

Large Deviations for the Rightmost Position in a Branching Brownian Motion

Bernard Derrida and Zhan Shi

Abstract We study the lower deviation probability of the position of the rightmost particle in a branching Brownian motion and obtain its large deviation function.

Keywords Branching Brownian motion · Lower deviation probability

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1 Introduction

The question of the distribution of the position $X_{\max}(t)$ of the rightmost particle in a branching Brownian motion (BBM) has a long history in probability theory [3–6, 9, 17, 20, 24, 26, 27] and in physics [15, 19, 22, 23].

By branching Brownian motion, we mean that the system starts with a single particle at the origin which performs a Brownian motion with variance σ^2 at time 1, and branches at rate 1 into two independent Brownian motions which themselves branch at rate 1 independently, and so on. For such a BBM, one knows since the work of McKean [20] that

$$u(x, t) := \mathbf{P}(X_{\max}(t) \leq x),$$

satisfies the F-KPP (Fisher–Kolmogorov–Petrovskii–Piskunov) equation

Dedicated to Professor Valentin Konakov on the occasion of his 70th birthday.

B. Derrida

Collège de France, 11 place Marcelin Berthelot, 75231 Paris Cedex 05, France
e-mail: derrida@lps.ens.fr

Z. Shi (✉)

Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France
e-mail: zhan.shi@upmc.fr

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$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + u^2 - u \tag{1}$$

with the initial condition $u(x, 0) = \mathbf{1}_{\{x \geq 0\}}$. It is also known since the works of Bramson [5, 6] that in the long time limit

$$u(x + m(t)\sigma, t) \rightarrow F(x), \tag{2}$$

where $F(z)$ is a traveling wave solution of

$$\frac{\sigma^2}{2} F'' + \sqrt{2\sigma^2} F' + F^2 - F = 0$$

and

$$m(t) := \sqrt{2} t - \frac{3}{2\sqrt{2}} \ln t. \tag{3}$$

This implies in particular that

$$\lim_{t \rightarrow \infty} \frac{X_{\max}(t)}{t} = \sqrt{2\sigma^2}, \quad \text{in probability.}$$

[The convergence also holds almost surely.]

In 1988, Chauvin and Rouault [9, 24] proved a large deviation result for $X_{\max}(t)/t > \sqrt{2\sigma^2}$, namely, that for $v > \sqrt{2\sigma^2}$

$$\ln \left[\mathbf{P} \left(\frac{X_{\max}(t)}{t} > v \right) \right] \sim t \left(1 - \frac{v^2}{2\sigma^2} \right). \tag{4}$$

In (4) and everywhere below, the symbol \sim means that

$$\lim_{t \rightarrow \infty} \frac{\ln \mathbf{P}(X_{\max}(t) > vt)}{t(1 - \frac{v^2}{2\sigma^2})} = 1. \tag{5}$$

Here, we are interested in the *lower deviation* probability $\mathbf{P}(X_{\max}(t) \leq vt)$ for each $v \in (-\infty, \sqrt{2\sigma^2})$. It turns out that $v/\sqrt{2\sigma^2}$ is an important parameter, so we fix $\alpha \in (-\infty, 1)$, and study

$$\mathbf{P}(X_{\max}(t) \leq \alpha\sqrt{2\sigma^2} t),$$

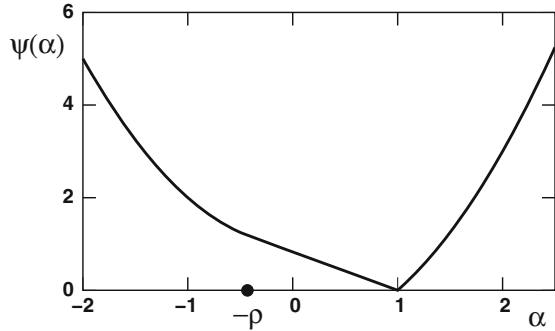
when $t \rightarrow \infty$.

Throughout the paper, we write

$$\rho := \sqrt{2} - 1. \tag{6}$$

Theorem 1 *Let $X_{\max}(t)$ denote the rightmost position of the BBM at time t . Then for all $\alpha \in (-\infty, 1)$,*

Fig. 1 The large deviation function of the position of the rightmost particle of a branching Brownian motion. The expression of $\psi(\alpha)$ is nonanalytic at $\alpha = -\rho = 1 - \sqrt{2}$ and at $\alpha = 1$



$$\ln \mathbf{P}(X_{\max}(t) \leq \alpha \sqrt{2\sigma^2 t}) \sim -t \psi(\alpha), \tag{7}$$

where

$$\psi(\alpha) = \begin{cases} 2\rho(1 - \alpha), & \text{if } \alpha \in [-\rho, 1), \\ 1 + \alpha^2, & \text{if } \alpha \in (-\infty, -\rho]. \end{cases} \tag{8}$$

Together with Theorem 1 and the upper large deviation probability in (4), a routine argument (proof of Theorem III.3.4 in den Hollander [16], proof of Theorem 2.2.3 in Dembo and Zeitouni [11]) yields the following formalism of large deviation principle: the family of the distributions of $\frac{X_{\max}(t)}{\sqrt{2\sigma^2 t}}$, for $t \geq 1$, satisfies the large deviation principle on \mathbf{R} , with speed t and with the rate function $\psi(\alpha)$ (shown in Fig. 1).

$$\psi(\alpha) = \begin{cases} 1 + \alpha^2, & \text{if } \alpha \leq -\rho, \\ 2\rho(1 - \alpha), & \text{if } -\rho \leq \alpha \leq 1, \\ \alpha^2 - 1, & \text{if } \alpha \geq 1, \end{cases} \tag{9}$$

i.e., for any closed set $F \subset \mathbf{R}$ and open set $G \subset \mathbf{R}$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}\left(\frac{X_{\max}(t)}{\sqrt{2\sigma^2 t}} \in F\right) &\leq -\inf_{\alpha \in F} \psi(\alpha), \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}\left(\frac{X_{\max}(t)}{\sqrt{2\sigma^2 t}} \in G\right) &\geq -\inf_{\alpha \in G} \psi(\alpha). \end{aligned}$$

Let us also mention that Proposition 2.5 of Chen [10] (recalled as Lemma 3 in Sect. 3 below) implies that for all $\alpha < 1$,

$$\psi(\alpha) \geq \frac{1 - \alpha}{6},$$

which is in agreement with (8).

The reason for the nonanalyticity of $\psi(\alpha)$ in (9) at $\alpha = -\rho$ is that, as we will see it in Sects. 2 and 3, for $\alpha < -\rho$ the events which dominate are those where the initial particle does not branch or branches at a very late time (at a time τ very close to t) while in the range $-\rho < \alpha < 1$ the first branching event occurs at a time $\tau \sim (1 - \alpha)t/\sqrt{2}$.

The rest of the paper is as follows. Sections 2 and 3 are devoted to the proof of the lower bound and the upper bound, respectively, for the probability in Theorem 1. In Sect. 4, we present some further remarks.

2 Lower Bound

Fix $v \in (-\infty, \sqrt{2\sigma^2})$. We prove the lower bound in the deviation probability, by considering a special event described as follows: The initial particle does not produce any offspring during time interval $[0, \tau]$ and is positioned at $y \in (-\infty, vt - \sqrt{2\sigma^2}(t - \tau) - 1]$ at time τ ; then, at time t , the maximal position lies in $(-\infty, vt)$. As such, we get

$$\begin{aligned} & \mathbf{P}(X_{\max}(t) \leq vt) \\ & \geq e^{-\tau} \int_{-\infty}^{vt - \sqrt{2\sigma^2}(t-\tau) - 1} \frac{dy}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{y^2}{2\sigma^2\tau}} \mathbf{P}(X_{\max}(t - \tau) < vt - y). \end{aligned} \quad (10)$$

Note that for $y \in (-\infty, vt - \sqrt{2\sigma^2}(t - \tau) - 1]$, we have $vt - y \geq \sqrt{2\sigma^2}(t - \tau) + 1$, so

$$\mathbf{P}(X_{\max}(t - \tau) \leq vt - y) \geq \mathbf{P}(X_{\max}(t - \tau) \leq \sqrt{2\sigma^2}(t - \tau) + 1).$$

Let $m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$ be as in (3). By (2), for any $z \in \mathbf{R}$, $\mathbf{P}(X_{\max}(s) \leq m(s)\sigma + z)$ converges, as $s \rightarrow \infty$, to a positive limit (which depends on z). This yields the existence of a constant $c > 0$ such that

$$\mathbf{P}(X_{\max}(t - \tau) \leq \sqrt{2\sigma^2}(t - \tau) + 1) \geq c,$$

for all $\tau \in [0, t]$. [The presence of $+1$ in $\sqrt{2\sigma^2}(t - \tau) + 1$ is only to ensure the positivity of the probability when τ equals t or is very close to t .] Going back to (10), we get that for all $\tau \in (0, t]$,

$$\mathbf{P}(X_{\max}(t) \leq vt) \geq c e^{-\tau} \int_{-\infty}^{vt - \sqrt{2\sigma^2}(t-\tau) - 1} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{y^2}{2\sigma^2\tau}} dy.$$

Hence,

$$\mathbf{P}(X_{\max}(t) \leq vt) \geq c \sup_{\tau \in (0, t]} \left\{ e^{-\tau} \int_{-\infty}^{vt - \sqrt{2\sigma^2}(t-\tau) - 1} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{y^2}{2\sigma^2\tau}} dy \right\}. \tag{11}$$

We now use the following result.

Lemma 2 For $v < \sqrt{2\sigma^2}$ and $t \rightarrow \infty$,

$$\ln \left(\sup_{\tau \in (0, t]} \left\{ e^{-\tau} \int_{-\infty}^{vt - \sqrt{2\sigma^2}(t-\tau) - 1} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{y^2}{2\sigma^2\tau}} dy \right\} \right) \sim -\varphi(v)t,$$

where

$$\varphi(v) := \begin{cases} 2\rho(1 - \alpha), & \text{if } \alpha \geq -\rho, \\ 1 + \alpha^2, & \text{if } \alpha \leq -\rho, \end{cases} \tag{12}$$

with $\alpha := \frac{v}{\sqrt{2\sigma^2}} < 1$ and $\rho := \sqrt{2} - 1$ as before.

The proof of Lemma 2 is quite elementary (as $\ln(\int_{-\infty}^z e^{-y^2} dy) \sim -z^2$ for $z \rightarrow -\infty$, and $\int_{-\infty}^z e^{-y^2} dy$ is greater than a positive constant if $z \geq 0$). We only indicate the optimal value of τ :

$$\tau = \begin{cases} \frac{1-\alpha}{\sqrt{2}} t + o(t), & \text{if } \alpha \geq -\rho, \\ t + o(t), & \text{if } \alpha \leq -\rho. \end{cases} \tag{13}$$

By (11) and Lemma 2, we obtain

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \mathbf{P}(X_{\max}(t) \leq vt) \geq -\varphi(v),$$

with $\varphi(v)$ as in (12). This yields the desired lower bound for the probability in the theorem, as $\varphi(v)$ coincides with $\psi(\alpha)$ defined in (9).

3 Upper Bound

We now look for the upper bound in the deviation probability. Fix $x = vt$ with $v < \sqrt{2\sigma^2}$. Let

$$u(x, t) := \mathbf{P}(X_{\max}(t) \leq x),$$

as before. Considering the event that the first branching time is τ , we have

$$u(x, t) = \int_{-\infty}^x \frac{dy}{\sqrt{2\pi\sigma^2 t}} e^{-t - \frac{y^2}{2\sigma^2 t}} + \int_0^t d\tau \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi\sigma^2 \tau}} e^{-\tau - \frac{y^2}{2\sigma^2 \tau}} u^2(x - y, t - \tau),$$

the first term on the right-hand side originating from the event that the first branching time is greater than t . [It is easy to check that this expression satisfies the F-KPP equation (1).] We also have a lower bound for $u(x, t)$ by considering only the event that there is no branching up to time τ : For any $\tau \in [0, t]$,

$$u(x, t) \geq \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi\sigma^2 \tau}} e^{-\tau - \frac{y^2}{2\sigma^2 \tau}} u(x - y, t - \tau).$$

Writing $(B_s, s \geq 0)$ for a standard Brownian motion (with variance of B_1 being 1), the last two displayed formulas can be expressed as follows:

$$u(x, t) = e^{-t} \mathbf{P}(\sigma B_t \leq x) + \int_0^t e^{-\tau} \mathbf{E}[u^2(x - \sigma B_\tau, t - \tau)] d\tau, \tag{14}$$

$$u(x, t) \geq e^{-\tau} \mathbf{E}[u(x - \sigma B_\tau, t - \tau)], \quad \forall \tau \in [0, t]. \tag{15}$$

Consider, for $\tau \in [0, t]$,

$$\Phi(\tau) := e^{-\tau} \mathbf{E}[u^2(x - \sigma B_\tau, t - \tau)].$$

Since Φ is a continuous function on $[0, t]$, there exists $\tau_0 = \tau_0(t, x)$ such that

$$\Phi(\tau_0) = \sup_{\tau \in [0, t]} \Phi(\tau).$$

On the other hand, since $u(\cdot, 0) = \mathbf{1}_{[0, \infty)}(\cdot)$, we have $e^{-t} \mathbf{P}(\sigma B_t \leq x) = \Phi(t)$. So (14) becomes $u(x, t) = \Phi(t) + \int_0^t \Phi(\tau) d\tau$, which is bounded by $(t + 1) \sup_{\tau \in [0, t]} \Phi(\tau)$. Taking $\tau = \tau_0$ in (15), it follows from (15) to (14) that

$$e^{-\tau_0} \mathbf{E}[u(x - \sigma B_{\tau_0}, t - \tau_0)] \leq u(x, t) \leq (t + 1)\Phi(\tau_0),$$

which can be represented as

$$e^{-\tau_0} \mathbf{E}(Y) \leq u(x, t) \leq (t + 1)e^{-\tau_0} \mathbf{E}(Y^2), \tag{16}$$

where

$$Y = Y(x, t, \sigma) := u(x - \sigma B_{\tau_0}, t - \tau_0).$$

Let us have a closer look at $\mathbf{E}(Y)$. We write

$$e^{-\tau_0} \mathbf{E}(Y) = A_1 + A_2,$$

with

$$A_1 = A_1(x, t, \sigma) := e^{-\tau_0} \mathbf{E}[Y \mathbf{1}_{\{Y < \frac{1}{2(t+1)}\}}],$$

$$A_2 = A_2(x, t, \sigma) := e^{-\tau_0} \mathbf{E}[Y \mathbf{1}_{\{Y \geq \frac{1}{2(t+1)}\}}].$$

Then

$$\begin{aligned} & (t + 1)e^{-\tau_0} \mathbf{E}(Y^2) \\ &= (t + 1)e^{-\tau_0} \mathbf{E}[Y^2 \mathbf{1}_{\{Y < \frac{1}{2(t+1)}\}}] + (t + 1)e^{-\tau_0} \mathbf{E}[Y^2 \mathbf{1}_{\{Y \geq \frac{1}{2(t+1)}\}}] \\ &\leq \frac{1}{2} e^{-\tau_0} \mathbf{E}[Y \mathbf{1}_{\{Y < \frac{1}{2(t+1)}\}}] + (t + 1)e^{-\tau_0} \mathbf{E}[Y \mathbf{1}_{\{Y \geq \frac{1}{2(t+1)}\}}], \end{aligned}$$

where, on the right-hand side, we have used the trivial inequality $Y^2 \leq Y$ when dealing with the event $\{Y \geq \frac{1}{2(t+1)}\}$. In other words,

$$(t + 1)e^{-\tau_0} \mathbf{E}(Y^2) \leq \frac{1}{2} A_1 + (t + 1)A_2.$$

So by (16), we obtain

$$A_1 + A_2 \leq u(x, t) \leq (t + 1)e^{-\tau_0} \mathbf{E}(Y^2) \leq \frac{1}{2} A_1 + (t + 1)A_2.$$

In particular, this implies $A_1 \leq 2tA_2$. As a consequence,

$$A_2 \leq u(x, t) \leq (2t + 1)A_2. \tag{17}$$

This yields that A_2 has the same asymptotic behavior as $u(x, t)$, as far as large deviation functions are concerned.

We now look for an upper bound for A_2 , which, multiplied by $2t + 1$, will be served as an upper bound for $u(x, t)$. Let us recall the following estimate:

Lemma 3 (Chen [10], Proposition 2.5) *Let $m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$ as in (3). There exist two constants $c_1 > 0$ and $c_2 > 0$ independent of σ , such that*

$$\mathbf{P}(X_{\max}(r) \leq \sigma m(r) - \sigma z \text{ for some } r \leq e^z) \leq c_1 e^{-c_2 z},$$

for all sufficiently large z . Moreover, one can take $c_2 = \frac{1}{6\sqrt{2}}$.

We apply the lemma to $z := t^{1/3}$, to see that when t is sufficiently large (say $t \geq t_0$), for any $\tau \in [0, t]$,

$$y < \sqrt{2\sigma^2 \tau} - t^{1/2} \Rightarrow u(y, \tau) < \frac{1}{2(t + 1)}.$$

As such, for $t \geq t_0$, we have

$$A_2 = e^{-\tau_0} \mathbf{E}[Y \mathbf{1}_{\{Y \geq \frac{1}{2(\sigma+1)}\}}] \leq e^{-\tau_0} \mathbf{E}[Y \mathbf{1}_{\{x - \sigma B_{\tau_0} \geq \sqrt{2\sigma^2}(t - \tau_0) - t^{1/2}\}}].$$

Since $Y \leq 1$, this yields, for $t \geq t_0$,

$$\begin{aligned} A_2 &\leq e^{-\tau_0} \mathbf{P}(x - \sigma B_{\tau_0} \geq \sqrt{2\sigma^2}(t - \tau_0) - t^{1/2}) \\ &\leq \sup_{\tau \in [0, t]} \left\{ e^{-\tau} \mathbf{P}(x - \sigma B_{\tau} \geq \sqrt{2\sigma^2}(t - \tau) - t^{1/2}) \right\} \\ &= \sup_{\tau \in (0, t]} \left\{ \int_{-\infty}^{x - \sqrt{2\sigma^2}(t - \tau) + t^{1/2}} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\tau - \frac{y^2}{2\sigma^2\tau}} dy \right\}. \end{aligned}$$

By (17), we have therefore, for all sufficiently large t ,

$$u(x, t) \leq (2t + 1) \sup_{\tau \in (0, t]} \left\{ \int_{-\infty}^{x - \sqrt{2\sigma^2}(t - \tau) + t^{1/2}} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\tau - \frac{y^2}{2\sigma^2\tau}} dy \right\}.$$

Recall that $x = vt$. The supremum on the right-hand side has already been estimated in Lemma 2 in Sect. 2: For $v < \sqrt{2\sigma^2}$ and $t \rightarrow \infty$,

$$\ln \left(\sup_{\tau \in (0, t]} \left\{ \int_{-\infty}^{x - \sqrt{2\sigma^2}(t - \tau) + t^{1/2}} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\tau - \frac{y^2}{2\sigma^2\tau}} dy \right\} \right) \sim -\varphi(v)t,$$

where $\varphi(v)$ is defined in (12). Note that we have $t^{1/2}$ here (in $x - \sqrt{2\sigma^2}(t - \tau) + t^{1/2}$) instead of -1 in the lemma; this makes in practice no difference because $t^{1/2} \leq \varepsilon t$ (for any $\varepsilon > 0$ and all sufficiently large t) and we can use the continuity of the function $v \mapsto \varphi(v)$. Consequently, for $x = vt$ with $v < \sqrt{2\sigma^2}$,

$$\limsup_{t \rightarrow \infty} \frac{\ln u(x, t)}{t} \leq -\varphi(v),$$

which yields the upper bound for the probability in the theorem because $\varphi(v)$ coincides with $\psi(\alpha)$ given in (8).

4 Conclusion and Remarks

The main result stated in (8) and (9) of the present work is the expression of the (lower) large deviation function $\psi(\alpha)$ of the position of the rightmost particle of a branching Brownian motion. One remarkable feature of this large deviation function is its nonanalyticity at some particular values $\alpha = -\sqrt{2} + 1$ and $\alpha = 1$ due to a change of scenario of the dominant contribution to the large deviation function: for $\alpha < -\sqrt{2} + 1$, the dominant event is a single Brownian particle which does not

branch up to time t ; for $-\sqrt{2} + 1 < \alpha < 1$, it corresponds to a particle which moves to position $-(\sqrt{2} - 1)(1 - \alpha)\sigma t$ without branching up to a time $t(1 - \alpha)/\sqrt{2}$, and then behaves like a normal BBM up to time t ; for $\alpha > 1$, the tree branches normally but one branch moves at the speed $\alpha\sqrt{2\sigma^2}$, faster than the normal speed $\sqrt{2\sigma^2}$.

Using more *heuristic* arguments as in [12], it is possible to determine the time dependence of the prefactor, for example, by showing [14] that for $-\rho < \alpha < 1$, there exists a constant $c \in (0, \infty)$ such that

$$\mathbf{P}(X_{\max}(t) \leq \alpha\sqrt{2\sigma^2}t) \sim ct^{\frac{3(\sqrt{2}-1)}{2}} e^{-\psi(\alpha)t} . \tag{18}$$

The result of the present work can also be easily extended to more general branching Brownian motions, where one includes the possibility that a particle branches into more than two particles (for example, one could consider that a particle branches into k particles with probability p_k). It can also be extended to branching random walks. In all these cases, one finds [14] as in (8) and (9) three different regimes with the same scenarios as described above.

It is, however, important to notice that expressions (8) and (9) of the large deviation function $\psi(\alpha)$ for $\alpha < 1$ depend crucially on the fact that one starts initially with a single particle and that branchings occur at random times according to Poisson processes. If instead one starts at time $t = 0$ with several particles in [21] or if the distribution of the branching times is not exponential (for example in the case of a branching random walk generated by a regular binary tree where at each (integer) time step each particle branches into two particles), $\mathbf{P}(X_{\max} \leq vt)$ might decay faster than an exponential of time.

Recently, there has been a renewed interest in the understanding of the extremal process and in particular of the measure seen at the tip of the branching Brownian motion [1, 2, 7, 8, 18, 25]. We think that it would be interesting to investigate how this extremal process is modified when it is conditioned on the position of the rightmost particle, i.e., how it depends on the parameter α .

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