Chapter 3
C^1 Interiors of Sets of Systems with Various Shadowing Properties

In this chapter, we study the structure of C^1 interiors of some basic sets of dynamical systems having various shadowing properties. We give either complete proofs or schemes of proof of the following main results:

- The C^1 interior of the set of diffeomorphisms having the standard shadowing property is a subset of the set of structurally stable diffeomorphisms (Theorem 3.1.1); this result and Theorem 1.4.1 (a) imply that the C^1 interior of the set of diffeomorphisms having the standard shadowing property coincides with the set of structurally stable diffeomorphisms;
- the set $\mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B})$ is a subset of the set of structurally stable vector fields (Theorem 3.3.1); similarly to the case of diffeomorphisms, this result and Theorem 1.4.1 (b) imply that the set $\mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B})$ coincides with the set of structurally stable vector fields;
- the set $\mathrm{Int}^1(\mathrm{OrientSP}_F)$ contains vector fields that are not structurally stable (Theorem 3.4.1).

The structure of the chapter is as follows.

Section 3.1 is devoted to the proof of Theorem 3.1.1:

$$\mathrm{Int}^1\,(\mathrm{SSP}_D) \subset \mathscr{S}_D.$$

Our proof of Theorem 3.1.1 is based on reduction to Theorem 1.3.6 (2) (the C^1 interior of the set of Kupka–Smale diffeomorphisms coincides with the set of structurally stable diffeomorphisms).

We give a detailed proof of the inclusion

$$\mathrm{Int}^1\,(\mathrm{SSP}_D) \subset \mathrm{HP}_D$$

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(thus, any periodic point of a diffeomorphism $f \in \text{Int}^1(\text{SSP}_D)$ is hyperbolic). Concerning the proof of transversality of stable and unstable manifolds of periodic points of a diffeomorphism $f \in \text{Int}^1(\text{SSP}_D)$, we refer the reader to Sect. 3.3 where a similar statement is proved in a more complicated case of flows on manifolds.

One of the necessary and sufficient conditions of structural stability of a diffeomorphism is Axiom A. In Sect. 3.2, we give an independent proof of the following statement, Theorem 3.2.1: If $f \in \text{Int}^1(\text{SSP}_D)$, then f satisfies Axiom A. Our proof uses neither Mañé's ergodic closing lemma [42] nor the techniques of creating homoclinic orbits developed in [44]. Instead, we refer to a sifting type lemma of Wen–Gan–Wen [109] influenced by Liao's work and apply it to Liao's closing lemma.

Sections 3.3 and 3.4 are devoted to the study of the C^1 interior of the set of vector fields having the oriented shadowing property. We introduce a special class $\mathscr{B}$ of vector fields having two rest points p and q for which there exists a trajectory of nontransverse intersection of the stable manifold $W^s(p)$ and $W^u(q)$. Of course, vector fields in $\mathscr{B}$ are not structurally stable.

In Sect. 3.3, we prove Theorem 3.3.1: The set

$$\text{Int}^1(\text{OrientSP}_F \setminus \mathscr{B})$$

is a subset of the set of structurally stable vector fields.

At the same time, we show in Sect. 3.4 that the set $\text{Int}^1(\text{OrientSP}_F)$ contains vector fields belonging to $\mathscr{B}$. The complete description of the corresponding example given in [69] is quite complicated, and we describe a "model" suggested in [100].

3.1 C^1 Interior of SSP_D

The main result of this section is the following theorem.

Theorem 3.1.1 *$\text{Int}^1(\text{SSP}_D) \subset \mathscr{S}_D$.*

It follows from Theorem 1.4.1 (a) that

$$\mathscr{S}_D \subset \text{LSP}_D \subset \text{SSP}_D.$$

Since the set of structurally stable diffeomorphisms is C^1-open,

$$\mathscr{S}_D = \text{Int}^1(\mathscr{S}_D) \subset \text{Int}^1(\text{SSP}_D).$$

Combining this with Theorem 3.1.1, we conclude that the C^1 interior of the set of diffeomorphisms having the standard shadowing property coincides with the set of structurally stable diffeomorphisms.

As was said at the beginning of this chapter, we reduce the proof of Theorem 3.1.1 to Theorem 1.3.6 (2). Thus, we have to show that

$$\mathrm{Int}^1\,(\mathrm{SSP}_D) \subset \mathrm{Int}^1\,(\mathrm{KS}_D).$$

Of course, for this purpose, it is enough to show that

$$\mathrm{Int}^1\,(\mathrm{SSP}_D) \subset \mathrm{KS}_D. \tag{3.1}$$

This means that we have to establish the inclusion

$$\mathrm{Int}^1\,(\mathrm{SSP}_D) \subset \mathrm{HP}_D \tag{3.2}$$

(i.e., every periodic point of a diffeomorphism in $\mathrm{Int}^1\,(\mathrm{SSP}_D)$ is hyperbolic) and to show that, for a diffeomorphism in $\mathrm{Int}^1\,(\mathrm{SSP}_D)$, stable and unstable manifolds of its periodic points are transverse.

We prove inclusion (3.2) in Lemma 3.1.2.

We do not give here a proof of transversality of stable and unstable manifolds of periodic points of a diffeomorphism in $\mathrm{Int}^1\,(\mathrm{SSP}_D)$. Instead, we refer the reader to Sect. 3.3 of this book; in this section, a similar statement is proved for the case of vector fields (which is technically really more complicated). We advice the reader to "transfer" the proof of Sect. 3.3 to the case of diffeomorphisms.

We start with a lemma proved by Franks in [19]; this lemma plays an essential role in proofs of several theorems below.

If U is a domain in $\mathbb{R}^m$ with compact closure and $f, g : U \to \mathbb{R}^m$ are diffeomorphisms of U onto their images such that $f(U) = g(U) = V$, then we define $\rho_{1,U}(f, g)$ as the maximum of the following values:

$$\sup_{x\in U} \left| f(x) - g(x) \right|, \quad \sup_{x\in U} \left\| Df(x) - Dg(x) \right\|,$$

$$\sup_{y\in V} \left| f^{-1}(y) - g^{-1}(y) \right|, \quad \sup_{y\in V} \left\| Df^{-1}(y) - Dg^{-1}(y) \right\|$$

(this definition corresponds to our definition of the C^1 topology of $\mathrm{Diff}^1(M)$, see Sect. 1.3).

Lemma 3.1.1 *Let U be a domain in $\mathbb{R}^m$ with compact closure, where $m \geq 1$, and let $f : U \to \mathbb{R}^m$ be a C^1 diffeomorphism of U onto its image.*

Consider a finite set of different points $\{x_1, x_2, \dots, x_n\} \subset U$.

Then for any $\varepsilon > 0$, any neighborhood N of the set $\{x_1, x_2, \dots, x_n\}$, and any linear isomorphisms

$$L_i : \mathbb{R}^m \to \mathbb{R}^m$$

such that

$$\|L_i - Df(x_i)\|, \|L_i^{-1} - (Df(x_i))^{-1}\| \leq \varepsilon/8, \quad 1 \leq i \leq n, \tag{3.3}$$

there exists a number $\delta > 0$ *and a* C^1 *diffeomorphism* $g : U \to \mathbb{R}^m$ *with* $f(U) = g(U)$ *and such that*

(a) $\rho_{1,U}(f, g) \leq \varepsilon$,
(b) $g(x) = f(x), \quad x \in U \setminus N$,
and
(c) $g(x) = f(x_i) + L_i(x - x_i), \quad x \in N(\delta/4, x_i),\ 1 \leq i \leq n$.

Proof Standard reasoning shows that since U is a domain with compact closure, there exists a number $\varepsilon_0 > 0$ such that if g is a C^1 mapping of U such that $f(U) = g(U)$ and

$$\sup_{x \in U} |f(x) - g(x)|, \quad \sup_{x \in U} \|Df(x) - Dg(x)\| < \varepsilon_0,$$

then g is a diffeomorphism of U onto $g(U)$.

For a positive $\delta > 0$, let

$$B_\delta(x_i) = \{y \in U : |y - x_i| \leq \delta\}, \quad 1 \leq i \leq n.$$

We assume that δ is small enough, so that the sets $B_\delta(x_i)$ with different i do not intersect. In what follows, we reduce δ if necessary.

Choose a C^∞ real-valued function $\sigma : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \sigma(x) \leq 1$,

$$\sigma(x) = \begin{cases} 0 \text{ if } |x| \geq \delta, \\ 1 \text{ if } |x| \leq \delta/4, \end{cases}$$

and $0 \leq |\sigma'(x)| < 2/\delta$ for all x.

Let $\rho : \bigcup_{i=1}^n B_\delta(x_i) \to \mathbb{R}$ be defined by

$$\rho(y) = \sigma\left(|y - x_i|\right), \quad y \in B_\delta(x_i),\ 1 \leq i \leq n.$$

Fix $\varepsilon \in (0, \varepsilon_0)$ and take $0 < \delta < \min(1, \varepsilon)$ so small that

$$\bigcup_{i=1}^n B_\delta(x_i) \subset N, \tag{3.4}$$

$$|f(x_i) + L_i(y - x_i) - f(y)| \leq \frac{\varepsilon}{4}|y - x_i|, \tag{3.5}$$

and

$$|L_i v - Df(y)v| \leq \frac{\varepsilon}{4}|v|, \quad v \in \mathbb{R}^m, \tag{3.6}$$

for $y \in B_\delta(x_i),\ 1 \leq i \leq n$ (clearly, this is possible due to estimates (3.3)).

Define a mapping $g : U \to \mathbb{R}^m$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \notin \bigcup_{i=1}^n B_\delta(x_i), \\ \rho(y)(f(x_i) + L_i(y - x_i)) + (1 - \rho(y))f(y) & \text{if } y \in \bigcup_{i=1}^n B_\delta(x_i). \end{cases}$$

It is easy to see that if $y \in \bigcup_{i=1}^n B_\delta(x_i)$, then

$$|f(y) - g(y)| = |\rho(y)(f(x_i) + L_i(y - x_i)) - \rho(y)f(y)| =$$

$$= \rho(y)|f(x_i) + L_i(y - x_i) - f(y)| \leq 1 \cdot \frac{\varepsilon}{4} \cdot \delta < \varepsilon.$$

Let us estimate the differences of the derivatives. If $y \in B_\delta(x_i)$ and $v \in \mathbb{R}^m$, then

$$Dg(y)v = \rho(y)L_i v + \langle D\rho(y), v\rangle (f(x_i) + L_i(y - x_i)) +$$

$$+ (1 - \rho(y))Df(y)v - \langle D\rho(y), v\rangle f(y),$$

where

$$\langle D\rho(y), v\rangle = \sum_{j=1}^m \frac{\partial \rho}{\partial y_j}(y)v_j.$$

Thus,

$$|Df(y)v - Dg(y)v| =$$

$$= |\rho(y)L_i v - \rho(y)Df(y)v + \langle D\rho(y), v\rangle (f(x_i) + L_i(y - x_i)) - \langle D\rho(y), v\rangle f(y)| \leq$$

$$\leq \rho(y)|L_i v - Df(y)v| + |\langle D\rho(y), v\rangle||f(x_i) + L_i(y - x_i) - f(y)|.$$

It is clear that if $|y - x_i| > \delta$, then $\rho(y) = 0$, and if $|y - x_i| \leq \delta$, then, by the choice of δ (see (3.6)),

$$\rho(y) \cdot |L_i v - Df(y)v| \leq |L_i v - Df(y)v| \leq \frac{\varepsilon}{4}|v|.$$

If $|y - x_i| > \delta$, then $D\rho(y) = 0$ (since $\rho(y) = 0$ for $|y - x_i| > \delta$). If $|y - x_i| \leq \delta$, then $|D\rho(y)| < 2/\delta$ and

$$|f(x_i) + L_i(y - x_i) - f(y)| \leq \frac{\varepsilon}{4}|y - x_i|$$

by the choice of δ (see (3.5)) and the definition of ρ. Thus,

$$|\langle D\rho(y), v\rangle||f(x_i) + L_i(y - x_i) - f(y)| \leq$$

$$\leq \frac{2}{\delta} \cdot \frac{\varepsilon}{4}|y - x_i||v| \leq \frac{2}{\delta} \cdot \frac{\varepsilon}{4}\delta|v| = \frac{\varepsilon}{2}|v|.$$

Hence,

$$|Df(y)v - Dg(y)v| \leq \frac{\varepsilon}{4}|v| + \frac{\varepsilon}{2}|v| \leq \varepsilon|v|.$$

It follows from the choice of $\varepsilon < \varepsilon_0$ that g is a diffeomorphism of U onto $g(U) = f(U)$.

Now a similar reasoning can be applied to estimate the values

$$|f^1(y) - g^{-1}(y)| \text{ and } \|Df^1(y) - Dg^{-1}(y)\|$$

(reducing δ, if necessary).

Inclusion (3.4) implies that g and f coincide outside N. The lemma is proved. □

Lemma 3.1.2 *Inclusion (3.2) holds.*

Proof Let us consider the case of an m-dimensional manifold M with $m \geq 1$. To get a contradiction, assume that there exists a diffeomorphism $f \in \text{Int}^1(\text{SSP}_D(M)) \setminus \text{HP}_D(M)$.

Then f has a nonhyperbolic periodic point p of period $\pi(p)$.

Take a C^1 neighborhood $\mathscr{U}(f)$ of f lying in $\text{SSP}_D(M)$.

To simplify presentation, we assume that $\pi(p) = 1$ (the case of a periodic point of minimal period $\pi(p) > 1$ is considered similarly). Moreover, since the argument is local, we assume further that f is defined on an open set of $\mathbb{R}^m$.

By the Franks lemma, it is possible to find a diffeomorphism $g \in \mathscr{U}(f)$ with the following properties:

– p is a fixed point of g,
– g is linear in a neighborhood of p.

Indeed, let us introduce local coordinates $x \in \mathbb{R}^m$ near p such that p is the origin. Then, by the Franks lemma, for any $r > 0$ there exists a diffeomorphism f_r such that

– $f_r(x) = f(x)$ for $x \notin N(4r, p)$,
– $f_r(x) = Df(p)x$ for $x \in N(r, p)$.

Note that f_r converges to f with respect to the C^1 topology as $r \to 0$. Fix $r_0 > 0$ such that $f_{r_0} \in \mathscr{U}(f)$ and write g instead of f_{r_0}.

Since the point $p = 0$ is not hyperbolic, the matrix $Dg(p)$ has an eigenvalue λ with $|\lambda| = 1$. To simplify presentation, we assume that $\lambda = 1$ (the proof in the general case can be found in [87]).

Applying a C^1-small perturbation of g (so that the perturbed g still is in $\mathscr{U}(f)$) and preserving the notation g for the perturbed diffeomorphism, we may assume further that $Dg(p)$ has an eigenvalue equal to 1, p is the origin with respect to some local coordinates $x = (x_1, \dots, x_m)$, and g maps a point $x = (x_1, y) \in N(r_0, p)$, where $y = (x_2, \dots, x_m)$, to the point (x_1, By), where B is a hyperbolic matrix.

In this case, the segment

$$\mathscr{I} = \{(x_1, 0, \dots, 0) : 0 \le |x_1| \le r_0\}$$

consists of fixed points of g.

Since it was assumed that $g \in \mathrm{SSP}_D(M)$, for $\varepsilon = r_0/2$ there is the corresponding $0 < d < \varepsilon$ from the definition of the standard shadowing property. Take a natural number l such that the sequence

$$\xi = \{x_k : k \in \mathbb{Z}\} \subset \mathscr{I},$$

where

$$x_k = \begin{cases} 0 & \text{for } k < 0; \\ \left(\frac{r_0 k}{2l}, 0, \dots, 0\right) & \text{for } 0 \le k \le l; \\ (r_0/2, 0, \dots, 0) & \text{for } k > l, \end{cases}$$

is a d-pseudotrajectory of g.

Let $x \in N(\varepsilon, x_0)$ be a point such that

$$|g^k(x) - x_k| < \epsilon \quad \text{for} \quad k \in \mathbb{Z}.$$

Since the matrix B is hyperbolic, for any point (x_1, y) with $y \neq 0$, its g-trajectory leaves the set $N(r_0, p)$. Hence, if

$$|g^k(x) - x_k| < \varepsilon, \quad k \in \mathbb{Z},$$

then $x = (b, 0, \dots, 0)$. Since

$$g(x) = g(b, 0, \dots, 0) = (b, 0, \dots, 0),$$

we see that $|b| < r_0/2$, and then $|b - r_0| < r_0/2$. The obtained contradiction proves our lemma. □

Historical Remarks One of the first results concerning C^1 interiors of sets of diffeomorphisms with properties similar to shadowing was proved by K. Moriyasu in [47].

Let us denote by TS_D the set of topologically stable diffeomorphisms. Recall that a diffeomorphism f of a smooth manifold M is called *topologically stable* if for any

$\varepsilon > 0$ there is a $d > 0$ such that for any homeomorphism g satisfying the inequality $\rho_0(f, g) < d$, there exists a continuous map h mapping M onto M and such that $\rho_0(h, id) < \varepsilon$ and $f \circ h = h \circ g$ (see [104]).

It is known that every topologically stable diffeomorphism has the standard shadowing property (see [46, 105]); thus, $\text{SSP}_D \subset \text{TS}_D$. In addition, every expansive diffeomorphism in SSP_D is in TS_D (see [64] for details).

K. Moriyasu proved in [47] that any diffeomorphism in $\text{Int}^1(\text{TS}_D)$ satisfies Axiom A. In fact, the paper [47] contains the proof of inclusion (3.2) (see Proposition 1 in [47]).

Theorem 3.1.1 was proved by the second author in [87].

Later, a more general result (in which the set SSP_D was replaced by a larger set OSP_D) was obtained by the first author, A. A. Rodionova, and the second author in [65] (the method of proving transversality of the stable and unstable manifolds of periodic points used in [65] was later applied in the case of vector fields [69]; see Sect. 3.3 of this book).

In [88], the second author introduced the notion of C^0 transversality and showed that for two-dimensional Axiom A diffeomorphisms, C^0 transversality of one-dimensional stable and unstable manifolds is equivalent to shadowing. Later, the authors related C^0 transversality to inverse shadowing in two-dimensional Axiom A diffeomorphisms [66].

Let us mention here one more result of that type related to shadowing properties. Let f be a homeomorphism of a metric space (M, dist). We say that f has the *weak shadowing property* if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory ξ of f there is a point $p \in M$ such that

$$\xi \subset N(\varepsilon, O(p, f)).$$

Denote by WSP_D the set of diffeomorphisms having the weak shadowing property.

It was shown by the second author in [89] that if M is a smooth two-dimensional manifold, then

$$\text{Int}^1(\text{WSP}_D(M)) \subset \Omega\mathcal{S}_D(M).$$

Let us note that the above inclusion is strict; it was shown by O. B. Plamenevskaya in [72] that there exist Ω-stable diffeomorphisms of the two-dimensional torus that do not have the weak shadowing property.

Let us also note that the result of [89] cannot be generalized to higher dimensions. R. Mañé constructed in [40] an example of a C^1-open subset $\mathcal{T}$ of the space of diffeomorphisms of the three-dimensional torus such that

- any diffeomorphism $f \in \mathcal{T}$ has a dense orbit (thus, any $f \in \mathcal{T}$ is in $\text{Int}^1(\text{WSP}_D)$);
- any diffeomorphism $f \in \mathcal{T}$ is not Anosov (and hence, it is not Ω-stable).

3.2 Diffeomorphisms in Int^1 (SSP_D) Satisfy Axiom A

As was said at the beginning of this chapter, in this section we prove the following statement.

Theorem 3.2.1 *If $f \in \mathrm{Int}^1$ (SSP_D), then f satisfies Axiom A.*

Remark 3.2.1

1. To get an independent proof of Theorem 3.1.1 using Theorem 3.2.1, one has to show that if a diffeomorphism $f \in \mathrm{Int}^1$ (SSP_D) satisfies Axiom A, then f also satisfies the strong transversality condition.
   This can be done by applying the following scheme. Assuming that the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ for two points $p, q \in \Omega(f)$ have a point r of nontransverse intersection, one can approximate r by points of intersection of periodic points of f and then, perturbing f, to get a point of nontransverse intersection of periodic points of a diffeomorphism $g \in \mathrm{Int}^1$ (SSP_D). After that, one can apply the techniques described in Sect. 3.3 to get a contradiction. We leave details to the reader.
2. Of course, it has shown by Mañé and Hayashi [25, 42, 45] that a diffeomorphism $f \in \mathrm{Int}^1$ (HP_D) satisfies Axiom A, but we give a simpler proof of this result under the assumption that $f \in \mathrm{Int}^1$ (SSP_D); this proof uses neither Mañé's ergodic closing lemma [42] nor the techniques creating homoclinic orbits developed in [44].

Let the phase space be a ν-dimensional manifold M.

Denote, as above, by $\mathrm{Per}(f)$ the set of periodic points of a diffeomorphism $f : M \to M$. Let $\pi(p)$ be the minimal period of a periodic point $p \in \mathrm{Per}(f)$.

It is proved in [40] that if $f \in \mathrm{Int}^1$ ($\mathrm{SSP}_D(M)$, then $\Omega(f) = \mathrm{Cl}(\mathrm{Per}(f))$.

Denote by $P_j(f)$, $0 \le j \le \nu$, the set of hyperbolic periodic points of f whose index (the dimension of the stable manifold) is equal to j. Let Λ_j be the closure of the set $P_j(f)$.

It has shown by Pliss [73] that the sets of sinks, $P_\nu(f)$, and of sources, $P_0(f)$, of a diffeomorphism $f \in \mathrm{Int}^1(\mathrm{SSP}_D(M))$ are finite sets (another proof can be found in [36]).

The following lemma is a "globalized" variant of Frank's lemma (Lemma 3.1.1) for C^1 diffeomorphisms of a smooth closed manifold using exponential mappings.

Lemma 3.2.1 *Let $f \in \mathrm{Diff}^1(M)$ and let $\mathscr{U}(f)$ be a neighborhood of f.*

Then there exists a number $\delta_0 > 0$ and a neighborhood $\mathscr{V}(f) \subset \mathscr{U}(f)$ such that for any $g \in \mathscr{V}(f)$, any finite set $\{x_1, x_2, \dots, x_m\}$ consisting of pairwise different points, any neighborhood U of the set $\{x_1, x_2, \dots, x_m\}$, and any linear isomorphisms $L_i : T_{x_i}M \to T_{g(x_i)}M$ such that

$$\|L_i - Dg(x_i)\|, \|L_i^{-1} - Dg^{-1}(x_i)\| \le \delta_0, \quad 1 \le i \le m,$$

there exist $\varepsilon_0 > 0$ *and* $\tilde{g} \in \mathscr{U}(f)$ *such that*

(*a*) $\tilde{g}(x) = g(x)$ *if* $x \in M \setminus U$, *and*
(*b*) $\tilde{g}(x) = \exp_{g(x_i)} \circ L_i \circ \exp_{x_i}^{-1}(x)$ *if* $x \in B_{\varepsilon_0}(x_i)$ *for all* $1 \le i \le m$.

Note that assertion (*b*) implies that $\tilde{g}(x) = g(x)$ if $x \in \{x_1, x_2, \dots, x_m\}$ and that $D_{x_i}\tilde{g} = L_i$ for all $1 \le i \le m$.

In what follows, we assume that $f \in \mathrm{Int}^1(\mathrm{SSP}_D)$; hence, by Lemma 3.1.1, $f \in \mathrm{Int}^1(\mathrm{HP}_D)$.

Thus, there exists a neighborhood $\mathscr{U}(f)$ of f in $\mathrm{Diff}^1(M)$ such that every periodic point $p \in \mathrm{Per}(g)$ is hyperbolic for any $g \in \mathscr{U}(f)$.

Then there exists a C^1 neighborhood $\mathscr{V}(f)$ of f such that the family of periodic sequences of linear isomorphisms of tangent spaces of M generated by the differentials Dg of diffeomorphisms $g \in \mathscr{V}(f)$ along hyperbolic periodic orbits of points $q \in \mathrm{Per}(g)$ is uniformly hyperbolic (see [42]).

To be exact, this means that there exists $\varepsilon > 0$ and a neighborhood $\mathscr{V}(f)$ of f such that for any $g \in \mathscr{V}(f)$, any $q \in \mathrm{Per}(g)$, and any sequence of linear maps

$$L_i : T_{g^i(q)}M \to T_{g^{i+1}(q)}M$$

with

$$\left\| L_i - Dg\left(g^i(q)\right)\right\| < \varepsilon, \quad i = 1, \dots, \pi(q) - 1,$$

$\prod_{i=0}^{\pi(q)-1} L_i$ is hyperbolic (here $\varepsilon > 0$ and $\mathscr{V}(f)$ correspond to $\mathscr{U}(f)$) according to Lemma 3.2.1.

The following result was proved by Mañé [42, Proposition II.1]. Denote by $E^s(q)(f)$ and $E^u(q)(f)$ the stable and unstable spaces of the hyperbolic structure at a point q of a hyperbolic periodic orbit of f, respectively.

Proposition 3.2.1 *Let* $f \in \mathit{Int}^1(\mathit{HP}_D)$.

In the above notation, there are constants $C > 0$, $m > 0$, *and* $0 < \lambda < 1$ *such that:*

(*a*) *if* $g \in \mathscr{V}(f)$, $q \in \mathit{Per}(g)$, *and* $\pi(q) \ge m$, *then*

$$\prod_{i=0}^{k-1} \left\| Dg^m_{|E^s\left(g^{im}(q)\right)(g)} \right\| \le C\lambda^k \text{ and } \prod_{i=0}^{k-1} \left\| Dg^{-m}_{|E^u\left(g^{-im}(q)\right)(g)} \right\| \le C\lambda^k,$$

where $k = [\pi(q)/m]$.

(*b*) *For any* $g \in \mathscr{V}(f)$ *and* $0 \le j \le \nu$, *the set* $\Lambda_j(g) = Cl(P_j(g))$ *admits a dominated splitting (see Definition 1.3.12)*

$$T_{\Lambda_j(g)}M = E(g) \oplus F(g)$$

with $\dim E(g) = j$, *i.e.,*

$$\left\| Dg^m_{|E(x)(g)} \right\| \cdot \left\| Dg^{-m}_{|F(g^m(x))(g)} \right\| \le \lambda$$

for all $x \in Cl(P_j(g))$ *(note that* $E(x)(g) = E^s(x)(g)$ *and* $F(x)(g) = E^u(x)(g)$ *if* $x \in P_j(g)$*).*

It is easy to see that the above proposition can be restated in the following way.

Proposition 3.2.2 *In the notation and assumptions of Proposition 3.2.1, there exist constants* $m > 0$, $0 < \lambda < 1$, *and* $L > 0$ *such that:*

(*a*) *If* $g \in \mathscr{V}(f)$, $q \in Per(g)$, *and* $\pi(q) \ge L$, *then*

$$\prod_{i=0}^{\pi(q)-1} \left\| Dg^m_{|E^s(g^{im}(q))(g)} \right\| < \lambda^{\pi(q)} \text{ and } \prod_{i=0}^{\pi(q)-1} \left\| Dg^{-m}_{|E^u(g^{-im}(q))(g)} \right\| < \lambda^{\pi(q)}.$$

(*b*) *For any* $g \in \mathscr{V}(f)$ *and* $0 \le j \le \nu$, *the set* $\Lambda_j(g)$ *admits a dominated splitting* $T_{\Lambda_j(g)}M = E(g) \oplus F(g)$ *with* $\dim E(g) = j$ *such that*

$$\left\| Dg^m_{|E(x)(g)} \right\| \cdot \left\| Dg^{-m}_{|F(g^m(x))(g)} \right\| < \lambda^2$$

for any $x \in \Lambda_j(g)$ *(note that* $E(x)(g) = E^s(x)(g)$ *and* $F(x)(g) = E^u(x)(g)$ *if* $x \in P_j(g)$*).*

In what follows, we need two technical lemmas (Lemmas 3.2.2 and 3.2.3).

Denote by Λ a set $\Lambda_j = \mathrm{Cl}(P_j(f))$, where $0 \le j \le \nu$.

Lemma 3.2.2 deals with extension of the dominated splitting to a small neighborhood of Λ in M. Assume that Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ for which there exist constants $m > 0$ and $0 < \lambda < 1$ such that

$$\left\| Df^m_{|E(x)} \right\| \cdot \left\| Df^{-m}_{|F(f^m(x))} \right\| \le \lambda$$

for all $x \in \Lambda$. To simplify notation, denote f^m by f.

It is known (see [27]) that there exists a neighborhood U of Λ, a constant $\hat{\lambda} > 0$, $\lambda < \hat{\lambda} < 1$, and a continuous splitting $T_U M = \hat{E} \oplus \hat{F}$ with $\dim \hat{E} = \dim E$ such that

- $\hat{E}|_\Lambda = E$ and $\hat{F}|_\Lambda = F$;
- $Df(x)\hat{E}(x) = \hat{E}(f(x))$ if $x \in U \cap f^{-1}(U)$;
- $Df^{-1}(x)\hat{F}(x) = \hat{F}(f^{-1}(x))$ if $x \in U \cap f(U)$;
- $\left\| Df^k_{|\hat{E}(x)} \right\| \cdot \left\| Df^{-k}_{|\hat{F}(f^k(x))} \right\| < \hat{\lambda}^k$ if $x \in \bigcap_{i=-k}^{k} f^i(U)$ for $k \ge 0$.

The continuity of the differential Df implies the following statement (in which we have to shrink the neighborhood U of Λ if necessary).

Lemma 3.2.2 *In the above notation and assumptions of Proposition 3.2.1, there exists a Df-invariant continuous splitting $T_{\Lambda_f(U)}M = \hat{E} \oplus \hat{F}$ with $\dim \hat{E} = \dim E$ and $0 < \hat{\lambda} < 1$ such that*

- $\hat{E}_{|\Lambda} = E$ *and* $\hat{F}_{|\Lambda} = F$;
- $\left\| Df^k_{|\hat{E}(x)} \right\| \cdot \left\| Df^{-k}_{|\hat{F}\left(f^k(x)\right)} \right\| < \hat{\lambda}^k$ *for any* $x \in \Lambda_f(U)$ *and* $k \geq 0$;
- *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $x \in \Lambda_f(U)$, $y \in \Lambda$, *and* $dist(x, y) < \delta$, *then*

$$\left| \log \left\| Df_{|\hat{E}(x)} \right\| - \log \left\| Df_{|E(y)} \right\| \right| < \varepsilon$$

and

$$\left| \log \left\| Df^{-1}_{|\hat{F}(x)} \right\| - \log \left\| Df^{-1}_{|F(y)} \right\| \right| < \varepsilon.$$

In the statement above,

$$\Lambda_f(U) = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

The second technical lemma (Lemma 3.2.3) is a variant of the so-called sifting lemma first proved by Liao (see [36]). The statement which we prove belongs to Wen–Gan–Wen [109].

Let $T_{\Lambda_f(U)}M = \hat{E} \oplus \hat{F}$ be as in Lemma 3.2.2 and let $0 < \lambda < 1$.

An orbit string

$$\{x, n\} = \{x, f(x), \dots, f^n(x)\} \subset \Lambda_f(U)$$

is called a λ-*quasi-hyperbolic string* with respect to the splitting $\hat{E} \oplus \hat{F}$ if the following conditions are satisfied:

(1)

$$\prod_{i=0}^{k-1} \left\| Df_{|\hat{E}\left(f^i(x)\right)} \right\| \leq \lambda^k \text{ for } k = 1, 2, \dots, n;$$

(2)

$$\prod_{i=k-1}^{n-1} m\left(Df_{|\hat{F}\left(f^i(x)\right)} \right) \geq \lambda^{k-n-1} \text{ for } k = 1, 2, \dots, n;$$

(3)

$$\left\| Df_{|\hat{E}\left(f^i(x)\right)} \right\| / m\left(Df_{|\hat{F}\left(f^i(x)\right)} \right) \leq \lambda^2 \text{ for every } i = 0, 1, \dots, n-1.$$

Here $m(A)$ is the minimum norm of a linear map A, i.e.,

$$m(A) = \inf_{\|v\|=1} \|Av\|.$$

Lemma 3.2.3 (Sifting Lemma, [36, 107, 109]) *Let $\{a_i\}_{i=0}^{\infty}$ be an infinite sequence for which there exists a constant K such that $|a_i| < K$. Assume that*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a_i = \xi \text{ and } \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a_i = \xi',$$

where $\xi' < \xi$. Then for any ξ_1 and ξ_2 with $\xi_1 < \xi < \xi_2$ there is an infinite sequence $\{m_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that

$$\frac{1}{k}\sum_{j=m_i}^{m_i+k-1} a_j \le \xi_2 \text{ and } \frac{1}{k}\sum_{j=m_{i+1}-k}^{m_{i+1}-1} a_j \ge \xi_1$$

for every $i = 1, 2, \ldots$ and every $k = 1, \ldots, m_i + 1 - m_i$.

Proof Let $S(n) = \sum_{i=0}^{n-1} a_i$.

Fix a small $\varepsilon > 0$ with

$$\frac{\xi - \xi'}{2} > \varepsilon.$$

(We determine ε at the end of the proof.)

Choose a large enough $N \in \mathbb{N}$ such that

$$\frac{1}{n}S(n) < \xi + \varepsilon$$

for any $n > N$.

By our assumption, the upper and lower limits are different; hence, there is an infinite sequence

$$N < n_1 < n_1' < n_2 < n_2' < n_3 < n_3' < \ldots$$

such that

$$\frac{1}{n_i}S(n_i) < \xi' + \varepsilon < \xi - \varepsilon < \frac{1}{n_i'}S(n_i')$$

for every $i = 1, 2, \ldots$.

Take an integer $n_i < m_i < n_{i+1}$ such that

$$\frac{S(k) - S(m_i)}{k - m_i} \le \xi - \varepsilon$$

for every $k = m_i + 1, m_+2 \dots, n_{i+1}$ and

$$\frac{S(m_i) - S(k)}{m_i - k} \ge \xi - \varepsilon$$

for every $k = n_i, n_i + 1, \dots, m_i - 1$.

This is a crucial point of the proof. Roughly speaking, m_i is the index at which $S(k) - S(n_i) - (k - n_i)(\xi - \varepsilon)$ attains maximum when k runs over the set $n_i + 1, n_i + 2, \dots, n_{i+1}$ (Fig. 3.1).

Claim

$$n_{i+1} - m_i > \frac{\xi - \xi' - 2\varepsilon}{K + \xi' + \varepsilon} m_i \text{ and } m_i - n_i > \frac{\xi - \xi' - 2\varepsilon}{K - \xi' - \varepsilon} m_i.$$

Proof (of the claim) By the choice of m_i, it is easy to see that

$$S(m_i) - S(n_i') \ge (m_i - n_i')(\xi - \varepsilon).$$

Hence,

$$S(m_i) \ge m_i(\xi - \varepsilon).$$

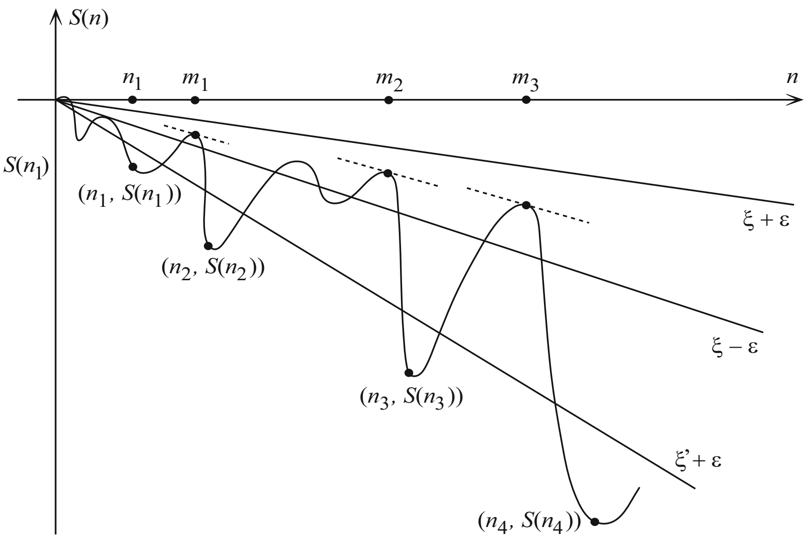

Fig. 3.1 The choice of m_i

Since $|a_i| < K$, we get the inequalities

$$n_{i+1}(\xi' + \varepsilon) > S(n_i + 1) > S(m_i) - K(n_{i+1} - m_i) \geq m_i(\xi - \varepsilon) - K(n_{i+1} - m_i)$$

and

$$n_i(\xi' + \varepsilon) + K(m_i - n_i) > S(n_i) + K(m_i - n_i) > S(m_i) \geq m_i(\xi - \varepsilon).$$

Hence,

$$K(n_{i+1} - m_i) > (\xi - \varepsilon)m_i - (\xi + \varepsilon)n_{i+1} =$$

$$= (\xi - \xi' - 2\varepsilon)m_i + (\xi' + \varepsilon)(m_i - n_{i+1})$$

and

$$K(m_i - n_i) > m_i(\xi - \varepsilon) - n_i(\xi' + \varepsilon) =$$

$$= m_i(\xi - \xi' - 2\varepsilon) + (\xi' + \varepsilon)(m_i - n_i).$$

Therefore,

$$n_{i+1} - m_i > \frac{\xi - \xi' - 2\varepsilon}{K + \xi' + \varepsilon} m_i \text{ and } m_i - n_i > \frac{\xi - \xi' - 2\varepsilon}{K - \xi' - \varepsilon} m_i.$$

Thus, the claim is proved. □

Let us pass to the proof of Lemma 3.2.3.

It is obvious that for $k = 1, 2, \ldots, n_{i+1} - m_i$,

$$\frac{1}{k}\left(S(m_i + k) - S(m_i)\right) \leq \xi - \varepsilon.$$

For $k = n_{i+1} - m_i + 1, \ldots, m_{i+1} - m_i$,

$$\frac{1}{k}\left(S(m_i + k) - S(m_i)\right) < \frac{1}{k}\left((m_i + k)(\xi + \varepsilon) - m_i(\xi - \varepsilon)\right) =$$

$$= \xi + \varepsilon + 2\varepsilon\frac{m_i}{k} < \xi + \left(1 + 2\frac{K + \xi' + \varepsilon}{\xi - \xi' - 2\epsilon}\right)\varepsilon.$$

Note that in the third inequality we have used the above claim.

Similarly, for $k = 1, 2, \ldots, m_i - n_i$,

$$\frac{1}{k}\left(S(m_i) - S(m_i - k)\right) \geq \xi - \varepsilon,$$

and for $k = m_i - n_i + 1, \dots, m_i - m_{i-1}$,

$$\frac{1}{k}\left(S(m_i) - S(m_i - k)\right) > \frac{1}{k}\left(m_i(\xi - \varepsilon) - (m_i - k)(\xi + \varepsilon)\right) =$$

$$= \xi + \varepsilon - 2\varepsilon\frac{m_i}{k} > \xi + \left(1 - 2\frac{K - \xi' - \varepsilon}{\xi - \xi' - 2\varepsilon}\right)\varepsilon.$$

Now choose ε small enough so that

$$\xi + \left(1 + 2\frac{K + \xi' + \varepsilon}{\xi - \xi' - 2\varepsilon}\right)\varepsilon < \xi_2$$

and

$$\min\left\{\xi - \varepsilon, \xi + \left(1 - 2\frac{K - \xi' - \varepsilon}{\xi - \xi' - 2\varepsilon}\right)\varepsilon\right\} > \xi_1.$$

This proves Lemma 3.2.3. □

A proof of the following lemma (in fact of its generalized version) is given at the end of this section (see Lemma 3.2.5).

Lemma 3.2.4 (Liao's Closing Lemma [36]) *Let $T_{\Lambda_f(U)}M = \hat{E} \oplus \hat{F}$ be a continuous Df-invariant splitting. For any $0 < \lambda < 1$ and any $\varepsilon > 0$ there is $\delta > 0$ such that for any λ-quasi-hyperbolic string $\{x, n\}$ of f in $\Lambda_f(U)$ with dist $(f^n(x), x) < \delta$, there is a periodic point $p \in M$ of f such that $f^n(p) = p$ and dist $\left(f^i(p), f^i(x)\right) \le \varepsilon$ for all $0 \le i \le n - 1$.*

In the following proposition, to simplify notation, we denote $\Lambda_f(U)$, $\hat{E} \oplus \hat{F}$, and $\hat{\lambda}$ by Λ, $E \oplus F$, and λ, respectively. The next proposition is proved by applying Lemmas 3.2.3 and 3.2.4.

Proposition 3.2.3 *Let Λ be a compact f-invariant set, let $0 < \lambda < 1$ be given, and assume that there is a continuous Df-invariant splitting $T_\Lambda M = E \oplus F$ such that*

$$\left\|Df_{|E(x)}\right\| \cdot \left\|Df^{-1}_{|F(f(x))}\right\| < \lambda^2$$

for any $x \in \Lambda$.

Assume that there exists a point $y \in \Lambda$ such that

$$\log\lambda < \log\lambda_1 = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left\|Df_{|E\left(f^i(y)\right)}\right\| < 0$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left\|Df_{|E\left(f^i(y)\right)}\right\| < \log\lambda_1.$$

Then for any λ_2 and λ_3 with $\lambda < \lambda_2 < \lambda_1 < \lambda_3 < 1$ and any neighborhood W of Λ there is a hyperbolic periodic point q of index $\dim E$ *such that $O(q,f) \subset W$,*

$$\prod_{i=0}^{k-1} \left\| Df_{|E^s(f^i(q))} \right\| \le \lambda_3^k, \;\; \text{and} \;\; \prod_{i=k-1}^{\pi(q)-1} \left\| Df_{|E^s(f^i(q))} \right\| > \lambda_2^{\pi(q)-k+1}$$

for all $k = 1, 2, \ldots, \pi(q)$.

Furthermore, q can be chosen so that the period $\pi(q)$ is arbitrarily large.

Our Theorem 3.2.1 follows from the next proposition (this kind of result was first obtained in [109]).

Proposition 3.2.4 *Let Λ be a compact f-invariant set, and let* $0 < \lambda < 1$ *and* $L > 1$ *be given. Assume that f has the following properties* (P.1)–(P.4).

(P.1) *There is a homogeneous Df-invariant splitting $T_\Lambda M = E \oplus F$ such that*

$$\left\| Df_{|E(x)} \right\| \cdot \left\| Df^{-1}_{|F(f(x))} \right\| < \lambda^2$$

for any $x \in \Lambda$.

(P.2) *There is a compact neighborhood U of Λ such that if $q \in \Lambda_f(U) \cap Per(f)$ and $\pi(q) \ge L$, then*

$$\prod_{i=0}^{\pi(q)-1} \left\| Df_{|E^s(f^i(q))} \right\| < \lambda^{\pi(q)} \; \text{and} \; \prod_{i=0}^{\pi(q)-1} \left\| Df^{-1}_{|E^u(f^{-i}(q))} \right\| < \lambda^{\pi(q)}.$$

(P.3) $\Lambda = \overline{P_{\dim E}(f)}$.

(P.4) *f has the standard shadowing property.*

Then Λ is hyperbolic.

Proof Let Λ be a compact f-invariant set, let $0 < \lambda < 1$ and $L > 0$ be given, and assume that f has properties (P.1)–(P.4). Let $T_\Lambda M = E \oplus F$ be a Df-invariant splitting as in (P.1) (recall that every dominated splitting is continuous). Thus, shrinking the neighborhood U of Λ, we may assume further that there exists an extension $T_{\Lambda_f(U)}M = \hat{E} \oplus \hat{F}$ of the dominated splitting $T_\Lambda M = E \oplus F$ (see Lemma 3.2.2).

Let us prove that Λ is hyperbolic. Assuming that E is not contracting, we show first that for any $\lambda < \eta < \eta' < 1$ there is $z \in \Lambda_f(U)$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Df_{|\hat{E}(f^j(z))} \right\| < \log \eta < \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Df_{|\hat{E}(f^j(z))} \right\| < \log \eta'.$$

After that, we derive a contradiction by applying Proposition 3.2.3.

It is known that if there exists $N > 0$ such that for any $x \in \Lambda$ there is $0 \le n(x) \le N$ such that $\left\| Df^{n(x)}_{|E(x)} \right\| < 1$, then E is contracting.

Since E is not contracting, it is easy to see that there is $y_0 \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \left\| Df_{|E(f^j(y_0))} \right\| \ge 1 \quad \text{for all} \quad n \ge 1$$

(recall that Λ is compact).

Choose $\varepsilon > 0$ small enough with $N(2\varepsilon, \Lambda) \subset U$ such that

(i) if $\operatorname{dist}(x, y) < \varepsilon$ for some $x, y \in N(\varepsilon, \Lambda)$, then

$$\left| \log \left\| Df_{|\hat{E}(x)} \right\| - \log \left\| Df_{|\hat{E}(y)} \right\| \right| < \min \left\{ \frac{1}{2}(\log \eta' - \log \eta), \frac{1}{3}(\log \eta - \log \lambda) \right\}.$$

Observe that item (i) follows from the continuity of E (recall that $\hat{E}_{|\Lambda} = E$).

Since f has the standard shadowing property, there is $0 < \delta \le \varepsilon$ such that any δ-pseudotrajectory of f in M can be ε-shadowed by a trajectory of f.

Denote the ω-limit set of y_0 by $\omega_f(y_0)$. It is well known that $\omega_f(y_0) \subset \Lambda$ is an f-invariant compact set, and for any neighborhood $V = V(\omega_f(y_0))$ of $\omega_f(y_0)$ there is $N > 0$ such that $f^n(y_0) \in V$ for any $n \ge N$. By the compactness, there exists a finite set of points $\{x_j\}_{j=1}^{\ell}$ in $\omega_f(y_0)$ such that

$$\omega_f(y_0) \subset \bigcup_{j=1}^{\ell} N(\delta/2, x_j).$$

Since $P_{\dim E}(f)$ is dense in Λ, it is easy to see that for the chosen δ there exists a finite set of periodic points $\{p_j\}_{j=1}^{\ell} \subset P_{\dim E}(f)$ with $\operatorname{dist}(x_j, p_j) < \frac{\delta}{2}$ such that

$$\omega_f(y_0) \subset \bigcup_{j=1}^{\ell} N(\delta, p_j)$$

and thus, there is $N' > 0$ such that

$$f^n(y_0) \in \bigcup_{j=1}^{\ell} N(\delta, p_j) \subset N(\varepsilon, \Lambda)$$

for any $n \ge N'$.

Assume that $n \ge N'$. Then

$$\prod_{j=0}^{n-1} \left\| Df_{|E(f^j(y_0))} \right\| = \prod_{j=N'}^{n-N'-1} \left\| Df_{|E(f^j(y_0))} \right\| \cdot \prod_{j=0}^{N'-1} \left\| Df_{|E(f^j(y_0))} \right\| \ge 1.$$

Thus,

$$\prod_{j=0}^{n-1} \left\| Df_{|E\left(f^{N'+j}(y_0)\right)} \right\| \geq \left(\prod_{j=0}^{N'-1} \left\| Df_{|E\left(f^j(y_0)\right)} \right\| \right)^{-1} \geq e^{-KN'},$$

so that

$$\frac{1}{n}\sum_{j=0}^{n-1} \log \left\| Df_{|E\left(f^{N'+j}(y_0)\right)} \right\| \geq -\frac{KN'}{n}.$$

Here $K = \max\left\{\left|\log \|Df(x)\|\right|, \left|\log \|Df^{-1}(x)\|\right| : x \in M\right\}$.

Hence,

$$(ii)\ \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \left\| Df_{|E\left(f^{N'+j}(y_0)\right)} \right\| \geq \lim_{n\to\infty}\left(-\frac{KN'}{n}\right) = 0.$$

We may assume that the period of p_j satisfies the inequality $\pi(p_j) \geq L$ for any j, and, finally, put

$$\pi = \prod_{j=1}^{\ell} \pi(p_j).$$

The set of periodic orbits

$$\mathscr{P}\mathscr{O} = \bigcup_{j=1}^{\ell} O(p_j, f)$$

forms a δ-net of $\omega_f(y_0)$, i.e., for any $w \in \omega_f(y_0)$, there is $q \in \mathscr{P}\mathscr{O}$ such that $\mathrm{dist}(w, q) < \delta$, and, conversely, for any $q \in \mathscr{P}\mathscr{O}$, there is $w \in \omega_f(y_0)$ such that $\mathrm{dist}(w, q) < \delta$.

Observe that for any for any $q \in \mathscr{P}\mathscr{O}$,

$$(iii)\ \frac{1}{\pi}\sum_{j=0}^{\pi-1} \log \left\| Df_{|E\left(f^j(q)\right)} \right\| < \frac{1}{2}(\log\lambda + \log\eta)$$

by the choice of δ (see (P.2)).

We construct a δ-pseudotrajectory $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f composed of points of the orbit $O(y_0, f)$ and of the set $\mathscr{P}\mathscr{O}$ by mimicking the procedure displayed in [109] (the construction is by induction). Denote $f^{N'}(y_0)$ by y_0 for simplicity.

Step I Since $y_0 \in \Lambda$, there is $q_{j_1} \in \mathscr{P}\mathscr{O}$ such that $\mathrm{dist}(y_0, q_{j_1}) < \delta$. Set

$$x_{-1} = q_{j_1-1},\ x_{-2} = q_{j_1-2},\ \ldots,\ x_{-\pi+1} = q_{j_1-\pi+1},$$

$$x_{-\pi} = q_{j_1},\ x_{-\pi-1} = q_{j_1-1},\ x_{-\pi-2} = q_{j_1-2}\ \ldots.$$

Then $\mathrm{dist}(f(x_{-i}), x_{-i+1}) < \delta$ for $i \geq 1$, so that the negative part $\{x_i\}_{i=-\infty}^{-1}$ of $\{x_i\}_{i\in\mathbb{Z}}$ is constructed.

Step II Let $n_1 = 1$. Then

$$\frac{1}{n_1\pi}\left(n_1\sum_{j=0}^{\pi-1}\log\left\|Df_{|E(q_{j_1}+j)}\right\|\right) < \frac{1}{2}(\log\lambda + \log\eta).$$

Obviously, this inequality is ensured by (*iii*).

Let $i_1 = n_1\pi$, put $x_j = q_{j_1+j}$ for $j = 0, 1, \ldots, i_1 - 1 = \pi - 1$, and put $x_{i_1} = y_0$. Then $\mathrm{dist}(f(x_j), x_{j+1}) < \delta$ for $j = 0, 1, \ldots, i_1 - 1$, and

$$\frac{1}{i_1}\sum_{j=0}^{i_1-1}\log\left\|Df_{|E(x_j)}\right\| < \frac{1}{2}(\log\lambda + \log\eta).$$

Put

$$a_j = \log\left\|Df_{|E(x_j)}\right\|$$

for $j = 0, 1, \ldots, i_1 - 1$, and choose l_1 so that

$$\frac{1}{i_1 + l_1}\left(\sum_{j=0}^{i_1-1}a_j + \sum_{j=0}^{l_1-1}\log\left\|Df_{|E(f^j(y_0))}\right\|\right) \geq \frac{1}{2}(\log\eta + \log\eta')$$

and

$$\frac{1}{i_1 + l}\left(\sum_{j=0}^{i_1-1}a_j + \sum_{j=0}^{l-1}\log\left\|Df_{|E(f^j(y_0))}\right\|\right) < \frac{1}{2}(\log\eta + \log\eta')$$

for any $l < l_1$.

The existence of l_1 is ensured by the choice of y_0 (recall the choice of y_0 and (*ii*)).

Set $j_1 = i_1 + l_1$, let $x_{i_1+1} = f(y_0), x_{i_1+2} = f^2(y_0), \ldots, x_{j_1-1} = f^{l_1-1}(y_0) \in O(y_0,f)$, and put

$$a_{i_1+j} = \log\left\|Df_{|E(x_{i_1+j})}\right\|$$

for $j = 0, 1, \ldots, l_1 - 1$.

Step III Let i_{k-1}, j_{k-1}, $\{x_i\}_{i=0}^{j_{k-1}-1}$, and $\{a_i\}_{i=0}^{j_{k-1}-1}$ have been constructed in the previous steps. Similarly with the choice of q_{j_1} and n_1, we can choose $q_{j_k} \in \mathscr{P}\mathscr{O}$ so that

$$\operatorname{dist}\left(f(x_{j_{k-1}}), q_{j_k}\right) < \delta,$$

and a positive number n_k such that

$$\frac{1}{i_k}\left(\sum_{j=0}^{j_{k-1}-1} a_j + n_k \sum_{j=0}^{\pi-1} \log \left\| Df_{|E(q_{j_k}+j)} \right\| \right) < \frac{1}{2}(\log \lambda + \log \eta),$$

where $i_k = j_{k-1} + n_k \pi$ (the existence of n_k is ensured by (iii)). Let

$$x_{j_{k-1}+1} = q_{j_k+1},\ x_{j_{k-1}+2} = q_{j_k+2},\ \dots,\ x_{j_{k-1}+\pi} = q_{j_k},$$

$$x_{j_{k-1}+\pi+1} = q_{j_k+1},\ x_{j_{k-1}+\pi+2} = q_{j_k+2},\ \dots,$$

and $x_{i_k} = f(x_{j_{k-1}-1}) \in O(y_0, f)$.

Obviously,

$$\operatorname{dist}(f(x_{j_{k-1}+j}), x_{j_{k-1}+j+1}) < \delta$$

for $j = 0, 1, \dots, n_k\pi - 1$. Put

$$a_{j_{k-1}+j} = \log \left\| Df_{|E(x_{j_{k-1}+j})} \right\|$$

for $j = 0, 1, \dots, n_k \cdot \pi - 1$, and choose l_k so that

$$\frac{1}{i_k + l_k}\left(\sum_{j=0}^{i_k-1} a_j + \sum_{j=0}^{l_k-1} \log \left\| Df_{|E(f^j(x_{i_k}))} \right\| \right) \geq \frac{1}{2}(\log \eta + \log \eta'),$$

and

$$\frac{1}{i_k + l}\left(\sum_{j=0}^{i_k-1} a_j + \sum_{j=0}^{l} \log \left\| Df_{|E(f^j(x_{i_k}))} \right\| \right) < \frac{1}{2}(\log \eta + \log \eta')$$

for any $l < l_k$.

The existence of l_k is ensured by the fact that $x_{i_k} \in O(y_0, f)$ (recall the choice of y_0 and (ii)).

Let $j_k = i_k + l_k$ and let $x_{i_k+1} = f(x_{i_k}), x_{i_k+2} = f^2(x_{i_k}), \dots, x_{j_k-1} = f^{l_k-1}(x_{i_k})$. Finally, we put

$$a_{j_{k-1}+j} = \log \left\| Df_{|E(f^j(x_{i_k}))} \right\|$$

for $j = 0, 1, \dots, l_k - 1$.

This completes the construction of $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$.

Roughly speaking, the δ-pseudotrajectory $\{x_i\}_{i\in\mathbb{Z}}$ looks as follows:

$$\Big\{\cdots,\ \mathscr{PO},\ \mathscr{PO},\ y_0,\ f(y_0),\ f^2(y_0),\ \ldots,\ f^{l_1}(y_0),\ \mathscr{PO},$$

$$\ldots,\ \mathscr{PO},\ f^{l_1+1}(y_0),\ \ldots,\ f^{l_1+l_2}(y_0),\ \mathscr{PO},\ \mathscr{PO},\ \ldots\Big\}.$$

Recall that $K = \max\big\{|\log\|Df(x)\||, |\log\|Df^{-1}(x)\|| : x \in M\big\}$.

It is easy to see that

$$\frac{1}{i_k}\sum_{j=0}^{i_{k-1}-1} a_j < \frac{1}{2}(\log\lambda + \log\eta) \quad\text{and}\quad \frac{1}{j_k}\sum_{j=0}^{j_k-1} a_j \ge \frac{1}{2}(\log\eta + \log\eta')$$

for every $k = 1, 2, \ldots$, and

$$\frac{1}{n}\sum_{j=0}^{n-1} a_j < \frac{1}{n}\left(\frac{1}{2}(\log\eta + \log\eta')\,(n-\pi) + K\cdot\pi\right)$$

for every $n \ge \pi$.

Hence,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} a_j = \frac{1}{2}(\log\eta + \log\eta') \quad\text{and}\quad \liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} a_j \le \frac{1}{2}(\log\lambda + \log\eta).$$

Let $z \in M$ be a point whose f-trajectory ε-shadows $\{x_i\}_{i\in\mathbb{Z}}$ (see (P.4)). Note that $O(z,f) \subset N(2\varepsilon, \Lambda) \subset U$. Thus, by the choice of ε (see (i)),

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\left\|Df_{|\hat{E}(f^j(z))}\right\| < \log\eta < \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\left\|Df_{|\hat{E}(f^j(z))}\right\| < \log\eta'.$$

By Proposition 3.2.3, there is a hyperbolic periodic point q of index $\dim E$ such that $O(q,f) \subset U$ and the derivatives along the trajectory $O(q,f)$ satisfy the inequalities

$$\prod_{i=0}^{k-1}\left\|Df_{|E^s(f^i(q))}\right\| \le \eta'^k \quad\text{and}\quad \prod_{i=k-1}^{\pi(q)-1}\left\|Df_{|E^s(f^i(q))}\right\| > \eta^{\pi(q)-k+1}$$

for all $k = 1, 2, \ldots, \pi(q)$.

Furthermore, q can be chosen so that $\pi(q)$ is arbitrarily large, and thus we may assume that $\pi(q) \geq L$. This is a contradiction because

$$\prod_{i=0}^{\pi(q)-1} \left\| Df_{|E^s(f^i(q))} \right\| < \lambda^{\pi(q)}$$

by (P.2). Applying a similar reasoning, we can show that F is expanding, and thus, Λ is hyperbolic. □

Now we give a proof of a generalization of Liao's closing lemma (Lemma 3.2.4) proved by Gan [20].

Recall that a definition of a λ-quasi-hyperbolic orbit string

$$\{x, f(x), f^2(x), \cdots, f^n(x)\}$$

with respect to a splitting of $T_xM = E(x) \oplus F(x)$ has been given before Lemma 3.2.3.

Let $\{x_i\}_{i=-\infty}^{\infty}$ be a sequence of points in M and let $\{n_i\}_{i=-\infty}^{\infty}$ be a sequence of positive integers. Denote

$$\{x_i, n_i\} = \{f^j(x_i) : 0 \leq j \leq n_i - 1\}.$$

The sequence $\{x_i, n_i\}_{i=-\infty}^{\infty}$ is called a *λ-quasi-hyperbolic δ-pseudotrajectory* with respect to splittings $T_{x_i}M = E(x_i) \oplus F(x_i)$ if for any i, $\{x_i, n_i\}$ is λ-quasi-hyperbolic with respect to $T_{x_i}M = E(x_i) \oplus F(x_i)$ and $\operatorname{dist}(f^{n_i}(x_i), x_{i+1}) \leq \delta$.

A point x *ε-shadows* $\{x_i, n_i\}_{i=-\infty}^{\infty}$ if

$$\operatorname{dist}\left(f^j(x), f^{j-N_i}(x_i)\right) \leq \varepsilon \quad \text{for} \quad N_i \leq j \leq N_{i+1} - 1,$$

where N_i is defined as follows:

$$N_i = \begin{cases} 0, & \text{if} \quad i = 0; \\ n_0 + n_1 + \cdots + n_{i-1}, & \text{if} \quad i > 0; \\ n_i + n_{i+1} + \cdots + n_{-1} & \text{if} \quad i < 0. \end{cases}$$

In the following result, it is assumed that Λ is a compact invariant set of $f \in \text{Diff}^1(M)$ and there is a continuous Df-invariant splitting $T_\Lambda M = E \oplus F$, i.e., $Df(x)(E(x)) = E(f(x))$ and $Df(x)(F(x)) = F(f(x))$.

Lemma 3.2.5 (Generalized Liao's Closing Lemma [20]) *For any $0 < \lambda < 1$ there exist $L > 0$ and $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ and any λ-quasi-hyperbolic δ-pseudotrajectory $\{x_i, n_i\}_{i=-\infty}^{\infty}$ with respect to the splitting $T_\Lambda M = E \oplus F$ there exists a point x that $L\delta$-shadows $\{x_i, n_i\}_{i=-\infty}^{\infty}$. Moreover, if the sequence $\{x_i, n_i\}_{i=-\infty}^{\infty}$ is periodic, i.e., there exists an $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i, then the point x can be chosen to be periodic with period N_m.*

Proof Let $(X, \|\cdot\|)$ be a Banach space and let

$$X(\eta) = \{v \in X : \|v\| \le \eta\}, \quad \eta > 0.$$

If X is the direct sum of two closed subspaces E and F, i.e., $X = E \oplus F$, then the angle between E and F is defined as

$$\angle(E, F) = \inf\{\|u - v\| : (u \in E, v \in F, \|u\| = 1) \text{ or } (u \in E, v \in F, \|v\| = 1)\}.$$

Since E and F are closed, $0 < \angle(E, F) \le 1$. □

The following lemma is well known (e.g., see [64]); we give a proof for completeness.

Lemma 3.2.6 *In the above notation, assume that* $X = E \oplus F$ *and* $\angle(E, F) \ge \alpha > 0$. *Let* $L : X \to X$ *be a linear automorphism of the form*

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : E \oplus F \to E \oplus F$$

such that

$$\max\{\|A\|, \|D^{-1}\|\} \le \lambda \quad \textit{and} \quad \max\{\|B\|, \|C\|\} \le \varepsilon$$

for some $0 < \lambda < 1$ *and* $\varepsilon > 0$.

If

$$\varepsilon_1 = \frac{2\varepsilon(1 + \lambda)}{\alpha^2(1 - \lambda)} < 1,$$

then $I - L$ *is invertible, and*

$$\left\|(I - L)^{-1}\right\| \le R = R(\lambda, \varepsilon, \alpha) = \frac{1 + \lambda}{\alpha(1 - \lambda)(1 - \varepsilon_1)}.$$

Proof Put $J = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $K = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Then

$$(I - J)^{-1} = \begin{pmatrix} (I - A)^{-1} & 0 \\ 0 & (I - D)^{-1} \end{pmatrix},$$

$$\left\|(I - A)^{-1}\right\| \le \frac{1}{1 - \lambda}, \text{ and } \left\|(I - D)^{-1}\right\| \le \frac{\lambda}{1 - \lambda}.$$

If $u \in E$, $v \in F$ and $\|u + v\| = 1$, then, by the definition of $\angle(E, F)$,

$$1 = \|u + v\| \geq \angle(E, F)\|u\| \geq \alpha\|u\| \text{ and } \|u + v\| \geq \alpha\|v\|.$$

Thus,

$$\|(I - J)^{-1}(u + v)\| \leq \|(I - A)^{-1}u\| + \|(I - D)^{-1}v\| \leq \frac{1 + \lambda}{\alpha(1 - \lambda)},$$

and hence,

$$\|(I - J)^{-1}\| \leq \frac{1 + \lambda}{\alpha(1 - \lambda)}.$$

A similar reasoning shows that

$$\|K\| \leq \frac{2\varepsilon}{\alpha}.$$

Since

$$\varepsilon_1 = \frac{2\varepsilon(1 + \lambda)}{\alpha^2(1 - \lambda)} < 1$$

by assumption, $I - L = (I - J) - K = (I - J)(I - (I - J)^{-1}K)$ and $\|(I - J)^{-1}K\| \leq \varepsilon_1$.

Hence, $I - L$ is invertible, and

$$\|(I - L)^{-1}\| = \|(I - J)^{-1}(I - (I - J)^{-1}K)^{-1}\| \leq R,$$

which proves our lemma. □

The sequence version of the shadowing lemma is derived from the following fixed point result. For completeness, we give a proof following the method of [64].

In the next proposition, we denote

$$R = R(\lambda, \epsilon, \alpha) = \frac{1 + \lambda}{\alpha(1 - \lambda)(1 - \epsilon_1)}, \; L = 2R, \; \text{ and } \; \delta_0 = \frac{\eta}{L}$$

for $0 < \lambda < 1$, $0 < \alpha \leq 1$, and $\epsilon > 0$ such that $\epsilon_1 = \frac{2\epsilon(1+\lambda)}{\alpha^2(1-\lambda)} < 1$ and $\eta > 0$.

The minimal Lipschitz constant of a map ϕ is denoted by $\operatorname{Lip}\phi$.

Proposition 3.2.5 *If* $0 < \delta \leq \delta_0$ *and* $\Phi = N + \phi : X(\eta) \to X$, *where* N *is a linear automorphism of the form*

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : E \oplus F \to E \oplus F$$

such that

$$\max\{\|A\|, \|D^{-1}\|\} \le \lambda,$$

$$\max\{\|B\|, \|C\|\} \le \epsilon,$$

$\angle(E, F) \ge \alpha$, $\operatorname{Lip}\phi \le \frac{1}{L}$, *and* $\|\phi(0)\| \le \delta$, *then* Φ *has a unique fixed point* z *in* $X(\eta)$ *such that* $\|z\| \le L\delta$.

Proof By Lemma 3.2.6, $I - N$ is invertible. Let

$$H = (I - N)^{-1}\phi : X(\eta) \to X.$$

The set of fixed points of H in $X(\eta)$ coincides with the set of fixed points of Φ in $X(\eta)$. If $x \in X(L\delta)$, then

$$\begin{aligned}\|H(x)\| &= \|H(0) + (H(x) - H(0))\| \le \\ &\le \|(I - N)^{-1}\phi(0)\| + \|(I - N)^{-1}(\phi(x) - \phi(0))\| \le \\ &\le R\delta + R\tfrac{1}{L}L\delta = L\delta.\end{aligned}$$

Thus, H maps $X(L\delta)$ to $X(L\delta)$.

If $x, y \in X(\eta)$, then

$$\|H(x) - H(y)\| = \|(I - N)^{-1}(\phi(x) - \phi(y))\| \le R\frac{1}{L}\|x - y\| = \frac{\|x - y\|}{2}. \tag{3.7}$$

Hence, the map $H : X(L\delta) \to X(L\delta)$ is contracting. Therefore, H has a unique fixed point z in $X(L\delta)$. Moreover, if $z' \in X(\eta)$ is another fixed point of H, then $z = z'$ by (3.7). □

In the following proposition, let $X_i = \mathbb{R}^\nu$ for integer i (where $\nu = \dim M$) and we assume that $X_i = E_i \oplus F_i$. Let

$$Y = \prod_{i=-\infty}^{\infty} X_i$$

be endowed with the supremum norm

$$\|v\| = \sup\{|v_i|\}, \quad v = (v_i).$$

Thus, Y is a Banach space.

We consider a map $\Phi : Y \to Y$ of the form $(\Phi(v))_{i+1} = \Phi_i(v_i)$, where $\Phi_i : X_i \to X_{i+1}$.

Applying Proposition 3.2.5 to $\Phi : Y \to Y$, we obtain the sequence version of the shadowing lemma for hyperbolic pseudotrajectories in the following way.

Proposition 3.2.6 *Let us assume that conditions of Proposition 3.2.5 are satisfied and use the above notation.*

If $0 < \delta \le \delta_0$ *and* $\Phi : Y(\eta) \to Y$ *has the form*

$$\Phi_i = L_i + \phi_i : X_i(\eta) \to X_{i+1},$$

where

$$L_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$$

with respect to the splitting $X_i = E_i \oplus F_i$ *such that* $\angle(E_i, F_i) \ge \alpha$,

$$\max\{\|A_i\|, \|D_i^{-1}\|\} \le \lambda, \quad \max\{\|B_i\|, \|C_i\|\} \le \epsilon, \quad \mathrm{Lip}\,\phi \le \frac{1}{L},$$

and $\|\phi_i(0)\| \le \delta$, *then* Φ *has a unique fixed point* $v \in Y(\eta)$, *and* $\|v\| \le L\delta$.

We need one more technical lemma. Fix $0 < \lambda < 1$.

A pair of sequences $\{a_i, b_i\}_{i=1}^n$ of positive numbers is called λ-*hyperbolic* if $a_k \le \lambda$ and $b_k \ge \lambda^{-1}$ for $k = 1, 2, \ldots, n$.

A pair of sequences $\{a_i, b_i\}_{i=1}^n$ of positive numbers is called λ-*quasi-hyperbolic* if the following three conditions are satisfied:

(1) $\prod_{j=1}^k a_j \le \lambda^k$;
(2) $\prod_{j=k}^n b_j \ge \lambda^{k-n-1}$;
(3) $b_k / a_k \ge \lambda^{-2}$
for $k = 1, 2, \ldots, n$.

A sequence $\{c_i\}_{i=1}^n$ of positive numbers is called a *balance sequence* if

$$\prod_{j=1}^k c_j \le 1 \quad \text{for} \quad k = 1, 2, \ldots, n-1 \quad \text{and} \quad \prod_{j=1}^n c_j = 1.$$

A balance sequence $\{c_i\}_{i=1}^n$ is called *adapted to a* λ-*quasi-hyperbolic sequence pair* $\{a_i, b_i\}_{i=1}^n$ if $\{a_i/c_i, b_i/c_i\}_{i=1}^n$ is still λ-quasi-hyperbolic. Moreover, if $\{a_i/c_i, b_i/c_i\}_{i=1}^n$ is λ-hyperbolic, then $\{c_i\}_{i=1}^n$ is called *well adapted*.

If a balance sequence $\{c_i\}_{i=1}^n$ is adapted to a λ-quasi-hyperbolic sequence pair $\{a_i, b_i\}_{i=1}^n$, then we say that $\{a_i/c_i, b_i/c_i\}_{i=1}^n$ *is derived from* $\{a_i, b_i\}_{i=1}^n$. If $\{\bar{a}_i, \bar{b}_i\}_{i=1}^n$ is derived from $\{a_i, b_i\}_{i=1}^n$ and $\{\bar{\bar{a}}_i, \bar{\bar{b}}_i\}_{i=1}^n$ is derived from $\{\bar{a}_i, \bar{b}_i\}_{i=1}^n$, then $\{\bar{\bar{a}}_i, \bar{\bar{b}}_i\}_{i=1}^n$ is derived from $\{a_i, b_i\}_{i=1}^n$ as well.

Lemma 3.2.7 *Let* $0 < \lambda < 1$. *Then any* λ-*quasi-hyperbolic pair of sequences* $\{a_i, b_i\}_{i=1}^n$ *has a well adapted sequence* $\{c_i\}_{i=1}^n$.

Proof First we show that $\{a_i, b_i\}_{i=1}^n$ has an adapted sequence $\{c_i\}_{i=1}^n$ such that $a_i/c_i \le \lambda$ for $1 \le i \le n$.

To get a contradiction, assume that

$$N = \max\{k : \text{ there exist } \{c_i\}_{i=1}^n \text{ such that } a_i/c_i \le \lambda, 1 \le i \le k\} < n.$$

Obviously, $N \ge 1$. Assume that $\{c_i\}_{i=1}^n$ is such an adapted sequence. Let $\bar{a}_i = a_i/c_i$, $\bar{b}_i = b_i/c_i$, $i = 1, 2, \dots, n$. Then $\bar{a}_{N+1} > \lambda$.

Since $\prod_{i=1}^{N+1} \bar{a}_i \le \lambda^{N+1}$, there exists $1 \le m < N+1$ such that

$$\prod_{i=k}^{N+1} \bar{a}_i > \lambda^{N+2-k} \quad \text{for} \quad k = m+1, \dots, N+1 \quad \text{and} \quad \prod_{i=m}^{N+1} \bar{a}_i \le \lambda^{N+2-m}.$$

Let $\bar{c}_i = \bar{a}_i/\lambda$ for $i = m+1, \dots, N+1$ and $\bar{c}_i = 1$ for $i < m$ and $i > N+1$.

Then $\{\bar{c}_i\}_{i=1}^n$ is a balance sequence. Let $\bar{\bar{a}}_i = \bar{a}_i/\bar{c}_i$ and $\bar{\bar{b}}_i = \bar{b}_i/\bar{c}_i$ for $1 \le i \le n$ and put $\bar{c}_m = \left(\prod_{i=m+1}^{N+1} \bar{c}_i\right)^{-1}$. Obviously, $\bar{\bar{a}}_i = \lambda$ for $m+1 \le i \le N+1$,

$$\begin{aligned}\bar{\bar{a}}_m = \bar{a}_m/\bar{c}_m = \bar{a}_m \left(\prod_{i=m+1}^{N+1} \bar{c}_i\right) = \\ = \bar{a}_m \left(\prod_{i=m+1}^{N+1} \bar{a}_i\right) \lambda^{-(N-m+1)} = \left(\prod_{i=m}^{N+1} \bar{a}_i\right) \lambda^{-(N-m+1)} \le \lambda,\end{aligned}$$

and $\bar{\bar{b}}_i = \bar{b}_i/\bar{c}_i = \lambda \bar{b}_i/\bar{a}_i \ge \lambda^{-1}$ for $m+1 \le i \le N+1$.

Thus, one can easily check that $\{\bar{\bar{a}}_i, \bar{\bar{b}}_i\}_{i=1}^n$ is a λ-quasi-hyperbolic pair which is derived from $\{a_i, b_i\}_{i=1}^n$. But $\bar{\bar{a}}_i \le \lambda$ for $1 \le i \le N+1$, which contradicts the maximality of N.

Similarly, $\{a_i, b_i\}_{i=1}^n$ has an adapted sequence $\{c_i\}_{i=1}^n$ such that $b_i/c_i \ge \lambda^{-1}$ for $1 \le i \le n$. In what follows, we assume that $\{a_i, b_i\}_{i=1}^n$ itself has the property that $b_i \ge \lambda^{-1}$ for $1 \le i \le n$. We will repeat the proof of the above paragraph to show that a well adapted sequence exists.

Let

$$N = \max\{k : \text{ there exist } \{c_i\}_{i=1}^n \text{ such that}$$

$$a_i/c_i \le \lambda, 1 \le i \le k, \text{ and } b_i/c_i \ge \lambda, 1 \le i \le n\} < n.$$

Now we can copy the proof of the first paragraph word by word and only have to show that $\bar{\bar{b}}_m \ge \lambda^{-1}$. Since $\bar{c}_m \le 1$, this is obvious. □

Remark 3.2.2 If $\{c_i\}_{i=1}^n$ is a well adapted sequence of $\{a_i, b_i\}_{i=1}^n$, then $a_i/c_i \le \lambda$ and $b_i/c_i \ge \lambda^{-1}$. Hence, $a_i < a_i/\lambda \le c_i \le b_i\lambda < b_i$ for $i = 1, 2, \dots, n$.

We prove Lemma 3.2.5 (the generalized Liao's closing lemma) by combining Proposition 3.2.6 and Lemma 3.2.7.

Let $G_k(x)$, $x \in M$, be the Grassmann manifold of k-dimensional subspaces of the tangent space $T_x(M)$. Denote by $G_k(M)$ the bundle $\{G_k(x) : x \in M\}$ and consider a metric ρ on $G_k(M)$ (we do not indicate the dependence of ρ on k).

The following lemma is an easy corollary of well-known properties of the exponential map.

Lemma 3.2.8 *For any α, ε, τ, $\gamma > 0$ there exists $\eta > 0$ such that if $x, y \in M$, $T_xM = E(x) \oplus F(x)$, $T_yM = E(y) \oplus F(y)$,*

$$\min\{\angle(E(x), F(x)), \angle(E_y, F_y)\} \geq \alpha,$$

and

$$\max\{\rho(Df(x)E(x), E(y)), \rho(Df(x)F(x), F(y)\} \leq \eta,$$

then the map

$$\Phi = \exp_y^{-1} \circ f \circ \exp_x : T_xM(\eta) \to T_yM$$

can be written as $\Phi = L + \phi$, where

$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *with respect to the splittings $E(x) \oplus F(x)$ and $E(y) \oplus F(y)$,*

$$1 - \tau \leq \frac{\|A\|}{\|Df|_{E(x)}\|} \leq 1 + \tau,$$

$$1 - \tau \leq \frac{\|D^{-1}\|^{-1}}{m\left(Df|_{F(x)}\right)} \leq 1 + \tau,$$

$$\max\{\|B\|, \|C\|\} \leq \varepsilon, \textit{ and } \operatorname{Lip} \phi \leq \gamma.$$

Proof of Lemma 3.2.5 Let $\{x_i, n_i\}_{-\infty}^{\infty}$ be a λ-quasi-hyperbolic pseudotrajectory with respect to the splitting $T_\Lambda M = E \oplus F$. Denote

$$K = \sup_{x \in M} \left\{\|Df(x)\|, \|Df^{-1}(x)\|\right\} \quad \text{and} \quad \alpha = \inf_{x \in \Lambda} \angle(E(x), F(x)) > 0.$$

We first show that there exists a point z that ε-shadows $\{x_i, n_i\}_{i=-\infty}^{\infty}$, i.e.,

$$\operatorname{dist}\left(f^j(z), f^{j-N_i}(x_i)\right) \leq \varepsilon \quad \text{for} \quad N_i \leq j \leq N_{i+1} - 1,$$

where

$$N_i = \begin{cases} 0 & \text{if} \quad i = 0; \\ n_0 + n_1 + \cdots + n_{i-1} & \text{if} \quad i > 0; \\ n_i + n_{i+1} + \cdots + n_{-1} & \text{if} \quad i < 0. \end{cases}$$

Let $y_j = f^{j-N_i}(x_i)$ for $N_i \le j < N_{i+1}$ and denote $X_j = T_{y_j}M$, $E_j = E(y_j)$, and $F_j = F(y_j)$.

Put $\mu = \frac{1+\lambda}{2}$ and $r = \mu/\lambda$ and take $\varepsilon > 0$ such that

$$\varepsilon_1 = \frac{2\varepsilon(1+\mu)}{\alpha^2(1-\mu)} < 1.$$

Let $R = R(\mu, \varepsilon, \alpha)$, $L = 2R$, and $\varepsilon_2 = \varepsilon/K$.

Since the splitting $T_\Lambda M = E \oplus F$ is continuous, it follows from Lemma 3.2.7 that if $\eta > 0$ is small enough and $\{y_j\}_{j=N_i}^{N_{i+1}}$ is λ-quasi-hyperbolic η-pseudotrajectory, then the map

$$\Phi_j = \exp_{y_j+1}^{-1} \circ f \circ \exp_{y_j} : X_j(\eta) \to X_{j+1}$$

has the form $\Phi_j = L_j + \phi_j$, where

$$L_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} : E_j \oplus F_j \to E_{j+1} \oplus F_{j+1}$$

and $\operatorname{Lip} \phi_j \le \frac{1}{KL}$.

If $N_i \le j < N_{i+1} - 1$, then $\phi_j(0) = 0$, $B_j = C_j = 0$, $A_j = Df|_{E_j}$, and $D_j = Df|_{F_j}$. If $j = N_{i+1} - 1$, then

$$\max\{\|B_j\|, \|C_j\|\} \le \varepsilon_2, \quad \|A_j\| \le r\|Df|_{E_j}\|, \text{ and } \|D_j^{-1}\| \le rm(Df|_{F_j})^{-1}.$$

Let $\delta_0 = \eta/L$ and fix $0 < \delta \le \delta_0$. If $\{x_i, n_i\}_{-\infty}^{\infty}$ is a quasi-hyperbolic δ-pseudotrajectory, then $\|\phi_j(0)\| \le \delta$. Thus, $\left\{\|A_j\|, m(D_j)\right\}_{j=N_i}^{N_{i+1}-1}$ is a μ-quasi-hyperbolic pair of sequences. Hence, there is a well adapted sequence $\{h_j\}_{j=N_i}^{N_{i+1}-1}$, i.e.,

$$\prod_{j=N_i}^{k} h_j \le 1 \quad \text{for} \quad k = N_i, \dots, N_{i+1} - 2 \quad \text{and} \quad \prod_{j=N_i}^{N_{i+1}-1} h_j = 1,$$

where $\frac{1}{K} \le h_j \le K$.

Let $g_j = \prod_{k=N_i}^{j} h_k$, $\tilde{L}_j = h_j^{-1} L_j$, $\tilde{\phi}_j(x) = g_j^{-1}\phi_j(g_{j-1}(x))$ (note that $g_{N_i-1} = 1$), and $\tilde{\Phi}_j = \tilde{L}_j + \tilde{\phi}_j$. Denote

$$\Psi_j = \Phi_j \circ \cdots \circ \Phi_{N_i} \quad \text{and} \quad \tilde{\Psi}_j = \tilde{\Phi}_j \circ \cdots \circ \tilde{\Phi}_{N_i}.$$

Then $\tilde{\Psi}_j = g_j^{-1}\Psi_j$.

Note that $g_{N_{i+1}-1} = 1$ and $\tilde{\Psi}_{N_{i+1}-1} = \Psi_{N_{i+1}-1}$. Thus,

$$\mathrm{Lip}\,\tilde{\phi}_j = g_j^{-1}\mathrm{Lip}\,\phi_j g_{j-1} = h_j^{-1}\mathrm{Lip}\,\phi_j \le K\frac{1}{KL} = \frac{1}{L},$$

$\tilde{\phi}_j(0) = \phi_j(0) = 0$ for $j = N_i, \dots, N_{i+1} - 2$, and $\tilde{\phi}_j(0) = g_j^{-1}\phi_j(0) = \phi_j(0)$ for $j = N_{i+1} - 1$ since $g_j = 1$.

Hence, by Proposition 3.2.6, $\tilde{\Phi} = \{\tilde{\Phi}_j\} : Y(\eta) \to Y$ (where $Y = \prod_{i=-\infty}^{\infty} X_i$) has a unique fixed point $\tilde{v} = \{\tilde{v}_j\}$, and $\|\tilde{v}\| \le L\delta$. Let $v_{N_i} = \tilde{v}_{N_i}$ and for $N_i < j < N_{i+1} - 1$, define $v_j = \Phi_{j-1}(v_{j-1})$ inductively.

To guarantee that this is possible, let us check that $\|v_j\| \le L\delta$. Since

$$v_j = \Psi_{j-1}(v_{N_i}) = g_{j-1}\tilde{\Psi}_{j-1} = g_{j-1}\tilde{v}_j,$$

we have the inequalities $\|v_j\| \le \|\tilde{v}_j\| \le L\delta$.

Since

$$v_{N_{i+1}} = \tilde{v}_{N_{i+1}} = \tilde{\Psi}_{N_{i+1}-1}(v_{N_i}) = \Psi_{N_{i+1}-1}(v_{N_i}) = \Phi_{N_{i+1}-1}(v_{N_{i+1}-1}),$$

v is a fixed point of Φ, and $\|v\| \le L\delta$. Then the f-trajectory of the point $z = \exp_{y_0}(v_0)$ $L\delta$-shadows $\{y_j\}$. This proves the first conclusion of Lemma 3.2.5.

Now we assume that the sequence $\{x_i, n_i\}_{i=-\infty}^{\infty}$ is periodic, i.e., there exists an $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i.

Define $\tilde{w}$ by $(\tilde{w})_i = (\tilde{v})_{N_{m+i}}$. Since $\tilde{v}$ and $\tilde{w}$ are fixed points of $\tilde{\Phi}$ in $Y(L\delta)$, $\tilde{v} = \tilde{w}$ by Proposition 3.2.6. Thus, $v = w$, and z has period N_m. □

Historical Remarks The theory involving a selection of some special kinds of λ-quasi-hyperbolic strings has its origins in the works of V. A. Pliss [73] and S. T. Liao [36].

The notion of λ-quasi-hyperbolic string and Liao's closing lemma played an essential part in the solution of the stability conjecture in [45].

3.3 Vector Fields in $\mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B})$

To formulate our main results in the last two sections of Chap. 3, we need one more definition.

Consider a smooth vector field X on a smooth closed manifold M.

Let us say that a vector field X belongs to the *class* $\mathscr{B}$ if X has two hyperbolic rest points p and q (not necessarily different) with the following properties:

(1) The Jacobi matrix $DX(q)$ has two complex conjugate eigenvalues $\mu_{1,2} = a_1 \pm ib_1$ of multiplicity one with $a_1 < 0$ such that if $\lambda \ne \mu_{1,2}$ is an eigenvalue of $DX(q)$ with $\mathrm{Re}\lambda < 0$, then $\mathrm{Re}\lambda < a_1$;

(2) the Jacobi matrix $DX(p)$ has two complex conjugate eigenvalues $\nu_{1,2} = a_2 \pm ib_2$ with $a_2 > 0$ of multiplicity one such that if $\lambda \neq \nu_{1,2}$ is an eigenvalue of $DX(p)$ with $\mathrm{Re}\lambda > 0$, then $\mathrm{Re}\lambda > a_2$;
(3) the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ have a trajectory of nontransverse intersection.

Clearly, vector fields $X \in \mathscr{B}$ are not structurally stable.

Condition (1) above means that the "weakest" contraction in $W^s(q)$ is due to the eigenvalues $\mu_{1,2}$ (condition (2) has a similar meaning).

The main result of this section is as follows.

Theorem 3.3.1

$$Int^1(OrientSP_F \setminus \mathscr{B}) \subset \mathscr{S}_F. \tag{3.8}$$

It follows from Theorem 1.4.1 (2) that $\mathscr{S}_F \subset \mathrm{SSP}_F$; since the set $\mathscr{S}_F$ is C^1-open and $\mathscr{S}_F \cap \mathscr{B} = \emptyset$,

$$\mathscr{S}_F \subset \mathrm{Int}^1(\mathrm{SSP}_F \setminus \mathscr{B}) \subset \mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B}).$$

Combining this inclusion with (3.8), we see that

$$\mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B}) = \mathscr{S}_F.$$

Proof The proof of inclusion (3.8) is based on Theorem 1.3.13 (2):

$$\mathrm{Int}^1(\mathrm{KS}_F) = \mathscr{S}_F$$

(recall that KS_F is the set of Kupka–Smale vector fields).

Thus, in fact, we are going to prove that

$$\mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B}) \subset \mathrm{KS}_F. \tag{3.9}$$

Before proving inclusion (3.9), we introduce some terminology and notation.

The term "transverse section" will mean a smooth open disk in M of codimension 1 that is transverse to the flow ϕ at any of its points.

Let, as above, $\mathrm{Per}(X)$ denote the set of rest points and closed orbits of a vector field X.

Recall (see Sect. 1.3) that we have denoted by HP_F the set of vector fields X for which any trajectory of the set $\mathrm{Per}(X)$ is hyperbolic. Our first lemma is valid for the set $\mathrm{OrbitSP}_F$ (which is, in general, larger than $\mathrm{OrientSP}_F$); we prove it in this, more general form, since it can be applied for other purposes.

Lemma 3.3.1

$$Int^1(OrbitSP_F) \subset HP_F. \tag{3.10}$$

Proof To get a contradiction, let us assume that there exists a vector field $X \in \mathrm{Int}^1(\mathrm{OrbitSP}_F)$ that does not belong to HP_F, i.e., the set $\mathrm{Per}(X)$ contains a trajectory p that is not hyperbolic.

Let us first consider the case where p is a rest point. Identify M with $\mathbb{R}^n$ in a neighborhood of p. Applying an arbitrarily C^1-small perturbation of the field X, we can find a field $Y \in \mathrm{Int}^1(\mathrm{OrbitSP}_F)$ that is linear in a neighborhood U of p (we also assume that p is the origin of U).

(Here and below in the proof of Lemma 3.3.1, all the perturbations are C^1-small perturbations that leave the field in $\mathrm{Int}^1(\mathrm{OrbitSP}_F)$; we denote the perturbed fields by the same symbol X and their flows by ϕ.)

Then trajectories of X in U are governed by a differential equation

$$\dot{x} = Px, \tag{3.11}$$

where the matrix P has an eigenvalue λ with $\mathrm{Re}\lambda = 0$.

Consider first the case where $\lambda = 0$. We perturb the field X (and change coordinates, if necessary) so that, in Eq. (3.11), the matrix P is block-diagonal,

$$P = \mathrm{diag}(0, P_1), \tag{3.12}$$

and P_1 is an $(n-1) \times (n-1)$ matrix.

Represent coordinate x in U as $x = (y, z)$ with respect to (3.12); then

$$\phi(t, (y, z)) = (y, \exp(P_1 t)z)$$

in U.

Take $\varepsilon > 0$ such that $N(4\varepsilon, p) \subset U$. To get a contradiction, assume that $X \in \mathrm{OrbitSP}$; let d correspond to the chosen ε.

Fix a natural number m and consider the following mapping from $\mathbb{R}$ into U:

$$g(t) = \begin{cases} y = -2\varepsilon, & z = 0; \quad t \le 0; \\ y = -2\varepsilon + t/m, & z = 0; \quad 0 < t < 4m\varepsilon; \\ y = 2\varepsilon, & z = 0; \quad 4m\varepsilon < t. \end{cases}$$

Since the mapping g is continuous, piecewise differentiable, and either $\dot{y} = 0$ or $\dot{y} = 1/m$, g is a d-pseudotrajectory for large m.

Any trajectory of X in U belongs to a plane $y = \mathrm{const}$; hence,

$$\mathrm{dist}_H\big(\mathrm{Cl}(O(q, \phi)), \mathrm{Cl}(\{g(t) : t \in \mathbb{R}\})\big) \ge 2\varepsilon$$

for any q. This completes the proof in the case considered.

A similar reasoning works if p is a rest point and the matrix P in (3.12) has a pair of eigenvalues $\pm ib$, $b \neq 0$.

Now we assume that p is a nonhyperbolic closed trajectory. In this case, we perturb the vector field X in a neighborhood of the trajectory p using the perturbation technique developed by Pugh and Robinson in [77]. Let us formulate their result (which will be used below several times).

Pugh-Robinson Pertubation *Assume that r_1 is not a rest point of a vector field X. Let $r_2 = \phi(\tau, r_1)$, where $\tau > 0$. Let Σ_1 and Σ_2 be two small transverse sections such that $r_i \in \Sigma_i, i = 1, 2$. Let σ be the local Poincaré transformation generated by these transverse sections.*

Consider a point $r' = \phi(\tau', r_1)$, where $\tau' \in (0, \tau)$, and let U be an arbitrary open set containing r'.

Fix an arbitrary C^1 neighborhood F of the field X.

There exist positive numbers ε_0 and Δ_0 with the following property: if σ' is a local diffeomorphism from the Δ_0-neighborhood of r_1 in Σ_1 into Σ_2 such that

$$dist_{C^1}(\sigma, \sigma') < \varepsilon_0,$$

then there exists a vector field $X' \in F$ such that

(1) *$X' = X$ outside U;*
(2) *σ' is the local Poincaré transformation generated by the sections Σ_1 and Σ_2 and trajectories of the field X'.*

Let ω be the least positive period of the nonhyperbolic closed trajectory p. We fix a point $\pi \in p$, local coordinates in which π is the center, and a hyperplane Σ of codimension 1 transverse to the vector $F(\pi)$. Let y be coordinate in Σ.

Let σ be the local Poincaré transformation generated by the transverse section Σ; denote $P = D\sigma(0)$. Our assumption implies that the matrix P is not hyperbolic. In an arbitrarily small neighborhood of the matrix P, we can find a matrix P' such that P' either has a real eigenvalue with unit absolute value of multiplicity 1 or a pair of complex conjugate eigenvalues with unit absolute value of multiplicity 1. In both cases, we can choose coordinates $y = (v, w)$ in Σ in which

$$P' = \mathrm{diag}(Q, P_1), \tag{3.13}$$

where Q is a 1×1 or 2×2 matrix such that $|Qv| = |v|$ for any v.

Now we can apply the Pugh-Robinson perturbation (taking $r_1 = r_2 = \pi$ and $\Sigma_1 = \Sigma_2 = \Sigma$) which modifies X in a small neighborhood of the point $\phi(\omega/2, \pi)$ and such that, for the perturbed vector field X', the local Poincaré transformation generated by the transverse section Σ is given by $y \mapsto P'y$.

Clearly, in this case, the trajectory of π in the field X' is still closed (with some period ω'). As was mentioned, we assume that X' has the orbital shadowing property (and write X, ϕ, ω instead of X', ϕ', ω').

We introduce in a neighborhood of the point π coordinates $x = (x', y)$, where x' is one-dimensional (with axis parallel to $X(\pi)$), and y has the above-mentioned property.

Of course, the new coordinates generate a new metric, but this new metric is equivalent to the original one; thus, the corresponding shadowing property (or its absence) is preserved.

We need below one more technical statement.

LE (Local Estimate) *There exists a neighborhood W of the origin in Σ and constants $l, \delta_0 > 0$ with the following property: If $z_1 \in \Sigma \cap W$ and $|z_2 - z_1| < \delta < \delta_0$, then we can represent z_2 as $\phi(\tau, z_2')$ with $z_2' \in \Sigma$ and*

$$|\tau|,\ |z_2' - z_1| < l\delta. \tag{3.14}$$

This statement is an immediate corollary of the theorem on local rectification of trajectories (see, for example, [8]): In a neighborhood of a point that is not a rest point, the flow of a vector field of class C^1 is diffeomorphic to the family of parallel lines along which points move with unit speed (and it is enough to note that a diffeomorphic image of Σ is a smooth submanifold transverse to lines of the family).

We may assume that the neighborhood W in LE is so small that for $y \in \Sigma \cap W$, the function $\alpha(y)$ (the time of first return to Σ) is defined, and that the point $\phi(\alpha(v, w), (0, v, w))$ has coordinates $(Qv, P_1 w)$ in Σ.

Let us take a neighborhood U of the trajectory p such that if $r \in U$, then the first point of intersection of the positive semitrajectory of r with Σ belongs to W.

Take $a > 0$ such that the $4a$-neighborhood of the origin in Σ is a subset of W. Fix

$$\varepsilon < \min\left(\delta_0, \frac{a}{4l}\right),$$

where δ_0 and l satisfy the LE. Let d correspond to this ε (in the definition of the orbital shadowing property).

Take $y_0 = (v_0, 0)$ with $|v_0| = a$. Fix a natural number ν and set

$$\alpha_k = \alpha\left(\left(\frac{k}{\nu} Q^k v_0, 0\right)\right), \quad k \in [0, \nu - 1),$$

$$\beta_0 = 0, \quad \beta_k = \alpha_1 + \dots + \alpha_k,$$

and

$$g(t) = \begin{cases} \phi(t, (0, 0, 0)), & t < 0; \\ \phi\left(t - \beta_k, \left(0, \frac{k}{\nu} Q^k v_0, 0\right)\right), & \beta_k \le t < \beta_{k+1},\ k \in [0, \nu - 1); \\ \phi\left(t - \beta_\nu, (0, Q^\nu v_0, 0)\right), & t \ge \beta_\nu. \end{cases}$$

Note that for any point $y = (v, 0)$ of intersection of the set $\{g(t) : t \in \mathbb{R}\}$ with Σ, the inequality $|v| \le a$ holds. Hence, we can take a so small that

$$N(2a, \mathrm{Cl}(\{g(t) : t \in \mathbb{R}\})) \subset U.$$

Since

$$\left|\frac{k}{\nu}Q^{k+1}v_0 - \frac{k+1}{\nu}Q^{k+1}v_0\right| = \frac{a}{\nu} \to 0, \quad \nu \to \infty,$$

$g(t)$ is a d-pseudotrajectory for large ν.

Assume that there exists a point q such that

$$\mathrm{dist}_H(\mathrm{Cl}(O(q,\phi)), \mathrm{Cl}(\{g(t) : t \in \mathbb{R}\})) < \epsilon.$$

In this case, $O(q,\phi) \subset U$, and there exist points $q_1, q_2 \in O(q,\phi)$ such that

$$|q_1| = |q_1 - (0,0,0)| < \varepsilon$$

and

$$|q_2 - (0, Q^\nu v_0, 0)| < \varepsilon.$$

By the choice of ε, there exist points $q_1', q_2' \in O(q,\phi) \cap \Sigma$ such that

$$|q_1'| < l\varepsilon < a/4 \quad \text{and} \quad |q_2' - Q^\nu v_0| < l\varepsilon < a/4.$$

Let $q_1' = (0, v_1, w_1)$ and $q_2' = (0, v_2, w_2)$. Since these points belong to the same trajectory that is contained in U, $|v_1| = |v_2|$. At the same time,

$$|v_1| < a/4, \quad |v_2 - Q^\nu v_0| < a/4, \quad \text{and} \quad |Q^\nu v_0| = a,$$

and we get a contradiction which proves Lemma 3.3.1. □

To complete the proof of Theorem 3.3.1, we show that any vector field

$$X \in \mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B})$$

has the second property from the definition of Kupka–Smale flows, i.e., stable and unstable manifolds of trajectories of the set $\mathrm{Per}(X)$ are transverse.

Then

$$\mathrm{Int}^1(\mathrm{OrientSP}_F \setminus \mathscr{B}) \subset \mathrm{KS}_F;$$

hence, inclusion (3.9) is valid.

To get a contradiction, let us assume that there exist trajectories $p, q \in \mathrm{Per}(X)$ for which the unstable manifold $W^u(q)$ and the stable manifold $W^s(p)$ have a point r of nontransverse intersection. We have to consider separately the following two cases.

Case (B1): p and q are rest points of the flow ϕ.
Case (B2): either p or q is a closed trajectory.

Case (B1) Since $X \notin \mathscr{B}$, we may assume (after an additional perturbation, if necessary) that the eigenvalues $\lambda_1, \dots, \lambda_u$ with $\mathrm{Re}\lambda_j > 0$ of the Jacobi matrix $DX(p)$ have the following property:

$$\mathrm{Re}\lambda_j > \lambda_1 > 0, \quad j = 2, \dots, u$$

(where u is the dimension of $W^u(p)$). This property means that there exists a one-dimensional "direction of weakest expansion" in $W^u(p)$.

If this is not the case, then our assumption that $X \notin \mathscr{B}$ implies that the eigenvalues $\mu_1, \dots, \mu_s$ with $\mathrm{Re}\mu_j < 0$ of the Jacobi matrix $DX(q)$ have the following property:

$$\mathrm{Re}\mu_j < \mu_1 < 0, \quad j = 2, \dots, s$$

(where s is the dimension of $W^s(q)$). If this condition holds, we reduce the problem to the previous case by passing from the field X to the field $-X$ (clearly, the fields X and $-X$ have the oriented shadowing property simultaneously).

Making a perturbation (in this part of the proof, we always assume that the perturbed field belongs to the set OrientSP $\setminus \mathscr{B}$), we may "linearize" the field X in a neighborhood U of the point p; thus, trajectories of X in U are governed by a differential equation

$$\dot{x} = Px,$$

where

$$P = \mathrm{diag}(P_s, P_u), \quad P_u = \mathrm{diag}(\lambda, P_1), \quad \lambda > 0, \tag{3.15}$$

P_1 is a $(u-1) \times (u-1)$ matrix for which there exist constants $K > 0$ and $\mu > \lambda$ such that

$$\| \exp(-P_1 t) \| \leq K^{-1} \exp(-\mu t), \quad t \geq 0, \tag{3.16}$$

and $\mathrm{Re}\lambda_j < 0$ for the eigenvalues λ_j of the matrix P_s.

Let us explain how to perform the above-mentioned perturbations preserving the nontransversality of $W^u(q)$ and $W^s(p)$ at the point r (we note that a similar reasoning can be used in "replacement" of a component of intersection of $W^u(q)$ with a transverse section Σ by an affine space, see the text preceding Lemma 3.3.2 below).

Consider points $r^* = \phi(\tau, r)$, where $\tau > 0$, and $r' = \phi(\tau', r)$, where $\tau' \in (0, \tau)$. Let Σ and Σ^* be small transverse sections that contain the points r and r^*. Take small neighborhoods V and U' of p and r', respectively, so that the set V does not intersect the "tube" formed by pieces of trajectories through points of U' whose endpoints belong to Σ and Σ^*. In this case, if we perturb the vector field X in V and apply the Pugh-Robinson perturbation in U', these perturbations are "independent."

We perturb the vector field X in V obtaining vector fields X' that are linear in small neighborhoods $V' \subset V$ and such that the values $\rho_1(X, X')$ are arbitrarily small.

Let γ_s and γ_s^* be the components of intersection of the stable manifold $W^s(p)$ (for the field X) with Σ and Σ^* that contain the points r and r^*, respectively.

Since the stable manifold of a hyperbolic rest point depends (on its compact subsets) C^1-smoothly on C^1-small perturbations, the stable manifolds $W^s(p)$ (for the perturbed fields X') contain components γ_s' of intersection with Σ^* that converge (in the C^1 metric) to γ_s^*.

Now we apply the Pugh-Robinson perturbation in U' and find a field X' in an arbitrary C^1 neighborhood of X such that the local Poincaré transformation generated by the field X' and sections Σ and Σ^* takes γ_s' to γ_s (which means that the nontransversality at r is preserved).

We introduce in U coordinates $x = (y; v, w)$ according to (3.15): y is coordinate in the s-dimensional "stable" subspace (denoted E^s); (v, w) are coordinates in the u-dimensional "unstable" subspace (denoted E^u). The one-dimensional coordinate v corresponds to the eigenvalue λ (and hence to the one-dimensional "direction of weakest expansion" in E^u).

In the neighborhood U,

$$\phi(t, (y, v, w)) = (\exp(P_s t)y; \exp(\lambda t)v, \exp(P_1 t)w),$$

and it follows from (3.16) that

$$|\exp(P_1 t)w| \geq K \exp(\mu t)|w|, \quad t \geq 0. \tag{3.17}$$

Denote by E_1^u the one-dimensional invariant subspace corresponding to λ.

We naturally identify $E^s \cap U$ and $E^u \cap U$ with the intersections of U with the corresponding local stable and unstable manifolds of p, respectively.

Let us construct a special transverse section for the flow ϕ. We may assume that the point r of nontransverse intersection of $W^u(q)$ and $W^s(p)$ belongs to U. Take a hyperplane Σ' in E^s of dimension $s - 1$ that is transverse to the vector $X(r)$. Set $\Sigma = \Sigma' + E^u$; clearly, Σ is transverse to $X(r)$.

By a perturbation of the field X outside U, we may get the following: in a neighborhood of r, the component of intersection $W^u(q) \cap \Sigma$ containing r (for the perturbed field) has the form of an affine space $r + L$, where L is the tangent space, $L = T_r(W^u(q) \cap \Sigma)$, of the intersection $W^u(q) \cap \Sigma$ at the point r for the unperturbed field (compare, for example, with [33]).

Let Σ_r be a small transverse disk in Σ containing the point r. Denote by γ the component of intersection of $W^u(q) \cap \Sigma_r$ containing r.

Lemma 3.3.2 *There exists $\varepsilon > 0$ such that if $x \in \Sigma_r$ and*

$$dist\,(\phi(t, x), O^-(r, \phi)) < \varepsilon, \quad t \leq 0, \tag{3.18}$$

then $x \in \gamma$.

Proof To simplify presentation, let us assume that q is a rest point; the case of a closed trajectory is considered using a similar reasoning.

By the Grobman–Hartman theorem, there exists $\varepsilon_0 > 0$ such that the flow of X in $N(2\varepsilon_0, q)$ is topologically conjugate to the flow of a linear vector field.

Denote by A the intersection of the local stable manifold of q, $W^s_{loc}(q)$, with the boundary of the ball $N(2\varepsilon_0, q)$.

Take a negative time T such that if $s = \phi(T, r)$, then

$$\phi(t, s) \in N(\varepsilon_0, q), \quad t \leq 0. \tag{3.19}$$

Clearly, if ε_0 is small enough, then the compact sets A and

$$B = \{\phi(t, r) : \ T \leq t \leq 0\}$$

are disjoint. There exists a positive number $\varepsilon_1 < \varepsilon_0$ such that the ε_1-neighborhoods of the sets A and B are disjoint as well.

Take $\varepsilon_2 \in (0, \varepsilon_1)$. There exists a neighborhood V of the point s with the following property: If $y \in V \setminus W^u_{loc}(q)$, then the first point of intersection of the negative semitrajectory of y with the boundary of $N(2\varepsilon_0, q)$ belongs to the ε_2-neighborhood of the set A (this statement is obvious for a neighborhood of a saddle rest point of a linear vector field; by the Grobman-Hartman theorem, it holds for X as well).

Clearly, there exists a small transverse disk Σ_s containing s and such that if $y \in \Sigma_s \cap W^u_{loc}(q)$, then the first point of intersection of the positive semitrajectory of y with the disk Σ_r belongs to γ (in addition, we assume that Σ_s belongs to the chosen neighborhood V).

There exists $\varepsilon \in (0, \varepsilon_1 - \varepsilon_2)$ such that the flow of X generates a local Poincaré transformation

$$\sigma : \ \Sigma_r \cap N(\varepsilon, r) \to \Sigma_s.$$

Let us show that this ε has the desired property. It follows from our choice of Σ_s and (3.18) with $t = 0$ that if $x \notin \gamma$, then

$$y := \sigma(x) \in \Sigma_s \setminus W^u_{loc}(q);$$

in this case, there exists $\tau < 0$ such that the point $z = \phi(\tau, y)$ belongs to the intersection of $N(\varepsilon_2, A)$ with the boundary of $N(2\varepsilon_0, q)$. By (3.19),

$$\text{dist}(z, \phi(t, s)) > \varepsilon_0, \quad t \leq 0. \tag{3.20}$$

At the same time,

$$\text{dist}(z, \phi(t, r)) > \varepsilon_1 - \varepsilon_2, \quad T \leq t \leq 0. \tag{3.21}$$

Inequalities (3.20) and (3.21) contradict condition (3.18). Our lemma is proved. □

Now let us formulate the property of nontransversality of $W^u(q)$ and $W^s(p)$ at the point r in terms of the introduced objects. Recall that we work in a small neighborhood U of the rest point p identified with the Euclidean space $\mathbb{R}^n$.

Let Π^u be the projection to E^u parallel to E^s.

The transversality of $W^u(q)$ and $W^s(p)$ at r means that

$$T_r W^u(q) + T_r W^s(p) = \mathbb{R}^n.$$

Since Σ is a transverse section to the flow ϕ at r, the above equality is equivalent to the equality

$$L + E^s = \mathbb{R}^n.$$

Thus, the nontransversality means that

$$L + E^s \neq \mathbb{R}^n,$$

which implies that

$$L' := \Pi^u L \neq E^u. \tag{3.22}$$

We claim that there exists a linear isomorphism J of Σ for which the norm $\|J - \mathrm{Id}\|$ is arbitrarily small and such that

$$\Pi^u J L \cap E_1^u = \{0\}. \tag{3.23}$$

Let e be a unit vector of the line E_1^u. If $e \notin L'$, we have nothing to prove (take $J = \mathrm{Id}$). Thus, we assume that $e \in L'$. Since $L' \neq E^u$, there exists a vector $v \in E^u \setminus L'$.

Fix a natural number N and consider a unit vector v_N that is parallel to $Ne + v$. Clearly, $v_N \to e$ as $N \to \infty$. There exists a sequence T_N of linear isomorphisms of E^u such that $T_N v_N = e$ and

$$\|T_N - \mathrm{Id}\| \to 0, \quad N \to \infty.$$

Note that $T_N^{-1} e$ is parallel to v_N; hence, $T_N^{-1} e$ does not belong to L', and

$$T_N \Pi^u L \cap E_1^u = \{0\}. \tag{3.24}$$

Define an isomorphism J_N of Σ by

$$J_N(y, z) = (y, T_N z)$$

and note that

$$\|J_N - \mathrm{Id}\| \to 0, \quad N \to \infty.$$

Let $L_N = J_N L$. Equality (3.24) implies that

$$\Pi^u L_N \cap E_1^u = \{0\}. \tag{3.25}$$

Our claim is proved.

First we consider the case where $\dim E^u \geq 2$. Since $\dim L' < \dim E^u$ by (3.22) and $\dim E_1^u = 1$, our reasoning above (combined with a Pugh-Robinson perturbation) shows that we may assume that

$$L' \cap E_1^u = \{0\}. \tag{3.26}$$

For this purpose, we take a small transverse section Σ' containing the point $r' = \phi(-1, r)$, denote by γ the component of intersection of $W^u(q)$ with Σ' containing r', and note that the local Poincaré transformation σ generated by Σ' and Σ takes γ to the linear space L (in local coordinates of Σ). The mapping $\sigma_N = J_N \sigma$ is C^1-close to σ for large N and takes γ to L_N for which equality (3.25) is valid. Thus, we get equality (3.26) for the perturbed vector field.

This equality implies that there exists a constant $C > 0$ such that if $(y; v, w) \in r + L$, then

$$|v| \leq C|w|. \tag{3.27}$$

Fix $a > 0$ such that $N(4a, p) \subset U$. Take a point $\alpha = (0; a, 0) \in E_1^u$ and a positive number T and set $\alpha_T = (r_y; a\exp(-\lambda T), 0)$, where r_y is the y-coordinate of r. Construct a pseudotrajectory as follows:

$$g(t) = \begin{cases} \phi(t, r), & t \leq 0; \\ \phi(t, \alpha_T), & t > 0. \end{cases}$$

Since

$$|r - \alpha_T| = a\exp(-\lambda T) \to 0$$

as $T \to \infty$, for any d there exists T such that g is a d-pseudotrajectory.

Lemma 3.3.3 *Assume that $b \in (0, a)$ satisfies the inequality*

$$\log K - \log C + \left(\frac{\mu}{\lambda} - 1\right)\left(\log\frac{a}{2} - \log b\right) \geq 0.$$

Then for any $T > 0$, reparametrization h, and a point $s \in r + L$ such that $|r - s| < b$ there exists $\tau \in [0, T]$ such that

$$|\phi(h(\tau), s) - g(\tau)| \geq \frac{a}{2}.$$

Proof To get a contradiction, assume that

$$|\phi(h(\tau), s) - g(\tau)| < \frac{a}{2}, \quad \tau \in [0, T]. \tag{3.28}$$

Let $s = (y_0; v_0, w_0) \in r + L$. Since $|r - s| < b$,

$$|v_0| < b. \tag{3.29}$$

By (3.28),

$$\phi(h(\tau), s) \in U, \quad \tau \in [0, T].$$

Take $\tau = T$ in (3.28) to show that

$$|v_0| \exp(\lambda h(T)) > \frac{a}{2}.$$

It follows that

$$h(T) > \lambda^{-1} \left(\log \frac{a}{2} - \log |v_0| \right). \tag{3.30}$$

Set $\theta(\tau) = |\exp(P_1 h(\tau)) w_0|$; then $\theta(0) = |w_0|$. By (3.27),

$$|v_0| \leq C\theta(0). \tag{3.31}$$

By (3.17),

$$\theta(T) \geq K \exp(\mu h(T)) \theta(0). \tag{3.32}$$

We deduce from (3.29)–(3.32) that

$$\log \left(\frac{2\theta(T)}{a} \right) \geq \log \theta(T) - \log |v_0 \exp(\lambda h(T))| \geq$$

$$\geq \log K + \log \theta(0) - \log |v_0| + (\mu - \lambda) h(T) \geq$$

$$\geq \log K - \log C + \left(\frac{\mu}{\lambda} - 1 \right) \left(\frac{a}{2} - \log |v_0| \right) \geq$$

$$\geq \log K - \log C + \left(\frac{\mu}{\lambda} - 1 \right) \left(\frac{a}{2} - \log b \right) \geq 0.$$

We get a contradiction with (3.28) for $\tau = T$ since the norm of the w-coordinate of $\phi(h(T), s)$ equals $\theta(T)$, while the w-coordinate of $g(T)$ is 0. The lemma is proved. □

Let us complete the proof of Theorem 3.3.1 in case (B1). Assume that $l, \delta_0 > 0$ are chosen for Σ so that the LE holds.

Take $\varepsilon \in (0, \min(\delta_0, \varepsilon_0, a/2))$ so small that if $\text{dist}(y, r) < \varepsilon$, then $\phi(t, y)$ intersects Σ at a point s such that

$$\text{dist}(\phi(t, s), r) < \varepsilon_0, \quad |t| \le l\varepsilon. \tag{3.33}$$

Consider the corresponding d and a d-pseudotrajectory g described above.

Assume that

$$\text{dist}(\phi(h(t), x), g(t)) < \varepsilon, \quad t \in \mathbb{R}, \tag{3.34}$$

for some point x and reparametrization h and set $y = \phi(h(0), x)$.

Then $\text{dist}(y, r) < \varepsilon$, and there exists a point $s = \phi(\tau, y) \in \Sigma$ with $|\tau| < l\varepsilon$.

If $-l\varepsilon \le t \le 0$, then

$$\text{dist}(\phi(t, s), O^-(r, \phi)) \le \varepsilon_0$$

by (3.33).

If $t < -l\varepsilon$, then $h(0) + \tau + t < h(0)$, and there exists $t' < 0$ such that $h(t') = h(0) + \tau + t$. In this case,

$$\phi(t, s) = \phi(h(0) + \tau + t, x) = \phi(h(t'), x),$$

and

$$\text{dist}\,(\phi(t, s), O^-(r, \phi)) \le \text{dist}\left(\phi(h(t'), x), \phi(t', r)\right) \le \varepsilon_0.$$

By Lemma 3.3.2, $s \in r + L$. If ε is small enough, then $\text{dist}(s, r) < b$, where b satisfies the condition of Lemma 3.3.3, whose conclusion contradicts (3.34).

This completes the consideration of case (B1) for $\dim W^u(p) \ge 2$. If $\dim W^u(p) = 1$, then the nontransversality of $W^u(q)$ and $W^s(p)$ implies that $L \subset E^s$. This case is trivial since any shadowing trajectory passing close to r must belong to the intersection $W^u(q) \cap W^s(p)$, while we can construct a pseudotrajectory "going away" from p along $W^u(p)$. If $\dim W^u(p) = 0$, $W^u(q)$ and $W^s(p)$ cannot have a point of nontransverse intersection.

Case (B2) Passing from the vector field X to $-X$, if necessary, we may assume that p is a closed trajectory. We "linearize" X in a neighborhood of p as described in the proof of Lemma 3.3.1 so that the local Poincaré transformation of the transverse section Σ is a linear mapping generated by a matrix P with the following properties: With respect to some coordinates in Σ,

$$P = \text{diag}(P_s, P_u), \tag{3.35}$$

where $|\lambda_j| < 1$ for the eigenvalues λ_j of the matrix P_s, $|\lambda_j| > 1$ for the eigenvalues λ_j of the matrix P_u, every eigenvalue has multiplicity 1, and P is in a Jordan form.

The same reasoning as in case (B1) shows that it is possible to perform such a "linearization" (and other perturbations of X performed below) so that the nontransversality of $W^u(q)$ and $W^s(p)$ is preserved.

Consider an eigenvalue λ of P_u such that $|\lambda| \le |\mu|$ for the remaining eigenvalues μ of P_u.

We treat separately the following two cases.

Case (B2.1): $\lambda \in \mathbb{R}$.
Case (B2.2): $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Case (B2.1) Applying a perturbation, we may assume that

$$P_u = \operatorname{diag}(\lambda, P_1),$$

where $|\lambda| < |\mu|$ for the eigenvalues μ of the matrix P_1 (thus, there exists a one-dimensional direction of "weakest expansion" in $W^u(p)$). In this case, we apply precisely the same reasoning as that applied to treat case (B1) (we leave details to the reader).

Case (B2.2) Applying one more perturbation of X, we may assume that

$$\lambda = \nu + i\eta = \rho \exp\left(\frac{2\pi m_1 i}{m}\right),$$

where m_1 and m are relatively prime natural numbers, and

$$P_u = \operatorname{diag}(Q, P_1),$$

where

$$Q = \begin{pmatrix} \nu & -\eta \\ \eta & \nu \end{pmatrix}$$

with respect to some coordinates (y, v, w) in Σ, where $\rho = |\lambda| < |\mu|$ for the eigenvalues μ of the matrix P_1.

Denote

$$E^s = \{(y, 0, 0)\}, \quad E^u = \{(0, v, w)\}, \quad E_1^u = \{(0, v, 0)\}.$$

Thus, E^s is the "stable subspace," E^u is the "unstable subspace," and E_1^u is the two-dimensional "unstable subspace of the weakest expansion."

Geometrically, the Poincaré transformation $\sigma : \Sigma \to \Sigma$ (extended as a linear mapping to E_1^u) acts on E_1^u as follows: the radius of a point is multiplied by ρ, while $2\pi m_1/m$ is added to the polar angle.

As in the proof of Lemma 3.3.1, we take a small neighborhood W of the origin of the transverse section Σ so that, for points $x \in W$, the function $\alpha(x)$ (the time of first return to Σ) is defined.

We assume that the point r of nontransverse intersection of $W^u(q)$ and $W^s(p)$ belongs to the section Σ. Similarly to case (B1), we perturb X so that, in a neighborhood of r, the component of intersection of $W^u(q) \cap \Sigma$ containing r has the form of an affine space, $r + L$.

Let Π^u be the projection in Σ to E^u parallel to E^s and let Π_1^u be the projection to E_1^u; thus,

$$\Pi^u(y, u, v) = (0, u, v) \text{ and } \Pi_1^u(y, u, v) = (0, u, 0).$$

The nontransversality of $W^u(q)$ and $W^s(p)$ at r means that

$$L' = \Pi^u L \neq E^u$$

(see case (B1)). Applying a reasoning similar to that in case (B1), we perturb X so that if $L'' = L' \cap E_1^u$, then

$$\dim L'' < \dim E_1^u = 2.$$

Hence, either $\dim L'' = 1$ or $\dim L'' = 0$. We consider only the first case, the second one is trivial.

Denote by A the line L''. Images of A under degrees of σ (extended to the whole plane E_1^u) are m different lines in E_1^u.

In what follows, we refer to an obvious geometric statement (given without a proof).

Proposition 3.3.1 *Consider coordinates $(x_1, \dots, x_n)$ in the Euclidean space $\mathbb{R}^n$. Let $x' = (x_1, x_2)$, $x'' = (x_3, \dots, x_n)$, and let G be the plane of coordinate x'. Let D be a hyperplane in $\mathbb{R}^n$ such that*

$$D \cap G = \{x_2 = 0\}.$$

For any $b > 0$ there exists $c > 0$ such that if $x = (x', x'') \in D$ and $x' = (x_1', x_2')$, then either $|x_2'| \leq b|x_1'|$ or $|x''| \geq c|x'|$.

Take $a > 0$ such that the $2a$-neighborhood of the origin in Σ belongs to W. We may assume that if $v = (v_1, v_2)$, then the line A is $\{v_2 = 0\}$.

Take $b > 0$ such that the images of the cone

$$C = \{v : \ |v_2| \leq b|v_1|\}$$

in E_1^u under degrees of σ intersect only at the origin (denote these images by $C_1, \dots, C_m$).

We apply Proposition 3.3.1 to find a number $c > 0$ such that if $(0, v, w) \in L'$, then either $(0, v, 0) \in C$ or

$$|w| \geq c|v|. \tag{3.36}$$

Take a point $\beta = (0, v, 0) \in \Sigma$, where $|v| = a$, such that $\beta \notin C_1 \cup \cdots \cup C_m$.

For a natural number N, set $\beta_N = (r_y, P_u^{-N}(v, 0)) \in \Sigma$ (we recall that equality (3.35) holds), where r_y is the y-coordinate of r. We naturally identify β and β_N with points of M and consider the following pseudotrajectory:

$$g(t) = \begin{cases} \phi(t, r), & t \leq 0; \\ \phi(t, \beta_N), & t > 0. \end{cases}$$

The following statement (similar to Lemma 3.3.2) holds: there exists $\varepsilon_0 > 0$ such that if

$$\text{dist}(\phi(t, s), O^-(r, \phi)) < \varepsilon_0, \quad t \leq 0,$$

for some point $s \in \Sigma$, then $s \in r + L$.

Since β does not belong to the closed set $C_1 \cup \cdots \cup C_m$, we may assume that the disk in E_1^u centered at β and having radius ε_0 does not intersect the set $C_1 \cup \cdots \cup C_m$.

Define numbers

$$\alpha_1(N) = \alpha(\beta_N), \ \alpha_2(N) = \alpha_1(N) + \alpha(\sigma(\beta_N)), \ldots,$$

$$\alpha_N(N) = \alpha_{N-1}(N) + \alpha(\sigma^{N-1}(\beta_N)).$$

Take δ_0 and l for which LE holds for the neighborhood W (reducing W, if necessary). Take $\varepsilon < \min(\varepsilon_0/l, \delta_0)$ and assume that there exists the corresponding d (from the definition of the OrientSP$_F$). Take N so large that g is a d-pseudotrajectory.

Let h be a reparametrization; assume that

$$|\phi(h(t), p_0) - g(t)| < \varepsilon, \quad 0 \leq t \leq \alpha_N(N),$$

for some point $p_0 \in \Sigma$.

Since $g\,(\alpha_k(N)) \in \Sigma$ for $0 \leq k \leq N$ by construction, there exist numbers χ_k such that

$$|\sigma^{\chi_k}(p_0) - g(\alpha_k(N))| < \varepsilon_0, \quad 0 \leq k \leq N.$$

To complete the proof of Theorem 3.3.1, let us show that for any $p_0 \in r + L$ and any reparametrization h there exists $t \in [0, \alpha_N(N)]$ such that

$$\text{dist}(\phi(h(t), p_0), g(t)) \geq \varepsilon.$$

Assuming the contrary, we see that

$$|\sigma^{\chi_k}(p_0) - g(\alpha_k(N))| < \varepsilon_0, \quad 0 \le k \le N,$$

where the numbers χ_k were defined above.

We consider two possible cases.

If

$$\Pi_1^u p_0 \in C$$

(C is the cone defined before estimate (3.36)), then

$$\Pi_1^u \sigma^{\chi_k}(p_0) \in C_1 \cup \dots \cup C_m.$$

By construction, $\Pi_1^u g(\alpha_N(N))$ is β. Hence,

$$|\Pi_1^u \sigma^{\chi_N}(p_0) - \Pi_1^u g(\alpha_N(N))| > \varepsilon_0,$$

and we get the desired contradiction.

If

$$\Pi_1^u p_0 \notin C$$

and $p_0 = (y_0, v_0, w_0)$, then $(0, v_0, w_0) \in L'$, and it follows from (3.36)) that $|w_0| \ge c|v_0|$. In this case, decreasing ε_0, if necessary, we apply the reasoning similar to Lemma 3.3.3.

Thus, we have proved inclusion (3.9), which completes the proof of Theorem 3.3.1. □

Historical Remarks The first result concerning C^1 interiors of sets of vector fields having some shadowing properties was obtained by K. Lee and the second author in [33]. Denote by $\mathscr{N}$ the set of nonsingular vector fields. It was shown in [33] that vector fields in the set

$$\mathrm{Int}^1(\mathrm{SSP}_F) \cap \mathscr{N}$$

are structurally stable.

The class $\mathscr{B}$ was introduced by S. B. Tikhomirov in [99].

Theorem 3.3.1 was proved by the first author and S. B. Tikhomirov in [69].

Let us also note that S. B. Tikhomirov proved in [99] the following result: If the dimension of the manifold does not exceed 3, then

$$\mathrm{Int}^1(\mathrm{OrientSP}_F) = \mathscr{S}_F.$$

3.4 Vector Fields of the Class $\mathscr{B}$

In the previous section, we defined the set $\mathscr{B}$ of vector fields. As was mentioned, vector fields of that class are not structurally stable. This section is devoted to the following result [69].

Theorem 3.4.1 $Int^1(OrientSP_F) \cap \mathscr{B} \neq \emptyset$.

This theorem states that there exist vector fields in $\text{Int}^1(\text{OrientSP}_F)$ that belong to the class $\mathscr{B}$. The complete proof of Theorem 3.4.1 given in [69] is quite complicated, and we do not give it here.

Instead, we explain the main idea of the proof. One constructs a vector field X of the class $\mathscr{B}$ on the four-dimensional manifold $M = S^2 \times S^2$ that has the following properties (F1)-(F3) (ϕ denotes the flow generated by X).

(F1) The nonwandering set of ϕ is the union of four rest points p, q, s, u.

(F2) We can introduce coordinates in the disjoint neighborhoods $U_p = N(1, p)$ and $U_q = N(1, q)$ so that

$$X(x) = J_p(x - p), \quad x \in U_p,$$

and

$$X(x) = J_q(x - q), \quad x \in U_q,$$

where

$$J_p = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and

$$J_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Since the eigenvalues of J_p are $-1, -2, 1 \pm i$ and the eigenvalues of J_p are $1, 2, -1 \pm i$, conditions (1) and (2) of the definition of the class $\mathscr{B}$ (see the previous section) are satisfied for the vector field X and its rest points q and p.

(F3) The point s is an attracting hyperbolic rest point. The point u is a repelling hyperbolic rest point. The following condition holds:

$$W^u(p) \setminus \{p\} \subset W^s(s), \quad W^s(q) \setminus \{q\} \subset W^u(u). \tag{3.37}$$

The intersection of $W^s(p) \cap W^u(q)$ consists of a single trajectory α, and for any $x \in \alpha$, the condition

$$\dim (T_x W^s(p) \oplus T_x W^u(q)) = 3 \tag{3.38}$$

holds.

These conditions imply that the two-dimensional manifolds $W^s(p)$ and $W^u(q)$ intersect along a one-dimensional curve in the four-dimensional manifold M. Thus, $W^s(p)$ and $W^u(q)$ are not transverse; hence, $X \in \mathscr{B}$.

Geometrically, condition (3.38) means the following. Fix a point $r \in \alpha$ and let Σ be a transverse section to the flow ϕ at r (as above, this means that Σ is a smooth open disk in M of codimension 1 containing r that is transverse to the flow ϕ at any of its points).

Denote by β_s and β_u the intersections of Σ with $W^s(p)$ and $W^u(q)$, respectively. Clearly, β_s and β_u are one-dimensional curves containing the point r. Condition (3.38) means that the curves β_s and β_u intersect at r at nonzero angle.

To prove Theorem 3.4.1, it is enough to show that any vector field X' that is C^1-close to X belongs to OrientSP_F.

The vector field X satisfies Axiom A$'$ and the no-cycle condition; hence, X is Ω-stable. Thus, there exists a neighborhood V of X in $\mathscr{X}^1(M)$ such that for any field $X' \in V$, its nonwandering set consists of four hyperbolic rest points p', q', s', u' that belong to small neighborhoods of p, q, s, u, respectively. We denote by ϕ' the flow of any $X' \in V$ and by $W^s(p'), W^u(p')$, etc. the corresponding stable and unstable manifolds.

Select compact subsets b_s and b_u of the curves β_s and β_u, respectively, such that the interiors of b_s and b_u (in the interior topology) contain the point r.

Let Δ_s and Δ_u be compact subsets of $W^s(p)$ and $W^u(q)$, respectively, such that $b_s \subset \Delta_s$ and $b_u \subset \Delta_u$.

It follows from the stable manifold theorem that if $x' \in V$, then the stable and unstable manifolds $W^s(p')$ and $W^u(q')$ of the hyperbolic rest points p' and q' contain compact subsets Δ'_s and Δ'_u that converge (in the C^1 topology) to Δ_s and Δ_u, respectively, as X' tends to X.

Hence, the corresponding curves b'_s and b'_u tend in the C^1 topology to b_s and b_u, respectively, as X' tends to X.

We have the following two possibilities for a vector field $X' \in V$:

- $b'_s \cap b'_u = \emptyset$;
- b'_s and b'_u have a point r' of intersection close to r, and they intersect at r' at nonzero angle.

Clearly, we can choose Σ so that in the first case,

$$W^u(p') \cap W^s(q') = \emptyset;$$

then the vector field X' is structurally stable, and $X' \in \text{OrientSP}_F$.

Thus, it remains to consider the second case. To simplify notation, we write X, ϕ, etc. instead of X', ϕ', etc.

In this case, we make several additional assumptions which help us to explain to the reader the main geometric ideas used in the proof of Theorem 3.4.1 and to avoid heavy technical constructions of [69]. Here we follow the reasoning of [100].

First, we assume that the vector field X is linear in neighborhoods U_p and U_q of the rest points p and q, respectively (see property (F2) above).

In addition, we assume that, in a sense, the shift at some fixed time along trajectories in a neighborhood of a compact part of the trajectory α of nontransverse intersection of $W^s(p)$ and $W^u(q)$ is a parallel translation (see property (F5) below).

Let us introduce some notation. For a point $x \in U_p$ denote $P_1 x = x_1$ and $P_{34}x = (x_3, x_4)$, where $x - p = (x_1, x_2, x_3, x_4)$; for a point $x \in U_q$, denote $P_1 x = x_1$ and $P_{24}x = (x_2, x_4)$, where $x - q = (x_1, x_2, x_3, x_4)$. For a small $m > 0$ we denote $W^u_{loc}(p, m) = W^u(p) \cap N(m, p)$ etc.

Our additional assumptions are as follows.

(F4) The trajectory α satisfies the following inclusions:

$$\alpha \cap U_p \subset \{p + (t, 0, 0, 0); t \in (0, 1)\} \text{ and } \alpha \cap U_q \subset \{q - (t, 0, 0, 0); t \in (0, 1)\}.$$

(F5) There exist numbers $\Delta \in (0, 1)$ and $T_a > 0$ such that

$$\phi(T_a, q + (-1, x_2, x_3, x_4)) = (p + (1, x_2, x_3, x_4)), \quad |x_2|, |x_3|, |x_4| < \Delta.$$

(F6) $\phi(t, x) \notin U_q$ for $x \in U_p,\ t \geq 0$.

In what follows, we need two simple geometric lemmas.

In the first lemma, we consider a planar linear system of differential equations

$$\frac{dx}{dt} = Jx, \quad x \in \mathbb{R}^2,$$

where

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and denote by $\psi(t, x)$ its flow on $\mathbb{R}^2$.

If a point $x \in \mathbb{R}^2$ has polar coordinates (r, θ) with $\theta \in [0, 2\pi)$ and $r \neq 0$, we put $\arg(x) = \theta$.

Lemma 3.4.1 *For any point $x_0 \in \mathbb{R}^2 \setminus 0$, angle $\Theta \in [0, 2\pi)$, and number T_0 there exists $t < T_0$ such that* $\arg(\psi(t, x_0)) = \Theta$.

The proof of this lemma is straightforward. Of course, a similar statement holds for the system

$$\frac{dx}{dt} = -Jx, \quad x \in \mathbb{R}^2,$$

with $t < T_0$ replaced by $t > T_0$.

Lemma 3.4.2 *Let S_1 and S_2 be three-dimensional vector spaces with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3), respectively. Let $Q : S_2 \to S_1$ be a linear map such that*

$$Q\{y_2 = y_3 = 0\} \neq \{x_2 = x_3 = 0\}.$$

Then for any $D > 0$ there exists $R > 0$ (depending on Q and D) such that if two sets $V_1 \subset S_1 \cap \{x_1 = 0\}$ and $V_2 \subset S_2 \cap \{y_1 = 0\}$ satisfy the following conditions:

- *$V_1 \subset N(R, 0)$ and $V_2 \subset N(R, 0)$;*
- *V_1 intersects any ray in $S_1 \cap \{x_1 = 0\}$ starting at 0;*
- *V_2 intersects any ray in $S_2 \cap \{y_1 = 0\}$ starting at 0;*

then

$$C_1 \cap QC_2 \neq \emptyset,$$

where

$$C_1 = \{(x_1, x_2, x_3) : \quad |x_1| < D, \ (0, x_2, x_3) \in V_1\}$$

and

$$C_2 = \{(y_1, y_2, y_3) : \quad |y_1| < D, \ (0, y_2, y_3) \in V_2\}.$$

Proof Let us fix a linear map Q and a number $D > 0$. Consider the lines $l_1 \subset S_1$ and $l_2 \subset S_2$ given by the equations $x_2 = x_3 = 0$ and $y_2 = y_3 = 0$, respectively.

By our assumption, $Ql_2 \neq l_1$. Let us consider the plane $\pi \subset S_1$ containing l_1 and Ql_2. Consider a parallelogram $P \subset \pi$ that is symmetric with respect to 0, has sides parallel to l_1 and Ql_2, and satisfies the relation

$$P \subset \{|x_1| < D\} \cap Q(\{|y_1| < D\}). \tag{3.39}$$

Find a number $R > 0$ such that the following inclusions hold:

$$B(R, 0) \cap \pi \subset P \quad \text{and} \quad Q(B(R, 0) \cap Q^{-1}\pi) \subset P. \tag{3.40}$$

Let z_1 be a point of intersection of V_1 and the line $\pi \cap \{x_1 = 0\}$. Condition (3.40) implies that $z_1 \in P$. Consider the line k_1 containing z_1 and parallel to l_1. Inclusion (3.39) implies that $k_1 \cap P \subset C_1$.

Similarly, let z_2 be a point of intersection of V_2 and the line $\pi \cap \{y_1 = 0\}$. Condition (3.40) implies the inclusion $Qz_2 \in P$. Let k_2 be the line containing Qz_2 and parallel to Ql_2. Inclusion (3.39) implies that $Q^{-1}(k_2 \cap V) \subset C_2$.

Since $k_1 \nparallel k_2$, there exists a point $z \in k_1 \cap k_2$. The inclusions $z_1, z_2 \in P$ imply that $z \in P$. Hence, $z \in C_1 \cap QC_2$. Our lemma is proved. □

Now let us prove that the vector field X has the oriented shadowing property.

Fix points $y_p = \alpha(T_p) \in U_p$ and $y_q = \alpha(T_q) \in U_q$ (note that in this case, $T_p > T_q$ by property (F5)) and a number $\delta > 0$.

We say that $g(t)$ is a pseudotrajectory of type Ps(δ) if

$$g(t) = \begin{cases} \phi(t - T_p, x_p), & t > T_p; \\ \phi(t - T_q, x_q), & t < T_q; \\ \alpha(t), & t \in [T_q, T_p], \end{cases}$$

for some points $x_p \in B(\delta, y_p)$ and $x_q \in B(\delta, y_q)$.

Fix an $\varepsilon > 0$. Let us say that a pseudotrajectory $g(t)$ can be ε-oriented shadowed if there exists a reparametrization $h \in \text{Rep}$ and a point z such that

$$\text{dist}(\phi(h(t), z), g(t)) < \varepsilon, \quad t \in \mathbb{R}.$$

Clearly, the required inclusion $X \in \text{OrientSP}_F$ is a corollary of the following two statements.

Proposition 3.4.1 *For any $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ there exists $d > 0$ such that if $g(t)$ is a d-pseudotrajectory of X, then either $g(t)$ can be ε-oriented shadowed or there exists a pseudotrajectory $g^*(t)$ of type Ps(δ) with these y_p and y_q and a number $t_0 \in \mathbb{R}$ such that*

$$\mathit{dist}\left(g(t), g^*(t + t_0)\right) < \varepsilon/2, \quad t \in \mathbb{R}.$$

Proposition 3.4.2 *There exist $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ such that any pseudotrajectory of type Ps(δ) with these y_p and y_q can be $\varepsilon/2$-oriented shadowed.*

Proposition 3.4.1 can be proved by a standard reasoning. Precisely the same statement was proved in [69] for a slightly different vector field (the only difference is in the structure of the matrices J_p and J_q). The proof can be literally repeated in our case.

The main idea of the proof is the following. Outside a neighborhood of the curve α, our vector field X coincides with a structurally stable one. Hence, pseudotrajectories that do not intersect a fixed neighborhood of α can be shadowed.

If $g(t)$ intersects a small neighborhood of α, then (after a proper shift of time), the points $g(t)$ with $t > T_p$ also belong to a set where X coincides with a structurally stable vector field; thus, for such t, $g(t)$ can be shadowed by $\phi(t - T_p, x_p)$. Similarly, the pseudotrajectory $g(t)$ can be shadowed by $\phi(t - T_q, x_q)$. For $t \in (T_q, T_p)$, the points $g(t)$ are close to α. We leave the rest of the proof to the reader.

Proof (of Proposition 3.4.2) Since the rest points s and u are a hyperbolic attractor and a hyperbolic repeller, we may assume, without loss of generality, that

$$O^+(N(\varepsilon/2, s), \phi) \subset N(\varepsilon, s) \quad \text{and} \quad O^-(N(\varepsilon/2, u), \phi) \subset N(\varepsilon, u),$$

where $O^+(A, \phi)$ and $O^-(A, \phi)$ are the positive and negative semitrajectories of a set A in the flow ϕ, respectively.

Take $m \in (0, \varepsilon/8)$. We fix points $y_p = \alpha(T_p) \in N(m/2, p) \cap \alpha$ and $y_q = \alpha(T_q) \in N(m/2, q) \cap \alpha$. Put $T = T_p - T_q$. Find a number $\delta > 0$ such that if $g(t)$ is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ (with y_p and y_q fixed above), $t_0 \in \mathbb{R}$, and $x_0 \in N(2\delta, g(t_0))$, then

$$\mathrm{dist}(\phi(t - t_0, x_0), g(t)) < \varepsilon/2, \quad |t - t_0| \leq T + 1. \tag{3.41}$$

Consider a number $\tau > 0$ such that if $x \in W^u(p) \setminus N(m/2, p)$, then $\phi(\tau, x) \in N(\varepsilon/8, s)$. Take $\varepsilon_1 \in (0, m/4)$ such that if two points $z_1, z_2 \in M$ satisfy the inequality $\mathrm{dist}(z_1, z_2) < \varepsilon_1$, then

$$\mathrm{dist}(\phi(t, z_1), \phi(t, z_2)) < \varepsilon/8, \quad |t| \leq \tau.$$

In this case, for any $y \in N(\varepsilon_1, x)$, the following inequalities hold:

$$\mathrm{dist}(\phi(t, x), \phi(t, y)) < \varepsilon/4, \quad t \geq 0. \tag{3.42}$$

Decreasing ε_1, we may assume that if $x' \in W^s(q) \setminus N(m/2, q)$ and $y' \in N(\varepsilon_1, x')$, then

$$\mathrm{dist}(\phi(t, x'), \phi(t, y')) < \varepsilon/4, \quad t \leq 0.$$

Let $g(t)$ be a pseudotrajectory of type $\mathrm{Ps}(\delta)$, where y_p, y_q, and δ satisfy the above-formulated conditions.

Let us consider several possible cases. t

Case (P1): $x_p \notin W^s(p)$ and $x_q \notin W^u(q)$. Let

$$T' = \inf\{t \in \mathbb{R} : \ \phi(t, x_p) \notin N(p, 3m/4)\}.$$

If δ is small enough, then $\mathrm{dist}(\phi(T', x_p), W^u(p)) < \varepsilon_1$. In this case, there exists a point $z_p \in W^u_{loc}(p, m) \setminus N(m/2, p)$ such that

$$\mathrm{dist}(\phi(T', x_p), z_p) < \varepsilon_1. \tag{3.43}$$

Applying a similar reasoning in a neighborhood of q (and reducing δ, if necessary), we find a point $z_q \in W^s_{loc}(q, m) \setminus N(m/2, q)$ and a number $T'' < 0$ such that $\mathrm{dist}(\phi(T'', x_q), z_q) < \varepsilon_1$.

Consider the hyperplanes $S_p := \{x_1 = P_1 y_p\}$ and $S_q := \{x_1 = P_1 y_q\}$. From our assumptions on the linearity of X in neighborhoods of p and q and from assumption (F5) it follows that the Poincaré map defined by $Q(x) = \phi(T, x)$ is a linear map $Q : S_q \to S_p$ such that $Q(\{(x_2, x_4) = 0\}) \neq \{(x_3, x_4) = 0\}$.

Apply Lemma 3.4.2 to the hyperplanes S_p and S_q, the map Q, and the number $D = \varepsilon/8$ and find the corresponding $R > 0$. Note that there exists a $T_R > 0$ such that

$$|\phi(t, P_{34}x_p)| < R,\ t < -T_R,\ \text{and}\ |\phi(t, P_{24}x_q)| < R,\ t > T_R.$$

Consider the sets

$$V^- = \left\{\phi(t, P_{34}x_p) :\ t < -T_R\right\} \text{ and } V^+ = \left\{\phi(t, P_{24}x_q) :\ t > T_R\right\}.$$

Due to Lemma 3.4.1, the sets $V^\pm$ satisfy the assumptions of Lemma 3.4.2; hence, the sets

$$C^- = \left\{x \in S_p : \quad P_{34}x \in V^-, |P_2 x| < D\right\}$$

and

$$C^+ = \left\{x \in S_q : \quad P_{24}x \in V^+, |P_3 x| < D\right\}$$

are such that $C^- \cap QC^+ \neq \emptyset$.

Let us consider a point

$$x_0 \in C^- \cap QC^+ \tag{3.44}$$

and numbers $t_p < -T_R$ and $t_q > T_R$ such that $P_{34}x_0 = \phi(t_p, P_{34}x_s)$ and $P_{24}Q^{-1}x_0 = \phi(t_q, P_{24}x_u)$. The following inclusions hold:

$$\phi(-T_Q - T_R - T'', x_0) \in N(2\varepsilon_1, z_q), \quad \phi(-T_Q, x_0) \in N(D, y_q),$$

$$\phi(0, x_0) \in N(D, y_p), \quad \phi(T_R + T', x_0) \in N(2\varepsilon_1, z_p).$$

Inequalities (3.41) imply that if δ is small enough, then

$$\operatorname{dist}(\phi(t_3 + t, x_0), g(T_p + t)) < \varepsilon/2, \quad t \in [-T, 0]. \tag{3.45}$$

Define a reparametrization $h(t)$ as follows:

$$h(t) = \begin{cases} h(T_q + T'' + t) = -T_Q - T_R - T'' + t, & t < 0; \\ h(T_p + T' + t) = T_R + T' + t, & t > 0; \\ h(T_p + t) = t, & t \in [-T, 0]; \\ h(t) \text{ increases}, & t \in [T_p, T_p + T'] \cup [T_q + T'', T_q]. \end{cases}$$

If $t \geq T_p + T'$, then inequality (3.42) implies that

$$\text{dist}(\phi(h(t), x_0), \phi(t - (T_p + T'), z_p)) < \varepsilon/4$$

and

$$\text{dist}(\phi(t - T_p, x_p), \phi(t - (T_p + T'), z_p)) < \varepsilon/4.$$

Hence, if $t \geq T_p + T'$, then

$$\text{dist}(\phi(h(t), x_0), g(t)) < \varepsilon/2. \tag{3.46}$$

For $t \in [T_p, T_p + T']$, the inclusions $\phi(h(t), x_0), g(t) \in N(m, p)$ hold, and inequality (3.46) holds for these t as well.

A similar reasoning shows that inequality (3.46) holds for $t \leq T_q$. If $t \in [T_q, T_p]$, then inequality (3.46) follows from (3.45). This completes the proof in case (P1).

Case (P2): $x_p \in W^s(p)$ and $x_q \notin W^u(q)$. In this case, the proof uses the same reasoning as in case (P1). The only difference is that instead of (3.44) we construct a point $x_0 \in N(D, y_p) \cap W^s_{loc}(p, m)$ such that

$$\phi(-T - T'', x_0) \in N(2\varepsilon_1, z_q) \text{ and } \phi(-T, x_0) \in N(\varepsilon/8, y_q).$$

The construction is straightforward and uses Lemma 3.4.1.

Case (P3): $x_p \notin W^s(p)$ and $x_q \in W^u(q)$. This case is similar to case (P2).

Case (P4): $x_p \in W^s(p)$ and $x_q \in W^u(q)$. In this case, we take α as the shadowing trajectory; the reparametrization is constructed similarly to case (P1).

Thus, we have shown that $X \in \text{OrientSP}_F$. □

Historical Remarks Theorem 3.4.1 was published by the first author and S. B. Tikhomirov in [69]. As was said at the beginning of Chap. 3, the complete proof given in this paper is technically very complicated, and we only describe a "model" published by S. B. Tikhomirov in the paper [100] devoted to the Komuro conjecture [29].