under transaction costs. Insur.: Math. Econ. **54**(1), 133–143 (2014)
56. Beard, R.E., Pentikäinen, T., Pesonen, E.: Risk Theory. The Stochastic Basis of Insurance, 3rd edn. Chapman and Hall, London (1984)
57. Belkina, T., Hipp, C., Luo, Sh., Taksar, M.: Optimal constrained investment in the Cramer-Lundberg model. Scand. Actuar. J. **2014**(5), 383–404 (2014)

58. Belkina, T., Kabanov, Yu.: Viscosity solutions of integro-differential equations for non-ruin probabilities. Theory Probab. Appl. **60**(4), 671–679 (2016)
59. Belkina, T., Konyukhova, N., Kurochkin, S.: Singular problems for integro-differential equations in dynamic insurance models. Springer Proceedings in Mathematics, Proceedings of the International Conference on Differential and Difference Equations and Applications (in honour of Prof. Ravi P. Agarval) (Azores University, Ponta Delgada, Portugal, 4–8 July 2011), vol. 47, pp. 27–44. Springer, Berlin (2013)
60. Belkina, T.: Risky investment for insurers and sufficiency theorems for the survival probability. Markov Process. Relat. Fields **20**(3), 505–525 (2014)
61. Belkina, T.A., Konyukhova, N.B., Kurochkin, S.V.: Dynamic insurance models with investment: constrained singular problems for integro-differential equations. Comput. Math. Math. Phys. **56**(1), 43–92 (2016)
62. Belkina, T.A., Konyukhova, N.B., Kurochkin, S.V.: Singular initial-value and boundary-value problems for integro-differential equations in dynamical insurance models with investments. J. Math. Sci. **218**(4), 369–394 (2016)
63. Bening, V.E., Korolev, V.Yu., Liu, L.X.: Asymptotic behavior of generalized risk processes. Acta Math. Sin. Engl. Ser. **20**(2), 349–356 (2004)
64. Bernstein, P.L.: Against the Gods: The Remarkable Story of Risk. Wiley, New York (1996)
65. Bertoin, J.: Lévy Processes. Cambridge Tracts in Mathematics, vol. 121. Cambridge University Press, Cambridge (1996)
66. Bi, X., Zhang, S.: Minimizing the risk of absolute ruin under a diffusion approximation model with reinsurance and investment. J. Syst. Sci. Complex. **28**(1), 144–155 (2015)
67. Biffis, E., Kyprianou, A.E.: A note on scale functions and the time value of ruin for Lévy insurance risk processes. Insur.: Math. Econ. **46**(1), 85–91 (2010)
68. Biffis, E., Morales, M.: On a generalization of the Gerber-Shiu function to path-dependent penalties. Insur.: Math. Econ. **46**, 92–97 (2010)
69. Billingsley, P.: Convergence of Probability Measures, 2nd edn. Wiley, New York (1999)
70. Blaževičius, K., Bieliauskienė, E., Šiaulys, J.: Finite-time ruin probability in the nonhomogeneous claim case. Lith. Math. J. **50**(3), 260–270 (2010)
71. Boland, P.J.: Statistical and Probabilistic Methods in Actuarial Science. Chapman and Hall, Boca Raton (2007)
72. Boonen, T.J., Tan, K.S., Zhuang, S.C.: The role of a representative reinsurer in optimal reinsurance. Insur.: Math. Econ. **70**, 196–204 (2016)
73. Booth, Ph., Chadburn, R., Haberman, S., James, D., Khorasanee, Z., Plumb, R.H., Rickayzen, B.: Modern Actuarial Theory and Practice, 2nd edn. Chapman and Hall/CRC, Boca Raton (2004)
74. Boudreault, M., Cossette, H., Landriault, D., Marceau, E.: On a risk model with dependence between interclaim arrivals and claim sizes. Scand. Actuar. J. **2005**, 265–285 (2006)
75. Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., Nesbitt, S.J.: Actuarial Mathematics, 2nd edn. Society of Actuaries, Schaumburg (1997)
76. Boykov, A.V.: Cramer-Lundberg model with stochastic premiums. Theory Probab. Appl. **47**(3), 549–553 (2002)
77. Brandtner, M., Kursten, W.: Solvency II, regulatory capital, and optimal reinsurance: how good are conditional value-at-risk and spectral risk measures? Insur.: Math. Econ. **59**, 156–167 (2014)
78. Breuer, L.: A quintuple law for Markov-additive processes with phase-type jumps. J. Appl. Probab. **47**(2), 441–458 (2010)
79. Breuer, L., Badescu, A.: A generalised Gerber-Shiu measure for Markov-additive risk processes with phase-type claims and capital injections. Scand. Actuar. J. **2014**(2), 93–115 (2014)
80. Browne, S.: Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Math. Oper. Res. **20**(4), 937–958 (1995)
81. Bühlmann, H.: Mathematical Methods in Risk Theory. Grundlehrenband, vol. 172. Springer, Heidelberg (1970)

82. Bühlmann, H.: Actuaries of the third kind. ASTIN Bull. **17**(2), 137–138 (1987)
83. Bühlmann, H., Gisler, A.: A Course in Credibility Theory and Its Applications. Springer, Berlin (2005)
84. Bühlmann, H., Moriconi, F.: Credibility claims reserving with stochastic diagonal effects. ASTIN Bull. **45**, 309–353 (2015)
85. Bulinskaya, E.V., Chepurin, E.V.: Actuarial education at the Moscow State University. Transactions of the 26th International Congress of Actuaries, 7–12 June 1998, Birmingham, vol. 1, pp. 41–52 (1998)
86. Bulinskaya, E., Gromov, A.: New approach to dynamic XL reinsurance. In: Proceedings of SMTDA 2010, Chania, Greece, June 8–11, pp. 147–154 (2010)
87. Bulinskaya, E., Muromskaya, A.: Optimization of multi-component insurance system with dividend payments. In: Manca, R., McClean, S., Skiadas, Ch.H. (eds.) New Trends in Stochastic Modeling and Data Analysis. ISAST, Athens (2015)
88. Bulinskaya, E., Sokolova, A.: Asymptotic behaviour of stochastic storage systems. Mod. Probl. Math. Mech. **10**(3), 37–62 (2015) (in Russian)
89. Bulinskaya, E., Yartseva, D.: Discrete time models with dividends and reinsurance. In: Proceedings of SMTDA 2010, Chania, Greece, June 8–11, pp. 155–162 (2010)
90. Bulinskaya, E.: On the cost approach in insurance. Rev. Appl. Ind. Math. **10**(2), 276–286 (2003) (in Russian)
91. Bulinskaya, E.V.: Some aspects of decision making under uncertainty. J. Stat. Plan. Inference. **137**(8), 2613–2632 (2007)
92. Bulinskaya, E.V.: Sensitivity analysis of some applied models. Pliska Stud. Math. Bulg. **18**, 57–90 (2007)
93. Bulinskaya, E.: Asymptotic analysis of insurance models with bank loans. In: Bozeman, J.R., Girardin, V., Skiadas, C.H. (eds.) New Perspectives on Stochastic Modeling and Data Analysis, pp. 255–270. ISAST, Athens (2014)
94. Bulinskaya, E., Gromov, A.: Asymptotic behavior of the processes describing some insurance models. Commun. Stat. Theory Methods **45**, 1778–1793 (2016)
95. Bulinskaya, E., Gusak, J.: Optimal control and sensitivity analysis for two risk models. Commun. Stat. Simul. Comput. **44**, 1–17 (2015)
96. Bulinskaya, E., Gusak, J., Muromskaya, A.: Discrete-time insurance model with capital injections and reinsurance. Methodol. Comput. Appl. Probab. **17**(4), 899–914 (2015)
97. Bunyakovskij, V.Ya.: One empirical expression of the law of mortality. St. Petersburg (1869) (in Russian)
98. Cai, J.: Discrete time risk models under rates of interest. Probab. Eng. Inf. Sci. **16**, 309–324 (2002)
99. Cai, J.: Ruin probabilities with dependent rates of interest. J. Appl. Probab. **39**, 312–323 (2002)
100. Cai, J.: Ruin probabilities and penalty functions with stochastic rates of interest. Stoch. Process. Appl. **112**, 53–78 (2004)
101. Cai, J.: On the time value of absolute ruin with debit interest. Adv. Appl. Probab. **39**(2), 343–359 (2007)
102. Cai, J., Dickson, D.C.M.: Ruin probabilities with a Markov chain interest model. Insur.: Math. Econ. **35**(3), 513–525 (2004)
103. Cai, J., Li, H.: Dependence properties and bounds for ruin probabilities in multivariate compound risk models. J. Multivar. Anal. **98**, 757–773 (2007)
104. Cai, J., Fang, Y., Li, Z., Willmot, G.E.: Optimal reciprocal reinsurance treaties under the joint survival probability and the joint profitable probability. J. Risk Insur. **80**, 145–168 (2012)
105. Campolieti, G., Makarov, R.N.: Financial Mathematics: A Comprehensive Treatment. Chapman and Hall/CRC, Boca Raton (2014)
106. Centeno, M.L., Simoes, O.: Optimal reinsurance. Revista de la Real Academia de Ciencias, RACSAM **103**(2), 387–405 (2009)
107. Chapman, R.J.: Simple Tools and Techniques for Enterprise Risk Management. Wiley, New York (2011)

108. Chen, S., Wang, X., Deng, Y., Zeng, Y.: Optimal dividend-financing strategies in a dual risk model with time-inconsistent preferences. Insur.: Math. Econ. **70**, 196–204 (2016)
109. Cheng, Y., Tang, Q.: Moments of the surplus before ruin and the deficit at ruin in the Erlang(2) risk process. N. Am. Actuar. J. **7**, 1–12 (2003)
110. Cheng, Y.B., Tang, Q.H., Yang, H.L.: Approximations for moments of deficit at ruin with exponential and subexponential claims. Stat. Probab. Lett. **59**, 367–378 (2002)
111. Chesney, M., Jeanblanc-Picque, M., Yor, M.: Brownian excursions and Parisian barrier options. Adv. Appl. Probab. **29**(1), 165–184 (1997)
112. Cheung, E.C.K., Feng, R.: A unified analysis of claim costs up to ruin in a Markovian arrival risk model. Insur.: Math. Econ. **53**, 98–109 (2013)
113. Cheung, E.C.K., Landriault, D.: A generalized penalty function with the maximum surplus prior to ruin in a MAP risk model. Insur.: Math. Econ. **46**(1), 127–134 (2010)
114. Cheung, E.C.K., Landriault, D.: On a risk model with surplus-dependent premium and tax rates. Appl. Stoch. Models Bus. Ind. **14**(2), 233–251 (2012)
115. Cheung, E.C.K., Liu, H.: On the joint analysis of the total discounted payments to policyholders and shareholders: threshold dividend strategy. Ann. Actuar. Sci. **10**(2), 236–269 (2016)
116. Cheung, E.C.K., Landriault, D., Willmot, G.E., Woo, J.-K.: Gerber-Shiu analysis with a generalized penalty function. Scand. Actuar. J. **2010**(3), 185–199 (2010)
117. Cheung, E.C.K., Landriault, D., Willmot, G.E., Woo, J.-K.: Structural properties of Gerber-Shiu functions in dependent Sparre Andersen models. Insur.: Math. Econ. **46**(1), 117–126 (2010)
118. Cheung, E.C.K., Landriault, D., Badescu, A.L.: On a generalization of the risk model with Markovian claim arrivals. Stoch. Models **27**(3), 407–430 (2011)
119. Cheung, E.C.K., Landriault, D., Willmot, G.E., Woo, J.-K.: On ordering and bounds in a generalized Sparre Andersen risk model. Appl. Stoch. Models Bus. Ind. **27**(1), 51–60 (2011)
120. Cheung, E.C.K., Liu, H., Woo, J.-K.: On the joint analysis of the total discounted payments to policyholders and shareholders: dividend barrier strategy. Risks **3**(4), 491–514 (2015)
121. Chi, Y.C., Lin, X.S.: On the threshold dividend strategy for a generalized jump-diffusion risk model. Insur.: Math. Econ. **48**, 326–337 (2011)
122. Chiu, S.N., Yin, C.C.: The time of ruin, the surplus prior to ruin and the deficit at ruin for the classical risk process perturbed by diffusion. Insur.: Math. Econ. **33**, 59–66 (2003)
123. Chiu, S.N., Yin, C.C.: On the complete monotonicity of the compound geometric convolution with applications to risk theory. Scand. Actuar. J. **2014**, 116–124 (2014)
124. Choi, M.C.H., Cheung, E.C.K.: On the expected discounted dividends in the Cramér-Lundberg risk model with more frequent ruin monitoring than dividend decisions. Insur.: Math. Econ. **59**, 121–132 (2014)
125. Choulli, T., Taksar, M.: Excess-of-loss reinsurance under taxes and fixed costs. J. Risk Decis. Anal. **2**, 85–102 (2010)
126. Cizek, P., Härdle, W., Weron, R. (eds.): Statistical Tools for Finance and Insurance. Springer, Berlin (2005)
127. Corazza, M., Pizzi, Cl. (eds.): Mathematical and Statistical Methods for Actuarial Sciences and Finance. Springer International Publishing, Switzerland (2014)
128. Cossette, H., Marceau, E., Marri, F.: On the compound Poisson risk model with dependence based on a generalized Farlie-Gumbel-Morgenstern copula. Insur.: Math. Econ. **43**, 444–455 (2009)
129. Cramér, H.: Historical review of Filip Lundberg's works on risk theory. Scand. Actuar. J. **1969**, 6–9 (1969)
130. Cruz, M.G., Peters, G.W., Shevchenko, P.V.: Fundamental Aspects of Operational Risk and Insurance Analytics: A Handbook of Operational Risk. Wiley, New York (2015)
131. Cruz, M.G.: Modeling, Measuring and Hedging Operational Risk. Wiley, New York (2002)
132. Czarna, I., Palmowski, Z.: Ruin probability with Parisian delay for a spectrally negative Lévy risk process. J. Appl. Probab. **48**, 984–1002 (2011)
133. D'Arcy, S.P.: On becoming an actuary of the fourth kind. Proc. Casualty Actuar. Soc. **XCII**(177), 745–754 (2005). http://www.casact.org/pubs/proceed/proceed05/05755.pdf. Accessed 5 July 2007

134. D'Arcy, S.P.: On becoming an actuary of the third kind. Proc. Casualty Actuar. Soc. **LXXVI**, 45–76 (1989). https://www.casact.org/pubs/proceed/proceed89/89045.pdf
135. D'Arcy, S.: Enterprise risk management. J. Risk Manag. Korea **12**(1), 207–228 (2001)
136. D'Arcy, S.P.: Risk appetite. Risk Manag. **15**, 38–41 (2009)
137. Dassios, A., Wu, Sh.: On barrier strategy dividends with Parisian implementation delay for classical surplus processes. Insur.: Math. Econ. **45**(2), 195–202 (2009)
138. Daykin, C.D., Pentikainen, T., Pesonen, M.: Practical Risk Theory for Actuaries. Chapman and Hall/CRC Press, Boca Raton (1993)
139. De Finetti, B.: Su un'impostazione alternativa della teoria collettiva del rischio. In: Transactions of the XV-th International Congress of Actuaries, vol. 2, pp. 433–443 (1957)
140. Decker, A., Galer, D.: Enterprise Risk Management - Straight to the Point: An Implementation Guide Function by Function (Viewpoints on ERM). CreateSpace Independent Publishing Platform (2013)
141. Deelstra, G., Plantin, G.: Risk Theory and Reinsurance. Springer, Berlin (2013)
142. Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R.: Actuarial Theory for Dependent Risks. Wiley, Chichester (2005)
143. Dickson, D.C.M.: On the distribution of the surplus prior to ruin. Insur.: Math. Econ. **11**, 191–207 (1992)
144. Dickson, D.C.M.: On the distribution of the claim causing ruin. Insur.: Math. Econ. **12**, 143–154 (1993)
145. Dickson, D.C.M.: Insurance Risk and Ruin. Cambridge University Press, Cambridge (2005)
146. Dickson, D.C.M., Drekic, S.: The joint distribution of the surplus prior to ruin and the deficit at ruin in some Sparre Andersen models. Insur.: Math. Econ. **34**, 97–107 (2004)
147. Dickson, D.C.M., Hipp, C.: Ruin problems for phase-type(2) risk processes. Scand. Actuar. J. **2000**, 147–167 (2000)
148. Dickson, D.C.M., Hipp, C.: On the time to ruin for Erlang(2) risk processes. Insur.: Math. Econ. **29**, 333–344 (2001)
149. Dickson, D.C.M., Waters, H.R.: Some optimal dividends problems. ASTIN Bull. **34**, 49–74 (2004)
150. Dobrushin, R.L.: Describing a system of random variables by conditional distributions. Theory Probab. Appl. **15**, 458–486 (1970)
151. Doherty, N.A.: Integrated Risk Management. McGraw-Hill, New York (2000)
152. Drekic, S., Mera, A.M.: Ruin analysis of a threshold strategy in a discrete-time Sparre Andersen model. Methodol. Comput. Appl. Probab. **13**, 723–747 (2011)
153. Drekic, S., Dickson, D.C.M., Stanford, D.A., Willmot, G.E.: On the distribution of the deficit at ruin when claims are phase-type. Scand. Actuar. J. **2004**, 105–120 (2004)
154. Dufresne, F., Gerber, H.U.: The surpluses immediately before and at ruin, and the amount of the claim causing ruin. Insur.: Math. Econ. **7**(3), 193–199 (1988)
155. Dufresne, F., Gerber, H.U.: Risk theory for the compound Poisson process that is perturbed by diffusion. Insur.: Math. Econ. **10**, 51–59 (1991)
156. Durrett, R.: Stochastic Calculus: A Practical Introduction. CRC Press, Boca Raton (1996)
157. Eisenberg, J., Schmidli, H.: Optimal control of capital injections by reinsurance in a diffusion approximation. Blätter der DGVFM **30**, 1–13 (2009)
158. Embrechts, P., Klüppelberg, C., Mikosch, T.: Modelling Extremal Events for Insurance and Finance. Springer, Berlin (2004)
159. Embrechts, P., Veraverbecke, N.: Estimates for the probability of ruin with special emphasis on the possibility of large claims. Insur.: Math. Econ. **1**, 55–72 (1982)
160. Fang, Y., Wu, R.: Optimal dividends in the Brownian motion risk model with interest. J. Comput. Appl. Math. **229**, 145–151 (2008)
161. Feng, R., Shimizu, Y.: Potential measures for spectrally negative Markov additive processes with applications in ruin theory. Insur.: Math. Econ. **59**, 11–26 (2014)
162. Feng, M.B., Tan, K.S.: Coherent distortion risk measures in portfolio selection. Syst. Eng. Procedia **4**, 25–34 (2012)

163. Fleming, W.H., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions. Applications of Mathematics, vol. 25. Springer, New York (1993)
164. Fraser, J., Simkins, B., Narvaez, K.: Implementing Enterprise Risk Management: Case Studies and Best Practices (Robert W. Kolb Series). Wiley, New York (2014)
165. Frees, E.W.: Stochastic life contingencies with solvency considerations. Trans. Soc. Actuar. **42**, 91–148 (1990)
166. Frolova, A., Kabanov, Yu., Pergamenshchikov, S.: In the insurance business risky investments are dangerous. Financ. Stoch. **6**, 227–235 (2002)
167. Fu, D., Guo, Y.: On the compound Poisson model with debit interest under absolute ruin. Int. J. Sci. Res. (IJSR) **5**(6), 1872–1875 (2016)
168. Funston, F., Wagner, S.: Surviving and Thriving in Uncertainty: Creating the Risk Intelligent Enterprise. Wiley, New York (2010)
169. Gaier, J., Grandits, P.: Ruin probabilities in the presence of regularly varying tails and optimal investment. Insur.: Math. Econ. **30**, 211–217 (2002)
170. Gaier, J., Grandits, P.: Ruin probabilities and investment under interest force in the presence of regularly varying tails. Scand. Actuar. J. **2004**(4), 256–278 (2004)
171. Gaier, J., Grandits, P., Schachermayer, W.: Asymptotic ruin probabilities and optimal investment. Ann. Appl. Probab. **13**, 1054–1076 (2003)
172. Gao, S., Liu, Z.M.: The perturbed compound Poisson risk model with constant interest and a threshold dividend strategy. J. Comput. Appl. Math. **233**, 2181–2188 (2010)
173. Gerber, H.U.: An Introduction to Mathematical Risk Theory. Huebner, Philadelphia (1979)
174. Gerber, H.U.: Der Einfluss von Zins auf die Ruinwahrscheinlichkeit. Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker. **71**(1), 63–70 (1971)
175. Gerber, H.U., Landry, B.: On a discounted penalty at ruin in a jump-diffusion and the perpetual put option. Insur.: Math. Econ. **22**, 263–276 (1998)
176. Gerber, H.U., Shiu, E.S.W.: The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. Insur.: Math. Econ. **21**(2), 129–137 (1997)
177. Gerber, H.U., Shiu, E.S.W.: On the time value of ruin. N. Am. Actuar. J. **2**(1), 48–78 (1998)
178. Gerber, H.U., Shiu, E.S.W.: From ruin theory to pricing reset guarantees and perpetual put options. Insur.: Math. Econ. **24**, 3–14 (1999)
179. Gerber, H.U., Shiu, E.S.W.: The time value of ruin in a Sparre Andersen model. N. Am. Actuar. J. **9**, 49–69 (2005)
180. Gerber, H., Shiu, E.: On optimal dividends: from reflection to refraction. J. Comput. Appl. Math. **186**, 4–22 (2006)
181. Gerber, H.U., Shiu, E.S.W., Smith, N.: Maximizing dividends without bankruptcy. ASTIN Bull. **36**(1), 5–23 (2006)
182. Gerber, H.U., Shiu, E.S.W., Yang, H.: The Omega model: from bankruptcy to occupation times in the red. Eur. Actuar. J. **2**, 259–272 (2012)
183. Gilina, L.C.: Estimation of ruin probability for some insurance model. Prykladna Statist. Actuarna ta Financ. Matematyka. **1**, 67–73 (2000)
184. Gong, L., Badescu, A.L., Cheung, E.: Recursive methods for a two-dimensional risk process with common shocks. Insur.: Math. Econ. **50**, 109–120 (2012)
185. Gosio, C., Lari, E., Ravera, M.: Optimal dynamic proportional and excess of loss reinsurance under dependent risks. Mod. Econ. **7**, 715–724 (2016)
186. Grandell, J.: Aspects of Risk Theory. Springer Series in Statistics: Probability and Its Applications. Springer, New York (1991)
187. Grandell, J.: Simple approximation of ruin probabilities. Insur.: Math. Econ. **26**, 157–173 (2000)
188. Grandits, P.: An analogue of the Cramér-Lundberg approximation in the optimal investment case. Appl. Math. Optim. **50**, 1–20 (2004)
189. Grandits, P.: Minimal ruin probabilities and investment under interest force for a class of subexponential distributions. Scand. Actuar. J. **2005**(6), 401–416 (2005)
190. Grigutis, A., Korvel, A., Šiaulys, J.: Ruin probability in the three-seasonal discrete-time risk model. Mod. Stoch.: Theory Appl. **2**, 421–441 (2015)

191. Guérin, H., Renaud, J.-F.: On distribution of cumulative Parisian ruin. arXiv:1509.06857v1 [math.PR]. 23 Sep 2015
192. Guo, W.: Optimal portfolio choice for an insurer with loss aversion. Insur.: Math. Econ. **58**, 217–222 (2014)
193. Guo, L., Landriault, D., Willmot, G.E.: On the analysis of a class of loss models incorporating time dependence. Eur. Actuar. J. **3**(1), 273–294 (2013)
194. Gup, B.E.: The Basics of Investing, 5th edn. Wiley, New York (1992)
195. Gusak, D.V.: Limit problems for processes with independent increments in risk theory. IM, Kyiv (2007)
196. Hampton, J.J.: Fundamentals of enterprise risk management: how top companies assess risk, manage exposure, and seize opportunity. In: AMACOM (2009)
197. Hartung, F., Pituk, M. (eds.): Recent Advances in Delay Differential and Difference Equations. Springer, Berlin (2014)
198. Heilpern, S.: Ruin measures for a compound Poisson risk model with dependence based on the Spearman copula and the exponential claim sizes. Insur.: Math. Econ. **59**, 251–257 (2014)
199. Hernandez, C., Junca, M.: Optimal dividend payments under a time of ruin constraint: exponential claims. Insur.: Math. Econ. **65**, 136–142 (2015)
200. Heywood, G.: Edmond Halley - actuary. Q. J. R. Astron. Soc. **35**(1), 151–154 (1994)
201. Hipp, C.: Dividend payment with ruin constraint: working paper (2016). https://www.researchgate.net/publication/291339576
202. Hipp, C.: Stochastic control for insurance: models, strategies and numerics (2016). https://www.researchgate.net/publication/308695862
203. Hipp, C., Plump, M.: Optimal investment for insurers. Insur.: Math. Econ. **27**(2), 215–228 (2000)
204. Hipp, C., Plump, M.: Optimal investment for investors with state dependent income, and for insurers. Financ. Stoch. **7**(3), 299–321 (2003)
205. Hipp, C., Schmidli, H.: Asymptotics of ruin probabilities for controlled risk processes in the small claims case. Scand. Actuar. J. **2004**(5), 321–335 (2004)
206. Hipp, C., Taksar, M.: Optimal non-proportional reinsurance control. Insur.: Math. Econ. **47**, 246–254 (2010)
207. Hipp, C., Vogt, M.: Optimal dynamic XL reinsurance. ASTIN Bull. **33**(2), 193–207 (2003)
208. Horcher, K.A.: Essentials of Financial Risk Management. Wiley, New York (2005)
209. Hu, X., Zhang, L.: Ruin probability in a correlated aggregate claims model with common Poisson shocks: application to reinsurance. Methodol. Comput. Appl. Probab. **18**(3), 675–689 (2016)
210. Hürlimann, W.: Mean-variance portfolio selection under portfolio insurance. In: Proceedings of the AFIR, Nürnberg, Germany, pp. 347–374 (1996)
211. Hürlimann, W.: A simple multi-state gamma claims reserving model. Int. J. Contemp. Math. Sci. **10**(2), 65–77 (2015)
212. Iglehart, D.L.: Diffusion approximations in collective risk theory. J. Appl. Probab. **6**, 285–292 (1969)
213. Ihedioha, S., Osu, B.: Optimal portfolios of an insurer and a reinsurer under proportional reinsurance and power utility preference. Open Access Libr. J. **2**(12), 1–11 (2015)
214. Ivanovs, J., Palmowski, Z.: Occupation densities in solving exit problems for Markov additive processes and their reflections. Stoch. Process. Appl. **122**(9), 3342–3360 (2012)
215. Jacobsen, M.: Point Processes, Theory and Applications. Birkhauser, Boston (2005)
216. Jacod, J., Shiryaev, A.: Limit Theorems for Stochastic Processes. Springer, Berlin (2003)
217. Janssen, J., Manca, R.: Semi-Markov Risk Models for Finance, Insurance and Reliability. Springer, Berlin (2007)
218. Jasiulewicz, H.: Discrete-time financial surplus models for insurance companies. Coll. Econ. Anal. Ann. **21**, 225–255 (2010)
219. Jeanblanc-Piqué, M., Shiryaev, A.N.: Optimization of the flow of dividends. Uspekhi Mat. Nauk. **50**(2(302)), 25–46 (1995)

220. Jiang, W.Y., Yang, Z.J., Li, X.P.: The discounted penalty function with multi-layer dividend strategy in the phase-type risk model. Insur.: Math. Econ. **82**, 1358–1366 (2012)
221. Kaas, R., Goovaerts, M., Dhaene, J., Denuit, M.: Modern Actuarial Risk Theory. Springer, Berlin (2008)
222. Kabanov, Yu., Pergamenshchikov, S.: In the insurance business risky investments are dangerous: the case of negative risk sums. Financ. Stoch. **20**(2), 355–379 (2016)
223. Kalashnikov, V., Norberg, R.: Power tailed ruin probabilities in the presence of risky investments. Stoch. Process. Appl. **98**, 221–228 (2002)
224. Karageyik, B.B., Şahin, Ş.: A review on optimal reinsurance under ruin probability constraint. J. Stat.: Stat. Actuar. Sci. **9**(1), 26–36 (2016)
225. Karapetyan, N.V.: Dividends and reinsurance. In: Proceedings of the 6th International Workshop on Simulation, VVM com.Ltd, pp. 47–52 (2009)
226. Karapetyan, N.V.: Optimization and limit distribution in a discrete insurance model. Mod. Probl. Math. Mech. **10**(3), 101–120 (2015) (in Russian)
227. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus. Springer, New York (1998)
228. Kennedy, D.: Stochastic Financial Models. Chapman and Hall/CRC, Boca Raton (2010)
229. Kim, S.S., Drekic, S.: Ruin analysis of a discrete-time dependent Sparre Andersen model with external financial activities and randomized dividends. Risks **4**(2), 1–15 (2016)
230. Klebaner, F.: Introduction to Stochastic Calculus with Applications. World Scientific Publishing Company, Singapore (1999)
231. Klugman, S.A., Panjer, H.H., Willmot, G.E.: Student Solutions Manual to Accompany Loss Models: From Data to Decisions, 4th edn. Wiley, New York (2012)
232. Klugman, S.A., Panjer, H.H., Willmot, G.E.: Loss Models: From Data to Decisions, 3rd edn. Wiley, Hoboken (2008)
233. Koller, G.: Risk Assessment and Decision Making in Business and Industry: A Practical Guide, 2nd edn. Chapman and Hall/CRC, Boca Raton (2005)
234. Koller, M.: Stochastic Models in Life Insurance. Springer, Berlin (2012)
235. Konstantinides, D.G., Li, J.: Asymptotic ruin probabilities for a multidimensional renewal risk model with multivariate regularly varying claims. Insur.: Math. Econ. **69**, 38–44 (2016)
236. Konstantopoulos, T., Papadakis, S.N., Walrand, J.: Functional approximation theorems for controlled renewal processes. J. Appl. Probab. **31**, 765–776 (1994)
237. Korolev, V., Bening, V., Shorgin, S.: Mathematical Foundations of Risk Theory. Fizmatlit, Moscow (2007)
238. Kostić, M.: Abstract Volterra Integro-Differential Equations. CRC Press, Boca Raton (2015)
239. Kulenko, N., Schmidli, H.: Optimal dividend strategies in a Cramér-Lundberg model with capital injections. Insur.: Math. Econ. **43**, 270–278 (2008)
240. Kulkarni, V.G.: A new class of multivariate phase type distributions. Oper. Res. **37**, 151–158 (1989)
241. Kyprianou, A.E.: Introductory Lectures on Fluctuations of Lévy Processes with Applications. Springer, Berlin (2006)
242. Kyprianou, A.: Gerber-Shiu Risk Theory. Springer International Publishing, Switzerland (2013)
243. Kyprianou, A.E., Loeffen, R.L.: Refracted Lévy processes. Ann. Inst. H. Poincaré Probab. Stat. **46**(1), 24–44 (2010)
244. Kyprianou, A.E., Zhou, X.W.: General tax structures and the Lévy insurance risk model. J. Appl. Probab. **46**(4), 1146–1156 (2009)
245. Kyprianou, A.E., Loeffen, R., Perez, J.-L.: Optimal control with absolutely continuous strategies for spectrally negative Lévy processes (2010). arXiv:1008.2363
246. Labbé, C.D., Sendova, K.P.: The expected discounted penalty function under a risk model with stochastic income. Appl. Math. Comput. **215**, 1852–1867 (2009)
247. Labbé, C.D., Sendov, H.S., Sendova, K.P.: The Gerber-Shiu function and the generalized Cramér-Lundberg model. Appl. Math. Comput. **218**, 3035–3056 (2011)
248. Lam, J.: Enterprise Risk Management: From Incentives to Controls, 2nd edn. Wiley, New York (2014)

249. Landriault, D., Li, B., Li, S.: Drawdown analysis for the renewal insurance risk process. Scand. Actuar. J. **2016**, 1–19. doi:10.1080/03461238.2015.1123174
250. Landriault, D.: On a generalization of the expected discounted penalty function in a discrete-time insurance risk model. Appl. Stoch. Models Bus. Ind. **24**(6), 525–539 (2008)
251. Landriault, D., Shi, T.: First passage time for compound Poisson processes with diffusion: ruin theoretical and financial applications. Scand. Actuar. J. **2014**(4), 368–382 (2014)
252. Landriault, D., Shi, T.: Occupation times in the MAP risk model. Insur.: Math. Econ. **60**(1), 75–82 (2015)
253. Landriault, D., Renaud, J.-F., Zhou, X.: Occupation times of spectrally negative Lévy processes with applications. Stoch. Process. Appl. **212**(11), 2629–2641 (2011)
254. Landriault, D., Shi, T., Willmot, G.E.: Joint density involving the time to ruin in the Sparre Andersen risk model under the exponential assumption. Insur.: Math. Econ. **49**(3), 371–379 (2011)
255. Landriault, D., Lemieux, C., Willmot, G.E.: An adaptive premium policy with a Bayesian motivation in the classical risk model. Insur.: Math. Econ. **51**(2), 370–378 (2012)
256. Landriault, D., Lee, W.Y., Willmot, G.E., Woo, J.-K.: A note on deficit analysis in dependency models involving Coxian claim amounts. Scand. Actuar. J. **2014**(5), 405–423 (2014)
257. Landriault, D., Renaud, J.-F., Zhou, X.: An insurance risk model with Parisian implementation delays. Methodol. Comput. Appl. Probab. **16**(3), 583–607 (2014)
258. Landriault, D., Willmot, G.E., Xu, D.: On the analysis of time dependent claims in a class of birth process claim count models. Insur.: Math. Econ. **58**, 168–173 (2014)
259. Landriault, D., Li, B., Li, S.: Analysis of a drawdown-based regime-switching Lévy insurance model. Insur.: Math. Econ. **60**(1), 98–107 (2015)
260. Lee, W.Y., Willmot, G.E.: On the moments of the time to ruin in dependent Sparre Andersen models with emphasis on Coxian interclaim times. Insur.: Math. Econ. **59**, 1–10 (2014)
261. Lefèvre, C., Loisel, S.: Finite-time ruin probabilities for discrete, possibly dependent, claim severities. Methodol. Comput. Appl. Probab. **11**(3), 425–441 (2009)
262. Li, S.: On a class of discrete time renewal risk models. Scand. Actuar. J. **2005**(4), 241–260 (2005)
263. Li, S.: Distributions of the surplus before ruin, the deficit at ruin and the claim causing ruin in a class of discrete time risk models. Scand. Actuar. J. **2005**(4), 271–284 (2005)
264. Li, S.: The distribution of the dividend payments in the compound Poisson risk model perturbed by diffusion. Scand. Actuar. J. **2006**(2), 73–85 (2006)
265. Li, S., Dickson, D.C.M.: The maximum surplus before ruin in an Erlang(n) risk process and related problems. Insur.: Math. Econ. **38**(3), 529–539 (2006)
266. Li, S., Garrido, J.: On ruin for the Erlang(n) risk process. Insur.: Math. Econ. **34**, 391–408 (2004)
267. Li, S., Garrido, J.: On a general class of renewal risk process: analysis of the Gerber-Shiu function. Adv. Appl. Probab. **37**(3), 836–856 (2005)
268. Li, S., Garrido, J.: The Gerber-Shiu function in a Sparre Andersen risk process perturbed by diffusion. Scand. Actuar. J. **2005**(3), 161–186 (2005)
269. Li, S., Garrido, J.: Ruin probabilities for two classes of risk processes. ASTIN Bull. **35**, 61–77 (2005)
270. Li, S.M., Lu, Y.: Moments of the dividend payments and related problems in a Markov-modulated risk model. N. Am. Actuar. J. **11**(2), 65–76 (2007)
271. Li, S., Lu, Y.: The decompositions of the discounted penalty functions and dividends-penalty identity in a Markov-modulated risk model. ASTIN Bull. **38**(1), 53–71 (2008)
272. Li, S., Lu, Y., Garrido, J.: A review of discrete-time risk models. Rev. R. Acad. Cien. Serie A. Mat. **103**, 321–337 (2009)
273. Li, S., Sendova, K.P.: The finite-time ruin probability under the compound binomial model. Eur. Actuar. J. **3**(1), 249–271 (2013)
274. Li, Z., Sendova, K.P.: On a ruin model with both interclaim times and premiums depending on claim sizes. Scand. Actuar. J. **2015**(3), 245–265 (2015)

275. Li, P., Yin, C., Zhou, M.: Dividend payments with a hybrid strategy in the compound Poisson risk model. Appl. Math. **5**, 1933–1949 (2014)
276. Li, D., Rong, X., Zhao, H.: Optimal investment problem for an insurer and a reinsurer. J. Syst. Sci. Complex. **28**(6), 1326–1348 (2015)
277. Li, S., Landriault, D., Lemieux, C.: A risk model with varying premiums: its risk management implications. Insur.: Math. Econ. **60**(1), 38–46 (2015)
278. Liang, Zh., Guo, J.: Optimal investment and proportional reinsurance in the Sparre Andersen model. J. Syst. Sci. Complex. **25**(5), 926–941 (2012)
279. Lin, X.S., Pavlova, K.P.: The compound Poisson risk model with a threshold dividend strategy. Insur.: Math. Econ. **38**, 57–80 (2006)
280. Lin, X.S., Sendova, K.P.: The compound Poisson risk model with multiple thresholds. Insur.: Math. Econ. **42**, 617–627 (2008)
281. Lin, X.S., Willmot, G.E.: The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. Insur.: Math. Econ. **27**, 19–44 (2000)
282. Liu, H.: On the joint analysis of discounted aggregate claim costs until ruin and other ruin-related quantities. Thesis. The University of Hong Kong Libraries, University of Hong Kong (2015)
283. Liu, Ch., Zhang, Zh.: A note on a generalized discounted penalty function in a Sparre Andersen risk model perturbed by diffusion. Abstr. Appl. Anal. **2013**, 6 p (2013)
284. Liu, Zh., Zhang, A., Li, C.: The expected discounted tax payments on dual risk model under a dividend threshold. Open J. Stat. **3**, 136–144 (2013)
285. Liu, J., Xu, J.C., Hu, H.C.: The Markov-dependent risk model with a threshold dividend strategy. Wuhan Univ. J. Nat. Sci. **16**(3), 193–198 (2011)
286. Liu, J., Xu, J.: A note on absolute ruin probability in a Markov risk model. In: 2011 International Conference on Electric Information and Control Engineering (ICEICE) (2011). doi:10.1109/ICEICE.2011.5777660
287. Liu, D.H., Liu, Z.M.: The perturbed compound Poisson risk model with linear dividend barrier. J. Comput. Appl. Math. **235**, 2357–2363 (2011)
288. Livshits, K., Yakimovich, K.: Cramér-Lundberg model with stochastic premiums and continuous non-insurance costs. In: Dudin, A., et al. (eds.) Information Technology and Mathematical Modelling, vol. 487, pp. 251–260. Springer, Berlin (2014)
289. Lkabous, M.A., Czarna, I., Renaud, J.-F.: Parisian ruin for a refracted Lévy process. arXiv:1603.09324v1 [math.PR]. Accessed 30 Mar 2016
290. Loeffen, R.L., Patie, P.: Absolute ruin in the Ornstein-Uhlenbeck type risk model (2010). arXiv:1006.2712
291. Loeffen, R.: On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. Ann. Appl. Probab. **18**, 1669–1680 (2008)
292. Loeffen, R., Renaud, J.F.: De Finetti's optimal dividends problem with an affine penalty function at ruin. Insur.: Math. Econ. **46**, 98–108 (2010)
293. Loeffen, R., Czarna, I., Palmowski, Z.: Parisian ruin probability for spectrally negative Lévy processes. Bernoulli **19**(2), 599–609 (2013)
294. Lu, Y., Li, S.: On the probability of ruin in a Markov-modulated risk model. Insur.: Math. Econ. **37**(3), 522–532 (2005)
295. Luo, S., Taksar, M.: Optimal excess-of-loss reinsurance under borrowing constraints. J. Risk Decis. Anal. **2**, 103–123 (2010)
296. Luo, S., Taksar, M.: On absolute ruin minimization under a diffusion approximation model. Insur.: Math. Econ. **48**(1), 123–133 (2011)
297. Markov, A.A.: Calculus of Probability. St. Petersburg (1908) (in Russian)
298. Markowitz, H.: Portfolio selection. J. Financ. **7**(1), 77–91 (1952)
299. Martin-Löf, A.: Harald Cramér and insurance mathematics. Appl. Stoch. Models Data Anal. **11**, 271–276 (1995)
300. McNeil, A.J., Frey, R., Embrechts, P.: Quantitative Risk Management: Concepts, Techniques and Tools. Princeton University Press, Princeton (2005)

301. Mehr, R.I., Hedges, B.A.: Risk Management in the Business Enterprise. Richard D. Irwin, Inc., Homewood (1963)
302. Melnikov, A.: Risk Analysis in Finance and Insurance, 2nd edn. Chapman and Hall/CRC, Boca Raton (2011)
303. Meyers, G.G.: Stochastic Loss Reserving Using Bayesian MCMC Models. Casualty Actuarial Society, New York (2015)
304. Mikosch, T.: Non-life Insurance Mathematics. Springer, Berlin (2004)
305. Ming, R.X., Wang, W.Y., Xiao, L.Q.: On the time value of absolute ruin with tax. Insur.: Math. Econ. **46**(1), 67–84 (2010)
306. Minkova, L.D.: The Polya-Aeppli process and ruin problems. J. Appl. Math. Stoch. Anal. **3**, 221–234 (2004)
307. Mircea, I., Covrig, M.: A discrete time insurance model with reinvested surplus and a fuzzy number interest rate. Procedia Econ. Financ. **32**, 1005–1011 (2015)
308. Mishura, Yu., Perestyuk, M., Ragulina, O.: Ruin probability in a risk model with variable premium intensity and risky investments. Opusc. Math. **35**(3), 333–352 (2015)
309. Mitov, K.V., Omey, E.: Renewal Processes. Springer, Berlin (2014)
310. Mitric, I.-R., Sendova, K.P.: On a multi-threshold compound Poisson surplus process with interest. Scand. Actuar. J. **2011**(2), 75–95 (2011)
311. Mitric, I.-R., Sendova, K.P., Tsai, C.C.-L.: On a multi-threshold compound Poisson process perturbed by diffusion. Stat. Probab. Lett. **80**, 366–375 (2010)
312. Mitric, I.R., Badescu, A.L., Stanford, D.: On the absolute ruin in a Sparre Andersen risk model with constant interest. Insur.: Math. Econ. **50**, 167–178 (2012)
313. Mnif, M., Sulem, A.: Optimal risk control and dividend policies under excess of loss reinsurance. Stochastics **77**(5), 455–476 (2005)
314. Muromskaya, A.: A generalization of Lundberg inequality for the case of a joint-stock insurance company. Vestnik Moskovskogo Universiteta. Seriya 1: Matematica, Mekhanica. **72**(1), 32–36 (2017) (in Russian)
315. Muromskaya, A.A.: Optimal reinsurance in the model with several risks within one insurance policy. Vestnik TvGU. Seriya: Prikladnaya Matematika [Herald of Tver State University. Series: Applied Mathematics], vol. 2016(4), pp. 79–97 (2016) (in Russian)
316. Muromskaya, A.: Discounted dividends in a strategy with a step barrier function. Moscow Univ. Math. Bull. **71**(5), 200–203 (2016)
317. Murphy, D.: Understanding Risk: The Theory and Practice of Financial Risk Management. Chapman and Hall/CRC, Boca Raton (2008)
318. Nakayama, H., Sawaragi, Y., Tanino, T.: Theory of Multiobjective Optimization. Academic Press, Waltham (1985)
319. Ng, A.C.Y.: On a dual model with a dividend threshold. Insur.: Math. Econ. **44**, 315–324 (2009)
320. Ng, A.C.Y.: On the upcrossing and downcrossing probabilities of a dual risk model with phase-type gains. ASTIN Bull. **40**, 281–306 (2010)
321. Nie, C., Dickson, D.C.M., Li, Sh.: Minimizing the ruin probability through capital injections. Ann. Actuar. Sci. **5**, 195–209 (2011)
322. Nie, C., Dickson, D.C.M., Li, Sh.: The finite time ruin probability in a risk model with capital injections. Scand. Actuar. J. **2015**(4), 301–318 (2015)
323. Note on enterprise risk management for capital and solvency purposes in the insurance industry. http://www.actuaries.org. Accessed 31 Mar 2009
324. Oakley, J.E., O'Hagan, A.: Probabilistic sensitivity analysis of complex models: a Bayesian approach. J. R. Stat. Soc. B. **66**, Part 3, 751–769 (2004)
325. Oh, K., Kang, H.: A discrete time pricing model for individual insurance contracts. J. Insur. Issues **27**(1), 41–65 (2004)
326. Ohlsson, E., Johansson, B.: Non-life Insurance Pricing with Generalized Linear Models. Springer, Berlin (2010)
327. Oksendal, B., Sulem, A.: Applied Stochastic Control of Jump Diffusions. Springer, Universitext (2005)

328. Oksendal, B.: Stochastic Differential Equations. Springer Verlag, Universitext (2013)
329. Olivieri, A., Pitacco, E.: Introduction to Insurance Mathematics. Technical and Financial Features of Risk Transfers. Springer, Berlin (2015)
330. Outreville, J.F.: Theory and Practice of Insurance. Springer, Berlin (1998)
331. Paulsen, J.: Ruin theory with compounding assets - a survey. Insur.: Math. Econ. **22**, 3–16 (1998)
332. Paulsen, J.: On Cramér-like asymptotics for risk processes with stochastic return on investments. Ann. Appl. Probab. **12**, 1247–1260 (2002)
333. Paulsen, J.: Ruin models with investment income. Probab. Surv. **5**, 416–434 (2008)
334. Paulsen, J., Gjessing, H.K.: Ruin theory with stochastic return on investments. Adv. Appl. Probab. **29**, 965–985 (1997)
335. Pavlova, K.P., Willmot, G.E.: The discrete stationary renewal risk model and the Gerber-Shiu discounted penalty function. Insur.: Math. Econ. **35**, 267–277 (2004)
336. Peng, L.: Joint tail of ECOMOR and LCR reinsurance treaties. Insur.: Math. Econ. **59**, 116–120 (2014)
337. Peng, X., Wang, W.: Optimal investment and risk control for an insurer under inside information. Insur.: Math. Econ. **69**, 104–116 (2016)
338. Peng, X., Chen, F., Hu, Y.: Optimal investment, consumption and proportional reinsurance under model uncertainty. Insur.: Math. Econ. **59**, 222–234 (2014)
339. Pentikainen, T., Bornsdorff, H., Pesonen, M., Rantala, J., Ruohonen, M.: Insurance Solvency and Financial Strength. Finnish Insurance Training and Publishing Company Ltd., Helsinki (1989)
340. Pergamenshchikov, S., Zeitouny, O.: Ruin probability in the presence of risky investments. Stoch. Process. Appl. **116**(2), 267–278 (2006)
341. Perna, C., Sibillo, M. (eds.): Mathematical and Statistical Methods for Actuarial Sciences and Finance. Springer, Berlin (2014)
342. Peters, G.W., Dong, A.X.D., Kohn, R.: A copula based Bayesian approach for paid-incurred claims models for non-life insurance reserving. Insur.: Math. Econ. **59**, 258–278 (2014)
343. Pham, H.: Continuous-Time Stochastic Control and Optimization with Financial Applications. Springer, Berlin (2009)
344. Picard, P., Lefèvre, C.: Probabilité de ruine éventuelle dans un modèle de risque à temps discret. J. Appl. Probab. **40**(3), 543–556 (2003)
345. Picard, P., Lefèvre, C., Coulibaly, I.: Problèmes de ruine en théorie du risque à temps discret avec horizon fini. J. Appl. Probab. **40**, 527–542 (2003)
346. Picard, P., Lefèvre, C., Coulibaly, I.: Multirisks model and finite-time ruin probabilities. Methodol. Comput. Appl. Probab. **5**, 337–353 (2003)
347. Pitacco, E.: Health Insurance. Basic Actuarial Models. Springer, Switzerland (2014)
348. Prabhu, N.U.: Stochastic Storage Processes. Queues, Insurance Risk, Dams and Data Communications. Springer, New York (2012)
349. Promyslow, D.S.: Fundamentals of Actuarial Mathematics, 2nd edn. Wiley, New York (2011)
350. Qi, X.: Analysis of the generalized Gerber-Shiu function in discrete-time dependent Sparre Andersen model. Thesis. The University of Hong Kong Libraries, University of Hong Kong (2016)
351. Quang, P.D.: Ruin probability in a generalized risk process under interest force with homogenous Markov chain premiums. Int. J. Stat. Probab. **2**(4), 85–92 (2013)
352. Rachev, S.T., Stoyanov, S.V., Fabozzi, F.J.: Advanced Stochastic Models, Risk Assessment, Portfolio Optimization. Wiley, Hoboken (2008)
353. Rachev, S.T., Klebanov, L., Stoyanov, S.V., Fabozzi, F.: The Methods of Distances in the Theory of Probability and Statistics. Springer, New York (2013)
354. Ramaswami, V.: Passage times in fluid models with application to risk processes. Methodol. Comput. Appl. Probab. **8**, 497–515 (2006)
355. Reinhard, J.M.: On a class of semi-Markov risk models obtained as classical risk models in a Markovian enviroment. ASTIN Bull. **14**, 23–43 (1984)

356. Ren, J.: The discounted joint distribution of the surplus prior to ruin and the deficit at ruin in a Sparre Andersen model. N. Am. Actuar. J. **11**, 128–136 (2007)
357. Renaud, J.-F.: On the time spent in the red by a refracted Lévy risk process. J. Appl. Probab. **51**(4), 1171–1188 (2014)
358. Renaud, J.-F., Zhou, X.: Distribution of the dividend payments in a general Lévy risk model. J. Appl. Probab. **44**, 420–427 (2007)
359. Řezáč, M., Řezáč, F.: How to measure the quality of credit scoring models. Finance a úvěr (Czech J. Econ. Financ.) **61**(5), 486–507 (2011)
360. Rockafellar, R.T., Uryasev, S., Zabarankin, M.: Generalized deviations in risk analysis. Financ. Stoch. **10**, 51–74 (2006)
361. Rolski, T., Schmidli, H., Schmidt, V., Teugels, J.L.: Stochastic Processes for Insuarance and Finance. Wiley, Chichester (1999)
362. Şahin, Ş., Cairns, A.J.G., Kleinow, T., Wilkie, A.D.: Comparison of the Wilkie model and a "Yield-Macro Model" for UK data. http://www.actuaries.org/lyon2013/papers/AFIR-Sahin-Cairns-Kleinow-Wilkie.pdf
363. Şahin, Ş., Cairns, A.J.G., Kleinow, T., Wilkie, A.D.: Revisiting the Wilkie investment model. http://www.actuaries.org/AFIR/Colloquia/Rome2/Cairns-Kleinow-Sahin-Wilkie.pdf
364. Saltelli, A., Tarantola, S., Campolongo, F.: Sensitivity analysis as an ingredient of modeling. Stat. Sci. **15**, 377–395 (2000)
365. Sandström, A.: Handbook of Solvency for Actuaries and Risk Managers: Theory and Practice. Chapman and Hall/CRC Press, Boca Raton (2011)
366. Sato, K.: Lévy Process and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)
367. Savich, S.E.: Elementary Theory of Life Insurance and Disability. SPb., 1900 (3rd edn. Yanus-K 2003) (in Russian)
368. Savitsch, S. von: Der Einfluss der Dimensionen des Feuerrisikos auf den Prämiensatz. Zeitschrift für die gesamte Versicherungswissenschaft (1907)
369. Schmidli, H.: On minimizing the ruin probability by investment and reinsurance. Ann. Appl. Probab. **12**, 890–907 (2002)
370. Schmidli, H.: Stochastic Control in Insurance. Springer, New York (2008)
371. Seal, H.L.: The Stochastic Theory of a Risk Business. Wiley, New York (1969)
372. Sendova, K.P.: Discrete Lundberg-type bounds with actuarial applications. ESAIM: Probab. Stat. **11**, 217–235 (2007)
373. Sendova, K.P., Zang, Y.: Interest bearing surplus model with liquid reserves. J. Insur. Issues **33**(2), 178–196 (2010)
374. Sendova, K.P., Zitikis, R.: The order-statistic claim process with dependent claim frequencies and severities. J. Stat. Theory Pract. **6**, 597–620 (2012)
375. Shang, H. (ed.): Actuarial Science: Theory and Methodology. World Scientific, Singapore (2006)
376. Shen, Y., Yin, C.: Optimal dividend problem for a compound Poisson risk model. Appl. Math. **5**, 1496–1502 (2014)
377. Shi, T., Landriault, D.: Distribution of the time to ruin in some Sparre Andersen risk models. Astin Bull. **43**(1), 39–59 (2013)
378. Shija, G., Jacob, M.J.: Expected discounted penalty function of Markov-modulated, constant barrier and stochastic income model. IOSR J. Math. **9**(6), 34–42 (2014)
379. Shiraishi, H.: Review of statistical actuarial risk modelling. Cogent Math. **3**, 1123945 (2016). http://dx.doi.org/10.1080/23311835.2015.1123945
380. Shiryaev, A.N., Grossinho, M.D.R., Oliveira, P.E., Esquível, M.L. (eds.): Stochastic Finance. Springer, Berlin (2006)
381. Shiryaev, A.N.: Actuarial and financial business: current state of art and perspectives of development. Rev. Appl. Ind. Math. **1**(5), 684–697 (1994) (in Russian)
382. Shorack, G.R., Wellner, J.A.: Empirical Processes with Application to Statistics. Wiley, New York (1986)

383. Silvestrov, D., Martin-Löf, A. (eds.): Modern Problems in Insurance Mathematics. Springer, Berlin (2014)
384. Simkins, B., Fraser, J.: Enterprise Risk Management: Today's Leading Research and Best Practices for Tomorrow's Executives. Wiley, New York (2010)
385. Smith Jr., C.W.: On the convergence of insurance and finance research. J. Risk Insur. **53**(4), 693–717 (1986)
386. Sobol, I.M.: Sensitivity analysis for nonlinear mathematical models. Math. Model. Comput. Exp. **1**, 407–414 (1993)
387. Song, R., Vondracek, Z.: On suprema of Lévy processes and application in risk theory. Ann. lnst. H. Poincare Probab. Stat. **44**, 977–986 (2008)
388. Stanford, D.A., Avram, F., Badescu, A.L., Breuer, L., Da Silva Soares, A., Latouche, G.: Phase-type approximations to finite-time ruin probabilities in the Sparre Andersen and stationary renewal risk models. ASTIN Bull. **35**, 131–144 (2005)
389. Sun, Y., Wei, L.: The finite-time ruin probability with heavy-tailed and dependent insurance and financial risks. Insur.: Math. Econ. **59**, 178–183 (2014)
390. Taksar, M., Markussen, C.: Optimal dynamic reinsurance policies for large insurance portfolios. Financ. Stoch. **7**, 97–121 (2003)
391. Taksar, M., Zeng, X.: Optimal non-proportional reinsurance control and stochastic differential games. Insur.: Math. Econ. **48**, 64–71 (2011)
392. Tankov, P., Cont, R.: Financial modelling with jump processes. Chapman and Hall/CRC, Boca Raton (2003)
393. Temnov, G.: Risk process with random income. J. Math. Sci. **123**(1), 3780–3794 (2004)
394. Temnov, G.: Risk models with stochastic premium and ruin probability estimation. J. Math. Sci. **196**(1), 84–96 (2014)
395. Tsai, C.C.-L.: A generalized defective renewal equation for the surplus process perturbed by diffusion. Insur.: Math. Econ. **30**(1), 51–66 (2002)
396. Tsanakas, A., Beck, M.B., Thompson, M.: Taming uncertainty: the limits to quantification. ASTIN Bull. **46**, 1–7 (2016)
397. Tuncel, A.: Survival probabilities for compound binomial risk model with discrete phase-type claims. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **65**(2), 11–22 (2016)
398. Turnbull, C.: A History of British Actuarial Thought. Springer, Berlin (2016)
399. Ungureanu, D., Vernic, R.: On a fuzzy cash flow model with insurance applications. Decis. Econ. Financ. **38**(1), 39–54 (2015)
400. Wan, N.: Dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion. Insur.: Math. Econ. **40**, 509–523 (2007)
401. Wang, C., Yin, C.: Dividend payments in the classical risk model under absolute ruin with debit interest. Appl. Stoch. Models Bus. Ind. **25**(3), 247–262 (2009)
402. Wang, C.W., Yin, C.C., Li, E.Q.: On the classical risk model with credit and debit interests under absolute ruin. Stat. Probab. Lett. **80**, 427–436 (2010)
403. Wang, X., Wang, W., Zhang, C.: Ornstein-Uhlenback type Omega model. Front. Math. China **11**(3), 735–751 (2016)
404. Wat, K.: Discrete-time insurance risk models with dependence structures. Thesis. The University of Hong Kong Libraries, University of Hong Kong (2012)
405. Wei, J., Wang, R., Yang, H.: On the optimal dividend strategy in a regime-switching diffusion model. Adv. Appl. Probab. **44**, 866–906 (2012)
406. Wilkie, A.D.: A stochastic investment model for actuarial use. Trans. Fac. Actuar. **39**, 341–403 (1986)
407. Wilkie, D.: Whither AFIR? Editorial. ASTIN Bull. **21**(1), 1–3 (1991)
408. Wilkie, A.D.: More on a stochastic asset model for actuarial use. Br. Actuar. J. **1**(5), 777–964 (1995)
409. Wilkie, A.D., Şahin, Ş., Cairns, A.J.G., Kleinow, T.: Yet more on a stochastic economic model: part 1: updating and refitting, 1995 to 2009. Ann. Actuar. Sci. **5**, 53–99 (2010)
410. Willmot, G.E.: A Laplace transform representation in a class of renewal queueing and risk process. J. Appl. Probab. **36**, 570–584 (1999)

411. Willmot, G.E.: Compound geometric residual lifetime distributions and the deficit at ruin. Insur.: Math. Econ. **30**, 421–438 (2002)
412. Willmot, G.E.: On the discounted penalty function in the renewal risk model with general interclaim times. Insur.: Math. Econ. **41**, 17–31 (2007)
413. Willmot, G.E., Lin, X.S.: Lundberg Approximations for Compound Distributions with Insurance Applications. Springer, New York (2001)
414. Wong, T.J.: On some Parisian problems in ruin theory. University of Hong Kong, Pokfulam, Hong Kong SAR (2014). http://dx.doi.org/10.5353/th-b5317068
415. Wong, J.T.Y., Cheung, E.C.K.: On the time value of Parisian ruin in (dual) renewal risk processes with exponential jumps. Insur.: Math. Econ. **65**, 280–290 (2015)
416. Wu, X., Li, S.: On a discrete-time Sparre Andersen model with phase-type claims. http://fbe.unimelb.edu.au/--data/assets/pdf-file/0005/806279/169.pdf
417. Wu, Y.: Bounds for the ruin probability under a Markovian modulated risk model. Commun. Stat. Stoch. Models **15**(1), 125–136 (1999)
418. Wüthrich, M.V., Merz, M.: Stochastic claims reserving manual: advances in dynamic modeling. Swiss Finance Institute Research Paper No. 15–34 (2015)
419. Wüthrich, M.V.: Market-Consistent Actuarial Valuation, 3rd edn. Springer, Berlin (2016)
420. Wüthrich, M.V., Merz, M.: Financial Modeling, Actuarial Valuation and Solvency in Insurance. Springer, Berlin (2013)
421. Wüthrich, M.V., Bühlmann, H., Furrer, H.: Market-Consistent Actuarial Valuation. Springer, Berlin (2010)
422. Xue, C.: Absolute ruin probability of a multi-type-insurance risk model. Int. J. Appl. Math. Stat. **53**(1), 74–81 (2015)
423. Yan, D.: Some results on a double compound Poisson-geometric risk model with interference. Theor. Econ. Lett. **2**(1), 45–49 (2012)
424. Yang, H., Li, J.: Asymptotic finite-time ruin probability for a bidimensional renewal risk model with constant interest force and dependent subexponential claims. Insur.: Math. Econ. **58**, 185–192 (2014)
425. Yang, Y., Yuen, K.C.: Asymptotics for a discrete-time risk model with gamma-like insurance risks. Scand. Actuar. J. **2016**(6), 565–579 (2016)
426. Yang, Y., Leipus, R., Siaulys, J.: On the ruin probability in a dependent discrete time risk model with insurance and financial risks. J. Comput. Appl. Math. **236**(13), 3286–3295 (2012)
427. Yang, H., Gao, W., Li, J.: Asymptotic ruin probabilities for a discrete-time risk model with dependent insurance and financial risks. Scand. Actuar. J. **2016**(1), 1–17 (2016)
428. Yin, C., Wen, Y., Zhao, Y.: On the optimal dividend problem for a spectrally positive Lévy process. arXiv:1302.2231v4 [q-fin.PM]. Accessed 10 Mar 2014
429. Yin, C.C., Wang, C.W.: Optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes: an alternative approach. J. Comput. Appl. Math. **233**, 482–491 (2009)
430. Yin, C.C., Wen, Y.Z.: An extension of Paulsen-Gjessing's risk model with stochastic return on investments. Insur.: Math. Econ. **52**, 469–476 (2013)
431. Yu, W., Huang, Y.: Dividend payments and related problems in a Markov-dependent insurance risk model under absolute ruin. Am. J. Ind. Bus. Manag. **1**(1), 1–9 (2011)
432. Yuen, K.C., Guo, J., Wu, X.: On a correlated aggregate claims model with Poisson and Erlang risk processes. Insur.: Math. Econ. **31**, 205–214 (2002)
433. Zhang, Z.: On a risk model with randomized dividend-decision times. J. Ind. Manag. Optim. **10**(4), 1041–1058 (2014)
434. Zhang, Z., Cheung, E.C.K.: The Markov additive risk process under an Erlangized dividend barrier strategy. Methodol. Comput. Appl. Probab. **18**(2), 275–306 (2016)
435. Zhang, Z., Yang, H.: Gerber-Shiu analysis in a perturbed risk model with dependence between claim sizes and interclaim times. J. Comput. Appl. Math. **235**(5), 1189–1204 (2011)
436. Zhang, Z., Yang, H.: Nonparametric estimation for the ruin probability in a Lévy risk model under low-frequency observation. Insur.: Math. Econ. **59**, 168–177 (2014)

437. Zhang, H.Y., Zhou, M., Guo, J.Y.: The Gerber-Shiu discounted penalty function for classical risk model with a two-step premium rate. Stat. Probab. Lett. **76**, 1211–1218 (2006)
438. Zhang, Z., Li, S., Yang, H.: The Gerber-Shiu discounted penalty functions for a risk model with two classes of claims. J. Comput. Appl. Math. **230**, 643–655 (2009)
439. Zhao, Q., Wei, J., Wang, R.: On dividend strategies with non-exponential discounting. Insur.: Math. Econ. **58**(1), 1–13 (2014)
440. Zheng, Y., Cui, W.: Optimal reinsurance with premium constraint under distortion risk measures. Insur.: Math. Econ. **59**, 109–120 (2014)
441. Zheng, Y., Cui, W., Yang, J.: Optimal reinsurance under distortion risk measures and expected value premium principle for reinsurer. J. Syst. Sci. Complex. **28**(1), 122–143 (2015)
442. Zhu, J., Yang, H.: Estimates for the absolute ruin probability in the compound Poisson risk model with credit and debit interest. J. Appl. Probab. **45**(3), 818–830 (2008)
443. Zhu, Y., Chi, Y., Weng, C.: Multivariate reinsurance designs for minimizing an insurer's capital requirement. Insur.: Math. Econ. **59**, 144–155 (2014)
444. Zinchenko, N., Andrusiv, A.: Risk process with stochastic premiums. Theory Stoch. Process. **14**(30), 189–208 (2008)
445. Zou, B., Cadenillas, A.: Optimal investment and risk control policies for an insurer: expected utility maximization. Insur.: Math. Econ. **58**, 57–67 (2014)
446. Zou, B., Cadenillas, A.: Optimal investment and liability ratio policies in a multidimensional regime switching model. Risks **5**(1), 6 (2017)