Chapter 2
Lipschitz and Hölder Shadowing and Structural Stability

In this chapter, we give either complete proofs or schemes of proof of the following main results:

- If a diffeomorphism f of a smooth closed manifold has the Lipschitz shadowing property, then f is structurally stable (Theorem 2.3.1);
- a diffeomorphism f has the Lipschitz periodic shadowing property if and only if f is Ω-stable (Theorem 2.4.1);
- if a diffeomorphism f of class C^2 has the Hölder shadowing property on finite intervals with constants $\mathscr{L}, C, d_0, \theta, \omega$, where $\theta \in (1/2, 1)$ and $\theta + \omega > 1$, then f is structurally stable (Theorem 2.5.1);
- there exists a homeomorphism of the interval that has the Lipschitz shadowing property and a nonisolated fixed point (Theorem 2.6.1);
- if a vector field X has the Lipschitz shadowing property, then X is structurally stable (Theorem 2.7.1).

The structure of the chapter is as follows.

We devote Sects. 2.1–2.3 to the proof of Theorem 2.3.1. In Sect. 2.1, we prove theorems of Maizel' and Pliss relating the so-called Perron property of difference equations and hyperbolicity of sequences of linear automorphisms, Sect. 2.2 is devoted to the Mañé theorem (Theorem 1.3.7), and in Sect. 2.3, we reduce the proof of Theorem 2.3.1 to results of the previous two sections.

Theorem 2.4.1 is proved in Sect. 2.4; Theorem 2.5.1 is proved in Sect. 2.5; Theorem 2.6.1 is proved in Sect. 2.6.

Finally, Sect. 2.7 is devoted to the proof of Theorem 2.7.1.

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2.1 Maizel' and Pliss Theorems

Let $I = \{k \in \mathbb{Z} : \ k \geq 0\}$. Let $\mathscr{A} = \{A_k, \ k \in I\}$ be a sequence of linear isomorphisms

$$A_k : \mathbb{R}^n \to \mathbb{R}^n.$$

We assume that there exists a constant $N \geq 1$ such that

$$\|A_k\|, \|A_k^{-1}\| \leq N, \quad k \in I. \tag{2.1}$$

We relate to this sequence two difference equations, the homogeneous one,

$$x_{k+1} = A_k x_k, \quad k \in I, \tag{2.2}$$

and the inhomogeneous one,

$$x_{k+1} = A_k x_k + f_{k+1}, \quad k \in I. \tag{2.3}$$

Definition 2.1.1 We say that the sequence $\mathscr{A}$ has the *Perron property* on I if for any bounded sequence f_k, Eq. (2.3) has a bounded solution.

Set

$$F(k,l) = \begin{cases} A_{k-1} \circ \cdots \circ A_l, & k > l; \\ \mathrm{Id}, & k = l; \\ A_k^{-1} \circ \cdots \circ A_{l-1}^{-1}, & k < l. \end{cases}$$

Definition 2.1.2 We say that the sequence $\mathscr{A}$ is *hyperbolic* on I if there exist constants $C > 0$ and $\lambda \in (0,1)$ and projections $P_k, Q_k, k \in I$, such that if $S_k = P_k\mathbb{R}^n$ and $U_k = Q_k\mathbb{R}^n$, then

$$S_k \oplus U_k = \mathbb{R}^n; \tag{2.4}$$

$$A_k S_k = S_{k+1}, \quad A_k U_k = U_{k+1}; \tag{2.5}$$

$$|F(k,l)v| \leq C\lambda^{k-l}|v|, \quad v \in S_l, \ k \geq l; \tag{2.6}$$

$$|F(k,l)v| \leq C\lambda^{l-k}|v|, \quad v \in U_l, \ k \leq l; \tag{2.7}$$

$$\|P_k\|, \|Q_k\| \leq C. \tag{2.8}$$

In the relations above, $k, l \in I$.

Our first main result in this section is the following statement.

Theorem 2.1.1 (Maizel') *If the sequence $\mathscr{A}$ has the Perron property on I, then this sequence is hyperbolic on I.*

Remark 2.1.1 Of course, it is well known that a hyperbolic sequence $\mathscr{A}$ has the Perron property on I (see Lemma 2.1.6 below), so the properties of $\mathscr{A}$ in the above theorem are equivalent. We formulate it in the above form since this implication is what we really need (and since precisely this statement was proved by Maizel').

Proof Thus, we assume that the sequence $\mathscr{A}$ has the Perron property on I.

Let us denote by $\mathscr{B}$ the Banach space of bounded sequences $x = \{x_k\}$, where $x_k \in \mathbb{R}^n$ and $k \in I$, with the usual norm

$$\|x\| = \sup_{k \in I} |x_k|.$$

A sequence $x \in \mathscr{B}$ that satisfies Eq. (2.2) (or (2.3)) will be called a $\mathscr{B}$-solution of the corresponding equation.

Denote

$$V_1 = \{x_0 : \ x = (x_0, x_1, \dots) \quad \text{is a } \mathscr{B} - \text{solution of} \quad (2.2)\}.$$

Since Eq. (2.2) is linear and $\mathscr{B}$ is a linear space, V_1 is a linear space as well. Denote by V_2 the orthogonal complement of V_1 in $\mathbb{R}^n$ and by P the orthogonal projection to V_1.

The difference of any two $\mathscr{B}$-solutions of Eq. (2.3) with a fixed $f \in \mathscr{B}$ is a $\mathscr{B}$-solution of Eq. (2.2). It is easily seen that for any $f \in \mathscr{B}$ there exists a unique $\mathscr{B}$-solution of Eq. (2.3) (we denote it $T(f)$) such that $(T(f))_0 \in V_2$.

The defined operator

$$T : \ \mathscr{B} \to \mathscr{B}$$

plays an important role in the proof. Clearly, the operator T is linear.

Lemma 2.1.1 *The operator T is continuous.*

Proof Since we know that the operator T is linear, it is enough to show that the graph of T is closed; then our statement follows from the closed graph theorem.

Thus, assume that

$$f_n = (f_0^n, \dots) \in \mathscr{B}, \quad y_n = (y_0^n, \dots) \in \mathscr{B},$$

$y_n = T(f_n), f_n \to f$, and $y_n \to y = (y_0, \dots)$ in $\mathscr{B}$.

Then, clearly, $y_0 \in V_2$.

Fix $k \in I$ and pass in the equality

$$y^n_{k+1} = A_k y^n_k + f^n_{k+1}$$

to the limit as $n \to \infty$ to show that

$$y_{k+1} = A_k y_k + f_{k+1}.$$

Hence, $y = T(f)$, and the graph of T is closed. □

Lemma 2.1.1 implies that there exists a constant $r > 0$ such that

$$\|T(f)\| \le r\|f\|, \quad f \in \mathscr{B}. \tag{2.9}$$

Without loss of generality, we assume that

$$rN \ge 1, \tag{2.10}$$

where N is the constant in (2.1).

Denote

$$X(k) = \begin{cases} F(k,0), & k > 0; \\ \mathrm{Id}, & k = 0; \\ F(0,-k), & k < 0. \end{cases}$$

Straightforward calculations show that the formula

$$y_k = \sum_{u=0}^{k} X(k)PX(-u)f_u - \sum_{u=k+1}^{\infty} X(k)(\mathrm{Id}-P)X(-u)f_u \tag{2.11}$$

represents a solution of Eq. (2.3) provided that the series in the second summand converges.

We can obtain a shorter variant of formula (2.11) by introducing the "Green function"

$$G(k,u) = \begin{cases} X(k)PX(-u), & 0 \le u \le k; \\ -X(k)(\mathrm{Id}-P)X(-u), & 0 \le k < u. \end{cases}$$

Then formula (2.11) becomes

$$y_k = \sum_{u=0}^{\infty} G(k,u)f_u. \tag{2.12}$$

Lemma 2.1.2 *Let $k_0, k_1, k \in I$ and let $\xi \in \mathbb{R}^n$ be a nonzero vector with $|\xi| \leq 1$. Then*

$$|X(k)P\xi| \sum_{u=k_0}^{k} |X(u)\xi|^{-1} \leq r, \quad 0 \leq k_0 \leq k, \tag{2.13}$$

and

$$|X(k)(Id - P)\xi| \sum_{u=k}^{k_1} |X(u)\xi|^{-1} \leq 2rN, \quad 0 \leq k \leq k_1. \tag{2.14}$$

Proof Without loss of generality, we may take $f_0 = 0$. Fix $l_0, l_1 \in I$ such that $l_0 \leq l_1$. Take a sequence f with $f_i = 0,\ i > l_1$. Then formula (2.12) takes the form

$$y_l = \sum_{u=0}^{l_1} G(l, u) f_u.$$

For $l \geq l_1$, all the indices u in this sum do not exceed l_1, and we apply the first line in the definition of G. Thus,

$$y_l = X(l)P \sum_{u=0}^{l_1} X(-u) f_u.$$

Hence, if $l \geq l_1$, then y_l is the image under $X(l)$ of a vector from V_1 that does not depend on l. It follows that the sequence y (with the exception of a finite number of entries) is a solution of Eq. (2.2) with initial value from V_1. Hence, $y \in \mathscr{B}$. Since $f_0 = 0$,

$$y_0 = -(\mathrm{Id} - P) \sum_{u=0}^{l_1} X(-u) f_u \in V_2.$$

Thus, $y = T(f)$, and $\| y\| \leq r\|f\|$.

Now we specify the choice of f. Let $x_i = X(i)\xi$; since $\xi \neq 0$, $x \neq 0$ as well. Set

$$f_i = \begin{cases} 0, & i < l_0; \\ x_i/|x_i|, & l_0 \leq i \leq l_1; \\ 0, & i > l_1. \end{cases}$$

Since $\|f\| = 1$, inequality (2.9) implies that

$$\left| \sum_{u=l_0}^{l_1} G(k, u) x_i/|x_i| \right| = |\, y_l| \leq r. \tag{2.15}$$

We take $l = l_1 = k$ and $l_0 = k_0$ in (2.15) and conclude that

$$r \geq \left| \sum_{u=k_0}^{k} G(k,u)x_u/|x_u| \right| = \left| \sum_{u=k_0}^{k} X(k)PX(-u)X(u)\xi/|X(u)\xi| \right| =$$

$$= |X(k)P\xi| \sum_{u=k_0}^{k} |X(u)\xi|^{-1},$$

which is precisely inequality (2.13).

We prove inequality (2.14) using a similar reasoning.

First we consider $0 < k \leq k_1$. We take $l = k-1$, $l_0 = k$, and $l_1 = k_1$ in (2.15) and get the estimates

$$r \geq \left| \sum_{u=k}^{k_1} G(k,u)x_u/|x_u| \right| = \left| \sum_{u=k}^{k_1} X(k-1)(\mathrm{Id}-P)X(-u)X(u)\xi/|X(u)\xi| \right| =$$

$$= |X(k-1)(\mathrm{Id}-P)\xi| \sum_{u=k}^{k_1} |X(u)\xi|^{-1} = |A_{k-1}^{-1}X(k)(\mathrm{Id}-P)\xi| \sum_{u=k}^{k_1} |X(u)\xi|^{-1} \geq$$

$$\geq \|A_{k-1}\|^{-1}|X(k)(\mathrm{Id}-P)\xi| \sum_{u=k}^{k_1} |X(u)\xi|^{-1}.$$

Applying inequality (2.1), we see that in this case,

$$|X(k)(\mathrm{Id}-P)\xi| \sum_{u=k}^{k_1} |X(u)\xi|^{-1} \leq rN.$$

Now we consider $0 = k < k_1$ and apply the previous estimate with $k = 1$:

$$|X(0)(\mathrm{Id}-P)\xi| \sum_{u=0}^{k_1} |X(u)\xi|^{-1} = |X(0)(\mathrm{Id}-P)\xi| \sum_{u=1}^{k_1} |X(u)\xi|^{-1} + |(\mathrm{Id}-P)\xi| \leq$$

$$\leq \|A_0\|^{-1}|X(1)(\mathrm{Id}-P)\xi| \sum_{u=1}^{k_1} |X(u)\xi|^{-1} + 1 \leq rN + 1 \leq 2rN$$

(recall that $|\xi| \leq 1$ and $rN \geq 1$).

For $k = k_1 = 0$, our inequality is trivial. □

Lemma 2.1.3 *Let $k_0, k_1, k, s \in I$ and let $\xi \in \mathbb{R}^n$ be a unit vector. Denote*

$$\mu = 1 - (2rN)^{-1}.$$

Then the following inequalities are satisfied:
if $P\xi \neq 0$, then

$$\sum_{u=k_0}^{s} |X(u)P\xi|^{-1} \leq \mu^{k-s} \sum_{u=k_0}^{k} |X(u)P\xi|^{-1}, \quad k_0 \leq s \leq k; \tag{2.16}$$

if $(Id - P)\xi \neq 0$, then

$$\sum_{u=s}^{k_1} |X(u)(Id - P)\xi|^{-1} \leq \mu^{s-k} \sum_{u=k}^{k_1} |X(u)(Id - P)\xi|^{-1}, \quad k \leq s \leq k_1. \tag{2.17}$$

Proof Denote

$$\phi_i = \sum_{u=k_0}^{i} |X(u)P\xi|^{-1}, \quad i \geq k_0,$$

and

$$\psi_i = \sum_{u=i}^{k_1} |X(u)(\mathrm{Id} - P)\xi|^{-1}, \quad i \leq k_1.$$

Let us prove inequality (2.16). Since $P\xi \neq 0$, $\phi_i > 0$. Clearly, $\phi_i - \phi_{i-1} = |X(i)P\xi|^{-1}$. Replacing ξ by $P\xi$ (and noting that $|P\xi| \leq 1$) in (2.13), we see that

$$\frac{\phi_i}{\phi_i - \phi_{i-1}} \leq r \leq 2rN.$$

Hence,

$$(2rN)^{-1} \leq \frac{\phi_i - \phi_{i-1}}{\phi_i} = 1 - \frac{\phi_{i-1}}{\phi_i},$$

and

$$\phi_{i-1} \leq (1 - (2rN)^{-1})\phi_i.$$

Iterating this inequality, we conclude that

$$\phi_s \leq (1 - (2rN)^{-1})^{k-s}\phi_k, \quad k \geq s.$$

We prove inequality (2.17) similarly. We note that $\psi_i > 0$ and that $\psi_i - \psi_{i+1} = |X(i)(\mathrm{Id} - P)\xi|^{-1}$. After that, we replace ξ by $(\mathrm{Id} - P)\xi$ in (2.14) and show that

$$\psi_{i+1} \leq (1 - (2rN)^{-1})\psi_i.$$

Iterating this inequality, we get (2.17). □

Now we prove that the sequence $\mathscr{A}$ is hyperbolic.

Lemma 2.1.4 *The following inequalities are satisfied:*

$$\|X(k)PX(-s)\| \leq r^2\mu^{k-s}, \quad 0 \leq s \leq k,$$

and

$$\|X(k)(Id - P)X(-s)\| \leq 2r^2N^2\mu^{s-k}, \quad 0 \leq k \leq s.$$

Proof Fix a natural s and a unit vector ξ. Define a sequence $y = \{y_k\}$ by

$$y_k = \begin{cases} -X(k)(\mathrm{Id} - P)X(-s)\xi, & 0 \leq k < s; \\ X(k)PX(-s)\xi, & k \geq s. \end{cases}$$

The sequence y coincides (up to a finite number of terms) with a solution of Eq. (2.2) with initial point from V_1; hence, $y \in \mathscr{B}$.

Now we define a sequence f by

$$f_k = \begin{cases} 0, & k \neq s; \\ \xi, & k = s. \end{cases}$$

It is easily seen that the above sequence y is a solution of Eq. (2.3) with inhomogeneity f. Hence, $y = T(f)$, and $\|y\| \leq r$.

The definition of y implies that

$$|X(k)PX(-s)\xi| = |y_k| \leq r, \quad 0 \leq s \leq k.$$

Since ξ is an arbitrary unit vector, $\|X(k)PX(-s)\| \leq r$ for $0 \leq s \leq k$.

We replace ξ by the solution of the equation $x_s = X(s)\xi$ to show that

$$|X(k)P\xi| = |X(k)PX(-s)x_s| \leq r|x_s|, \quad 0 \leq s \leq k. \tag{2.18}$$

Using inequalities (2.13), (2.16) with $k_0 = s$, and (2.18) with $k = s$, we see that

$$|X(k)PX(-s)x_s| = |X(k)P\xi| \leq r\left(\sum_{u=s}^{k} |X(u)P\xi|^{-1}\right)^{-1} \leq$$

$$\leq r\left(\mu^{-(k-s)}|X(s)P\xi|^{-1}\right)^{-1} = r\mu^{k-s}|X(s)P\xi| \leq r^2\mu^{k-s}|x_s|.$$

If $P\xi = 0$, then the resulting estimate is obvious. Since $x_s = X(s)\xi$ and $X(s)$ is an isomorphism, we get the following estimate for the operator norm:

$$\|X(k)PX(-s)\| \leq r^2\mu^{k-s}, \quad 1 \leq s \leq k.$$

In this reasoning, we have used inequality (2.18) with $s = k$. It is also true for $s = k = 0$ since $\| P\| \leq 1$. Therefore, the first estimate of our lemma is proved for $0 \leq s \leq k$.

The proof of the second estimate is quite similar. The only difference is as follows. We cannot use an analog of (2.18) with $k = s$ since $k \neq s$ in the definition of the sequence y. The following inequality is proved by the same reasoning as above:

$$|X(k)(\mathrm{Id} - P)\xi| = |X(k)(\mathrm{Id} - P)X(-s)x_s| \leq r|x_s|, \quad s > k.$$

In the case $k = s - 1$, we write

$$|X(s)(\mathrm{Id} - P)\xi| = |A_{s-1}X(s-1)(\mathrm{Id} - P)X(-s)x_s| \leq$$

$$\leq \|A_{s-1}\||X(s-1)(\mathrm{Id} - P)X(-s)x_s| \leq rN|x_s|,$$

and then repeat the reasoning of the first case. □

Lemma 2.1.4 shows that if we take constants $C_0 = r^2N$ and $\lambda = \mu$ and projections

$$P_k = X(k)PX(-k) \quad \text{and} \quad Q_k = X(k)(\mathrm{Id} - P)X(-k),$$

then the operators $F(k, l)$ generated by the sequence $\mathscr{A}$ satisfy estimates (2.6) and (2.7) with $C = C_0$ and λ. Clearly, relations (2.4) and (2.5) are valid.

Thus, to show that $\mathscr{A}$ is hyperbolic on I, it remains to prove the following statement.

Lemma 2.1.5 *There exists a constant $C = C(N, C_0, \lambda) \geq C_0$ such that inequalities (2.8) are fulfilled.*

Proof Let L_1 and L_2 be two linear subspaces of $\mathbb{R}^n$. Introduce the value

$$\angle(L_1, L_2) = \min |v_1 - v_2|,$$

where the minimum is taken over all pairs of unit vectors $v_1 \in L_1, v_2 \in L_2$.

We claim that there exists a constant $C_1 = C_1(N, C_0, \lambda)$ such that

$$\angle(S_k, U_k) \geq C_1, \quad k \in I. \tag{2.19}$$

Fix an index $k \in I$, take unit vectors $v_1 \in S_k$ and $v_2 \in U_k$ for which $\angle(S_k, U_k) = |v_1 - v_2|$, and denote

$$\alpha_l = |F(l,k)(v_1 - v_2)|, \quad l \geq k.$$

Inequalities (2.6) and (2.7) imply that

$$\alpha_l \geq |F(l,k)v_2| - |F(l,k)v_1| \geq \lambda^{k-l}/C_0 - C_0\lambda^{l-k}.$$

Hence, there exists a constant $m = m(C_0, \lambda)$ such that

$$\alpha_{k+m} \geq 1.$$

At the same time, it follows from (2.1) that

$$\alpha_{k+m} \leq N^m \alpha_k.$$

Combining the above two inequalities, we see that

$$\angle(S_k, U_k) = \alpha_k \geq C_1(N, C_0, \lambda) := N^{-m(C_0,\lambda)},$$

which proves (2.19).

Clearly, if v_1 and v_2 are two unit vectors, then the usual angle $\langle v_1, v_2 \rangle$ satisfies the relation

$$|v_1 - v_2| = 2\sin(\langle v_1, v_2 \rangle/2),$$

and we see that estimate (2.19) implies the existence of $\beta = \beta(N, C_0, \lambda)$ such that if γ is the usual angle between S_k and U_k, then

$$\sin(\gamma) \geq \beta.$$

Now we take an arbitrary unit vector $v \in \mathbb{R}^n$ and denote $v_s = P_k v$. If γ_s is the angle between v and v_s, then the sine law implies that

$$\frac{|v|}{\sin(\gamma)} = \frac{|v_s|}{\sin(\gamma_0)} \geq |v_s|,$$

and we conclude that

$$|v_s| = |P_k v| \leq 1/\beta,$$

which implies that

$$\|P_k\| \le C = \max(C_0, 1/\beta).$$

A similar estimate holds for $\|Q_k\|$. □

As we said above, the following statement holds.

Lemma 2.1.6 *A hyperbolic sequence $\mathscr{A}$ has the Perron property on I.*

Proof Assume that the sequence $\mathscr{A}$ has properties stated in relations (2.4)–(2.8). Take a sequence

$$f = \{f_k \in \mathbb{R}^n : k \in I\}$$

such that $\|f\| = \nu < \infty$ and consider the sequence y defined by formula (2.11).

Then

$$|X(k)PX(-u)f_u| \le C\lambda^{k-u}\nu, \quad 0 \le u \le k,$$

and

$$|X(k)(\mathrm{Id} - P)X(-u)f_u| \le C\lambda^{u-k}\nu, \quad k+1 \le u < \infty,$$

which implies that the second term in (2.11) is a convergent series (hence, the sequence y is a solution of (2.3)) and the estimate

$$\|y\| \le C(1 + \lambda + \lambda^2 \dots)\nu + C(\lambda + \lambda^2 \dots)\nu = \frac{1+\lambda}{1-\lambda}C\nu$$

holds. □

Now we pass to the Pliss theorem.

This time, $I = \mathbb{Z}$, and we denote $I_+ = \{k \in \mathbb{Z} : k \ge 0\}$ and $I_- = \{k \in \mathbb{Z} : k \le 0\}$.

Now $\mathscr{A}$ is a sequence of linear isomorphisms

$$A_k : \mathbb{R}^n \to \mathbb{R}^n, \quad k \in I = \mathbb{Z}.$$

It is again assumed that an analog of inequalities (2.1) holds, and we consider difference equations (2.2) and (2.3).

The Perron property of (2.2) on $\mathbb{Z}$ is defined literally as in the case of $I = \{k \in \mathbb{Z} : k \ge 0\}$.

It follows from the Maizel' theorem and its obvious analog for the case of $I = \{k \in \mathbb{Z} : k \le 0\}$ that the sequence $\mathscr{A}$ is hyperbolic on both I_+ and I_- (the definition of hyperbolicity in the case of I_- is literally the same).

Without loss of generality, we assume that C and λ are the same for the hyperbolicity on I_+ and I_- and denote by $S_k^+, U_k^+, k \in I_+$, and $S_k^-, U_k^-, k \in I_-$, the corresponding subspaces of $\mathbb{R}^n$.

Theorem 2.1.2 (Pliss) *If $\mathscr{A}$ has the Perron property on $I = \mathbb{Z}$, then the subspaces U_0^- and S_0^+ are transverse.*

Remark 2.1.2 In fact, Pliss proved in [74] that the transversality of U_0^- and S_0^+ is equivalent to the Perron property of $\mathscr{A}$ on $I = \mathbb{Z}$, but we need only the implication stated above.

Remark 2.1.3 Note that there exist sequences $\mathscr{A}$ that are separately hyperbolic on I_+ and I_- for which the subspaces U_0^- and S_0^+ are transverse and such that these sequences are not hyperbolic on $I = \mathbb{Z}$. It is easy to construct such a sequence with $S_k^+ = \mathbb{R}^n, U_k^+ = \{0\}, k \in I_+$, and $S_k^- = \{0\}, U_k^- = \mathbb{R}^n, k \in I_-$ (we leave details to the reader).

Proof To get a contradiction, assume that the subspaces U_0^- and S_0^+ are not transverse. Then there exists a vector $x \in \mathbb{R}^n$ such that

$$x \neq y_1 + y_2 \tag{2.20}$$

for any $y_1 \in U_0^-$ and $y_2 \in S_0^+$.

Since the subspaces U_0^+ and S_0^+ are complementary (see (2.4)), we can represent

$$x = \xi + \eta, \quad \xi \in S_0^+, \; \eta \in U_0^+.$$

Then it follows from (2.20) that

$$\eta \neq z_1 + z_2 \tag{2.21}$$

for any $z_1 \in S_0^+$ and $z_2 \in U_0^-$. We may assume that $|\eta| = 1$.

Consider the sequence

$$a_k = \begin{cases} 0, \; k \le 0; \\ 1, \; k > 0. \end{cases}$$

Since $\eta \neq 0$ in (2.21), $X(k)\eta \neq 0$ for $k \in I$. Define a sequence $f = \{f_k, \; k \in I\}$ by

$$f_k = \frac{X(k)\eta}{|X(k)\eta|} a_k, \quad k \in I. \tag{2.22}$$

Clearly, $\|f\| = 1$. We claim that the corresponding Eq. (2.3) does not have bounded solutions.

Consider the sequence

$$\phi_k = -\sum_{u=k+1}^{\infty} X(k)(\mathrm{Id} - P)X(-u)f_u, \quad k \geq 0.$$

In this formula, P is the projection defined for Eq. (2.2).

The sequence $\{\phi_k\}$ is bounded for $k \geq 0$. Indeed, $f_u \in U_u^+$ for $u \geq 0$; hence,

$$|\phi_k| = \left| \sum_{u=k+1}^{\infty} X(k)(\mathrm{Id} - P)X(-u)f_u \right| \leq$$

$$\leq \sum_{u=k+1}^{\infty} C\lambda^{u-k} = C\frac{\lambda}{1-\lambda}.$$

We know that since the series defining ϕ_k is convergent, the sequence $\{\phi_k\}$ is a solution of the homogeneous equation (2.2) for $k \geq 0$.

Clearly,

$$\phi_0 = -\sum_{u=1}^{\infty}(\mathrm{Id} - P)X(-u)f_u = -\sum_{u=1}^{\infty} \frac{\eta}{|X(u)\eta|} = \nu\eta,$$

where

$$\nu = -\sum_{u=1}^{\infty} \frac{1}{|X(u)\eta|}.$$

Deriving these relations, we take into account the definition of f and the equality $(\mathrm{Id} - P)\eta = \eta$. In addition, the value ν is finite since

$$\frac{1}{|X(k)\eta|} \leq C\lambda^k, \quad k \geq 0,$$

due to inequalities (2.7).

It follows from (2.21) that

$$\phi_0 \neq y_1 + y_2 \tag{2.23}$$

for any $y_1 \in S_0^+$ and $y_2 \in U_0^-$.

Now let us assume that Eq. (2.2) has a solution $\psi = \{\psi_k\}$ that is bounded on $I = \mathbb{Z}$. Then $\psi_0 \in U_0^-$.

On the other hand,

$$\psi_k = X(k)(\psi_0 - \phi_0) + \phi_0.$$

Since ϕ_k are bounded for $k \geq 0$, ψ_k can be bounded for $k \geq 0$ only if

$$X(k)(\psi_0 - \phi_0)$$

are bounded for $k \geq 0$, which implies that

$$\psi_0 - \phi_0 \in S_0^+.$$

Set

$$y_1 = \phi_0 - \psi_0 \in S_0^+ \quad \text{and} \quad y_2 = \psi_0 \in U_0^-.$$

Then $\phi_0 = y_1 + y_2$, and we get a contradiction with (2.23). □

Remark 2.1.4 We will apply the Maizel' and Pliss theorems proved in this section in a slightly different situation.

We consider a diffeomorphism f of a smooth closed manifold M, fix a point $x \in M$ and the trajectory $\{x_k = f^k(x) : \ k \in \mathbb{Z}\}$ of this point and define linear isomorphisms

$$A_k = Df(x_k) : \ T_{x_k}M \to T_{x_{k+1}}M.$$

To the sequence $\mathscr{A} = \{A_k\}$ we assign difference equations

$$v_{k+1} = A_k v_k, \quad v_k \in T_{x_k}M,$$

and

$$v_{k+1} = A_k v_k + f_{k+1}, \quad v_k \in T_{x_k}M, \ f_{k+1} \in T_{x_{k+1}}M.$$

Clearly, these difference equations are completely similar to Eqs. (2.2) and (2.3), and analogs of the Maizel' and Pliss theorems are valid for them.

Historical Remarks Theorem 2.1.1 was proved by A. D. Maizel' in [38]. See also the classical W. A. Coppel's book [13].

The Pliss theorem (Theorem 2.1.2) was published in [74]. Later, it was generalized by many authors; let us mention, for example, K. Palmer [55] who studied Fredholm properties of the corresponding operators.

2.2 Mañé Theorem

In this section, we prove Theorem 1.3.7.

Remark 2.2.1 In several papers, the analytic strong transversality condition is formulated in the following form, which is obviously stronger than the condition formulated in Definition 1.3.11: it is assumed that

$$\tilde{B}^+(x) + \tilde{B}^-(x) = T_xM, \quad x \in M,$$

where the subspaces $\tilde{B}^+(x)$ and $\tilde{B}^-(x)$ are defined by the equalities

$$\tilde{B}^+(x) = \left\{ v \in T_xM : \lim_{k\to\infty} \left|Df^k(x)v\right| = 0 \right\}$$

and

$$\tilde{B}^-(x) = \left\{ v \in T_xM : \lim_{k\to-\infty} \left|Df^k(x)v\right| = 0 \right\}.$$

In fact, it is easily seen from our proof below that the structural stability of f implies this form of the analytic strong transversality condition as well, so that both conditions are equivalent.

The main part of our proof of Theorem 1.3.7 is contained in the following statement.

Theorem 2.2.1 *The analytic strong transversality condition implies Axiom A.*

First we prove that the analytic strong transversality condition implies the hyperbolicity of the nonwandering set Ω.

We assign to a diffeomorphism $f : M \to M$ the mapping $\pi : TM \to TM$ (where TM is the tangent bundle of M) which maps a pair $(x, v) \in TM$ (where $x \in M$ and $v \in T_xM$) to the pair $(f(x), Df(x)v)$.

A subbundle Y of TM is a set of pairs (x, Y_x), where $x \in M$ and Y_x is a linear subspace of T_xM.

Definition 2.2.1 A subbundle Y is called *π-invariant* if

$$Df(x)Y_x = Y_{f(x)} \quad \text{for} \quad x \in M.$$

Assuming that f satisfies the analytic strong transversality condition, we define two subbundles B^+ and B^- of TM by setting

$$B^+_x = B^+(x) \quad \text{and} \quad B^-_x = B^-(x) \quad \text{for} \quad x \in M.$$

Since

$$\liminf_{k\to\infty} \left|Df^k(x)v\right| = 0$$

if and only if

$$\liminf_{k\to\infty} \left|Df^k(f(x))Df(x)v\right| = 0,$$

the subbundle B^+ is π-invariant. A similar reasoning shows that the subbundle B^- is π-invariant as well.

The main object in the proof is the mapping π^*, dual to the mapping π.

Denote by $<,>$ the scalar product in T_xM. Let $D^*f(x) : T_{f(x)}M \to T_xM$ be defined as follows:

$$< \xi, Df(x)v >=< D^*f(x)\xi, v >$$

for all $v \in T_xM$ and $\xi \in T_{f(x)}M$ (thus, $D^*f(x)$ is the adjoint of $Df(x)$). We define π^* as follows: a pair $(f(x), \xi)$, $\xi \in T_{f(x)}M$, is mapped to

$$\pi^* (f(x), \xi) = \left(x, D^*f(x)\xi\right).$$

If $p : TM \to M$ is the projection to the first coordinate (i.e., $p(x, v) = x$), then $p(\pi(x, v)) = f(x)$ (in this case, one says that π covers f); since $p(\pi^*(x, v)) = f^{-1}(x)$, π^* covers f^{-1}.

Clearly, the definition of π^* implies the following statement.

Lemma 2.2.1

$$(\pi^*)^* = \pi.$$

If Y is a subbundle of TM, we define the orthogonal subbundle $Y^\perp$ as follows:

$$Y_x^\perp = \{\xi : < \xi, v >= 0 \quad \text{for all} \quad v \in Y_x\}, \quad x \in M.$$

Lemma 2.2.2 *If a subbundle Y is π-invariant, then $Y^\perp$ is π^*-invariant.*

Proof Consider vectors $\xi \in Y_{f(x)}^\perp$ and $D^*f(x)\xi \in T_xM$. If $v \in Y_x$, then

$$< v, D^*f(x)\xi >=< \xi, Df(x)v >= 0$$

since $Df(x)v \in Y_{f(x)}$, which means that $D^*f(x)\xi \in Y_x^\perp$. □

We call two subbundles Y^1 and Y^2 complementary if

$$Y_x^1 \oplus Y_x^2 = T_xM \quad \text{for any} \quad x \in M. \tag{2.24}$$

Lemma 2.2.3 *If Y^1 and Y^2 are complementary subbundles that are π-invariant, then $\left(Y^1\right)^\perp$ and $\left(Y^2\right)^\perp$ are complementary subbundles that are π^*-invariant.*

Proof The subbundles $\left(Y^1\right)^\perp$ and $\left(Y^2\right)^\perp$ are π^*-invariant by Lemma 2.2.2. If $\dim Y_x^1 = k$, then equality (2.24) implies that $\dim Y_x^2 = n - k$. Clearly,

$$\dim\left(Y^1\right)_x^\perp = n - k \quad \text{and} \quad \dim\left(Y^2\right)_x^\perp = k. \tag{2.25}$$

Consider a vector $\xi \in \left(Y^1\right)_x^\perp \cap \left(Y^2\right)_x^\perp$. Due to (2.24), any vector $v \in T_xM$ is representable as

$$v = v_1 + v_2, \quad v_1 \in Y_x^1,\ v_2 \in Y_x^2.$$

Then $< \xi, v >=< \xi, v_1 > + < \xi, v_2 >= 0$. Since v is arbitrary, $\xi = 0$. The equality

$$\left(Y^1\right)_x^\perp \cap \left(Y^2\right)_x^\perp = \{0\}$$

and (2.25) imply the statement of our lemma. □

Let $M_0 \subset M$ be a hyperbolic set of f. Then S and U defined by $S_x = S(x)$ and $U_x = U(x)$ for $x \in M_0$ are two complementary π-invariant subbundles on M_0 such that inequalities (HSD2.3) and (HSD2.4) hold (see Definition 1.3.1). In this case, we say that M_0 is hyperbolic with respect to π with subbundles S and U and constants C and λ.

Lemma 2.2.4 *If a set M_0 is hyperbolic with respect to π with subbundles S and U and constants C and λ, then M_0 is hyperbolic with respect to π^* with subbundles $U^\perp$ and $S^\perp$ and the same constants C and λ.*

Proof If A and B are linear operators, then $(AB)^* = B^*A^*$; hence,

$$\left(Df(f(x))Df(x)\right)^* = D^*f(x)D^*f(f(x)).$$

If we take $v \in T_xM$ and $\xi \in T_{f^2(x)}M$, then

$$< Df^2(x)v, \xi >=< Df(f(x))Df(x)v, \xi >=$$

$$=< Df(x)v, D^*f(f(x))\xi >=< v, D^*f(x)D^*f(f(x))\xi >=< v, D^*f^2(x)\xi > .$$

Applying induction, it is easy to show that

$$< v, D^*f^k(x)\xi >=< \xi, Df^k(x)v >, \quad k \in \mathbb{Z}, \tag{2.26}$$

for $v \in T_xM$ and $\xi \in T_{f^k(x)}M$, where

$$D^*f^k(x) = D^*f(x)D^*f(f(x))\dots D^*f(f^{k-1}(x))$$

and

$$Df^k(x) = Df(f^{k-1}(x))Df(f^{k-2}(x))\dots Df(x).$$

By Lemma 2.2.3, the subbundles $S^\perp$ and $U^\perp$ are complementary and π^*-invariant.

Fix $k \geq 0$ and a vector $\xi \in \left(U_{f^k(x)}\right)^\perp$. Then $D^*f^k(x)\xi \in T_xM$. The obvious equality

$$|\eta| = \max_{|v|=1} < \eta, v >, \quad \eta, v \in T_xM,$$

implies that

$$\left|D^*f^k(x)\xi\right| = \max_{|v|=1} < v, D^*f^k(x)\xi > .$$

Represent $v = v_1 + v_2$, where $v_1 \in S_x$ and $v_2 \in U_x$.

Since $U^\perp$ is π^*-invariant,

$$D^*f^k(x)\xi \in (U_x)^\perp,$$

and $< v_2, D^*f^k(x)\xi >= 0$. It follows that

$$\left|D^*f^k(x)\xi\right| = \max_{|v_1|=1} < v_1, D^*f^k(x)\xi >= \max_{|v_1|=1} < \xi, Df^k(x)v_1 >\leq C\lambda^k|\xi|.$$

In the last inequality, we used inequality (HSD2.3) and the obvious relation

$$< \xi, v >\leq |\xi||v|.$$

A similar reasoning shows that

$$\left|D^*f^{-k}(x)\xi\right| \leq C\lambda^{-k}|\xi|$$

for $\xi \in \left(S_{f^k(x)}\right)^\perp$ and $k \leq 0$. □

Now we prove that the analytic strong transversality condition implies that, in a sense, π^* does not have nontrivial bounded trajectories. Fix a point $(x, v) \in TM$ and define the sequence $(x_k, v_k) = (\pi^*)^k(x, v)$.

Lemma 2.2.5 *If*

$$\sup_{k\in\mathbb{Z}} |v_k| < \infty, \tag{2.27}$$

then $v = 0$.

Proof The obvious equalities

$$x = f^{-k}\left(f^k(x)\right) \quad \text{and} \quad u = Df^{-k}\left(f^k(x)\right)Df^k(x)u$$

which are valid for all $x \in M$, $u \in T_xM$, and $k \in \mathbb{Z}$ imply that

$$< \xi, u > = < \xi, Df^{-k}\left(f^k(x)\right)Df^k(x)u > = < D^*f^{-k}\left(f^k(x)\right)\xi, Df^k(x)u >$$

for all $\xi, u \in T_xM$ and k.

Assume that a point (x, v) satisfies condition (2.27).

By the analytic strong transversality condition, we can represent any vector $\xi \in T_xM$ in the form $\xi = \xi_1 + \xi_2$ for which there exist sequences $l_n \to \infty$ and $m_n \to -\infty$ as $n \to \infty$ such that

$$\left|Df^{l_n}(x)\xi_1\right| \to 0 \quad \text{and} \quad \left|Df^{m_n}(x)\xi_2\right| \to 0, \quad n \to \infty.$$

Let us write

$$< v, \xi > = < v, \xi_1 + \xi_2 > = < v, Df^{-l_n}\left(f^{l_n}(x)\right)Df^{l_n}(x)\xi_1 > +$$

$$+ < v, Df^{-m_n}\left(f^{m_n}(x)\right)Df^{m_n}(x)\xi_1 > =$$

$$= < D^*f^{-l_n}\left(f^{l_n}(x)\right)v, Df^{l_n}(x)\xi_1 > + < D^*f^{-m_n}\left(f^{m_n}(x)\right)v, Df^{m_n}(x)\xi_2 > . \tag{2.28}$$

By condition (2.27), both values $\left|D^*f^{-l_n}(f^{l_n}(x))v\right|$ and $\left|D^*f^{-m_n}(f^{m_n}(x))v\right|$ are bounded; hence, both terms in (2.28) tend to 0 as $n \to \infty$. Thus, $< \xi, v > = 0$ for any ξ, which means that $v = 0$. □

To simplify notation, let us denote π^* by ρ and write

$$\rho(x, v) = (\phi(x), \Phi(x)v),$$

so that $\phi(x) = f^{-1}(x)$ and $\Phi(x)$ is the linear mapping $T_xM \to T_{\phi(x)}M$, $\Phi(x) = D^*f(x)$. Let

$$F(0, x) = \text{Id},$$

$$F(k, x) = \Phi(\phi^{k-1}(x)) \cdots \Phi(x), \quad k > 0,$$

and

$$F(-k,x) = \Phi^{-1}(\phi^{1-k}(x)) \cdots \Phi^{-1}(x), \quad k > 0.$$

Obviously, the mapping ρ is continuous. By Lemma 2.2.5, it satisfies the following *Condition B*: If

$$\sup_{k \in \mathbb{Z}} |F(k,x)v| < \infty$$

for some $(x,v) \in TM$, then $v = 0$.

Let us define the following two subbundles in TM: $V = \{(x, V_x)\}$ and $W = \{(x, W_x)\}$. We agree that

- $v \in T_xM$ belongs to V_x if $|F(k,x)v| \to 0$ as $k \to \infty$
  and
- $v \in T_xM$ belongs to W_x if $|F(k,x)v| \to 0$ as $k \to -\infty$.

Clearly, the subbundles V and W are ρ-invariant.

Lemma 2.2.6 *Let a sequence $(x_m, v_m) \in TM$ be such that*

(1) $(x_m, v_m) \to (x, v)$ *as* $m \to \infty$*;*
(2) there exists a number $L > 0$ *and a sequence* $k_m \to \infty$ *as* $m \to \infty$ *such that*

$$|F(k, x_m)v_m| \le L, \quad 0 \le k \le k_m. \tag{2.29}$$

Then $(x, v) \in V$.

Proof Fix an arbitrary $l \ge 0$. There exists an m_0 such that $k_m > l$ for $m \ge m_0$. Then it follows from (2.29) that

$$|F(l, x_m)v_m| \le L. \tag{2.30}$$

Since $F(l,y)w$ is continuous in y and w, we may pass to the limit in (2.30) as $m \to \infty$; thus,

$$|F(l,x)v| \le L.$$

Since l is arbitrary, this means that

$$|F(k,x)v| \le L, \quad k \ge 0. \tag{2.31}$$

Let (x_0, v_0) be a limit point of the sequence $\left(\phi^k(x), F(k,x)v\right)$, i.e., the limit of the sequence

$$\left(\phi^{t_m}(x), F(t_m,x)v\right) \tag{2.32}$$

for some sequence $t_m \to \infty$.

Take an arbitrary $k \in \mathbb{Z}$. Since

$$\phi^{t_m}(x) \to x_0 \quad \text{and} \quad F(t_m, x)v \to v_0, \quad m \to \infty,$$

$$\phi^{k+t_m}(x) \to \phi^k(x_0) \quad \text{and} \quad F(k + t_m, x)v \to F(k, x_0)v_0, \quad m \to \infty. \tag{2.33}$$

For large m, $k + t_m > 0$, and it follows from (2.31) and the second relation in (2.33) that

$$|F(k, x_0)v_0| \leq L. \tag{2.34}$$

Since (2.34) is valid for any $k \in \mathbb{Z}$, Condition B implies that $v_0 = 0$. Thus, in any convergent sequence of the form (2.32) with $t_m \to \infty$,

$$|F(t_m, x)v| \to 0,$$

which means that $(x, v) \in V$. □

Remark 2.2.2 A similar reasoning shows that if we take $k_m \to -\infty$ and $k_m \leq k \leq 0$ in condition (2) of Lemma 2.2.6, then $(x, v) \in W$. In what follows, we do not make such comments and only consider the case of the subbundle V.

Define the set

$$A = \{(x, v) \in TM : \; |F(k, x)v| \leq 1 \quad \text{for} \quad k \geq 1\}.$$

Clearly, the set A is positively ρ-invariant, i.e., if $(x, v) \in A$ and $k \geq 0$, then $\left(\phi^k(x), F(k, x)v\right) \in A$.

Let us say that a set $C = \{(x, v) \in TM\}$ is bounded if

$$\sup_{(x,v)\in C} |v| < \infty.$$

Since the manifold M is compact, any closed and bounded subset C of TM is (sequentially) compact, i.e., any sequence in C has a convergent subsequence, and the limit of this subsequence belongs to C.

Lemma 2.2.7 *The set A is a compact subset of V.*

Proof It was shown in the proof of Lemma 2.2.6 that inequality (2.31) implies the inclusion $(x, v) \in V$; thus, $A \subset V$. Since $F(0, x)v = v$, A is bounded. Consider a sequence $(x_m, v_m) \in A$ such that $(x_m, v_m) \to (x, v), \; m \to \infty$. For any fixed $k \geq 0$,

$$|F(k, x)v| = \lim_{m\to\infty} |F(k, x_m)v_m| \leq 1.$$

Hence, $(x, v) \in A$, and A is closed. □

Lemma 2.2.8 *For any* $\mu > 0$ *there exists a* $K > 0$ *such that if* $(x, v) \in A$*, then*

$$|F(k, x)v| < \mu, \quad k \geq K. \tag{2.35}$$

Proof Assuming the converse, let us find sequences $(x_m, v_m) \in A$ and $k_m \to \infty$ and a number $\mu > 0$ such that

$$|F(k_m, x_m)v| \geq \mu. \tag{2.36}$$

Since A is positively ρ-invariant,

$$\left(\phi^{k_m}(x_m), F(k_m, x_m)v_m\right) \in A;$$

since A is compact, the above sequence has a convergent subsequence. Assume, for definiteness, that

$$\left(\phi^{k_m}(x_m), F(k_m, x_m)v_m\right) \to (x, v).$$

Then it follows from (2.36) that $|v| \geq \mu$. Fix a number $k \in \mathbb{Z}$. Since $k + k_m > 0$ for large m,

$$\left(\phi^{k+k_m}(x_m), F(k + k_m, x_m)v_m\right) \to \left(\phi^k(x), F(k, x)v\right), \quad m \to \infty,$$

and

$$|F(k + k_m, x_m)v_m)| \leq 1,$$

we conclude that

$$|F(k, x)v)| \leq 1, \quad k \in \mathbb{Z}.$$

Condition B implies that $v = 0$. The contradiction with (2.36) completes the proof. □

Lemma 2.2.9 *There exists a number* $\mu > 0$ *such that if* $(x, v) \in V$ *and* $|v| \leq \mu$*, then* $(x, v) \in A$.

Proof Assuming the contrary, we can find a sequence $(x_m, v_m) \in V$ such that $|v_m| \to 0, \ m \to \infty$, and $(x_m, v_m) \notin A$.

Then

$$\mu_m = \max_{k \geq 0} |F(k, x_m)v_m| > 1$$

(we take into account that $|F(k, x_m)v_m| \to 0, \ k \to \infty$).

Find numbers $k_m > 0$ such that

$$|F(k_m, x_m)v_m| = \mu_m.$$

Since

$$|F(k, x_m)(v_m/\mu_m)| \le 1, \quad k \ge 0,$$

$(x_m, v_m/\mu_m) \in A$.

The mapping ρ is continuous and $F(k, x)0 = 0$; hence,

$$\max_{0 \le k \le K} |F(k, x_m)(v_m/\mu_m)| \to 0, \quad m \to \infty,$$

for any fixed K (note that $x_m \in M$, M is compact, $|v_m| \to 0$, and $\mu_m > 1$).

Hence, $k_m \to \infty,\ m \to \infty$. Lemma 2.2.8 implies now that the relations

$$(x_m, v_m/\mu_m) \in A \quad \text{and} \quad |F(k_m, x_m)(v_m/\mu_m)| = 1$$

are contradictory. □

Lemma 2.2.10 *There exists a number* $K > 0$ *such that if* $(x, v) \in V$*, then*

$$|F(k, x)v| \le (1/2)|v|, \quad k \ge K. \tag{2.37}$$

Proof Apply Lemma 2.2.8 to find a number K such that

$$|F(k, x)v'| < \mu/2, \quad k \ge K,$$

for any $(x, v') \in A$ (where μ is the number from Lemma 2.2.9).

Take any $(x, v) \in V$. If $v \ne 0$, set $v' = \mu(v/|v|)$. Then $(x, v') \in A$ by Lemma 2.2.9, and it follows from Lemma 2.2.8 that

$$\left|F(k, x)v'\right| = (\mu/|v|)\,|F(k, x)v| \le \mu/2, \quad k \ge K,$$

which obviously implies the desired relation (2.37). If $v = 0$, we have nothing to prove. □

Lemma 2.2.11

(1) The subbundles V *and* W *are closed.*
(2) There exist numbers $C > 0$ *and* $\lambda \in (0, 1)$ *such that*
if $(x, v) \in V$*, then*

$$|F(k, x)v| \le C\lambda^k|v|, \quad k \ge 0; \tag{2.38}$$

if $(x, v) \in W$, *then*

$$|F(k, x)v| \le C\lambda^{-k}|v|, \quad k \le 0. \tag{2.39}$$

Proof We prove the statements for the subbundle V; for W, the proofs are similar.

To prove statement (1), consider a sequence $(x_k, v_k) \in V$ such that $(x_k, v_k) \to (x, v)$ as $k \to \infty$.

If $v = 0$, then, obviously, $(x, v) \in V$. Assume that $v \ne 0$; then $v_k \ne 0$ for large k, and, by Lemma 2.2.9 there exists a $\mu > 0$ such that

$$(x_k, \mu v_k/|v_k|) \in A.$$

Since A is closed (see Lemma 2.2.7),

$$(x, \mu v/|v|) \in A,$$

and $(x, v) \in V$ by Lemma 2.2.7. This proves the first statement of our lemma.

To prove the second one, apply Lemma 2.2.10 and find a number K such that

$$|F(k, x)v| \le (1/2)|v|, \quad k \ge K, \tag{2.40}$$

for any $(x, v) \in V$.

It follows from (2.40) and from the ρ-invariance of V that

$$|F(2K, x)v| \le (1/2)^2|v|, \ldots, |F(kK, x)v| \le (1/2)^k|v|, \quad k \ge 0. \tag{2.41}$$

There exists a number $C_0 > 0$ such that

$$\max_{0 \le k < K,\, x \in M} \| F(k, x)\| \le C_0. \tag{2.42}$$

Let us show that inequality (2.38) holds with $C = 2C_0$ and $\lambda = 2^{1/K}$. We can represent any $k \ge 0$ in the form $k = k_0K + k_1$, where $k_0 \ge 0$ and $0 \le k_1 < K$. If $(x, v) \in V$, then it follows from (2.41) and (2.42) that

$$|F(k, x)v| = |F(k_1, \phi^{k_0K}(x))F(k_0K, x)v| \le C_0(1/2)^{k_0}|v|,$$

but since $k_0 + 1 > k/K$, $-k_0 < -k/K + 1$, and $2^{-k_0} < 2\lambda^k$, we conclude that

$$|F(k, x)v| \le C\lambda^k|v|,$$

as required. □

Remark 2.2.3 Inequalities (2.38) and (2.39) have the same form as inequalities (HSD2.3) and (HSD2.4) in the definition of a hyperbolic set. Thus, if we want to show that some compact, ρ-invariant subset M_0 of M is a hyperbolic set of ρ with subbundles V and W, we only have to show that

$$V_x + W_x = T_x M, \quad x \in M_0. \tag{2.43}$$

Lemma 2.2.12 *Assume that for a sequence $(x_m, v_m) \in TM$ there exists a sequence $k_m \to \infty$ as $m \to \infty$ and a number $r > 0$ such that*

$$|v_m| \leq r \quad \textit{and} \quad |F(k_m, x_m)v_m| \leq r.$$

Then there exists a number $R > 0$ such that

$$|F(k, x_m)v_m| \leq R, \quad 0 \leq k \leq k_m.$$

Proof Assume the contrary, and let there exist $(x_m, v_m) \in TM$ and $k_m \to \infty$ such that

$$b_m := \max_{0 \leq k \leq k_m} |F(k, x_m)v_m| \to \infty, \quad m \to \infty.$$

Find numbers $l_m \in [0, k_m]$ such that $b_m = |F(l_m, x_m)v_m|$. Since ρ is continuous, it is obvious that

$$l_m \to \infty \quad \text{and} \quad k_m - l_m \to \infty, \quad m \to \infty. \tag{2.44}$$

Set

$$w_m = F(l_m, x_m)(v_m/b_m).$$

Let (x, v) be a limit point of the sequence $(\phi^{l_m}(x_m), w_m)$; then $|v| = 1$. The inequality

$$\left| F(k, \phi^{l_m}(x_m))w_m \right| \leq 1$$

holds for $k \in [-l_m, 0] \cup [0, k_m - l_m]$. We apply relations (2.44) and Lemma 2.2.6 (and its analog for W) to conclude that $v \in V_x \cap W_x$, but then $v = 0$ by Condition B. □

Remark 2.2.4 A similar statement is valid if $k_m \to -\infty$. In this case,

$$|F(k, x_m)v_m| \leq R, \quad k_m \leq k \leq 0.$$

Lemma 2.2.13 *If x is a nonwandering point of the diffeomorphism f, then equality (2.43) holds.*

Proof By the definition of a nonwandering point, there exist sequences of points $x_m \in M$ and numbers k_m such that

$$x_m \to x, \quad f^{k_m}(x_m) \to x, \quad |k_m| \to \infty$$

as $m \to \infty$. We may assume that $k_m \to -\infty$.

Consider the linear subspace W_x and let Q be its orthogonal complement. Let $\dim Q = s$. Fix an orthonormal base $v_1, \dots, v_s$ in Q. Clearly, we can find s orthonormal vectors $v_1^m, \dots, v_s^m$ in $T_{x_m}M$ such that $v_j^m \to v_j$ as $m \to \infty$ for $j = 1, \dots, s$.

Let Q_m be the subspace of $T_{x_m}M$ spanned by $v_1^m, \dots, v_s^m$. Introduce the numbers

$$\mu_m = \min \{ |F(k_m, x_m)v| : \ v \in Q_m, \ |v| = 1 \}.$$

We claim that

$$\mu_m \to \infty, \quad m \to \infty. \tag{2.45}$$

If we assume the contrary, we can find a number $r > 0$ and sequences $w_m \in Q_m$, $|w_m| = 1$, and $k_m \to -\infty$ such that

$$|F(k_m, x_m)w_m| \le r.$$

By the remark to Lemma 2.2.12, there exists a number R such that

$$|F(k, x_m)w_m| \le R, \quad k \in [k_m, 0].$$

By Lemma 2.2.6, in this case, any limit point (x, v) of the sequence (x_m, w_m) belongs to W, i.e., $v \in W_x$. This relation contradicts our construction since $w_m \in Q_m$, which implies that v is orthogonal to Q (note that $|v| = 1$). This proves (2.45).

Consider the linear space

$$K_m = F(k_m, x_m)Q_m.$$

Clearly, $K_m \subset T_{y_m}M$, where $y_m = f^{k_m}(x_m)$, and $\dim K_m = s$.

Consider a vector $w \in K_m$, $|w| = 1$. Let $w = F(k_m, x_m)v$. It follows from the definition of the numbers μ_m that

$$|v| \le \mu_m |w| = \mu_m. \tag{2.46}$$

Inequalities (2.46), relations (2.45), and Lemma 2.2.12 imply that for any sequence (y_m, w_m), where $w_m \in K_m$ and $|w_m| = 1$, there exists a number R such that

$$|F(k, x_m)w_m| \le R, \quad k \in [0, -k_m].$$

Now Lemma 2.2.6 implies that any limit point (x, w) of such a sequence (y_m, w_m) belongs to V, i.e., $w \in V_x$.

Select an orthonormal basis $w_1^m, \dots, w_s^m$ in K_m. We may assume that all the sequences $w_1^m, \dots, w_s^m$ converge for some sequence of indices. For definiteness, let

$$w_1^m \to w_1, \dots, w_s^m \to w_s, \quad m \to \infty.$$

The vectors $w_1, \dots, w_s$ are pairwise orthogonal unit vectors in V_x; hence,

$$\dim V_x \geq s. \tag{2.47}$$

By the definition of the spaces Q and Q_m,

$$\dim W_x = n - s.$$

Combining this with inequality (2.47), we see that

$$\dim V_x + \dim W_x \geq n.$$

Since $V_x \cap W_x = \{0\}$ by Condition B, we conclude that

$$V_x + W_x = T_x M,$$

as claimed. □

The nonwandering set of the diffeomorphism f coincides with the nonwandering set of the diffeomorphism $\phi = f^{-1}$. Combining Lemma 2.2.1 with Lemma 2.2.4 applied to the mapping ρ, we conclude that the following statement holds.

Theorem 2.2.2 *If a diffeomorphism f satisfies the analytic strong transversality condition, then the nonwandering set of f is hyperbolic.*

Now we show that the analytic strong transversality condition implies the second part of Axiom A, the density of periodic points in the nonwandering set $\Omega(f)$ of the diffeomorphism f.

Since we are going to use the Mañé theorem in the proof of the implication (the analytic strong transversality condition) ⇒ (structural stability) for a diffeomorphism f having the Lipschitz shadowing property, we can essentially simplify this proof (compared to the original Mañé proof) assuming that f has the shadowing property.

Thus, now we prove the following statement.

Theorem 2.2.3 *If a diffeomorphism f has the shadowing property and the nonwandering set $\Omega(f)$ of f is hyperbolic, then periodic points are dense in $\Omega(f)$.*

In this proof, we apply the following two well-known results (see, for example, [71] for their proofs).

First we recall a known definition.

Definition 2.2.2 A homeomorphism f of a metric space (M, dist) is called *expansive* on a set A with *expansivity constant* $a > 0$ if the relations

$$f^k(x), f^k(y) \in A, \quad k \in \mathbb{Z},$$

and

$$\text{dist}\left(f^k(x), f^k(y)\right) \le a, \quad k \in \mathbb{Z},$$

imply that $x = y$.

Theorem 2.2.4 *If Λ is a hyperbolic set of a diffeomorphism f, then there exists a neighborhood of Λ on which f is expansive.*

Denote by card$A$ the cardinality of a finite or countable set A.

Theorem 2.2.5 (The Birkhoff Constant Theorem) *If the phase space X of a homeomorphism f is compact and U is a neighborhood of the nonwandering set $\Omega(f)$ of f, then there exists a constant $T = T(U)$ such that for any point $x \in X$, the inequality*

$$card\left\{k \in \mathbb{Z} : f^k(x) \notin U\right\} \le T$$

holds.

Proof (of Theorem 2.2.3) Fix an arbitrary point $z \in \Omega(f)$. There exist sequences of points z_n and numbers $l_n \to \infty$ such that

$$z_n \to z \quad \text{and} \quad f^{l_n}(z_n) \to z, \quad n \to \infty.$$

Let U be a neighborhood of the set $\Omega(f)$ on which f is expansive and let a be the corresponding expansivity constant.

Fix an $\varepsilon > 0$ such that the 3ε-neighborhood of $\Omega(f)$ is a subset of U. Denote by U' the 2ε-neighborhood of $\Omega(f)$. We assume, in addition, that $2\varepsilon < a$.

For this ε there exists a $d > 0$ such that any d-pseudotrajectory of f is ε-shadowed by an exact trajectory.

Fix an index n such that

$$\text{dist}(z, z_n),\ \text{dist}(z, f^{l_n}(z_n)) < d/2.$$

Construct a sequence $\{x_k\}$ as follows. Represent $k \in \mathbb{Z}$ in the form $k = k_0 + k_1 l_n$, where $k_1 \in \mathbb{Z}$ and $0 \le k_0 < l_n$, and set $x_k = f^{k_0}(z_n)$.

Clearly, the sequence $\{x_k\}$ is periodic with period l_n; the choice of n implies that this sequence is a d-pseudotrajectory of f.

We claim that

$$\{x_k\} \subset U'. \tag{2.48}$$

Assuming the contrary, we can find an index m such that $x_m \notin U'$, i.e.,

$$\text{dist}\,(x_m, \Omega(f)) \geq 2\varepsilon,$$

but then

$$\text{dist}\,(x_{m+kl_n}, \Omega(f)) \geq 2\varepsilon, \quad k \in \mathbb{Z}. \tag{2.49}$$

Let $p \in M$ be a point whose trajectory ε-shadows $\{x_k\}$, i.e.,

$$\text{dist}\left(f^k(p), x_k\right) < \varepsilon, \quad k \in \mathbb{Z};$$

let $p_k = f^k(p)$.

Then it follows from inequalities (2.49) that

$$\text{dist}\,(p_{m+kl_n}, \Omega(f)) \geq \varepsilon, \quad k \in \mathbb{Z},$$

which contradicts Theorem 2.2.5. Thus, we have established inclusion (2.48).

Set $r = f^{l_n}(p)$. Since $x_k = x_{k+l_n}$, the following inequalities hold:

$$\text{dist}\left(f^k(r), x_k\right) = \text{dist}\left(f^{k+l_n}(p), x_{k+l_n}\right) < \varepsilon, \quad k \in \mathbb{Z}.$$

Then

$$\text{dist}\left(f^k(r), f^k(p)\right) < 2\varepsilon < a, \quad k \in \mathbb{Z};$$

in addition, inclusion (2.48) implies that

$$f^k(r), f^k(p) \in U, \quad k \in \mathbb{Z}.$$

Since f is expansive on U, $r = p$.

Thus, p is a periodic point of f.

Since ε and d can be taken arbitrarily small, there is such a point p in an arbitrarily small neighborhood of the point z. □

Thus, it remains to show that the analytic strong transversality condition implies the strong transversality condition (stable and unstable manifolds of nonwandering points are transverse).

For this purpose, we apply the following well-known theorem on the behavior of trajectories of a diffeomorphism in a neighborhood of a hyperbolic set (its proof can be easily reduced to Theorem 6.4.9 in the book [28]).

Theorem 2.2.6 *Let Λ be a hyperbolic set of a diffeomorphism f with hyperbolicity constants C, λ. For any $C_1 > C$ and $\lambda_1 \in (\lambda, 1)$ there exists a neighborhood U of Λ with the following property. If $x \in W^s(p)$, $p \in \Lambda$, and $f^k(x) \in U$ for $k \geq 0$,* *then there exist two complementary linear subspaces* $L^+(x)$ *and* $L^-(x)$ *of* T_xM *such that*

(1)

$$L^+(x) = T_xW^s(p),\ L^-(x) = T_xW^u(p);$$

(2)

$$\left|Df^k(x)v\right| \le C_1\lambda_1^k|v|, \quad k \ge 0,\ v \in L^+(x),$$

and

$$\left|Df^k(x)v\right| \ge (1/C_1)\lambda_1^{-k}|v|, \quad k \ge 0,\ v \in L^-(x).$$

Remark 2.2.5 Of course, a similar statement holds if $x \in W^u(p)$, $p \in \Lambda$, and $f^k(x)$ belongs to a small neighborhood of Λ for $k \le 0$.

Clearly, it is enough for us to prove that if $r \in W^s(p) \cap W^u(q)$, where $p, q \in \Omega(f)$, then

$$B^+(r) \subset T_rW^s(p) \text{ and } B^-(r) \subset T_rW^u(q). \tag{2.50}$$

We prove the first inclusion in (2.50) by proving that

$$B^+(r) \subset L^+(r) \tag{2.51}$$

and applying Theorem 2.2.6; the second inclusion is proved in a similar way.

Any trajectory of a diffeomorphism satisfying Axiom A tends to one of the basic sets as time tends to $\pm\infty$ (see Theorem 1.3.2).

Take as Λ the basic set to which $f^k(r)$ tends as $k \to \infty$; obviously, p belongs to this basic set. Of course, we may assume that the positive semitrajectory of r belongs to a neighborhood of Λ having the properties described in Theorem 2.2.6.

Assume that inclusion (2.51) does not hold; take $v \in B^+(r) \setminus L^+(r)$ and represent

$$v = v^s + v^u, \quad v^s \in L^+(r),\ v^u \in L^-(r);$$

then $v^u \ne 0$.

Then

$$\left|Df^kv\right| \ge \left|Df^kv_u\right| - \left|Df^kv_s\right| \ge (1/C_1)\lambda^{-k}|v^u| - C_1\lambda^k|v^s| \to \infty, \quad k \to \infty,$$

which contradicts the relation defining $B^+(r)$.

We have completely proved the Mañé theorem.

Historical Remarks In his paper [39], R. Mañé gave several equivalent characterizations of structural stability of a diffeomorphism; Theorem 1.3.7 of this book is just one of them.

The property of expansivity of a dynamical system with discrete time is now one of the classical properties studied in the global theory of dynamical systems. Theorem 2.2.4 is folklore. Let us mention J. Ombach's paper [49] in which it was shown (see Proposition 9) that a compact invariant set Λ of a diffeomorphism f is hyperbolic if and only if $f|_\Lambda$ is expansive and has the (standard) shadowing property (compare with Sect. 4.1).

Theorem 2.2.5 was proved in G. Birkhoff's book [10].

2.3 Diffeomorphisms with Lipschitz Shadowing

Our main result in this section is as follows.

Theorem 2.3.1 *If a diffeomorphism of class C^1 of a smooth closed n-dimensional manifold M has the Lipschitz shadowing property, then f is structurally stable.*

As stated in Theorem 1.4.1 (1), a structurally stable diffeomorphism f has the Lipschitz shadowing property. Combining this statement with Theorem 2.3.1, we conclude that for diffeomorphisms, structural stability is equivalent to Lipschitz shadowing.

Proof (of Theorem 2.3.1) Let us first explain the main idea of the proof.

Fix an arbitrary point $p \in M$, consider its trajectory $\{p_k = f^k(p) : k \in \mathbb{Z}\}$, and denote $A_k = Df(p_k)$. Consider the sequence $\mathscr{A} = \{A_k : k \in \mathbb{Z}\}$.

In Sect. 2.1 devoted to the Maizel' and Pliss theorems, we worked with sequences $\mathscr{A}$ of isomorphisms of Euclidean spaces. Here we apply these theorems (and all the corresponding notions of the Perron property etc.) to the sequences $\mathscr{A} = \{Df(p_k)\}$ (see the remark concluding Sect. 3.1).

We claim that if f has the Lipschitz shadowing property, then $\mathscr{A}$ has the Perron property on $\mathbb{Z}$.

By the Maizel' theorem, the Perron property on $\mathbb{Z}$ implies that the sequence $\mathscr{A}$ is hyperbolic on both "rays" $\mathbb{Z}_-$ and $\mathbb{Z}_+$. Denote by $S_k^-, U_k^-, k \in \mathbb{Z}_-$ and $S_k^+, U_k^+, k \in \mathbb{Z}_+$ the corresponding stable and unstable subspaces.

Then, by the Pliss theorem, the subspaces U_0^- and S_0^+ are transverse.

Clearly,

$$|A_k \circ \cdots \circ A_0 v| \to 0, \quad v \in S_0^+, k \to \infty,$$

and

$$\left|(A_k)^{-1} \circ \cdots \circ (A_0)^{-1} v\right| \to 0, \quad v \in U_0^-, k \to -\infty,$$

which means that $U_0^- \subset B^-(p)$ and $S_0^+ \subset B^+(p)$, where $B^-(p)$ and $B^+(p)$ are the subspaces from the analytic transversality condition.

The transversality of the subspaces U_0^- and S_0^+ implies the transversality of the subspaces $B^-(x)$ and $B^+(x)$. Since x is arbitrary, f is structurally stable by the Mañé theorem.

Now we prove our claim.

To clarify the reasoning, we first prove an analog of this result, Lemma 2.3.2, for a diffeomorphism of the Euclidean space $\mathbb{R}^n$. Of course, $\mathbb{R}^n$ is not compact, but we avoid the appearing difficulty making the following additional assumption (and noting that an analog of this assumption is certainly valid for a diffeomorphism of class C^1 of a closed smooth manifold). We call the condition below *Condition S*.

Thus, we assume that for any $\mu > 0$ we can find a $\delta = \delta(\mu) > 0$ (independent of k) such that if $|v| \le \delta$, then

$$|f(p_k + v) - A_k v - p_{k+1}| \le \mu |v|, \quad k \in \mathbb{Z}. \tag{2.52}$$

The basic technical part of the proof of Lemma 2.3.2 is the following statement (Lemma 2.3.1). In the following two Lemmas, 2.3.1 and 2.3.2, f is a diffeomorphism of $\mathbb{R}^n$ that has the Lipschitz shadowing property with constants $\mathscr{L}, d_0 > 0$, $\{p_k = f^k(p)\}$ is an arbitrary trajectory of f, $A_k = Df(p_k)$, and it is assumed that Condition S is satisfied.

Lemma 2.3.1 *Fix a natural number N. For any sequence*

$$w_k \in \mathbb{R}^n, \quad k \in \mathbb{Z},$$

with $|w_k| < 1$ there exists a sequence

$$z_k \in \mathbb{R}^n, \quad k \in \mathbb{Z},$$

such that

$$|z_k| \le \mathscr{L} + 1, \quad k \in \mathbb{Z}, \tag{2.53}$$

and

$$z_{k+1} = A_k z_k + w_{k+1}, \quad -N \le k \le N. \tag{2.54}$$

Proof Thus, we assume that f has the Lipschitz shadowing property with constants $\mathscr{L}, d_0 > 0$.

Define vectors

$$\Delta_k \in \mathbb{R}^n, \quad -N \le k \le N + 1,$$

by the following relations:

$$\Delta_{-N} = 0 \quad \text{and} \quad \Delta_{k+1} = A_k \Delta_k + w_{k+1}, \quad -N \le k \le N. \tag{2.55}$$

Clearly, there exists a number Q (depending on N, $\mathscr{A}$, and w_k) such that

$$|\Delta_k| \le Q, \quad -N \le k \le N + 1. \tag{2.56}$$

Fix a small number $d \in (0, d_0)$ (we will reduce this number during the proof) and consider the following sequence $\xi = \{x_k \in \mathbb{R}^n : \ k \in \mathbb{Z}\}$:

$$x_k = \begin{cases} f^{k+N}(p_{-N}), & k < -N; \\ p_k + d\Delta_k, & -N \le k \le N+1; \\ f^{k-N-1}(p_{N+1} + d\Delta_{N+1}), & k > N+1. \end{cases}$$

Note that if $-N \le k \le N$, then

$$|x_{k+1} - f(x_k)| = |p_{k+1} + d\Delta_{k+1} - f(p_k + d\Delta_k)| \le$$

$$\le d\,|\Delta_{k+1} - A_k\Delta_k| + |f(p_k + d\Delta_k) - p_{k+1} - dA_k\Delta_k|\,.$$

Since we consider a finite number of w_k, the condition $|w_k| < 1$ implies that there is a $\mu \in (0, 1)$ such that the first term above does not exceed μd; by Condition S, the second term is less than $(1 - \mu)d$ if d is small. Hence, in this case, the sum is less than d.

For the remaining values of k,

$$|x_{k+1} - f(x_k)| = 0.$$

Thus, we may take $d \le d_0$ so small that ξ is a d-pseudotrajectory of f. Then there exists a trajectory $\eta = \{y_k : \ k \in \mathbb{Z}\}$ of f such that

$$|x_k - y_k| \le \mathscr{L}d, \quad k \in \mathbb{Z}. \tag{2.57}$$

Denote $t_k = (y_k - p_k)/d$. Since $\Delta_k = (x_k - p_k)/d$, it follows from (2.57) that

$$|\Delta_k - t_k| = |x_k - y_k|/d \le \mathscr{L}, \quad k \in \mathbb{Z}. \tag{2.58}$$

It follows from (2.56) and (2.57) that

$$|y_k - p_k| \le |y_k - x_k| + |x_k - p_k| \le (\mathscr{L} + Q)d, \quad k \in \mathbb{Z}.$$

Hence,

$$|t_k| \le \mathscr{L} + Q, \quad k \in \mathbb{Z}. \tag{2.59}$$

Now we define a finite sequence

$$b_k \in \mathbb{R}^n, \quad -N \le k \le N+1,$$

by the following relations:

$$b_{-N} = t_{-N} \quad \text{and} \quad b_{k+1} = A_k b_k, \quad -N \le k \le N. \tag{2.60}$$

Take $\mu_1 \in (0,1)$ such that

$$\left((K+1)^{2N} + (K+1)^{2N-1} + \cdots + 1\right)\mu_1 < 1, \tag{2.61}$$

where $K = \sup \|A_k\|$. Set

$$\mu = \frac{\mu_1}{\mathscr{L} + Q}$$

and consider d so small that inequality (2.52) holds for $|v| \le \delta$ with $\delta = (\mathscr{L} + Q)d$.

The definition of the vectors t_k implies that

$$dt_{k+1} = y_{k+1} - p_{k+1} = f(y_k) - f(p_k) = f(p_k + dt_k) - f(p_k).$$

Since $|dt_k| \le (\mathscr{L} + Q)d$ by (2.59), it follows from Condition S and from the above choice of d that

$$|dt_{k+1} - dA_k t_k| = |f(p_k + dt_k) - f(p_k) - dA_k t_k| \le$$

$$\le \mu |dt_k| \le \mu(\mathscr{L} + Q)d = \mu_1 d.$$

Hence,

$$t_{k+1} = A_k t_k + \theta_k, \quad \text{where} \quad |\theta_k| < \mu_1. \tag{2.62}$$

Consider the vectors

$$c_k = t_k - b_k.$$

Note that $c_{-N} = 0$ by (2.60) and

$$c_{k+1} = A_k c_k + \theta_k, \quad \text{where} \quad |\theta_k| < \mu_1$$

by (2.62).

Thus,

$$|c_{-N+1}| \le |\theta_{-N}| < \mu_1,$$

$$|c_{-N+2}| \le |A_{-N+1}c_{-N+1} + \theta_{-N+1}| \le (K+1)\mu_1,$$

and so on, which implies the estimate

$$|c_k| \le \left((K+1)^{2N} + (K+1)^{2N-1} + \cdots + 1\right)\mu_1 < 1, \quad -N \le k \le N.$$

Hence,

$$|t_k - b_k| \le 1, \quad -N \le k \le N. \tag{2.63}$$

Finally, we consider the sequence

$$z_k = \begin{cases} 0, & k < -N; \\ \Delta_k - b_k, & -N \le k \le N+1; \\ 0, & k > N+1. \end{cases}$$

Relations (2.55) and (2.60) imply relations (2.54); estimates (2.58) and (2.63) imply estimate (2.53). □

Lemma 2.3.2 *The sequence $\mathscr{A} = \{A_k\}$ has the Perron property.*

Proof Take an arbitrary sequence

$$w_k \in \mathbb{R}^n, \quad k \in \mathbb{Z},$$

with $|w_k| < 1$ and prove that an analog of Eq. (2.54) has a solution

$$z_k \in \mathbb{R}^n, \quad k \in \mathbb{Z},$$

with

$$|z_k| \le \mathscr{L} + 1, \quad k \in \mathbb{Z}.$$

Fix a natural N and consider the sequence

$$w_k^{(N)} = \begin{cases} w_k, & -N \le k \le N; \\ 0, & |k| \ge N+1. \end{cases}$$

By Lemma 2.3.1, there exists a sequence $\left\{z_k^{(N)},\ k \in \mathbb{Z}\right\}$ such that

$$z_{k+1}^{(N)} = A_k z_k^{(N)} + w_k^{(N)}, \quad -N \le k \le N, \tag{2.64}$$

and

$$\left|z_k^{(N)}\right| \le \mathscr{L} + 1, \quad k \in \mathbb{Z}. \tag{2.65}$$

Passing to a subsequence of $\left\{z_k^{(N)}\right\}$, we can find a sequence $\{v_k\}$ such that

$$v_k = \lim_{N\to\infty} z_k^{(N)}, \quad k \in \mathbb{Z}.$$

(Note that do not assume uniform convergence.) Passing to the limit in (2.64) and (2.65) as $N \to \infty$, we see that

$$v_{k+1} = A_k y_k + w_k, \quad k \in \mathbb{Z},$$

and

$$|v_k| \le \mathscr{L} + 1, \quad k \in \mathbb{Z}.$$

Thus, we have shown that the sequence $\mathscr{A}$ has the Perron property. □

Now let us explain how to prove the required statement in the case of a smooth closed manifold M.

Lemma 2.3.3 *If a diffeomorphism of class C^1 of a smooth closed n-dimensional manifold M has the Lipschitz shadowing property, $\left\{p_k = f^k(p)\right\}$ is an arbitrary trajectory of f, and $A_k = Df(p_k)$, then the sequence $\mathscr{A} = \{A_k\}$ has the Perron property.*

Proof Let exp be the standard exponential mapping on the tangent bundle of M generated by the fixed Riemannian metric dist. Let

$$\exp_x : T_xM \to M$$

be the corresponding exponential mapping at a point $x \in M$.

Denote (just for this proof) by $B(r,x)$ the ball in M of radius r centered at a point x; let $B_T(r,x)$ be the ball in T_xM of radius r centered at the origin.

It is well known that there exists an $r > 0$ such that for any $x \in M$, $\exp_x$ is a diffeomorphism of $B_T(r,x)$ onto its image and $\exp_x^{-1}$ is a diffeomorphism of $B(r,x)$ onto its image; in addition, $D\exp_x(0) = \mathrm{Id}$.

Thus, we may assume that r is chosen so that the following inequalities hold for any $x \in M$:

$$\operatorname{dist}(\exp_x(v), \exp_x(w)) \le 2|v - w|, \quad v, w \in B_T(r,x), \tag{2.66}$$

and

$$\left|\exp_x^{-1}(y) - \exp_x^{-1}(z)\right| \le 2\operatorname{dist}(y,z), \quad y, z \in B(r,x). \tag{2.67}$$

These inequalities mean that distances are distorted not more than twice when we pass from the manifold to its tangent space or from the tangent space to the manifold (if we work in a small neighborhood of a point of the manifold or in a small neighborhood of the origin of the tangent space).

In our reasoning below, we always assume that d is so small that the corresponding points belong to such small neighborhoods.

Now we fix a trajectory $\{p_k = f^k(p)\}$ of our diffeomorphism f and introduce the mappings

$$F_k = \exp_{p_{k+1}}^{-1} \circ f \circ \exp_{p_k} : \; T_{p_k}M \to T_{p_{k+1}}M.$$

Clearly,

$$DF_k(0) = A_k.$$

The analog of Condition S is as follows: For any $\mu > 0$ we can find a $\delta > 0$ (independent of k) such that if $|v| < \delta$, then

$$|F_k(v) - A_k v| \le \mu |v|, \quad k \in \mathbb{Z}. \tag{2.68}$$

Of course, this condition is satisfied automatically since f is of class C^1 and the manifold M is compact.

To prove that the sequence $\mathscr{A}$ has the Perron property, let us consider the difference equations

$$v_{k+1} = A_k y_k + w_k, \quad k \in \mathbb{Z}, \tag{2.69}$$

where $v_k \in T_{p_k}M$ and $w_k \in T_{p_{k+1}}M$.

We assume that $|w_k| < 1, \; k \in \mathbb{Z}$. Let us "translate" the reasoning of Lemma 2.3.1 to the "manifold language."

We fix a natural N and consider the sequence

$$\Delta_k \in T_{p_k}M, \quad -N \le k \le N+1,$$

defined by relations (2.55). Let Q satisfy (2.56).

We fix a small d and define the sequence $\xi = \{x_k \in M : \; k \in \mathbb{Z}\}$ by

$$x_k = \begin{cases} f^{k+N}(p_{-N}), & k < -N; \\ \exp_{p_k}(d\Delta_k), & -N \le k \le N+1; \\ f^{k-N-1}(\exp_{p_{N+1}}(d\Delta_{N+1})), & k > N+1. \end{cases}$$

This definition and inequalities (2.66) imply that if d is small enough, then

$$\operatorname{dist}\left(x_{k+1}, \exp_{p_{k+1}}(dA_k\Delta_k)\right) < 2d.$$

Since

$$f(x_k) = \exp_{p_{k+1}}(F_k(d\Delta_k)),$$

condition (2.68) with $\mu < 1$ implies that

$$\text{dist}\left(\exp_{p_{k+1}}(dA_k\Delta_k), f(x_k)\right) < 2d,$$

and we see that

$$\text{dist}\,(f(x_k), x_{k+1}) < 4d.$$

Thus, there exists an exact trajectory $\eta = \{y_k : k \in \mathbb{Z}\}$ of f such that

$$\text{dist}(x_k, y_k) \le 4\mathscr{L}d, \quad k \in \mathbb{Z}. \tag{2.70}$$

Now we consider the finite sequence

$$t_k = \frac{1}{d}\exp_{p_k}^{-1}(y_k), \quad -N \le k \le N.$$

Inequalities (2.70) and (2.67) imply that

$$|\Delta_k - t_k| \le 8\mathscr{L}, \quad k \in \mathbb{Z}. \tag{2.71}$$

Note that

$$\text{dist}(y_k, p_k) \le \text{dist}(y_k, x_k) + \text{dist}(x_k, p_k) \le (4\mathscr{L} + 2Q)d, \quad k \in \mathbb{Z}.$$

Hence,

$$|t_k| \le 8\mathscr{L} + 4Q, \quad k \in \mathbb{Z}.$$

Now we define a finite sequence

$$b_k \in T_{p_k}M, \quad -N \le k \le N + 1,$$

by relations (2.60) and repeat the reasoning of Lemma 2.3.1 with

$$\mu = \frac{\mu_1}{8\mathscr{L} + 4Q},$$

where μ_1 is the same as above (see relation (2.61)).

The rest of the proof is literally the same (with natural replacement of $\mathbb{R}^n$ by the corresponding tangent spaces), and we get the relation

$$|t_k - b_k| < 1$$

similar to (2.63).

Finally, we get the estimate

$$|z_k| \leq 8\mathscr{L} + 1,$$

which completes the proof of the analog of Lemma 2.3.1.

The rest of the proof of the implication "Lipschitz shadowing property implies the Perron property of the sequence $\mathscr{A}$" almost literally repeats the proof of Lemma 2.3.2. □

Historical Remarks Theorem 2.3.1 was published by the first author and S. B. Tikhomirov in the paper [68]. Let us mention that the paper [67] contained the first proof of the fact that structural stability follows from certain shadowing property based on a combination of the Maizel', Pliss, and Mañé theorems.

2.4 Lipschitz Periodic Shadowing for Diffeomorphisms

The main result of this section is as follows.

Theorem 2.4.1 *A diffeomorphism f of class C^1 of a smooth closed n-dimensional manifold M has the Lipschitz periodic shadowing property if and only if f is Ω-stable.*

First we prove the "if" statement of Theorem 2.4.1.

Theorem 2.4.2 *If a diffeomorphism f is Ω-stable, then f has the Lipschitz periodic shadowing property.*

Let us give one more definition.

Definition 2.4.1 We say that a diffeomorphism f has the *Lipschitz shadowing property* on a set U if there exist positive constants $\mathscr{L}, d_0$ such that if $\xi = \{x_i : i \in \mathbb{Z}\} \subset U$ is a d-pseudotrajectory with $d \leq d_0$, then there exists a point $p \in U$ such that inequalities (1.5) hold.

Remark 2.4.1 It follows from Theorems 1.4.2 and 2.2.4 that we can find a neighborhood U of a hyperbolic set Λ of a diffeomorphism f having the above-formulated property and such that f is expansive on U.

We start by proving several auxiliary results.

Lemma 2.4.1 *Let f be a homeomorpism of a compact metric space $(M, dist)$. For any neighborhood U of the nonwandering set $\Omega(f)$ there exist positive numbers*

T, δ_1 such that if $\xi = \{x_i : i \in \mathbb{Z}\}$ is a d-pseudotrajectory of f with $d \le \delta_1$ and

$$x_k, x_{k+1}, \dots, x_{k+l} \notin U$$

for some $k \in \mathbb{Z}$ and $l > 0$, then $l \le T$.

Proof Take a neighborhood U of the nonwandering set $\Omega(f)$ and let T be the Birkhoff constant for the homeomorphism f given for this neighborhood by Theorem 2.2.5. Assume that there does not exist a number δ_1 with the desired property; then there exists a sequence $d_j \to 0$ as $j \to \infty$ and a sequence of d_j-pseudotrajectories $\{x_k^{(j)} : k \in \mathbb{Z}\}$ of f such that

$$\left\{x_k^{(j)} : 0 \le k \le T-1\right\} \cap U = \emptyset$$

for all j.

The set $M' = M \setminus U$ is compact. Passing to a subsequence, if necessary, we may assume that $x_0^{(j)} \to x_0$ as $j \to \infty$. In this case,

$$x_k^{(j)} \to f^k(x_0) \in M', \quad 0 \le k \le T-1,$$

and we get a contradiction with the choice of T. □

Now let us recall some basic properties of Ω-stable diffeomorphisms. It was noted in Sect. 1.3 that a diffeomorphism f is Ω-stable if and only if f satisfies Axiom A and the no cycle condition (Theorem 1.3.3).

Let $\Omega_1, \dots, \Omega_m$ be the basic sets in decomposition (1.15) of the nonwandering set of an Ω-stable diffeomorphism f.

Below we need one folklore technical statement. Recall that we write $\Omega_i \to \Omega_j$ if there is a point $x \notin \Omega(f)$ such that

$$f^{-k}(x) \to \Omega_i \text{ and } f^k(x) \to \Omega_j, \quad k \to \infty.$$

Theorem 2.4.3 *Assume that a diffeomorphism f is Ω-stable. For any family of neighborhoods U_i of the basic sets Ω_i one can find neighborhoods $V_i \subset U_i$ such that if a point x belongs to some V_i and there exist indices $0 < l \le m$ such that*

$$f^l(x) \notin U_i \text{ and } f^m(x) \in V_j,$$

then there exist basic sets $\Omega_{i_1}, \dots, \Omega_{i_t}$ such that

$$\Omega_i \to \Omega_{i_1} \to \dots \to \Omega_{i_t} \to \Omega_j. \tag{2.72}$$

Proof Reducing the given neighborhoods U_i, we may assume that the compact sets $U_i' = f(\mathrm{Cl}(U_i)) \cup \mathrm{Cl}(U_i)$ are disjoint.

Assume that our statement does not hold. In this case, there exist sequences of points x_k, $k \geq 0$, and indices $l(k) \leq m(k)$ such that

$$x_k \to \Omega_i, \quad f^{l(k)}(x_k) \notin U_i, \quad f^{m(k)}(x_k) \to \Omega_i, \quad k \to \infty.$$

Clearly, we may assume that

$$x_k, f(x_k), \dots, f^{l(k)-1}(x_k) \in U_i$$

while

$$y_k := f^{l(k)}(x_k) \notin U_i.$$

Then $y_k \in U_i'$, and, passing to a subsequence, if necessary, we may assume that $y_k \to y \in U_i'$ as $k \to \infty$.

Since Ω_i is a compact f-invariant set, $l(k) \to \infty$ as $k \to \infty$. Thus, for any $t < 0$, $f^t(y_k) \in U_i$ for large k, and it follows that $f^t(y) \in \mathrm{Cl}(U_i)$ for any $t < 0$. We note that the set $\mathrm{Cl}(U_i)$ intersects a single basic set, Ω_i, and refer to (1.16) to conclude that

$$y \in W^u(\Omega_i). \tag{2.73}$$

By the same relation (1.16), there exists a basic set Ω_{i_1} such that

$$y \in W^s(\Omega_{i_1}). \tag{2.74}$$

By our choice of U_i, the sets $\mathrm{Cl}(f(U_i)) \setminus U_i$ do not contain nonwandering points. Thus, if $i_1 = i$, inclusions (2.73) and (2.74) mean the existence of a 1-cycle, and we get the desired contradiction.

Hence, $i_1 \neq i$ and $\Omega_i \to \Omega_{i_1}$. Consider the compact set

$$Y = \left\{f^k(y) : k \geq 0\right\} \cup \Omega_{i_1}.$$

Clearly, the set Y has a neighborhood Z such that $U_{i_1} \subset Z$ and Z does not intersect a small neighborhood of Ω_i.

Since $y_k = f^{l_k}(x_k) \to y$, there exist indices $l_1(k)$ such that

$$f^t(y_k) = f^{l(k)+t}(x_k) \in Z, \quad 0 \leq t \leq l_1(k),$$

for large k, and

$$x_{1,k} = f^{l_1(k)}(y_k) = f^{l(k)+l_1(k)}(x_k) \to \Omega_{i_1}, \quad k \to \infty.$$

At the same time, the positive trajectories of the points y_k (and hence, of the points $x_{1,k}$) must leave Z (and hence, U_{i_1}) since the sequence

$$f^{m(k)-l(k)}(y_k) = f^{m(k)}(x_k)$$

tends to Ω_i.

Thus, we can repeat the above reasoning with the points $x_{1,k}$ and the basic set Ω_{i_1} instead of x_k and Ω_i.

Such a process will produce basic sets $\Omega_{i_1}, \Omega_{i_2}, \dots$ such that

$$\Omega_i \to \Omega_{i_1} \to \Omega_{i_2} \to \dots.$$

Since f has no cycles, this process is finite, and, as a result, we conclude that there exist basic sets $\Omega_{i_1}, \dots, \Omega_{i_t}$ such that relations (2.72) hold. □

Now we apply the above theorem to prove a statement concerning periodic pseudotrajectories of Ω-stable diffeomorphisms.

Lemma 2.4.2 *Assume that a diffeomorphism f is Ω-stable. For any family of disjoint neighborhoods W_i of the basic sets Ω_i there exists a number $\delta_2 > 0$ such that any periodic d-pseudotrjectory ξ of f with $d \le \delta_2$ belongs to a single neighborhood W_i.*

Proof Fix arbitrary disjoint neighborhoods W_i of the basic sets Ω_i and find a number $\varepsilon > 0$ and neighborhoods U_i of Ω_i such that

$$N(\varepsilon, U_i) \subset W_i, \quad i = 1, \dots, m.$$

Apply Theorem 2.4.3 to find for U_i the corresponding neighborhoods V_i of Ω_i. Reducing ε, if necessary, we can find neighborhoods V_i' of Ω_i such that

$$N(\varepsilon, V_i') \subset V_i, \quad i = 1, \dots, m.$$

By Lemma 2.4.1, there exist positive numbers T, δ_1 such that if $\xi = \{x_k\}$ is a d-pseudotrajectory of f with $d \le \delta_1$ and

$$x_k, x_{k+1}, \dots, x_{k+l} \notin V := \bigcup_{i=1}^{m} V_i'$$

for some $k \in \mathbb{Z}$ and $l > 0$, then $l \le T$.

Find a number $\delta_2 \in (0, \delta_1)$ such that if $\xi = \{x_k\}$ is a d-pseudotrajectory of f with $d \le \delta_2$, then

$$\operatorname{dist}(f^l(x_k), x_{k+l}) < \varepsilon, \quad 0 \le l \le T + 1,$$

for any $k \in \mathbb{Z}$.

Now let $\xi = \{x_k\}$ be a periodic d-pseudotrajectory of f of period μ with $d \le \delta_2$.

Let us call a V-block of ξ a finite segment

$$\xi_{k,m} = \{x_k, x_{k+1}, \dots, x_{k+m}\}, \quad k \in \mathbb{Z},\ m > 0,$$

such that $x_k, x_{k+m} \in V$ while $x_{k+l} \notin V$ for $0 < l < m$. Note that in this case, $m \le T + 1$.

Let us note simple properties of V-blocks.

It follows from the choice of δ_2 that if $\xi_{k,m}$ is a V-block for which there exist indices $i, j \in \{1, \dots, m\}$ such that $x_k \in V_i'$ and $x_{k+m} \in V_j'$, then $\mathrm{dist}(f^m(x_k), x_{k+m}) < \varepsilon$; hence, $f^m(x_k) \in V_j$.

At the same time, if for such a V-block there exists an index $l \in (0, m)$ such that $x_{k+l} \notin W_i$, then $\mathrm{dist}(f^l(x_k), x_{k+l}) < \varepsilon$; hence, $f^l(x_k) \notin U_i$.

It follows from Theorem 2.4.3 that in this case, there exists a relation of the form (2.72); the absence of cycles implies that $j \ne i$.

Since $\delta_2 < \delta_1$, there exists a neighborhood V_i' such that ξ intersects V_i'.

Changing indices of ξ, we may assume that $x_0 \in V_i'$.

If either $x_k \in W_i$ for $k \ge 0$ or any V-block $\xi_{k,m}$ with $k \ge 0$ belongs to W_i, then the statement of our lemma follows from the periodicity of ξ.

It was noted above that if $\xi_{k,m}$ be a V-block with $x_k \in V_j$ for $k \ge 0$ for which there exists an index $l \in (0, m)$ such that $x_{k_l} \notin W_j$, then there exists an index $j' \ne j$ for which we have a relation

$$\Omega_j \to \cdots \to \Omega_{j'}$$

of the form (2.72).

Thus, if we assume that there exists a V-block $\xi_{k,m}$ with $k \ge 0$ such that $\xi_{k,m} \setminus W_i \ne \emptyset$, then we get an index $j_1 \ne i$ such that we have a relation

$$\Omega_i \to \cdots \to \Omega_{j_1}$$

of the form (2.72).

Going to "the right" of this V-block $\xi_{k,m}$ and continuing this process, we construct a sequence of pairs of indices $(i, j_1), (j_1, j_2), \dots$ such that

$$\Omega_i \to \cdots \to \Omega_{j_1}, \quad \Omega_{j_1} \to \cdots \to \Omega_{j_2}, \quad \dots .$$

In this case, it follows from the absence of cycles that all the indices $i, j_1, j_2, \dots$ are different.

But the μ-periodicity of ξ implies that if $\xi_{k,m}$ is a V-block and n is a natural number, then $\xi_{k+n\mu,m}$ is an identical V-block, and the existence of the above sequence with different $i, j_1, j_2, \dots$ is impossible.

Now we prove Theorem 2.4.2.

By Remark 2.4.1, there exist disjoint neighborhoods $U_1, \dots, U_m$ of the basic sets $\Omega_1, \dots, \Omega_m$ such that

(i) f has the Lipschitz shadowing property on any of U_j with the same constants $\mathscr{L}, d_0^*$;
(ii) f is expansive on any of U_j with the same expansivity constant a.

Find neighborhoods W_j of Ω_j (and reduce d_0^*, if necessary) so that the $\mathscr{L} d_0^*$-neighborhoods of W_j belong to U_j. Apply Lemma 2.4.2 to find the corresponding constant δ_2.

We claim that f has the Lipschitz periodic shadowing property with constants $\mathscr{L}, d_0$, where

$$d_0 = \min\left(d_0^*, \delta_2, \frac{a}{2\mathscr{L}}\right).$$

Take a μ-periodic d-pseudotrajectory $\xi = \{x_k\}$ of f with $d \le d_0$. Lemma 2.4.2 implies that there exists a neighborhood W_i such that $\xi \subset W_i \subset U_i$.

Thus, there exists a point p such that inequalities (1.5) hold. Let us show that p is a periodic point of f. By the choice of U_i and W_i, $f^k(p) \in U_i$ for all $k \in \mathbb{Z}$. Let $q = f^\mu(p)$. Inequalities (1.5) and the periodicity of ξ imply that

$$\operatorname{dist}\left(f^k(q), x_k\right) = \operatorname{dist}\left(f^{k+\mu}(p), x_k\right) = \operatorname{dist}\left(f^{k+\mu}(p), x_{k+\mu}\right) \le \mathscr{L} d, \quad k \in \mathbb{Z}.$$

Thus,

$$\operatorname{dist}\left(f^k(q), f^k(p)\right) \le 2\mathscr{L} d \le a, \quad k \in \mathbb{Z},$$

which implies that $f^\mu(p) = q = p$. This completes the proof. □

Now we prove the "only if" statement of Theorem 2.4.1.

Theorem 2.4.4 *If a diffeomorphism f has the Lipschitz periodic shadowing property, then f is Ω-stable.*

Thus, let us assume that f has the Lipschitz periodic shadowing property (with constants $\mathscr{L} \ge 1, d_0 > 0$). Clearly, in this case f^{-1} has the Lipschitz periodic shadowing property as well (and we assume that the constants $\mathscr{L}, d_0$ are the same for f and f^{-1}).

To clarify the presentation, in the construction of pseudotrajectories in the following Lemmas 2.4.3 and 2.4.4, we assume that f is a diffeomorphism of $\mathbb{R}^n$ (and leave to the reader consideration of the case of a manifold).

We also assume that there exists a number $N > 0$ such that $\|Df(x)\| \le N$ for all considered points x (an analog of this assumption is satisfied in the case of a closed manifold).

Recall that we denote by Per(f) the set of periodic points of f.

Lemma 2.4.3 *Every point $p \in Per(f)$ is hyperbolic.*

Proof To get a contradiction, let us assume that f has a nonhyperbolic periodic point p (to simplify notation, we assume that p is a fixed point; literally the same reasoning can be applied to a periodic point of period $m > 1$). In addition, we assume that $p = 0$.

In this case, we can represent

$$f(v) = Av + F(v),$$

where $A = Df(0)$ and $F(v) = o(v)$ as $v \to 0$.

By our assumption, A is a nonhyperbolic matrix. The following two cases are possible:

Case 1: A has a real eigenvalue λ with $|\lambda| = 1$;
Case 2: A has a complex eigenvalue λ with $|\lambda| = 1$.

We treat in detail only Case 1 and give a comment concerning Case 2. To simplify presentation, we assume that 1 is an eigenvalue of A; the case of eigenvalue -1 is treated similarly.

We can introduce coordinate v such that, with respect to this coordinate, the matrix A has block-diagonal form,

$$A = \operatorname{diag}(B, P), \tag{2.75}$$

where B is a Jordan block of size $l \times l$:

$$B = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Of course, introducing new coordinates, we have to change the constants $\mathcal{L}$ and d_0; we denote the new constants by the same symbols. In addition, we assume that $\mathcal{L}$ is integer.

We start considering the case $l = 2$; in this case,

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$e_1 = (1, 0, 0, \dots, 0) \text{ and } e_2 = (0, 1, 0, \dots, 0)$$

be the first two vectors of the standard orthonormal basis.

Let $K = 7\mathcal{L}$.

Take a small $d > 0$ and construct a finite sequence $y_0, \dots, y_Q$ of points (where Q is determined later) as follows: $y_0 = 0$ and

$$y_{k+1} = Ay_k + de_2, \quad k = 0, \dots, K-1. \tag{2.76}$$

Then

$$y_K = (Z_1(K)d, Kd, 0, \dots, 0),$$

where the natural number $Z_1(K)$ is determined by K (we do not write $Z_1(K)$ explicitly). Now we set

$$y_{k+1} = Ay_k - de_2, \quad k = K, \dots, 2K-1.$$

Then

$$y_{2K} = (Z_2(K)d, 0, 0, \dots, 0),$$

where the natural number $Z_2(K)$ is determined by K as well. Take $Q = 2K + Z_2(K)$; if we set

$$y_{k+1} = Ay_k - de_1, \quad k = 2K, \dots, Q-1,$$

then $y_Q = 0$. Let us note that both numbers Q and

$$Y := \frac{\max_{0 \le k \le Q-1} |y_k|}{d}$$

are determined by K (and hence, by $\mathscr{L}$).

Now we construct a Q-periodic sequence x_k, $k \in \mathbb{Z}$, that coincides with the above sequence for $k = 0, \dots, Q$.

We claim that if d is small enough, then $\xi = \{x_k\}$ is a $2d$-pseudotrajectory of f (and this pseudotrajectory is Q-periodic by construction).

Indeed, we know that $|x_k| \le Yd$ for $k \in \mathbb{Z}$. Since $F(v) = o(|v|)$ as $|v| \to 0$,

$$|F(x_k)| < d, \quad k \in \mathbb{Z}, \tag{2.77}$$

if d is small enough.

The definition of x_k implies that

$$|x_{k+1} - Ax_k| = d, \quad k \in \mathbb{Z}. \tag{2.78}$$

It follows from (2.77) and (2.78) that

$$|x_{k+1} - f(x_k)| \le |x_{k+1} - Ax_k| + |F(x_k)| < 2d,$$

which implies that $\xi = \{x_k\}$ is a $2d$-pseudotrajectory of f if d is small enough.

Now we estimate the distances between points of trajectories of the diffeomorphism f and its linearization at zero.

Let us take a vector p_0 and assume that the sequence $p_k = f^k(p_0)$ belongs to the ball $|v| \le (Y + 2\mathscr{L})d$ for $0 \le k \le K$. Let $r_k = A^k p_0$ (we impose no conditions on r_k since below we estimate F at points q_k only).

Take a small number $\mu \in (0, 1)$ (to be chosen later) and assume that d is small enough, so that the inequality

$$|F(v)| \le \mu |v|$$

holds for $|v| \le (Y + 2\mathscr{L})d$.

By our assumption, $\|A\| = \|Df(0)\| \le N$. Then

$$|p_1| \le |Ap_0| + |F(p_0)| \le (N + 1)|p_0|, \dots,$$

$$|p_k| \le |Ap_{k-1}| + |F(p_{k-1})| \le (N + 1)^k |p_0|$$

for $1 \le k \le K$, and

$$|p_1 - r_1| = |Ap_0 + F(p_0) - Ap_0| \le \mu |p_0|,$$

$$|p_2 - r_2| = |Ap_1 + F(p_1) - Ar_1| \le N|p_1 - r_1| + \mu |p_1| \le \mu(2N + 1)|p_0|,$$

$$|p_3 - r_3| \le N|p_2 - r_2| + \mu |p_2| \le \mu(N(2N + 1) + (N + 1)^2)|p_0|,$$

and so on.

Thus, there exists a number $\nu = \nu(K, N)$ such that

$$|p_k - r_k| \le \mu\nu |p_0|, \quad 0 \le k \le K.$$

We take $\mu = 1/\nu$, note that $\mu = \mu(K, N)$, and get the inequalities

$$|p_k - r_k| \le |p_0|, \quad 0 \le k \le K, \tag{2.79}$$

for d small enough.

Since f has the Lipschitz periodic shadowing property, for d small enough, the Q-periodic $2d$-pseudotrajectory ξ is $2\mathscr{L}d$-shadowed by a periodic trajectory. Let p_0 be a point of this trajectory such that

$$|p_k - x_k| \le \mathscr{L}d, \quad k \in \mathbb{Z}, \tag{2.80}$$

where $p_k = f^k(p_0)$.

The inequalities $|x_k| \le Yd$ and (2.80) imply that

$$|p_k| \le |x_k| + |p_k - x_k| \le (Y + 2\mathscr{L})d, \quad k \in \mathbb{Z}. \tag{2.81}$$

Note that $|p_0| \le 2\mathscr{L}d$.

Set $r_k = A^k p_0$; we deduce from estimate (2.79) that if d is small enough, then

$$|p_K - r_K| \le |p_0| \le 2\mathscr{L}d. \tag{2.82}$$

Denote by $v^{(2)}$ the second coordinate of a vector v.

It follows from the structure of the matrix A that

$$\left|r_K^{(2)}\right| = \left|p_0^{(2)}\right| \le 2\mathscr{L}d. \tag{2.83}$$

The relations

$$\left|y_K^{(2)}\right| = Kd \text{ and } |p_K - y_K| \le 2\mathscr{L}d$$

imply that

$$\left|p_K^{(2)}\right| \ge Kd - 2\mathscr{L}d = 5\mathscr{L}d \tag{2.84}$$

(recall that $K = 7\mathscr{L}$).

Estimates (2.82)–(2.84) are contradictory. Our lemma is proved in Case 1 for $l = 2$.

If $l = 1$, then the proof is simpler; the first coordinate of $A^k v$ equals the first coordinate of v, and we construct the periodic pseudotrajectory perturbing the first coordinate only.

If $l > 2$, the reasoning is parallel to that above; we first perturb the lth coordinate to make it Kd, and then produce a periodic sequence consequently making zero the lth coordinate, the $(l-1)$st coordinate, and so on.

If λ is a complex eigenvalue, $\lambda = a + bi$, we take a real 2×2 matrix

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and assume that in representation (2.75), B is a real $2l \times 2l$ Jordan block:

$$B = \begin{pmatrix} R & E_2 & 0 & \dots & 0 \\ 0 & R & E_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R \end{pmatrix},$$

where E_2 is the 2×2 identity matrix.

After that, almost the same reasoning works; we note that $|Rv| = |v|$ for any 2-dimensional vector v and construct periodic pseudotrajectories replacing, for example, formulas (2.76) by the formulas

$$y_{k+1} = Ay_k + dw_k, \quad k = 0, \dots, K-1,$$

where jth coordinates of the vector w_k are zero for $j = 1, \dots, 2l-2, 2l+1, \dots, n$, while the 2-dimensional vector corresponding to $(2l-1)$st and $2l$th coordinates has the form $R^k w$ with $|w| = 1$, and so on. We leave details to the reader. The lemma is proved. □

Lemma 2.4.4 *There exist constants $C > 0$ and $\lambda \in (0, 1)$ depending only on N and $\mathscr{L}$ and such that, for any point $p \in Per(f)$, there exist complementary subspaces $S(p)$ and $U(p)$ of $\mathbb{R}^n$ that are Df-invariant, i.e.,*

(H1) $Df(p)S(p) = S(f(p))$ and $Df(p)U(p) = U(f(p))$,
and the inequalities
(H2.1) $\left|Df^j(p)v\right| \le C\lambda^j |v|, \quad v \in S(p), j \ge 0$,
and
(H2.2) $\left|Df^{-j}(p)v\right| \le C\lambda^j |v|, \quad v \in U(p), j \ge 0$,
hold.

Remark 2.4.2 This lemma means that the set $\mathrm{Per}(f)$ has all the standard properties of a hyperbolic set, with the exception of compactness.

Proof Take a periodic point $p \in \mathrm{Per}(f)$; let m be the minimal period of p.

Denote $p_i = f^i(p), A_i = Df(p_i)$, and $B = Df^m(p)$. It follows from Lemma 2.4.3 that the matrix B is hyperbolic. Denote by $S(p)$ and $U(p)$ the invariant subspaces of B corresponding to parts of its spectrum inside and outside the unit disk, respectively. Clearly, $S(p)$ and $U(p)$ are invariant with respect to Df, they are complementary subspaces of $\mathbb{R}^n$, and the following relations hold:

$$\lim_{n\to+\infty} B^n v_s = \lim_{n\to+\infty} B^{-n} v_u = 0, \quad v_s \in S(p), v_u \in U(p). \tag{2.85}$$

We prove that inequalities (H2.2) hold with $C = 4\mathscr{L}$ and $\lambda = 1 + 1/(2\mathscr{L})$ (inequalities (H2.1) are established by similar reasoning applied to f^{-1} instead of f).

Consider an arbitrary nonzero vector $v_u \in U(p)$ and an integer $j \ge 0$. Define sequences of vectors v_i, e_i and numbers $\lambda_i > 0$ for $i \ge 0$ as follows:

$$v_0 = v_u, \quad v_{i+1} = A_i v_i, \quad e_i = \frac{v_i}{|v_i|}, \quad \lambda_i = \frac{|v_{i+1}|}{|v_i|} = |A_i e_i|.$$

Let

$$\tau = \frac{\lambda_{m-1} \cdot \ldots \cdot \lambda_1 + \lambda_{m-1} \cdot \ldots \cdot \lambda_2 + \ldots + \lambda_{m-1} + 1}{\lambda_{m-1} \cdot \ldots \cdot \lambda_0}.$$

Consider the sequence $\{a_i \in \mathbb{R} : \ i \geq 0\}$ defined by the following formulas:

$$a_0 = \tau, \quad a_{i+1} = \lambda_i a_i - 1. \tag{2.86}$$

Note that

$$a_m = 0 \quad \text{and} \quad a_i > 0, \quad i \in [0, m-1]. \tag{2.87}$$

Indeed, if $a_i \leq 0$ for some $i \in [0, m-1]$, then $a_k < 0$ for $k \in [i+1, m]$.

It follows from (2.85) that there exists an $n > 0$ such that

$$|B^{-n} \tau e_0| < 1. \tag{2.88}$$

Consider the finite sequence of vectors $\{w_i : \ i \in [0, m(n+1)]\}$ defined as follows:

$$\begin{cases} w_i = a_i e_i, & i \in [0, m-1]; \\ w_m = B^{-n} \tau e_0; & \\ w_{m+1+i} = A_i w_{m+i}, & i \in [0, mn-1]. \end{cases}$$

Clearly,

$$w_{km} = B^{k-1-n} \tau e_0, \quad k \in [1, n+1],$$

which means that we can consider $\{w_i\}$ as an $m(n+1)$-periodic sequence defined for $i \in \mathbb{Z}$.

Let us note that

$$A_i w_i = a_i A_i e_i = a_i \frac{v_{i+1}}{|v_i|}, \quad i \in [0, m-2],$$

$$w_{i+1} = (\lambda_i a_i - 1) \frac{v_{i+1}}{|v_{i+1}|} = a_i \frac{v_{i+1}}{|v_i|} - e_{i+1}, \quad i \in [0, m-2],$$

and

$$A_{m-1} w_{m-1} = a_{m-1} \frac{v_m}{|v_{m-1}|} = \frac{v_m}{\lambda_{m-1} |v_{m-1}|} = e_m$$

(in the last relation, we take into account that $a_{m-1} \lambda_{m-1} = 1$ since $a_m = 0$).

The above relations and condition (2.88) imply that

$$|w_{i+1} - A_i w_i| < 2, \quad i \in \mathbb{Z}. \tag{2.89}$$

Now we take a small $d > 0$ and consider the $m(n+1)$-periodic sequence

$$\xi = \{x_i = p_i + dw_i : \ i \in \mathbb{Z}\}.$$

We claim that if d is small enough, then ξ is a $2d$-pseudotrajectory of f.

Represent

$$f(x_i) = f(p_i) + Df(p_i)dw_i + F_i(dw_i) = p_{i+1} + A_i dw_i + F_i(dw_i),$$

where $F_i(v) = o(|v|)$ as $v \to 0$.

It follows from estimates (2.77) that

$$|f(x_i) - x_{i+1}| < 2d$$

for small d.

By Lemma 2.4.3, the m-periodic trajectory $\{p_i\}$ is hyperbolic; hence, $\{p_i\}$ has a neighborhood in which $\{p_i\}$ is the unique periodic trajectory. It follows that if d is small enough, then the pseudotrajectory $\{x_i\}$ is $2\mathscr{L}d$-shadowed by $\{p_i\}$.

The inequalities $|x_i - p_i| \leq 2\mathscr{L}d$ imply that $|a_i| = |w_i| \leq 2\mathscr{L}$ for $0 \leq i \leq m-1$.

Now the equalities $\lambda_i = (a_{i+1}+1)/a_i$ imply that if $0 \leq i \leq m-1$, then

$$\lambda_0 \cdot \ldots \cdot \lambda_{i-1} = \frac{a_1+1}{a_0}\frac{a_2+1}{a_1} \cdots \frac{a_i+1}{a_{i-1}} =$$

$$= \frac{a_i+1}{a_0}\left(1+\frac{1}{a_1}\right) \cdots \left(1+\frac{1}{a_{i-1}}\right) \geq$$

$$\geq \frac{1}{2\mathscr{L}}\left(1+\frac{1}{2\mathscr{L}}\right)^{i-1} > \frac{1}{4\mathscr{L}}\left(1+\frac{1}{2\mathscr{L}}\right)^{i}$$

(we take into account that $1 + 1/(2\mathscr{L}) < 2$ since $\mathscr{L} \geq 1$).

It remains to note that

$$\left|Df^i(p)v_u\right| = \lambda_{i-1} \cdots \lambda_0 |v_u|, \quad 0 \leq i \leq m-1,$$

and that we started with an arbitrary vector $v_u \in U(p)$.

This proves our statement for $j \leq m-1$. If $j \geq m$, we take an integer $k > 0$ such that $km > j$ and repeat the above reasoning for the periodic trajectory $p_0, \ldots, p_{km-1}$ (note that we have not used the condition that m is the minimal period). The lemma is proved. □

In the following lemmas, we return to the case of a diffeomorphism f of a smooth closed manifold M since the reasoning becomes “global.” We still assume that f has the Lipschitz periodic shadowing property and apply analogs of Lemmas 2.4.3 and 2.4.4 for the case of a manifold.

Lemma 2.4.5 *The diffeomorphism f satisfies Axiom A.*

Proof Denote by P_l the set of points $p \in \mathrm{Per}(f)$ of index l (as usual, the index of a hyperbolic periodic point is the dimension of its stable manifold).

Let R_l be the closure of P_l. Clearly, R_l is a compact f-invariant set. We claim that any R_l is a hyperbolic set. Let $n = \dim M$.

Consider a point $q \in R_l$ and fix a sequence of points $p_m \in P_l$ such that $p_m \to q$ as $m \to \infty$. By an analog of Lemma 2.4.4, there exist complementary subspaces $S(p_m)$ and $U(p_m)$ of $T_{p_m}M$ (of dimensions l and $n - l$, respectively) for which estimates (H2.1) and (H2.2) hold.

Standard reasoning shows that, introducing local coordinates in a neighborhood of (q, T_qM) in the tangent bundle of M, we can select a subsequence p_{m_k} for which the sequences $S(p_{m_k})$ and $U(p_{m_k})$ converge (in the Grassmann topology) to subspaces of T_qM (let S_0 and U_0 be the corresponding limit subspaces).

The limit subspaces S_0 and U_0 are complementary in T_qM. Indeed, consider the "angle" β_{m_k} between the subspaces $S(p_{m_k})$ and $U(p_{m_k})$ which is defined (with respect to the introduced local coordinates in a neighborhood of (q, T_qM)) as follows:

$$\beta_{m_k} = \min |v^s - v^u|,$$

where the minimum is taken over all possible pairs of unit vectors $v^s \in S(p_{m_k})$ and $v^u \in U(p_{m_k})$.

The same reasoning as in the proof of Lemma 2.1.5 shows that the values β_{m_k} are estimated from below by a positive constant $\alpha = \alpha(N, C, \lambda)$. Clearly, this implies that the subspaces S_0 and U_0 are complementary.

It is easy to show that the limit subspaces S_0 and U_0 are unique (which means, of course, that the sequences $S(p_m)$ and $U(p_m)$ converge). For the convenience of the reader, we prove this statement.

To get a contradiction, assume that there is a subsequence p_{m_i} for which the sequences $S(p_{m_i})$ and $U(p_{m_i})$ converge to complementary subspaces S_1 and U_1 different from S_0 and U_0 (for definiteness, we assume that $S_0 \setminus S_1 \neq \emptyset$).

Due to the continuity of Df, the inequalities

$$\left|Df^j(q)v\right| \le C\lambda^j|v|, \quad v \in S_0 \cup S_1,$$

and

$$\left|Df^j(q)v\right| \ge C^{-1}\lambda^{-j}|v|, \quad v \in U_0 \cup U_1,$$

hold for $j \ge 0$.

Since

$$T_qM = S_0 \oplus U_0 = S_1 \oplus U_1,$$

our assumption implies that there is a vector $v \in S_0$ such that

$$v = v^s + v^u, \quad v^s \in S_1, v^u \in U_1, v^u \neq 0.$$

Then

$$|Df^j(q)v| \leq C\lambda^j|v| \to 0, \quad j \to \infty,$$

and

$$\left|Df^j(q)v\right| \geq C^{-1}\lambda^{-j}|v^u| - C\lambda^j|v^s| \to \infty, \quad j \to \infty,$$

and we get the desired contradiction.

It follows that there are uniquely defined complementary subspaces $S(q)$ and $U(q)$ for $q \in R_l$ with proper hyperbolicity estimates; the Df-invariance of these subspaces is obvious. We have shown that each R_l is a hyperbolic set with $\dim S(q) = l$ and $\dim U(q) = n - l$ for $q \in R_l$.

If $r \in \Omega(f)$, then there exists a sequence of points $r_m \to r$ as $m \to \infty$ and a sequence of indices $k_m \to \infty$ as $m \to \infty$ such that $f^{k_m}(r_m) \to r$.

Clearly, if we continue the sequence

$$r_m, f(r_m), \dots, f^{k_m - 1}(r_m)$$

periodically with period k_m, we get a periodic d_m-pseudotrajectory of f with $d_m \to 0$ as $m \to \infty$.

Since f has the Lipschitz periodic shadowing property, for large m there exist periodic points p_m such that $\mathrm{dist}(p_m, r_m) \to 0$ as $m \to \infty$. Thus, periodic points are dense in $\Omega(f)$.

Since hyperbolic sets with different dimensions of the subspaces $U(q)$ are disjoint, we get the equality

$$\Omega(f) = R_0 \cup \dots \cup R_n,$$

which implies that $\Omega(f)$ is hyperbolic. The lemma is proved. □

Thus, to prove Theorem 2.4.4, it remains to prove the following lemma.

Lemma 2.4.6 *If f has the Lipschitz periodic shadowing property, then f satisfies the no cycle condition.*

Proof To simplify presentation, we prove that f has no 1-cycles (in the general case, the idea is literally the same, but the notation is heavy; we leave this case to the reader).

To get a contradiction, assume that

$$p \in (W^u(\Omega_i) \cap W^s(\Omega_i)) \setminus \Omega(f).$$

In this case, there are sequences of indices $j_m, k_m \to \infty$ as $m \to \infty$ such that

$$f^{-j_m}(p), f^{k_m}(p) \to \Omega_i, \quad m \to \infty.$$

Since the set Ω_i is compact, we may assume that

$$f^{-j_m}(p) \to q \in \Omega_i \text{ and } f^{k_m}(p) \to r \in \Omega_i.$$

Since Ω_i contains a dense positive semitrajectory, there exist points $s_m \to r$ and indices $l_m > 0$ such that $f^{l_m}(s_m) \to q$ as $m \to \infty$.

Clearly, if we continue the sequence

$$p, f(p), \dots, f^{k_m-1}(p), s_m, \dots, f^{l_m-1}(s_m), f^{-j_m}(p), \dots, f^{-1}(p)$$

periodically with period $k_m + l_m + j_m$, we get a periodic d_m-pseudotrajectory of f with $d_m \to 0$ as $m \to \infty$.

Since f has the Lipschitz periodic shadowing property, there exist periodic points p_m (for m large enough) such that $p_m \to p$ as $m \to \infty$, and we get the desired contradiction with the assumption that $p \notin \Omega(f)$. The lemma is proved. □

Historical Remarks Theorem 2.4.1 was published by A. V. Osipov, the first author, and S. B. Tikhomirov in [50].

2.5 Hölder Shadowing for Diffeomorphisms

In this section, we explain the main ideas of the proof of the following result.

Theorem 2.5.1 *Assume that a diffeomorphism f of class C^2 of a smooth closed manifold has the Hölder shadowing property on finite intervals with constants $\mathscr{L}, C, d_0, \theta, \omega$ and that*

$$\theta \in (1/2, 1) \text{ and } \theta + \omega > 1. \tag{2.90}$$

Then f is structurally stable.

The proof of Theorem 2.5.1 is quite complicated. For that reason, we try to simplify the presentation and omit inessential technical details; the reader can find the original Tikhomirov's proof in the paper [101].

The main two steps of the proof of Theorem 2.5.1 are as follows.

First one considers a trajectory $\{p_k = f^k(p)\}$ of f, denotes $A_k = Df(p_k)$, and shows that under conditions of Theorem 2.5.1, the sequence $\mathscr{A} = \{A_k\}$ has a weak analog of the Perron property (in which the existence of bounded solutions of the inhomogeneous difference equations is replaced by the existence of "slowly growing" solutions).

We reproduce this part of the proof in Theorem 2.5.2 in which we restrict our consideration to the case of a diffeomorphism f of the Euclidean space $\mathbb{R}^n$.

After that, it is shown that the above-mentioned weak analog of the Perron property implies then f satisfies the analytic strong transversality condition (with exponential estimates) and, hence, by the Mañé theorem, f is structurally stable. To explain the basic techniques of that part of the proof, we prove the above statement in Theorem 2.5.3 in the case of a one-dimensional phase space (and note that the reasoning in the proof of Theorem 2.5.3 reproduces the most important part of the proof given by Tikhomirov). We again refer the reader to [101] for the proof of the general case.

Theorem 2.5.2 *Assume that a diffeomorphism f of the Euclidean space $\mathbb{R}^n$ has the Hölder shadowing property on finite intervals with constants $\mathscr{L}, C, d_0, \theta, \omega$ and that condition (2.90) is satisfied.*

Assume, in addition, that there exist constants $S, \varepsilon > 0$ such that

$$|f(p_k + v) - p_{k+1} - A_k v| \le S|v|^2, \quad k \in \mathbb{Z}, \ |v| \le \varepsilon. \tag{2.91}$$

Then there exist constants $L > 0$ and $\gamma \in (0, 1)$ such that for any $i \in \mathbb{Z}$ and $N > 0$ and any sequence

$$W = \{w_k \in \mathbb{R}^n : \ i + 1 \le k \le i + N + 1\} \tag{2.92}$$

with $|w_k| \le 1$, the difference equations

$$v_{k+1} = A_k v_k + w_{k+1}, \quad i \le k \le i + N, \tag{2.93}$$

have a solution

$$V = \{v_k : \ i \le k \le i + N + 1\}, \tag{2.94}$$

such that the value

$$\|V\| := \max_{i \le k \le i+N+1} |v_k| \tag{2.95}$$

satisfies the estimate

$$\|V\| \le L N^\gamma. \tag{2.96}$$

Remark 2.5.1 Clearly, an analog of condition (2.91) is satisfied if we consider a diffeomorphism of class C^2 for which the trajectory $\{p_k\}$ is contained in a bounded subset of $\mathbb{R}^n$ (or a diffeomorphism of class C^2 of a smooth closed manifold studied in the original paper [101]). In fact, it was noted by Tikhomirov that one can prove a similar result in the case where exponent 2 in (2.91) is replaced by any $\nu > 1$. The reasoning remains almost the same, but calculations become very cumbersome. For that reason, we follow the proof given in [101] (with exponent 2).

Proof (of Theorem 2.5.2) Denote

$$\alpha = \theta - 1/2.$$

Inequalities (2.90) imply that

$$\alpha \in (0, 1/2) \text{ and } 1/2 - \alpha < \omega. \tag{2.97}$$

Consider two auxiliary linear functions of $\beta \geq 0$,

$$g_1(\beta) = (2 + \beta)(1/2 - \alpha) \text{ and } g_1(\beta) = (2 + \beta)\omega.$$

By inequalities (2.97),

$$g_2(0) = 2\omega > 1 - 2\alpha = g_1(0) \in (0, 1)$$

and

$$g_2'(\beta) = \omega > 1/2 - \alpha = g_1'(\beta).$$

Hence, there exists a $\beta > 0$ such that

$$g_1(\beta) \in (0, 1) \text{ and } g_2(\beta) > 1.$$

We fix such a β and write the above relations in the form

$$0 < (2 + \beta)(1/2 - \alpha) < 1 \text{ and } (2 + \beta)\omega > 1. \tag{2.98}$$

Introduce the values

$$\gamma = ((2 + \beta)\omega)^{-1} \text{ and } \gamma_1 = 1 - (2 + \beta)(1/2 - \alpha).$$

Then it follows from (2.98) that

$$0 < \gamma < 1 \text{ and } \gamma_1 > 0. \tag{2.99}$$

Now we fix a sequence W of the form (2.92) and denote by $E(W)$ the set of all sequences V of the form (2.94) that satisfy Eqs. (2.93). The function $\|V\|$ is continuous on the linear space of sequences V; the set $E(W)$ is closed. Hence, the value

$$F(W) = \min_{V \in E(W)} \|V\| \tag{2.100}$$

is defined.

The set of finite sequences W of the form (2.92) with

$$\|W\| = \max_{i+1 \le k \le N+1} |w_k| \le 1$$

is compact. The function $F(W)$ is continuous in W; thus, there exists the number

$$Q = \max_W F(W).$$

Let us fix sequences W_0 and $V_0 \in E(W_0)$ such that

$$Q = F(W_0) = \|V_0\|. \tag{2.101}$$

Note the following two properties of the number Q. They follow from the definition of Q and from the linearity of Eqs. (2.93).

(Q1) Any sequence $V \in E(W_0)$ satisfies the inequality

$$\|V\| \ge Q.$$

(Q2) For any sequence W of the form (2.92) there exists a sequence $V \in E(W)$ such that

$$\|V\| \le Q\|W\|.$$

It follows from property (Q2) that to complete the proof of our theorem, it is enough to prove the following statement:

There exists a number L independent of i and N such that

$$Q \le LN^{\gamma}. \tag{2.102}$$

Define the number

$$d = \varepsilon Q^{-(2+\beta)}. \tag{2.103}$$

Let us consider the following two cases.

Case 1: $C((S+1)d)^{-\omega} < N$. In this case,

$$Q < (\varepsilon^{\omega}(S+1)^{\omega}/C)^{\gamma} N^{\gamma},$$

which proves inequalities (2.103) with

$$L = (\varepsilon^{\omega}(S+1)^{\omega}/C)^{\gamma}\ .$$

Case 2: $C((S+1)d)^{-\omega} \geq N$. In this case, we prove a stronger statement: There exists a number L independent of i and N such that

$$Q \leq L. \tag{2.104}$$

Treating Case 2, we assume without loss of generality that $i = 0$.

Also, without loss of generality, we assume that $\varepsilon < 1$ and $Q > 2$. Concerning the latter assumption, we note that if there exists a fixed number L independent of N such that $Q \leq L$, then estimate (2.104) is obviously valid. Thus, we may assume that Q is larger than any prescribed number independent of N. Applying the same reasoning, we assume that Q is so large that

$$Q > ((S+1)\varepsilon/d_0)^{1/(2+\beta)} \tag{2.105}$$

and

$$\mathscr{L}((S+1)\varepsilon/Q^{2+\beta})^{\theta} < \varepsilon/2. \tag{2.106}$$

Fix sequences W_0 and V_0 for which relation (2.101) is valid. To simplify notation, write $V_0 = \{v_k\}$.

Consider the sequence of points

$$y_k = p_k + dv_k, \quad 0 \leq k \leq N+1.$$

We claim that this sequence is an $(S+1)d$-pseudotrajectory of f.

Let us first note that $|v_k| \leq Q$; hence,

$$|dv_k| \leq \varepsilon Q^{-(2+\beta)} Q = \varepsilon Q^{-(1+\beta)} < \varepsilon/2. \tag{2.107}$$

In addition,

$$(dQ)^2 = (\varepsilon Q^{-(1+\beta)})^2 < \varepsilon Q^{-(2+\beta)} = d. \tag{2.108}$$

Now we estimate

$$|f(y_k) - y_{k+1}| = |f(p_k + dv_k) - (p_{k+1} + dv_{k+1})| =$$

$$= |f(p_k + dv_k) - (p_{k+1} + dA_k v_k + dw_{k+1})| \leq$$

$$\leq |f(p_k + dv_k) - (p_{k+1} + dA_k v_k)| + d|w_{k+1}| \leq$$

$$\leq S|dv_k|^2 + d \leq (S+1)d.$$

We estimate the first term of the third line taking into account condition (2.91) and inequality (2.107); estimating the first term of the last line, we refer to inequality (2.108).

Inequality (2.105) implies that

$$Q^{2+\beta} > (S+1)\varepsilon/d_0;$$

hence,

$$(S+1)d = (S+1)\varepsilon Q^{-(2+\beta)} < d_0.$$

Since we treat Case 2,

$$N \leq C((S+1)d)^{-\omega} < Cd^{-\omega},$$

and we can apply the Hölder shadowing property on finite intervals to conclude that there exists an exact trajectory $\{x_k\}$ of f such that

$$|y_k - x_k| \leq \mathscr{L}((S+1)d)^{\theta}, \quad 0 \leq k \leq N+1.$$

Denote $x_k = p_k + c_k$ and $\mathscr{L}_1 = \mathscr{L}(S+1)^{\theta}$. Then

$$|dv_k - c_k| \leq |y_k - x_k| \leq \mathscr{L}_1 d^{\theta}, \quad 0 \leq k \leq N+1, \tag{2.109}$$

and

$$|c_k| \leq Qd + \mathscr{L}_1 d^{\theta}, \quad 0 \leq k \leq N+1. \tag{2.110}$$

Inequalities (2.107) and (2.106) imply that

$$|c_k| < \varepsilon.$$

By the first inequality in (2.98),

$$Q > Q^{(1/2-\alpha)(2+\beta)} = (\varepsilon/d)^{1/2-\alpha} = \varepsilon^{1/2-\alpha} d^{\alpha-1/2}.$$

Hence,

$$Qd > \varepsilon^{1/2-\alpha} d^{\alpha+1/2} = \varepsilon^{1/2-\alpha} d^{\theta}.$$

Now it follows from (2.110) that there exists an $\mathscr{L}_2$ independent of N such that

$$|c_k| \le \mathscr{L}_2 Qd.$$

Since $p_{k+1} + c_{k+1} = f(p_k + c_k)$, we can estimate

$$|c_{k+1} - A_k c_k| = |f(p_k + c_k) - (p_{k+1} + A_k c_k| \le S|c_k)|^2 \le S\mathscr{L}_2(Qd)^2.$$

Denote $t_{k+1} = c_{k+1} - A_k c_k$; then

$$|t_k| \le S|c_k|^2 \le \mathscr{L}_3(Qd)^2,$$

where the constant $\mathscr{L}_3$ does not depend on N. By property (Q2), there exists a sequence z_k such that

$$z_{k+1} = A_k z_k + t_{k+1} \text{ and } |z_k| \le Q\mathscr{L}_3(Qd)^2, \quad 0 \le k \le N.$$

Consider the sequence $r_k = c_k - z_k$. Clearly,

$$r_{k+1} = A_k r_k \text{ and } |r_k - c_k| \le Q\mathscr{L}_3(Qd)^2, \quad 0 \le k \le N. \tag{2.111}$$

Now we define the sequence $e_k = (dv_k - r_k)/d$. Relations (2.109) and (2.111) imply that

$$e_{k+1} = A_k e_k + w_{k+1}, \quad 0 \le k \le N, \tag{2.112}$$

and

$$|e_k| = |((dv_k - c_k) - (r_k - c_k))/d| \le \mathscr{L}_1 d^{\theta-1} + \mathscr{L}_3 Q^3 d, \quad 0 \le k \le N.$$

Property (Q1) implies that

$$\mathscr{L}_1 d^{\theta-1} + \mathscr{L}_3 Q^3 d = \mathscr{L}_1 d^{\alpha-1/2} + \mathscr{L}_3 Q^3 d \ge Q.$$

We can apply (2.103) and find $\mathscr{L}_4, \mathscr{L}_5 > 0$ independent of N and such that this inequality takes the form

$$\mathscr{L}_4 Q^{-(2+\beta)(\alpha-1/2)} + \mathscr{L}_5 Q^{1-\beta} \ge Q,$$

or

$$\mathscr{L}_4 Q^{1-\gamma_1} + \mathscr{L}_5 Q^{1-\beta} \ge Q.$$

It follows that either

$$\mathscr{L}_4 Q^{1-\gamma_1} \ge Q/2$$

or

$$\mathscr{L}_5 Q^{1-\beta} \geq Q/2,$$

which implies that

$$Q \leq \max\left((2\mathscr{L}_4)^{1/\gamma_1}, (2\mathscr{L}_5)^{1/\beta}\right).$$

The theorem is proved. □

Now we assume, in addition, that there exists a constant $R > 0$ such that

$$\|A_k\| \leq R, \quad k \in \mathbb{Z}. \tag{2.113}$$

Remark 2.5.2 Of course, an estimate of the form (2.113) holds for $A_k = Df(p_k)$ in the case of a diffeomorphism f of a closed manifold.

Theorem 2.5.3 *Let f be a diffeomorphism of the line $\mathbb{R}$ having the Hölder shadowing property on finite intervals. Assume that conditions (2.91) and (2.113) are satisfied for a trajectory $\{p_k = f^k(p)\}$. There exists a constant $\mu \in (0,1)$ with the following property.*

For any $k \in \mathbb{Z}$ there exists a constant $C > 0$ and subspaces $S(p_k)$ and $U(p_k)$ of $\mathbb{R}$ such that

$$S(p_k) + U(p_k) = \mathbb{R}, \tag{2.114}$$

$$|A_{k+l-1} \cdots A_k v| \leq C\mu^l |v|, \quad v \in S(p_k),\ l \geq 0, \tag{2.115}$$

$$|A_{k-l}^{-1} \cdots A_{k-1}^{-1} v| \leq C\mu^l |v|, \quad v \in U(p_k),\ l \geq 0. \tag{2.116}$$

The essential part of the proof of Theorem 2.5.3 is contained in the following lemma.

Let us first introduce some notation. Consider a one-dimensional vector (i.e., a real number) e_0 with $|e_0| = 1$ and define a sequence $\{e_k : \ k \in \mathbb{Z}\}$ as follows:

$$e_{k+1} = A_k e_k / |A_k e_k|, \quad e_{-k-1} = A_{-k-1}^{-1} e_{-k} / |A_{-k-1}^{-1} e_{-k}|, \quad k \geq 0. \tag{2.117}$$

Set

$$\lambda_k = |A_k e_k|.$$

It follows from inequalities (2.113) that

$$\lambda_k \in [1/R, R], \quad k \in \mathbb{Z}. \tag{2.118}$$

Set also

$$\Pi(k,l) = \lambda_k \cdots \lambda_{k+l-1}, \quad k \in \mathbb{Z},\ l \geq 1. \tag{2.119}$$

Lemma 2.5.1 *If the sequence $\mathscr{A}$ satisfies the conclusion of Theorem 2.5.2, then there exists a number N depending only on L, γ, and R (see inequality (2.113)) and such that, for any $i \in \mathbb{Z}$, one of the following alternatives is valid:*

$$\text{either } \Pi(i,N) > 2 \text{ or } \Pi(i+N,N) < 1/2. \tag{2.120}$$

Proof Fix numbers $i \in \mathbb{Z}$ and $N > 0$ and consider the sequence

$$w_k = -e_k, \quad i \le k \le i + 2N + 1.$$

It follows from the conclusion of Theorem 2.5.2 that there exists a sequence

$$\{v_k : \ i \le k \le i + 2N\}$$

such that

$$v_{k+1} = A_k v_k + w_{k+1} \text{ and } |v_k| \le L(2N+1)^\gamma, \quad i \le k \le i + 2N.$$

Set $v_k = a_k e_k$, where $a_k \in \mathbb{R}$. Then

$$a_{k+1} = \lambda_k a_k - 1 \text{ and } |a_k| \le L(2N+1)^\gamma, \quad i \le k \le i + 2N. \tag{2.121}$$

Now we show that there exists a large enough number N (depending only on L, γ, and R) such that if $a_{i+N} \ge 0$, then $\Pi(i,N) > 2$, and if $a_{i+N} < 0$, then $\Pi(i+N,N) < 1/2$.

Let us prove the existence of N for the first case (i.e., for the case where $a_{i+N} \ge 0$).

Since $\lambda_k > 0$, it follows from relations (2.121) that if $a_k \le 0$ for some $k \in [i, i+2N-1]$, then $a_{k+1} < 0$. Thus, if $a_{i+N} \ge 0$, then $a_i, \dots, a_{i+N-1} > 0$.

Relations (2.121) imply that in this case,

$$\lambda_k = \frac{a_{k+1}+1}{a_k}, \quad i \le k \le i + N - 1.$$

Hence,

$$\Pi(i,N) = \frac{a_{i+1}+1}{a_i}\frac{a_{i+2}+1}{a_{i+1}} \cdots \frac{a_{i+N}+1}{a_{i+N-1}} =$$

$$= \frac{1}{a_i}\frac{a_{i+1}+1}{a_{i+1}}\frac{a_{i+2}+1}{a_{i+2}} \cdots \frac{a_{i+N-1}+1}{a_{i+N-1}}(a_{i+N}+1) =$$

$$= \frac{a_{i+N}+1}{a_i}\prod_{k=i+1}^{i+N-1}\frac{a_k+1}{a_k},$$

and it follows from relations (2.121) that

$$\Pi(i,N) \geq \frac{1}{L(2N+1)^{\gamma}} \left(1 + \frac{1}{L(2N+1)^{\gamma}}\right)^{N-1}. \tag{2.122}$$

Denote the expression on the right in (2.122) by $G_1(\gamma, N)$. Since

$$\log G_1(\gamma,N) = -\gamma \log(L(2N+1)) + (N-1)\log\left(1 + \frac{1}{L(2N+1)^{\gamma}}\right),$$

$$\log\left(1 + \frac{1}{L(2N+1)^{\gamma}}\right) \simeq (L(2N+1))^{-\gamma}$$

for large N, and $\gamma \in (0,1)$, we conclude that

$$G_1(\gamma,N) \to \infty, \quad N \to \infty.$$

Hence, there exists an N_1 depending only on L and γ such that $G_1(\gamma,N) > 2$ for $N \geq N_1$.

Now we consider the second case, i.e., we assume that $a_{i+N} < 0$. In this case, it follows from relations (2.121) that

$$a_k \in (-L(2N+1)^{\gamma}, -1), \quad i+N < k \leq i+2N. \tag{2.123}$$

As above, we set

$$\lambda_k = \frac{a_{k+1}+1}{a_k}.$$

Now we write

$$\Pi(i+N+1,N-1) = \frac{a_{i+N+2}+1}{a_{i+N+1}} \frac{a_{i+N+3}+1}{a_{i+N+2}} \cdots \frac{a_{i+2N}+1}{a_{i+2N-1}} =$$

$$= \frac{1}{a_{i+N+1}} \frac{a_{i+N+2}+1}{a_{i+N+2}} \frac{a_{i+N+3}+1}{a_{i+N+3}} \cdots \frac{a_{i+2N-1}+1}{a_{i+2N-1}} (a_{i+2N}+1)$$

and conclude that

$$\Pi(i+N+1,N-1) = \frac{a_{i+2N}+1}{a_{i+N+1}} \prod_{k=i+N+2}^{i+2N-1} \frac{a_k+1}{a_k}. \tag{2.124}$$

Inclusions (2.123) imply that

$$0 < \frac{a_k+1}{a_k} < 1 - \frac{1}{L(2N+1)^{\gamma}}, \quad i+N+2 \leq k \leq i+2N-1,$$

and

$$0 < \frac{a_{i+2N} + 1}{a_{i+N+1}} < L(2N+1)^{\gamma}.$$

Combining these inequalities with (2.124), we conclude that

$$\Pi(i+N+1, N-1) < L(2N+1)^{\gamma}\left(1 - \frac{1}{L(2N+1)^{\gamma}}\right)^{N-2}.$$

Denote the right-hand side of the above inequality by $G_2(\gamma, N)$. Clearly, $G_2(\gamma, N) \to 0$ as $N \to \infty$; hence, there exists an N_2 depending only on L, γ, and R such that

$$G(\gamma, N) < \frac{1}{2R}, \quad N \geq N_2.$$

If $N \geq N_2$, then

$$\Pi(i+N, N) = \lambda_{i+N}\Pi(i+N+1, N-1) < R\frac{1}{2R} = 1/2.$$

Hence, the conclusion of our lemma holds for $N = \max(N_1, N_2)$.

□

Proof (of Theorem 2.5.3) Take an arbitrary $i \in \mathbb{Z}$ and the number N given by Lemma 2.5.1. The following statements hold:

(a) If $\Pi(i, N) > 2$, then $\Pi(i-N, N) > 2$;
(b) If $\Pi(i, N) < 1/2$, then $\Pi(i+N, N) < 1/2$.

Let us prove statement (a); the proof of statement (b) is similar.

By Lemma 2.5.1 applied to i–N, either $\Pi(i–N, N) > 2$ or $\Pi(i, N) < 1/2$. By the assumption of statement (a), the second case is impossible; thus, $\Pi(i-N, N) > 2$.

It follows from these statements that only one of the following cases is realized:

Case 1. $\Pi(i, N) > 2$ for all $i \in \mathbb{Z}$.
Case 2. $\Pi(i, N) < 1/2$ for all $i \in \mathbb{Z}$.
Case 3. There exist indices $i, j \in \mathbb{Z}$ such that $\Pi(i, N) > 2$ and $\Pi(j, N) < 1/2$.

Now we show that Theorem 2.5.3 is valid with $\mu = 2^{-1/N}$.

Consider Case 1. Take e_0 with $|e_0| = 1$ and define e_k, $k \in \mathbb{Z}$, by formulas (2.117). Represent any integer $l \geq 0$ in the form

$$l = nN + l_1, \quad n \in \mathbb{Z}_+,\ 0 \leq l_1 < N.$$

Then

$$\Pi(i,l) = \Pi(i,nN)\Pi(i+nN,l_1) > 2^n R^{-l_1}$$

(in the last estimate, we take into account inequalities (2.118)).

Hence, in Case 1,

$$\Pi(i,l) > R^{-l_1}\left(2^{-l_1/N}\right)\left(2^{1/N}\right)^l > C_0\mu^{-l}, \quad i \in \mathbb{Z},\ l \geq 0, \tag{2.125}$$

where

$$C_0 = R^{-N}/2.$$

Now we fix a point p_k of the trajectory $\{p_k\}$ and set $S(p_k) = \{0\}$ and $U(p_k)=\mathbb{R}$. Clearly, in this case, relations (2.114) and (2.115) are satisfied. Let us prove inequalities (2.116). Take any $v \in \mathbb{R} = U(p_k)$ and $l \geq 0$. Let

$$w = A_{k-l}^{-1} \cdots A_{k-1}^{-1} v.$$

Then

$$v = A_{k-1} \cdots A_{k-l} w.$$

Hence,

$$|v| = \lambda_{k-l} \cdots \lambda_{k-1}|w| = \Pi(k-l,l)|w|,$$

and it follows from (2.125) that

$$|w| \leq C\mu^l |v|,$$

where $C = (C_0)^{-1}$, as required.

In Case 2, we set $U(p_k) = \{0\}$ and $S(p_k) = \mathbb{R}$ and apply a similar reasoning.

Let us now consider Case 3. By our remark at the beginning of the proof,

$$\Pi(i-nN,N) > 2 \text{ and } \Pi(j+nN,N) < 1/2, \quad n \in \mathbb{Z}_+.$$

In this case, we set $S(p_k) = U(p_k) = \mathbb{R}$. Clearly, in this case, relation (2.114) is satisfied. Let us show how to prove inequalities (2.115).

We treat in detail two cases:

Case (I). $k + l \leq j$

and

Case (II). $k < j$ and $k + l > j$

(the remaining cases and the proof of inequalities (2.116) are left to the reader).

In Case (I), we note that $l \le j - k$ and estimate

$$\Pi(k, l) \le R^{j-k} = R^{j-k} 2^{l/N} 2^{-l/N} \le C\mu^l,$$

where $C = R^{j-k} 2^{(j-k)/N}$. Hence,

$$|A_{k+l-1} \cdots A_k v| \le C\mu^l |v|, \quad v \in S(p_k).$$

In Case (II), we represent $k + l = j + nN + l_1$, where $n \in \mathbb{Z}_+$ and $0 \le l_1 < N$. Then

$$\Pi(k, l) = \Pi(k, j-k)\Pi(j, nN)\Pi(j + nN, l_1).$$

We note that $\Pi(k, j-k) \le R^{j-k}$,

$$\Pi(j, nN) < 2^{-n} = 2^{l_1/N}\mu^l,$$

and

$$\Pi(j + nN, l_1) \le R^{l_1} < R^N,$$

which gives us the desired estimate $\Pi(k, l) < C\mu^l$ (and, hence, inequalities (2.115)) with $C = 2R^{j-k+N}$. □

Historical Remarks Theorem 2.5.1 was published by S. B. Tikhomirov in [101].

Let us mention that earlier S. M. Hammel, J. A. Yorke, and C. Grebogi, based on results of numerical experiments, conjectured that a generic dissipative mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ belongs to a class $\mathrm{FHSP}_D(\mathscr{L}, C, d_0, 1/2, 1/2)$ [23, 24]. If this conjecture is true, then, in a sense, Theorem 2.5.1 cannot be improved.

2.6 A Homeomorphism with Lipschitz Shadowing and a Nonisolated Fixed Point

Consider the segment

$$I_0 = [-7/6, 4/3]$$

and a mapping $f_0 : I_0 \to I_0$ defined as follows:

$$f_0(x) = \begin{cases} 1 + (x-1)/2, & x \in [1/3, 4/3]; \\ 2x, & x \in (-1/3, 1/3); \\ -1 + (x+1)/2, & x \in [-7/6, -1/3]. \end{cases}$$

Clearly, the restriction f^* of f_0 to $[-1, 1]$ is a homeomorphism of $[-1, 1]$ having three fixed points: the points $x = \pm 1$ are attracting and the point $x = 0$ is repelling (and this homeomorphism f^* is an example of the so-called "North Pole – South Pole" dynamical system; every trajectory starting at a point $x \neq 0, \pm 1$ tends to an attractive fixed point as time tends to $+\infty$ and to the repelling fixed point as time tends to $-\infty$).

Now we define a homeomorphism $f : [-1, 1] \to [-1, 1]$. For an integer $n \geq 0$, denote $\mathscr{N}_n = 2^{-(n+2)}$ and set

$$f(x) = \mathscr{N}_n f_0(\mathscr{N}_n^{-1}(x - 3\mathscr{N}_n)) + 3\mathscr{N}_n, \quad x \in (2\mathscr{N}_n, 4\mathscr{N}_n]. \tag{2.126}$$

This defines f on $(0, 1]$. Set $f(0) = 0$ and $f(x) = -f(-x)$ for $x \in [-1, 0)$.

Clearly, f is a homeomorphism with a nonisolated fixed point $x = 0$ (for example, every point $x = \pm 2^{-n}$ is fixed). Let us note that in a neighborhood of any fixed point (with the exception of $x = 0$), f is either linearly expanding with coefficient 2 or linearly contracting with coefficient 1/2.

Theorem 2.6.1 *The homeomorphism f has the Lipschitz shadowing property.*

Before proving Theorem 2.6.1, we prove two auxiliary lemmas.

Lemma 2.6.1 *The mapping f_0 has the Lipschitz shadowing property on I_0.*

Proof Let

$$G_0 = (-1/3, 1/3)$$

and

$$G_1 = (-7/6, -1/3) \cup (1/3, 4/3).$$

We take d_0 small enough and $d \leq d_0$; in fact, we write below several explicit conditions on d and assume that they are satisfied.

There exist trivial cases where ξ is a subset of one of the segments $J_1 = [-7/6, -1/3]$, $J_2 = [-1/3, 1/3]$, or $J_3 = [1/3, 4/3]$.

Let, for example, $\xi \subset J_3$. The inequalities $1/3 \leq x_k \leq 4/3$ imply that

$$1/2 < 2/3 - d \leq f_0(x_k) - d \leq x_{k+1} \leq f_0(x_k) + d \leq 7/6 + d < 15/12.$$

These inequalities are satisfied for an arbitrary k; hence, ξ belongs to a domain in which f_0 is a hyperbolic diffeomorphism (and ξ is uniformly separated from the boundaries of the domain); by Theorem 1.4.2 (which, of course, is valid for infinite pseudotrajectories as well), there exist $\mathscr{L}, d_0 > 0$ such that if $d \leq d_0$, then ξ is $\mathscr{L}d$-shadowed by an exact trajectory of f_0.

A similar reasoning can be applied if $\xi \subset J_1$ or $\xi \subset J_2$.

To consider "nontrivial" cases, let us first describe possible positions of d-pseudotrajectories ξ of f_0 with small d with respect to $J_1, \dots, J_3$.

First we show that such a pseudotrajectory cannot intersect both J_1 and J_3. Indeed, if we assume that $\xi \cap J_3 \neq \emptyset$, i.e., there exists an index m such that $x_m \geq 1/3$, then

$$x_{m+1} \geq f_0(1/3) - d = 2/3 - d > 1/3$$

and, consequently,

$$x_{m+i} > 1/3, \quad i > 0.$$

Similarly, if there exists an index l such that $x_l \leq -1/3$, then

$$x_{l+1} \leq -2/3 + d < -1/3$$

and

$$x_{l+i} < -1/3, \quad i > 0,$$

and we get a contradiction.

Thus, it remains to consider the cases where either

$$\xi \subset J_2 \cup J_3, \quad \xi \cap \mathrm{Int}(J_2) \neq \emptyset, \quad \xi \cap \mathrm{Int}(J_3) \neq \emptyset,$$

or

$$\xi \subset J_1 \cup J_2, \quad \xi \cap \mathrm{Int}(J_1) \neq \emptyset, \quad \xi \cap \mathrm{Int}(J_2) \neq \emptyset.$$

We consider the first case; the reasoning in the second case is similar.

We claim that in the case considered, ξ contains two points x_k, x_l such that

$$0 < x_k < 1/3 < x_l. \tag{2.127}$$

The existence of the point x_l follows directly from our assumption; it is easily seen that

$$x_{l+i} \geq 2/3 - d > 1/2, \quad i > 0. \tag{2.128}$$

Thus, either the set

$$\{m : \ x_m \in \mathrm{Int}(J_2), x_m \leq 0\}$$

is empty (which implies that there exists an index k for which inequality (2.127) is valid) or it is nonempty and bounded from above. In the latter case, let m_0 be its maximal element. Then

$$x_{m_0+1} \leq f_0(x_{m_0}) + d \leq d$$

(i.e., $x_{m_0+1} \in J_2$) and $x_{m_0+1} > 0$; thus, we get the required $k = m_0 + 1$.

Obviously, $l > k$ (see (2.128)). Consider the finite set of indices

$$\kappa = \{ i \in [k, l-1] : \ x_i \leq 1/3 \}.$$

This set is nonempty ($k \in \kappa$) and finite; hence, it contains the maximal element. Let it be x_{k_0}; clearly,

$$x_{k_0} \leq 1/3 < x_{k_0+1}.$$

To simplify notation, let us assume that $k_0 = 0$. Thus,

$$x_0 \leq 1/3 < x_1.$$

In this case,

$$x_i \geq 2/3 - d > 1/2, \quad i \geq 2. \tag{2.129}$$

On the other hand,

$$x_1 \leq 2/3 + d < 1,$$

and one easily shows that

$$x_i \leq 1 + 2d, \quad i \geq 2. \tag{2.130}$$

Since f_0^{-1} has Lipschitz constant 2, ξ is a $2d$-pseudotrajectory of f_0^{-1}; hence,

$$x_{-1} \leq 1/6 + 2d < 2/9,$$

and, applying the same reasoning as above, we conclude that

$$-4d < x_i < 1/6 + 2d < 2/9, \quad i < 0. \tag{2.131}$$

Now we show that there exists a d_0 such that if $d \leq d_0$ and $p = x_0$, then

$$\left| f_0^k(p) - x_k \right| < 3d, \quad k \in \mathbb{Z}. \tag{2.132}$$

First, clearly,

$$|f_0(p) - x_1| < d.$$

Since the Lipschitz constant of f_0 is 2,

$$\left| f_0^2(p) - x_2 \right| \leq |f(f(p)) - f(x_1)| + |f(x_1) - x_2| < 2d + d = 3d.$$

It follows from (2.129) that

$$f_0^2(p) > 1/2 - 3d > 1/3,$$

and then

$$f_0^k(p) > 1/3, \quad k \geq 2.$$

Hence,

$$\left|f_0^3(p) - x_3\right| \leq \left|f_0(f_0^2(p)) - f_0(x_2)\right| + |f_0(x_2) - x_3| < 3d/2 + d < 3d.$$

Repeating these estimates, we establish inequalities (2.132) for $k \geq 2$.

On the other hand, the inclusion $p \in J_2$ implies that $f_0^k(p) \in J_2$ for $k \leq 0$. Since $f_0^{-1}(x) = x/2$ for $x \in J_2$ and (2.131) holds, the inequality

$$|f_0(x_1) - p| < d$$

implies that

$$|x_1 - f_0^{-1}(p)| < d/2.$$

After that, we estimate

$$\left|x_2 - f_0^{-2}(p)\right| \leq \left|x_2 - f_0^{-1}(x_{-1})\right| + \left|f_0^{-1}(x_{-1}) - f_0^{-1}(f_0^{-1}(p))\right| < d/2 + d/2,$$

and so on, which shows that an analog of (2.132) with $3d$ replaced by d holds for $k < 0$. □

The following statement is almost obvious.

Lemma 2.6.2 *Let g be a mapping of a segment J and let numbers $M > 0$ and m be given. Consider the mapping*

$$g'(y) = M^{-1}g(M(y - m)) + m$$

on the set

$$J' = \{y : \; M(y - m) \in J\}.$$

If g has the Lipschitz shadowing property with constants $\mathscr{L}, d_0$, then g' has the Lipschitz shadowing property with constants $\mathscr{L}, M^{-1}d_0$.

Proof First we note that if $\{y_k\}$ is a d-pseudotrajectory of g' with $d \le d_0/M$ and $x_k = M(y_k - m)$, then

$$g(x_k) - x_{k+1} = M(g'(y_k) - y_{k+1}).$$

Hence, $\{x_k\}$ is an Md-pseudotrajectory of g.

Since $Md \le d_0$, there exists a point p such that

$$\left|g^k(p) - x_k\right| \le \mathscr{L}Md.$$

Set $p' = M^{-1}p + m$. Then, obviously,

$$\left|(g')^k(p') - y_k\right| = M^{-1}\left|g^k(p) - x_k\right| \le \mathscr{L}d.$$

□

Let us prove Theorem 2.6.1.

Proof For a natural n, define the segment

$$I_n = [\alpha_n, \beta_n] = [11\mathscr{N}_n/6, 13\mathscr{N}_n/3]$$

and note that formula (2.126) defining f for $x \in (2\mathscr{N}_n, 4\mathscr{N}_n]$ is, in fact, valid for $x \in I_n$.

To prove Theorem 2.6.1, we first claim that there exists a constant c independent of n such that if d satisfies a condition of the form

$$d \le c\mathscr{N}_n \tag{2.133}$$

and $\xi = \{x_k\}$ is a d-pseudotrajectory of f that intersects I_n, then ξ is a subset of one of the segments I_{n-1}, I_n, I_{n+1}.

In fact, all the conditions imposed below on d have the form (2.133).

It follows from the inequalities

$$f(\alpha_n) = 23\mathscr{N}_n/12 > \alpha_n, \quad f(\beta_n) = 25\mathscr{N}_n/6 < \beta_n$$

that if c is small enough (we do not repeat this assumption below), then

$$\mathrm{Cl}(N(d, f(I_m))) \subset I_m, \quad m = n-1, n, n+1. \tag{2.134}$$

Thus, if $x_k \in I_m$ for some $m = n-1, n, n+1$, then it follows from (2.134) that

$$x_{k+i} \in I_m, \quad i \ge 0. \tag{2.135}$$

Let $x_0 \in I_n$.

We assume that

$$\mathrm{Cl}(N(2d, f^{-1}(I_n))) \subset I_{n-1} \cup I_n \cup I_{n+1}$$

(note that this condition on d has precisely form (2.133)).

By (2.135), $x_k \in I_n$ for $k \geq 0$. Thus, if the inclusion $\xi \subset I_n$ does not hold, there exists an index $l < 0$ such that

$$x_l \in (I_{n-1} \cup I_{n+1}) \setminus I_n$$

(recall that ξ is a $2d$-pseudotrajectory of f^{-1}).

Assume, for definiteness, that $x_l \in I_{n-1}$ (the remaining case is treated similarly). In this case, the same inclusions (2.135) imply that

$$x_{l+i} \in I_{n-1}, \quad i \geq 0.$$

To show that

$$x_{l+i} \in I_{n-1}, \quad i < 0,$$

take an index $m < l$ and assume that $x_m \in I_\nu$. Then inclusions (2.135) imply that

$$x_0, x_l \in I_\nu;$$

hence,

$$I_\nu \cap I_n \neq \emptyset \text{ and } I_\nu \cap I_{n-1} \neq \emptyset,$$

from which it follows that either $\nu = n$ or $\nu = n - 1$. But since $x_l \notin I_n$, $\nu \neq n$, and we conclude that $\xi \subset I_{n-1}$, as claimed.

Of course, a similar statement holds for the segments $I'_n = [-\beta_n, -\alpha_n]$.

Without loss of generality, we assume that

$$c \leq d_0/2, \tag{2.136}$$

where d_0 is given by Lemma 2.6.1. Let $\delta(m) = c\mathcal{N}_m$.

Consider a d-pseudotrajectory $\xi = \{x_k\} \subset [-1, 1]$ of f with $d \leq d_0$. If

$$d \geq \delta(0) = c\mathcal{N}_0 = c/4,$$

then $1 \leq 4d/c$, and ξ is $4d/c$-shadowed by the fixed point $x = 0$.

Otherwise, we find the maximal index m_0 for which $d < \delta(m_0)$. In this case,

$$d \geq \delta(m_0 + 1) = \delta(m_0)/2. \tag{2.137}$$

First we assume that

$$\xi \cap I_m \neq \emptyset \text{ for some } m \leq m_0 \tag{2.138}$$

(the case of I'_m is similar).

In this case, the inequalities

$$d < \delta(m_0) \leq \delta(m)$$

imply that ξ is a subset of one of the segments I_{m-1}, I_m, I_{m+1}. We assume that $\xi \subset I_{m+1}$; in the remaining cases, the same estimates work.

Since

$$d \leq \delta(m) = c\mathcal{N}_m \leq d_0\mathcal{N}_m/2 = d_0\mathcal{N}_{m+1}$$

(we refer to (2.136)), Lemma 2.6.2 implies that ξ is $\mathcal{L}$-shadowed.

If relation (2.138) does not hold, then

$$|x_k| \leq \alpha_{m_0} = \frac{11N_{m_0}}{6} = \frac{11\delta(m_0)}{6c} \leq \frac{11}{3c}d$$

(we take into account inequality (2.137) in the last estimate). Thus, in this case, ξ is $11d/(3c)$-shadowed by the fixed point $x = 0$. □

Historical Remarks In this section, we give a simplified proof of Theorem 2.6.1 compared to the original variant published by A. A. Petrov and the first author in [59].

2.7 Lipschitz Shadowing Implies Structural Stability: The Case of a Vector Field

Let M be a smooth closed manifold with Riemannian metric dist and let X be a vector field on M of class C^1. Denote by $\phi(t, x)$ the flow on M generated by the vector field X.

Our main goal in this section is to prove the following statement.

Theorem 2.7.1 *If a vector field X has the Lipschitz shadowing property, then X is structurally stable.*

In the proof of Theorem 2.7.1, we refer to Theorem 1.3.14.

Define a diffeomorphism f on M by setting $f(x) = \phi(1, x)$.

It is an easy exercise to show that the chain recurrent set $\mathcal{R}(\phi)$ of the flow ϕ (see Definition 1.3.22) coincides with the chain recurrent set of the diffeomorphism f.

2.7.1 Discrete Lipschitz Shadowing for Flows

In this section, we introduce the notion of discrete Lipschitz shadowing for a vector field in terms of the diffeomorphism $f(x) = \phi(1, x)$ introduced above and show that the Lipschitz shadowing property of ϕ implies the discrete Lipschitz shadowing.

Definition 2.7.1 A vector field X has the *discrete Lipschitz shadowing property* if there exist $d_0, L > 0$ such that if $y_k \in M$ is a sequence with

$$\mathrm{dist}(y_{k+1}, f(y_k)) \le d \le d_0, \quad k \in \mathbb{Z}, \tag{2.139}$$

then there exist sequences $x_k \in M$ and $t_k \in \mathbb{R}$ such that

$$|t_k - 1| \le Ld, \ \mathrm{dist}(x_k, y_k) \le Ld, \ x_{k+1} = \phi(t_k, x_k), \quad k \in \mathbb{Z}. \tag{2.140}$$

Lemma 2.7.1 *The Lipschitz shadowing property of ϕ implies the discrete Lipschitz shadowing of X.*

Proof First we note that since M is compact and X is C^1-smooth, there exists a $\nu > 0$ such that

$$\mathrm{dist}(\phi(t, x), \phi(t, y)) \le \nu \mathrm{dist}(x, y), \quad x, y \in M, \ t \in [0, 1]. \tag{2.141}$$

Consider a sequence y_k that satisfies inequalities (2.139) and define a mapping $y : \mathbb{R} \to M$ by setting

$$y(t) = \phi(t - k, y_k), \quad k \le t < k + 1, \ k \in \mathbb{Z}.$$

Fix a $\tau \in [k, k + 1)$. If $t \in [0, 1]$ and $\tau + t < k + 1$, then

$$\mathrm{dist}\,(y(\tau + t), \phi(t, y(\tau))) = \mathrm{dist}\,(\phi(\tau + t - k, y_k), \phi(t, \phi(\tau - k, y_k))) = 0.$$

If $k + 1 \le \tau + t$, then

$$\mathrm{dist}(y(\tau + t), \phi(t, y(\tau))) = \mathrm{dist}(\phi(\tau + t - k - 1, y_{k+1}), \phi(\tau + t - k, y_k)) =$$

$$= \mathrm{dist}(\phi(\tau + t - k - 1, y_{k+1}), \phi(\tau + t - k - 1, \phi(1, y_k))) \le \nu d.$$

Thus, $y(t)$ is a $(\nu + 1)d$-pseudotrajectory of ϕ. Hence, if $d \le d_0/(\nu + 1)$, where d_0 is from the definition of the Lipschitz shadowing property for ϕ, then there exists a trajectory $x(t)$ of X and a reparametrization

$$\alpha \in \mathrm{Rep}(\mathscr{L}(\nu + 1)d)$$

such that

$$\mathrm{dist}(y(t), x(\alpha(t))) \le \mathscr{L}(\nu + 1)d, \quad t \in \mathbb{R}.$$

If we set

$$x_k = x(\alpha(k)) \text{ and } t_k = \alpha(k+1) - \alpha(k),$$

then

$$x_{k+1} = x(\alpha(k+1)) = \phi(\alpha(k+1) - \alpha(k), x(\alpha(k)) = \phi(t_k, x_k),$$

$$\mathrm{dist}(x_k, y_k) = \mathrm{dist}(x(\alpha(k)), y_k) \le \mathscr{L}(\nu + 1)d,$$

and

$$|t_k - 1| = \left| \frac{\alpha(k+1) - \alpha(k)}{k + 1 - k} - 1 \right| \le \mathscr{L}(\nu + 1)d.$$

Taking $L = \mathscr{L}(\nu + 1)$ and d_0 in Definition 2.7.1 as $d_0/(\nu + 1)$, we complete the proof of the lemma. □

As in Sect. 2.3, we reduce our shadowing problem to the problem of existence of bounded solutions of certain difference equations. To clarify the presentation, we again first take $M = \mathbb{R}^n$, assume that the considered vector field X defines a flow (every trajectory is defined for $t \in \mathbb{R}$), and assume that the diffeomorphism f satisfies Condition S formulated in Sect. 2.3 (see estimate (2.52)). To treat the general case of a compact manifold M, one has to apply exponential mappings (see Remark 2.7.1 below); we leave details to the reader.

As above, we denote

$$\|V\| = \sup_{k \in \mathbb{Z}} |v_k|$$

for a bounded sequence of vectors $V = \{v_k : \ k \in \mathbb{Z}\}$.

Lemma 2.7.2 *Assume that X has the discrete Lipschitz shadowing property with constant L. Let $x(t)$ be an arbitrary trajectory of X, let $p_k = x(k)$, and set $A_k = Df(p_k)$ (recall that $f(x) = \phi(1, x)$). Assume that f satisfies Condition S formulated in Sect. 2.3. Let $B = \{b_k \in \mathbb{R}^n\}$ be a bounded sequence and denote $\beta_0 = \|B\|$.*

Then there exists a sequence of scalars s_k with

$$|s_k| \le \beta = L(\beta_0 + 1)$$

such that the difference equation

$$v_{k+1} = A_k v_k + X(p_{k+1})s_k + b_{k+1} \tag{2.142}$$

has a solution $V = \{v_k\}$ *with*

$$\|V\| \le \beta. \tag{2.143}$$

Proof Fix a natural number N and define $\Delta_k \in \mathbb{R}^n$ as the solution of

$$v_{k+1} = A_k v_k + b_{k+1}, \quad k = -N, \dots, N-1,$$

with $\Delta_{-N} = 0$. Then

$$|\Delta_k| \le C, \quad k = -N, \dots, N, \tag{2.144}$$

where C depends on N, β_0, and an upper bound of $\|A_k\|$ for $k = -N, \dots, N-1$.
Fix a small number $d > 0$ and fix μ in (2.52) so that

$$\mu C < 1. \tag{2.145}$$

Consider the sequence of points $y_k \in \mathbb{R}^n$ defined as follows: $y_k = p_k$ for $k \le -N$,

$$y_k = p_k + d\Delta_k, \quad k = -N, \dots, N-1,$$

and $y_{N+k} = f^k(y_N)$ for $k > 0$.
Then $y_{k+1} = f(y_k)$ for $k \le -N-1$ and $k \ge N$.
Since

$$y_{k+1} = p_{k+1} + d\Delta_{k+1} = p_{k+1} + dA_k\Delta_k + db_{k+1},$$

$$|y_{k+1} - p_{k+1} - dA_k\Delta_k| \le d|b_{k+1}| \le d\beta_0. \tag{2.146}$$

On the other hand, if $dC \le \delta(\mu)$, then it follows from (2.52) that

$$|f(y_k) - p_{k+1} - dA_k\Delta_k| = |f(p_k + d\Delta_k) - f(p_k) - dA_k\Delta_k| \le$$

$$\le \mu|d\Delta_k| \le \mu dC < d \tag{2.147}$$

(see (2.145)).
Combining (2.146) and (2.147), we see that

$$|y_{k+1} - f(y_k)| < d(\beta_0 + 1), \quad k \in \mathbb{Z},$$

if d is small enough. Let us emphasize that the required smallness of d depends on β_0, N, and estimates on $\|A_k\|$.

Now the assumptions of our lemma imply that there exist sequences x_k and t_k such that

$$|t_k - 1| \le d\beta,\ |x_k - y_k| \le d\beta,\ x_{k+1} = \phi(t_k, x_k), \quad k \in \mathbb{Z}.$$

If we represent

$$x_k = p_k + dc_k \text{ and } t_k = 1 + ds_k,$$

then

$$|dc_k - d\Delta_k| = |x_k - y_k| \le d\beta.$$

Thus,

$$|c_k - \Delta_k| \le \beta, \quad -N \le k \le N. \tag{2.148}$$

Clearly,

$$|s_k| \le \beta, \quad k \in \mathbb{Z}. \tag{2.149}$$

Define mappings

$$G_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad k \in \mathbb{Z},$$

by

$$G_k(t, v) = \phi(1 + t, p_k + v) - p_{k+1}.$$

Then

$$G_k(0,0) = 0,\ D_t G_k(t, v)|_{t=0, v=0} = X(p_{k+1}),\ D_v G_k(t, v)|_{t=0, v=0} = A_k.$$

We can write the equality

$$x_{k+1} = \phi(1 + ds_k, x_k)$$

in the form

$$p_{k+1} + dc_{k+1} = \phi(1 + ds_k, p_k + dc_k),$$

which is equivalent to

$$dc_{k+1} = G_k(ds_k, dc_k). \tag{2.150}$$

Now we fix a sequence of values $d = d^{(m)} \to 0,\ m \to \infty$. Let us denote by $c_k^{(m)}$, $t_k^{(m)}$, and $s_k^{(m)}$ the values c_k, t_k, and s_k defined above and corresponding to $d = d^{(m)}$.

It follows from estimates (2.148) and (2.149) that $|c_k^{(m)}| \le C + \beta$ and $|s_k^{(m)}| \le \beta$ for all m and $-N \le k \le N-1$. The second inequality implies that $\left|t_k^{(m)}\right| \le 1$ for large m. Hence (passing to a subsequence, if necessary), we can assume that

$$c_k^{(m)} \to \tilde{c}_k,\ t_k^{(m)} \to \tilde{t}_k,\ s_k^{(m)} \to \tilde{s}_k, \quad m \to \infty,$$

for $-N \le k \le N-1$.

Applying relations (2.150) and (2.149), we can write

$$d_m c_{k+1}^{(m)} = G_k\left(d_m s_k^{(m)}, d_m c_k^{(m)}\right) = A_k d_m c_{k+1}^{(m)} + X(p_{k+1}) d_m s_k^{(m)} + o(d_m).$$

Dividing these equalities by d_m, we get the relations

$$c_{k+1}^{(m)} = A_k c_{k+1}^{(m)} + X(p_{k+1}) s_k^{(m)} + o(1), \quad -N \le k \le N-1.$$

Letting $m \to \infty$, we arrive at the relations

$$\tilde{c}_{k+1} = A_k \tilde{c}_k + X(p_{k+1}) \tilde{s}_k, \quad -N \le k \le N-1,$$

where

$$|\Delta_k - \tilde{c}_k|, |\tilde{s}_k| \le \beta, \quad -N \le k \le N-1,$$

due to (2.148) and (2.149).

Recall that N was fixed in the above reasoning. Denote the obtained $\tilde{s}_k$ by $s_k^{(N)}$. Then $v_k^{(N)} = \Delta_k - \tilde{c}_k$ is a solution of the equations

$$v_{k+1}^{(N)} = A_k v_k^{(N)} + X(p_{k+1}) s_k^{(N)} + b_{k+1}, \quad -N \le k \le N-1,$$

such that $\left|v_k^{(N)}\right| \le \beta$.

There exist subsequences $v_k^{(j_N)} \to v_k'$ and $s_k^{(j_N)} \to s_k'$ as $N \to \infty$ (we do not assume uniform convergence) such that

$$v_{k+1}' = A_k v_k' + X(p_{k+1}) s_k' + b_{k+1}, \quad k \in \mathbb{Z},$$

and $|v_k'|, |s_k'| \le \beta$. The lemma is proved. □

Remark 2.7.1 An analog of Lemma 2.7.2 is valid in the case of a smooth closed manifold M. In this case, we denote $\mathscr{M}_k = T_{p_k} M$ and consider the difference equation (2.142) in which $v_k, b_k \in \mathscr{M}_k$.

Proving an analog of Lemma 2.7.2 in the case of a closed manifold (and replacing, for example, the formula $y_k = p_k + d\Delta_k$ by $y_k = \exp_{p_k}(d\Delta_k)$, compare with the proof of Lemma 2.3.3 in Sec 2.3), one gets a similar statement with the estimates $|s_k| \le \beta := L(2\beta_0 + 1)$ and $\|V\|_\infty \le 2\beta$ (see the original paper [57]).

Thus, in what follows, we refer to Lemma 2.7.2 in the case of a vector field X on a smooth closed manifold M (with $B = \{b_k \in \mathbb{R}^n\}$ replaced by $B = \{b_k \in \mathscr{M}_k\}$ and properly corrected estimates).

2.7.2 Rest Points

In this section, we show that if a vector field has the Lipschitz shadowing property, then its rest points are hyperbolic and isolated in the chain recurrent set. Thus, in what follows we assume that we work with a vector field X on a smooth closed manifold M having the Lipschitz shadowing property.

Lemma 2.7.3 *Every rest point of X is hyperbolic.*

Proof Let x_0 be a rest point. Applying an analog of Lemma 2.7.2 for the case of a manifold with $p_k = x_0$ and noting that $X(p_k) = 0$, we conclude that the difference equation

$$v_{k+1} = Df(x_0)v_k + b_{k+1}$$

has a bounded solution for any bounded sequence $b_k \in \mathscr{M}_k$ (recall that $\mathscr{M}_k = T_{p_k}M$).

Then it follows from the Maizel' theorem (see Theorem 2.1.1 of Sect. 2.1) that the constant sequence $\mathscr{A} = \{A_k = Df(x_0)\}$ is hyperbolic on $\mathbb{Z}_+$; in particular, every bounded solution of the equation

$$v_{k+1} = Df(x_0)v_k$$

tends to 0 as $k \to \infty$.

However, if the rest point x_0 is not hyperbolic, then the matrix $Df(x_0)$ has an eigenvalue on the unit circle, in which case the above equation has a nontrivial solution with constant norm. Thus, x_0 is hyperbolic. □

Lemma 2.7.4 *Rest points are isolated in the chain recurrent set $\mathscr{R}(\phi)$.*

Proof Let us assume that there exists a rest point x_0 that is not isolated in $\mathscr{R}(\phi)$. First we want to show that there is a homoclinic trajectory $x(t)$ associated with x_0.

Since x_0 is hyperbolic by the previous lemma, there exists a small $d > 0$ and a number $a > 0$ such that if $\text{dist}(\phi(t, y), x_0) \le \mathscr{L}d$ for $|t| \ge a$, then $\phi(t, y) \to x_0$ as $|t| \to \infty$.

Assume that there exists a point $y \in \mathscr{R}(\phi)$ such that y is arbitrarily close to x_0 and $y \ne x_0$. Given any $\varepsilon_0, \theta > 0$ we can find points $y_1, \dots, y_N$ and numbers

$T_0, \dots, T_N > \theta$ such that

$$\text{dist}(\phi(T_0, y), y_1) < \varepsilon_0,$$

$$\text{dist}(\phi(T_i, y_i), y_{i+1}) < \varepsilon_0, \quad i = 1, \dots, N,$$

and

$$\text{dist}(\phi(T_N, y_N), y_1) < \varepsilon_0.$$

Set $T = T_0 + \dots + T_N$ and define g^* on $[0, T]$ by

$$g^*(t) = \begin{cases} \phi(t, y), \ 0 \le t \le T_0; \\ \phi(t, y_i), \ T_0 + \dots + T_{i-1} < t < T_0 + \dots + T_i; \\ \quad y, \quad\ t = T. \end{cases}$$

Clearly, for any $\varepsilon > 0$ we can find ε_0 depending only on ε and ν (see (2.141)) such that $g^*(t)$ is an ε-pseudotrajectory of ϕ on $[0, T]$.

Now we define

$$g(t) = \begin{cases} \quad x_0, \quad t \le 0; \\ g^*(t), \ 0 < t \le T; \\ \quad x_0, \quad t > T. \end{cases}$$

We want to choose y and ε in such a way that $g(t)$ is a d-pseudotrajectory of ϕ. We have to show that

$$\text{dist}(\phi(t, g(\tau)), g(t + \tau)) < d \tag{2.151}$$

for all τ and $t \in [0, 1]$.

Clearly, (2.151) holds for (i) $\tau \le -1$, (ii) $\tau \ge T$, (iii) $\tau, \tau + t \in [-1, 0]$, and (iv) $\tau, \tau + t \in [0, T]$ and $\varepsilon < d$.

If $\tau \in [-1, 0]$ and $\tau + t > 0$, then

$$\text{dist}(\phi(t, g(\tau)), g(t + \tau)) = \text{dist}(x_0, g^*(t + \tau)) \le$$

$$\le \text{dist}(x_0, \phi(t + \tau, y)) + \text{dist}(\phi(t + \tau, y), g^*(t + \tau)) \le \nu\text{dist}(x_0, y) + \varepsilon,$$

where ν is as in (2.141). The last value is less than d if $\text{dist}(x_0, y)$ and ε are small enough. Note that, for a fixed y, we can decrease ε and increase $N, T_0, \dots, T_N$ arbitrarily so that $g(t)$ remains a d-pseudotrajectory.

Similarly, (2.151) holds if $\tau \in [0, T]$ and $\tau + t > T$.

Thus, $g(t)$ is $\mathscr{L}d$ shadowed by a trajectory $x(t)$ such that $\text{dist}(x(t), x_0) \le \mathscr{L}d$ if $|t|$ is sufficiently large; hence, $x(t) \to x_0$ as $|t| \to \infty$.

Now we want to show that $x(t)$ is a homoclinic trajectory if d is small enough. For this purpose, we have to show that $x(t) \neq x_0$.

There exists an $\varepsilon_1 > \mathscr{L}d$ (provided that d is small enough) such that if y does not belong to the local stable manifold of x_0, then $\text{dist}(\phi(t_0), y) \geq \varepsilon_1$ for some $t_0 > 0$. We can choose $T_0 > t_0$ (not changing the point y). Then $g(t)$ contains the point $g^*(t_0) = \phi(t_0, y)$ whose distance to x_0 is more than $\mathscr{L}d$. Hence, $x(t)$ contains a point different from x_0, as was claimed.

If y belongs to the local stable manifold of x_0, then it does not belong to the local unstable manifold of x_0. In this case, considering the flow $\psi(t, x) = \phi(-t, x)$, we can apply the above reasoning to ψ noting that $\mathscr{R}(\psi) = \mathscr{R}(\phi)$ and ψ has the Lipschitz shadowing property as well.

Now we show that the existence of this homoclinic trajectory $x(t)$ leads to a contradiction. Set $p_k = x(k)$. Since $A_k X(p_k) = X(p_{k+1})$, it is easily verified that if we consider two sequences β_k and s_k such that

$$\beta_{k+1} = \beta_k + s_k, \quad k \in \mathbb{Z},$$

then $u_k = \beta_k X(p_k)$ is a solution of

$$u_{k+1} = A_k u_k + X(p_{k+1}) s_k, \quad k \in \mathbb{Z}. \tag{2.152}$$

In addition, if the sequence s_k is bounded, then the sequence $\beta_k X(p_k)$ is bounded as well since $X(p_k) \to 0$ exponentially as $|k| \to \infty$ (the trajectory $x(t)$ tends to a hyperbolic rest point as time goes to $\pm\infty$) and the sequence $|\beta_k|/|k|$ is bounded).

By Lemma 2.7.2, for any bounded sequence $b_k \in \mathscr{M}_k$ there exists a bounded scalar sequence s_k such that Eqs. (2.142) have a bounded solution v_k. We have shown that Eqs. (2.152) have a bounded solution u_k. Then the sequence $w_k = v_k - u_k$ is bounded and satisfies the equations

$$w_k = A_k w_k + b_{k+1}, \quad k \in \mathbb{Z}.$$

Thus, the sequence $\mathscr{A} = \{A_k\}$ has the Perron property on $\mathbb{Z}$. It follows from Theorems 2.1.1 and 2.1.2 that the sequence $\mathscr{A}$ is hyperbolic both on $\mathbb{Z}_+$ and $\mathbb{Z}_-$ and the corresponding spaces S_0^+ and U_0^- are transverse. But this leads to a contradiction since

$$\dim S_0^+ + \dim U_0^- = \dim M$$

(because $\dim S_0^+$ equals the dimension of the stable manifold of the hyperbolic rest point x_0 and $\dim U_0^-$ equals the dimension of its unstable manifold), while any of the spaces S_0^+ and U_0^- contains the nonzero vector $X(p_0)$. The lemma is proved. □

2.7.3 Hyperbolicity of the Chain Recurrent Set

We have shown that rest points of ϕ are hyperbolic and isolated in the chain recurrent set $\mathscr{R}(\phi)$. Since M is compact, this implies that the set $\mathscr{R}(\phi)$ is the union of a finite set of hyperbolic rest points and a compact set (let us denote it Σ) on which the vector field X is nonzero.

To show that $\mathscr{R}(\phi)$ is hyperbolic, it remains to show that the set Σ is hyperbolic.

Consider the subbundle $\mathscr{V}(\Sigma)$ of the tangent bundle $TM|_{\Sigma}$ defined in Sect. 1.3 before Theorem 1.3.15.

Let $x(t)$ be a trajectory in Σ. Let us introduce the following notation. Put $p_k = x(k)$ and let $P_k = P(p_k)$ and $V_k = V(p_k)$ (recall that $P(x)$ is the orthogonal projection in T_xM with kernel spanned by $X(x)$ and $V(x)$ is the orthogonal complement to $X(x)$ in T_xM). Introduce the operators

$$B_k = P_{k+1}A_k : \ V_k \to V_{k+1}$$

(recall that $A_k = Df(p_k)$).

Lemma 2.7.5 *For every bounded sequence $b_k \in V_k$ there exists a bounded solution $v_k \in V_k$ of*

$$v_{k+1} = B_k v_k + b_{k+1}, \quad k \in \mathbb{Z}. \tag{2.153}$$

Proof Fix a bounded sequence $b_k \in V_k$. There exist bounded sequences s_k of scalars and w_k of vectors in $T_{p_k}M$ such that

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k + b_{k+1}, \quad k \in \mathbb{Z}, \tag{2.154}$$

(see the remark after Lemma 2.7.2).

Note that $A_k X(p_k) = X(p_{k+1})$. Since $(\mathrm{Id} - P_k)v \in \{X(p_k)\}$ for $v \in \mathscr{M}_k$, we see that

$$P_{k+1}A_k(\mathrm{Id} - P_k) = 0,$$

which gives us the equality

$$P_{k+1}A_k = P_{k+1}A_k P_k. \tag{2.155}$$

The properties of the set Σ imply that the projections P_k are uniformly bounded.

Multiplying (2.154) by P_{k+1}, taking into account the equalities $P_{k+1}X(p_{k+1}) = 0$ and $P_{k+1}b_{k+1} = b_{k+1}$, and applying (2.155), we conclude that $v_k = P_k w_k$ is the required bounded solution of (2.153). The lemma is proved. □

It follows from the above lemma that if we fix a trajectory $x(t)$ in Σ and consider the corresponding sequence of operators $\mathscr{B} = \{B_k\}$, then $\mathscr{B}$ has the Perron property.

By Theorems 2.1.1 and 2.1.2, the sequence $\mathscr{B}$ is hyperbolic both on $\mathbb{Z}_-$ and $\mathbb{Z}_+$ and the corresponding spaces $U_0^-(\mathscr{B})$ and $S_0^+(\mathscr{B})$ are transverse.

Consider the mapping π on the normal bundle $\mathscr{V}(\Sigma)$ defined in Sect. 1.3. Recall that

$$\pi(x, v) = (f(x), B(x)v), \text{ where } B(x) = P(f(x))Df(x)$$

(see Sect. 1.3).

In fact, we have shown that π satisfies an analog of the strong transversality condition.

The same reasoning as in the proof of Lemma 2.2.5 shows that the dual mapping π^* does not have nontrivial bounded trajectories. It is easy to show that if the flow ϕ has the shadowing property, then its nonwandering set coincides with its chain recurrent set.

Hence, we can repeat the reasoning of the proof of Theorem 2.2.2 to conclude that the mapping π is hyperbolic.

It remains to refer to Theorem 1.3.15 to conclude that Σ is a hyperbolic set of the flow ϕ.

2.7.4 Transversality of Stable and Unstable Manifolds

Let $x(t)$ be a trajectory that belongs to the intersection of the stable and unstable manifolds of two trajectories, $x_+(t)$ and $x_-(t)$, respectively, lying in the chain recurrent set of ϕ.

Without loss of generality, we may assume that

$$\text{dist}(x(0), x_+(0)) \to 0, \quad t \to \infty,$$

and

$$\text{dist}(x(0), x_-(0)) \to 0, \quad t \to -\infty.$$

Denote $p_k = x(k)$, $k \in \mathbb{Z}$; let $W^s(p_0)$ and $W^u(p_0)$ be the stable and unstable manifolds of p_0, respectively. Then, of course, $W^s(p_0) = W^s(x_+(0))$ and $W^u(p_0) = W^u(x_-(0))$. Denote by E^s and E^u the tangent spaces of $W^s(p_0)$ and $W^u(p_0)$ at p_0.

We use the notation introduced before Lemma 2.7.5.

By Lemma 2.7.5, for any bounded sequence $b_k \in V_k$ there exists a bounded solution $v_k \in V_k$ of (2.153). By the Maizel' theorem (Theorem 2.1.1), the sequence B_k is hyperbolic on $\mathbb{Z}_-$ and $\mathbb{Z}_+$.

By the Pliss theorem (Theorem 2.1.2),

$$\mathscr{E}^s + \mathscr{E}^u = V_0, \tag{2.156}$$

where

$$\mathscr{E}^s = \{w_0 : \ w_{k+1} = B_k w_k, \ |w_k| \to 0, \ k \to \infty\}$$

and

$$\mathscr{E}^u = \{w_0 : \ w_{k+1} = B_k w_k, \ |w_k| \to 0, k \to -\infty\}.$$

Clearly, it follows from the hyperbolicity of the sequence B_k on $\mathbb{Z}_-$ and $\mathbb{Z}_+$ that the following equalities hold:

$$\mathscr{E}^s = \{w_0 : \ w_{k+1} = B_k w_k, \ \sup_{k \geq 0} |w_k| < \infty\}$$

and

$$\mathscr{E}^u = \{w_0 : \ w_{k+1} = B_k w_k, \ \sup_{k \leq 0} |w_k| < \infty\}.$$

We claim that

$$\mathscr{E}^s \subset E^s \text{ and } E^u \subset \mathscr{E}^u. \tag{2.157}$$

First we note that (2.157) implies the desired transversality of $W^s(p_0)$ and $W^u(p_0)$ at p_0.

Indeed, combining equality (2.156) with inclusions (2.157) and the trivial relations

$$E^s = V_0 \cap E^s + \{X(p_0)\} \text{ and } E^u = V_0 \cap E^u + \{X(p_0)\},$$

we conclude that

$$E^s + E^u = T_{p_0}M,$$

which gives us the transversality of $W^s(p_0)$ and $W^u(p_0)$ at p_0.

Thus, it remains to prove inclusions (2.157). We prove the first inclusion; for the second one, the proof is similar.

Case 1: The limit trajectory $x_0(t) = x_0$ is a rest point of X. In this case, the stable manifold of the rest point x_0 in the flow ϕ coincides with the stable manifold of the fixed point x_0 for the time-one diffeomorphism $f(x) = \phi(1, x)$.

It is clear that if p_k is a trajectory of f belonging to the stable manifold of x_0, then the tangent space to the stable manifold at p_0 is the subspace E^s of the initial values of bounded solutions of

$$v_{k+1} = A_k v_k, \quad k \geq 0. \tag{2.158}$$

Let us prove that $\mathscr{E}^s \subset E^s$. Fix an arbitrary sequence w_k such that $w_{k+1} = B_k w_k$ and $w_0 \in \mathscr{E}^s$. Consider the sequence

$$v_k = \lambda_k X(p_k)/|X(p_k)| + w_k,$$

where the λ_k satisfy the relations

$$\lambda_{k+1} = \frac{|X(p_{k+1})|}{|X(p_k)|}\lambda_k + \frac{X(p_{k+1})^*}{|X(p_{k+1})|}A_k w_k \tag{2.159}$$

(we denote by X^* the row-vector corresponding to the column-vector X) and $\lambda_0 = 0$. It is easy to see that the sequence v_k satisfies (2.158).

Since $x(t)$ is in the stable manifold of the hyperbolic rest point x_0, there exist positive constants K and α such that

$$\left|\frac{dx}{dt}(t)\right| \leq K\exp(\alpha(t-s))\left|\frac{dx}{dt}(s)\right|, \quad 0 \leq s \leq t.$$

This implies that

$$|X(p_k)| \leq K\exp(\alpha(k-m))\,|X(p_m)|\,, \quad 0 \leq m \leq k.$$

Thus, the scalar difference equation

$$\lambda_{k+1} = \frac{|X(p_{k+1})|}{|X(p_k)|}\lambda_k$$

is hyperbolic on $\mathbb{Z}_+$ and is, in fact, stable. Since the second term on the right in (2.159) is bounded as $k \to \infty$ (recall that we take $w_0 \in \mathscr{E}^s$), it follows that the λ_k are bounded for any choice of λ_0.

We conclude that v_k is a bounded solution of (2.158), and $v_0 = w_0 \in E^s$. Thus, we have shown that $\mathscr{E}^s \subset E^s$, which completes the proof in Case 1.

Case 2: The limit trajectory is in the set Σ (the chain recurrent set minus rest points). We know that the set Σ is hyperbolic. Our goal is to find the intersection of its stable manifold near $p_0 = x(0)$ with the cross-section at p_0 orthogonal to the vector field (in local coordinates generated by the exponential mapping). To do this, we discretize the problem and note that there exists a number $\sigma > 0$ such that a point p close to p_0 belongs to $W^s(p_0)$ if and only if the distances of the consecutive points of intersections of the positive semitrajectory of p to the points p_k do not exceed σ.

For suitably small $\mu > 0$ we find all the sequences of numbers t_k and vectors $z_k \in V_k$ (recall that V_k is the orthogonal complement to $\{X(p_k)\}$ at p_k) such that

$$|t_k - 1| \leq \mu, \ |z_k| \leq \mu, \ y_{k+1} = \phi(t_k, y_k), \quad k \geq 0,$$

where $y_k = p_k + z_k$.

Thus, we have to solve the equations

$$p_{k+1} = \phi(t_k, p_k + v_k), \quad k \geq 0,$$

for numbers t_k and vectors $z_k \in V_k$ such that $|t_k - 1| \leq \mu$ and $|z_k| \leq \mu$.

We reduce this problem to an equation in a Banach space. It was mentioned above that the sequence $\{B_k\}$ generating the difference equation

$$z_k = B_k z_k, \quad k \geq 0,$$

(where $B_k = P_{k+1}A_k$ and P_k is the orthogonal projection with range V_k) is hyperbolic on $\mathbb{Z}_+$. Denote by $Q_k : V_k \to V_k$ the corresponding projections to the stable subspaces and by $\mathscr{R}(Q_0)$ the range of Q_0 (note that $\mathscr{R}(Q_0) = \mathscr{E}^s$).

Fix a positive number μ_0 and denote by $\mathscr{V}$ the space of sequences

$$\mathscr{V} = \{z_k \in V_k : \ |z_k| \leq \mu_0, \ k \in \mathbb{Z}_+\}.$$

Let $l^\infty(\mathbb{Z}_+, \{\mathscr{M}_{k+1}\})$ be the space of sequences $\{\zeta_k \in \mathscr{M}_{k+1} : \ k \in \mathbb{Z}_+\}$ with the usual norm.

Define a C^1 function

$$G : \ [1 - \mu_0, 1 + \mu_0]^{\mathbb{Z}_+} \times \mathscr{V} \times \mathscr{R}(Q_0) \to l^\infty(\mathbb{Z}_+, \{\mathscr{M}_{k+1}\}) \times \mathscr{R}(Q_0)$$

by

$$G(t, z, \eta) = (\{p_{k+1} + z_{k+1} - \phi(t_k, p_k + z_k)\}, Q_0 z_0 - \eta).$$

This function is defined if μ_0 is small enough.

We want to solve the equation

$$G(t, z, \eta) = 0$$

for (t, z) as a function of η. It is clear that

$$G(1, 0, 0) = 0,$$

where the first argument of G is $\{1, 1, \ldots\}$, the second argument is $\{0, 0, \ldots\}$, and the right-hand side is $(\{0, 0, \ldots\}, 0)$.

To apply the implicit function theorem, we must verify that the operator

$$T = \frac{\partial G}{\partial(t,z)}(1,0,0)$$

is invertible.

First we note that if $(s,w) \in l^\infty(\mathbb{Z}_+, \{\mathscr{M}_{k+1}\}) \times \mathscr{V}$, then

$$T(s,w) = (\{w_{k+1} - X(p_{k+1})s_k - A_k w_k\}, Q_0 w_0).$$

To show that T is invertible, we have to show that the equation

$$T(s,w) = (g,\eta)$$

has a unique solution for any $(g,\eta) \in l^\infty(\mathbb{Z}_+, \{\mathscr{M}_{k+1}\}) \times \mathscr{R}(Q_0)$. Thus, we have to solve the equation

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k = g_k, \quad k \geq 0, \tag{2.160}$$

subject to the condition

$$Q_0 w_0 = \eta.$$

If we multiply Eq. (2.160) by $X(p_{k+1})^*$ and solve for s_k, we get the equalities

$$s_k = -\frac{X(p_{k+1})^*}{|X(p_{k+1})|^2}[A_k w_k + g_k], \quad k \geq 0,$$

and if we multiply Eq. (2.160) by P_{k+1}, we get the equalities

$$w_{k+1} = P_{k+1}A_k w_k + P_{k+1}g_k = B_k w_k + P_{k+1}g_k, \quad k \geq 0.$$

Now we know that the last equations have a unique bounded solution $w_k \in V_k,\ k \geq 0$, that satisfies $Q_0 w_0 = \eta$. Thus, T is invertible.

Hence, we can apply the implicit function theorem to show that there exists a $\mu > 0$ such that if $|\eta|$ is sufficiently small, then the equation $G(t,z,\eta) = 0$ has a unique solution $(t(\eta), z(\eta))$ such that $\|t-1\|_\infty \leq \mu$ and $\|z\|_\infty \leq \mu$. Moreover, $t(0) = 1$, $z(0) = 0$, and the functions $t(\eta)$ and $z(\eta)$ are of class C^1.

The points $p_0 + z_0(\eta)$ form a submanifold containing p_0 and contained in $W^s(p_0)$. Thus, the range of the derivative $z_0'(0)$ is contained in E^s.

Take an arbitrary vector $\xi \in \mathscr{E}^s$ and consider $\eta = \tau\xi,\ \tau \in \mathbb{R}$. Differentiating the equalities

$$p_{k+1} + z_{k+1}(\tau\xi) = \phi(t_k(\tau\xi), p_k + z_k(\tau\xi)), \quad k \geq 0,$$

and

$$Q_0(\tau\xi) = \tau\xi$$

with respect to τ at $\tau = 0$, we see that

$$s_k = \frac{\partial t_k}{\partial \eta}|_{\eta=0}\xi \text{ and } w_k = \frac{\partial z_k}{\partial \eta}|_{\eta=0}\xi \in V_k$$

are bounded sequences satisfying the equalities

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k \text{ and } Q_0 w_0 = \xi.$$

Multiplying by P_{k+1}, we conclude that

$$w_{k+1} = B_k w_k \text{ and } Q_0 w_0 = \xi.$$

It follows that $w_0 \in \mathscr{E}^s = \mathscr{R}(Q_0)$. Then $w_0 = Q_0 w_0 = \xi$.

We have shown that the range of $z_0'(0)$ is exactly $\mathscr{E}^s$. Thus, $\mathscr{E}^s \subset E^s$.

Historical Remarks Theorem 2.7.1 was published by K. Palmer, the first author, and S. B. Tikhomirov in [57].